UNIFORM-IN-TIME ESTIMATES ON THE SIZE OF CHAOS FOR INTERACTING BROWNIAN PARTICLES

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Abstract. We consider a system of classical Brownian particles interacting via a smooth long-range potential in the mean-field regime, and we analyze the propagation of chaos in form of sharp, uniform-in-time estimates on many-particle correlation functions. Our results cover both the kinetic Langevin setting and the corresponding overdamped Brownian dynamics. The approach is mainly based on so-called Lions expansions, which we combine with new diagrammatic tools to capture many-particle cancellations, as well as with fine ergodic estimates on the linearized mean-field equation, and with discrete stochastic calculus with respect to initial data. In the process, we derive some new ergodic estimates for the linearized Vlasov–Fokker–Planck kinetic equation that are of independent interest. Our analysis also leads to uniform-in-time concentration estimates and to a uniform-in-time quantitative central limit theorem for the empirical measure associated with the particle dynamics.

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1. Introduction

1.1. **General overview.** We consider the Langevin dynamics for a system of N Brownian particles with mean-field interactions, moving in a confining potential in $\mathbb{R}^d$, $d \geq 1$, as described by the following system of coupled SDEs: for $1 \leq i \leq N$,

$$\begin{cases} \mathrm{d}X_t^{i,N} = V_t^{i,N}\mathrm{d}t, \\ \mathrm{d}V_t^{i,N} = -\frac{\kappa}{N}\sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N})\,\mathrm{d}t - \frac{\beta}{2}V_t^{i,N}\mathrm{d}t - \nabla A(X_t^{i,N})\,\mathrm{d}t + \mathrm{d}B_t^i, \qquad t \geq 0, \\ (X_t^{i,N}, V_t^{i,N})|_{t=0} = (X_\circ^{i,N}, V_\circ^{i,N}), \end{cases} \tag{1.1}$$

where $\{Z^{i,N} := (X^{i,N}, V^{i,N})\}_{1\leq i\leq N}$ is the set of particle positions and velocities in the phase space $\mathbb{D}^d := \mathbb{R}^d \times \mathbb{R}^d$, where $W : \mathbb{R}^d \to \mathbb{R}$ is a long-range interaction potential, where A is a uniformly convex confining potential, where $\{B^i\}_i$ are i.i.d. d-dimensional Brownian motions, and where $\kappa, \beta > 0$

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are given constants. The interaction potential W is assumed to satisfy the action-reaction condition $W(x) = W(-x)$, and we assume that it is smooth, $W \in C_b^\infty(\mathbb{R}^d)$. Regarding the confining potential A, we choose it to be quadratic for simplicity,

$$A(x) := \tfrac{1}{2}a|x|^2, \qquad \text{for some } a > 0, \tag{1.2}$$

although this is not essential for our results; see Remark 1.5 below. Next to this Langevin dynamics, we also consider its overdamped limit, that is, the following inertialess Brownian dynamics: for $1 \le i \le N$,

$$\begin{cases} \mathrm{d}Y_t^{i,N} = -\frac{\kappa}{N}\sum_{j=1}^N \nabla W(Y_t^{i,N} - Y_t^{j,N})\,\mathrm{d}t - \nabla A(Y_t^{i,N})\,\mathrm{d}t + \mathrm{d}B_t^i, \qquad t \ge 0, \\ Y_t^{i,N}|_{t=0} = Y_\circ^{i,N}, \end{cases} \tag{1.3}$$

where $\{Y^{i,N}\}_{1\le i\le N}$ is now the corresponding set of particle positions in $\mathbb{R}^d$. For presentation purposes in this introduction, we restrict to the more delicate setting of the Langevin dynamics (1.1), but we emphasize that all our results hold in both cases.

In the regime of a large number $N \gg 1$ of particles, let us turn to a statistical description of the system and consider the evolution of a random ensemble of particles. In terms of a probability density F^N on the N-particle phase space $(\mathbb{D}^d)^N$, the Langevin dynamics (1.1) is equivalent to the Liouville equation

$$\partial_t F^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} F^N = \tfrac{1}{2}\sum_{i=1}^N \mathrm{div}_{v_i}((\nabla_{v_i} + \beta v_i)F^N) + \tfrac{\kappa}{N}\sum_{i,j=1}^N \nabla W(x_i - x_j)\cdot\nabla_{v_i}F^N + \sum_{i=1}^N \nabla_{x_i}A\cdot\nabla_{v_i}F^N. \tag{1.4}$$

Particles are assumed to be exchangeable, which amounts to the symmetry of F^N in its N variables $z_i = (x_i, v_i) \in \mathbb{D}^d$, $1 \le i \le N$. More precisely, we assume for simplicity that particles are initially chaotic, meaning that the initial data $\{Z_\circ^{i,N} := (X_\circ^{i,N}, V_\circ^{i,N})\}_{1\le i\le N}$ are i.i.d. with some common phase-space density $\mu_\circ \in \mathcal{P}(\mathbb{D}^d)$: in other words, F^N is initially tensorized,

$$F_t^N|_{t=0} = \mu_\circ^{\otimes N}. \tag{1.5}$$

In the large-N limit, we aim at an averaged description of the system and we focus on the evolution of a finite number of "typical" particles as described by the marginals of F^N,

$$F_t^{m,N}(z_1,\ldots,z_n) := \int_{(\mathbb{D}^d)^{N-m}} F_t^N(z_1,\ldots,z_N)\,\mathrm{d}z_{m+1}\ldots\mathrm{d}z_N, \qquad 1 \le m \le N.$$

In view of Boltzmann's chaos assumption, correlations between particles are expected to be negligible to leading order, hence the chaotic behavior of initial data would remain approximately satisfied: this is the so-called propagation of chaos,

$$F_t^{m,N} - (F_t^{1,N})^{\otimes m} \to 0, \qquad \text{as } N \uparrow \infty, \tag{1.6}$$

for any fixed $m \ge 1$ and $t \ge 0$. If this holds, it automatically implies the validity of the mean-field limit

$$F_t^{m,N} \to \mu_t^{\otimes m}, \qquad \text{as } N \uparrow \infty,$$

where μ_t is the solution of the Vlasov–Fokker–Planck mean-field equation

$$\begin{cases} \partial_t \mu + v\cdot\nabla_x\mu = \frac{1}{2}\mathrm{div}_v((\nabla_v + \beta v)\mu) + (\nabla A + \kappa\nabla W * \mu)\cdot\nabla_v\mu, \qquad t \ge 0, \\ \mu|_{t=0} = \mu_\circ, \end{cases} \tag{1.7}$$

with the short-hand notation $\nabla W * \mu(x) := \int_{\mathbb{D}^d} \nabla W(x-y)\,\mu(y,v)\,\mathrm{d}y\,\mathrm{d}v$. This topic has been extensively investigated since the 1990s, starting in particular with [52, 85]; see e.g. [63, 23] for a review.

On the formal level, corrections to the propagation of chaos and to the mean-field limit are naturally unravelled by means of the BBGKY approach, which goes back to the work of Bogolyubov [7]. This starts by noting that the Liouville equation (1.4) is equivalent to the following hierarchy of coupled equations for marginals: for $1 \le m \le N$,

$$\partial_t F^{m,N} + \sum_{i=1}^m v_i \cdot \nabla_{x_i} F^{m,N} \,=\, \tfrac12 \sum_{i=1}^m \operatorname{div}_{v_i}((\nabla_{v_i} + \beta v_i) F^{m,N}) \\ + \sum_{i=1}^m \nabla_{x_i} A \cdot \nabla_{v_i} F^{m,N} + \kappa \tfrac{N-m}{N} \sum_{i=1}^m \int_{\mathbb{D}^d} \nabla W(x_i - x_*) \cdot \nabla_{v_i} F^{m+1,N}(\cdot, z_*)\, dx_*\, dv_* \\ + \tfrac{\kappa}{N} \sum_{i,j=1}^m \nabla W(x_i - x_j) \cdot \nabla_{v_i} F^{m,N}, \quad (1.8)$$

with the convention $F^{m,N} \equiv 0$ for $m > N$. In each of those equations, the last right-hand side term is precisely the one that disrupts the chaotic structure: it creates correlations between initially independent particles, hence leads to deviations from the mean-field approximation. As this term is formally of order $O(m^2 N^{-1})$, we are led to conjecture the following error estimate for the propagation of chaos,

$$F_t^{m,N} - (F_t^{1,N})^{\otimes m} \,=\, O(m^2 N^{-1}). \quad (1.9)$$

This was first made rigorous in [79] for several related particle systems, and it is referred to as estimating the *size of chaos*. For the particle systems of interest in the present work, (1.1) or (1.3), a rigorous BBGKY analysis can be performed at least to some extent to deduce similar estimates, cf. [67, 12, 62]. In case of non-Brownian interacting particles ($\beta = 0$), the problem is more difficult and was solved in [40] by means of different techniques.

A variant of the above estimates on the size of chaos is given by so-called *weak propagation of chaos* estimates: for any sufficiently well-behaved functional Φ defined on the space $\mathcal{P}(\mathbb{D}^d)$ of probability measures on $\mathbb{D}^d$, one expects

$$\mathbb{E}\left[\Phi(\mu_t^N)\right] - \Phi(\mu_t) \,=\, O(N^{-1}), \quad (1.10)$$

in terms of the empirical measure

$$\mu_t^N := \tfrac{1}{N} \textstyle\sum_{i=1}^N \delta_{Z_t^{i,N}} \quad \in \mathcal{P}(\mathbb{D}^d), \quad (1.11)$$

where we recall that the limit μ_t is the solution of the mean-field equation (1.7) and where the expectation $\mathbb{E}$ is taken with respect to both the initial data and the Brownian forces. Such an estimate is essentially equivalent to (1.9) (up to the precise dependence on m and Φ), and we refer to [64, 73, 75, 5, 25] for results in that direction. Note that the rate $O(N^{-1})$ in (1.10) is only expected for Φ smooth enough. For the specific choice $\Phi = \mathcal{W}_2(\cdot, \mu_t)$, for instance, the question amounts to estimating the expectation of the 2-Wasserstein distance between μ_t^N and μ_t: this is referred to as *strong propagation of chaos* and is known to lead only to a weaker convergence rate $\mathbb{E}\left[\mathcal{W}_2(\mu_t^N, \mu_t)\right] = O(N^{-1/2})$ in link with random fluctuations of the empirical measure; see [3, 72, 10, 22, 49, 67].

In recent years, there has been increasing interest in uniform-in-time versions of the above chaos estimates (1.9) or (1.10). This happens to be an important question both in theory and for practical applications: it amounts to describing the long-time behavior of particle systems uniformly in the limit $N \uparrow \infty$, thus showing in particular the proximity of corresponding equilibria. This is naturally related to the long-time behavior of the mean-field equation (1.7), which has itself been an intense topic of research for more than two decades. While the mean-field equilibrium is not unique in general, cf. [44], the long-time convergence of the mean-field density has been established under several types of assumptions guaranteeing uniqueness [88, 9, 61, 76, 54, 2]; see also [4, 20, 8] for the Brownian dynamics. In contrast, uniform-in-time propagation of chaos is a more subtle question and is indeed not ensured by the uniqueness of the mean-field equilibrium [72, 6]. Uniform-in-time weak chaos estimates with optimal rate $O(N^{-1})$ were first obtained by Delarue and Tse [35] for the Brownian dynamics, and we refer to [72, 10, 22, 9, 45, 81, 54, 27] for corresponding uniform-in-time *strong* chaos estimates with weaker convergence rates. We also refer to [56, 80, 55, 29] for some uniform-in-time chaos estimates in case of singular interactions.

In the present work, we aim to go beyond uniform-in-time chaos estimates by further estimating many-particle correlation functions, which provide finer information on the propagation of chaos in the system. The two-particle correlation function is defined as

$$G^{2,N} := F^{2,N} - (F^{1,N})^{\otimes 2},$$

which captures the defect to propagation of chaos (1.6) at the level of two-particle statistics. From the BBGKY hierarchy (1.8), we note that proving the mean-field limit $F_t^{1,N} \to \mu_t$ amounts to proving $G_t^{2,N} \to 0$, which is precisely ensured by standard chaos estimates, cf. (1.9). Yet, two-particle correlations do not allow to reconstruct the full particle density F^N: in particular, understanding corrections to the mean-field limit requires to further estimate higher-order correlation functions $\{G^{k,N}\}_{2\le k\le N}$. Those are defined as suitable polynomial combinations of marginals of F^N in such a way that the full particle distribution F^N be recovered in form of a cluster expansion,

$$F^N(z_1,\ldots,z_N) = \sum_{\pi\vdash[\![N]\!]} \prod_{A\in\pi} G^{\sharp A,N}(z_A), \tag{1.12}$$

where π runs through the list of all partitions of the index set $[\![N]\!] := \{1,\ldots,N\}$, where A runs through the list of blocks of the partition π, where $\sharp A$ is the cardinality of A, and where for $A = \{i_1,\ldots,i_l\} \subset [\![N]\!]$ we write $z_A = (z_{i_1},\ldots,z_{i_l})$. As is easily checked, correlation functions are fully determined by prescribing (1.12) together with the "maximality" requirement $\int_{\mathbb{D}} G^{m,N}(z_1,\ldots,z_m)\,dz_l = 0$ for $1 \le l \le m$. More explicitly, we can write

$$\begin{aligned} G^{3,N} &= \operatorname{sym}\big(F^{3,N} - 3F^{2,N}\otimes F^{1,N} + 2(F^{1,N})^{\otimes 3}\big),\\ G^{4,N} &= \operatorname{sym}\big(F^{4,N} - 4F^{3,N}\otimes F^{1,N} - 3F^{2,N}\otimes F^{2,N} + 12F^{2,N}\otimes(F^{1,N})^{\otimes 2} - 6(F^{1,N})^{\otimes 4}\big), \end{aligned}$$

and so on, where the symbol 'sym' stands for the symmetrization of coordinates. More generally, we can write for all $2\le m\le N$,

$$G^{m,N}(z_1,\ldots,z_m) = \sum_{\pi\vdash[\![m]\!]} (\sharp\pi-1)!(-1)^{\sharp\pi-1} \prod_{A\in\pi} F^{\sharp A,N}(z_A), \tag{1.13}$$

where we use a similar notation as in (1.12) and where $\sharp\pi$ stands for the number of blocks in a partition π. While standard propagation of chaos leads to $G_t^{2,N} = O(N^{-1})$, cf. (1.9), and in fact $G_t^{m,N} = O(N^{-1})$ for all $2\le m\le N$, a formal analysis of the BBGKY hierarchy (1.8) further leads to expect

$$G_t^{m,N} = O(N^{1-m}), \qquad 2\le m\le N. \tag{1.14}$$

We shall refer to this as *higher-order propagation of chaos*: such estimates provide a much deeper understanding of the structure of propagation of chaos and are key tools to describe deviations from mean-field theory, cf. [39, 40, 62, 43]. Such estimates have been obtained in several settings with an exponential time growth: non-Brownian particle systems ($\beta = 0$) were covered in [40], while in [62] the Brownian dynamics (1.3) was covered in the more general case of bounded non-smooth interactions. In the present work, we obtain for the first time corresponding *uniform-in-time* estimates, both for the Langevin and Brownian dynamics. Along the way, we also establish uniform-in-time concentration estimates and a uniform-in-time central limit theorem for the empirical measure.

From the technical perspective, we mainly take inspiration from a recent work of Delarue and Tse [35] (see also [19, 25, 24]), where the uniform-in-time weak propagation of chaos (1.10) was established for the Brownian dynamics. The key idea of the analysis is to consider the mean-field semigroup induced on functionals $\mu\mapsto\Phi(\mu)$ on the space of probability measures, and then appeal to so-called Lions calculus on this space to expand the expectation $\mathbb{E}[\Phi(\mu_t^N)]$ of functionals along the particle dynamics (see Lemma 2.1 below). As noted in [35], the resulting so-called Lions expansions can be combined with ergodic properties of the linearized mean-field equation to deduce uniform-in-time estimates. In the present contribution, in order to control correlation functions $\{G^{m,N}\}_{2\le m\le N}$, we reduce the problem to estimating cumulants of functionals of the empirical measure. We apply Lions expansions to cumulants and we develop suitable diagrammatic tools to efficiently capture cancellations and derive the desired estimates (1.14). To account for the effect of initial correlations, we further combine Lions expansions with the so-called Glauber calculus that we developed in [40]. While only the case of the Brownian dynamics was considered in [35], note that we need to further appeal to hypocoercivity techniques to establish the relevant ergodic estimates for the linearized mean-field equation in case of the kinetic Langevin dynamics: for that purpose, we mainly draw inspiration from the work of Mischler and Mouhot [74], which we are led to revisit in several ways (see Theorem 3.1).

1.2. **Main results.** We start with uniform-in-time higher-order propagation of chaos estimates (1.14). We focus on the Langevin dynamics (1.1), but we emphasize that all our results also hold in the simpler case of the Brownian dynamics (1.3).[1] The smallness assumption $\kappa \ll 1$ for the interaction strength ensures the uniqueness of the steady state for the mean-field equation (1.7), which is useful to ensure strong ergodic properties.

Theorem 1.1 (Uniform-in-time higher-order propagation of chaos)**.** *There exists $\kappa_0 > 0$ (only depending on d, β, W, A) such that the following holds for any $\kappa \in [0, \kappa_0]$. Assume that the initial law $\mu_\circ \in \mathcal{P}(\mathbb{D}^d)$ satisfies $\int_{\mathbb{D}^d} |z|^{p_0} \mu_\circ(\mathrm{d}z) < \infty$ for some $p_0 > 0$, and consider the Langevin dynamics (1.1) and the associated correlation functions $\{G^{m,N}\}_{2 \le m \le N}$ as defined in (1.13). For all $2 \le m \le N$, there exist $\ell_m > 0$ (only depending on m) and $C_m > 0$ (only depending on $d, \beta, W, A, \mu_\circ, m$) such that we have for all $t \ge 0$,*

$$\|G_t^{m,N}\|_{W^{-\ell_m,1}(\mathbb{D}^d)} \le C_m N^{1-m}. \tag{1.15}$$

In particular, the uniform-in-time smallness of the two-particle correlation function $G_t^{2,N} = O(N^{-1})$ allows to truncate the BBGKY hierarchy (1.8) and to recover the uniform-in-time validity of propagation of chaos: for all $1 \le m \le N$ and $t \ge 0$,

$$\|F_t^{m,N} - \mu_t^{\otimes m}\|_{W^{-\ell_m,1}(\mathbb{D}^d)} \le C_m N^{-1}. \tag{1.16}$$

Corrections to this mean-field approximation can be further captured by truncating the BBGKY hierarchy (1.8) to higher orders as e.g. in [40]. In fact, by a suitable analysis of those corrections, it is possible to deduce the following improvement of (1.16), which we state here for simplicity in case $m = 1$: for all $k \ge 1$, there exist $\lambda_k, \ell_k, C_k > 0$ such that we have for all $N \ge 1$ and $t \ge 0$,

$$\|F_t^{1,N} - \mu_t\|_{W^{-\ell_k,1}(\mathbb{D}^d)} \le C_k(e^{-\lambda_k t} N^{-1} + N^{-k}). \tag{1.17}$$

To our knowledge, this $O(e^{-\lambda t} N^{-1} + N^{-\infty})$ estimate constitutes a new type of result in mean-field theory, which can be viewed as combining the mean-field approximation (1.16) quantitatively with the convergence of the particle system to Gibbs equilibrium. The proof of this refined estimate (1.17) as an application of Theorem 1.1 requires detailed computations of corrections to mean field and is postponed to a forthcoming work.

The strategy of the proof of Theorem 1.1 can be taken further to derive uniform-in-time concentration estimates and a quantitative central limit theorem for the empirical measure. We start with concentration estimates. For the Langevin dynamics (1.1), the following result completes the concentration estimates obtained in [9, Theorem 5]. In the simpler setting of the Brownian dynamics (1.3), corresponding results were already well-known: a uniform-in-time concentration estimate was first deduced in [71] from a logarithmic Sobolev inequality in the case when W is convex, and it was largely extended more recently in [70].

Theorem 1.2 (Uniform-in-time concentration)**.** *There exists $\kappa_0 > 0$ (only depending on d, β, W, A) such that the following holds for any $\kappa \in [0, \kappa_0]$. Assume that the initial law $\mu_\circ \in \mathcal{P}(\mathbb{D}^d)$ is compactly supported, and consider the Langevin dynamics (1.1) and the associated empirical measure μ_t^N,*

[1] In fact, in case of the Brownian dynamics (1.3), we can choose $p_0 = 0$ in our different results, meaning that no moment assumption is needed on the initial law $\mu_\circ$.

cf. (1.11). *For all* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *(only depending on* $d, \beta, W, A, \varepsilon, \mu_\circ$*) such that the following holds: for all* $\phi \in C_c^\infty(\mathbb{D}^d)$ *and* $N, t, r \geq 0$,

$$\mathbb{P}\left[\int_{\mathbb{D}^d} \phi \mu_t^N - \mathbb{E}\Big[\int_{\mathbb{D}^d} \phi \mu_t^N\Big] \geq r\right] \leq \exp\left(-\frac{Nr^2}{C_\varepsilon \|\phi\|_{W^{2+\varepsilon,\infty}(\mathbb{D}^d)}}\right),$$

provided that $r \leq (eC_\varepsilon \|\phi\|_{W^{2+\varepsilon,\infty}(\mathbb{D}^d)})^{1/2}$ *and that* $r(Nr^2)^\varepsilon \leq e^{1/2}(eC_\varepsilon \|\phi\|_{W^{2+\varepsilon,\infty}(\mathbb{D}^d)})^\varepsilon$.

Finally, we state our uniform-in-time quantitative central limit theorem (CLT) for leading fluctuations of the empirical measure. As expected from formal computations, leading fluctuations are described by the Gaussian linearized Dean–Kawasaki SPDE, cf. (1.19) below. A qualitative CLT has actually been known to hold since the early days of mean-field theory [86, 84, 48], and it has recently been extended to some singular interaction potentials as well [90, 28]. In case of smooth interactions, as considered in the present work, an optimal quantitative estimate for fluctuations already follows from [40], but we provide here the first uniform-in-time result. To our knowledge, this is new both in the Langevin and in the Brownian cases.

Theorem 1.3 (Uniform-in-time CLT)**.** *There exist* $\kappa_0, \lambda_0 > 0$ *(only depending on* d, β, W, A*) such that the following holds for any* $\kappa \in [0, \kappa_0]$. *Assume that the initial law* $\mu_\circ \in \mathcal{P}(\mathbb{D}^d)$ *satisfies* $\int_{\mathbb{D}^d} |z|^{p_0} \mu_\circ(dz) < \infty$ *for some* $0 < p_0 \leq 1$, *and consider the Langevin dynamics* (1.1) *and the associated empirical measure* μ_t^N, *cf.* (1.11). *For all* $\phi \in C_c^\infty(\mathbb{D}^d)$, *there exists* $C_\phi > 0$ *(only depending on* $d, \beta, W, A, \phi, \mu_\circ$*) such that for all* $N, t \geq 0$ *we have*

$$\mathrm{d}_2\bigg(N^{\frac12} \int_{\mathbb{X}} \phi\,(\mu_t^N - \mu_t)\ ;\ \int_{\mathbb{X}} \phi\,\nu_t\bigg) \leq C_\phi\Big(N^{-\frac12} + e^{-p_0\lambda_0 t} N^{-\frac13}\Big),$$

where:

— d_2 *stands for the second-order Zolotarev distance between random variables,*

$$\mathrm{d}_2(X;Y) := \sup\Big\{\mathbb{E}[g(X)] - \mathbb{E}[g(Y)]\ :\ g \in C_b^2(\mathbb{R}),\ g'(0) = 0,\ \|g''\|_{\mathrm{L}^\infty(\mathbb{R})} = 1\Big\}; \tag{1.18}$$

— *the limit fluctuation* ν_t *is the centered Gaussian process that is the unique almost sure distributional solution of the Gaussian linearized Dean–Kawasaki SPDE (see Section 7.1 for details),*

$$\begin{cases} \partial_t \nu_t + v \cdot \nabla_x \nu_t = \mathrm{div}_v(\sqrt{\mu_t}\xi_t) + \mathrm{div}_v((\nabla_v + \beta v)\nu_t) \\ \qquad\qquad +\nabla A \cdot \nabla_v \nu_t + \kappa(\nabla W * \nu_t) \cdot \nabla_v \mu_t + \kappa(\nabla W * \mu_t) \cdot \nabla_v \nu_t, \\ \nu_t|_{t=0} = \nu_\circ, \end{cases} \tag{1.19}$$

where ξ *is a vector-valued space-time white noise on* $\mathbb{R}^+ \times \mathbb{D}^d$, *and where* $\nu_\circ$ *is the Gaussian field describing the fluctuations of the initial empirical measure in the sense that* $N^{1/2} \int_{\mathbb{D}^d} \phi\,(\mu_\circ^N - \mu_\circ)$ *converges in law to* $\int_{\mathbb{D}^d} \phi\,\nu_\circ$ *for all* $\phi \in C_c^\infty(\mathbb{D}^d)$.[2]

Remark 1.4 (Higher-order fluctuations)**.** In recent years, much work has been devoted to the justification of the non-Gaussian nonlinear Dean–Kawasaki equation, which is a highly singular SPDE formally expected to capture higher-order fluctuations; see in particular [65, 30, 36, 31]. In contrast, the above result only focuses on *Gaussian leading fluctuations*, but it provides the first uniform-in-time justification. Extensions to non-Gaussian corrections and the uniform-in-time justification of the nonlinear Dean–Kawasaki equation is postponed to a future work.

Remark 1.5 (Confining potential)**.** Although we focus for simplicity on particle systems in $\mathbb{R}^d$ with quadratic confinement (1.2), we emphasize that this requirement is not essential.

(a) *Non-quadratic confinement:* The same results hold if instead of the quadratic confinement (1.2) we choose $A(x) = a|x|^2 + A'(x)$ for some $a > 0$ and some smooth potential $A' \in C_b^\infty(\mathbb{R}^d)$, provided that $\|\nabla^2 A'\|_{\mathrm{L}^\infty(\mathbb{R}^d)}$ is small enough (depending on β, W, a). In that case, we can still appeal to [9]

[2] In other words, this means that $\nu_\circ$ is the random tempered distribution on $\mathbb{D}^d$ characterized by having Gaussian law with $\mathbb{E}[\int_{\mathbb{D}^d} \phi\nu_\circ] = 0$ and $\mathrm{Var}[\int_{\mathbb{D}^d} \phi\nu_\circ] = \int_{\mathbb{D}^d} (\phi - \int_{\mathbb{D}^d} \phi\mu_\circ)^2 \mu_\circ$ for all $\phi \in C_c^\infty(\mathbb{D}^d)$.

to ensure the validity of Theorem 3.1(i), while the rest of our approach can be adapted directly without major difficulties.

(b) *Periodic setting:* Our approach is easily adapted to particle systems on the torus $\mathbb{T}^d$ with $A \equiv 0$. The above results still hold in the same form in that case, and the only difference in the proof appears when investigating the ergodic properties of the linearized mean-field operator. We refer to Remark 3.2 for details.

Remark 1.6 (Expansions of functionals along the flow)**.** Along the way, we also extend the work of Chassagneux, Szpruch, and Tse [25] to the case of the kinetic Langevin dynamics: more precisely, in the setting of Theorem 1.1 with $\kappa \in [0, \kappa_0]$, for all smooth functionals Φ,[3] we obtain a truncated expansion of the following form, for all $k \geq 0$,

$$\mathbb{E}\left[\Phi(\mu_t^N)\right] - \Phi(\mu_t) = \sum_{j=1}^{k} \frac{C_{j,\Phi}(t,\mu_\circ)}{N^j} + O(N^{-k-1}),$$

with exact expressions for the coefficients $\{C_{j,\Phi}(t,\mu_\circ)\}_j$ independent of N. As explained in [25, Section 1.1], by means of Romberg extrapolation, such an expansion can be used to accelerate the convergence of numerical schemes to estimate $\Phi(\mu_t)$ through the particle method. Only the case of Brownian dynamics was previously covered in [25].

1.3. **Strategy and plan of the paper.** We start by describing the strategy of the proof of Theorem 1.1. It is well known that the estimation of correlation functions $\{G^{m,N}\}_{2\leq m\leq N}$ can be reduced to the estimation of cumulants $\{\kappa^n(\int_{\mathbb{D}^d} \phi\mu_t^N)\}_{n\geq 1}$ of linear functionals of the empirical measure μ_t^N; see Lemma 2.6. As the probability space is a product space accounting both for initial data and for Brownian forces, cumulants can be split through the law of total cumulance: we are led to consider separately "initial" and "Brownian" cumulants. To estimate initial cumulants, we appeal to the machinery that we developed in [40] based on so-called Glauber calculus; see Section 2.3. In order to estimate Brownian cumulants, we might try to appeal similarly to Malliavin calculus in the form of [77]. Unfortunately, representations of cumulants through Malliavin calculus do not seem easy to combine with ergodic properties of the linearized mean-field equation to deduce uniform-in-time estimates. Instead, we draw inspiration from the recent literature on mean-field games using the master equation formalism and the so-called Lions calculus on the space of probability measures, cf. [19, 25, 24]. In a nutshell, the key idea is to consider the mean-field semigroup induced on functionals $\mu \mapsto \Phi(\mu)$ on the space of probability measures, and then to use Lions calculus on that space to expand the Brownian expectation $\mathbb{E}_B[\Phi(\mu_t^N)]$ of functionals along the particle dynamics; see Lemma 2.1. As noted by Delarue and Tse [35], such expansions can be combined with ergodic properties of the linearized mean-field equation to obtain uniform-in-time estimates. Yet, this does not immediately lead to the desired cumulant estimates $G_t^{m,N} = O(N^{1-m})$: we further need to capture underlying cancellations, which we achieve by developing new diagrammatic techniques in form of so-called Lions graphs; see Section 4.

As explained, for uniform-in-time estimates, we rely on ergodic properties of the linearized mean-field equation. While ergodic estimates follow from the standard parabolic theory in the case of the Brownian dynamics, cf. [35], we have to further appeal to hypocoercivity techniques in the kinetic Langevin setting. For ergodic estimates on the weighted space $\mathrm{L}^2(M^{-1/2})$, where the weight is given by the steady state M for the mean-field equation, we can simply appeal to hypocoercivity in form of the theory of Dolbeault, Mouhot, and Schmeiser [38]. Since estimating cumulants costs derivatives, we rather need ergodic estimates on negative Sobolev spaces, and we easily check that the estimates on $\mathrm{L}^2(M^{-1/2})$ can be upgraded to estimates on $H^{-k}(M^{-1/2})$ for all $k \geq 0$. Yet, we would ideally rather need ergodic estimates on the larger space $W^{-k,1}(\mathbb{D}^d)$. Unfortunately, even the enlargement theory of Gualdani, Mischler, and Mouhot [53, 74] does not allow to reach such spaces. In Section 3, we revisit

[3] For our purposes in this work, we actually focus on linear functionals Φ, but we emphasize that this is not essential in the proofs and nonlinear functionals could be considered as well under suitable smoothness assumptions as in [25].

enlargement techniques and show that we can actually reach $W^{-k,q}(\langle z\rangle^p)$ with arbitrarily small $q>1$ and $p>0$ provided $pq' \gg 1$, which happens to be just enough for our purposes.

Finally, in order to deduce the concentration estimates and the quantitative CLT stated in Theorems 1.2 and 1.3, we combine the same Lions expansions with the Herbst argument and with Stein's method, respectively. We believe this combination of techniques to be of independent interest for applications to other settings. Note that the proof of Theorems 1.2 and 1.3 is actually much simpler than the proof of cumulant estimates in Theorem 1.1 as it does not require to capture arbitrarily fine cancellations. For the quantitative CLT, for instance, the proof essentially boils down to the convergence of the variance and to the smallness of the third cumulant of the empirical measure, thus requiring no fine information on higher cumulants.

Plan of the paper. We start in Section 2 with the presentation and development of the main technical tools that are used to prove our main results, namely Lions and Glauber calculus. In Section 3, we establish suitable ergodic estimates for the linearized mean-field equation, which are key to our uniform-in-time results. In Section 4, we develop suitable diagrammatic representations for iterated Lions expansions of Brownian cumulants of the empirical measure, which allows us to systematically capture the needed cancellations. Finally, the correlation estimates of Theorem 1.1 are concluded in Section 5, the concentration estimates of Theorem 1.2 are established in Section 6, and the quantitative CLT of Theorem 1.3 is proven in Section 7.

1.4. **Notation.** For notational convenience, we consider a general framework that covers both the Langevin and the Brownian dynamics (1.1) and (1.3) as special cases. More precisely, we denote by $\{Z_t^{i,N}\}_{1\le i\le N}$ the set of particle trajectories in the space $\mathbb{X} := \mathbb{D}^d$ or $\mathbb{R}^d$, as given by the following system of coupled SDEs: for $1\le i\le N$,

$$\begin{cases} \mathrm{d}Z_t^{i,N} = b(Z_t^{i,N},\mu_t^N)\mathrm{d}t + \sigma_0 \mathrm{d}B_t^i, & t\ge 0,\\ Z_t^{i,N}|_{t=0} = Z_\circ^{i,N}, \end{cases} \tag{1.20}$$

where μ_t^N stands for the empirical measure

$$\mu_t^N := \tfrac{1}{N}\textstyle\sum_{i=1}^N \delta_{Z_t^{i,N}} \in \mathcal{P}(\mathbb{X}),$$

where $b:\mathbb{X}\times\mathcal{P}(\mathbb{X})\to\mathbb{R}^d$ is a smooth functional (in a sense that will be made clear later on), where $\{B^i\}_i$ are i.i.d. Brownian motions in $\mathbb{X}$, and where σ_0 is a constant matrix. We assume that initial data $\{Z_\circ^{i,N}\}_{1\le i\le N}$ are i.i.d. with some density $\mu_\circ\in\mathcal{P}(\mathbb{X})$. The associated mean-field equation takes form of the following McKean–Vlasov equation,

$$\begin{cases} \partial_t\mu + \operatorname{div}(b(\cdot,\mu)\,\mu) = \frac12 \operatorname{div}(a_0\nabla\mu), & \text{in } \mathbb{R}^+\times\mathbb{X},\\ \mu|_{t=0} = \mu_\circ, & \text{in } \mathbb{X}, \end{cases} \tag{1.21}$$

with $a_0 := \sigma_0\sigma_0^T$, and we denote the well-posed solution operator on $\mathcal{P}(\mathbb{X})$ by

$$\mu_t := m(t;\mu_\circ). \tag{1.22}$$

This general framework allows us to consider both systems of interest (1.1) and (1.3) at once: the Langevin dynamics (1.1) is given by

$$\mathbb{X}=\mathbb{D}^d,\quad b((x,v),\mu) = \big(v, -\tfrac{\beta}{2}v - (\nabla A + \kappa\nabla W * \mu)(x)\big),\quad \sigma_0 = \begin{pmatrix} 0_{\mathbb{R}^d} & 0_{\mathbb{R}^d}\\ 0_{\mathbb{R}^d} & \mathrm{Id}_{\mathbb{R}^d}\end{pmatrix}, \tag{1.23}$$

and the Brownian dynamics (1.3) by

$$\mathbb{X}=\mathbb{R}^d,\quad b(x,\mu) = -(\nabla A + \kappa\nabla W*\mu)(x),\quad \sigma_0 = \mathrm{Id}_{\mathbb{R}^d}. \tag{1.24}$$

Note that the diffusion matrix $a_0=\sigma_0\sigma_0^T$ is degenerate in the Langevin case, which is why specific hypocoercivity techniques are then needed. Most of our work can actually be performed in the general framework (1.20) without any structural assumption on $\mathbb{X}, b, \sigma_0$, except when establishing ergodic estimates in Section 3. More precisely, our different main results hold for any system of the form (1.20), under suitable smoothness assumptions for b, provided that the ergodic estimates of Theorem 3.1 are available. For the latter, we restrict to the setting of the Langevin or Brownian dynamics in the weak coupling regime $\kappa \ll 1$. Under mere smoothness assumptions on b, if ergodic estimates are not available, we note that our analysis can at least be repeated to obtain non-uniform estimates with exponential time growth.

Finally, let us briefly list the main notation used throughout this work:

— We denote by $C \geq 1$ any constant that only depends on the space dimension d. We use the notation $\lesssim$ (resp. $\gtrsim$) for $\leq C\times$ (resp. $\geq \frac{1}{C}\times$) up to such a multiplicative constant C. We write $\simeq$ when both $\lesssim$ and $\gtrsim$ hold. We add subscripts to $C, \lesssim, \gtrsim, \simeq$ to indicate dependence on other parameters.

— The underlying probability space $(\Omega, \mathbb{P})$ splits as a product $(\Omega, \mathbb{P}) = (\Omega_\circ, \mathbb{P}_\circ) \times (\Omega_B, \mathbb{P}_B)$, where the first factor accounts for random initial data and where the second factor accounts for Brownian forces. The space Ω° is endowed with the σ-algebra $\mathcal{F}_\circ = \sigma(Z_\circ^{1,N}, \ldots, Z_\circ^{N,N})$ generated by initial data, while Ω_B is endowed with the σ-algebra $\sigma((B_t^1, \ldots, B_t^N)_{t\geq 0})$. We also denote by $\mathcal{F}_t^B := \sigma((B_s^1, \ldots, B_s^N)_{0\leq s\leq t})$ the Brownian filtration. We use $\mathbb{E}[\cdot]$ and $\kappa^m[\cdot]$ to denote the expectation and the cumulant of order m with respect to $\mathbb{P}$, and we similarly denote by $\mathbb{E}_\circ[\cdot], \kappa_\circ^m[\cdot]$ and by $\mathbb{E}_B[\cdot], \kappa_B^m[\cdot]$ the expectation and cumulants with respect to $\mathbb{P}_\circ$ and $\mathbb{P}_B$, respectively.

— For any two integers $b \geq a \geq 0$, we use the short-hand notation $[\![a, b]\!] := \{a, a+1, \ldots, b\}$, and in addition for any integer $a \geq 1$ we set $[\![a]\!] := [\![1, a]\!]$.

— For all $z \in \mathbb{R}^d$, we use the notation $\langle z \rangle := (1 + |z|^2)^{\frac{1}{2}}$.

2. Preliminary

This section is devoted to the presentation and development of the main technical tools used in this work. We start with an account of the master equation formalism and of Lions calculus for functionals on the space of probability measures, then we turn to the study of correlation functions by means of cumulants of the empirical measure, and finally we recall useful tools from Glauber calculus.

2.1. Lions calculus.

We recall several notions of derivatives for functionals defined on the space $\mathcal{P}(\mathbb{X})$ of probability measures, and how they can be used to expand functionals along the particle dynamics.

2.1.1. *Linear derivative.* We start with the notion of *linear derivative*, as used for instance by Lions in his course at Collège de France [16]; see also [19, Chapter 5] for a slightly different exposition. A functional $\mathcal{V} : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ is said to be continuously differentiable if there exists a continuous map $\frac{\delta \mathcal{V}}{\delta \mu} : \mathcal{P}(\mathbb{X}) \times \mathbb{X} \to \mathbb{R}$ such that, for all $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{X})$,

$$\mathcal{V}(\mu_0) - \mathcal{V}(\mu_1) \;=\; \int_0^1 \int_{\mathbb{X}} \frac{\delta \mathcal{V}}{\delta \mu}\big(s\mu_0 + (1-s)\mu_1, y\big)\,(\mu_0 - \mu_1)(\mathrm{d}y)\,\mathrm{d}s, \tag{2.1}$$

and we then call $\frac{\delta \mathcal{V}}{\delta \mu}$ the linear functional derivative of $\mathcal{V}$. This definition holds up to a constant, which we fix by setting

$$\int_{\mathbb{X}} \frac{\delta \mathcal{V}}{\delta \mu}(\mu, y)\,\mu(\mathrm{d}y) \;=\; 0, \qquad \text{for all } \mu \in \mathcal{P}(\mathbb{X}).$$

The denomination "linear derivative" is understood as it is precisely defined to satisfy for all $\mu \in \mathcal{P}(\mathbb{X})$ and $y \in \mathbb{X}$,

$$\lim_{h\to 0} \frac{\mathcal{V}((1-h)\mu + h\delta_y) - \mathcal{V}(\mu)}{h} \;=\; \frac{\delta \mathcal{V}}{\delta \mu}(\mu, y). \tag{2.2}$$

Higher-order linear derivatives are defined by induction: for all integers $p \geq 1$, if the functional $\mathcal{V}$ is p-times continuously differentiable, we say that it is $(p+1)$-times continuously differentiable if there exists a continuous map $\frac{\delta^{p+1}\mathcal{V}}{\delta \mu^{p+1}} : \mathcal{P}(\mathbb{X}) \times \mathbb{X}^{p+1} \to \mathbb{R}$ such that for all μ, μ' in $\mathcal{P}(\mathbb{X})$ and $y \in \mathbb{X}^p$,

$$\frac{\delta^p \mathcal{V}}{\delta \mu^p}(\mu, y) - \frac{\delta^p \mathcal{V}}{\delta \mu^p}(\mu', y) \;=\; \int_0^1 \int_{\mathbb{X}} \frac{\delta^{p+1}\mathcal{V}}{\delta \mu^{p+1}}\big((s\mu + (1-s)\mu', y, y'\big)\,(\mu - \mu')(\mathrm{d}y')\,\mathrm{d}s.$$

Once again, to ensure the uniqueness of the $(p+1)$th linear functional derivative $\frac{\delta^{p+1}\mathcal{V}}{\delta\mu^{p+1}}$, we choose the convention

$$\int_{\mathbb{X}} \frac{\delta^{p+1}\mathcal{V}}{\delta\mu^{p+1}}(\mu, y_1, \ldots, y_{p+1})\,\mu(\mathrm{d}y_{p+1}) \,=\, 0, \qquad \text{for all } \mu \in \mathcal{P}(\mathbb{X}) \text{ and } y_1, \ldots, y_p \in \mathbb{X}.$$

2.1.2. *L-derivative.* We further recall the notion of so-called *L-derivatives* (or Lions derivatives, or intrinsic derivatives), as developed in [69]. We refer e.g. to [17, Section 2.2] for the link to the Otto calculus on Wasserstein space [78, 1]. For a continuously differentiable functional $\mathcal{V} : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, if the map $y \mapsto \frac{\delta\mathcal{V}}{\delta\mu}(\mu, y)$ is of class C^1 on $\mathbb{X}$, the L-derivative of $\mathcal{V}$ is defined as

$$\partial_\mu \mathcal{V}(\mu, y) \,:=\, \nabla_y \frac{\delta\mathcal{V}}{\delta\mu}(\mu, y). \tag{2.3}$$

We also define corresponding higher-order derivatives: for all $\mu \in \mathcal{P}(\mathbb{X})$ and $y_1, \ldots, y_p \in \mathbb{X}$, we define, provided that it makes sense,

$$\partial_\mu^p \mathcal{V}(\mu, y_1, \ldots, y_p) \,:=\, \nabla_{y_1} \ldots \nabla_{y_p} \frac{\delta^p\mathcal{V}}{\delta\mu^p}(\mu, y_1, \ldots, y_p).$$

2.1.3. *Master equation formalism.* In terms of the above calculus on the space $\mathcal{P}(\mathbb{X})$ of probability measures, we now introduce the so-called master equation formalism to describe the evolution of functionals on $\mathcal{P}(\mathbb{X})$ along the mean-field flow. For a smooth functional $\Phi : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, we define

$$U(t, \mu) \,:=\, P_t\Phi(\mu) \,:=\, \Phi(m(t, \mu)), \qquad t \geq 0, \quad \mu \in \mathcal{P}(\mathbb{X}), \tag{2.4}$$

where we recall the notation $m(t, \mu)$ for the mean-field solution operator (1.22). This defines a semi-group $(P_t)_{t\geq 0}$ acting on bounded measurable functionals on $\mathcal{P}(\mathbb{X})$. From [14, Theorem 7.2], using the regularity of b, and assuming corresponding regularity of Φ, we find that $U(t, \mu)$ satisfies the following master equation, which is viewed as an evolution equation for functionals on $\mathcal{P}(\mathbb{X})$,

$$\begin{cases} \partial_t U(t, \mu) = \int_{\mathbb{X}} \Big[b(x, \mu) \cdot \nabla_x \frac{\delta U}{\delta\mu}(t, \mu, x) + \frac{1}{2} a_0 : \nabla_x^2 \frac{\delta U}{\delta\mu}(t, \mu, x) \Big] \mu(\mathrm{d}x), \\ U(0, \mu) = \Phi(\mu), \end{cases} \tag{2.5}$$

where we recall $a_0 = \sigma_0\sigma_0^T$. For the Langevin dynamics (1.1), this takes on the following guise,

$$\begin{cases} \partial_t U(t, \mu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \Big[v \cdot \big(\nabla_x - \frac{\beta}{2}\nabla_v\big) \frac{\delta U}{\delta\mu}(t, \mu, x, v) + \frac{1}{2}\triangle_v \frac{\delta U}{\delta\mu}(t, \mu, x, v) \\ \qquad\qquad -\big(\nabla A(x) + \kappa\nabla W * \mu(x)\big) \cdot \nabla_v \frac{\delta U}{\delta\mu}(t, \mu, x, v) \Big] \mu(\mathrm{d}x\mathrm{d}v), \\ U(0, \mu) = \Phi(\mu). \end{cases}$$

2.1.4. *Expansions along the particle dynamics.* We recall the following useful result that allows to expand functionals along the particle dynamics in terms of the corresponding mean-field flow, cf. [19, (5.131)] or [25, Lemma 2.8]. Note that the proof in [25] only relies on the master equation (2.5) and on [24, Proposition 3.1], so that in particular there is no uniform ellipticity requirement for the diffusivity $a_0 = \sigma_0\sigma_0^T$.

Lemma 2.1 (see [19, 25])**.** *Let $\Phi : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ be a smooth functional and let $U(t, \mu)$ be defined in (2.4). Then for all $0 \leq s \leq t$ we have*

$$U(t-s, \mu_s^N) \,=\, U(t, \mu_0^N) \,+\, \frac{1}{2N}\int_0^s \int_{\mathbb{X}} \mathrm{tr}\, \Big[a_0\, \partial_\mu^2 U(t-u, \mu_u^N)(z, z) \Big]\, \mu_u^N(\mathrm{d}z)\,\mathrm{d}u \,+\, M_{t,s}^N, \tag{2.6}$$

where $(M_{t,s}^N)_{s\geq 0}$ is a square-integrable $(\mathcal{F}_s^B)_{s\geq 0}$-martingale with $M_{t,0}^N = 0$, which is explicitly given by

$$M_{t,s}^N \,:=\, \frac{1}{N}\sum_{i=1}^N \int_0^{s\wedge t} \partial_\mu U(t-u, \mu_u^N)(Z_u^{i,N}) \cdot \sigma_0\,\mathrm{d}B_u^i.$$

This expansion will be used throughout this work to compare the empirical measure to the corresponding mean-field semigroup. More precisely, we shall abundantly use the following immediate consequences.

Corollary 2.2. *Let $\Phi : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ be a smooth functional and let $U(t,\mu)$ be defined in (2.4).*

(i) For all $t \geq 0$, we have

$$\big|\mathbb{E}[\Phi(\mu_t^N)] - \mathbb{E}_\circ[\Phi(m(t,\mu_0^N))]\big| \lesssim N^{-1}\,\mathbb{E}\bigg[\int_0^t \int_{\mathbb{X}} \big|\partial_\mu^2 U(t-u,\mu_u^N)(z,z)\big|\,\mu_u^N(\mathrm{d}z)\,\mathrm{d}u\bigg].$$

(ii) For all $t \geq 0$, we have

$$\begin{aligned}\|\Phi(\mu_t^N) - \Phi(m(t,\mu_0^N))\|_{\mathrm{L}^2(\Omega_B)} \lesssim\; & N^{-\frac12}\,\mathbb{E}_B\bigg[\int_0^t \int_{\mathbb{X}} \big|\partial_\mu U(t-u,\mu_u^N)(z)\big|^2\,\mu_u^N(\mathrm{d}z)\,\mathrm{d}u\bigg]^{\frac12}\\ &+ N^{-1}\,\mathbb{E}_B\bigg[\Big(\int_0^t \int_{\mathbb{X}} \big|\partial_\mu^2 U(t-u,\mu_u^N)(z,z)\big|\,\mu_u^N(\mathrm{d}z)\,\mathrm{d}u\Big)^2\bigg].\end{aligned}$$

Proof. Taking the expectation $\mathbb{E} = \mathbb{E}_\circ\mathbb{E}_B$ in (2.6), using $\mathbb{E}_B[M_{t,s}^N] = 0$, and setting $s = t$, we are led in particular to the following expansion for the expectation of a functional of the empirical measure,

$$\mathbb{E}[\Phi(\mu_t^N)] \,=\, \mathbb{E}_\circ[\Phi(m(t,\mu_0^N))] \,+\, \frac{1}{2N}\int_0^t \mathbb{E}\bigg[\int_{\mathbb{X}} \operatorname{tr}\Big[a_0\,\partial_\mu^2 U(t-u,\mu_u^N)(z,z)\Big]\,\mu_u^N(\mathrm{d}z)\bigg]\mathrm{d}u,$$

and item (i) immediately follows. Next, taking the $\mathrm{L}^2(\Omega)$ norm in (2.6), noting that Jensen's inequality yields

$$\mathbb{E}_B[(M_{t,s}^N)^2] \,\leq\, N^{-1}\mathbb{E}_B\bigg[\int_0^{s\wedge t} \int_{\mathbb{X}} \big|\sigma_0^T(\partial_\mu U)(t-u,\mu_u^N)(z)\big|^2\,\mu_u^N(\mathrm{d}z)\,\mathrm{d}u\bigg],$$

and setting $s = t$, we similarly obtain item (ii). ☐

Due to the above result, as emphasized in [19, 25, 35], Lions calculus provides a natural starting point for propagation of chaos, which was indeed successfully used in particular in [35] to establish uniform-in-time weak propagation of chaos estimates for the Brownian dynamics. More precisely, in order to obtain a weak propagation of chaos estimate of the form (1.10),

$$\mathbb{E}[\Phi(\mu_t^N)] - \Phi(m(t,\mu_\circ)) \,=\, O(\tfrac{1}{N}),$$

we can appeal to item (i) above and it remains to compare $\mathbb{E}_\circ[\Phi(m(t,\mu_0^N))]$ to $\Phi(m(t,\mu_\circ))$. The missing estimate is provided by the following general result; see [25, Theorem 2.11].

Lemma 2.3 (see [25])**.** *For any smooth functional $\Phi : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, we have*

$$\mathbb{E}[\Phi(\mu_0^N)] - \Phi(\mu_\circ) \,=\, \frac{1}{N}\int_0^1\int_0^1\int_{\mathbb{X}} \mathbb{E}\bigg[\frac{\delta^2\Phi}{\delta\mu^2}(\tilde\mu_{s,u,z}^N, z, z) - \frac{\delta^2\Phi}{\delta\mu^2}(\tilde\mu_{s,u,z}^N, z, Z_\circ^{1,N})\bigg]\,\mu_\circ(\mathrm{d}z)\,\mathrm{d}u\,s\,\mathrm{d}s,$$

in terms of

$$\tilde\mu_{s,u,z}^N \,:=\, \frac{su}{N}(\delta_z - \delta_{Z_\circ^{1,N}}) + \mu_\circ + s(\mu_0^N - \mu_\circ).$$

2.2. **Cumulants.** In order to estimate the many-particle correlation functions $\{G^{k,N}\}_{1\leq k\leq N}$ defined in (1.13), we shall proceed by estimating cumulants of the empirical measure, which have a more exploitable probabilistic content. We recall that the mth cumulant of a bounded random variable X is defined by

$$\kappa^m[X] := \Big((\tfrac{\mathrm{d}}{\mathrm{d}t})^m \log\mathbb{E}\big[e^{tX}\big]\Big)\Big|_{t=0},$$

hence in particular,

$$\kappa^1[X] = \mathbb{E}[X],$$
$$\kappa^2[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathrm{Var}[X],$$
$$\kappa^3[X] = \mathbb{E}[X^3] - 3\mathbb{E}[X^2]\mathbb{E}[X] + 2\mathbb{E}[X]^3,$$
$$\kappa^4[X] = \mathbb{E}[X^4] - 4\mathbb{E}[X^3]\mathbb{E}[X] - 3\mathbb{E}[X^2]^2 + 12\mathbb{E}[X^2]\mathbb{E}[X]^2 - 6\mathbb{E}[X]^4,$$

and so on. Using similar notation as in (1.13), the following general formula holds for all $m \geq 1$,

$$\kappa^m[X] \;=\; \sum_{\pi \vdash [\![m]\!]} (-1)^{\sharp\pi-1}(\sharp\pi-1)! \prod_{A\in\pi} \mathbb{E}[X^{\sharp A}]. \tag{2.7}$$

We can also define the joint cumulant of a family of bounded random variables $X_1, \ldots, X_m$ as

$$\kappa[X_1, \ldots, X_m] \;:=\; \Big(\tfrac{\mathrm{d}^m}{\mathrm{d}t_1 \ldots \mathrm{d}t_m} \log \mathbb{E}\big[e^{\sum_{j=1}^m t_j X_j}\big]\Big)\Big|_{t_1=\ldots=t_m=0}.$$

Since we consider in this work a product probability space $(\Omega, \mathbb{P}) = (\Omega_\circ, \mathbb{P}_\circ) \times (\Omega_B, \mathbb{P}_B)$, where the first factor accounts for random initial data and where the second factor accounts for Brownian forces, we shall appeal to the following law of total cumulance in order to split cumulants accordingly.

Lemma 2.4 (see [13]). *For all $m \geq 2$ and all bounded random variables X, we have*

$$\kappa^m[X] \;=\; \sum_{\pi \vdash [\![m]\!]} \kappa_\circ^{\sharp\pi}\Big[\big(\kappa_B^{\sharp A}[X]\big)_{A\in\pi}\Big],$$

where we recall that $\kappa_\circ$ and κ_B stand for cumulants with respect to $\mathbb{P}_\circ$ and $\mathbb{P}_B$, respectively.

2.2.1. *Moments and cumulants.* While cumulants are defined as polynomial expressions involving moments, cf. (2.7), those relations are easily inverted: similarly as in (1.12), moments can be recovered from cumulants in form of a cluster expansion,

$$\mathbb{E}[X^m] \;=\; \sum_{\pi \vdash [\![m]\!]} \prod_{A\in\pi} \kappa^{\sharp A}[X]. \tag{2.8}$$

For later purposes, we state the following recurrence relation between moments and cumulants: it immediately implies the above cluster expansion by induction, and it will be useful in this form in the sequel. A short proof is included for convenience.

Lemma 2.5. *For all $m \geq 2$ and all bounded random variables $X_1, \ldots, X_m$, we have*

$$\mathbb{E}[X_1 \ldots X_m] \;=\; \sum_{J \subset [\![2,m]\!]} \kappa[X_1, X_J]\, \mathbb{E}\Big[\prod_{j\in[\![2,m]\!]\setminus J} X_j\Big], \tag{2.9}$$

where we use the standard convention $\prod_{j\in\varnothing} X_j = 1$ for the empty product. In particular, for all $m \geq 1$ and all bounded random variables X, we have

$$\mathbb{E}[X^m] \;=\; \sum_{j=1}^m \binom{m-1}{j-1} \kappa^j[X]\, \mathbb{E}[X^{m-j}].$$

Proof. We follow [77, Proposition 2.2], extending it to the present multivariate setting. Let

$$M_{X_1,\ldots,X_m}(t_1, \ldots, t_m) \;:=\; \mathbb{E}\big[e^{\sum_{j=1}^m t_j X_j}\big]$$

be the multivariate moment generating function of $X_1, \ldots, X_m$. We can write

$$\begin{aligned}
\mathbb{E}[X_1 \ldots X_m] \;&=\; \frac{\mathrm{d}^m}{\mathrm{d}t_1 \ldots \mathrm{d}t_m} M_{X_1,\ldots,X_m}(t_1, \ldots, t_m)\Big|_{t_1=\ldots=t_m=0} \\
&=\; \frac{\mathrm{d}^{m-1}}{\mathrm{d}t_2 \ldots \mathrm{d}t_m} \bigg(\Big(\frac{\mathrm{d}}{\mathrm{d}t_1} \log M_{X_1,\ldots,X_m}(t_1, \ldots, t_m)\Big) M_{X_1,\ldots,X_m}(t_1, \ldots, t_n)\bigg)\bigg|_{t_1=\ldots=t_m=0},
\end{aligned}$$

and thus, by the Leibniz rule,

$$\mathbb{E}[X_1\dots X_m] \;=\; \sum_{J\subset[\![2,m]\!]}\Big(\frac{\mathrm{d}^{\sharp J+1}}{\mathrm{d}t_1\mathrm{d}t_J}\log M_{X_1\dots X_m}(t_1,\dots,t_m)\Big)\Big|_{t_1=\dots=t_m=0} \times\Big(\frac{\mathrm{d}^{m-1-\sharp J}}{\mathrm{d}t_{[\![2,m]\!]\setminus J}}M_{X_1\dots X_m}(t_1,\dots,t_m)\Big)\Big|_{t_1=\dots=t_m=0},$$

where $\mathrm{d}t_J$ stands for $\mathrm{d}t_{j_1}\dots\mathrm{d}t_{j_s}$ if $J=\{j_1,\dots,j_s\}$. By definition of cumulants and of the moment generating function, this yields the conclusion. □

2.2.2. *From cumulants to correlations.* We work out the standard link between cumulants of the empirical measure and correlation functions. We state it in form of an inequality that can be directly iterated to bound successive correlation functions in terms of cumulants of the empirical mesure. This was used for instance in [40, Section 4], but we provide a self-contained statement and a short proof for convenience.

Lemma 2.6. *For all $1\le m\le N$ and $t\ge0$, we have*

$$\Big|\int_{\mathbb{X}^m}\phi^{\otimes m}G_t^{m,N}\Big| \;\le\; \Big|\kappa^m\Big[\int_{\mathbb{X}}\phi\,\mathrm{d}\mu_t^N\Big]\Big| + C_m\sum_{\substack{\pi\vdash[\![m]\!]\\ \sharp\pi<m}}\sum_{\rho\vdash\pi}N^{\sharp\pi-\sharp\rho-m+1}\Big|\int_{\mathbb{X}^{\sharp\pi}}\Big(\bigotimes_{B\in\pi}\phi^{\sharp B}\Big)\Big(\bigotimes_{D\in\rho}G_t^{\sharp D,N}(z_D)\Big)\,\mathrm{d}z_\pi\Big|.$$

Proof. We start from the relation between cumulants and moments, cf. (2.7), applied to a linear functional of the empirical measure: given $\phi\in C_c^\infty(\mathbb{X})$, we have

$$\kappa^m\Big[\int_{\mathbb{X}}\phi\,\mathrm{d}\mu_t^N\Big] \;=\; \sum_{\pi\vdash[\![m]\!]}(-1)^{\sharp\pi-1}(\sharp\pi-1)!\prod_{A\in\pi}\mathbb{E}\Big[\Big(\int_{\mathbb{X}}\phi\,\mathrm{d}\mu_t^N\Big)^{\sharp A}\Big].$$

Now moments of the empirical measure $\mu^N=\frac1N\sum_{i=1}^N\delta_{Z^{i,N}}$ can be computed as follows,

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_{\mathbb{X}}\phi\,\mathrm{d}\mu_t^N\Big)^n\Big] &= \frac{1}{N^n}\sum_{i_1,\dots,i_n=1}^N\mathbb{E}\Big[\prod_{\ell=1}^n\phi(Z_t^{i_\ell,N})\Big]\\ &= \frac{1}{N^n}\sum_{\pi\vdash[\![n]\!]}N(N-1)\dots(N-\sharp\pi+1)\int_{\mathbb{X}^{\sharp\pi}}\Big(\bigotimes_{A\in\pi}\phi^{\sharp A}\Big)F_t^{\sharp\pi,N},\end{aligned}$$

while marginals of F_N can be expressed in terms of correlations via the cluster expansion (1.12),

$$F_t^{n,N}(z_{[\![n]\!]}) \;=\; \sum_{\pi\vdash[\![n]\!]}\prod_{A\in\pi}G_t^{\sharp A,N}(z_A).$$

Combining those different identities, after straightforward simplifications, we obtain the following expression for cumulants of the empirical measure in terms of correlation functions,

$$\kappa^m\Big[\int_{\mathbb{X}}\phi\,\mathrm{d}\mu_t^N\Big] \;=\; \sum_{\pi\vdash[\![m]\!]}N^{\sharp\pi-m}\sum_{\rho\vdash\pi}K_N(\rho)\int_{\mathbb{X}^{\sharp\pi}}\Big(\bigotimes_{B\in\pi}\phi^{\sharp B}\Big)\Big(\bigotimes_{D\in\rho}G_t^{\sharp D,N}(z_D)\Big)\,\mathrm{d}z_\pi, \tag{2.10}$$

where the coefficients are given by

$$K_N(\rho) \;:=\; \sum_{\sigma\vdash\rho}(-1)^{\sharp\sigma-1}(\sharp\sigma-1)!\Big(\prod_{C\in\sigma}(1-\tfrac1N)\dots\big(1-\tfrac{(\sum_{D\in C}\sharp D)-1}{N}\big)\Big).$$

Isolating $\int_{\mathbb{X}^m}\phi^{\otimes(m)}G_t^{m,N}$ in the right-hand side of (2.10) (this term is obtained for the choice $\pi=\{\{1\},\dots,\{m\}\}$ and $\rho=\{\pi\}$), and noting that $|K_N(\rho)|\le C_mN^{1-\sharp\rho}$, the conclusion follows. □

2.3. **Glauber calculus.** We recall some useful tools from the so-called Glauber calculus on $(\Omega_\circ, \mathbb{P}_\circ)$, as developed in particular by the second-named author in [40] (see also [33, 42]). Given that initial data $(Z_\circ^{j,N})_{1\le j\le N}$ are i.i.d., the probability measure $\mathbb{P}_\circ$ is a product measure and we denote by $\mathbb{E}_{\circ,j}$ the expectation with respect to the jth variable $Z_\circ^{j,N}$ only. Given a random variable $X \in \mathrm{L}^2(\Omega_\circ)$, we then define its Glauber derivative at $j \in [\![N]\!]$ as

$$D_j^\circ X \,:=\, X - \mathbb{E}_{\circ,j}[X].$$

The full gradient $D^\circ X = (D_j^\circ X)_{j\in[\![N]\!]}$ is viewed as an element of $\ell^2([\![N]\!]; \mathrm{L}^2(\Omega_\circ))$. A straightforward computation shows that D_j° is self-adjoint on $\mathrm{L}^2(\Omega_\circ)$ and satisfies

$$D_j^\circ D_j^\circ = D_j^\circ, \qquad D_j^\circ D_k^\circ = D_k^\circ D_j^\circ, \qquad \text{for all } j,k \in [\![N]\!].$$

We then define the Glauber Laplacian

$$\mathcal{L}_\circ \,:=\, (D^\circ)^* D^\circ \,=\, \sum_{j=1}^N (D_j^\circ)^* D_j^\circ \,=\, \sum_{j=1}^N D_j^\circ,$$

which is a nonnegative self-adjoint operator on $\mathrm{L}^2(\Omega_\circ)$. We recall some fundamental properties of this operator; see [40, Lemmas 2.5 and 2.6].

Lemma 2.7 (see [40])**.**

(i) The kernel of $\mathcal{L}_\circ$ is reduced to constants, $\ker \mathcal{L}_\circ = \mathbb{R}$. *Moreover, $\mathcal{L}_\circ$ has a unit spectral gap above* 0, *and its spectrum is the set* $\mathbb{N}$.

(ii) The restriction of $\mathcal{L}_\circ$ to $(\ker \mathcal{L}_\circ)^\perp = \{X \in \mathrm{L}^2(\Omega_\circ) : \mathbb{E}_\circ[X] = 0\}$ *admits a well-defined inverse $\mathcal{L}_\circ^{-1}$, which is a nonnegative self-adjoint contraction on* $(\ker \mathcal{L}_\circ)^\perp$. *Moreover, this inverse operator satisfies for all $1 < p < \infty$ and $X \in \mathrm{L}^p(\Omega_\circ)$ with $\mathbb{E}_\circ[X] = 0$,*

$$\|\mathcal{L}_\circ^{-1} X\|_{\mathrm{L}^p(\Omega_\circ)} \,\lesssim\, \tfrac{p^2}{p-1} \|X\|_{\mathrm{L}^p(\Omega_\circ)}. \tag{2.11}$$

(iii) The following Helffer-Sjöstrand representation holds for covariances: for all $X, Y \in \mathrm{L}^2(\Omega_\circ)$,

$$\mathrm{Cov}_\circ[X,Y] \,=\, \sum_{j=1}^N \mathbb{E}\big[(D_j^\circ X) \mathcal{L}_\circ^{-1} (D_j^\circ Y)\big]. \tag{2.12}$$

Combining the spectral gap for $\mathcal{L}_\circ$ and the Helffer–Sjöstrand inequality (2.12), we recover in particular the following well-known variance inequality due to Efron and Stein [47]: for all $X \in \mathrm{L}^2(\Omega_\circ)$,

$$\mathrm{Var}_\circ[X] \,\le\, \sum_{j=1}^N \mathbb{E}_\circ[|D_j^\circ X|^2]. \tag{2.13}$$

2.3.1. *Cumulant estimates via Glauber calculus.* It was shown in [40] how cumulants can be expressed as polynomials of Glauber derivatives. We further show now that this can be extended to joint cumulants of families of random variables. For that purpose, we first introduce some notation and recall a suitable notion of so-called Stein kernels $\{\Gamma_n\}_n$ generalizing the one in [40]. For all $n \ge 1$, given bounded $\sigma((Z_\circ^{j,N})_j)$-measurable random variables $X_1, \ldots, X_n$, we define for all $j \in [\![N]\!]$,

$$\delta_j^n(X_1, \ldots, X_n) \,:=\, \mathbb{E}_\circ^{j'}\bigg[\prod_{i=1}^n (X_i - (X_i)^{j'})\bigg],$$

where for all i, j the random variable $(X_i)^{j'}$ is obtained from X_i by replacing the underlying variable $Z_\circ^{j,N}$ by an i.i.d. copy, and where $\mathbb{E}_\circ^{j'}$ stands for expectation with respect to this i.i.d. copy. Note in particular that $\delta_j^1(X_1) = D_j^\circ X_1$, while $\delta_j^n(X_1,\ldots,X_n)$ should be compared to $\prod_{i=1}^n (D_j^\circ X_i)$. In these terms, we now define the Stein kernels

$$\Gamma_0(X_1) := \Gamma_0^0(X_1) := X_1,$$
$$\Gamma_1(X_1,X_2) := \Gamma_1^1(X_1,X_2) := \sum_{j=1}^N (D_j^\circ X_2)\mathcal{L}_\circ^{-1}(D_j^\circ X_1) = \sum_{j=1}^N \delta_j^1(X_2)\mathcal{L}_\circ^{-1}(D_j^\circ X_1),$$

and iteratively, for all $n \ge 1$, $m \ge 0$, and $\sharp J = m$,

$$\Gamma_n^{n+m}(X_1,\ldots,X_{n+1},X_J) := \sum_{j=1}^N \Big(\delta_j^{m+1}(X_{n+1},X_J)\mathcal{L}_\circ^{-1}D_j\Gamma_{n-1}^{n-1}(X_1,\ldots,X_n) - \tfrac{1}{m+2}\mathbb{1}_{n>1}\Gamma_{n-1}^{n+m}(X_1,\ldots,X_{n+1},X_J)\Big),$$

where we let $X_J = (X_{j_1},\ldots,X_{j_s})$ for $J = \{j_1,\ldots,j_s\}$, and we then set

$$\Gamma_n(X_1,\ldots,X_{n+1}) := \Gamma_n^n(X_1,\ldots,X_{n+1}).$$

Note that $\Gamma_n(X_1,\ldots,X_{n+1})$ is not symmetric in its arguments $X_1,\ldots,X_{n+1}$ (we could choose to consider instead its symmetrization, but it does not matter). In these terms, we can now state the following representation formula for cumulants.

Lemma 2.8. *For all $n \ge 0$ and all bounded $\sigma((Z_\circ^{j,N})_j)$-measurable random variables $X_1,\ldots,X_{n+1}$, we have*

$$\kappa_\circ^{n+1}[X_1,\ldots,X_{n+1}] = \mathbb{E}_\circ\big[\Gamma_n(X_1,\ldots,X_{n+1})\big].$$

Proof. We omit the subscript '$\circ$' for notational simplicity. By the Hellfer-Sjöstrand representation formula (2.12), we can write

$$\begin{aligned}\mathbb{E}\Big[\prod_{i=1}^m X_i\Big] &= \mathbb{E}[X_1]\,\mathbb{E}\Big[\prod_{i=2}^m X_i\Big] + \operatorname{Cov}\Big[X_1,\prod_{i=2}^m X_i\Big]\\ &= \mathbb{E}[X_1]\,\mathbb{E}\Big[\prod_{i=2}^m X_i\Big] + \sum_{j=1}^N \mathbb{E}\Big[D_j\Big(\prod_{i=2}^m X_i\Big)\mathcal{L}_\circ^{-1}D_jX_1\Big].\end{aligned} \tag{2.14}$$

Now note that the following formula is easily obtained by induction for differences of products: for all $(a_i)_{2\le i\le m}, (b_i)_{2\le i\le m} \subset \mathbb{R}$,

$$\prod_{i=2}^m a_i - \prod_{i=2}^m b_i = \sum_{\substack{J\subset[\![2,m]\!]\\ J\ne\varnothing}} (-1)^{\sharp J+1}\Big(\prod_{i\notin J} a_i\Big)\Big(\prod_{i\in J}(a_i-b_i)\Big),$$

and this obviously implies

$$D_j\Big(\prod_{i=2}^m X_i\Big) = \mathbb{E}^{j'}\Big[\prod_{i=2}^m X_i - \prod_{i=2}^m (X_i)^{j'}\Big] = \sum_{\substack{J\subset[\![2,m]\!]\\ J\ne\varnothing}} (-1)^{\sharp J+1}\Big(\prod_{i\in[\![2,m]\!]\setminus J} X_i\Big)\,\delta_j^{\sharp J}(X_J). \tag{2.15}$$

Inserting this into (2.14), separating the contributions of singletons in the sum, and recognizing the definition of Γ_0, Γ_1, we get

$$\begin{aligned}\mathbb{E}\Big[\prod_{i=1}^m X_i\Big] &= \mathbb{E}[\Gamma_0(X_1)]\,\mathbb{E}\Big[\prod_{i=2}^m X_i\Big] + \sum_{\substack{J\subset[\![2,m]\!]\\ J\neq\varnothing}} (-1)^{\sharp J+1} \sum_{j=1}^N \mathbb{E}\Big[\Big(\prod_{i\in[\![2,m]\!]\setminus J} X_k\Big)\, \delta_j^{\sharp J}(X_J)\mathcal{L}_\circ^{-1} D_j X_1\Big]\\ &= \mathbb{E}[\Gamma_0(X_1)]\,\mathbb{E}\Big[\prod_{i=2}^m X_i\Big] + \sum_{\ell\in[\![2,m]\!]} \mathbb{E}\Big[\Big(\prod_{i\in[\![2,m]\!]\setminus\{\ell\}} X_i\Big)\Gamma_1(X_1,X_\ell)\Big]\\ &\quad + \sum_{\ell\in[\![2,m]\!]}\sum_{\substack{J\subset[\![2,m]\!]\setminus\{\ell\}\\ \sharp J\ge 1}} \tfrac{(-1)^{\sharp J}}{\sharp J+1} \sum_{j=1}^N \mathbb{E}\Big[\Big(\prod_{i\in[\![2,m]\!]\setminus(\{\ell\}\cup J)} X_i\Big)\Gamma_1^{\sharp J+1}(X_1,X_\ell,X_J)\Big]. \end{aligned} \tag{2.16}$$

Using again the Hellfer–Sjöstrand representation formula (2.12) to handle the second right-hand side term, we can decompose for all $\ell\in[\![2,m]\!]$,

$$\begin{aligned}\mathbb{E}\Big[\Big(\prod_{i\in[\![2,m]\!]\setminus\{\ell\}} X_i\Big)\Gamma_1(X_1,X_\ell)\Big] &= \mathbb{E}[\Gamma_1(X_1,X_\ell)]\,\mathbb{E}\Big[\prod_{i\in[\![2,m]\!]\setminus\{\ell\}} X_i\Big]\\ &\quad + \sum_{j=1}^N \mathbb{E}\Big[D_j\Big(\prod_{i\in[\![2,m]\!]\setminus\{\ell\}} X_i\Big)\mathcal{L}_\circ^{-1}D_j\Gamma_1(X_1,X_\ell)\Big],\end{aligned}$$

and thus, appealing again to (2.15) to reformulate the last term,

$$\begin{aligned}\mathbb{E}\Big[\Big(\prod_{i\in[\![2,m]\!]\setminus\{\ell\}} X_i\Big)\Gamma_1(X_1,X_\ell)\Big] &= \mathbb{E}[\Gamma_1(X_1,X_\ell)]\,\mathbb{E}\Big[\prod_{i\in[\![2,m]\!]\setminus\{\ell\}} X_i\Big]\\ &\quad + \sum_{\substack{J\subset[\![2,m]\!]\setminus\{\ell\}\\ J\neq\varnothing}} (-1)^{\sharp J+1}\sum_{j=1}^N \mathbb{E}\Big[\Big(\prod_{i\in[\![2,m]\!]\setminus(\{\ell\}\cup J)} X_i\Big)\delta_j^{\sharp J}(X_J)\mathcal{L}_\circ^{-1}D_j\Gamma_1(X_1,X_\ell)\Big].\end{aligned}$$

Inserting this into (2.16) and recognizing the definition of Γ_2, we find

$$\begin{aligned}\mathbb{E}\Big[\prod_{i=1}^m X_i\Big] &= \mathbb{E}[\Gamma_0(X_1)]\,\mathbb{E}\Big[\prod_{i=2}^m X_i\Big] + \sum_{\ell\in[\![2,m]\!]}\mathbb{E}[\Gamma_1(X_1,X_\ell)]\,\mathbb{E}\Big[\prod_{i\in[\![2,m]\!]\setminus\{\ell\}} X_i\Big]\\ &\quad + \sum_{\ell,\ell'\in[\![2,m]\!]}\mathbb{E}\Big[\Big(\prod_{i\in[\![2,m]\!]\setminus\{\ell,\ell'\}} X_i\Big)\Gamma_2(X_1,X_\ell,X_{\ell'})\Big]\\ &\quad + \sum_{\ell,\ell'\in[\![2,m]\!]}\sum_{\substack{J\subset[2,m]\setminus\{\ell,\ell'\}\\ J\neq\varnothing}} \tfrac{(-1)^{\sharp J}}{\sharp J+1}\mathbb{E}\Big[\Big(\prod_{i\in[\![2,m]\!]\setminus(\{\ell,\ell'\}\cup J)} X_i\Big)\Gamma_2^{\sharp J+2}(X_1,X_\ell,X_{\ell'},X_J)\Big],\end{aligned}$$

and the claim follows by iteration and a direct comparison with the formula (2.9). ☐

The above representation formula for cumulants implies in particular that cumulants can be controlled in terms of higher-order Glauber derivatives. This provides a generalization of [40, Theorem 2.2] to the multivariate case and can be viewed as a higher-order version of Poincaré's inequality (2.13) on $\mathrm{L}^2(\Omega_\circ)$ with respect to Glauber calculus.

Proposition 2.9. *For all $n \geq 0$ and bounded $\sigma((Z_\circ^{j,N})_j)$-measurable random variables $X_1, \ldots, X_{n+1}$, we have*

$$\kappa_\circ^{n+1}[X_1, \ldots, X_{n+1}] \lesssim_n \sum_{k=0}^{n-1} N^{k+1} \sum_{\substack{a_1,\ldots,a_{n+1}\geq 1\\ \sum_j a_j = n+k+1}} \prod_{j=1}^{n+1} \big\|(D^\circ)^{a_j} X_j\big\|_{\ell^\infty_{\neq}\left(\mathrm{L}^{\frac{1}{a_j}(n+k+1)}(\Omega_\circ)\right)},$$

where we have set

$$\big\|(D^\circ)^m Z\|_{\ell^\infty_{\neq}(\mathrm{L}^p(\Omega_\circ))} := \sup_{\substack{j_1\cdots j_m\\ \text{distinct}}} \big\|D^\circ_{j_1} \ldots D^\circ_{j_m} Z\big\|_{\mathrm{L}^p(\Omega_\circ)}.$$

Proof. First note that Jensen's inequality yields for all $m \geq 0$ and $r \geq 1$,

$$\|\delta_j^m(X_1, \ldots, X_m)\|_{\mathrm{L}^r(\Omega_\circ)} = \mathbb{E}_\circ\bigg[\bigg|\mathbb{E}_\circ^{j'}\bigg[\prod_{i=1}^m (X_i - (X_i)^{j'})\bigg]\bigg|^r\bigg]^{\frac{1}{r}} \leq \mathbb{E}_\circ \mathbb{E}_\circ^{j'}\bigg[\prod_{i=1}^m |X_i - (X_i)^{j'}|^r\bigg]^{\frac{1}{r}},$$

and thus, decomposing $X_i - (X_i)^{j'} = D_j^\circ X_i - (D_j^\circ X_i)^{j'}$ and using Hölder's inequality

$$\|\delta_j^m(X_1, \ldots, X_m)\|_{\mathrm{L}^r(\Omega_\circ)} \leq \prod_{i=1}^m \mathbb{E}_\circ \mathbb{E}_\circ^{j'}\Big[\big|D_j^\circ X_i - (D_j^\circ X_i)^{j'}\big|^{rm}\Big]^{\frac{1}{rm}} \leq 2^m \prod_{i=1}^m \|D_j^\circ X_i\|_{\mathrm{L}^{rm}(\Omega_\circ)}.$$

By induction, using this estimate along with (2.11), we find for all m, n, r, for all bounded $\sigma((Z_\circ^{j,N})_j)$-measurable random variables $X_1, \ldots, X_{n+m+1}$,

$$\begin{aligned}&\|\Gamma_n^{m+n}(X_1, \ldots, X_{m+n+1})\|_{\mathrm{L}^r(\Omega)} \\ &\qquad\lesssim_{m,n,r} \sum_{k=0}^{n-1} N^{k+1} \sum_{\substack{a_1,\ldots,a_{m+n+1}\geq 1\\ \sum_j a_j = m+n+k+1}} \prod_{j=1}^{m+n+1} \|(D^\circ)^{a_j} X_j\|_{\ell^\infty_{\neq}\left(\mathrm{L}^{\frac{r}{a_j}(m+n+k+1)}(\Omega_\circ)\right)}.\end{aligned}$$

Combined with the representation formula of Lemma 2.8, this yields the conclusion. ☐

2.3.2. *Asymptotic normality via Glauber calculus.* As the approximate normality of a random variable essentially follows from the smallness of its cumulants of order ≥ 3, there is no surprise that it can be quantified as well by means of Glauber calculus. The following result is typically known in the literature as a "second-order Poincaré inequality" for approximate normality. It was first established by Chatterjee [26, Theorem 2.2] based on Stein's method for the 1-Wasserstein distance, while the corresponding bound on the Kolmogorov distance is due to [66, Theorem 4.2]. We include a short proof for convenience to show that the same result also holds for the Zolotarev distance.

Proposition 2.10 (Second-order Poincaré inequality [26, 66])**.** *For all bounded $\sigma((Z_\circ^{j,N})_j)$-measurable random variable Y, setting $\sigma_Y^2 := \mathrm{Var}\,[Y]$, there holds*

$$\begin{aligned}&\mathrm{d}_2\bigg(\frac{Y - \mathbb{E}_\circ[Y]}{\sigma_Y}; \mathcal{N}\bigg) + \mathrm{d}_{\mathrm{W}}\bigg(\frac{Y - \mathbb{E}_\circ[Y]}{\sigma_Y}; \mathcal{N}\bigg) + \mathrm{d}_{\mathrm{K}}\bigg(\frac{Y - \mathbb{E}_\circ[Y]}{\sigma_Y}; \mathcal{N}\bigg) \\ &\qquad\lesssim \frac{1}{\sigma_Y^3} \sum_{j=1}^N \mathbb{E}_\circ[|D_j^\circ Y|^6]^{\frac{1}{2}} + \frac{1}{\sigma_Y^2}\bigg(\sum_{j=1}^N \bigg(\sum_{l=1}^N \mathbb{E}_\circ[|D_l^\circ Y|^4]^{\frac{1}{4}} \mathbb{E}_\circ[|D_j^\circ D_l^\circ Y|^4]^{\frac{1}{4}}\bigg)^2\bigg)^{\frac{1}{2}},\end{aligned}$$

where $\mathrm{d}_{\mathrm{W}}(\cdot; \mathcal{N})$ and $\mathrm{d}_{\mathrm{K}}(\cdot; \mathcal{N})$ stand for the 1-Wasserstein and the Kolmogorov distances to a standard Gaussian random variable respectively, and where we recall that $\mathrm{d}_2(\cdot; \mathcal{N})$ stands for the corresponding second-order Zolotarev distance (1.18).

Proof. By homogeneity, it suffices to consider a bounded random variable Y with

$$\mathbb{E}_\circ[Y] = 0, \qquad \sigma_Y^2 = \mathrm{Var}_\circ[Y] = 1.$$

Given $g \in C_b^1(\mathbb{R})$, we define its Stein transform S_g as the solution of Stein's equation

$$S_g'(x) - xS_g(x) = g(x) - \mathbb{E}_{\mathcal{N}}[g(\mathcal{N})]. \tag{2.17}$$

As shown in [83], the latter can be computed as

$$S_g(x) \;=\; -\int_0^1 \frac{1}{2\sqrt{t}}\,\mathbb{E}_{\mathcal{N}}\Big[g'\big(\sqrt{t}\,x + \sqrt{1-t}\,\mathcal{N}\big)\Big]\,dt,$$

Using this formula and a Gaussian integration by parts, we easily obtain the following bound,

$$\|S_g'\|_{W^{1,\infty}(\mathbb{R})} \;\lesssim\; \|g''\|_{\mathrm{L}^\infty(\mathbb{R})}. \tag{2.18}$$

Evaluating equation (2.17) at Y and taking the expectation, we find

$$\mathbb{E}_\circ[g(Y)] - \mathbb{E}_{\mathcal{N}}[g(\mathcal{N})] \;=\; \mathbb{E}_\circ\big[S_g'(Y) - YS_g(Y)\big].$$

Now appealing to the Helffer–Sjöstrand representation formula of Lemma 2.7(iii) for the covariance $\mathbb{E}_\circ[YS_g(Y)] = \mathrm{Cov}_\circ[Y, S_g(Y)]$, this yields

$$\mathbb{E}_\circ[g(Y)] - \mathbb{E}_{\mathcal{N}}[g(\mathcal{N})] \;=\; \mathbb{E}_\circ\bigg[S_g'(Y) - \sum_{j=1}^N (D_j^\circ S_g(Y))\mathcal{L}_\circ^{-1}(D_j^\circ Y)\bigg]. \tag{2.19}$$

A Taylor expansion gives for all $p \ge 1$,

$$\big\|D_j^\circ S_g(Y) - S_g'(Y)D_j^\circ Y\big\|_{\mathrm{L}^p(\Omega_\circ)} \;\le\; \|S_g''\|_{\mathrm{L}^\infty(\mathbb{R})}\|D_j^\circ Y\|_{\mathrm{L}^{2p}(\Omega_\circ)}^2.$$

Using this to replace $D_j^\circ S_g(Y)$ in (2.19), using Hölder's inequality with $p = \frac{3}{2}$ to bound the error, using the boundedness of $\mathcal{L}_\circ^{-1}$ in $\mathrm{L}^3(\Omega_\circ)$, cf. Lemma 2.7(ii), and recalling the bound (2.18) on the Stein transform, we are led to

$$\mathbb{E}_\circ[g(Y)] - \mathbb{E}_{\mathcal{N}}[g(\mathcal{N})] \;\lesssim\; \|g''\|_{\mathrm{L}^\infty(\mathbb{R})}\bigg(\mathbb{E}_\circ\bigg[\bigg|1 - \sum_{j=1}^N (D_j^\circ Y)\mathcal{L}_\circ^{-1}(D_j^\circ Y)\bigg|\bigg] + \sum_{j=1}^N \mathbb{E}_\circ[|D_j^\circ Y|^3]\bigg).$$

Now recalling the Helffer–Sjöstrand representation formula of Lemma 2.7(iii) in form of

$$1 \;=\; \mathrm{Var}_\circ[Y] \;=\; \mathbb{E}_\circ\Big[\sum_{j=1}^N (D_j^\circ Y)\mathcal{L}_\circ^{-1}(D_j^\circ Y)\Big],$$

we deduce by the Cauchy–Schwarz inequality,

$$\mathbb{E}_\circ[g(Y)] - \mathbb{E}_{\mathcal{N}}[g(\mathcal{N})] \;\lesssim\; \|g''\|_{\mathrm{L}^\infty(\mathbb{R})}\bigg(\mathrm{Var}_\circ\Big[\sum_{j=1}^N (D_j^\circ Y)\mathcal{L}_\circ^{-1}(D_j^\circ Y)\Big]^{\frac{1}{2}} + \sum_{j=1}^N \mathbb{E}_\circ[|D_j^\circ Y|^3]\bigg).$$

Taking the supremum over $g \in C_b^2(\mathbb{R})$, the conclusion follows in the second-order Zolotarev distance d_2. Notice that the proof in 1-Wasserstein distance can actually obtained in the same way by noting that on top of (2.18) the Stein transform also satisfies $\|S_g'\|_{W^{1,\infty}(\mathbb{R})} \lesssim \|g'\|_{\mathrm{L}^\infty(\mathbb{R})}$, cf. [83]. The proof in Kolmogorov distance is more delicate and we refer to [66, Theorem 4.2]. ☐

2.3.3. *Concentration via Glauber calculus.* We establish the following concentration estimate for random variables in $\mathrm{L}^2(\Omega_\circ)$. It follows from some degraded version of a log-Sobolev inequality, combined with the Herbst argument. Note however that we do not have an exact log-Sobolev inequality with respect to Glauber calculus, cf. [68], which is why we need to require an *almost sure* a priori bound on the Glauber derivative.

Proposition 2.11. *Let $X \in \mathrm{L}^2(\Omega_\circ)$ be $\sigma((Z_\circ^{j,N})_{1\le j\le N})$-measurable with $\mathbb{E}_\circ[X] = 0$ and $|D_j^\circ X| \le \frac{1}{2}L$ almost surely for all $1 \le j \le N$, for some constant $L > 0$. Then for all $\lambda > 0$ we have*

$$\mathbb{E}_\circ[e^{\lambda X}] \;\le\; \exp\Big(\tfrac{N}{2}\lambda L(e^{\lambda L} - 1)\Big).$$

In particular, this entails

$$\mathbb{P}_\circ[X>r]\,\le\,\exp\Big(-\tfrac{r}{4L}\log\big(1+\tfrac{r}{NL}\big)\Big),$$

where the right-hand side is $\le\exp(-\frac{r^2}{8NL^2})$ *as long as* $r\le NL$.

Proof. We appeal to the following degraded version of a log-Sobolev inequality as obtained in [41, Proposition 2.4]: for all random variables $Y\in\mathrm{L}^2(\Omega_\circ)$, we have

$$\mathrm{Ent}_\circ[Y^2]\,\le\,2\sum_{j=1}^N\mathbb{E}_\circ\Big[\sup_j\,(Y-Y^j)^2\Big],$$

where we recall that Y^j stands for the random variable obtained from Y by replacing the underlying variable $Z_\circ^{j,N}$ by an i.i.d. copy, and where $\sup_j$ stands for the essential supremum with respect to this i.i.d. copy. Applying this inequality to $Y=e^{\frac12X}$, using the bound

$$|e^{\frac12X}-e^{\frac12X^j}|\,\le\,\tfrac12|X-X^j|\int_0^1e^{\frac12(X-t(X-X^j))}\,\mathrm{d}t\,\le\,\tfrac12e^{\frac12X}|X-X^j|\,e^{\frac12|X-X^j|},$$

we find

$$\mathrm{Ent}_\circ[e^X]\,\le\,\tfrac12\sum_{j=1}^N\mathbb{E}_\circ\Big[e^X\sup_j\,(X-X^j)^2e^{|X-X^j|}\Big]\,\le\,\tfrac{N}{2}M_X^2e^{M_X}\mathbb{E}_\circ[e^X],$$

in terms of

$$M_X\,:=\,\sup_{1\le j\le N}\,\operatorname{sup\,ess}|X-X^j|\,\le\,2\sup_{1\le j\le N}\,\operatorname{sup\,ess}|D_j^\circ X|\,\le\,L.$$

We are now in position to appeal to the Herbst argument in the form of [68, Proposition 2.9 and Corollary 2.12] and the conclusion follows. ☐

2.3.4. *Link to linear derivatives.* As the following lemma shows, Glauber derivatives can be estimated in terms of linear derivatives. This is particularly convenient in the sequel to unify notations when both Glauber and Lions derivatives are involved.

Lemma 2.12. *Given a smooth functional* $\Phi:\mathcal{P}(\mathbb{X})\to\mathbb{R}$, *we have almost surely for all* $k\ge1$ *and all distinct indices* $j_1,\ldots,j_k\in[\![N]\!]$,

$$\big|D_{j_1}^\circ\ldots D_{j_k}^\circ\Phi(\mu_0^N)\big|\,\le\,N^{-k}\,2^k\sup_{\mu\in\mathcal{P}(\mathbb{X})}\Big\|\frac{\delta^k\Phi}{\delta\mu^k}(\mu,\cdot)\Big\|_{\mathrm{L}^\infty(\mathbb{X}^k)}.$$

Proof. For all $j\in[\![N]\!]$, by definition of the Glauber derivative and of the linear derivative, we can compute

$$\begin{aligned}D_j^\circ\Phi(\mu_0^N)&=\Phi(\mu_0^N)-\int_{\mathbb{X}}\Phi\Big(\mu_0^N+\tfrac1N(\delta_z-\delta_{Z_0^{j,N}})\Big)\,\mu_\circ(\mathrm{d}z)\\&=N^{-1}\int_0^1\int_{\mathbb{X}}\int_{\mathbb{X}}\frac{\delta\Phi}{\delta\mu}\Big(\mu_0^N+\tfrac{1-s}{N}(\delta_z-\delta_{Z_0^{j,N}})\,,\,y\Big)\,(\delta_{Z_0^{j,N}}-\delta_z)(\mathrm{d}y)\,\mu_\circ(\mathrm{d}z)\,\mathrm{d}s.\end{aligned}\tag{2.20}$$

By induction, we are led to the following representation formula for iterated Glauber derivatives: for all $k\ge1$ and all distinct indices $j_1,\ldots,j_k\in[\![N]\!]$,

$$\begin{aligned}D_{j_1}^\circ\ldots D_{j_k}^\circ\Phi(\mu_0^N)\,=\,N^{-k}\int_{([0,1]\times\mathbb{X}\times\mathbb{X})^k}\frac{\delta^k\Phi}{\delta\mu^k}\Big(\mu_0^N+\sum_{i=1}^k\tfrac{1-s_i}{N}(\delta_{z_i}-\delta_{Z_0^{j_i,N}})\,,\,y_1,\ldots,y_k\Big)&\\\times\prod_{i=1}^k(\delta_{Z_0^{j_i,N}}-\delta_{z_i})(\mathrm{d}y_i)\,\mu_\circ(\mathrm{d}z_i)\,\mathrm{d}s_i.&\end{aligned}\tag{2.21}$$

Recalling $Z_0^{j,N}\sim\mu_\circ$ for all j, the conclusion immediately follows. ☐

3. Ergodic Sobolev estimates for mean field

In this section, we establish ergodic estimates for the linearized mean-field equation, which will be the key tool for our uniform-in-time results in the spirit of [35]. Given $\mu \in \mathcal{P}(\mathbb{X})$, the linearized mean-field McKean–Vlasov operator at μ is defined as follows: for all $h \in C_c^\infty(\mathbb{X})$ with $\int_{\mathbb{X}} h = 0$,

$$L_\mu h \,:=\, \tfrac{1}{2}\mathrm{div}(a_0 \nabla h) - \mathrm{div}\big(b(\cdot,\mu)h\big) - \mathrm{div}\Big(\mu \int_{\mathbb{X}} \frac{\delta b}{\delta \mu}(\cdot,\mu,z)\, h(z)\, \mathrm{d}z\Big). \tag{3.1}$$

In the Langevin setting (1.23), this means for all h on $\mathbb{X} = \mathbb{R}^d \times \mathbb{R}^d$,

$$L_\mu h \,=\, \tfrac{1}{2}\mathrm{div}_v((\nabla_v + \beta v)h) + \nabla A \cdot \nabla_v h - v \cdot \nabla_x h + \kappa(\nabla W * \mu) \cdot \nabla_v h + \kappa(\nabla W * h) \cdot \nabla_v \mu, \tag{3.2}$$

and in the Brownian setting (1.24), this means for all h on $\mathbb{X} = \mathbb{R}^d$,

$$L_\mu h \,=\, \tfrac{1}{2}\triangle h + \mathrm{div}(h \nabla A) + \kappa \mathrm{div}(h \nabla W * \mu) + \kappa \mathrm{div}(\mu \nabla W * h).$$

For our purposes in this work, we shall establish ergodic estimates in a weighted Sobolev framework with arbitrary integrability, negative regularity, and polynomial weight: more precisely, for all $1 \le q \le 2$ and $p \ge 0$, we consider the space $\mathrm{L}^q(\langle z \rangle^p)$ as the weighted Lebesgue space with the norm

$$\|h\|_{\mathrm{L}^q(\langle z\rangle^p)} \,:=\, \|\langle z\rangle^p h\|_{\mathrm{L}^q(\mathbb{X})} \,=\, \Big(\int_{\mathbb{X}} |h(z)|^q \,\langle z\rangle^{pq}\, \mathrm{d}z\Big)^{\frac{1}{q}},$$

and, for all $k \ge 0$, we consider the space $W^{-k,q}(\langle z\rangle^p)$ as the weighted negative Sobolev space associated with the dual norm

$$\|h\|_{W^{-k,q}(\langle z\rangle^p)} \,:=\, \sup\Big\{ \int_{\mathbb{X}} h h' \langle z\rangle^p \,:\, \|h'\|_{W^{k,q'}(\mathbb{X})} = 1 \Big\}, \tag{3.3}$$

where $q' := \frac{q}{q-1}$ is the dual integrability exponent and where $W^{k,q'}(\mathbb{X})$ is the standard Sobolev space with norm

$$\|h\|_{W^{k,q'}(\mathbb{X})} \,:=\, \sup_{0 \le j \le k} \|\nabla^j h\|_{\mathrm{L}^{q'}(\mathbb{X})}.$$

In these terms, the main result of this section takes on the following guise. While item (i) is well known (see e.g. [9] and the discussion below), our main contribution here is to prove the Sobolev ergodic estimates of item (ii). Note that the restriction $pq' \gg 1$ in the Langevin setting is fairly natural: indeed, we note for instance that the restriction $pq' > d$ precisely ensures the spatial density $\rho_h(x) := \int_{\mathbb{R}^d} h(x,v)\, \mathrm{d}v$ to be defined in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ for all $h \in W^{-k,q}(\langle z\rangle^p)$.

Theorem 3.1. *There exist constants $\kappa_0, \lambda_0 > 0$ (only depending on d, β, a, and $\|W\|_{W^{1,\infty}(\mathbb{R}^d)}$), such that the following results hold for any $\kappa \in [0, \kappa_0]$.*

(i) There is a unique steady state M for the mean-field evolution (1.21), and the solution operator (1.22) satisfies for all $\mu_\circ \in \mathcal{P}(\mathbb{X})$ and $t \ge 0$,

$$\mathcal{W}_2\big(m(t,\mu_\circ), M\big) \,\lesssim_{W,\beta,a}\, e^{-\lambda_0 t}\, \mathcal{W}_2(\mu_\circ, M), \tag{3.4}$$

where $\mathcal{W}_2$ stands for the 2-Wasserstein distance and where the multiplicative constant only depends on d, β, a, and $\|W\|_{W^{1,\infty}(\mathbb{R}^d)}$.

(ii) Let $1 < q \le 2$, $k \ge 1$, and $0 < p \le 1$, and assume in the Langevin setting (1.23) that $pq' \gg_{\beta,a} 1$ is large enough (only depending on d, β, a). For all $\mu_\circ \in \mathcal{P}(\mathbb{X}) \cap W^{1-k,q}(\langle z\rangle^p)$, $f_\circ \in W^{-k,q}(\langle z\rangle^p)$, and $r \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W^{-k,q}(\langle z\rangle^p))$ with $\int_{\mathbb{X}} f_\circ = 0$ and $\int_{\mathbb{X}} r_t = 0$ for $t \ge 0$, there is a unique weak solution $f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W^{-k,q}(\langle z\rangle^p))$ to the Cauchy problem

$$\left\{\begin{array}{l} \partial_t f_t = L_{m(t,\mu_\circ)} f_t + r_t, \\ f_t|_{t=0} = f_\circ, \end{array}\right. \tag{3.5}$$

and it satisfies for all $\lambda \in [0, \lambda_0)$ and $t \ge 0$,

$$\sup_{0 \le s \le t} \Big(e^{p\lambda s} \|f_s\|_{W^{-k,q}(\langle z\rangle^p)} \Big) \,\lesssim_{W,\beta,\lambda,k,p,q,a,\mu_\circ}\, \|f_\circ\|_{W^{-k,q}(\langle z\rangle^p)} + \int_0^t e^{p\lambda s} \|r_s\|_{W^{-k,q}(\langle z\rangle^p)}\, \mathrm{d}s, \tag{3.6}$$

where the multiplicative constant only depends on $d, \beta, \lambda, k, p, q, a, \|W\|_{W^{(2k)\vee(k+d+1),\infty}(\mathbb{R}^d)}$*, and on* $\|\mu_\circ\|_{W^{1-k,q}(\langle z\rangle^p)}$.

The convergence to equilibrium stated in item (i) is well known: it was proven for instance by Bolley, Guillin, and Malrieu [9] in the Langevin setting (see also [71, 59, 11] for earlier results), and their elementary coupling argument is immediately adapted to the Brownian setting as well. For corresponding results relying on convexity rather than on smallness of the interaction, we refer to [76, 57] in the Langevin setting, and to [71, 72, 20, 21, 22] in the Brownian setting. Perturbations of the strictly convex case have also been investigated e.g. in [8, 15, 46].

Regarding the ergodic estimates stated in item (ii), in the Brownian setting, they easily follow from classical parabolic theory [50, 51]. In the periodic case with $A \equiv 0$, such estimates can be found in [18, Lemma 7.4] on the space $\mathrm{L}^\infty(\mathbb{T}^d)$, and in [35] on the space $W^{-k,1}(\mathbb{T}^d)$ with $0 \le k < 2$. Those results are easily generalized to the case of a nontrivial confinement in the whole space $\mathbb{R}^d$, and they can be checked to hold on the space $W^{-k,q}(\langle z\rangle^p)$ for all $1 \le q \le 2$, $k \ge 1$, and $0 \le p \le 1$. We emphasize in particular that they also hold on the unweighted space $W^{-k,1}(\mathbb{R}^d)$ for all $k \ge 1$. We skip the detail as it is similar to [35]. Note that the control of higher-order correlation functions indeed requires ergodic estimates in Sobolev spaces with *arbitrary* negative regularity $k \ge 1$.

The main challenge is to obtain the corresponding ergodic estimates in the kinetic Langevin setting, where parabolic tools are no longer available due to hypocoercivity. This has been a very active area of research over the last two decades and it is the focus of the rest of this section. In the PDE community, the convergence to equilibrium for linear kinetic equations was first studied in [58, 60]. General hypocoercivity techniques were developed in [87, 37, 38], where the linear kinetic Fokker–Planck equation served as a prototypical example and where the exponential convergence to equilibrium was obtained both on the spaces $\mathrm{L}^2(M^{-1/2})$ and $H^1(M^{-1/2})$. Combining hypocoercivity techniques with so-called enlargement theory, Gualdani, Mischler and Mouhot [53, 74] later obtained corresponding estimates on larger spaces. While ergodic estimates in the Brownian setting hold on $W^{-k,1}(\mathbb{X})$ for all $k \ge 1$, hypocoercivity techniques in the kinetic Langevin setting actually require working on weighted spaces $W^{-k,q}(\langle z\rangle^p)$ with integrability exponent $q > 1$ and with $p > 0$. More precisely, enlargement theory as developed in [74] leads to estimates on $W^{-k,q}(\langle z\rangle^p)$ for all $q > 1$, $k \in \{-1, 0, 1\}$, and for large enough weight exponents $p \gg 1$. Yet, it is critical for our concentration results in Theorem 1.2 to be able to cover arbitrarily small $p > 0$ when the integrability exponent q is close enough to 1. This has led us to revisit and partially improve the work of Mischler and Mouhot [74]: our ergodic estimates are proven to hold for all $q > 1$ and $p > 0$ under the sole restriction that pq' be large enough, which is of independent interest. In addition, the control of higher-order correlation functions requires to cover arbitrary negative regularity $k \ge 1$.

Remark 3.2 (Periodic setting)**.** As mentioned in the introduction, cf. Remark 1.5(b), the above result can essentially be adapted to the corresponding periodic setting on the torus $\mathbb{T}^d$ with $A \equiv 0$, but some special care is then needed in the Langevin setting. Indeed, the nonlinear hypocoercivity result available in that case is slightly weaker, cf. [88, Theorem 56]: it only yields a convergence rate $t^{-\infty}$ in (3.4), thus leading to a similar decay rate $t^{-\infty}$ instead of exponential in (3.6). Fortunately, the resulting non-exponential estimates are still enough to repeat the proofs of Theorems 1.1, 1.2, and 1.3, which can be checked to hold in the very same form.

3.1. **Exponential relaxation for modified linearized operators.**

We focus on the proof of the ergodic estimates of Theorem 3.1(ii) in the kinetic Langevin setting (1.23), while the same arguments can be repeated and substantially simplified in the Brownian setting. We start by considering the following modified version of the linearized operator L_μ defined in (3.2), where we remove the (compact) convolution term: given a measure $\mu \in \mathcal{P}(\mathbb{X})$, we define for all $h \in C_c^\infty(\mathbb{X})$,

$$R_\mu h := \tfrac{1}{2}\mathrm{div}_v((\nabla_v + \beta v)h) - v \cdot \nabla_x h + (\nabla A + \kappa \nabla W * \mu) \cdot \nabla_v h. \tag{3.7}$$

Given $\mu_\circ \in \mathcal{P}(\mathbb{X})$, $s \geq 0$, and $h_s \in C_c^\infty(\mathbb{X})$, recalling that $\mu_t := m(t,\mu_\circ)$ stands for the solution of the mean-field equation (1.7), we consider the following (non-autonomous) equation,

$$\begin{cases} \partial_t h_t = R_{\mu_t} h_t, & \text{for } t \geq s, \\ h_t|_{t=s} = h_s. \end{cases} \tag{3.8}$$

It is easily checked that this linear parabolic equation is well-posed with $h \in C_{\mathrm{loc}}([s,\infty);\mathrm{L}^2(M^{-1/2}))$ whenever the initial condition h_s belongs to $\mathrm{L}^2(M^{-1/2})$. We then consider the associated fundamental solution operators $\{V_{t,s}\}_{t\geq s\geq 0}$ on $\mathrm{L}^2(M^{-1/2})$ defined by

$$h_t \,=\, V_{t,s} h_s.$$

Note that $\int_\mathbb{X} V_{t,s} h_s = \int_\mathbb{X} h_s$ for all $t \geq s$. We establish the following exponential convergence result to the steady state.

Proposition 3.3. *Let κ_0 be as in Theorem 3.1(i) and let $\kappa \in [0,\kappa_0]$. There exists a constant $\lambda_0 > 0$ (only depending on d,β,a, and $\|W\|_{W^{1,\infty}(\mathbb{R}^d)}$), such that the following holds: given $1 < q \leq 2$ and $0 < p \leq 1$ with $pq' \gg_{\beta,a} 1$ large enough (only depending on d,β,a), we have for all $\lambda \in [0,\lambda_0)$, $k \geq 0$, $h_s \in C_c^\infty(\mathbb{X})$, and $t \geq s \geq 0$,*

$$\Big\| V_{t,s} h_s - M \int_\mathbb{X} h_s \Big\|_{W^{-k,q}(\langle z\rangle^p)} \lesssim_{W,\beta,\lambda,k,p,q,a} e^{-p\lambda(t-s)} \|h_s\|_{W^{-k,q}(\langle z\rangle^p)},$$

where M is the unique steady state given by Theorem 3.1(i), and where the multiplicative constant only depends on d,β,λ,k,p,q,a, and $\|W\|_{W^{k+d+1,\infty}(\mathbb{R}^d)}$.

As stated in Theorem 3.1(i), recall that the mean-field evolution (1.7) has a unique steady state M for $\kappa \in [0,\kappa_0]$, which can actually be characterized as the unique solution of the fixed-point Gibbs equation

$$M(x,v) \,=\, c_M \exp\Big[-\beta\Big(\tfrac12|v|^2 + A(x) + \kappa W * M(x)\Big)\Big], \qquad (x,v) \in \mathbb{X},$$

where c_M is the normalizing constant such that $\int_\mathbb{X} M(x,v)\,\mathrm{d}x\mathrm{d}v \,=\, 1$. Note that this fixed-point equation has indeed a unique solution provided that $\kappa\beta\|W\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < 1$. In order to prove Proposition 3.3, we shall first establish the exponential decay on negative Sobolev spaces with this Gibbs weight M, that is, the exponential decay on the smaller spaces $H^{-k}(M^{-1/2})$, and next we shall appeal to the enlargement theory of Gualdani, Mischler and Mouhot [53, 74] to conclude with the desired result on $W^{-k,q}(\langle z\rangle^p)$. Here, for all $k \geq 0$, the space $H^{-k}(M^{-1/2})$ is defined as the weighted negative Sobolev space associated with the dual norm

$$\|h\|_{H^{-k}(M^{-1/2})} \,:=\, \sup\bigg\{ \int_\mathbb{X} hh' M^{-1} \,:\, \|h'\|_{H^k(M^{-1/2})} = 1 \bigg\}, \tag{3.9}$$

where $H^k(M^{-1/2})$ is the standard weighted Sobolev space with norm

$$\|h\|_{H^k(M^{-1/2})} \,:=\, \sup_{i,j\geq 0,\, i+j\leq k} \|\nabla_x^i \nabla_v^j h\|_{\mathrm{L}^2(M^{-1/2})}, \qquad \|h\|_{\mathrm{L}^2(M^{-1/2})} \,:=\, \Big(\int_\mathbb{X} |h|^2 M^{-1}\Big)^{\frac12}.$$

Note that the treatment of the weight in the definition of those weighted spaces differs slightly from the one in the definition of $W^{-k,q}(\langle z\rangle^p)$, cf. (3.3), but for convenience we stick to this slight inconsistency in the choice of definitions.

In order to appeal to enlargement theory, we start by introducing a suitable decomposition of the operator R_μ. Let a cut-off function $\chi \in C_c^\infty(\mathbb{X})$ be fixed with $\chi(z) = 1$ for $|z| \leq 1$, and set

$$\chi_R(z) := \chi(\tfrac1R z). \tag{3.10}$$

In those terms, let us split the operator R_μ as follows,

$$\begin{aligned} R_\mu &:= A + B_\mu, \\ Ah &:= \Lambda\chi_R h, \\ B_\mu h &:= \tfrac12 \mathrm{div}_v((\nabla_v + \beta v)h) - v\cdot\nabla_x h + (\nabla A + \kappa\nabla W * \mu)\cdot\nabla_v h - \Lambda\chi_R h, \end{aligned} \tag{3.11}$$

for some constants $\Lambda, R > 0$ to be properly chosen later on (see Lemmas 3.5 and 3.6 below). Let us denote by $\{W_{t,s}\}_{t\ge s\ge 0}$ the fundamental solution operators for the (non-autonomous) evolution equation associated with B_μ: for all $s\ge 0$ and $h_s\in C_c^\infty(\mathbb{X})$, we define $h_t = W_{t,s}h_s$ as the solution of

$$\begin{cases} \partial_t h_t = B_{\mu_t} h_t, & \text{for } t\ge s,\\ h_t|_{t=s} = h_s. \end{cases}$$

Again, it is easily checked that this equation is well-posed with $h\in C_{\mathrm{loc}}([s,\infty);\mathrm{L}^2(M^{-1/2}))$ whenever $h_s\in \mathrm{L}^2(M^{-1/2})$. Our proof of Proposition 3.3 is based on the following three preliminary lemmas, the proofs of which are postponed to Sections 3.3, 3.4, 3.5, and 3.6 below.

Lemma 3.4 (Exponential decay on restricted space). *Let κ_0 be as in Theorem 3.1(i) and let $\kappa\in[0,\kappa_0]$. There exists a constant $\lambda_1>0$ (only depending on d,β,a, and $\|W\|_{W^{1,\infty}(\mathbb{R}^d)}$), such that the following holds: given $\lambda\in[0,\lambda_1)$ and $k\ge 0$, we have for all $h_s\in C_c^\infty(\mathbb{X})$ and $t\ge s\ge 0$,*

$$\Big\|V_{t,s}h_s - M\int_{\mathbb{X}} h_s\Big\|_{H^{-k}(M^{-1/2})} \lesssim_{W,\beta,\lambda,k,a} e^{-\lambda(t-s)}\|h_s\|_{H^{-k}(M^{-1/2})},$$

where the multiplicative constant only depends on d,β,λ,k,a, and $\|W\|_{W^{k+2,\infty}(\mathbb{R}^d)}$.

Lemma 3.5 (Exponential decay for modified operator). *Let κ_0 be as in Theorem 3.1(i) and let $\kappa\in[0,\kappa_0]$. There exists a constant $\lambda_2>0$ (only depending on d,β,a, and $\|W\|_{W^{1,\infty}(\mathbb{R}^d)}$), such that the following holds: given $1<q\le 2$ and $0<p\le 1$ with $pq'\gg_{\beta,a}1$ large enough (only depending on d,β,a), choosing Λ,R large enough (only depending on d,β,p,a, and $\|W\|_{W^{1,\infty}(\mathbb{R}^d)}$), we have for all $\lambda\in[0,\lambda_2)$, $k\ge 0$, $h_s\in C_c^\infty(\mathbb{X})$, and $t\ge s\ge 0$,*

$$\|W_{t,s}h_s\|_{W^{-k,q}(\langle z\rangle^p)} \lesssim_{W,\beta,\lambda,k,p,q,a} e^{-\lambda(t-s)}\|h_s\|_{W^{-k,q}(\langle z\rangle^p)}, \tag{3.12}$$

$$\|W_{t,s}h_s\|_{H^{-k}(M^{-1/2})} \lesssim_{W,\beta,\lambda,k,p,q,a} e^{-\lambda(t-s)}\|h_s\|_{H^{-k}(M^{-1/2})}, \tag{3.13}$$

where the multiplicative constants only depend on d,β,λ,k,p,q,a, and $\|W\|_{W^{k+1,\infty}(\mathbb{R}^d)}$.

Lemma 3.6 (Regularization estimate). *Let κ_0,λ_2 be as in Theorem 3.1(i) and Lemma 3.5, respectively, and let $\kappa\in[0,\kappa_0]$. There is some $n\ge 1$ large enough (only depending on d) such that the following holds: given $1<q\le 2$ and $0<p\le 1$ with $pq'\gg_{\beta,a}1$ large enough (only depending on d,β,a), choosing Λ,R large enough (only depending on d,β,p,a, and $\|W\|_{W^{1,\infty}(\mathbb{R}^d)}$), we have for all $\lambda\in[0,\lambda_2)$, $k\ge0$, $h_s\in C_c^\infty(\mathbb{X})$, and $t\ge s\ge 0$,*

$$\int_{s\le s_1\le\ldots\le s_n\le t}\|AW_{t,s_n}AW_{s_n,s_{n-1}}\ldots AW_{s_1,s}h_s\|_{H^{-k}(M^{-1/2})}\,\mathrm{d}s_1\ldots\mathrm{d}s_n \lesssim_{W,\beta,\lambda,k,p,q,a} e^{-p\lambda(t-s)}\|h_s\|_{W^{-k,q}(\langle z\rangle^p)},$$

where the multiplicative constant only depends on d,β,λ,k,p,q,a, and $\|W\|_{W^{k+d+1,\infty}(\mathbb{R}^d)}$.

With those lemmas at hand, we are now in position to conclude the proof of Proposition 3.3 based on the enlargement theory of Gualdani, Mischler, and Mouhot [53, 74].

Proof of Proposition 3.3. Let λ_1,λ_2 be defined in Lemmas 3.4 and 3.5, respectively, and let $1<q\le 2$, $k\ge 0$, and $0<p\le 1$ with $pq'\gg 1$ large enough in the sense of Lemmas 3.5 and 3.6. We note that the space $H^{-k}(M^{-1/2})$ is continuously embedded in $W^{-k,q}(\langle z\rangle^p)$: by definition of dual norms, we find for all $h\in C_c^\infty(\mathbb{X})$,

$$\|h\|_{W^{-k,q}(\langle z\rangle^p)} \lesssim_{W,\beta,k,a} \|h\|_{H^{-k}(M^{-1/2})}, \tag{3.14}$$

where the constant only depends on d,β,k,a, and $\|W\|_{W^{k,\infty}(\mathbb{R}^d)}$. In this setting, we can appeal to enlargement theory to extend the estimates of Lemma 3.4 to $W^{-k,q}(\langle z\rangle^p)$: by Lemmas 3.4, 3.5, and 3.6, we can apply [74, Theorem 1.1] and the conclusion precisely follows. For completeness, we include a short proof of enlargement as the present situation does not exactly fit in the semigroup setting of [74].

Starting point is the following form of the Duhamel formula: based on the decomposition (3.11), the fundamental solution operators $\{V_{t,s}\}_{0\le s\le t}$ can be expanded around $\{W_{t,s}\}_{0\le s\le t}$ via

$$V_{t,s} \,=\, W_{t,s} + \int_s^t V_{t,u} A W_{u,s}\, du.$$

By iteration, we get for all $n \ge 1$,

$$V_{t,s} \,=\, W_{t,s} + \sum_{j=1}^{n-1} \int_s^t W_{t,u} (AW)^j_{u,s}\, du + \int_s^t V_{t,u} (AW)^n_{u,s}\, du,$$

where we have set for abbreviation, for all $j \ge 1$ and $0 \le s_0 \le s_j$,

$$(AW)^j_{s_j,s_0} \,:=\, \int_{(\mathbb{R}^+)^{j-1}} AW_{s_j,s_{j-1}} AW_{s_{j-1},s_{j-2}} \ldots AW_{s_1,s_0} \mathbb{1}_{s_0\le s_1\le\ldots\le s_j}\, ds_1 \ldots ds_{j-1}.$$

Given $h_s \in C_c^\infty(\mathbb{X})$, taking norms, applying the exponential decay of Lemma 3.5 for $\{W_{t,s}\}_{0\le s\le t}$ on the space $W^{-k,q}(\langle z\rangle^p)$, noting that A is bounded on $W^{-k,q}(\langle z\rangle^p)$ and that $|\int_{\mathbb{X}} h| \lesssim_{k,p,q} \|h\|_{W^{-k,q}(\langle z\rangle^p)}$ provided $pq' > 2d$, we get for all $n \ge 1$ and $\lambda \in [0, \lambda_2)$,

$$\begin{aligned} \Big\|V_{t,s} h_s - M \int_{\mathbb{X}} h_s\Big\|_{W^{-k,q}(\langle z\rangle^p)} \,&=\, \Big\|V_{t,s} h_s - M \int_{\mathbb{X}} V_{t,s} h_s\Big\|_{W^{-k,q}(\langle z\rangle^p)} \\ &\lesssim_{W,\beta,\lambda,k,p,q,n,a}\, \big(1 + (t-s)^n\big) e^{-p\lambda(t-s)} \|h_s\|_{W^{-k,q}(\langle z\rangle^p)} \\ &\qquad + \int_s^t \Big\|V_{t,u}(AW)^n_{u,s} h_s - M \int_{\mathbb{X}} (AW)^n_{u,s} h_s\Big\|_{W^{-k,q}(\langle z\rangle^p)} du. \end{aligned}$$

In order to estimate the last right-hand side term, we recall the embedding (3.14), we use the exponential relaxation of Lemma 3.4 for $\{V_{t,s}\}_{0\le s\le t}$ on the space $H^{-k}(M^{-1/2})$, and we use the regularization estimate of Lemma 3.6 for $n \ge 1$ large enough (only depending on d): for $\lambda \in [0, \lambda_1 \wedge \lambda_2)$, this leads us to

$$\begin{aligned} &\Big\|V_{t,s} h_s - M \int_{\mathbb{X}} h_s\Big\|_{W^{-k,q}(\langle z\rangle^p)} \\ &\quad\lesssim_{W,\beta,\lambda,k,p,q,a}\, \big(1 + (t-s)^n\big) e^{-p\lambda(t-s)} \|h_s\|_{W^{-k,q}(\langle z\rangle^p)} + \int_s^t e^{-\lambda(t-u)} \|(AW)^n_{u,s} h_s\|_{H^{-k}(M^{-1/2})}\, du \\ &\quad\lesssim_{W,\beta,\lambda,k,p,q,a}\, \big(1 + (t-s)^n\big) e^{-p\lambda(t-s)} \|h_s\|_{W^{-k,q}(\langle z\rangle^p)}, \end{aligned}$$

and the conclusion follows with $\lambda_0 = \lambda_1 \wedge \lambda_2$. ☐

3.2. **Proof of Theorem 3.1(ii).** In this section, we establish Theorem 3.1(ii) as a consequence of Proposition 3.3. As a preliminary, we start by noting that the convergence of the mean-field evolution (1.21) to equilibrium as stated in Theorem 3.1(i) also holds on the spaces $W^{-k,q}(\langle z\rangle^p)$.

Lemma 3.7. *Let κ_0, λ_0 be as in Proposition 3.3 and let $\kappa \in [0, \kappa_0]$. Given $1 < q \le 2$ and $0 < p \le 1$ with $pq' \gg_{\beta,a} 1$ large enough (only depending on d, β, a), the mean-field evolution (1.21) satisfies for all $\lambda \in [0, \lambda_0)$, $k \ge 0$, $\mu_\circ \in \mathcal{P} \cap C_c^\infty(\mathbb{X})$, and $t \ge 0$,*

$$\|m(t, \mu_\circ) - M\|_{W^{-k,q}(\langle z\rangle^p)} \,\lesssim_{W,\beta,\lambda,k,p,q,a}\, e^{-p\lambda t} \|\mu_\circ\|_{W^{-k,q}(\langle z\rangle^p)},$$

hence, in particular,

$$\|m(t, \mu_\circ)\|_{W^{-k,q}(\langle z\rangle^p)} \,\lesssim_{W,\beta,\lambda,k,p,q,a}\, \|\mu_\circ\|_{W^{-k,q}(\langle z\rangle^p)}, \tag{3.15}$$

where the multiplicative constants only depend on $d, \beta, \lambda, k, p, q, a$, and $\|W\|_{W^{k+d+1,\infty}(\mathbb{R}^d)}$.

Proof. The mean-field equation (1.7) for $\mu_t := m(t, \mu_\circ)$ can be written as $\partial_t m(t, \mu_\circ) = R_{\mu_t} m(t, \mu_\circ)$, which entails $m(t, \mu_\circ) = V_{t,0}\mu_\circ$, and the conclusion then follows from Proposition 3.3. ☐

With the above estimate at hand, we can finally conclude the proof of Theorem 3.1(ii) in the Langevin setting.

Proof of Theorem 3.1(ii). By a standard approximation argument, it suffices to consider $f_\circ \in C_c^\infty(\mathbb{X})$ and $r \in C_c^\infty(\mathbb{R}^+ \times \mathbb{X})$, with $\int_\mathbb{X} f_\circ = 0$ and $\int_\mathbb{X} r_t = 0$ for all t. In that case, the well-posedness of the Cauchy problem (3.5) is standard and it remains to establish the stability estimate (3.6). In terms of the modified linearized operator R_μ defined in (3.7), setting $\mu_t := m(t,\mu_\circ)$, equation (3.5) can be reformulated as

$$\partial_t f_t \,=\, R_{\mu_t} f_t + \kappa(\nabla W * f_t)\cdot\nabla_v \mu_t + r_t,$$

hence, by Duhamel's formula,

$$f_t = V_{t,0} f_\circ + \int_0^t V_{t,s}\Big(\kappa(\nabla W * f_s)\cdot\nabla_v\mu_s + r_s\Big)\,\mathrm{d}s.$$

Appealing to the exponential decay of Proposition 3.3 for $\{V_{t,s}\}_{t\ge s\ge 0}$ with $\int_\mathbb{X} f_\circ = 0$ and $\int_\mathbb{X} r_t = 0$, noting that for $pq' > 2d$ we have

$$\begin{aligned}
&\|(\nabla W * f_s)\cdot\nabla_v\mu_s\|_{W^{-k,q}(\langle z\rangle^p)}\\
&\quad\lesssim_k \quad \|W * f_s\|_{W^{k,\infty}(\mathbb{X})}\|\mu_s\|_{W^{1-k,q}(\langle z\rangle^p)}\\
&\quad\lesssim_{k,p,q} \quad \|W\|_{W^{2k,\infty}(\mathbb{R}^d)}\|f_s\|_{W^{-k,q}(\langle z\rangle^p)}\|\mu_s\|_{W^{1-k,q}(\langle z\rangle^p)},
\end{aligned}$$

and further appealing to the a priori estimate (3.15) in Lemma 3.7 above, we deduce for all $\lambda\in[0,\lambda_0)$ and $k\ge 1$,

$$e^{p\lambda t}\|f_t\|_{W^{-k,q}(\langle z\rangle^p)} \lesssim_{W,\beta,\lambda,k,p,q,a,\mu_\circ} \|f_\circ\|_{W^{-k,q}(\langle z\rangle^p)} + \int_0^t e^{p\lambda s}\Big(\|f_s\|_{W^{-k,q}(\langle z\rangle^p)} + \|r_s\|_{W^{-k,q}(\langle z\rangle^p)}\Big)\,\mathrm{d}s.$$

The conclusion follows from Grönwall's inequality. □

3.3. **Proof of Lemma 3.4: ergodic estimates with Gibbs weight.** This section is devoted to the proof of Lemma 3.4. We start by considering the standard kinetic Fokker–Planck operator

$$R_M h \,:=\, \tfrac{1}{2}\mathrm{div}_v((\nabla_v + \beta v)h) - v\cdot\nabla_x h + (\nabla A + \kappa\nabla W * M)\cdot\nabla_v h.$$

The exponential relaxation of the associated semigroup $\{e^{tR_M}\}_{t\ge0}$ on $\mathrm{L}^2(M^{-1/2})$ was established in the seminal work of Dolbeault, Mouhot, and Schmeiser [38] based on hypocoercivity techniques. We post-process this well-known result to further derive estimates on Sobolev spaces with arbitrary negative regularity. For that purpose, we appeal to a duality argument and argue by induction using parabolic estimates.

Lemma 3.8. *Let κ_0 be as in Theorem 3.1(i) and let $\kappa\in[0,\kappa_0]$. There exists $\lambda_1>0$ (only depending on $d,\beta,a,$ and $\|W\|_{W^{1,\infty}(\mathbb{X})}$) such that for all $\lambda\in[0,\lambda_1)$, $t\ge0$ and $h\in C_c^\infty(\mathbb{X})$,*

$$\Big\|e^{tR_M}h - M\int_\mathbb{X} h\Big\|_{H^{-k}(M^{-1/2})} \lesssim_{W,\beta,k,a} e^{-\lambda t}\|h\|_{H^{-k}(M^{-1/2})}, \tag{3.16}$$

where the multiplicative factor only depends on $d,\beta,k,a,$ and $\|W\|_{W^{k+1,\infty}(\mathbb{R}^d)}$.

Proof. We set for abbreviation $\pi_M^\perp h := h - M\int_\mathbb{X} h$, and we note that $\int_\mathbb{X} e^{tR_M}h = \int_\mathbb{X} h$ for all $t\ge0$. By definition of dual norms, cf. (3.9), also recalling the definition of the steady state M, it suffices to show that there is some $\lambda_1>0$ such that for all $k\ge0$, $\lambda\in[0,\lambda_1)$, $t\ge0$, and $h\in C_c^\infty(\mathbb{X})$ we have

$$\|\pi_M^\perp e^{tR_M^*}h\|_{H^k(M^{-1/2})} \lesssim_{W,\beta,k,a} e^{-\lambda t}\|h\|_{H^k(M^{-1/2})},$$

where R_M^* stands for the dual Fokker–Planck operator

$$R_M^* h \,=\, \tfrac{1}{2}\mathrm{div}_v((\nabla_v+\beta v)h) + v\cdot\nabla_x h - (\nabla A + \kappa\nabla W * M)\cdot\nabla_v h. \tag{3.17}$$

We shall actually prove the following more detailed estimate, further capturing the dissipation: there is some $\lambda_1 > 0$ such that for all $k \ge 0$, $\lambda \in [0, \lambda_1)$, $t \ge 0$, and $h \in C_c^\infty(\mathbb{X})$ we have

$$e^{\lambda t}\|\pi_M^\perp e^{tR_M^*}h\|_{H^k(M^{-1/2})} + \Big(\int_0^t e^{2\lambda s}\|(\nabla_v + \beta v)e^{sR_M^*}h\|^2_{H^k(M^{-1/2})}\,\mathrm{d}s\Big)^{\frac12} \lesssim_{W,\beta,\lambda,k,a} \|h\|_{H^k(M^{-1/2})}. \quad (3.18)$$

We split the proof into two steps.

Step 1. Case $k = 0$: there exists $\lambda_1 > 0$ (only depending on d, β, a, and $\|W\|_{W^{1,\infty}(\mathbb{X})}$) such that for all $t \ge 0$ and $h \in C_c^\infty(\mathbb{X})$,

$$\|\pi_M^\perp e^{tR_M^*}h\|_{\mathrm{L}^2(M^{-1/2})} \lesssim_{W,\beta,a} e^{-\lambda_1 t}\|\pi_M^\perp h\|_{\mathrm{L}^2(M^{-1/2})}. \quad (3.19)$$

This was precisely established by Dolbeault, Mouhot, and Schmeiser in [38, Theorem 10].

Step 2. Conclusion: proof of (3.18).
Given $h \in C_c^\infty(\mathbb{X})$, we set for shortness $J_t^{\alpha,\gamma} := \nabla_x^\alpha\nabla_v^\gamma\pi_M^\perp e^{tR_M^*}h$ for multi-indices $\alpha, \gamma \in \mathbb{N}^d$. By definition, it satisfies

$$\begin{cases} \partial_t J_t^{\alpha,\gamma} = R_M^* J_t^{\alpha,\gamma} + r_t^{\alpha,\gamma}, & \text{for } t \ge 0,\\ J_t^{\alpha,\gamma}|_{t=0} = \nabla_x^\alpha\nabla_v^\gamma(\pi_M^\perp h), \end{cases} \quad (3.20)$$

where the source term $r_t^{\alpha,\gamma}$ is given by

$$r_t^{\alpha,\gamma} := [\nabla_x^\alpha\nabla_v^\gamma, R_M^*]\pi_M^\perp e^{tR_M^*}h. \quad (3.21)$$

On the one hand, by Duhamel's formula in form of

$$J_t^{\alpha,\gamma} = e^{tR_M^*}\nabla_x^\alpha\nabla_v^\gamma(\pi_M^\perp h) + \int_0^t e^{(t-s)R_M^*}r_s^{\alpha,\gamma}\,\mathrm{d}s,$$

the exponential decay (3.19) yields

$$\|J_t^{\alpha,\gamma}\|_{\mathrm{L}^2(M^{-1/2})} \lesssim_{W,\beta,a} e^{-\lambda_1 t}\|\nabla_x^\alpha\nabla_v^\gamma(\pi_M^\perp h)\|_{\mathrm{L}^2(M^{-1/2})} + \int_0^t e^{-\lambda_1(t-s)}\|r_s^{\alpha,\gamma}\|_{\mathrm{L}^2(M^{-1/2})}\,\mathrm{d}s. \quad (3.22)$$

On the other hand, integrating by parts, the energy identity for equation (3.20) takes the form

$$\begin{aligned}\partial_t\|J_t^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})} &= 2\int_{\mathbb{X}} J_t^{\alpha,\gamma}(R_M^*J_t^{\alpha,\gamma})M^{-1} + 2\int_{\mathbb{X}} J_t^{\alpha,\gamma}r_t^{\alpha,\gamma}M^{-1}\\ &\le -\|(\nabla_v + \beta v)J_t^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})} + 2\|J_t^{\alpha,\gamma}\|_{\mathrm{L}^2(M^{-1/2})}\|r_t^{\alpha,\gamma}\|_{\mathrm{L}^2(M^{-1/2})}. \quad (3.23)\end{aligned}$$

Regarding the dissipation term in this last estimate, we make the following observation: integrating by parts and using $\nabla_v M^{-1} = \beta v M^{-1}$, we find for all $f \in C_c^\infty(\mathbb{X})$,

$$\begin{aligned}\int_{\mathbb{X}} |\nabla_v f|^2 M^{-1} &= -\int_{\mathbb{X}} M^{-1} f\big((\nabla_v + \beta v)\cdot\nabla_v f\big)\\ &= -\int_{\mathbb{X}} M^{-1} f\,\mathrm{div}_v\big((\nabla_v + \beta v)f\big) + \beta d\int_{\mathbb{X}} |f|^2 M^{-1}\\ &= \int_{\mathbb{X}} |(\nabla_v + \beta v)f|^2 M^{-1} + \beta d\int_{\mathbb{X}} |f|^2 M^{-1}. \quad (3.24)\end{aligned}$$

Using this to replace half of the dissipation term in (3.23), we get

$$\partial_t\|J_t^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})} + \tfrac12\Big(\|\nabla_v J_t^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})} + \|(\nabla_v + \beta v)J_t^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big) \le \tfrac{\beta d}{2}\|J_t^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})} + 2\|J_t^{\alpha,\gamma}\|_{\mathrm{L}^2(M^{-1/2})}\|r_t^{\alpha,\gamma}\|_{\mathrm{L}^2(M^{-1/2})},$$

and thus, by Grönwall's inequality, for all $\lambda \geq 0$,

$$\sup_{0\le s\le t}\Big(e^{2\lambda s}\|J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big)+\int_0^t e^{2\lambda s}\Big(\|\nabla_v J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}+\|(\nabla_v+\beta v)J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big)\,ds$$
$$\lesssim_{\beta,\lambda}\ \|\nabla_x^\alpha\nabla_v^\gamma(\pi_M^\perp h)\|^2_{\mathrm{L}^2(M^{-1/2})}+\Big(\int_0^t e^{\lambda s}\|r_s^{\alpha,\gamma}\|_{\mathrm{L}^2(M^{-1/2})}\,ds\Big)^2+\int_0^t e^{2\lambda s}\|J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}\,ds.$$

Now using (3.22) to bound the last term, we obtain for all $0\le\lambda<\lambda'<\lambda_1$,

$$\sup_{0\le s\le t}\Big(e^{2\lambda s}\|J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big)+\int_0^t e^{2\lambda s}\Big(\|\nabla_v J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}+\|(\nabla_v+\beta v)J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big)\,ds$$
$$\lesssim_{W,\beta,\lambda,\lambda',a}\ \|\nabla_x^\alpha\nabla_v^\gamma(\pi_M^\perp h)\|^2_{\mathrm{L}^2(M^{-1/2})}+\Big(\int_0^t e^{\lambda' s}\|r_s^{\alpha,\gamma}\|_{\mathrm{L}^2(M^{-1/2})}\,ds\Big)^2.$$

By definition of R_M^*, cf. (3.17), the source term $r^{\alpha,\gamma}$ defined in (3.21) takes the form

$$r_t^{\alpha,\gamma}=\sum_{i:e_i\le\gamma}\binom{\gamma}{e_i}J_t^{\alpha+e_i,\gamma-e_i}+\sum_{(\alpha',\gamma')<(\alpha,\gamma)}\binom{\alpha}{\alpha'}\binom{\gamma}{\gamma'}\nabla_x^{\alpha-\alpha'}\nabla_v^{\gamma-\gamma'}\big(\tfrac{\beta}{2}v-\nabla A-\kappa\nabla W*M\big)\cdot\nabla_v J_t^{\alpha',\gamma'},$$

so the above yields for all $0\le\lambda<\lambda'<\lambda_1$,

$$\sup_{0\le s\le t}\Big(e^{2\lambda s}\|J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big)+\int_0^t e^{2\lambda s}\Big(\|\nabla_v J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}+\|(\nabla_v+\beta v)J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big)\,ds$$
$$\lesssim_{W,\beta,\lambda,\lambda',\alpha,\gamma,a}\ \|\nabla_x^\alpha\nabla_v^\gamma(\pi_M^\perp h)\|^2_{\mathrm{L}^2(M^{-1/2})}+\max_{i:e_i\le\gamma}\sup_{0\le s\le t}\Big(e^{2\lambda' s}\|J_s^{\alpha+e_i,\gamma-e_i}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big)$$
$$+\max_{(\alpha',\gamma')<(\alpha,\gamma)}\int_0^t e^{2\lambda' s}\|\nabla_v J_s^{\alpha',\gamma'}\|^2_{\mathrm{L}^2(M^{-1/2})}\,ds.$$

A direct induction then yields for all $\alpha,\gamma\ge0$, $\lambda\in[0,\lambda_1)$, and $t\ge0$,

$$e^{2\lambda t}\|J_t^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}+\int_0^t e^{2\lambda s}\Big(\|\nabla_v J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}+\|(\nabla_v+\beta v)J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big)\,ds$$
$$\lesssim_{W,\beta,\lambda,\alpha,\gamma,a}\ \|\nabla_x^\alpha\nabla_v^\gamma(\pi_M^\perp h)\|^2_{\mathrm{L}^2(M^{-1/2})}.$$

Recalling $J_t^{\alpha,\gamma}=\nabla_x^\alpha\nabla_v^\gamma\pi_M^\perp e^{tR_M^*}h$ and $\pi_M^\perp h=h-M\int_{\mathbb X}h$, this proves the claim (3.18). ☐

With the above exponential decay for the Fokker–Planck semigroup, we can easily conclude the proof of Lemma 3.4 by means of a perturbation argument.

Proof of Lemma 3.4. Let λ_0,λ_1 be as in Theorem 3.1(i) and in Lemma 3.8, respectively. We split the proof into two steps.

Step 1. Proof that for all $k\ge0$, $\lambda\in[0,\lambda_0\wedge\lambda_1)$, $t\ge s\ge0$, and $g_s\in C_c^\infty(\mathbb X)$,

$$e^{\lambda(t-s)}\|V_{t,s}\pi_M^\perp g_s\|_{H^{-k}(M^{-1/2})}\ \lesssim_{W,\beta,\lambda,k,a}\ \|g_s\|_{H^{-k}(M^{-1/2})}. \tag{3.25}$$

Duhamel's formula yields

$$V_{t,s}\pi_M^\perp g_s\ =\ e^{(t-s)R_M}\pi_M^\perp g_s+\kappa\int_s^t e^{(t-u)R_M}\big((\nabla W*(\mu_u-M))\cdot\nabla_v(V_{u,s}\pi_M^\perp g_s)\big)\,du,$$

and thus, for all $h_t\in C_c^\infty(\mathbb X)$, integrating by parts and using $\pi_M^\perp e^{tR_M}=e^{tR_M}\pi_M^\perp$,

$$\int_{\mathbb X}(V_{t,s}\pi_M^\perp g_s)h_tM^{-1}\ =\ \int_{\mathbb X}\big(e^{(t-s)R_M^*}\pi_M^\perp h_t\big)g_sM^{-1}$$
$$-\kappa\int_s^t\Big(\int_{\mathbb X}\big((\nabla_v+\beta v)e^{(t-u)R_M^*}h_t\big)\cdot(\nabla W*(\mu_u-M))(V_{u,s}\pi_M^\perp g_s)M^{-1}\Big)\,du.$$

Applying the exponential decay estimate (3.18) of Lemma 3.8, and taking the supremum over h_t in $H^k(M^{-1/2})$, we deduce for all $k\ge 0$, $\lambda\in[0,\lambda_1)$, and $t\ge0$,

$$e^{\lambda(t-s)}\|V_{t,s}\pi_M^\perp g_s\|_{H^{-k}(M^{-1/2})}\lesssim_{W,\beta,\lambda,k,a}\|g_s\|_{H^{-k}(M^{-1/2})}+\kappa\Big(\int_s^t e^{2\lambda(u-s)}\|V_{u,s}\pi_M^\perp g_s\|^2_{H^{-k}(M^{-1/2})}\|\nabla W*(\mu_u-M)\|^2_{W^{k,\infty}(\mathbb{R}^d)}\mathrm{d}u\Big)^{\frac12}.$$

Now, by Theorem 3.1(i), we have

$$\|\nabla W*(\mu_t-M)\|_{W^{k,\infty}(\mathbb{R}^d)}\lesssim_{W,k}\mathcal{W}_2(\mu_t,M)\lesssim_{W,\beta,a}e^{-\lambda_0 t}\mathcal{W}_2(\mu_\circ,M),\tag{3.26}$$

and the claim (3.25) then follows from Grönwall's inequality.

Step 2. Conclusion.
It remains to replace $V_{t,s}\pi_M^\perp$ by $\pi_M^\perp V_{t,s}$ in the result (3.25) of Step 1. For that purpose, let us decompose

$$\pi_M^\perp V_{t,s}g_s=V_{t,s}\pi_M^\perp g_s+\Big(\int_{\mathbb{X}}g_s\Big)r_{t,s},\qquad r_{t,s}:=\pi_M^\perp V_{t,s}M.\tag{3.27}$$

By definition, $r_{t,s}$ satisfies

$$\begin{aligned}\partial_t r_{t,s}&=R_MV_{t,s}M+\kappa(\nabla W*(\mu_t-M))\cdot\nabla_vV_{t,s}M\\&=R_Mr_{t,s}+\kappa(\nabla W*(\mu_t-M))\cdot\nabla_vr_{t,s}+\kappa(\nabla W*(\mu_t-M))\cdot\nabla_vM,\end{aligned}$$

with $r_{t,s}|_{t=s}=0$. By Duhamel's formula, this yields

$$r_{t,s}=\kappa\int_s^t V_{t,u}\big(\nabla W*(\mu_u-M)\cdot\nabla_vM\big)\,\mathrm{d}u.$$

Applying the relaxation estimate (3.25) of Step 1, and using Theorem 3.1(i) again in form of (3.26), we deduce for all $k\ge0$, $\lambda\in[0,\lambda_0\wedge\lambda_1)$, and $t\ge s\ge0$,

$$\|r_{t,s}\|_{H^{-k}(M^{-1/2})}\lesssim_{W,\beta,\lambda,k,a}e^{-\lambda t}.$$

Together with (3.25) and (3.27), this yields the conclusion (up to renaming λ_1). ☐

3.4. **Proof of Lemma 3.5 on** $W^{-k,q}(\langle z\rangle^p)$**.** This section is devoted to the proof of (3.12). Instead of $\langle z\rangle^p$, we shall consider deformed weights of the form ω^p in terms of

$$\omega(x,v):=1+\beta\Big(\tfrac12|v|^2+a|x|^2+\eta x\cdot v\Big),\tag{3.28}$$

where the parameter $0<\eta\ll1$ will be properly chosen later on. We naturally restrict to $\eta\le\frac12\sqrt a$, which ensures

$$\omega(x,v)\simeq_{\beta,a}\langle z\rangle^2.$$

Note that those weights differ from the choice used in [74] and are critical for the improved result we establish in this work. We define the weighted negative Sobolev spaces $W^{-k,q}(\omega^p)$ exactly as the spaces $W^{-k,q}(\langle z\rangle^p)$ in (3.3), simply replacing the weight $\langle z\rangle^p$ by ω^p in the definition. Comparing ω and $\langle z\rangle^2$, the definition of dual norms easily ensures for all $h\in C_c^\infty(\mathbb{X})$,

$$\|h\|_{W^{-k,q}(\langle z\rangle^{2p})}\simeq_{\beta,k,p,a}\|h\|_{W^{-k,q}(\omega^p)}.\tag{3.29}$$

For a densely-defined operator X on $\mathrm{L}^q(\omega^p):=W^{0,q}(\omega^p)$, we denote by $X^{*,p}$ its adjoint on $\mathrm{L}^{q'}(\mathbb{X})$ with respect to the weighted duality product $(g,h)\mapsto\int_{\mathbb{X}}gh\omega^p$: more precisely, $X^{*,p}$ stands for the closed operator on $\mathrm{L}^{q'}(\mathbb{X})$ defined by the relation

$$\int_{\mathbb{X}}g(Xh)\omega^p=\int_{\mathbb{X}}h(X^{*,p}g)\omega^p.$$

In particular, for $\mu \in \mathcal{P}(\mathbb{X})$, we consider the weighted adjoint $B_\mu^{*,p}$ of B_μ, which takes the explicit form

$$B_\mu^{*,p} h \;=\; \tfrac12 \triangle_v h + v\cdot\nabla_x h - \Big(\tfrac12\beta v + \nabla A + \kappa\nabla W*\mu - \omega^{-p}\nabla_v\omega^p\Big)\cdot\nabla_v h \\ + \Big(\tfrac12\omega^{-p}\triangle_v\omega^p + \omega^{-p} v\cdot\nabla_x\omega^p - \omega^{-p}\big(\tfrac12\beta v + \nabla A + \kappa\nabla W*\mu\big)\cdot\nabla_v\omega^p - \Lambda\chi_R\Big)h. \quad (3.30)$$

By the equivalence of norms (3.29) and by the definition of dual norms, it suffices to prove that there is some $0<\eta\le\frac12\sqrt a$ and some $\lambda_2>0$ (only depending on d,β,a) such that the following result holds: given $1<q\le 2$ and $0<p\le 1$ with $pq'\gg_{\beta,a}1$ large enough (only depending on d,β,a), choosing Λ,R large enough (only depending on d,β,p,a, and $\|W\|_{W^{1,\infty}(\mathbb{R}^d)}$), if $g\in C([0,t];\mathrm{L}^{q'}(\mathbb{X}))$ satisfies the backward Cauchy problem

$$\begin{cases} \partial_s g_s = -B_{\mu_s}^{*,p} g_s, & \text{for } 0\le s\le t,\\ g_s|_{s=t} = g_t, \end{cases} \quad (3.31)$$

for some $t\ge 0$ and $g_t\in C_c^\infty(\mathbb{X})$, then we have for all $\lambda\in[0,\lambda_2)$, $k\ge 0$, and $0\le s\le t$,

$$\|g_s\|_{W^{k,q'}(\mathbb{X})} \lesssim_{W,\beta,\lambda,k,p,q,a} e^{-\lambda(t-s)}\|g_t\|_{W^{k,q'}(\mathbb{X})}. \quad (3.32)$$

We split the proof into two steps, starting with the case $k=0$ before treating all $k\ge0$ by induction.

Step 1. Proof of (3.32) for $k=0$.
By definition of $B_\mu^{*,p}$ and by integration by parts, we find

$$\partial_s\|g_s\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} \;=\; -q'\int_{\mathbb{X}}|g_s|^{q'-2}g_s B_{\mu_s}^{*,p}g_s \\ =\; \tfrac12 q'(q'-1)\int_{\mathbb{X}}|g_s|^{q'-2}|\nabla_v g_s|^2 - \int_{\mathbb{X}}|g_s|^{q'}\Big(q'\omega^{-p}v\cdot\nabla_x\omega^p - q'\omega^{-p}\big(\tfrac12\beta v+\nabla A+\kappa\nabla W*\mu\big)\cdot\nabla_v\omega^p \\ - q'\Lambda\chi_R + \tfrac12 q'\omega^{-p}\triangle_v\omega^p + \tfrac{\beta d}{2} - \operatorname{div}_v(\omega^{-p}\nabla_v\omega^p)\Big).$$

Now inserting the form of the weight ω and explicitly computing its derivatives, we get after straightforward simplifications,

$$\partial_s\|g_s\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} \;\ge\; \tfrac12 q'(q'-1)\int_{\mathbb{X}}|g_s|^{q'-2}|\nabla_v g_s|^2 \\ + \int_{\mathbb{X}}|g_s|^{q'}\Big(pq'\beta\big(\tfrac\beta2-\eta(1+\tfrac{1}{32a}\beta^2)\big)\omega^{-1}|v|^2 + 2a\eta\beta pq'\omega^{-1}|x|^2 - \tfrac{\beta d}{2} + q'\big(\Lambda\chi_R - C_{W,\beta,a}\omega^{-\frac12}\big)\Big).$$

We show that we can choose our parameters in such a way that the last bracket be bounded below by a positive constant, which is the key to the desired exponential decay. More precisely, choosing

$$\eta := \min\big\{\tfrac14\beta(1+\tfrac{1}{32a}\beta^2)^{-1},\tfrac12\sqrt a\big\}, \qquad 4\lambda_2 := \min\{\tfrac12\beta, 2\eta\},$$

we get

$$\partial_s\|g_s\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} \;\ge\; \tfrac12 q'(q'-1)\int_{\mathbb{X}}|g_s|^{q'-2}|\nabla_v g_s|^2 \\ + \int_{\mathbb{X}}|g_s|^{q'}\Big(4pq'\beta\lambda_2\omega^{-1}(\tfrac12|v|^2+a|x|^2) - \tfrac{\beta d}{2} + q'\big(\Lambda\chi_R - C_{W,\beta,a}\omega^{-\frac12}\big)\Big).$$

As the choice $\eta\le\frac12\sqrt a$ ensures $\frac12|v|^2+a|x|^2\ge\frac{1}{2\beta}(\omega-1)$, this actually means

$$\partial_s\|g_s\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} \;\ge\; \tfrac12 q'(q'-1)\int_{\mathbb{X}}|g_s|^{q'-2}|\nabla_v g_s|^2 + \int_{\mathbb{X}}|g_s|^{q'}\Big(2pq'\lambda_2 - \tfrac{\beta d}{2} + q'\big(\Lambda\chi_R - C_{W,\beta,a}\omega^{-\frac12}\big)\Big).$$

Now, recalling the definition of the cut-off function χ_R, we note that we can choose $\Lambda,R>0$ large enough (only depending on d,W,β,p,a) such that

$$\Lambda\chi_R - C_{W,\beta,a}\omega^{-\frac12} \;\ge\; -\tfrac12 p\lambda_2.$$

Provided that $pq' \geq 2\beta d\lambda_2^{-1}$, we then obtain

$$\partial_s \|g_s\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} \;\geq\; \tfrac{1}{2}q'(q'-1)\int_{\mathbb{X}} |g_s|^{q'-2}|\nabla_v g_s|^2 + pq'\lambda_2\|g_s\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'}, \tag{3.33}$$

hence, by Grönwall's inequality,

$$\|g_s\|_{\mathrm{L}^{q'}(\mathbb{X})} \;\leq\; e^{-p\lambda_2(t-s)}\|g_t\|_{\mathrm{L}^{q'}(\mathbb{X})}. \tag{3.34}$$

that is, (3.32) for $k=0$.

Step 2. Proof of (3.32) for all $k \geq 0$.
For multi-indices $\alpha,\gamma \in \mathbb{N}^d$, we set $J_s^{\alpha,\gamma} := \nabla_x^\alpha \nabla_v^\gamma g_s$. Differentiating equation (3.31), we get

$$\begin{cases} \partial_s J_s^{\alpha,\gamma} = -B_{\mu_s}^{*,p} J_s^{\alpha,\gamma} - r_s^{\alpha,\gamma}, & \text{for } 0 \leq s \leq t,\\ J_s^{\alpha,\gamma}|_{s=t} = \nabla_x^\alpha \nabla_v^\gamma g_t, \end{cases}$$

where the remainder is given by

$$r_s^{\alpha,\gamma} \;:=\; [\nabla_x^\alpha \nabla_v^\gamma, B_{\mu_s}^{*,p}]g_s.$$

Repeating the proof of (3.33), for the choice of $\eta, \lambda_2, \Lambda, R$ in Step 1, we get

$$\partial_s \|J_s^{\alpha,\gamma}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} \;\geq\; \tfrac{1}{2}q'(q'-1)\int_{\mathbb{X}} |J_s^{\alpha,\gamma}|^{q'-2}|\nabla_v J_s^{\alpha,\gamma}|^2 + pq'\lambda_2\|J_s^{\alpha,\gamma}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} - q'\int_{\mathbb{X}} |J_s^{\alpha,\gamma}|^{q'-2}J_s^{\alpha,\gamma} r_s^{\alpha,\gamma}, \tag{3.35}$$

and it remains to analyze the last contribution. By definition of $B_\mu^{*,p}$, cf. (3.30), we can compute

$$\begin{aligned} r_s^{\alpha,\gamma} \;=\;& \sum_{i:e_i\leq\gamma} \binom{\gamma}{e_i} J_s^{\alpha+e_i,\gamma-e_i} \\ &- \sum_{(\alpha',\gamma')<(\alpha,\gamma)} \binom{\alpha}{\alpha'}\binom{\gamma}{\gamma'} \operatorname{div}_v\Big[J_s^{\alpha',\gamma'}\, \nabla_x^{\alpha-\alpha'}\nabla_v^{\gamma-\gamma'}\Big(\tfrac{1}{2}\beta v + \nabla A + \kappa\nabla W * \mu - \omega^{-p}\nabla_v\omega^p\Big)\Big] \\ &+ \sum_{(\alpha',\gamma')<(\alpha,\gamma)} \binom{\alpha}{\alpha'}\binom{\gamma}{\gamma'} J_s^{\alpha',\gamma'}\, \nabla_x^{\alpha-\alpha'}\nabla_v^{\gamma-\gamma'}\Big(\tfrac{1}{2}\omega^{-p}\triangle_v\omega^p + \omega^{-p}v\cdot\nabla_x\omega^p - \operatorname{div}_v(\omega^{-p}\nabla_v\omega^p) \\ &\qquad\qquad -\omega^{-p}\big(\tfrac{1}{2}\beta v + \nabla A + \kappa\nabla W*\mu\big)\cdot\nabla_v\omega^p + \tfrac{\beta d}{2} - \Lambda\chi_R\Big). \end{aligned} \tag{3.36}$$

Integrating by parts, this allows us to estimate

$$\begin{aligned} \int_{\mathbb{X}} |J_s^{\alpha,\gamma}|^{q'-2}J_s^{\alpha,\gamma} r_s^{\alpha,\gamma} \;\lesssim_{W,\beta,\alpha,\gamma,a}\;& \|J_s^{\alpha,\gamma}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'-1}\Big(\max_{i:e_i\leq\gamma}\|J_s^{\alpha+e_i,\gamma-e_i}\|_{\mathrm{L}^{q'}(\mathbb{X})} + \max_{(\alpha',\gamma')<(\alpha,\gamma)}\|J_s^{\alpha',\gamma'}\|_{\mathrm{L}^{q'}(\mathbb{X})}\Big) \\ &+ q'\max_{(\alpha',\gamma')<(\alpha,\gamma)}\int_{\mathbb{X}} |J_s^{\alpha,\gamma}|^{q'-2}|J_s^{\alpha',\gamma'}||\nabla_v J_s^{\alpha,\gamma}|. \end{aligned}$$

Inserting this estimate into (3.35) and appealing to Young's inequality to absorb $\nabla_v J^{\alpha,\gamma}$ into the dissipation term, we are led to

$$\begin{aligned} \partial_s\|J_s^{\alpha,\gamma}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} \;\geq\;& pq'\lambda_2\|J_s^{\alpha,\gamma}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} - (q')^2 C_{W,\beta,\alpha,\gamma,a}\|J_s^{\alpha,\gamma}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'-2}\Big(\max_{(\alpha',\gamma')<(\alpha,\gamma)}\|J_s^{\alpha',\gamma'}\|_{\mathrm{L}^{q'}(\mathbb{X})}^2\Big) \\ &- q' C_{W,\beta,\alpha,\gamma,a}\|J_s^{\alpha,\gamma}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'-1}\Big(\max_{i:e_i\leq\gamma}\|J_s^{\alpha+e_i,\gamma-e_i}\|_{\mathrm{L}^{q'}(\mathbb{X})} + \max_{(\alpha',\gamma')<(\alpha,\gamma)}\|J_s^{\alpha',\gamma'}\|_{\mathrm{L}^{q'}(\mathbb{X})}\Big). \end{aligned}$$

Further appealing to Young's inequality, we get for all $\lambda < \lambda_2$,

$$\begin{aligned} \partial_s\|J_s^{\alpha,\gamma}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} \;\geq\;& pq'\lambda\|J_s^{\alpha,\gamma}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} \\ &- C_{W,\beta,\lambda,\alpha,\gamma,p,q,a}\Big(\max_{i:e_i\leq\gamma}\|J_s^{\alpha+e_i,\gamma-e_i}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} + \max_{(\alpha',\gamma')<(\alpha,\gamma)}\|J_s^{\alpha',\gamma'}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'}\Big), \end{aligned}$$

and thus, by Grönwall's inequality,

$$e^{pq'\lambda(t-s)}\|J_s^{\alpha,\gamma}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} \lesssim_{W,\beta,\lambda,\alpha,\gamma,p,q,a} \|\nabla_x^\alpha\nabla_v^\gamma g_t\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} + \int_s^t e^{pq'\lambda(t-u)}\Big(\max_{i:e_i\le\gamma}\|J_u^{\alpha+e_i,\gamma-e_i}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'} + \max_{(\alpha',\gamma')<(\alpha,\gamma)}\|J_u^{\alpha',\gamma'}\|_{\mathrm{L}^{q'}(\mathbb{X})}^{q'}\Big)du.$$

Iterating this inequality and starting from the result (3.34) of Step 1 for $J_s^{0,0} = g_s$, the conclusion follows. ☐

3.5. **Proof of Lemma 3.5 on $H^{-k}(M^{-1/2})$.** This section is devoted to the proof of (3.13). Taking inspiration from the work of Mischler and Mouhot [74, Section 4.2], we consider deformed weights of the form $M^{-1}\zeta$ with the factor ζ given by

$$\zeta(x,v) := 1+\tfrac12\Big(\frac{x\cdot v}{1+\frac{\eta}{2}|x|^2+\frac{1}{2\eta}|v|^2}\Big) \tag{3.37}$$

where the parameter $\eta>0$ will be properly chosen later on. Note that for any $\eta>0$ we have

$$\tfrac12 \le \zeta \le \tfrac32.$$

In these terms, we define the weighted negative Sobolev spaces $H^{-k}(\sqrt{M^{-1}\zeta})$ exactly as $H^{-k}(M^{-1/2})$ in (3.9), simply replacing the weight M^{-1} by $M^{-1}\zeta$ in the definition. Comparing $M^{-1}\zeta$ to M^{-1}, the definition of dual norms easily yields the equivalence, for all $h\in C_c^\infty(\mathbb{X})$,

$$\|h\|_{H^{-k}(M^{-1/2})} \simeq_{k,\eta} \|h\|_{H^{-k}(\sqrt{M^{-1}\zeta})}. \tag{3.38}$$

For a densely-defined operator X on $\mathrm{L}^2(\sqrt{M^{-1}\zeta})$, we denote by $X^{*,\zeta}$ its adjoint on $\mathrm{L}^2(\sqrt{M^{-1}\zeta})$ with respect to the weighted duality product $(g,h)\mapsto\int_{\mathbb{X}} ghM^{-1}\zeta$: more precisely, $X^{*,\zeta}$ stands for the closed operator on $\mathrm{L}^2(\sqrt{M^{-1}\zeta})$ defined by the relation

$$\int_{\mathbb{X}} g(Xh)M^{-1}\zeta = \int_{\mathbb{X}} h(X^{*,\zeta}h)M^{-1}\zeta.$$

By the equivalence of norms (3.38) and by definition of dual norms, it suffices to prove that there is some $0<\eta\le1$ and $\lambda_2>0$ (only depending on d,β,a, and $\|W\|_{W^{1,\infty}(\mathbb{R}^d)}$) such that the following result holds: choosing Λ,R large enough (only depending on d,β,a, and $\|W\|_{W^{1,\infty}(\mathbb{R}^d)}$), if $g\in C([0,t];\mathrm{L}^2(M^{-1/2}))$ satisfies the backward Cauchy problem

$$\begin{cases} \partial_s g_s = -B_{\mu_s}^{*,\zeta} g_s, & \text{for } 0\le s\le t,\\ g_s|_{s=t} = g_t, \end{cases} \tag{3.39}$$

for some $t\ge0$ and $g_t\in C_c^\infty(\mathbb{X})$, then we have for all $\lambda\in[0,\lambda_2)$, $k\ge0$, and $0\le s\le t$,

$$\|g_s\|_{H^k(M^{-1/2})} \lesssim_{W,\beta,\lambda,k,a} e^{-\lambda(t-s)}\|g_t\|_{H^k(M^{-1/2})}. \tag{3.40}$$

We split the proof into two steps, starting with the case $k=0$ before treating all $k\ge0$ by induction.

Step 1. Proof of (3.40) for $k=0$.
By definition of B_μ and by integration by parts, we find

$$\begin{aligned}\partial_s\|g_s\|_{\mathrm{L}^2(\sqrt{M^{-1}\zeta})}^2 &= -2\int_{\mathbb{X}} g_s(B_{\mu_s}^{*,\zeta}g_s)M^{-1}\zeta = -2\int_{\mathbb{X}} g_s(B_{\mu_s}g_s)M^{-1}\zeta\\ &= \int_{\mathbb{X}}|\nabla_v(\sqrt{M^{-1}\zeta}g_s)|^2 + \int_{\mathbb{X}}|g_s|^2M^{-1}\Big(\tfrac14\zeta|\beta v|^2 + (\nabla A+\kappa\nabla W*\mu_s)\cdot\nabla_v\zeta - v\cdot\nabla_x\zeta\\ &\qquad + 2\Lambda\chi_R\zeta - \tfrac{\beta d}{2}\zeta + \kappa\beta\zeta v\cdot(\nabla W*(\mu_s-M)) - |\nabla_v\sqrt{\zeta}|^2\Big).\end{aligned} \tag{3.41}$$

We show that we can choose parameters in such a way that the last bracket be bounded below by a positive constant, which is the key to the desired exponential decay. By definition of ζ, cf. (3.37), and by Young's inequality, we find

$$\begin{aligned}
&\tfrac14\zeta|\beta v|^2+\nabla A\cdot\nabla_v\zeta-v\cdot\nabla_x\zeta\\
&\quad=\ \tfrac12|v|^2\Big(\tfrac12\zeta\beta^2-\frac{1}{1+\frac{\eta}{2}|x|^2+\frac{1}{2\eta}|v|^2}\Big)+a\frac{|x|^2}{1+\frac{\eta}{2}|x|^2+\frac{1}{2\eta}|v|^2}-(a\eta^{-1}-\tfrac12\eta)\frac{(x\cdot v)^2}{(1+\frac{\eta}{2}|x|^2+\frac{1}{2\eta}|v|^2)^2}\\
&\quad\geq\ \tfrac12|v|^2\Big(\tfrac14\beta^2-\frac{1+a\eta^{-2}}{1+\frac{\eta}{2}|x|^2+\frac{1}{2\eta}|v|^2}\Big)+\tfrac12a\frac{|x|^2}{1+\frac{\eta}{2}|x|^2+\frac{1}{2\eta}|v|^2}\\
&\quad\geq\ \tfrac12|v|^2\Big(\tfrac14\beta^2-\frac{1+2a\eta^{-2}}{1+\frac{\eta}{2}|x|^2+\frac{1}{2\eta}|v|^2}\Big)+a\eta^{-1}\Big(1-\frac{1}{1+\frac{\eta}{2}|x|^2+\frac{1}{2\eta}|v|^2}\Big).
\end{aligned}$$

Inserting this into (3.41), and further using $|\nabla_v\zeta|\lesssim\eta^{-1}\langle z\rangle^{-1}$, we obtain

$$\begin{aligned}
\partial_s\|g_s\|^2_{\mathrm{L}^2(\sqrt{M^{-1}\zeta})}\ \geq\ &\int_{\mathbb{X}}|\nabla_v(\sqrt{M^{-1}\zeta}g_s)|^2\\
&+\int_{\mathbb{X}}|g_s|^2M^{-1}\Big(a\eta^{-1}-C_{W,\beta,a}+\Lambda\chi_R+\tfrac{1}{10}|\beta v|^2\big(1-\eta^{-3}\langle z\rangle^{-2}C_{W,\beta,a}\big)-C_{W,\beta,a}\eta^{-2}\langle z\rangle^{-1}\Big).
\end{aligned}$$

Noting that the dissipation term can be bounded below as

$$\begin{aligned}
|\nabla_v(\sqrt{M^{-1}\zeta}g)|^2\ &\geq\ \tfrac12|\zeta|^2|\nabla_v(\sqrt{M^{-1}}g)|^2-2|\nabla_v\zeta|^2|g|^2M^{-1}\\
&\geq\ \tfrac18|(\nabla_v+\tfrac{\beta}{2}v)g|^2M^{-1}-C\eta^{-2}\langle z\rangle^{-2}|g|^2M^{-1},
\end{aligned}$$

the above becomes

$$\begin{aligned}
\partial_s\|g_s\|^2_{\mathrm{L}^2(\sqrt{M^{-1}\zeta})}\ \geq\ &\tfrac18\|(\nabla_v+\tfrac{\beta}{2}v)g_s\|^2_{\mathrm{L}^2(M^{-1/2})}\\
&+\int_{\mathbb{X}}|g_s|^2M^{-1}\Big(a\eta^{-1}-C_{W,\beta,a}+\Lambda\chi_R+\tfrac{1}{10}|\beta v|^2\big(1-\eta^{-3}\langle z\rangle^{-2}C_{W,\beta,a}\big)-C_{W,\beta,a}\eta^{-2}\langle z\rangle^{-1}\Big).
\end{aligned}$$

Now let us choose $\eta:=\frac12aC_{W,\beta,a}^{-1}$, and note that, by definition of the cut-off function χ_R, we may then choose $\Lambda,R>0$ large enough (only depending on d,W,β,a) such that

$$\Lambda\chi_R+\tfrac{1}{10}|\beta v|^2\big(1-\eta^{-3}\langle z\rangle^{-2}C_{W,\beta,a}\big)-C_{W,\beta,a}\eta^{-2}\langle z\rangle^{-1}\ \geq\ \tfrac{1}{20}|\beta v|^2-\tfrac14a\eta^{-1}.$$

Further setting $\lambda_2:=\frac{1}{12}a\eta^{-1}$, this choice leads us to

$$\partial_s\|g_s\|^2_{\mathrm{L}^2(\sqrt{M^{-1}\zeta})}\ \geq\ \tfrac18\|(\nabla_v+\tfrac{\beta}{2}v)g_s\|^2_{\mathrm{L}^2(M^{-1/2})}+\tfrac{1}{20}\|\beta vg_s\|^2_{\mathrm{L}^2(M^{-1/2})}+2\lambda_2\|g_s\|^2_{\mathrm{L}^2(\sqrt{M^{-1}\zeta})}.\qquad(3.42)$$

In particular, by Grönwall's inequality,

$$\|g_s\|_{\mathrm{L}^2(M^{-1/2})}\ \lesssim\ e^{-\lambda_2(t-s)}\|g_t\|_{\mathrm{L}^2(M^{-1/2})},\qquad(3.43)$$

that is, (3.40) for $k=0$.

Step 2. Proof of (3.40) for all k.
For multi-indices $\alpha,\gamma\in\mathbb{N}^d$, we set $J_s^{\alpha,\gamma}:=\nabla_x^\alpha\nabla_v^\gamma g_s$. Differentiating equation (3.39), we get

$$\begin{cases}\partial_sJ_s^{\alpha,\gamma}=-B_{\mu_s}^{*,\zeta}J_s^{\alpha,\gamma}-r_s^{\alpha,\gamma},\quad\text{for }0\leq s\leq t,\\ J_s^{\alpha,\gamma}|_{s=t}=\nabla_x^\alpha\nabla_v^\gamma g_t,\end{cases}$$

where the remainder is given by

$$r_s^{\alpha,\gamma}\ :=\ [\nabla_x^\alpha\nabla_v^\gamma,B_{\mu_s}^{*,\zeta}]g_s.$$

Repeating the proof of (3.42), for the choice of $\eta, \lambda_2, \Lambda, R$ in Step 1, we get

$$\partial_s \|J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(\sqrt{M^{-1}\zeta})} \ge \tfrac18 \|(\nabla_v + \tfrac{\beta}{2}v)J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})} + \tfrac{1}{20}\|\beta v J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})} + 2\lambda_2 \|J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(\sqrt{M^{-1}\zeta})} - 2\int_{\mathbb{X}} J_s^{\alpha,\gamma} r_s^{\alpha,\gamma} M^{-1}\zeta. \quad (3.44)$$

and it remains to analyze the last contribution. By definition of B_μ, the weighted adjoint $B_\mu^{*,\zeta}$ takes the explicit form

$$\begin{aligned} B_\mu^{*,\zeta} h \;=\; & \tfrac12 \triangle_v h + v\cdot\nabla_x h + \Big(\tfrac{\beta}{2}v - \nabla A - \kappa \nabla W * \mu + \tfrac{1}{\zeta}\nabla_v \zeta\Big)\cdot \nabla_v h \\ & + \Big(\tfrac{\beta d}{2} + \tfrac{1}{2\zeta}\triangle_v\zeta - \kappa\beta v\cdot(\nabla W*(\mu - M)) \\ & + \tfrac{1}{\zeta} v\cdot\nabla_x\zeta + \tfrac{1}{\zeta}\big(\tfrac{\beta}{2}v - \nabla A - \kappa\nabla W * \mu\big)\cdot\nabla_v\zeta - \Lambda\chi_R\Big)h, \end{aligned}$$

and we may then compute

$$\begin{aligned} r_s^{\alpha,\gamma} \;=\; [\nabla_x^\alpha \nabla_v^\gamma, B_{\mu_s}^{*,\zeta}]g_s \;=\; & \sum_{i:e_i\le\gamma} \binom{\gamma}{e_i} J_s^{\alpha+e_i,\gamma-e_i} \\ & + \sum_{(\alpha',\gamma')<(\alpha,\gamma)} \binom{\alpha}{\alpha'}\binom{\gamma}{\gamma'} \nabla_x^{\alpha-\alpha'}\nabla_v^{\gamma-\gamma'}\Big(\tfrac{\beta}{2}v - \nabla A - \kappa\nabla W*\mu + \tfrac{1}{\zeta}\nabla_v\zeta\Big)\cdot\nabla_v J_s^{\alpha',\gamma'} \\ & + \sum_{(\alpha',\gamma')<(\alpha,\gamma)} \binom{\alpha}{\alpha'}\binom{\gamma}{\gamma'} J_s^{\alpha',\gamma'}\,\nabla_x^{\alpha-\alpha'}\nabla_v^{\gamma-\gamma'}\Big(\tfrac{\beta d}{2} + \tfrac{1}{2\zeta}\triangle_v\zeta - \kappa\beta v\cdot(\nabla W*(\mu-M)) \\ & + \tfrac{1}{\zeta}v\cdot\nabla_x\zeta + \tfrac{1}{\zeta}\big(\tfrac{\beta}{2}v - \nabla A - \kappa\nabla W*\mu\big)\cdot\nabla_v\zeta - \Lambda\chi_R\Big), \end{aligned}$$

from which we easily estimate

$$\begin{aligned} \int_{\mathbb{X}} J_s^{\alpha,\gamma} r_s^{\alpha,\gamma} M^{-1}\zeta \;\lesssim_{W,\beta,\alpha,\gamma,a}\; & \Big(\|J_s^{\alpha,\gamma}\|_{\mathrm{L}^2(\sqrt{M^{-1}\zeta})} + \|\nabla_v J_s^{\alpha,\gamma}\|_{\mathrm{L}^2(M^{-1/2})} + \|\beta v J_s^{\alpha,\gamma}\|_{\mathrm{L}^2(M^{-1/2})}\Big) \\ & \times \Big(\max_{i:e_i\le\gamma} \|J_s^{\alpha+e_i,\gamma-e_i}\|_{\mathrm{L}^2(M^{-1/2})} + \max_{(\alpha',\gamma')<(\alpha,\gamma)} \|J_s^{\alpha',\gamma'}\|_{\mathrm{L}^2(M^{-1/2})}\Big). \end{aligned}$$

Inserting this into (3.44), and appealing to Young's inequality to absorb $J^{\alpha,\gamma}$, $\nabla_v J^{\alpha,\gamma}$, and $\beta v J^{\alpha,\gamma}$ into the dissipation terms, we deduce for all $\lambda < \lambda_2$,

$$\begin{aligned} \partial_s \|J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(\sqrt{M^{-1}\zeta})} \;\ge\; & 2\lambda \|J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(\sqrt{M^{-1}\zeta})} \\ & - C_{W,\beta,\lambda,\alpha,\gamma,a}\Big(\max_{i:e_i\le\gamma} \|J_s^{\alpha+e_i,\gamma-e_i}\|^2_{\mathrm{L}^2(M^{-1/2})} + \max_{(\alpha',\gamma')<(\alpha,\gamma)} \|J_s^{\alpha',\gamma'}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big), \end{aligned}$$

and thus, by Grönwall's inequality,

$$\begin{aligned} e^{2\lambda(t-s)}\|J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(M^{-1/2})} \;\lesssim_{W,\beta,\lambda,\alpha,\gamma,a}\; & \|\nabla_x^\alpha\nabla_v^\gamma g_t\|^2_{\mathrm{L}^2(M^{-1/2})} \\ & + \int_s^t e^{2\lambda(t-u)}\Big(\max_{i:e_i\le\gamma} \|J_u^{\alpha+e_i,\gamma-e_i}\|^2_{\mathrm{L}^2(M^{-1/2})} + \max_{(\alpha',\gamma')<(\alpha,\gamma)} \|J_u^{\alpha',\gamma'}\|^2_{\mathrm{L}^2(M^{-1/2})}\Big)du. \end{aligned}$$

Iterating this inequality, and starting from the result (3.43) of Step 1 for $J_s^{0,0} = g_s$, the conclusion follows. □

3.6. **Proof of Lemma 3.6.** In this section, we appeal again to duality but we shall use a slightly different notation than in Section 3.4: for a densely-defined operator X on $\mathrm{L}^q(\langle z\rangle^p)$ we now denote by $X^{*,p}$ its adjoint on $\mathrm{L}^{q'}(\mathbb{X})$ with respect to the weighted duality product $(g,h)\mapsto\int_{\mathbb{X}}gh\langle z\rangle^p$. In other words, we use the same notation as in Section 3.4 for $X^{*,p}$, but now with the weight ω replaced by $\langle z\rangle$. We consider in particular the weighted adjoints $\{W_{t,s}^{*,p}\}_{t\ge s\ge0}$ of the fundamental operators $\{W_{t,s}\}_{t\ge s\ge 0}$, and we note that for all $t\ge0$ and $g_t\in C_c^\infty(\mathbb{X})$ the flow $g_s:=W_{t,s}^{*,p}g_t$ is such that $g\in C([0,t];L^{q'}(\mathbb{X}))$ satisfies the backward Cauchy problem

$$\left\{\begin{array}{l}\partial_s g_s=-B_{\mu_s}^{*,p}g_s,\quad\text{for }0\le s\le t,\\ g_s|_{s=t}=g_t,\end{array}\right.\tag{3.45}$$

where $B_\mu^{*,p}$ is the weighted adjoint of B_μ. This operator $B_\mu^{*,p}$ takes the explicit form (3.30) with ω now replaced by $\langle z\rangle$. The proof of Lemma 3.6 ultimately relies on the following result.

Lemma 3.9. *For all $k\ge0$, $0\le p\le1$, $0\le t-s\le1$, and $g_t\in C_c^\infty(\mathbb{X})$, we have*

$$\big\|W_{t,s}^{*,p}g_t\big\|_{H^{k+1}(\mathbb{X})}\lesssim_{W,\beta,k,a}(t-s)^{-\frac32}\|g_t\|_{H^k(\mathbb{X})},\tag{3.46}$$

where the constant only depends on d,β,k,a, and $\|W\|_{W^{k+2,\infty}(\mathbb{R}^d)}$.

We postpone the proof of this result for a moment and start by showing that Lemma 3.6 follows as a straightforward consequence.

Proof of Lemma 3.6. We start by applying the interpolation argument of [74, Lemma 2.4]: thanks to the exponential decay estimates of Lemma 3.5, it suffices to find some $\theta\ge0$ (only depending on d) such that for all $1<q\le2$, $k\ge0$, $0<p\le1$, $0\le t-s\le1$, and $h_s\in C_c^\infty(\mathbb{X})$ we have

$$\|AW_{t,s}h_s\|_{H^{-k}(M^{-1/2})}\lesssim_{W,\beta,k,p,q,a}(t-s)^{-\theta}\|h_s\|_{W^{-k,q}(\langle z\rangle^p)}.$$

In order to prove this, we argue by duality: more precisely, recalling $A=\Lambda\chi_R$, it suffices to find some $\theta\ge0$ such that for all $1<q\le2$, $k\ge0$, $0<p\le1$, $0\le t-s\le1$, and $g_t\in C_c^\infty(\mathbb{X})$,

$$\|W_{t,s}^{*,p}(\chi_RM^{-1}\langle z\rangle^{-p}g_t)\|_{W^{k,q'}(\mathbb{X})}\lesssim_{W,\beta,k,p,q,a}(t-s)^{-\theta}\|g_t\|_{H^k(M^{-1/2})}.\tag{3.47}$$

By the Sobolev inequality with $2\le q'<\infty$, the left-hand side can be estimated as follows,

$$\|W_{t,s}^{*,p}(\chi_RM^{-1}\langle z\rangle^{-p}g_t)\|_{W^{k,q'}(\mathbb{X})}\lesssim_{k,q}\|W_{t,s}^{*,p}(\chi_RM^{-1}\langle z\rangle^{-p}g_t)\|_{H^{k+d}(\mathbb{X})},$$

and the desired bound (3.47) then follows with $\theta=\frac{3d}{2}$ by iterating the result of Lemma 3.9. ☐

The rest of this section is devoted to the proof of Lemma 3.9. For $k=0$, this is in fact a standard consequence of the theory of hypoellipticity as in [60, 88, 74]. For $k>0$, we argue by induction, further using parabolic estimates similarly as in Section 3.4.

Proof of Lemma 3.9. Given $t\ge0$ and $g_t\in C_c^\infty(\mathbb{X})$, let $g_s:=W_{t,s}^{*,p}g_t$ be the solution of the backward Cauchy problem (3.45), and recall that $B_\mu^{*,p}$ takes the explicit form (3.30) with ω replaced by $\langle z\rangle$,

$$B_\mu^{*,p}h=\tfrac12\triangle_vh+v\cdot\nabla_xh+A_{\mu,p}^0h-A_{\mu,p}^1\cdot\nabla_vh,$$

in terms of

$$\begin{aligned}A_{\mu,p}^0&:=\tfrac12\langle z\rangle^{-p}\triangle_v\langle z\rangle^p+\langle z\rangle^{-p}v\cdot\nabla_x\langle z\rangle^p-\langle z\rangle^{-p}\big(\tfrac12\beta v+\nabla A+\kappa\nabla W*\mu\big)\cdot\nabla_v\langle z\rangle^p-\Lambda\chi_R,\\ A_{\mu,p}^1&:=\tfrac12\beta v+\nabla A+\kappa\nabla W*\mu-\langle z\rangle^{-p}\nabla_v\langle z\rangle^p.\end{aligned}$$

We split the proof into two steps.

Step 1. Case $k=0$: proof that for all $0\le t-s\le1$ we have

$$\|\nabla_xg_s\|_{\mathrm{L}^2(\mathbb{X})}\lesssim_{W,\beta,a}(t-s)^{-\frac32}\|g_s\|_{\mathrm{L}^2(\mathbb{X})}.\tag{3.48}$$

Integrating by parts, we can compute

$$\partial_s \int_{\mathbb{X}} |g_s|^2 \;=\; -2\int_{\mathbb{X}} g_s B^{*,p}_{\mu_s} g_s \;=\; \int_{\mathbb{X}} |\nabla_v g_s|^2 - \int_{\mathbb{X}} |g_s|^2 \Big(2A^0_{\mu_s,p} + \operatorname{div}_v(A^1_{\mu_s,p})\Big),$$

and thus, by definition of $A^0_{\mu,p}, A^1_{\mu,p}$,

$$\partial_s \int_{\mathbb{X}} |g_s|^2 \;\ge\; \int_{\mathbb{X}} |\nabla_v g_s|^2 - C_{W,\beta,a} \int_{\mathbb{X}} |g_s|^2.$$

Similarly, we can easily estimate

$$\begin{aligned}
\partial_s \int_{\mathbb{X}} |\nabla_x g_s|^2 \;&\ge\; \int |\nabla_{xv} g|^2 - C_{W,\beta,a} \int_{\mathbb{X}} \Big(|\nabla_x g_s|^2 + |g_s|^2 + |\nabla_x g_s||\nabla_v g_s|\Big),\\
\partial_s \int_{\mathbb{X}} |\nabla_v g_s|^2 \;&\ge\; \int_{\mathbb{X}} |\nabla_v^2 g_s|^2 - 2\int_{\mathbb{X}} \nabla_v g_s \cdot \nabla_x g_s - C_{W,\beta,a} \int_{\mathbb{X}} \Big(|\nabla_v g_s|^2 + |g_s|^2\Big),\\
-\partial_s \int_{\mathbb{X}} \nabla_x g_s \cdot \nabla_v g_s \;&\ge\; \tfrac12 \int_{\mathbb{X}} |\nabla_x g_s|^2 - \int_{\mathbb{X}} |\nabla_{xv} g_s||\nabla_v^2 g_s| - C_{W,\beta,a} \int_{\mathbb{X}} \Big(|\nabla_v g_s|^2 + |g_s|^2\Big).
\end{aligned}$$

Let us consider the functional

$$F_s(g) \;:=\; a_0 \int_{\mathbb{X}} |g|^2 + a_1(t-s) \int_{\mathbb{X}} |\nabla_v g|^2 + a_2(t-s)^3 \int_{\mathbb{X}} |\nabla_x g|^2 - 2a_3(t-s)^2 \int_{\mathbb{X}} \nabla_x g \cdot \nabla_v g, \qquad (3.49)$$

where the constants $a_0, a_1, a_2, a_3 > 0$ will be suitably chosen in a moment. In these terms, using Young's inequality, the above inequalities lead us to deduce for all $0 \le t-s \le 1$ and $0 < \varepsilon, \delta \le 1$,

$$\begin{aligned}
\partial_s F_s(g_s) \;\ge\; &\Big(a_0 - C_{W,\beta,a}(\delta^{-1}a_1 + a_2 + a_3)\Big) \int_{\mathbb{X}} |\nabla_v g_s|^2 + \Big(\tfrac12 a_3 - \delta a_1 - C_{W,\beta,a} a_2\Big)(t-s)^2 \int_{\mathbb{X}} |\nabla_x g_s|^2\\
&+ \big(a_1 - \varepsilon^{-1} a_3\big)(t-s) \int_{\mathbb{X}} |\nabla_v^2 g_s|^2 + \big(a_2 - \varepsilon a_3\big)(t-s)^3 \int_{\mathbb{X}} |\nabla_{xv} g_s|^2\\
&- C_{W,\beta,a}\big(a_0 + a_1 + a_2 + a_3\big) \int_{\mathbb{X}} |g_s|^2.
\end{aligned}$$

Choosing for instance $a_0 = 2C_{W,\beta,a}(\delta^{-1}a_1 + a_2 + a_3)$, $a_1 = 4a_3^2$, $a_2 = 1$, $a_3 = 4C_{W,\beta,a}$, $\varepsilon = (2a_3)^{-1}$, and $\delta = (32a_3)^{-1}$, we obtain

$$\begin{aligned}
\partial_s F_s(g_s) \;&\ge\; \tfrac12 a_0 \int_{\mathbb{X}} |\nabla_v g_s|^2 + \tfrac18 a_3 (t-s)^2 \int_{\mathbb{X}} |\nabla_x g_s|^2\\
&\quad + \tfrac12 a_1 (t-s) \int_{\mathbb{X}} |\nabla_v^2 g_s|^2 + \tfrac12 (t-s)^3 \int_{\mathbb{X}} |\nabla_{xv} g_s|^2 - 2a_0 C_{W,\beta,a} \int_{\mathbb{X}} |g_s|^2\\
&\ge\; -2a_0 C_{W,\beta,a} \int_{\mathbb{X}} |g_s|^2. \qquad (3.50)
\end{aligned}$$

By definition of F_s, as the choice of a_1, a_2, a_3 satisfies $2a_3 = \sqrt{a_1 a_2}$, we have

$$F_s(g) \;\ge\; a_0 \|g\|^2_{\mathrm{L}^2(\mathbb{X})} + \tfrac12 a_1 (t-s) \|\nabla_v g\|^2_{\mathrm{L}^2(\mathbb{X})} + \tfrac12 (t-s)^3 \|\nabla_x g\|^2_{\mathrm{L}^2(\mathbb{X})}, \qquad (3.51)$$

so that the above estimate (3.50) entails

$$\partial_s F_s(g_s) \;\gtrsim_{W,\beta,a}\; -F_s(g_s).$$

By Grönwall's inequality with $F_t(g) = a_0 \|g\|^2_{\mathrm{L}^2(\mathbb{X})}$, this yields for all $0 \le t-s \le 1$,

$$F_s(g_s) \;\lesssim_{W,\beta,a}\; \|g_t\|_{\mathrm{L}^2(\mathbb{X})},$$

and the claim (3.48) then follows from (3.51).

Step 2. Conclusion.
Given multi-indices $\alpha,\gamma\in\mathbb{N}^d$, we set for abbreviation $J_s^{\alpha,\gamma}:=\nabla_x^\alpha\nabla_v^\gamma g_s$, which satisfies

$$\left\{\begin{array}{ll}\partial_s J_s^{\alpha,\gamma}=-B_{\mu_s}^{*,p}J_s^{\alpha,\gamma}-r_s^{\alpha,\gamma}, & \text{for } 0\le s\le t,\\ J_s^{\alpha,\gamma}|_{s=t}=\nabla_x^\alpha\nabla_v^\gamma g_t, &\end{array}\right.$$

where the remainder term is given by

$$r_s^{\alpha,\gamma}\,:=\,[\nabla_x^\alpha\nabla_v^\gamma,B_{\mu_s}^{*,p}]g_s.$$

Repeating the proof of (3.50), for F_s defined in (3.49) with the same choice of constants a_0,a_1,a_2,a_3 as in Step 1, we get for all $0\le t-s\le 1$,

$$\begin{aligned}\partial_s F_s(J_s^{\alpha,\gamma})\,\ge\,&\tfrac12 a_0\int_{\mathbb{X}}|\nabla_v J_s^{\alpha,\gamma}|^2+\tfrac18 a_3(t-s)^2\int_{\mathbb{X}}|\nabla_x J_s^{\alpha,\gamma}|^2\\ &+\tfrac12 a_1(t-s)\int_{\mathbb{X}}|\nabla_v^2 J_s^{\alpha,\gamma}|^2+\tfrac12(t-s)^3\int_{\mathbb{X}}|\nabla_{xv}J_s^{\alpha,\gamma}|^2-2a_0C_{W,\beta,a}\int_{\mathbb{X}}|J_s^{\alpha,\gamma}|^2\\ &-2a_0\int_{\mathbb{X}}J_s^{\alpha,\gamma}r_s^{\alpha,\gamma}-2a_1(t-s)\int_{\mathbb{X}}\nabla_v J_s^{\alpha,\gamma}\cdot\nabla_v r_s^{\alpha,\gamma}-a_2(t-s)^3\int_{\mathbb{X}}\nabla_x J_s^{\alpha,\gamma}\cdot\nabla_x r_s^{\alpha,\gamma}\\ &+2a_3(t-s)^2\int_{\mathbb{X}}\nabla_x J_s^{\alpha,\gamma}\cdot\nabla_v r_s^{\alpha,\gamma}+2a_3(t-s)^2\int_{\mathbb{X}}\nabla_v J_s^{\alpha,\gamma}\cdot\nabla_x r_s^{\alpha,\gamma}.\end{aligned}$$

Recalling that the remainder term $r_s^{\alpha,\gamma}$ can be written as in (3.36) with ω replaced by $\langle z\rangle$, integrating by parts, and using Young's inequality to absorb all factors involving $J_s^{\alpha,\gamma}$ into the dissipation terms, we deduce for all $0\le t-s\le 1$,

$$\begin{aligned}\partial_s F_s(J_s^{\alpha,\gamma})\,\gtrsim_{W,\beta,\alpha,\gamma,a}\,&-\|J_s^{\alpha,\gamma}\|^2_{\mathrm{L}^2(\mathbb{X})}\\ &-\max_{(\alpha',\gamma')<(\alpha,\gamma)}\Big(\|J_s^{\alpha',\gamma'}\|^2_{\mathrm{L}^2(\mathbb{X})}+(t-s)\|\nabla_v J_s^{\alpha',\gamma'}\|^2_{\mathrm{L}^2(\mathbb{X})}+(t-s)^3\|\nabla_x J_s^{\alpha',\gamma'}\|^2_{\mathrm{L}^2(\mathbb{X})}\Big)\\ &-\max_{i:e_i\le\gamma}\Big(\|J_s^{\alpha+e_i,\gamma-e_i}\|^2_{\mathrm{L}^2(\mathbb{X})}+(t-s)\|\nabla_v J_s^{\alpha+e_i,\gamma-e_i}\|^2_{\mathrm{L}^2(\mathbb{X})}+(t-s)^3\|\nabla_x J_s^{\alpha+e_i,\gamma-e_i}\|^2_{\mathrm{L}^2(\mathbb{X})}\Big).\end{aligned}$$

By (3.51), this entails

$$\partial_s F_s(J_s^{\alpha,\gamma})\,\gtrsim_{W,\beta,\alpha,\gamma,a}\,-F_s(J_s^{\alpha,\gamma})-\max_{(\alpha',\gamma')<(\alpha,\gamma)}F_s(J_s^{\alpha',\gamma'})-\max_{i:e_i\le\gamma}F_s(J_s^{\alpha+e_i,\gamma-e_i}),$$

and thus, by Grönwall's inequality with $F_t(g)=a_0\|g\|^2_{\mathrm{L}^2(\mathbb{X})}$, we deduce for all $0\le t-s\le 1$,

$$F_s(J_s^{\alpha,\gamma})\,\lesssim_{W,\beta,\alpha,\gamma,a}\,\|\nabla_x^\alpha\nabla_v^\gamma g_t\|_{\mathrm{L}^2(\mathbb{X})}+\int_s^t\Big(\max_{(\alpha',\gamma')<(\alpha,\gamma)}F_u(J_u^{\alpha',\gamma'})+\max_{i:e_i\le\gamma}F_u(J_u^{\alpha+e_i,\gamma-e_i})\Big)\,\mathrm{d}u.$$

By a direct iteration, this proves for all $0\le t-s\le 1$,

$$F_s(J_s^{\alpha,\gamma})\,\lesssim_{W,\beta,\alpha,\gamma,a}\,\|g_t\|_{H^{|\alpha|+|\gamma|}(\mathbb{X})},$$

and the conclusion (3.46) then follows from (3.51). □

4. Representation of Brownian cumulants

This section is devoted to the representation of Brownian cumulants by means of Lions calculus. More precisely, our starting point is the Lions expansion of Lemma 2.1: following [25], it leads to an expansion of quantities of the form $\mathbb{E}_B[\Phi(\mu_t^N)]$ as power series in $\frac1N$. We introduce so-called *L-graphs* (or Lions graphs) as a new diagram representation that allows to efficiently capture cancellations in moment computations, leading us to a useful representation of Brownian cumulants. (Note this is unrelated to the Lions forests in [34].) In the sequel, the nth time-integration simplex is denoted by

$$\triangle_t^n\,:=\,\big\{(t_1,\ldots,t_n)\in[0,t]^n:0<t_n<\ldots<t_1<t\big\},$$

and we also define

$$\triangle^n := \big\{(t,\tau) : t>0,\, \tau\in\triangle_t^n\big\}.$$

4.1. **Lions expansion along the flow.** We appeal to Lemma 2.1 similarly as in [25] to expand of the Brownian expectation $\mathbb{E}_B[\Phi(\mu_t^N)]$ as a power series in $\frac{1}{N}$. To this aim, we start with the following iterative definition, which describes the natural quantities that appear in the expansion.

Definition 4.1. Given $n\ge 1$ and a smooth functional $\Phi:\Delta^n\times\mathcal{P}(\mathbb{X})\to\mathbb{R}$, we define the sequence $(\mathcal{U}_\Phi^{(m)},\Phi^{(m)})_{m\ge1}$ as follows:

- For $m=1$, we set for all $t>0$, $\tau=(\tau_1,\ldots,\tau_n)\in\triangle_t^n$, $0<s<\tau_n$, and $\mu\in\mathcal{P}(\mathbb{X})$,

$$\mathcal{U}_\Phi^{(1)}\big((t,\tau,s),\mu\big) := \Phi\big((t,\tau),m(\tau_n-s,\mu)\big),$$

 and

$$\Phi^{(1)}\big((t,\tau,s),\mu\big) := \int_{\mathbb{X}}\operatorname{tr}\Big[a_0\,\partial_\mu^2\mathcal{U}_\Phi^{(1)}\big((t,\tau,s),\mu\big)(z,z)\Big]\,\mu(\mathrm{d}z).$$

- For $m\ge2$, we iteratively define for all $t>0$, $\tau=(\tau_1,\ldots,\tau_{n+m-1})\in\triangle_t^{n+m-1}$, $0<s<\tau_{n+m-1}$, and $\mu\in\mathcal{P}(\mathbb{X})$,

$$\mathcal{U}_\Phi^{(m)}\big((t,\tau,s),\mu\big) := \Phi^{(m-1)}\big((t,\tau),m(\tau_{n+m-1}-s,\mu)\big),$$

 and

$$\Phi^{(m)}((t,\tau,s),\mu) := \int_{\mathbb{X}}\operatorname{tr}\Big[a_0\,\partial_\mu^2\mathcal{U}_\Phi^{(m)}\big((t,\tau,s),\mu\big)(z,z)\Big]\,\mu(\mathrm{d}z).$$

By convention, for $n=0$, given a smooth functional $\Phi:\mathcal{P}(\mathbb{X})\to\mathbb{R}$, we identify it with the functional $\tilde\Phi:\triangle^0\times\mathcal{P}(\mathbb{X})\to\mathbb{R}$ given by $\tilde\Phi(t,\mu):=\Phi(\mu)$, and we then set

$$\mathcal{U}_\Phi^{(1)}\big((t,s),\mu\big) := \Phi\big(m(t-s,\mu)\big),\qquad 0<s<t,$$

from which we can define $\mathcal{U}_\Phi^{(m)},\Phi^{(m)}$ iteratively as above.

This definition is a minor extension of [25], where only the case $n=0$ was considered. By a straightforward adaptation of [25, Theorems 2.15–2.16], we emphasize that this definition always makes sense with our smoothness assumptions. Moreover, for all $n\ge0$ and all smooth functionals $\Phi:\triangle^n\times\mathcal{P}(\mathbb{X})\to\mathbb{R}$, it is clear from the definition that for all $m,p\ge1$, $t>0$, $\tau\in\triangle_t^{n+m+p}$, and $\mu\in\mathcal{P}(\mathbb{X})$ we have

$$\mathcal{U}_\Phi^{(m+p)}\big((t,\tau),\mu\big) = \mathcal{U}_{\Phi^{(m)}}^{(p)}\big((t,\tau),\mu\big), \tag{4.1}$$

and similarly,

$$\Phi^{(m+p)}\big((t,\tau),\mu\big) = (\Phi^{(m)})^{(p)}\big((t,\tau),\mu\big).$$

In these terms, we can now state the following expansion result for functionals of the empirical measure along the particle dynamics. This is similar to the so-called weak error expansion in [25, Theorem 2.9]; we include a short proof for completeness.

Proposition 4.2. *Given a smooth functional $\Phi:\mathcal{P}(\mathbb{X})\to\mathbb{R}$, we have for all $n\ge0$ and $t>0$,*

$$\mathbb{E}_B[\Phi(\mu_t^N)] = \sum_{m=0}^n\frac{1}{(2N)^m}\int_{\triangle_t^m}\mathcal{U}_\Phi^{(m+1)}\big((t,\tau,0),\mu_0^N\big)\,\mathrm{d}\tau \\ +\frac{1}{(2N)^{n+1}}\int_{\triangle_t^{n+1}}\mathbb{E}_B\Big[\mathcal{U}_\Phi^{(n+2)}\big((t,\tau,\tau_{n+1}),\mu_{\tau_{n+1}}^N\big)\Big]\,\mathrm{d}\tau. \tag{4.2}$$

Proof. We proceed by induction and split the proof into two steps.

Step 1. Case $n = 0$.
By Lemma 2.1, we find for all $0 \le s \le t$ and $\mu \in \mathcal{P}(\mathbb{X})$,

$$\Phi\big(m(t-s,\mu_s^N)\big) \;=\; \Phi\big(m(t,\mu_0^N)\big) + \frac{1}{2N}\int_0^s \int_{\mathbb{X}} \operatorname{tr}\Big[a_0\,\partial_\mu^2 \mathcal{U}_\Phi^{(1)}\big((t,u),\mu_u^N\big)(z,z)\Big]\,\mu_u^N(\mathrm{d}z)\,\mathrm{d}u + M_{t,s}^N,$$

for some square-integrable martingale $(M_{t,s}^N)_s$ with $M_{t,0}^N = 0$, where we use the notation $\mathcal{U}_\Phi^{(1)}$ from Definition 4.1. Hence, taking the expectation with respect to $\mathbb{P}_B$ and choosing $s = t$, we get

$$\mathbb{E}_B[\Phi(\mu_t^N)] \;=\; \Phi\big(m(t,\mu_0^N)\big) + \frac{1}{2N}\int_0^t \mathbb{E}_B\Big[\int_{\mathbb{X}} \operatorname{tr}\big[a_0\,\partial_\mu^2\mathcal{U}_\Phi^{(1)}\big((t,u),\mu_u^N\big)(z,z)\big]\,\mu_u^N(\mathrm{d}z)\Big]\,\mathrm{d}u.$$

With the notation of Definition 4.1, this means

$$\mathbb{E}_B[\Phi(\mu_t^N)] \;=\; \mathcal{U}_\Phi^{(1)}\big((t,0),\mu_0^N\big) + \frac{1}{2N}\int_0^t \mathbb{E}_B\Big[\Phi^{(1)}\big((t,s),\mu_s^N\big)\Big]\,\mathrm{d}s.$$

As $\Phi^{(1)}\big((t,s),\mu_s^N\big) = \mathcal{U}_\Phi^{(2)}\big((t,s,s),\mu_s^N\big)$, this proves (4.2) with $n = 0$.

Step 2. General case.
We argue by induction. Suppose that (4.2) has been established for some $n \ge 0$. Let $t > 0$ and $\tau = (\tau_1,\ldots,\tau_{n+1}) \in \triangle_t^{n+1}$ be fixed. Applying Lemma 2.1 with $\mu \mapsto \Phi(\mu)$ replaced by $\mu \mapsto \Phi^{(n+1)}((t,\tau),\mu)$, we find similarly as in Step 1,

$$\begin{aligned}\mathbb{E}_B\big[\Phi^{(n+1)}\big((t,\tau),\mu_{\tau_{n+1}}^N\big)\big] \;&=\; \Phi^{(n+1)}\big((t,\tau),m(\tau_{n+1},\mu_0^N)\big)\\ &\quad + \frac{1}{2N}\int_0^{\tau_{n+1}} \mathbb{E}_B\Big[\int_{\mathbb{X}} \operatorname{tr}\Big[a_0\,\partial_\mu^2\mathcal{U}_{\Phi^{(n+1)}}^{(1)}\big((t,\tau,u),\mu_u^N\big)(z,z)\Big]\,\mu_u^N(\mathrm{d}z)\Big]\,\mathrm{d}u.\end{aligned}$$

Using this to further decompose the remainder term in (4.2), using the notation of Definition 4.1, and recalling (4.1), we precisely deduce that (4.2) also holds with n replaced by $n + 1$. ☐

4.2. **Graphical notation and definition of L-graphs.** We introduce a graphical notation associated with Definition 4.1, defining the notion of L-graphs (or Lions graphs), which will considerably simplify combinatorial manipulations in the sequel. Let $\Phi : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ be a reference smooth functional.

• *Base point.* Given $n \ge 0$ and a smooth functional $\Psi : \triangle^n \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, we set for all $t > 0$, $\tau = (\tau_1,\ldots,\tau_n) \in \triangle_t^n$, and $0 < s < \tau_n$,

$$\bullet_{[\Psi]}\big((t,\tau,s),\mu\big) \;:=\; \mathcal{U}_\Psi^{(1)}\big((t,\tau,s),\mu\big) \;=\; \Psi\big((t,\tau),m(\tau_n-s,\mu)\big).$$

In case of the reference smooth functional $\Psi = \Phi$, we drop the subscript and simply set

$$\bullet\big((t,s),\mu\big) \;:=\; \bullet_{[\Phi]}\big((t,s),\mu\big) \;=\; \Phi\big(m(t-s,\mu)\big).$$

• *Round edge.* In view of Proposition 4.2, the key operation that we want to account for in our graphical representation is

$$\mathcal{U}_\Psi^{(k)} \;\mapsto\; \mathcal{U}_\Psi^{(k+1)},$$

cf. Definition 4.1. This will be represented with the symbol $\boxed{}$, which we henceforth call "round edge". More precisely, given $n \ge 0$ and a smooth functional $\Psi : \triangle^n \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, we define for all $(t,\tau) \in \triangle^n$, $0 < s < \tau_n$, and $\mu \in \mathcal{P}(\mathbb{X})$,

$$\boxed{\Psi}\big((t,\tau,s),\mu\big) \;:=\; \bigg(\int_{\mathbb{X}} \operatorname{tr}\Big[a_0\,\partial_\mu^2\Psi\big((t,\tau),\nu\big)(z,z)\Big]\,\nu(\mathrm{d}z)\bigg)\bigg|_{\nu = m(\tau_n - s,\mu)}, \tag{4.3}$$

hence for instance

$$\boxed{\bullet} = \mathcal{U}_\Phi^{(2)}, \qquad \boxed{\boxed{\bullet}} = \mathcal{U}_\Phi^{(3)},$$

and so on. In particular, we emphasize that

$$\boxed{\Psi} \neq \boxed{\bullet_{[\Psi]}}.$$

When iterating this operation, we add a subscript (m) to indicate the number m of iterations, that is,

$$\odot_{(0)} := \bullet, \qquad \odot_{(1)} := \odot, \qquad \odot_{(m+1)} := \boxed{\odot_{(m)}}.$$

With this notation, the identity (4.1) takes on the following guise, for all $m, p \geq 0$,

$$\odot_{(m+p)} = \boxed{\odot_{(m)}}_{(p)}.$$

• *Time integration.* We introduce a short-hand notation for ordered time integrals: given $n, m \geq 0$ and given a smooth functional $\Psi : \triangle^{n+m} \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, we define for all $(t, \tau) \in \triangle^n$ and $\mu \in \mathcal{P}(\mathbb{X})$,

$$\Big(\int_{\triangle^m} \Psi\Big)\big((t,\tau),\mu\big) := \Big(\int_{\triangle^m_t} \Psi\Big)(\tau,\mu) := \int_{\triangle^m_{\tau_n}} \Psi\big((t,\tau,\sigma),\mu\big)\, d\sigma,$$

and also, for $m \geq 1$,

$$\Big(\int_{\triangle^{m-1}\times 0} \Psi\Big)\big((t,\tau),\mu\big) := \Big(\int_{\triangle^{m-1}_t\times 0} \Psi\Big)(\tau,\mu) := \int_{\triangle^{m-1}_{\tau_n}} \Psi\big((t,\tau,\sigma,0),\mu\big)\, d\sigma.$$

In these terms, the result of Proposition 4.2 takes on the following guise: for all $n \geq 0$ and $t > 0$,

$$\mathbb{E}_B[\Phi(\mu^N_t)] = \sum_{m=0}^{n} \frac{1}{(2N)^m} \Big(\int_{\triangle^m_t \times 0} \odot_{(m)}\Big)(\mu^N_0) + \frac{1}{(2N)^{n+1}} \int_{\triangle^{n+1}_t} \mathbb{E}_B\Big[\odot_{(n+1)}\big((t,\tau,\tau_{n+1}),\mu^N_{\tau_{n+1}}\big)\Big]\, d\tau. \tag{4.4}$$

In order to compute cumulants of functionals of the empirical measure along the flow, we shall need to apply (4.4) with Φ replaced by powers Φ^k with $k \geq 1$, and try to recognize the structure of the relation between moments and cumulants, cf. Lemma 2.5. This will be performed in Section 4.3 and motivates the further notation that we introduce below.

• *Products.* Products of different functionals are simply denoted by juxtaposing the corresponding graphs. Some care is however needed on how to identify respective time variables. For that purpose, we include subscripts with angular brackets indicating labels of the time variables for the different subgraphs: given $n, m \geq 0$, given functionals $\Psi : \triangle^n \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ and $\Theta : \triangle^m \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, and given $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_m$ with $\{i_1, \ldots, i_n\} \cup \{j_1, \ldots, j_m\} = [\![p]\!]$ (possibly not disjoint), we set for all $(t, \tau) \in \triangle^p$ and $\mu \in \mathcal{P}(\mathbb{X})$,

$$\Psi_{\langle i_1,\ldots,i_n\rangle}\Theta_{\langle j_1,\ldots,j_m\rangle}\big((t,\tau),\mu\big) := \Psi\big((t,\tau_{i_1},\ldots,\tau_{i_n}),\mu\big)\Theta\big((t,\tau_{j_1},\ldots,\tau_{j_m}),\mu\big). \tag{4.5}$$

For instance,

$$\odot_{\langle 1,4\rangle}\odot_{(2)\langle 2,3,4\rangle}\big((t,\tau_1,\tau_2,\tau_3,\tau_4),\mu\big) = \mathcal{U}^{(2)}_\Phi\big((t,\tau_1,\tau_4),\mu\big)\,\mathcal{U}^{(3)}_\Phi\big((t,\tau_2,\tau_3,\tau_4),\mu\big).$$

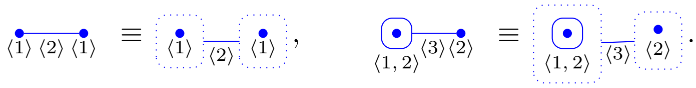
We also occasionally use indices to label time variables in subgraphs, for instance

$$\boxed{\bullet_{\langle 1\rangle}}_{\langle 4\rangle}\boxed{\odot_{\langle 2,3\rangle}}_{\langle 4\rangle} \equiv \odot_{\langle 1,4\rangle}\odot_{(2)\langle 2,3,4\rangle}.$$

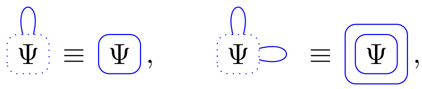
• *Straight edges.* When applying the round edge (4.3) to a power of a given functional, we are led to defining another type of operation, which we represent by a straight edge between subgraphs. More precisely, given $n, m \geq 0$, given smooth functionals $\Psi : \triangle^{n+1} \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ and $\Theta : \triangle^{m+1} \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, and given a partition $\{i_1,\ldots,i_n\} \uplus \{j_1,\ldots,j_m\} = [\![n+m]\!]$ with $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_m$, we set for all $(t,\tau,s,s') \in \triangle^{n+m+2}$,

$$\boxed{\Psi_{\langle i_1,\ldots,i_n,m+n+1\rangle}} \underset{\langle m+n+2\rangle}{\text{———}} \boxed{\Theta_{\langle j_1,\ldots,j_m,m+n+1\rangle}} \big((t,\tau,s,s',\mu)\big)$$
$$:= \bigg(\int_{\mathbb{X}} \Big(\partial_\mu \Psi\big((t,\tau_{i_1},\ldots,\tau_{i_n},s),\nu\big)(z) \cdot a_0 \partial_\mu \Theta\big((t,\tau_{j_1},\ldots,\tau_{j_m},s),\nu\big)(z) \Big)\, \nu(\mathrm{d}z) \bigg)\bigg|_{\nu=m(s-s',\mu)}. \qquad (4.6)$$

Note that the dotted boxes around Ψ and Θ above have no particular meaning in the graphical notation: they are just meant to emphasize that the straight edge is between the subgraphs corresponding to Ψ and Θ; we remove them when no confusion is possible, simply writing for instance



• *Graphical notation and terminology.* Starting from a number of base points, surrounding some subgraphs by round edges, and connecting some subgraphs via straight edges, we are led to a class of diagrams that we shall call *L-graphs* (or Lions graphs). Viewing round edges as loops,



we can view L-graphs as (undirected) multi-hypergraphs that satisfy a number of properties. First recall that a multi-hypergraph is a pair (V,E) where:

— V is a set of elements called base points or vertices;

— E is a multiset of elements called edges, which are pairs of non-empty subsets of V. The two subsets that are connected by an edge are called the ends of the edge.

We then formally define an *(unlabeled) L-graph* as a multi-hypergraph $\Psi = (V,E)$ such that the following three properties hold:

— an edge in E is either a loop (so-called round edge) or it connects *disjoint* vertex subsets (so-called straight edge): in other words, for all $\{A,B\} \in E$, we have either $A = B$ or $A \cap B = \varnothing$;

— a vertex subset $S \subset V$ can only be the end of at most one straight edge, but it can at the same time be the end of several round edges; in particular, each straight edge is simple, but round edges can be multiple;

— if a vertex subset $S \subset V$ is the end of some edge (round or straight), then strict subsets of S can only be connected to other strict subsets of S: in other words, for all $\{A,B\} \in E$ and $A' \subsetneq A$, the condition $\{A',A''\} \in E$ implies $A'' \subsetneq A$.

A *labeled L-graph* is an unlabeled L-graph endowed with a time labeling $V \sqcup E \to \mathbb{N}$, which associates a time label to each vertex and each edge. Note that each labeled L-graph is uniquely associated to a functional that can be obtained by iterating the round edge operation (4.3), the straight edge operation (4.6), and by taking products (4.5). We introduce some further useful terminology:

— *Induced subgraphs.* Given an L-graph (V,E) and a subset $S \subset V$, we define the *L-graph induced by* S as the pair (S,E_S) where E_S is the multiset of all edges $\{A,B\} \in E$ with $A,B \subset S$. We define the *L-graph strictly induced by* S as the pair (S,E'_S) where E'_S is now the multiset of all edges $\{A,B\} \in E$ with $A,B \subsetneq S$. By definition, we note that induced L-graphs are indeed L-graphs themselves, and moreover E'_S coincides with E_S after removing all occurrences of the round edge $\{S\}$. We call *L-subgraph* of (V,E) any L-graph (S,F) with $S \subset V$ and $E'_S \subset F \subset E_S$.

— *Stability.* Given an L-graph (V,E), an L-subgraph (S,F) is said to be *stable* if for all $\{A,B\} \in E$ with $A \subsetneq S$ we also have $B \subsetneq S$. In particular, by definition of an L-graph, a vertex subset that is the end of an edge is automatically inducing a stable subgraph.

— *Connectedness.* An L-graph (V, E) is said to be *connected* if there is no partition $V = A \cup B$ with $A, B \neq \varnothing$, $A \cap B = \varnothing$, and $E = E_A \cup E_B$. An L-graph can be uniquely decomposed into its connected components.

— *Irreducibility.* An L-graph (V, E) is said to be *irreducible* if for all straight edges $\{A, B\} \in E$ the induced subgraphs (A, E_A) and (B, E_B) are both connected and if for all round edge $\{A\} \in E$ the strictly induced subgraph (A, E'_A) is connected.

• *Rules for time ordering and symmetrization.* Given an L-graph (V, E), we consider the following four rules that restrict the possibilities for the time labeling $V \sqcup E \to \mathbb{N}$:

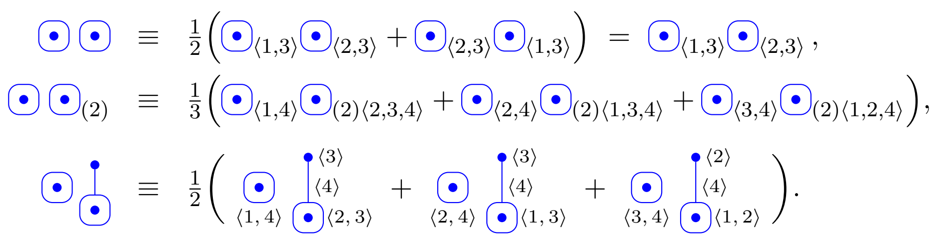
(R1) *Round edges.* In accordance with definition (4.3), the time label of a round edge $\{A\} \in E$ must always be larger than time labels of the strictly induced subgraph (A, E'_A); in other words, it must be larger than time labels of the subgraph that the round edge 'surrounds'.

(R2) *Straight edges.* In accordance with definition (4.6), the time label of a straight edge $\{A, B\} \in E$ must always be larger than time labels of the two induced subgraphs (A, E_A) and (B, E_B). In addition, the last time label of the two subgraphs must coincide.

(R3) *Products.* In any stable subgraph (S, F), decomposing it into its connected components, the last time label of each component coincides.

(R4) *No other repetition and no gap.* Apart from equalities of time labels imposed by the above three rules (R1)–(R3), all time labels must be different. In addition, the set of time labels, that is, the image of the time labeling map $V \sqcup E \to \mathbb{N}$, must be of the form $[\![n]\!] = \{1, \ldots, n\}$ for some $n \geq 1$ (that is, without gap).

In our notation, an unlabeled L-graph will be understood as the arithmetic average of all the labeled L-graphs that can be obtained by endowing the graph with a time labeling that satisfies the above four rules (R1)–(R4). For instance,



As we shall see in the next section, the whole point of this graphical notation is that it allows for quick and easy computations to derive representation formulas for Brownian cumulants. We summarize the main graphical computation rules in the following lemma. Note in particular that item (ii) below implies that any L-graph is equal to a linear combination of *irreducible* L-graphs with the same number of vertices and edges.

Lemma 4.3 (Graphical computation rules)**.**

(i) For all $k \geq 1$, a base point associated with the power Φ^k is equivalent to the product of k copies of the basepoint associated with Φ,

$$\bullet_{[\Phi]} = \bullet, \qquad \bullet_{[\Phi^2]} = \bullet\,\bullet, \qquad \bullet_{[\Phi^k]} = (\bullet)^k.$$

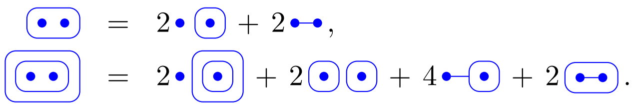
(ii) For any L-graph $\Psi : \triangle^n \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, for all $t > 0$, $\tau = (\tau_1, \ldots, \tau_n) \in \triangle^n_t$ and $0 < s < \tau_n$, for all $\mu \in \mathcal{P}(\mathbb{X})$,

$$\Psi\big((t, \tau), m(\tau_n - s, \mu)\big) = \Psi\big((t, \tau_1, \ldots, \tau_{n-1}, s), \mu\big). \tag{4.7}$$

(iii) *Given two L-graphs $\Psi : \triangle^{n+1} \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ and $\Theta : \triangle^{m+1} \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, and given $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_m$ with $\{i_1, \ldots, i_n\} \cup \{j_1, \ldots, j_m\} = [\![p]\!]$ for some $0 \le p \le n+m$, we find*

$$\boxed{\Psi_{\langle i_1,\ldots,i_n,p+1\rangle} \quad \Theta_{\langle j_1,\ldots,j_m,p+1\rangle}}_{\langle p+2\rangle} = \boxed{\Psi}_{\langle i_1,\ldots,i_n,p+1,p+2\rangle}\Theta_{\langle j_1,\ldots,j_m,p+2\rangle} + \Psi_{\langle i_1,\ldots,i_n,p+2\rangle}\boxed{\Theta}_{\langle j_1,\ldots,j_m,p+1,p+2\rangle} + 2\,\Psi_{\langle i_1,\ldots,i_n,p+1\rangle}\underset{\langle p+2\rangle}{\text{———}}\Theta_{\langle j_1,\ldots,j_m,p+1\rangle}. \tag{4.8}$$

In particular, in the first right-hand side term, we note that the penultimate time label $p+1$ of Ψ is larger than all the time labels $j_1, \ldots, j_m$ in Θ (and conversely in the second term), thus adding nontrivial time ordering not implied by the basic rules (R1)–(R4). *Using symmetrized notations, we find for instance*



(4.9)

(4.10)

This naturally generalizes to products of more than two functionals; we skip the details for conciseness.

(iv) *Given functionals $\Psi : \triangle^{n+1} \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ and $\Theta : \triangle^{m+1} \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, the time integral of their symmetrized product can be factorized as*

$$\binom{n+m}{n} \int_{\triangle^{n+m}\times 0} (\,\Psi\;\Theta\,) = \Big(\int_{\triangle^n\times 0} \Psi\Big)\Big(\int_{\triangle^m\times 0} \Theta\Big).$$

Proof. All four items are direct consequences of the definitions. First, the definition of the basepoint in the graphical notation means

$$\bullet_{[\Phi^k]}\big((t,s),\mu\big) = \mathcal{U}^{(1)}_{\Phi^k}\big((t,s),\mu\big) = (\Phi^k)\big(t, m(t-s,\mu)\big) = \big(\Phi\big(t, m(t-s,\mu)\big)\big)^k,$$

which proves item (i). Second, recalling the semigroup property

$$m(\tau_{n-1} - \tau_n, m(\tau_n - s, \mu)) = m(\tau_{n-1} - s, \mu),$$

we find from item (i) that $\bullet_{[\Phi^k]}$ satisfies (4.7). From the definitions (4.3) and (4.6), this property is clearly conserved when surrounding an L-graph satisfying (4.7) by a round edge, and when connecting two such L-graphs by a straight edge, which yields item (ii). The proof of items (iii) and (iv) is divided into the following two steps.

Step 1. Proof of (iii).
Given smooth functionals $\Psi : \triangle^{n+1} \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ and $\Theta : \triangle^{m+1} \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, and given $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_m$ with $\{i_1, \ldots, i_n\} \cup \{j_1, \ldots, j_m\} = [\![p]\!]$, the definition of the round edge notation yields for all $(t,\tau) \in \triangle^{p+2}$ and $\mu \in \mathcal{P}(\mathbb{X})$,

$$\begin{aligned}
&\boxed{\Psi_{\langle i_1,\ldots,i_n,p+1\rangle} \;\; \Theta_{\langle j_1,\ldots,j_m,p+1\rangle}}_{\langle p+2\rangle}\big((t,\tau),\mu\big) \\
&\qquad = \bigg(\int_{\mathbb{X}} \operatorname{tr}\Big[a_0\,\partial_\mu^2\Big(\Psi\big((t,\tau_{i_1},\ldots,\tau_{i_n},\tau_{p+1}),\nu\big) \\
&\qquad\qquad \times \Theta\big((t,\tau_{j_1},\ldots,\tau_{j_m},\tau_{p+1}),\nu\big)\Big)(z,z)\Big]\,\nu(\mathrm{d}z)\bigg)\bigg|_{\nu = m(\tau_{p+1}-\tau_{p+2},\mu)},
\end{aligned}$$

and thus, using the chain rule for the Lions derivative,

$$\begin{aligned}
&\boxed{\Psi_{\langle i_1,\ldots,i_n,p+1\rangle}\ \Theta_{\langle j_1,\ldots,j_m,p+1\rangle}}_{\langle p+2\rangle}\big((t,\tau),\mu\big)\\
&= \bigg(\int_{\mathbb{X}} \operatorname{tr}\Big[a_0\,(\partial_\mu^2\Psi)\big((t,\tau_{i_1},\ldots,\tau_{i_n},\tau_{p+1}),\nu\big)(z,z)\Big]\\
&\qquad\qquad \times\Theta\big((t,\tau_{j_1},\ldots,\tau_{j_m},\tau_{p+1}),\nu\big)\,\nu(\mathrm{d}z)\bigg)\bigg|_{\nu=m(\tau_{p+1}-\tau_{p+2},\mu)}\\
&+ \bigg(\int_{\mathbb{X}} \operatorname{tr}\Big[a_0\,(\partial_\mu^2\Theta)\big((t,\tau_{j_1},\ldots,\tau_{j_m},\tau_{p+1}),\nu\big)(z,z)\Big]\\
&\qquad\qquad \times\Psi\big((t,\tau_{i_1},\ldots,\tau_{i_n},\tau_{p+1}),\nu\big)\,\nu(\mathrm{d}z)\bigg)\bigg|_{\nu=m(\tau_{p+1}-\tau_{p+2},\mu)}\\
&+ 2\bigg(\int_{\mathbb{X}} \Big((\partial_\mu\Psi)\big((t,\tau_{i_1},\ldots,\tau_{i_n},\tau_{p+1}),\nu\big)(z)\\
&\qquad\qquad \cdot a_0(\partial_\mu\Theta)\big((t,\tau_{j_1},\ldots,\tau_{j_m},\tau_{p+1}),\nu\big)(z)\Big]\,\nu(\mathrm{d}z)\bigg)\bigg|_{\nu=m(\tau_{p+1}-\tau_{p+2},\mu)},
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\boxed{\Psi_{\langle i_1,\ldots,i_n,p+1\rangle}\quad \Theta_{\langle j_1,\ldots,j_m,p+1\rangle}}_{\langle p+2\rangle}\big((t,\tau,\tau_{p+2}),\mu\big)\\
&= \Theta\big((t,\tau_{j_1},\ldots,\tau_{j_m},\tau_{p+1}),m(\tau_{p+1}-\tau_{p+2},\mu)\big)\\
&\qquad \times\bigg(\int_{\mathbb{X}} \operatorname{tr}\Big[a_0\,(\partial_\mu^2\Psi)\big((t,\tau_{i_1},\ldots,\tau_{i_n},\tau_{p+1}),\nu\big)(z,z)\Big]\,\nu(\mathrm{d}z)\bigg)\bigg|_{\nu=m(\tau_{p+1}-\tau_{p+2},\mu)}\\
&+ \Psi\big((t,\tau_{i_1},\ldots,\tau_{i_n},\tau_{p+1}),m(\tau_{p+1}-\tau_{p+2},\mu)\big)\\
&\qquad \times\bigg(\int_{\mathbb{X}} \operatorname{tr}\Big[a_0\,(\partial_\mu^2\Theta)\big((t,\tau_{j_1},\ldots,\tau_{j_m},\tau_{p+1}),\nu\big)(z,z)\Big]\,\nu(\mathrm{d}z)\bigg)\bigg|_{\nu=m(\tau_{p+1}-\tau_{p+2},\mu)}\\
&+ 2\bigg(\int_{\mathbb{X}} \Big((\partial_\mu\Psi)\big((t,\tau_{i_1},\ldots,\tau_{i_n},\tau_{p+1}),\nu\big)(z)\\
&\qquad\qquad \cdot a_0(\partial_\mu\Theta)\big((t,\tau_{j_1},\ldots,\tau_{j_m},\tau_{p+1}),\nu\big)(z)\Big]\,\nu(\mathrm{d}z)\bigg)\bigg|_{\nu=m(\tau_{p+1}-\tau_{p+2},\mu)}.
\end{aligned}$$

Further using item (ii), the identity (4.8) follows.

Step 2. Proof of (iv).
By definition of the symmetrized product, we have for all $(t,\tau)\in\triangle^{n+m}$ and $0<s<\tau_{n+m}$,

$$\begin{aligned}
(\Psi\,\Theta)\big((t,\tau,s),\mu\big) &= \binom{n+m}{n}^{-1}\sum_{\sigma\in\mathrm{sym}(n+m)} \mathbb{1}_{\sigma(1)<\ldots<\sigma(n)}\,\mathbb{1}_{\sigma(n+1)<\ldots<\sigma(n+m)}\\
&\qquad\times\Psi\big((t,\tau_{\sigma(1)},\ldots,\tau_{\sigma(n)},s),\mu\big)\,\Theta\big((t,\tau_{\sigma(n+1)},\ldots,\tau_{\sigma(n+m)},s),\mu\big),
\end{aligned}$$

and thus, taking the time integral,

$$\begin{aligned}
\int_{\triangle_t^{n+m}}(\Psi\,\Theta)\,\big((t,\tau,0),\mu\big)\,\mathrm{d}\tau &= \binom{n+m}{n}^{-1}\sum_{\sigma\in\mathrm{sym}(n+m)} \mathbb{1}_{\sigma(1)<\ldots<\sigma(n)}\,\mathbb{1}_{\sigma(n+1)<\ldots<\sigma(n+m)}\\
&\qquad\times\int_{\triangle_t^{n+m}}\Psi\big((t,\tau_{\sigma(1)},\ldots,\tau_{\sigma(n)},0),\mu\big)\,\Theta\big((t,\tau_{\sigma(n+1)},\ldots,\tau_{\sigma(n+m)},0),\mu\big)\,\mathrm{d}\tau.
\end{aligned}$$

Noting that the sum over permutations allows to reconstruct the full product of integrals, we obtain

$$\int_{\triangle_t^{n+m}}(\Psi\,\Theta)\,\big((t,\tau,0),\mu\big)\,\mathrm{d}\tau = \binom{n+m}{n}^{-1}\Big(\int_{\triangle_t^{n}}\Psi\big((t,\tau,0),\mu\big)\,\mathrm{d}\tau\Big)\Big(\int_{\triangle_t^{m}}\Theta\big((t,\tau,0),\mu\big)\,\mathrm{d}\tau\Big),$$

which is precisely the statement of item (iv). ☐

4.3. **Graphical representation of Brownian cumulants.** Starting from Proposition 4.2 in form of (4.4), we can use the above graphical notation to easily compute cumulants of functionals of the empirical measure along the particle dynamics. We first illustrate this by a direct computation of the leading contribution to the variance and to the third cumulant.

Lemma 4.4. *Given a smooth functional $\Phi : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, we can represent the variance and the third cumulant along the particle dynamics as*

$$\begin{aligned}\mathrm{Var}_B[\Phi(\mu_t^N)] &= \frac{1}{N}\int_{\triangle_t\times 0} \bullet\!\!-\!\!\bullet\,(\mu_0^N) + \frac{E_{N,2}}{(2N)^2},\\ \kappa_B^3[\Phi(\mu_t^N)] &= \frac{3}{N^2}\int_{\triangle_t^2\times 0} \bullet\!-\!\boxed{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}}\,(\mu_0^N) + \frac{E_{N,3}}{(2N)^3},\end{aligned}$$

where the error terms $E_{N,2}, E_{N,3}$ are given explicitly by

$$\begin{aligned}E_{N,2} &:= A_{2,2} - 2A_{1,2}\,\mathbb{E}_B[\Phi(\mu_t^N)] + \frac{(A_{1,2})^2}{(2N)^2} - \Big(\int_{\triangle_t\times 0}\boxed{\bullet}\,(\mu_0^N)\Big)^2,\\ E_{N,3} &:= A_{3,3} - 3A_{2,2}\,\mathbb{E}_B[\Phi(\mu_t^N)] - A_{1,3}\,\mathbb{E}_B[\Phi(\mu_t^N)^2] + 4A_{1,3}\,\mathbb{E}_B[\Phi(\mu_t^N)]^2 - \frac{2}{(2N)^3}A_{1,3}^2\mathbb{E}_B[\Phi(\mu_t^N)]\\ &\quad - \int_{\triangle_t^3\times 0}\Big(4\bullet\boxed{\bullet}\boxed{\boxed{\bullet}} + \begin{smallmatrix}\boxed{\bullet}\\ \boxed{\bullet}\boxed{\bullet}\end{smallmatrix} + 6\boxed{\bullet}\boxed{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} + 12\boxed{\bullet}\begin{smallmatrix}\bullet\\ |\\ \boxed{\bullet}\end{smallmatrix} + 6\boxed{\boxed{\bullet}\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}}\Big)(\mu_0^N)\\ &\quad - \frac{1}{2N}\int_{\triangle_t^4\times 0}\Big(\begin{smallmatrix}\boxed{\bullet}\\ \boxed{\bullet}\end{smallmatrix}\boxed{\boxed{\bullet}} + 2\bullet\boxed{\boxed{\bullet}}\boxed{\boxed{\bullet}} + 12\boxed{\boxed{\bullet}}\begin{smallmatrix}\bullet\\ |\\ \boxed{\bullet}\end{smallmatrix} + 6\boxed{\boxed{\bullet}}\boxed{\boxed{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}}}\Big)(\mu_0^N)\\ &\quad + \mathbb{E}_B[\Phi(\mu_t^N)]\Big(\int_{\triangle_t^3\times 0} 4\boxed{\bullet}\boxed{\boxed{\bullet}}\,(\mu_0^N) + \frac{1}{N}\boxed{\boxed{\bullet}}\boxed{\boxed{\bullet}}\,(\mu_0^N)\Big)\end{aligned}$$

where for shortness we have defined

$$A_{k,m} := \int_{\triangle_t^m}\mathbb{E}_B\Big[\boxed{\begin{smallmatrix}\bullet\\ [\Phi^k]\end{smallmatrix}}_{(m)}\big((t,\tau,\tau_m),\mu_{\tau_m}^N\big)\Big]\,d\tau.$$

Proof. We split the proof into three steps.

Step 1. Formula for variance.
Using Proposition 4.2 in form of (4.4) to accuracy $O(N^{-2})$, we find

$$\mathbb{E}_B[\Phi(\mu_t^N)^2] = \bullet\;\bullet\,((t,0),\mu_0^N) + \frac{1}{2N}\int_{\triangle_t\times 0}\boxed{\bullet\;\bullet}\,(\mu_0^N) + \frac{A_{2,2}}{(2N)^2}, \tag{4.11}$$

with the notation for $A_{2,2}$ in the statement. By Lemma 4.3(iii) in form of (4.9), we get

$$\begin{aligned}\mathbb{E}_B[\Phi(\mu_t^N)^2] &= \Big(\bullet\,((t,0),\mu_0^N)\Big)^2\\ &\quad + \frac{2}{2N}\Big(\bullet\,((t,0),\mu_0^N)\Big)\Big(\int_{\triangle_t\times 0}\boxed{\bullet}\,(\mu_0^N)\Big) + \frac{2}{2N}\int_{\triangle_t\times 0}\bullet\!\!-\!\!\bullet\,(\mu_0^N) + \frac{A_{2,2}}{(2N)^2}.\end{aligned} \tag{4.12}$$

On the other hand, using again Proposition 4.2 in form of (4.4) to accuracy $O(N^{-2})$, we also have

$$\mathbb{E}_B[\Phi(\mu_t^N)] = \bullet\,((t,0),\mu_0^N) + \frac{1}{2N}\int_{\triangle_t\times 0}\boxed{\bullet}\,(\mu_0^N) + \frac{A_{1,2}}{(2N)^2}.$$

Taking the square of this identity, and comparing it to (4.12), the formula for the variance follows after straightforward simplifications.

Step 2. Next-order formula for variance.
Before turning to the third cumulant, we expand the formula for the variance to the next order. Instead of (4.11), we start from Proposition 4.2 with accuracy $O(N^{-3})$, in form of

$$\mathbb{E}_B[\Phi(\mu_t^N)^2] \;=\; \bullet\;\bullet\,((t,0),\mu_0^N) + \frac{1}{2N}\int_{\triangle_t\times 0} \boxed{\bullet\;\bullet}\,(\mu_0^N) + \frac{1}{(2N)^2}\int_{\triangle_t^2\times 0} \boxed{\boxed{\bullet\;\bullet}}\,(\mu_0^N) + \frac{A_{2,3}}{(2N)^3}.$$

Then further appealing to Lemma 4.3(iii) in form of (4.9), as well as to Lemma 4.3(iv), we obtain, instead of (4.12),

$$\begin{aligned}\mathbb{E}_B[\Phi(\mu_t^N)^2] \;=\; & \Big(\bullet\,((t,0),\mu_0^N)\Big)^2 + \frac{2}{2N}\Big(\bullet\,((t,0),\mu_0^N)\Big)\Big(\int_{\triangle_t\times 0}\boxed{\bullet}\,(\mu_0^N)\Big) + \frac{2}{2N}\int_{\triangle_t\times 0}\bullet\!\!-\!\!\bullet\,(\mu_0^N)\\ & + \frac{2}{(2N)^2}\Big(\bullet\,((t,0),\mu_0^N)\Big)\Big(\int_{\triangle_t^2\times 0}\boxed{\boxed{\bullet}}\,(\mu_0^N)\Big) + \frac{1}{(2N)^2}\Big(\int_{\triangle_t\times 0}\boxed{\bullet}\,(\mu_0^N)\Big)^2\\ & + \frac{1}{(2N)^2}\int_{\triangle_t^2\times 0}\Big(4\,\bullet\!\!-\!\!\boxed{\bullet} + 2\,\boxed{\bullet\!\!-\!\!\bullet}\Big)(\mu_0^N) + \frac{A_{2,3}}{(2N)^3}.\end{aligned}$$

On the other hand, using again Proposition 4.2 in form of (4.4) to accuracy $O(N^{-3})$, we also have

$$\mathbb{E}_B[\Phi(\mu_t^N)] \;=\; \bullet\,((t,0),\mu_0^N) + \frac{1}{2N}\int_{\triangle_t\times 0}\boxed{\bullet}\,(\mu_0^N) + \frac{1}{(2N)^2}\int_{\triangle_t\times 0}\boxed{\boxed{\bullet}}\,(\mu_0^N) + \frac{A_{1,3}}{(2N)^3}. \tag{4.13}$$

Taking the square of this identity, and comparing it to the previous one for $\mathbb{E}_B[\Phi(\mu_t^N)^2]$, we are led to

$$\mathrm{Var}_B[\Phi(\mu_t^N)] \;=\; \frac{2}{2N}\int_{\triangle_t\times 0}\bullet\!\!-\!\!\bullet\,(\mu_0^N) + \frac{1}{(2N)^2}\int_{\triangle_t^2\times 0}\Big(4\,\bullet\!\!-\!\!\boxed{\bullet} + 2\,\boxed{\bullet\!\!-\!\!\bullet}\Big)(\mu_0^N) + \frac{R_N}{(2N)^3}. \tag{4.14}$$

where the error term R_N is given by

$$R_N \;:=\; A_{2,3} - 2A_{1,3}\,\mathbb{E}_B[\Phi(\mu_t^N)] + \frac{(A_{1,3})^2}{(2N)^3} - 2\int_{\triangle_t^3\times 0}\boxed{\bullet}\,\boxed{\boxed{\bullet}}\,(\mu_0^N) - \frac{1}{2N}\Big(\int_{\triangle_t^2\times 0}\boxed{\boxed{\bullet}}\,(\mu_0^N)\Big)^2.$$

Step 3. Formula for third cumulant.
Using Proposition 4.2 in form of (4.4), we find

$$\mathbb{E}_B[\Phi(\mu_t^N)^3] \;=\; \therefore\,((t,0),\mu_0^N) + \frac{1}{2N}\Big(\int_{\triangle_t\times 0}\boxed{\therefore}\,(\mu_0^N)\Big) + \frac{1}{(2N)^2}\Big(\int_{\triangle_t^2\times 0}\boxed{\boxed{\therefore}}\,(\mu_0^N)\Big) + \frac{A_{3,3}}{(2N)^3}.$$

To compute the different right-hand side terms, we appeal to Lemma 4.3 in form of

$$\begin{aligned}\boxed{\therefore} \;&=\; 3\,\boxed{\bullet}\,:\; + 6\,\bullet\,\underset{\bullet}{\overset{\bullet}{|}}\,,\\ \boxed{\boxed{\therefore}} \;&=\; 3\,\boxed{\boxed{\bullet}}\,: + 3\,\big(\boxed{\bullet}\big)^2\bullet + 12\,\bullet\,\underset{\boxed{\bullet}}{\overset{\bullet}{|}} + 6\,\boxed{\bullet}\,\underset{\bullet}{\overset{\bullet}{|}} + 6\,\bullet\,\boxed{\underset{\bullet}{\overset{\bullet}{|}}} + 12\,\bullet\!\!-\!\!\boxed{\underset{\bullet}{\overset{\bullet}{|}}}\end{aligned}$$

Inserting these identities into the above, comparing with (4.13) and (4.14), and recalling that the third cumulant is given by

$$\kappa_3[\Phi(\mu_t^N)] \;=\; \mathbb{E}_B[\Phi(\mu_t^N)^3] - 3\,\mathrm{Var}_B[\Phi(\mu_t^N)]\mathbb{E}_B[\Phi(\mu_t^N)] - \mathbb{E}_B[\Phi(\mu_t^N)]^3,$$

the formula in the statement follows after straightforward simplifications. □

We now show how the above explicit diagrammatic computation can be pursued systematically to higher orders. First note that starting from Proposition 4.2 in form of (4.4) and appealing to the computation rules of Lemma 4.3 to expand each L-graph into a sum of irreducible graphs, the kth moment of a smooth functional along the flow can be expanded as a power series in N^{-1}, where the term of order $O(N^{-m})$ is given by a sum of all irreducible L-graphs with k vertices and m edges (see indeed (4.15) below). In the above lemma, for the first cumulants, we manage to capture cancellations showing that the power series for the variance starts at order $O(N^{-1})$ and that the power series for the third cumulant starts at $O(N^{-2})$. In the following result, we unravel the underlying combinatorial structure and show how cancellations can be systematically captured to higher orders: in a nutshell, the power series for cumulants takes the same form as for moments, except that the sum over irreducible L-graphs is restricted to *connected* graphs, cf. (4.17). In particular, given that for $m < k-1$ there is no connected L-graph with k vertices and only m edges, we deduce that the power series for the kth cumulant must start at order $O(N^{1-k})$.

Proposition 4.5. *Given a smooth functional $\Phi : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, for all $k \geq 1$, we can expand as follows the kth Brownian moment along the flow: for all $n \geq 0$,*

$$\mathbb{E}_B[\Phi(\mu_t^N)^k] \;=\; \sum_{m=0}^{n} \frac{1}{(2N)^m} \sum_{\Psi \in \Gamma(k,m)} \gamma(\Psi) \int_{\triangle_t^m \times 0} \Psi \;+\; \frac{R_{N,n}^k(t)}{(2N)^{n+1}}, \tag{4.15}$$

where $\Gamma(k,m)$ stands for the set of all (unlabeled) irreducible L-graphs with k vertices and m edges, where γ is some map $\Gamma(k,m) \to \mathbb{N}$, and where the remainder is given by

$$R_{N,n}^k(t) \;:=\; \int_{\triangle_t^{n+1}} \mathbb{E}_B\Big[\boxed{\overset{\bullet}{[\Phi^k]}}_{(n+1)}\big((t,\tau,\tau_{n+1}),\mu_{\tau_{n+1}}^N\big)\Big]\,\mathrm{d}\tau. \tag{4.16}$$

Moreover, with this notation, the kth cumulant can be expanded as follows: for all $n \geq 0$,

$$\kappa_B^k[\Phi(\mu_t^N)] \;=\; \mathbb{1}_{n \geq k-1} \sum_{m=k-1}^{n} \frac{1}{(2N)^m} \sum_{\Psi \in \Gamma_\circ(k,m)} \gamma(\Psi) \int_{\triangle_t^m \times 0} \Psi \;+\; \frac{\tilde{R}_{N,n}^k(t)}{(2N)^{n+1}}, \tag{4.17}$$

where the sum is now restricted to $\Gamma_\circ(k,m) \subset \Gamma(k,m)$, which stands for the subset of all connected *(unlabeled) irreducible L-graphs with k vertices and m edges, and where the remainder $\tilde{R}_{N,n}^k(t)$ can be expressed as a linear combination of elements of the set*

$$\bigg\{ S_{N,n}^{k-j}(t) \prod_{i=1}^{j} \mathbb{E}_B[\Phi(\mu_t^N)^i]^{\alpha_i} \,:\, 0 \leq j \leq k \text{ and } \alpha_1,\ldots,\alpha_j \in \mathbb{N} \text{ with } \sum_{i=1}^{j} i\alpha_i = j \bigg\},$$

with bounded coefficients independent of N, Φ, μ_0^N, t, where the factors $\{S_{N,n}^k(t)\}_k$ are defined by

$$S_{N,n}^k(t) \;:=\; R_{N,n}^k(t) - \sum_{j=1}^{k} \binom{k-1}{j-1} \sum_{m=0}^{n} R_{N,n-m}^{k-j}(t) \sum_{\Psi \in \Gamma_\circ(j,m)} \gamma(\Psi) \int_{\triangle_t^m \times 0} \Psi(\mu_0^N). \tag{4.18}$$

Proof. We split the proof into three steps.

Step 1. Proof of (4.15).
By Proposition 4.2 in form of (4.4), we recall that we have for all $k \geq 1$ and $n \geq 0$,

$$\mathbb{E}_B[\Phi(\mu_t^N)^k] \;=\; \sum_{m=0}^{n} \frac{1}{(2N)^m} \Big(\int_{\triangle_t^m \times 0} \boxed{\overset{\bullet}{[\Phi^k]}}_{(m)} (\mu_0^N) \Big) \;+\; \frac{R_{N,n}^k(t)}{(2N)^{n+1}}, \tag{4.19}$$

with remainder $R_{N,n}^k(t)$ defined in (4.16). In order to prove (4.15), it remains to use Lemma 4.3(iii) to expand the L-graphs in the above right-hand side as sums of *irreducible* L-graphs. By a direct induction argument, we note that all time labelling satisfying the basic rules (R1)–(R4) appear symmetrically in the expansion, thus proving that for all $k \geq 1$ and $m \geq 0$ we can expand

$$\boxed{\overset{\bullet}{[\Phi^k]}}_{(m)} \;=\; \sum_{\Psi \in \Gamma(k,m)} \gamma(\Psi)\Psi, \tag{4.20}$$

for some map $\gamma : \Gamma(k,m) \to \mathbb{N}$, where as in the statement $\Gamma(k,m)$ stands for the set of all (unlabeled) irreducible L-graphs with k vertices and m edges. This already proves (4.15).

Step 2. Proof that for all $k \geq 1$ and $m \geq 0$ the map $\gamma : \Gamma(k,m) \to \mathbb{N}$ in (4.20) satisfies for all L-graphs $\Psi \in \Gamma(k,m)$,

$$\gamma(\Psi) \;=\; \sum_{\substack{\Theta\subset\Psi\\ \text{connected}\\ \text{component}}} \binom{k-1}{V(\Theta)-1}\binom{m}{E(\Theta)}\,\gamma(\Theta)\,\gamma(\Psi\setminus\Theta), \tag{4.21}$$

where the sum runs over all connected components Θ of the L-graph Ψ, where $V(\Theta)$ and $E(\Theta)$ stand for the number of vertices and the number of edges in Θ, respectively, and where $\Psi \setminus \Theta$ stands for the L-subgraph obtained by removing the component Θ from Ψ.

To prove this identity, we start by noting that, when appealing to Lemma 4.3(iii) to iteratively prove (4.20), the map γ can be given an explicit interpretation: for all $\Psi \in \Gamma(k,m)$, the coefficient $\gamma(\Psi)$ is the positive integer given by

$$\gamma(\Psi) \;:=\; 2^{SE(\Psi)}N(\Psi),$$

where $SE(\Psi)$ is the number of straight edges in Ψ and where $N(\Psi)$ is the number of ways to obtain the graph Ψ by starting from k labeled vertices and by iteratively adding round or straight edges between stable subgraphs. Conditioning on the connected component that the vertex with the first label belongs to, the identity (4.21) immediately follows from this interpretation.

Step 3. Proof of (4.17).
Given $k \geq 1$ and $m \geq 0$, the result (4.21) of Step 2 implies

$$\sum_{\Psi\in\Gamma(k,m)} \gamma(\Psi)\Psi \;=\; \sum_{j=1}^{k}\binom{k-1}{j-1}\sum_{p=0}^{m}\binom{m}{p}\Bigg(\sum_{\substack{\Psi\in\Gamma(j,p)\\ \text{connected}}}\gamma(\Psi)\Psi\Bigg)\Bigg(\sum_{\Psi\in\Gamma(k-j,m-p)}\gamma(\Psi)\Psi\Bigg),$$

and thus, by (4.20),

$$\boxed{\bullet\,[\Phi^k]}_{(m)} \;=\; \sum_{j=1}^{k}\binom{k-1}{j-1}\sum_{p=0}^{m}\binom{m}{p}\Bigg(\sum_{\substack{\Psi\in\Gamma(j,p)\\ \text{connected}}}\gamma(\Psi)\Psi\Bigg)\boxed{\bullet\,[\Phi^{k-j}]}_{(m-p)}.$$

Taking the time integral and appealing to Lemma 4.3(iii), this leads us to

$$\int_{\triangle_t^m\times 0}\boxed{\bullet\,[\Phi^k]}_{(m)}(\mu) \;=\; \sum_{j=1}^{k}\binom{k-1}{j-1}\sum_{p=0}^{m}\Bigg(\sum_{\substack{\Psi\in\Gamma(j,p)\\ \text{connected}}}\gamma(\Psi)\int_{\triangle_t^p\times 0}\Psi(\mu)\Bigg)\Bigg(\int_{\triangle_t^{m-p}\times 0}\boxed{\bullet\,[\Phi^{k-j}]}_{(m-p)}(\mu)\Bigg).$$

Now recalling (4.19), and defining

$$L^k_{N,n}(t,\mu_0^N) \;:=\; \sum_{m=0}^{n}\frac{1}{(2N)^m}\sum_{\substack{\Psi\in\Gamma(k,m)\\ \text{connected}}}\gamma(\Psi)\int_{\triangle_t^m\times 0}\Psi(\mu_0^N),$$

we deduce

$$\mathbb{E}_B[\Phi(\mu_t^N)^k] \;=\; \sum_{j=1}^{k}\binom{k-1}{j-1}L^j_{N,n}(t,\mu_0^N)\,\mathbb{E}_B[\Phi(\mu_t^N)^{k-j}] + \frac{S^k_{N,n}(t)}{(2N)^{n+1}}, \tag{4.22}$$

with remainder $S^k_{N,n}(t)$ defined in (4.18). Finally, we recall the recurrence relation of Lemma 2.5 between moments and cumulants: for all $k \geq 1$, we have

$$\mathbb{E}_B[\Phi(\mu_t^N)^k] \;=\; \sum_{j=1}^{k}\binom{k-1}{j-1}\kappa_B^j[\Phi(\mu_t^N)]\,\mathbb{E}_B[\Phi(\mu_t^N)^{k-j}].$$

Comparing this with the identity (4.22) above, the conclusion follows by a direct induction. □

4.4. **Error estimates.** We turn to the uniform-in-time estimation of error terms in expansions such as (4.15) or (4.17). We start with the following lemma describing Lions derivatives of the solution of the mean-field McKean–Vlasov equation (1.21).

Lemma 4.6. *Let $m(\cdot,\mu):\mathbb{R}^+\times\mathbb{X}\to\mathbb{R}^+$ denote the solution operator (1.22) for the mean-field McKean–Vlasov equation (1.21). For all $t\ge0$ and $\phi\in C_c^\infty(\mathbb{X})$, the functional $\mu\mapsto\int_{\mathbb{X}}\phi\, m(t,\mu)$ is smooth and its linear functional derivatives can be represented as follows: for all $k\ge1$, $\mu\in\mathcal{P}(\mathbb{X})$, and $y_1,\ldots,y_k\in\mathbb{X}$,*

$$\frac{\delta^k}{\delta\mu^k}\Big(\mu\mapsto\int_{\mathbb{X}}\phi\, m(t,\mu)\Big)(\mu,y_1,\ldots,y_k)\;=\;\int_{\mathbb{X}}\phi\, m^{(k)}(t,\mu,y_1,\ldots,y_k),\tag{4.23}$$

where $m^{(k)}(\cdot,\mu,y_1,\ldots,y_k):\mathbb{R}^+\times\mathbb{X}\to\mathbb{R}$ is a distributional solution of the linear Cauchy problem

$$\left\{\begin{array}{l}\partial_t m^{(k)}(t,\mu,y_1,\ldots,y_k)-L_{m(t,\mu)}m^{(k)}(t,\mu,y_1,\ldots,y_k)=F_k(t,\mu,y_1,\ldots,y_k),\\ m^{(k)}(t,\mu,y_1,\ldots,y_k)|_{t=0}=(-1)^{k-1}(\delta_{y_k}-\mu),\end{array}\right.\tag{4.24}$$

where for all $\mu\in\mathcal{P}(\mathbb{X})$ we recall that L_μ stands for the linearized McKean–Vlasov operator at μ, cf. (3.1), and where the source term F_k is given by

$$F_k(t,\mu,y_1,\ldots,y_k)\;:=\;\sum_{j=1}^{k-1}\sum_{\substack{I_0\cup I_1=[\![k]\!]\\ \textit{disjoint}}}\mathbb{1}_{\sharp I_0=j,\sharp I_1=k-j}\\ \times\operatorname{div}\Big(m^{(j)}(t,\mu,y_{I_0})\int_{\mathbb{X}}\frac{\delta b}{\delta\mu}(\cdot,m(t,\mu),z)\,m^{(k-j)}((t,z),\mu,y_{I_1})\,dz\Big),\tag{4.25}$$

where for $I=\{i_1,\ldots,i_r\}$ with $1\le i_1<\ldots<i_r\le k$ we set $y_I:=(y_{i_1},\ldots,y_{i_r})$. Note that for $k=1$ we have $F_1\equiv0$. In addition, given κ_0,λ_0 as in Theorem 3.1, we have the following uniform-in-time estimates: given $\kappa\in[0,\kappa_0]$, $1<q\le2$, and $0<p\le1$, further assuming in the Langevin setting that $pq'\gg_{\beta,a}1$ is large enough (only depending on d,β,a), we have for all $\lambda\in[0,\lambda_0)$, $k\ge1$, $\alpha_1,\ldots,\alpha_k\ge0$, $y_1,\ldots,y_k\in\mathbb{X}$, $t\ge0$, and $\ell_0>\frac{1}{q'}\dim\mathbb{X}$,

$$\|\nabla_{y_1}^{\alpha_1}\ldots\nabla_{y_k}^{\alpha_k}\,m^{(k)}(t,\mu,y_1,\ldots,y_k)\|_{W^{-(\ell_0+k-1+\max_j\alpha_j),q}(\langle z\rangle^p)}\;\lesssim\;e^{-p\lambda t}\,\langle y_1\rangle^p\ldots\langle y_k\rangle^p,\tag{4.26}$$

where the multiplicative constant only depends on $d,W,\beta,\lambda,k,\ell_0,p,q,a$, and $\max_j\alpha_j$.

Proof. By successively taking linear derivatives in the McKean–Vlasov equation (1.21), the representation (4.23)–(4.24) in terms of linearized equations is straightforward with source term given by

$$F_k(t,\mu,y_1,\ldots,y_k)\;:=\;\sum_{l=1}^{k}\sum_{a_0=0}^{k-1}\sum_{\substack{a_1+\ldots+a_l=k-a_0\\1\le a_1\le\ldots\le a_l<k}}\sum_{\substack{I_0\cup\ldots\cup I_l=[\![k]\!]\\ \text{disjoint}}}\mathbb{1}_{\forall 0\le r\le l:\sharp I_r=a_r}\\ \times\operatorname{div}\Big(m^{(j)}(t,\mu,y_{I_0})\int_{\mathbb{X}^l}\frac{\delta^l b}{\delta\mu^l}(\cdot,m(t,\mu),z_1,\ldots,z_l)\\ \times m^{(a_1)}((t,z_1),\mu,y_{I_1})\ldots m^{(a_l)}((t,z_l),\mu,y_{I_l})\,dz_1\ldots dz_l\Big),$$

where we recall the notation $y_I=(y_{i_1},\ldots,y_{i_r})$ for $I=\{i_1,\ldots,i_r\}$. The pairwise structure of interactions actually yields various simplifications: noting that

$$\frac{\delta^l(W*\mu)}{\delta\mu^l}(\cdot,\mu,z_1,\ldots,z_l)=(-1)^lW*(\delta_{z_l}-\mu),$$

and noting that $\int_{\mathbb{X}} m^{(k)}(t,\mu,y_1,\ldots,y_k)=0$ for all $k\geq 1$, the above expression for F_k reduces precisely to (4.25). We emphasize that this simplification is not essential, but it slightly simplifies the computations. It remains to deduce the uniform-in-time estimate (4.26): we argue by induction and split the proof into two steps. Let p,q be fixed as in the statement.

Step 1. Preliminary: we prove the following properties of the spaces $W^{-k,q}(\langle z\rangle^p)$ and their duals $W^{k,q'}(\mathbb{X})$,

— for all $\ell\geq 0$ and $h\in C_c^\infty(\mathbb{X})$,

$$\|\nabla h\|_{W^{\ell,q'}(\mathbb{X})} \leq \|h\|_{W^{\ell+1,q'}(\mathbb{X})}; \tag{4.27}$$

$$\|\nabla h\|_{W^{-\ell,q}(\langle z\rangle^p)} \lesssim_{W,\beta,\ell,a} \|h\|_{W^{1-\ell,q}(\langle z\rangle^p)}; \tag{4.28}$$

— for all $\ell>\frac{1}{q'}\dim(\mathbb{X})$ and $y\in\mathbb{X}$ we have

$$\|\delta_y\|_{W^{-\ell,q}(\langle z\rangle^p)} \lesssim_{\ell,q} \langle y\rangle^p. \tag{4.29}$$

The claim (4.27) is a direct consequence of the definition of $W^{k,q'}(\mathbb{X})$. The claim (4.28) is a direct consequence of (4.27) by definition of dual norms. We turn to the proof of (4.29). By the Sobolev embedding, we have for all $y\in\mathbb{X}$ and $h\in C_c^\infty(\mathbb{X})$,

$$\Big|\int_{\mathbb{X}} h\delta_y\langle z\rangle^p\Big| = \langle y\rangle^p|h(y)| \leq \langle y\rangle^p\,\|h\|_{L^\infty(\mathbb{X})} \lesssim_{\ell,q} \langle y\rangle^p\|h\|_{W^{\ell,q'}(\mathbb{X})},$$

provided that $\ell>\frac{1}{q'}\dim\mathbb{X}$. By definition of dual norms, the claim (4.29) follows.

Step 2. Conclusion.

Let $\ell_0>\frac{1}{q'}\dim(\mathbb{X})$. Applying $\nabla_{y_1}^{\alpha_1}\ldots\nabla_{y_k}^{\alpha_k}$ to both sides of equation (4.24), and then appealing to Theorem 3.1(ii), we obtain for all $\ell\geq 0$, $\lambda\in[0,\lambda_0)$ and $t\geq 0$

$$\begin{aligned}&e^{p\lambda t}\|\nabla_{y_1}^{\alpha_1}\ldots\nabla_{y_k}^{\alpha_k}m^{(k)}(t,\mu,y_1,\ldots,y_k)\|_{W^{-\ell,q}(\langle z\rangle^p)}\\&\lesssim_{W,\beta,\lambda,\ell,p,q,a} \|\nabla_{y_1}^{\alpha_1}\ldots\nabla_{y_k}^{\alpha_k}(\delta_{y_k}-\mu)\|_{W^{-\ell,q}(\langle z\rangle^p)}+\int_0^t e^{p\lambda s}\|\nabla_{y_1}^{\alpha_1}\ldots\nabla_{y_k}^{\alpha_k}F_k(s,\mu,y_1,\ldots,y_k)\|_{W^{-\ell,q}(\langle z\rangle^p)}\,\mathrm{d}s.\end{aligned}$$

Note that the first right-hand side term is equal to $\mathbb{1}_{\alpha_1=\ldots=\alpha_{k-1}=0}\|\nabla^{\alpha_k}\delta_{y_k}\|_{W^{-\ell,q}(\langle z\rangle^p)}$. Using (4.28) and (4.29) to bound this term, we then get for all $\ell\geq\ell_0+\alpha_k$,

$$\begin{aligned}&e^{p\lambda t}\|\nabla_{y_1}^{\alpha_1}\ldots\nabla_{y_k}^{\alpha_k}m^{(k)}(t,\mu,y_1,\ldots,y_k)\|_{W^{-\ell,q}(\langle z\rangle^p)}\\&\qquad\lesssim_{W,\beta,\lambda,\ell,\ell_0,p,q,a} \langle y_k\rangle^p+\int_0^t e^{p\lambda s}\|\nabla_{y_1}^{\alpha_1}\ldots\nabla_{y_k}^{\alpha_k}F_k(s,\mu,y_1,\ldots,y_k)\|_{W^{-\ell,q}(\langle z\rangle^p)}\,\mathrm{d}s.\end{aligned}$$

To shorten notation, let us introduce the following norms: given $k\geq 1$, we define for all $\ell,n\geq 0$ and $H:\mathbb{X}\times\mathbb{X}^k\to\mathbb{R}$,

$$[\![H]\!]_{\ell,n} := \sup_{0\leq\alpha_1,\ldots,\alpha_k\leq n}\ \sup_{y_1,\ldots,y_k\in\mathbb{X}}\Big(\langle y_1\rangle^{-p}\ldots\langle y_k\rangle^{-p}\|\nabla_{y_1}^{\alpha_1}\ldots\nabla_{y_k}^{\alpha_k}H(\cdot,y_1,\ldots,y_k)\|_{W^{-\ell,q}(\langle z\rangle^p)}\Big).$$

In these terms, the above reads as follows, for all $\ell\geq\ell_0+n$,

$$e^{p\lambda t}[\![m^{(k)}(t,\mu)]\!]_{\ell,n} \lesssim_{W,\beta,\lambda,\ell,\ell_0,p,q,a} 1+\int_0^t e^{p\lambda s}[\![F_k(s,\mu)]\!]_{\ell,n}\,\mathrm{d}s. \tag{4.30}$$

We turn to the estimation of the source term F_k as defined in (4.25). Recalling the choice of b, cf. (1.23) and (1.24), and using again (4.27), we easily find for all $\ell,\ell',n\geq 0$,

$$[\![F_k(t,\mu)]\!]_{\ell+1,n} \lesssim_{W,\beta,\ell,\ell',k,a} \max_{1\leq j\leq k-1}[\![m^{(j)}(t,\mu)]\!]_{\ell,n}[\![m^{(k-j)}(t,\mu)]\!]_{\ell',n}.$$

Inserting this into (4.30), and recalling that $F_1=0$, we deduce by induction for all $n\geq 0$, $\ell\geq\ell_0+n+k-1$, and $\lambda\in[0,\lambda_0)$, for all $t\geq 0$

$$e^{p\lambda t}[\![m^{(k)}(t,\mu)]\!]_{\ell,n} \lesssim_{W,\beta,\lambda,\ell,\ell_0,k,p,q,a} 1,$$

which concludes the proof of (4.26). □

In order to compensate for the polynomial growth in (4.26), we shall appeal to the following uniform-in-time moment estimates both for the particle dynamics and for the mean-field dynamics.

Lemma 4.7 (Uniform moment estimates). *For all $t \geq 0$, $N, k \geq 1$, and $\mu \in \mathcal{P}(\mathbb{X})$, we have*

$$\mathbb{E}_B\Big[\int_{\mathbb{X}} |z|^k \,\mu_t^N(\mathrm{d}z)\Big] \;\leq\; (Ck)^k \int_{\mathbb{X}} \langle e^{-t/C} z\rangle^k \,\mu_0^N(\mathrm{d}z), \tag{4.31}$$

$$\int_{\mathbb{X}} |z|^k \, m(t,\mu) \;\leq\; (Ck)^k \int_{\mathbb{X}} \langle e^{-t/C} z\rangle^k \,\mu(\mathrm{d}z), \tag{4.32}$$

and for all $0 \leq \theta \leq \frac{1}{C}$,

$$\mathbb{E}_B\Big[\int_{\mathbb{X}} e^{\theta|z|^2} \mu_t^N(\mathrm{d}z)\Big] \;\leq\; C \int_{\mathbb{X}} e^{C\theta e^{-t/C}|z|^2} \mu_0^N(\mathrm{d}z), \tag{4.33}$$

$$\int_{\mathbb{X}} e^{\theta|z|^2} \, m(t,\mu) \;\leq\; C \int_{\mathbb{X}} e^{C\theta e^{-t/C}|z|^2} \,\mu(\mathrm{d}z), \tag{4.34}$$

for some constant $C < \infty$ only depending on d, W, β, a.

Proof. We focus on the Langevin setting for shortness. We split the proof into two steps, separately proving (4.31) and (4.33), while the proof of (4.32) and (4.34) for the mean-field dynamics is identical and is skipped.

Step 1. Proof of (4.31).
In the spirit of [9], we consider the random process

$$G_t^N \;:=\; a|X_t^{1,N}|^2 + |V_t^{1,N}|^2 + \eta X_t^{1,N} \cdot V_t^{1,N},$$

for some $\eta > 0$ to be suitable chosen. By Itô's formula, the particle dynamics (1.1) yields

$$\begin{aligned} \mathrm{d}G_t^N \;=&\; -a\eta|X_t^{1,N}|^2 \,\mathrm{d}t - (\beta-\eta)|V_t^{1,N}|^2 \,\mathrm{d}t - \tfrac{\eta\beta}{2} X_t^{1,N}\cdot V_t^{1,N}\,\mathrm{d}t \\ & - \big(2V_t^{1,N} + \eta X_t^{1,N}\big)\cdot\Big(\tfrac{\kappa}{N}\sum_{j=1}^N \nabla W(X_t^{1,N} - X_t^{j,N})\Big)\,\mathrm{d}t + \big(2V_t^{1,N}+\eta X_t^{1,N}\big)\cdot \mathrm{d}B_t^1. \end{aligned} \tag{4.35}$$

From this equation and Itô's formula, we then find for all $k \geq 1$,

$$\begin{aligned} \partial_t \mathbb{E}_B[(G_t^N)^k] \;\leq\; & -k\mathbb{E}_B\Big[(G_t^N)^{k-1}\Big(\tfrac14 a\eta|X_t^{1,N}|^2 + \big(\tfrac{\beta}{2} - \eta(1+\tfrac{\beta^2}{8a})\big)|V_t^{1,N}|^2\Big)\Big] \\ & + k\big((\tfrac{\eta}{a}+\tfrac{2}{\beta})\|\nabla W\|^2_{\mathrm{L}^\infty(\mathbb{R}^d)}\big)\mathbb{E}_B[(G_t^N)^{k-1}] \\ & + \tfrac12 k(k-1)\mathbb{E}_B[(G_t^N)^{k-2}|2V_t^{1,N}+\eta X_t^{1,N}|^2]. \end{aligned}$$

Provided that $0 < \eta \ll_{\beta,a} 1$ is small enough (only depending on β, a), there exist $\lambda, C > 0$ (only depending on d, W, β, a) such that we get for all $k \geq 1$,

$$\partial_t \mathbb{E}_B[(G_t^N)^k] \;\leq\; -\lambda pk\mathbb{E}_B[(G_t^N)^k] + Ck^2\mathbb{E}_B[(G_t^N)^{k-1}],$$

and thus, by Grönwall's inequality,

$$\mathbb{E}_B[(G_t^N)^k] \;\leq\; e^{-\lambda kt}\mathbb{E}_B[(G_0^N)^k] + Ck^2\int_0^t e^{-\lambda k(t-s)}\mathbb{E}_B[(G_s^N)^{k-1}]\,\mathrm{d}s.$$

A direct induction then yields for all $k \geq 1$,

$$\mathbb{E}_B[(G_t^N)^k] \;\leq\; \sum_{j=0}^k e^{-\lambda jt}(\tfrac{1}{\lambda}C)^{k-j}\frac{(k!)^2}{(k-j)!}\,\mathbb{E}_B[(G_0^N)^j],$$

and the conclusion follows.

Step 2. Proof of (4.33).
From (4.35) and Itô's formula, arguing similarly as in Step 1, provided that $0<\eta\ll_{\beta,a}1$ is small enough (only depending on β,a), there exist $\lambda,C>0$ (only depending on d,W,β,a) such that we have for all $\theta>0$,

$$\partial_t\mathbb{E}[e^{\theta G_t^N}]\,\le\,-2\lambda\theta\mathbb{E}[G_t^N e^{\theta G_t^N}]+C\theta\mathbb{E}[e^{\theta G_t^N}]+C\theta^2\mathbb{E}[G_t^N e^{\theta G_t^N}].$$

Hence, for $\theta\le\tilde\theta:=\frac{\lambda}{2C}$,

$$\partial_t\mathbb{E}[e^{\theta G_t^N}]\,\le\,-\lambda\theta\mathbb{E}[G_t^N e^{\theta G_t^N}]+C\theta\mathbb{E}[e^{\theta G_t^N}].$$

This amounts to the following differential inequality for the Laplace transform $F(t,\theta):=\mathbb{E}[e^{\theta G_t^N}]$: for all $t\ge0$ and $0\le\theta\le\tilde\theta$,

$$\partial_t F+\lambda\theta\partial_\theta F\le C\theta F,$$

which can be rewritten as follows, for all $t\ge0$ and $0\le e^{\lambda t}\theta\le\tilde\theta$,

$$\partial_t[F(t,e^{\lambda t}\theta)]\le Ce^{\lambda t}\theta F(t,e^{\lambda t}\theta).$$

By integration, this yields $F(t,\theta)\le e^{\theta C/\lambda}F(0,e^{-\lambda t}\theta)$ for all $t\ge0$ and $0\le\theta\le\tilde\theta$, that is, (4.33). ☐

With the above estimates at hand, we may now turn to the estimation of the Lions derivative of smooth functionals along the particle dynamics. For that purpose, we define the following hierarchy of norms: for any smooth functional $\Phi:\mathcal{P}(\mathbb{X})\to\mathbb{R}$, we define for all $\mu\in\mathcal{P}(\mathbb{X})$, $k\ge1$, $n\ge0$, and $p\ge0$,

$$|\!|\!|\,\Phi\,|\!|\!|_{k,n,p,\mu}\;:=\;\max_{1\le j\le k}\;\max_{0\le\alpha_1,\ldots,\alpha_j\le n}\bigg(\sup_{y_1,\ldots,y_j\in\mathbb{X}}\langle y_1\rangle^{-p}\ldots\langle y_j\rangle^{-p}\Big|\nabla_{y_1}^{\alpha_1}\ldots\nabla_{y_j}^{\alpha_j}\frac{\delta^j\Phi}{\delta\mu^j}(\mu,y_1,\ldots,y_j)\Big|\bigg).$$

The following result can be iterated to estimate arbitrary Lions graphs.

Lemma 4.8. *Let κ_0,λ_0 be as in Theorem 3.1 and let $\kappa\in[0,\kappa_0]$. Given $m\ge0$ and a smooth functional $\Psi:\triangle^m\times\mathcal{P}(\mathbb{X})\mapsto\mathbb{R}$, we have for all $\lambda\in[0,\lambda_0)$, $k\ge1$, $n\ge0$, $0<p\le1$, $(t,\tau,s)\in\triangle^{m+1}$, $\mu\in\mathcal{P}(\mathbb{X})$, and $\ell_0>0$,*

$$|\!|\!|\,\mathcal{U}_\Psi^{(1)}((t,\tau,s),\cdot)\,|\!|\!|_{k,n,p,\mu}\;\lesssim_{W,\beta,\lambda,\ell_0,k,p,n,a}\;e^{-p\lambda(\tau_m-s)}\;|\!|\!|\,\Psi((t,\tau),\cdot)\,|\!|\!|_{k,\ell_0+n+k-1,\frac p2,m(\tau_m-s,\mu)},$$

and in addition,

$$\Big|\!\Big|\!\Big|\boxed{\Psi}((t,\tau,s),\cdot)\Big|\!\Big|\!\Big|_{k,n,p,\mu}\;\lesssim_{W,\beta,\lambda,\ell_0,k,p,n,a}\;e^{-p\lambda(\tau_m-s)}\Big(\int_{\mathbb{X}}\langle z\rangle^p\mu(dz)\Big)\;|\!|\!|\,\Psi((t,\tau),\cdot)\,|\!|\!|_{k+2,\ell_0+n+k,\frac p3,m(\tau_m-s,\mu)}\,.$$

Given $m,m'\ge0$, smooth functionals $\Psi:\triangle^{m+1}\times\mathbb{X}\mapsto\mathbb{R}$ and $\Theta:\triangle^{m'+1}\times\mathbb{X}\mapsto\mathbb{R}$, and given a partition $\{i_1,\ldots,i_m\}\cup\{j_1,\ldots,j_{m'}\}=[\![m+m']\!]$ with $i_1<\ldots<i_m$ and $j_1<\ldots<j_{m'}$, we further have for all $\lambda\in[0,\lambda_0)$, $k\ge1$, $n\ge0$, $0<p\le1$, $(t,\tau,s,s')\in\triangle^{m+m'+2}$, $\mu\in\mathcal{P}(\mathbb{X})$, and $\ell_0>0$,

$$\Big|\!\Big|\!\Big|\;\boxed{\Psi_{\langle i_1,\ldots,i_m,m+m'+1\rangle}}\underset{\langle m+m'+2\rangle}{\overline{\qquad}}\boxed{\Theta_{\langle j_1,\ldots,j_{m'},m+m'+1\rangle}}((t,\tau,s,s'),\cdot)\Big|\!\Big|\!\Big|_{k,n,p,\mu}$$
$$\lesssim_{W,\beta,\lambda,\ell_0,k,p,n,a}\;e^{-p\lambda(s-s')}\Big(\int_{\mathbb{X}}\langle z\rangle^p\mu(dz)\Big)\;|\!|\!|\,\Psi\big((t,\tau_{i_1},\ldots,\tau_{i_m},s),\cdot\big)\,|\!|\!|_{k+1,\ell_0+n+k,\frac p3,m(s-s',\mu)}$$
$$\times\,|\!|\!|\,\Theta\big((t,\tau_{j_1},\ldots,\tau_{j_{m'}},s),\cdot\big)\,|\!|\!|_{k+1,\ell_0+n+k,\frac p3,m(s-s',\mu)}\,.$$

Proof. Given $m\ge0$ and a smooth functional $\Psi:\triangle^m\times\mathcal{P}(\mathbb{X})\to\mathbb{R}$, recalling that $\mathcal{U}_\Psi^{(1)}$ is defined in Definition 4.1, we can compute

$$\frac{\delta\mathcal{U}_\Psi^{(1)}}{\delta\mu}((t,\tau,s),\mu,y)\;=\;\int_{\mathbb{X}}\frac{\delta\Psi}{\delta\mu}\big((t,\tau),m(\tau_m-s,\mu),\cdot\big)\,m^{(1)}(\tau_m-s,\mu,y),\tag{4.36}$$

with $m^{(1)}$ as defined in Lemma 4.6. Recalling the definition of dual norms and using Lemma 4.6, for any $1<q\le 2$ with $pq'\gg_{\beta,a} 1$ large enough and with $\ell_0 q'>\dim\mathbb{X}$, we deduce for all $\lambda\in[0,\lambda_0)$, $(t,\tau,s)\in\triangle^{m+1}$, $\mu\in\mathcal{P}(\mathbb{X})$, and $y\in\mathbb{X}$,

$$\Big|\frac{\delta\mathcal{U}_\Psi^{(1)}}{\delta\mu}\big((t,\tau,s),\mu,y\big)\Big| \,\le\, \|m^{(1)}(\tau_m-s,\mu,y)\|_{W^{-\ell_0,q}(\langle\cdot\rangle^p)}\Big\|\langle\cdot\rangle^{-p}\frac{\delta\Psi}{\delta\mu}\big((t,\tau),m(\tau_m-s,\mu),\cdot\big)\Big\|_{W^{\ell_0,q'}(\mathbb{X})}$$
$$\lesssim_{W,\beta,\lambda,\ell_0,p,a}\,\langle y\rangle^p e^{-p\lambda(\tau_m-s)}\,|||\,\Psi\,|||_{1,\ell_0,\frac{p}{2},m(\tau_m-s,\mu)},$$

where in the last estimate we further used $pq'>2\dim\mathbb{X}$. By induction, on top of (4.36), we find for all $k\ge1$ and $y_1,\ldots,y_k\in\mathbb{X}$,

$$\frac{\delta^k\mathcal{U}_\Psi^{(1)}}{\delta\mu^k}\big((t,\tau,s),\mu,y_1,\ldots,y_k\big) \,=\, \sum_{l=1}^k\sum_{\substack{a_1+\ldots+a_l=k\\1\le a_1\le\ldots\le a_l\le k}}\sum_{\substack{I_1\cup\ldots\cup I_l=[\![k]\!]\\ \text{disjoint}}}\mathbb{1}_{\forall 1\le r\le l:\sharp I_r=a_r}$$
$$\times\int_{\mathbb{X}^l}\frac{\delta^l\Psi}{\delta\mu^l}\big((t,\tau),m(\tau_m-s,\mu),z_1,\ldots,z_l\big)\,m^{(a_1)}\big((\tau_m-s,z_1),\mu,y_{I_1}\big)$$
$$\ldots m^{(a_l)}\big((\tau_m-s,z_l),\mu,y_{I_l}\big)\,dz_1\ldots dz_l,\qquad(4.37)$$

from which we then get the following conclusion, using Lemma 4.6,

$$|||\,\mathcal{U}_\Psi^{(1)}((t,\tau,s),\cdot)\,|||_{k,n,p,\mu}\,\lesssim_{W,\beta,\lambda,\ell_0,k,q,p,n,a}\,e^{-p\lambda(\tau_m-s)}\,|||\,\Psi((t,\tau),\cdot)\,|||_{k,\ell_0+n+k-1,\frac{p}{2},m(\tau_m-s,\mu)}\,.$$

We turn to the estimation of the round edge. By definition (4.3), we can write

$$\boxed{\Psi}\big((t,\tau,s),\mu\big)\,=\,\mathcal{U}_{\Psi'}^{(1)}\big((t,\tau,s),\mu\big),$$

in terms of

$$\Psi'\big((t,\tau),\mu\big):=\int_{\mathbb{X}}\operatorname{tr}\Big[a_0\partial_\mu^2\Psi\big((t,\tau),\mu\big)(z,z)\Big]\mu(dz).$$

By a similar induction as the one performed to get (4.37), we find for all $k\ge1$ and $y_1,\ldots,y_k\in\mathbb{X}$,

$$\frac{\delta^k\mathcal{U}_{\Psi'}^{(1)}}{\delta\mu^k}\big((t,\tau,s),\mu,y_1,\ldots,y_k\big)=\sum_{j=0}^k\sum_{l=1}^j\sum_{\substack{a_1+\ldots+a_l=j\\1\le a_1\le\ldots\le a_l\le j}}\sum_{\substack{I_1\cup\ldots\cup I_l\subset[\![k]\!]\\ \text{disjoint}}}\mathbb{1}_{\forall 1\le r\le l:\sharp I_r=a_r}$$
$$\times\int_{\mathbb{X}^{l+1}}\frac{\delta^l}{\delta\mu^l}\operatorname{tr}\Big[a_0\partial_\mu^2\Psi\big((t,\tau),m(\tau_m-s,\mu),z,z\big)\Big](z_1,\ldots,z_l)\,m^{(a_1)}\big((\tau_m-s,z_1),\mu,y_{I_1}\big)$$
$$\ldots m^{(a_l)}\big((\tau_m-s,z_l),\mu,y_{I_l}\big)\,m^{(k-j)}\big((\tau_m-s,z),\mu,y_{[\![k]\!]\setminus(I_1\cup\ldots\cup I_l)}\big)\,dz_1\ldots dz_l\,dz.$$

For $j<k$, the terms can be estimated as before using Lemma 4.6. For $j=k$, we use the following: for any bounded function φ, by Lemma 4.7, we have

$$\int_{\mathbb{X}}\varphi(z,z)\,m\big((\tau_m-s,z),\mu\big)\,dz\,\lesssim_{W,\beta,a}\,\sup_{z\in\mathbb{X}}\Big(\langle z\rangle^{-p}\varphi(z,z)\Big)\int_{\mathbb{X}}\langle z\rangle^p\,\mu(dz),$$

and the conclusion then follows. The argument for the straight edge is similar and we skip the detail for shortness. ∎

The above result can be iterated to estimate arbitrary Lions graphs. Combining it with the diagrammatic representation of moments and cumulants in Proposition 4.5, we obtain the following.

Corollary 4.9 (Truncated Lions expansions). *Let κ_0 be as in Theorem 3.1, let $\kappa\in[0,\kappa_0]$, and further assume that the initial law $\mu_\circ\in\mathcal{P}(\mathbb{X})$ satisfies $\int_{\mathbb{X}}|z|^{p_0}\mu_\circ(dz)<\infty$ for some $p_0>0$. Given a smooth* *functional $\Phi : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, for all $k \ge 1$, we can expand as follows the kth Brownian moment and the kth Brownian cumulant along the particle dynamics: for all $n \ge 0$,*

$$\mathbb{E}_\circ\bigg[\Big|\mathbb{E}_B[\Phi(\mu_t^N)^k] \;-\; \sum_{m=0}^{n} \frac{1}{(2N)^m} \sum_{\Psi\in\Gamma(k,m)} \gamma(\Psi) \int_{\triangle_t^m\times 0} \Psi\Big|\bigg] \;\lesssim\; N^{-n-1}, \tag{4.38}$$

$$\mathbb{E}_\circ\bigg[\Big|\kappa_B^k[\Phi(\mu_t^N)] \;-\; \mathbb{1}_{n\ge k-1} \sum_{m=k-1}^{n} \frac{1}{(2N)^m} \sum_{\Psi\in\Gamma_\circ(k,m)} \gamma(\Psi) \int_{\triangle_t^m\times 0} \Psi\Big|\bigg] \;\lesssim\; N^{-n-1}, \tag{4.39}$$

where we recall that $\Gamma(k,m)$ stands for the set of all (unlabeled) irreducible L-graphs with k vertices and m edges and that $\Gamma_\circ(k,m)$ stands for the subset of all connected *(unlabeled) irreducible L-graphs with k vertices and m edges, where γ is some map $\Gamma(k,m) \to \mathbb{N}$, and where multiplicative constants only depend on $d, W, \beta, \ell_0, k, n, a, \int_{\mathbb{X}} |z|^{p_0} \mu_\circ(\mathrm{d}z)$, and on*

$$\sup_{\mu\in\mathcal{P}(\mathbb{X})} |||\, \Phi \,|||_{2(n+1),(n+1)(\ell_0+n+2),3^{-n-3}p_0,\mu},$$

for any $\ell_0 > 0$.

Proof. Let κ_0, λ_0 be as in Theorem 3.1 and let $\kappa \in [0,\kappa_0]$. By definition (4.3) of the round edge, and by Lemma 4.7, given $m \ge 0$ and a smooth functional $\Psi : \triangle^m \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$, we have for all $(t,\tau,s) \in \triangle^{m+1}$, $0 < p \le 1$, and $\mu \in \mathcal{P}(\mathbb{X})$,

$$\Big|\boxed{\Psi}((t,\tau,s),\mu)\Big| \;\lesssim_{W,\beta,p,a}\; |||\, \Psi((t,\tau),\cdot) \,|||_{2,1,\frac{p}{2},m(\tau_m-s,\mu)} \int_{\mathbb{X}} \langle z\rangle^p \mu(\mathrm{d}z).$$

Using this, we get in particular, for all $m \ge 0$, $(t,\tau,s) \in \triangle^{m+2}$, and $0 < p \le 1$,

$$\Big|\boxed{\overset{\bullet}{[\Phi^k]}}_{(m+1)}((t,\tau,s),\mu_s^N)\Big| \;\lesssim_{W,\beta,p,a}\; \Big|\Big|\Big|\boxed{\overset{\bullet}{[\Phi^k]}}_{(m)}((t,\tau),\cdot)\Big|\Big|\Big|_{2,1,\frac{p}{2},m(\tau_m-s,\mu_s^N)} \int_{\mathbb{X}} \langle z\rangle^p \mu_s^N(\mathrm{d}z).$$

Now repeatedly applying Lemma 4.8 to control the right-hand side, and using Jensen's inequality, we get for all $\lambda \in [0,\lambda_0)$ and $\ell_0 > 0$,

$$\begin{aligned}\Big|\boxed{\overset{\bullet}{[\Phi^k]}}_{(m+1)}((t,\tau,s),\mu_s^N)\Big| \;&\lesssim_{W,\beta,\lambda,\ell_0,m,p,a}\; e^{-3^{-m}p\lambda(\tau_1-\tau_{m+1})}\\ &\times \Big|\Big|\Big| \bullet_{[\Phi^k]}((t,\tau_1),\cdot)\Big|\Big|\Big|_{2(m+1),m\ell_0+m(m+1)+1,3^{-m-1}p,m(\tau_1-s,\mu_s^N)} \int_{\mathbb{X}} \langle z\rangle^{(2-3^{-m})p} \mu_s^N(\mathrm{d}z).\end{aligned}$$

Recalling $\bullet_{[\Phi^k]} = \mathcal{U}^{(1)}_{\Phi^k} = (\mathcal{U}^{(1)}_{\Phi})^k$, using again Lemma 4.8, taking the expectation, and using the moment bounds of Lemma 4.7 with $\mathbb{E}_\circ[\mu_0^N] = \mu_\circ$, we get

$$\begin{aligned}\mathbb{E}\bigg[\Big|\boxed{\overset{\bullet}{[\Phi^k]}}_{(m+1)}((t,\tau,s),\mu_s^N)\Big|\bigg] \;&\lesssim_{W,\beta,\lambda,\ell_0,k,m,p,a}\; e^{-3^{-m}p\lambda(t-\tau_{m+1})}\\ &\times |||\, \Phi \,|||^k_{2(m+1),(m+1)(\ell_0+m+2),3^{-m-2}p,m(t-s,\mu_s^N)} \int_{\mathbb{X}} \langle z\rangle^{2p} \mu_\circ(\mathrm{d}z).\end{aligned}$$

Inserting this into (4.15) and recalling the moment assumption for $\mu_\circ$, the conclusion (4.38) follows. Noting that similar a priori bounds on any irreducible L-graph can be obtained iteratively from Lemmas 4.7 and 4.8, the proof of (4.39) follows similarly from (4.17). ☐

5. Higher-order propagation of chaos

This section is devoted to the proof of Theorem 1.1. Let κ_0, λ_0 be as in Theorem 3.1 and let $\kappa \in [0,\kappa_0]$ be fixed. For $t \ge 0$ and $\phi \in C_c^\infty(\mathbb{X})$, consider the random variables

$$X_t^N(\phi) := \int_{\mathbb{X}} \phi\, \mu_t^N.$$

In the spirit of Lemma 2.6, we start by estimating cumulants of $X_t^N(\phi)$. By the law of total cumulance, cf. Lemma 2.4, they can be decomposed as follows, for all $m\ge 2$,

$$\kappa^m[X_t^N(\phi)] = \sum_{\pi\vdash[\![m]\!]} \kappa_\circ^{\sharp\pi}\Big[\big(\kappa_B^{\sharp A}[X_t^N(\phi)]\big)_{A\in\pi}\Big]. \tag{5.1}$$

We appeal to Corollary 4.9 with $\Phi(\mu):=\int_{\mathbb{X}}\phi\mu$ to expand Brownian cumulants of $X_t^N(\phi)=\Phi(\mu_t^N)$: for $k\le m$, we find

$$\mathbb{E}_\circ\bigg[\Big|\kappa_B^k[X_t^N(\phi)] - \mathbb{1}_{k<m}\sum_{p=k-1}^{m-2}\frac{1}{(2N)^p}\sum_{\Psi\in\Gamma_\circ(k,p)}\gamma(\Psi)\int_{\triangle_t^p\times 0}\Psi\Big|\bigg] \lesssim_{W,\beta,\phi,m,a,\mu_\circ} N^{1-m},$$

where we recall that $\Gamma_\circ(k,m)$ stands for the set of all connected irreducible L-graphs with k vertices and m edges built from the reference base point Φ, and where the multiplicative constant only depends on $d,W,\beta,m,a,\int_{\mathbb{X}}|z|^{p_0}\mu_\circ(\mathrm{d}z)$, and on the $W^{r,\infty}(\mathbb{X})$ norm of ϕ for some r only depending on m. Inserting this approximation into (5.1), we deduce

$$\begin{aligned}\bigg|\kappa^m[X_t^N(\phi)] - \sum_{s=2}^m\sum_{\{A_1,\ldots,A_s\}\vdash[\![m]\!]}\sum_{p_1=\sharp A_1-1}^{m-2}\cdots\sum_{p_s=\sharp A_s-1}^{m-2}\frac{1}{(2N)^{p_1+\ldots+p_s}}\\ \times\sum_{\Psi_1\in\Gamma_\circ(\sharp A_1,p_1)}\cdots\sum_{\Psi_s\in\Gamma_\circ(\sharp A_s,p_s)}\gamma(\Psi_1)\ldots\gamma(\Psi_s)\\ \times\kappa_\circ^s\Big[\int_{\triangle_t^{p_1}\times 0}\Psi_1,\ldots,\int_{\triangle_t^{p_s}\times 0}\Psi_s\Big]\bigg| \lesssim_{W,\beta,\phi,m,a,\mu_\circ} N^{1-m}.\end{aligned} \tag{5.2}$$

It remains to estimate the joint Glauber cumulants in this expression. For that purpose, we appeal to the higher-order Poincaré inequality of Proposition 2.9: recalling that Glauber derivatives can be bounded by linear derivatives, cf. (2.21), we get

$$\begin{aligned}&\kappa_\circ^s\Big[\int_{\triangle_t^{p_1}\times 0}\Psi_1,\ldots,\int_{\triangle_t^{p_s}\times 0}\Psi_s\Big]\\ &\quad\lesssim_s N^{1-s}\sum_{k=0}^{s-2}\sum_{\substack{a_1,\ldots,a_s\ge 1\\ \sum_j a_j=s+k}}\prod_{j=1}^s\bigg\|\int_{([0,1]\times\mathbb{X}\times\mathbb{X})^{a_j}}\frac{\delta^{a_j}}{\delta\mu^{a_j}}\Big(\int_{\triangle_t^{p_j}\times 0}\Psi_j\Big)(m_0^{N,s_1,\ldots,s_{a_j}},y_1,\ldots,y_{a_j})\\ &\qquad\qquad\times\prod_{l=1}^{a_j}(\delta_{Z_0^{l,N}}-\delta_{z_l})(\mathrm{d}y_l)\,\mu_\circ(\mathrm{d}z_l)\,\mathrm{d}s_l\bigg\|_{\mathrm{L}^{\frac{s+k}{a_j}}(\Omega_\circ)},\end{aligned}$$

where we have set for abbreviation

$$m_0^{N,s_1,\ldots,s_{a_j}} := \mu_0^N + \sum_{l=1}^{a_j}\frac{1-s_l}{N}(\delta_{z_l}-\delta_{Z_0^{l,N}}).$$

Norms of linear derivatives of each $\Psi_j\in\Gamma_\circ(\sharp A_j,p_j)$ can be estimated using Lemmas 4.7 and 4.8, together with the moment assumption on $\mu_\circ$. Inserting the result into (5.2), we conclude for all $m\ge 1$,

$$\kappa^m[X_t^N(\phi)] \lesssim_{W,\beta,\phi,m,a,\mu_\circ} N^{1-m}.$$

We now appeal to Lemma 2.6 to turn this into an estimate on correlation functions: the above cumulant estimate implies for all $1\le m\le N$,

$$\Big|\int_{\mathbb{X}^m}\phi^{\otimes m}G^{m,N}\Big| \lesssim_{W,\beta,\phi,m,a,\mu_\circ} N^{1-m} + \sum_{\substack{\pi\vdash[\![m]\!]\\ \sharp\pi<m}}\sum_{\rho\vdash\pi} N^{\sharp\pi-\sharp\rho-m+1}\Big|\int_{\mathbb{X}^{\sharp\pi}}\Big(\bigotimes_{B\in\pi}\phi^{\sharp B}\Big)\Big(\bigotimes_{D\in\rho}G^{\sharp D,N}(z_D)\Big)\,\mathrm{d}z_\pi\Big|,$$

and a direct induction argument then yields

$$\Big|\int_{\mathbb{X}^m}\phi^{\otimes m}G^{m,N}\Big| \lesssim_{W,\beta,\phi,m,a,\mu_\circ} N^{1-m}.$$

As the multiplicative constant only depends on ϕ via its $W^{r,\infty}(\mathbb{X})$ norm for some r only depending on m, the conclusion of Theorem 1.1 follows by duality. □

6. Concentration estimates

This section is devoted to the proof of Theorem 1.2. Let κ_0,λ_0 be as in Theorem 3.1, and let $\kappa\in[0,\kappa_0]$ be fixed. For $t\geq 0$ and $\phi\in C_c^\infty(\mathbb{X})$, consider the centered random variables

$$Y_t^N(\phi) := X_t^N(\phi)-\mathbb{E}[X_t^N(\phi)] = \int_{\mathbb{X}}\phi\,\mu_t^N-\mathbb{E}\Big[\int_{\mathbb{X}}\phi\,\mu_t^N\Big].$$

We shall establish concentration by means of moment estimates. By the Lions expansion of Lemma 2.1, we can decompose

$$Y_t^N(\phi) = \tilde Y_t^N(\phi)+M_t^N(\phi)+\frac{1}{2N}\big(E_t^N(\phi)-\mathbb{E}[E_t^N(\phi)]\big),$$

in terms of

$$\begin{aligned}
\tilde Y_t^N(\phi) &:= \int_{\mathbb{X}}\phi\,m(t,\mu_0^N)-\mathbb{E}\Big[\int_{\mathbb{X}}\phi\,m(t,\mu_0^N)\Big],\\
M_t^N(\phi) &:= \frac1N\sum_{i=1}^N\int_0^t\partial_\mu U(t-s,\mu_s^N)(Z_s^{i,N})\cdot\sigma_0\mathrm{d}B_s^i,\\
E_t^N(\phi) &:= \int_0^t\int_{\mathbb{X}}\mathrm{tr}\Big[a_0\partial_\mu^2U(t-s,\mu_s^N)(z,z)\Big]\mu_s^N(\mathrm{d}z)\,\mathrm{d}s,
\end{aligned}$$

where we use the short-hand notation

$$U(t-s,\mu) := \int_{\mathbb{X}}\phi\,m(t-s,\mu).$$

For all $k\geq1$, we may then decompose the kth moment of $Y_t^N(\phi)$ as

$$\mathbb{E}\big[|Y_t^N(\phi)|^k\big] \leq 3^k\mathbb{E}_\circ\big[|\tilde Y_t^N(\phi)|^k\big]+3^k\mathbb{E}\big[|M_t^N(\phi)|^k\big]+3^kN^{-k}\mathbb{E}\big[|E_t^N(\phi)|^k\big]. \tag{6.1}$$

We separately analyze the three right-hand side terms and split the proof into four steps. In the sequel, constants C are implicitly allowed to depend on W,β,a, on the compact support of $\mu_\circ$, as well as on the further parameters ℓ_0,p,q used below.

Step 1. Proof that for all $1\leq k\leq N$, $1<q\leq2$, $0<p\leq1$ with $pq'\gg_{\beta,a}1$, and $\ell_0>\frac{1}{q'}\dim\mathbb{X}$,

$$\mathbb{E}_\circ\big[|\tilde Y_t^N(\phi)|^k\big] \leq \Big(\frac{Ck}{N}\Big)^{\frac k2}\|\langle z\rangle^{-p}\phi\|^k_{W^{\ell_0,q'}(\mathbb{X})}. \tag{6.2}$$

Using (2.21) to estimate the Glauber derivative by means of a linear derivative, and using Lemma 4.6 to control the latter, we get for all $\lambda\in[0,\lambda_0)$, $1<q\leq2$, $0<p\leq1$ with $pq'\gg_{\beta,a}1$, and $\ell_0>\frac{1}{q'}\dim\mathbb{X}$,

$$|D_j^\circ\tilde Y_t^N(\phi)| \leq CN^{-1}e^{-p\lambda t}\Big(\langle Z_0^{j,N}\rangle^p+\int_{\mathbb{X}}\langle z\rangle^p\mu_\circ(\mathrm{d}z)\Big)\|\langle z\rangle^{-p}\phi\|_{W^{\ell_0,q'}(\mathbb{X})}.$$

By the compact support assumption for $\mu_\circ$, this yields almost surely

$$|D_j^\circ\tilde Y_t^N(\phi)| \leq CN^{-1}\|\langle z\rangle^{-p}\phi\|_{W^{\ell_0,q'}(\mathbb{X})}.$$

We may then appeal to Proposition 2.11, to the effect of

$$\begin{aligned}
\mathbb{E}_\circ\big[|\tilde Y_t^N(\phi)|^k\big] &\leq \inf_{\lambda>0}\Big\{k!\lambda^{-k}\mathbb{E}_\circ\big[e^{\lambda\tilde Y_t^N(\phi)}\big]\Big\}\\
&\leq \inf_{\lambda>0}\Big\{k!\lambda^{-k}\exp\Big(C\lambda\|\langle z\rangle^{-p}\phi\|_{W^{\ell_0,q'}(\mathbb{X})}\big(e^{2\lambda CN^{-1}\|\langle z\rangle^{-p}\phi\|_{W^{\ell_0,q'}(\mathbb{X})}}-1\big)\Big)\Big\}.
\end{aligned}$$

Choosing $\lambda = (kN)^{1/2}(2C\|\langle z\rangle^{-p}\phi\|_{W^{\ell_0,q'}(\mathbb{X})})^{-1}$, the claim (6.2) follows.

Step 2. Proof that for all $k\ge 1$, $1<q\le 2$, $0<p\le 1$ with $pq'\gg_{\beta,a} 1$, and $\ell_0>\frac{1}{q'}\dim\mathbb{X}$,

$$\mathbb{E}[|M_t^N(\phi)|^k] \,\le\, \Big(\frac{Ck}{N}\Big)^{\frac{k}{2}}\|\langle z\rangle^{-p}\phi\|^k_{W^{1+\ell_0,q'}(\mathbb{X})}\bigg(1+\min\Big\{(Ck^{\frac{p}{2}})^k\,;\,\max_{0\le s\le t}\mathbb{E}\big[|Y_s^N(\langle z\rangle^{2p})|^{\frac{k}{2}}\big]\Big\}\bigg). \tag{6.3}$$

We recall that $(M_t^N(\phi))_t$ is a martingale with quadratic variation given by

$$\langle M_t^N(\phi)\rangle \,=\, \frac{1}{N^2}\sum_{i=1}^N\int_0^t \big|\partial_\mu U(t-s,\mu_s^N)(Z_s^{i,N})\big|^2\,\mathrm{d}s.$$

Appealing to Lemma 4.6 to estimate the L-derivative, we get for all $\lambda\in[0,\lambda_0)$, $1<q\le 2$, $0<p\le 1$ with $pq'\gg_{\beta,a} 1$, and $\ell_0>\frac{1}{q'}\dim\mathbb{X}$,

$$\langle M_t^N(\phi)\rangle \,\le\, CN^{-1}\|\langle z\rangle^{-p}\phi\|^2_{W^{1+\ell_0,q'}(\mathbb{X})}\int_0^t e^{-2p\lambda(t-s)}\Big(\int_{\mathbb{X}}\langle z\rangle^{2p}\mu_s^N(\mathrm{d}z)\Big)\mathrm{d}s.$$

By the Burkholder–Davis–Gundy inequality, see e.g. [89, Theorem 1], we have

$$\mathbb{E}[|M_t^N(\phi)|^k] \,\le\, (Ck)^{\frac{k}{2}}\mathbb{E}[\langle M_t^N(\phi)\rangle^{\frac{k}{2}}],$$

and thus

$$\mathbb{E}[|M_t^N(\phi)|^k] \,\le\, \Big(\frac{Ck}{N}\Big)^{\frac{k}{2}}\|\langle z\rangle^{-p}\phi\|^k_{W^{1+\ell_0,q'}(\mathbb{X})}\max_{0\le s\le t}\mathbb{E}\bigg[\Big(\int_{\mathbb{X}}\langle z\rangle^{2p}\,\mu_s^N(\mathrm{d}z)\Big)^{\frac{k}{2}}\bigg].$$

Subtracting from μ_s^N its expectation, recognizing the definition of Y_s^N, and appealing to Lemma 4.7 together with the compact support assumption for $\mu_\circ$ to control the expectation $\mathbb{E}[\int_{\mathbb{X}}\langle z\rangle^{2p}\,\mu_s^N(\mathrm{d}z)]$, we get

$$\mathbb{E}[|M_t^N(\phi)|^k] \,\le\, \Big(\frac{Ck}{N}\Big)^{\frac{k}{2}}\|\langle z\rangle^{-p}\phi\|^k_{W^{1+\ell_0,q'}(\mathbb{X})}\Big(1+\max_{0\le s\le t}\mathbb{E}\big[|Y_s^N(\langle z\rangle^{2p})|^{\frac{k}{2}}\big]\Big).$$

By Jensen's inequality and the compact support assumption for $\mu_\circ$, we note that the Gaussian bounds of Lemma 4.7 imply

$$\mathbb{E}\big[|Y_s^N(\langle z\rangle^{2p})|^{\frac{k}{2}}\big] \,\le\, (Ck^{\frac{p}{2}})^k,$$

and the claim (6.3) follows.

Step 3. Proof that for all $k\ge 1$, $1<q\le 2$, $0<p\le 1$ with $pq'\gg_{\beta,a} 1$, and $\ell_0>\frac{1}{q'}\dim\mathbb{X}$,

$$\mathbb{E}[|E_t^N(\phi)|^k] \,\le\, (Ck^p)^k\|\langle z\rangle^{-p}\phi\|^k_{W^{2+\ell_0,q'}(\mathbb{X})}. \tag{6.4}$$

Recalling the definition of $E_t^N(\phi)$, and appealing to Lemma 4.6 to estimate multiple L-derivatives, we get for all $\lambda\in[0,\lambda_0)$, $1<q\le 2$, $0<p\le 1$ with $pq'\gg_{\beta,a} 1$, and $\ell_0>\frac{1}{q'}\dim\mathbb{X}$,

$$|E_t^N(\phi)| \,\lesssim\, \|\langle z\rangle^{-p}\phi\|_{W^{2+\ell_0,q'}(\mathbb{X})}\int_0^t e^{-p\lambda(t-s)}\Big(\int_{\mathbb{X}}\langle z\rangle^{2p}\mu_s^N(\mathrm{d}z)\Big)\,\mathrm{d}s.$$

Now appealing to the moment bounds of Lemma 4.7, together with Jensen's inequality and with the compact support assumption for $\mu_\circ$, the claim (6.4) follows.

Step 4. Conclusion.
Inserting the results of the first three steps into (6.1), we obtain for all $1\le k\le N$, $1<q\le 2$, $0<p\le 1$ with $pq'\gg_{\beta,a} 1$, and $\ell_0>\frac{1}{q'}\dim\mathbb{X}$,

$$\mathbb{E}[|Y_t^N(\phi)|^k] \,\le\, \Big(\frac{Ck}{N}\Big)^{\frac{k}{2}}\|\langle z\rangle^{-p}\phi\|^k_{W^{2+\ell_0,q'}(\mathbb{X})}\bigg(1+\min\Big\{(Ck^{\frac{p}{2}})^k\,;\,\max_{0\le s\le t}\mathbb{E}\big[|Y_s^N(\langle z\rangle^{2p})|^{\frac{k}{2}}\big]\Big\}\bigg).$$

Further applying this same estimate with $\phi = \langle z\rangle^{2p}$ and with p replaced by $3p$ to control the last factor, we are led to the following: for all $1 \le k \le N$ and $p, \ell_0 > 0$,

$$\mathbb{E}[|Y_t^N(\phi)|^k] \le \bigg(\Big(\frac{Ck}{N}\Big)^{\frac{k}{2}} + \Big(\frac{Ck^{1+p}}{N}\Big)^k\bigg)\|\phi\|^k_{W^{2+\ell_0,\infty}(\mathbb{X})}.$$

For $r \ge 0$, this entails by Markov's inequality, for all $1 \le k \le N$ and $p, \ell_0 > 0$,

$$\mathbb{P}\left[Y_t^N(\phi) \ge r\right] \le r^{-k}\,\mathbb{E}[|Y_t^N(\phi)|^k] \le \bigg(\Big(\frac{Ck}{Nr^2}\Big)^{\frac{k}{2}} + \Big(\frac{Ck^{1+p}}{Nr}\Big)^k\bigg)\|\phi\|^k_{W^{2+\ell_0,\infty}(\mathbb{X})}.$$

Choosing

$$k = Nr^2\big(eC\|\phi\|_{W^{2+\ell_0,\infty}(\mathbb{X})}\big)^{-1}, \qquad \text{for } 0 \le r \le \big(eC\|\phi\|_{W^{2+\ell_0,\infty}(\mathbb{X})}\big)^{1/2},$$

the conclusion follows. □

7. Quantitative central limit theorem

This section is devoted to the proof of Theorem 1.3. For $t \ge 0$ and $\phi \in C_c^\infty(\mathbb{R}^d)$, consider the centered random variables

$$S_t^N(\phi) := \sqrt{N}Y_t^N(\phi) := \sqrt{N}\bigg(\int_{\mathbb{X}}\phi\mu_t^N - \mathbb{E}\Big[\int_{\mathbb{X}}\phi\mu_t^N\Big]\bigg).$$

We shall start by using a Lions expansion to split the contributions from initial data and from Brownian forces in the fluctuations. From there, we separately analyze initial and Brownian fluctuations, using tools from Glauber and Lions calculus, respectively.

7.1. **Gaussian Dean–Kawasaki equation.** We consider the Gaussian Dean–Kawasaki SPDE (1.19). With our general notation, covering the Langevin and Brownian settings at the same time, this reads as follows,

$$\begin{cases} \partial_t\nu_t = L_{\mu_t}\nu_t + \operatorname{div}(\sqrt{\mu_t}\sigma_0\xi_t), & \text{for } t \ge 0,\\ \nu_t|_{t=0} = \nu_\circ, \end{cases} \tag{7.1}$$

where:

— L_μ is the linearized mean-field operator defined in (3.1);

— $\mu_t := m(t, \mu_\circ)$ is the solution of the mean-field McKean–Vlasov equation (1.21);

— $\nu_\circ$ is the Gaussian field describing the fluctuations of the initial empirical measure, in the sense that $\sqrt{N}\int_{\mathbb{X}}\phi(\mu_0^N - \mu_\circ)$ converges in law to $\int_{\mathbb{X}}\phi\,\nu_\circ$ for all $\phi \in C_c^\infty(\mathbb{X})$; in other words, $\nu_\circ$ is the random tempered distribution on $\mathbb{X}$ characterized by having Gaussian law with

$$\operatorname{Var}\Big[\int_{\mathbb{X}}\phi\,\nu_\circ\Big] = \int_{\mathbb{X}}\Big(\phi - \int_{\mathbb{X}}\phi\,\mu_\circ\Big)^2\mu_\circ, \qquad \mathbb{E}\Big[\int_{\mathbb{X}}\phi\,\nu_\circ\Big] = 0, \qquad \text{for all } \phi \in C_c^\infty(\mathbb{X}); \tag{7.2}$$

— $\xi = (\xi_t)_{t\in\mathbb{R}}$ is a Gaussian white noise on $\mathbb{R} \times \mathbb{X}$ and is taken independent of $\nu_\circ$.

As $\nu_\circ$ and ξ are random tempered distributions on $\mathbb{X}$ and $\mathbb{R}\times\mathbb{X}$, respectively, equation (7.1) is naturally understood in the distributional sense almost surely, and its solution ν must itself be sought as a random tempered distribution on $\mathbb{R}^+ \times \mathbb{X}$.

In order to solve equation (7.1), we start by introducing some notation. Given $1 < q \le 2$, $k \ge 1$, and $0 < p \le 1$ with $pq' \gg_{\beta,a} 1$ large enough, for all $h \in W^{-k,q}(\langle z\rangle^p)$, we can define $\{U_{t,s}[h]\}_{s\le t}$ as the unique weak solution in $C_{\mathrm{loc}}([s,\infty); W^{-k,q}(\langle z\rangle^p))$ of

$$\begin{cases} \partial_t U_{t,s}[h] = L_{\mu_t}U_{t,s}[h], & \text{for } t \ge s,\\ U_{t,s}[h]|_{t=s} = h, \end{cases} \tag{7.3}$$

see Theorem 3.1(ii). We also consider the dual evolution $\{U_{t,s}^*\}_{t\ge s}$ on $W^{-k,q}(\langle z\rangle^p)^*$, where for all $t \ge s$ we define $U_{t,s}^*$ as the adjoint of $U_{t,s}$,

$$\int_{\mathbb{X}} U_{t,s}^*[g]h = \int_{\mathbb{X}} U_{t,s}[h]g.$$

Recall that the condition $pq' > d$ ensures that the dual space $W^{-k,q}(\langle z\rangle^p)^*$ contains $W^{k,\infty}(\mathbb{X})$. For any $g \in W^{-k,q}(\langle z\rangle^p)^*$ and $t \ge 0$, the dual flow $s \mapsto U^*_{t,s}[g]$ naturally belongs to $C([0,t]; W^{-k,q}(\langle z\rangle^p)^*)$, where $W^{-k,q}(\langle z\rangle^p)^*$ is endowed with the weak topology. Denoting by L^*_μ the adjoint of the linearized mean-field operator L_μ, we find that the dual flow satisfies the backward Cauchy problem

$$\begin{cases} \partial_s U^*_{t,s}[g] = -L^*_{\mu_s} U^*_{t,s}[g], & \text{for } 0 \le s \le t, \\ U^*_{t,s}[g]|_{s=t} = g. \end{cases} \tag{7.4}$$

In these terms, using the theory of Da Prato and Zabczyk [32], and more specifically its non-autonomous extension by Seidler [82], we can check that the Gaussian Dean–Kawasaki equation (7.1) admits a unique weak solution that is a random element in $C(\mathbb{R}^+; \mathcal{S}'(\mathbb{X}))$, and it can be expressed by Duhamel's principle

$$\nu_t := U_{t,0}[\nu_\circ] + \int_0^t U_{t,s}\big[\operatorname{div}\big(\sqrt{\mu_s}\sigma_0 \xi_s\big)\big]\, \mathrm{d}s.$$

In particular, the solution is characterized by its covariance structure

$$\begin{aligned} \operatorname{Var}\Big[\int_{\mathbb{X}} \phi \nu_t\Big] &= \operatorname{Var}\Big[\int_{\mathbb{X}} U^*_{t,0}[\phi]\,\nu_\circ\Big] + \operatorname{Var}\Big[\int_0^t \Big(\int_{\mathbb{X}} \sqrt{\mu_s}\sigma_0^T \nabla U^*_{t,s}[\phi]\cdot \xi_s\Big)\,\mathrm{d}s\Big] \\ &= \int_{\mathbb{X}} \Big(U^*_{t,0}[\phi] - \int_{\mathbb{X}} U^*_{t,0}[\phi]\,\mu_\circ\Big)^2 \mu_\circ + \int_0^t \Big(\int_{\mathbb{X}} \big|\sigma_0^T \nabla U^*_{t,s}[\phi]\big|^2 \mu_s\Big)\,\mathrm{d}s. \end{aligned} \tag{7.5}$$

7.2. **Splitting fluctuations.** By means of a Lions expansion, we start by showing that fluctuations can be split neatly into contributions from initial data and from Brownian forces.

Lemma 7.1. *Let κ_0 be as in Theorem 3.1, let $\kappa \in [0, \kappa_0]$, and assume that the initial law $\mu_\circ$ satisfies $\int_{\mathbb{X}} |z|^{p_0}\mu_\circ(\mathrm{d}z) < \infty$ for some $p_0 > 0$. We have for all $\phi \in C^\infty_c(\mathbb{X})$ and $t \ge 0$,*

$$\big\|S^N_t(\phi) - C^N_t(\phi) - D^N_t(\phi)\big\|_{\mathrm{L}^2(\Omega)} \lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-\frac12},$$

in terms of

$$\begin{aligned} C^N_t(\phi) &:= \sqrt{N}\Big(\int_{\mathbb{X}} \phi\, m(t,\mu^N_0) - \mathbb{E}_\circ\Big[\int_{\mathbb{X}} \phi\, m(t,\mu^N_0)\Big]\Big), \\ D^N_t(\phi) &:= \frac{1}{\sqrt{N}} \sum_{i=1}^N \int_0^t \partial_\mu U(t-s,\mu^N_s)(Z^{i,N}_s)\cdot \sigma_0 \mathrm{d}B^i_s, \end{aligned}$$

where we have set for abbreviation $U(t,\mu) := \int_{\mathbb{X}} \phi\, m(t,\mu)$.

Proof. Let κ_0, λ_0 be as in Theorem 3.1 and let $\kappa \in [0,\kappa_0]$ be fixed. Starting point is the Lions expansion of Lemma 2.1,

$$\begin{aligned} \int_{\mathbb{X}} \phi\, \mu^N_t = \int_{\mathbb{X}} \phi\, m(t,\mu^N_0) + \frac{1}{N}\sum_{i=1}^N \int_0^t \partial_\mu U(t-s,\mu^N_s)(Z^{i,N}_s)\cdot\sigma_0 \mathrm{d}B^i_s \\ + \frac{1}{2N}\int_0^t \int_{\mathbb{X}} \operatorname{tr}\Big[a_0\, \partial^2_\mu U(t-s,\mu^N_s)(z,z)\Big]\mu^N_s(\mathrm{d}z)\,\mathrm{d}s. \end{aligned}$$

Multiplying by $\sqrt{N}$, subtracting the expectation to both sides of the identity, taking the $\mathrm{L}^2(\Omega)$ norm, and recognizing the definition of $C^N_t(\phi)$ and $D^N_t(\phi)$, we are led to

$$\big\|S^N_t(\phi) - C^N_t(\phi) - D^N_t(\phi)\big\|_{\mathrm{L}^2(\Omega)} \le \frac{1}{2\sqrt{N}}\Big\|\int_0^t \int_{\mathbb{X}} \operatorname{tr}\Big[a_0\,\partial^2_\mu U(t-s,\mu^N_s)(z,z)\Big]\mu^N_s(\mathrm{d}z)\,\mathrm{d}s\Big\|_{\mathrm{L}^2(\Omega)}. \tag{7.6}$$

By Lemma 4.6, for all $\lambda \in [0,\lambda_0)$, $0<p\le 1$, and $\ell_0>0$, we get

$$\int_0^t \int_{\mathbb{X}} \operatorname{tr}\Big[a_0\, \partial_\mu^2 U(t-s,\mu_s^N)(z,z)\Big]\mu_s^N(\mathrm{d}z)\,\mathrm{d}s \lesssim_{W,\beta,\lambda,\ell_0,p,a} \|\phi\|_{W^{2+\ell_0,\infty}(\mathbb{X})} \int_0^t e^{-p\lambda(t-s)}\Big(\int_{\mathbb{X}} \langle z\rangle^{2p}\mu_s^N(\mathrm{d}z)\Big)\,\mathrm{d}s.$$

Appealing to Lemma 4.7 together with Jensen's inequality and with the moment assumption for $\mu_\circ$, we deduce

$$\bigg\|\int_0^t \int_{\mathbb{X}} \operatorname{tr}\Big[a_0\, \partial_\mu^2 U(t-s,\mu_s^N)(z,z)\Big]\mu_s^N(\mathrm{d}z)\,\mathrm{d}s\bigg\|_{\mathrm{L}^2(\Omega)} \lesssim_{W,\beta,\ell_0,p_0,a} \|\phi\|_{W^{2+\ell_0,\infty}(\mathbb{X})}.$$

Combined with (7.6), this yields the conclusion. ☐

7.3. **Initial fluctuations.** We establish the following quantitative central limit theorem for initial fluctuations $C_t^N(\phi)$. The proof is split into two separate parts: the asymptotic normality of $C_t^N(\phi)$ and the convergence of its variance structure. For both parts, we exploit tools from Glauber calculus: more precisely, the first part follows from Stein's method in form of the so-called second-order Poincaré inequality of Proposition 2.10, while for the second part our starting point is the Helffer–Sjöstrand representation for the variance in Lemma 2.7(iii).

Lemma 7.2. *Let κ_0,λ_0 be as in Theorem 3.1, let $\kappa\in[0,\kappa_0]$, and assume that the initial law $\mu_\circ$ satisfies $\int_{\mathbb{X}}|z|^{p_0}\mu_\circ(\mathrm{d}z)<\infty$ for some $0<p_0\le 1$. The random variable $C_t^N(\phi)$ defined in Lemma 7.1 satisfies for all $\phi\in C_c^\infty(\mathbb{X})$, $t\ge 0$, and $\lambda\in[0,\frac12 p_0\lambda_0)$,*

$$\mathrm{d}_2\Big(C_t^N(\phi)\,,\,\sigma_t^C(\phi,\mu_\circ)\mathcal{N}\Big) \lesssim_{W,\beta,\lambda,\phi,a,\mu_\circ} N^{-\frac12}e^{-\lambda t}\bigg(1+\Big(\sigma_t^C(\phi,\mu_\circ)+(N^{-\frac13}e^{-\lambda t})^{\frac12}\Big)^{-1}\bigg),$$

where the limit variance is defined by

$$\sigma_t^C(\phi,\mu_\circ)^2 \,:=\, \operatorname{Var}_\circ\big[(U_{t,0}^*[\phi])(Z_\circ^{1,N})\big] \,=\, \int_{\mathbb{X}}\Big(U_{t,0}^*[\phi]-\int_{\mathbb{X}}U_{t,0}^*[\phi]\,\mu_\circ\Big)^2\mu_\circ, \tag{7.7}$$

where we recall that $U_{t,0}^*$ is defined in (7.4), that d_2 is the second-order Zolotarev metric (1.18), and that $\mathcal{N}$ stands for a standard normal random variable.

Proof. Let κ_0,λ_0 be as in Theorem 3.1 and let $\kappa\in[0,\kappa_0]$ be fixed. We split the proof into three steps.

Step 1. Asymptotic normality: proof that for all $t\ge0$ and $\lambda\in[0,\frac12 p_0\lambda_0)$,

$$\mathrm{d}_2\bigg(\frac{C_t^N(\phi)}{\operatorname{Var}_\circ[C_t^N(\phi)]^{\frac12}}\,,\mathcal{N}\bigg) + \mathrm{d}_{\mathrm{W}}\bigg(\frac{C_t^N(\phi)}{\operatorname{Var}_\circ[C_t^N(\phi)]^{\frac12}}\,,\mathcal{N}\bigg) + \mathrm{d}_{\mathrm{K}}\bigg(\frac{C_t^N(\phi)}{\operatorname{Var}_\circ[C_t^N(\phi)]^{\frac12}}\,,\mathcal{N}\bigg) \lesssim_{W,\beta,\lambda,\phi,a,\mu_\circ} N^{-\frac12}e^{-\lambda t}\operatorname{Var}_\circ[C_t^N(\phi)]^{-1}\Big(1+\operatorname{Var}_\circ[C_t^N(\phi)]^{-\frac12}\Big). \tag{7.8}$$

Set for abbreviation

$$\hat C_t^N(\phi) \,:=\, \frac{C_t^N(\phi)}{\operatorname{Var}_\circ[C_t^N(\phi)]^{\frac12}}.$$

By Proposition 2.10, we can estimate

$$\mathrm{d}_2\big(\hat C_t^N(\phi),\mathcal{N}\big) + \mathrm{d}_{\mathrm{W}}\big(\hat C_t^N(\phi),\mathcal{N}\big) + \mathrm{d}_{\mathrm{K}}\big(\hat C_t^N(\phi),\mathcal{N}\big) \lesssim \operatorname{Var}_\circ[C_t^N(\phi)]^{-\frac32}\sum_{j=1}^N \mathbb{E}_\circ\big[|D_j^\circ C_t^N(\phi)|^6\big]^{\frac12} + \operatorname{Var}_\circ[C_t^N(\phi)]^{-1}\bigg(\sum_{j=1}^N\Big(\sum_{l=1}^N \mathbb{E}_\circ\big[|D_l^\circ C_t^N(\phi)|^4\big]^{\frac14}\mathbb{E}_\circ\big[|D_j^\circ D_l^\circ C_t^N(\phi)|^4\big]^{\frac14}\Big)^2\bigg)^{\frac12}.$$

Recalling the definition of $C_t^N(\phi)$ in Lemma 7.1, using (2.20) and (2.21) to bound Glauber derivatives by means of linear derivatives, appealing to Lemma 4.6 to estimate the latter, using the moment assumption for $\mu_\circ$, and distinguishing between the cases $l=j$ and $l\neq j$ in the second right-hand side term, we deduce for all $\lambda\in[0,\lambda_0)$ and $\ell_0>0$,

$$\begin{aligned}\mathrm{d}_2\big(\hat C_t^N(\phi),\mathcal N\big)\,+\,\mathrm{d}_{\mathrm W}\big(\hat C_t^N(\phi),\mathcal N\big)\,+\,\mathrm{d}_{\mathrm K}\big(\hat C_t^N(\phi),\mathcal N\big)\\ \lesssim_{W,\beta,\lambda,\ell_0,p_0,a}\;\mathrm{Var}_\circ[C_t^N(\phi)]^{-\frac32}N^{-\frac12}\|\phi\|^3_{W^{\ell_0,\infty}(\mathbb X)}e^{-\frac12 p_0\lambda t}\Big(\int_{\mathbb X}\langle z\rangle^{p_0}\mu_\circ(\mathrm{d}z)\Big)^{\frac12}\\ +\,\mathrm{Var}_\circ[C_t^N(\phi)]^{-1}N^{-\frac12}\|\phi\|^2_{W^{1+\ell_0,\infty}(\mathbb X)}e^{-\frac12 p_0\lambda t}\Big(\int_{\mathbb X}\langle z\rangle^{p_0}\mu_\circ(\mathrm{d}z)\Big)^{\frac34},\end{aligned}$$

and the claim follows.

Step 2. Convergence of the variance: proof that for all $t\ge0$ and $\lambda\in[0,\frac12 p_0\lambda_0)$,

$$\Big|\mathrm{Var}_\circ[C_t^N(\phi)]-\sigma_t^C(\phi,\mu_\circ)^2\Big|\;\lesssim_{W,\beta,\lambda,\phi,a,\mu_\circ}\;N^{-1}e^{-\lambda t},\tag{7.9}$$

where the limit variance is defined in (7.7).

By the definition of $C_t^N(\phi)$ in Lemma 7.1, we have

$$\mathrm{Var}_\circ[C_t^N(\phi)]\;=\;N\,\mathrm{Var}_\circ\Big[\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big].$$

Appealing to the Helffer–Sjöstrand representation for the variance in terms of Glauber calculus, cf. Lemma 2.7(iii), we get

$$\mathrm{Var}_\circ[C_t^N(\phi)]\;=\;N\sum_{j=1}^N\mathbb E_\circ\Big[\Big(D_j^\circ\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big)\mathcal L_\circ^{-1}\Big(D_j^\circ\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big)\Big].$$

By exchangeability, this is equivalently written as

$$\mathrm{Var}_\circ[C_t^N(\phi)]\;=\;N^2\,\mathbb E_\circ\Big[\Big(D_1^\circ\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big)\mathcal L_\circ^{-1}\Big(D_1^\circ\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big)\Big].$$

Denoting by $\mathbb E_\circ^{\neq1}:=\mathbb E_\circ[\,\cdot\,|Z_\circ^{1,N}]$ and $\mathrm{Var}_\circ^{\neq1}:=\mathrm{Var}_\circ[\,\cdot\,|Z_\circ^{1,N}]$ the expectation and variance with respect to $\{Z_\circ^{j,N}\}_{j:j\neq1}$, and noting that $\mathbb E_\circ^{\neq1}\mathcal L_\circ^{-1}D_1^\circ=\mathcal L_\circ^{-1}\mathbb E_\circ^{\neq1}D_1^\circ$, we deduce from the triangle inequality

$$\begin{aligned}\bigg|\mathrm{Var}_\circ[C_t^N(\phi)]-N^2\,\mathbb E_\circ\Big[\Big(\mathbb E_\circ^{\neq1}\Big[D_1^\circ\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big]\Big)\mathcal L_\circ^{-1}\Big(\mathbb E_\circ^{\neq1}\Big[D_1^\circ\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big]\Big)\Big]\bigg|\\ \le\;N^2\,\mathbb E_{\circ,1}\Big[\mathrm{Var}_\circ^{\neq1}\Big[D_1^\circ\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big]\Big].\end{aligned}$$

Since we have $\mathcal L_\circ X=X$ for any $\sigma(Z_\circ^{1,N})$-measurable random variable X with $\mathbb E_\circ[X]=0$, the operator $\mathcal L_\circ^{-1}$ can be replaced by Id in the left-hand side. Further appealing to the variance inequality (2.13) for Glauber calculus, we are led to

$$\bigg|\mathrm{Var}_\circ[C_t^N(\phi)]-N^2\,\mathbb E_\circ\Big[\mathbb E_\circ^{\neq1}\Big[D_1^\circ\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big]^2\Big]\bigg|\;\le\;N^3\,\mathbb E_\circ\Big[\Big|D_2^\circ D_1^\circ\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big|^2\Big].$$

Using (2.21) to bound Glauber derivatives by means of linear derivatives, appealing to Lemma 4.6 to estimate the latter, and recalling the moment assumption for $\mu_\circ$, we obtain for all $\lambda\in[0,p_0\lambda_0)$,

$$\bigg|\mathrm{Var}_\circ[C_t^N(\phi)]-N^2\,\mathbb E_\circ\Big[\mathbb E_\circ^{\neq1}\Big[D_1^\circ\int_{\mathbb X}\phi\,m(t,\mu_0^N)\Big]^2\Big]\bigg|\;\lesssim_{W,\beta,\lambda,\phi,a,\mu_\circ}\;N^{-1}e^{-\lambda t}.\tag{7.10}$$

It remains to evaluate the Glauber derivative in the left-hand side. Recalling again the link between Glauber and linear derivatives, cf. (2.20), and appealing to Lemma 4.6 for the computation of the linear derivative, we get

$$D_1^\circ \int_{\mathbb{X}} \phi\, m(t,\mu_0^N) \\ = N^{-1} \int_0^1 \int_{\mathbb{X}} \int_{\mathbb{X}} \Big(\int_{\mathbb{X}} \phi\, m^{(1)}\big(t, \mu_0^N + \tfrac{1-s}{N}(\delta_z - \delta_{Z_\circ^{1,N}}), y\big)\Big)\, (\delta_{Z_\circ^{1,N}} - \delta_z)(\mathrm{d}y)\, \mu_\circ(\mathrm{d}z)\, \mathrm{d}s. \quad (7.11)$$

Let us further appeal to the definition of linear derivative to replace the measure $\mu_0^N + \frac{1-s}{N}(\delta_z - \delta_{Z_\circ^{1,N}})$ in the argument of $m^{(1)}$ by

$$\mu_{0,z'}^N := \mu_0^N + \tfrac{1}{N}(\delta_{z'} - \delta_{Z_\circ^{1,N}}) = \tfrac{1}{N}\delta_{z'} + \tfrac{1}{N}\textstyle\sum_{j=2}^N \delta_{Z_\circ^{j,N}},$$

where z' is a new variable integrated over with respect to $\mu_\circ$. Using Lemma 4.6 to estimate the additional linear derivative that constitutes the resulting error, we get for all $\lambda \in [0, \frac{1}{2}p_0\lambda_0)$,

$$\mathbb{E}_\circ\bigg[\Big| D_1^\circ \int_{\mathbb{X}} \phi\, m(t,\mu_0^N) \\ - N^{-1} \int_{\mathbb{X}}\int_{\mathbb{X}}\int_{\mathbb{X}} \Big(\int_{\mathbb{X}} \phi\, m^{(1)}(t, \mu_{0,z'}^N, y)\Big)\, (\delta_{Z_\circ^{1,N}} - \mu_\circ)(\mathrm{d}y)\, \mu_\circ(\mathrm{d}z')\Big|^2\bigg]^{\frac12} \lesssim_{W,\beta,\lambda,\phi,a,\mu_\circ} N^{-2} e^{-\lambda t}.$$

Inserting this into (7.10) and reorganizing expectations and integrals, we obtain for all $\lambda \in [0, \frac{1}{2}p_0\lambda_0)$,

$$\bigg| \mathrm{Var}_\circ[C_t^N(\phi)] - \int_{\mathbb{X}} \mathbb{E}_\circ\bigg[\int_{\mathbb{X}} \Big(\int_{\mathbb{X}} \phi\, m^{(1)}(t,\mu_0^N, y)\Big)\, (\delta_z - \mu_\circ)(\mathrm{d}y)\bigg]^2 \mu_\circ(\mathrm{d}z)\bigg| \lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-1} e^{-\lambda t}.$$

We are now in position to appeal to Lemma 2.3 to replace μ_0^N by $\mu_\circ$ in the expectation. Using as before Lemma 4.6 to estimate linear derivatives, this leads us to

$$\bigg| \mathrm{Var}_\circ[C_t^N(\phi)] - \int_{\mathbb{X}} \bigg(\int_{\mathbb{X}} \Big(\int_{\mathbb{X}} \phi\, m^{(1)}(t,\mu_\circ, y)\Big)\, (\delta_z - \mu_\circ)(\mathrm{d}y)\bigg)^2 \mu_\circ(\mathrm{d}z)\bigg| \lesssim_{W,\beta,\lambda,\phi,a,\mu_\circ} N^{-1} e^{-\lambda t}. \quad (7.12)$$

Using the notation (7.3), the definition of $m^{(1)}$ in Lemma 4.6 amounts to

$$m^{(1)}(t,\mu_\circ,y) = U_{t,0}[\delta_y - \mu_\circ],$$

and the claim (7.9) follows with the limit variance $\sigma_t^C(\phi,\mu_\circ)$ defined in (7.7).

Step 3. Conclusion.
By homogeneity of d_2 and by the triangle inequality, we can estimate

$$\mathrm{d}_2\Big(C_t^N(\phi)\,,\, \sigma_t^C(\phi,\mu_\circ)\mathcal{N}\Big) \\ = \mathrm{Var}[C_t^N(\phi)]\, \mathrm{d}_2\bigg(\frac{C_t^N(\phi)}{\mathrm{Var}[C_t^N(\phi)]^{1/2}}\,,\, \frac{\sigma_t^C(\phi,\mu_\circ)}{\mathrm{Var}[C_t^N(\phi)]^{1/2}}\mathcal{N}\bigg) \\ \le \mathrm{Var}[C_t^N(\phi)]\, \mathrm{d}_2\bigg(\frac{C_t^N(\phi)}{\mathrm{Var}[C_t^N(\phi)]^{1/2}}\,,\, \mathcal{N}\bigg) + \tfrac12\Big| \mathrm{Var}[C_t^N(\phi)] - \sigma_t^C(\phi,\mu_\circ)^2\Big|.$$

By the asymptotic normality (7.8) and by the convergence result (7.9) for the variance, we then get for all $\lambda \in [0, \frac{1}{2}p_0\lambda_0)$

$$\mathrm{d}_2\Big(C_t^N(\phi)\,,\, \sigma_t^C(\phi,\mu_\circ)\mathcal{N}\Big) \lesssim_{W,\beta,\lambda,\phi,a,\mu_\circ} N^{-\frac12} e^{-\lambda t}\Big(1 + \mathrm{Var}[C_t^N(\phi)]^{-\frac12}\Big). \quad (7.13)$$

It remains to deal with the last factor involving the inverse of the variance. For that purpose, we distinguish between two cases:

— Case 1: assume that $\sigma_t^C(\phi,\mu_\circ)^2 \ge L$.
In this case, the convergence result (7.9) for the variance yields

$$\mathrm{Var}[C_t^N(\phi)]^{-\frac12} \le \Big(\sigma_t^C(\phi,\mu_\circ)^2 - C_\phi N^{-1}e^{-\lambda t}\Big)^{-\frac12} \le \big(L - C_\phi N^{-1}e^{-\lambda t}\big)^{-\frac12},$$

so that (7.13) becomes

$$\mathrm{d}_2\Big(C_t^N(\phi)\,,\,\sigma_t^C(\phi,\mu_\circ)\mathcal{N}\Big) \lesssim_\phi N^{-\frac12}e^{-\lambda t}\Big(1 + \big(L - C_\phi N^{-1}e^{-\lambda t}\big)^{-\frac12}\Big).$$

— Case 2: assume that $\sigma_t^C(\phi,\mu_\circ)^2 \le L$.
In this case, the convergence result (7.9) for the variance yields

$$\begin{aligned}\mathrm{d}_2\Big(C_t^N(\phi)\,,\,\sigma_t^C(\phi,\mu_\circ)\mathcal{N}\Big) &\le \sigma_t^C(\phi,\mu_\circ)^2 + \mathrm{Var}_\circ[C_t^N(\phi)]\\ &\le 2L + C_\phi N^{-1}e^{-\lambda t}.\end{aligned}$$

Optimizing between those two cases, the conclusion follows. □

7.4. **Brownian fluctuations.** We establish the following quantitative central limit theorem for Brownian fluctuations $D_t^N(\phi)$. The proof is based on the Lions expansion combined with a simple heat-kernel PDE argument. The additional estimate (7.16) in the statement below ensures that the central limit theorem for $D_t^N(\phi)$ also holds given $C_t^N(\phi)$, which is key to deduce a joint result.

Lemma 7.3 (Brownian fluctuations). *Let κ_0 be as in Theorem 3.1, let $\kappa \in [0,\kappa_0]$, and assume that the initial law $\mu_\circ$ satisfies $\int_{\mathbb{X}} |z|^{p_0}\mu_\circ(\mathrm{d}z) < \infty$ for some $p_0 > 0$. The random variable $D_t^N(\phi)$ defined in Lemma 7.1 satisfies for all $t \ge 0$,*

$$\mathrm{d}_2\big(\,D_t^N(\phi)\,,\,\sigma_t^D(\phi,\mu_\circ)\mathcal{N}\,\big) \lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-\frac12}, \tag{7.14}$$

where the limit variance is given by

$$\sigma_t^D(\phi,\mu_\circ)^2 := \int_0^t \Big(\int_{\mathbb{X}} \big|\sigma_0^T\nabla U_{t,s}^*[\phi]\big)\big|^2\, m(s,\mu_\circ)\Big)\,\mathrm{d}s, \tag{7.15}$$

where we recall that $U_{t,s}^*$ is defined in (7.4), that d_2 is the second-order Zolotarev metric (1.18), and that $\mathcal{N}$ stands for a standard normal random variable. In addition, for all $h \in C_b^2(\mathbb{R}^2)$ and $t \ge 0$,

$$\Big|\mathbb{E}\big[h\big(C_t^N(\phi), D_t^N(\phi)\big)\big] - \mathbb{E}_\circ\mathbb{E}_{\mathcal{N}}\big[h\big(C_t^N(\phi), \sigma_t^D(\phi,\mu_\circ)\mathcal{N}\big)\big]\Big| \lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-\frac12}\|\partial_2^2 h\|_{\mathrm{L}^\infty(\mathbb{R}^2)}, \tag{7.16}$$

where the standard normal variable $\mathcal{N}$ is taken independent both of initial data and of Brownian forces, and where we denote by $\mathbb{E}_{\mathcal{N}}$ the expectation with respect to $\mathcal{N}$.

Proof. Let κ_0, λ_0 be as in Theorem 3.1 and let $\kappa \in [0,\kappa_0]$ be fixed. We focus on the proof of (7.14), while the additional statement (7.16) can be obtained along the exact same lines — simply replacing the test function g below by $h(C_t^N(\phi),\cdot)$ and recalling that $C_t^N(\phi)$ is independent of Brownian forces. We split the proof into two steps.

Step 1. Proof that for all $g \in C_b^2(\mathbb{R})$ and $t \ge 0$,

$$\Big|\mathbb{E}\big[g(D_t^N(\phi))\big] - \mathbb{E}_\circ\mathbb{E}_{\mathcal{N}}\Big[g\Big(\sigma_t^D(\phi,\mu_0^N)\mathcal{N}\Big)\Big]\Big| \lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-\frac12}\|g''\|_{\mathrm{L}^\infty(\mathbb{R})}, \tag{7.17}$$

where the limit variance is defined in (7.15), and where as in the statement $\mathcal{N}$ stands for a standard normal random variable taken independent both of initial data and of Brownian forces.

To prove this result, let us consider the $(\mathcal{F}_s^B)_{0\le s\le t}$-martingale $(D_{s,t}^N(\phi))_{0\le s\le t}$ given by

$$D_{s,t}^N(\phi) := \frac{1}{\sqrt{N}}\sum_{i=1}^N \int_0^s (\partial_\mu U)(t-u,\mu_u^N)(Z_u^{i,N})\cdot\sigma_0\mathrm{d}B_u^i,$$

which satisfies

$$D_{0,t}^N(\phi) = 0, \qquad D_{t,t}^N(\phi) = D_t^N(\phi).$$

Let $g \in C^2_b(\mathbb{R})$ be fixed. By definition of $D^N_{s,t}(\phi)$, Itô's lemma yields for all $\theta \in \mathbb{R}$ and $0 \le s \le t$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}_B\big[g(\theta + D^N_{s,t}(\phi))\big] \,=\, \tfrac12 \mathbb{E}_B\Big[g''(\theta + D^N_{s,t}(\phi)) \int_{\mathbb{X}} \big|\sigma_0^T(\partial_\mu U)(t-s,\mu^N_s)\big|^2 \mu^N_s\Big]. \tag{7.18}$$

Appealing to the Lions expansion in form of Corollary 2.2(ii), we find for all $0 \le s \le t$,

$$\begin{aligned}\bigg\| \int_{\mathbb{X}} \big|\sigma_0^T(\partial_\mu U)(t-s,\mu^N_s)\big|^2 \mu^N_s - \int_{\mathbb{X}} \big|\sigma_0^T(\partial_\mu U)(t-s,m(s,\mu^N_0))\big|^2 m(s,\mu^N_0)\bigg\|_{\mathrm{L}^2(\Omega_B)}&\\ \lesssim\, N^{-\frac12}\mathbb{E}_B\bigg[\int_0^s \int_{\mathbb{X}} \big|\partial_\mu H_{t-s}(s-u,\mu^N_u)(z)\big|^2 \mu^N_u(\mathrm{d}z)\,\mathrm{d}u\bigg]^{\frac12}&\\ + N^{-1}\mathbb{E}_B\bigg[\Big(\int_0^s \int_{\mathbb{X}} \big|\partial^2_\mu H_{t-s}(s-u,\mu^N_u)(z,z)\big|\, \mu^N_u(\mathrm{d}z)\,\mathrm{d}u\Big)^2\bigg]&,\end{aligned}$$

in terms of

$$H_{t-s}(u,\mu) \,:=\, \int_{\mathbb{X}} \Big|\sigma_0^T \partial_\mu U(t-s,m(u,\mu))\Big|^2 m(u,\mu).$$

Appealing to Lemma 4.6 to estimate the multiple linear derivatives, and combining it with the moment bounds of Lemma 4.7, we get after straightforward computations for all $0 \le s \le t$ and $\lambda \in [0, \frac18 p_0\lambda_0)$,

$$\begin{aligned}\bigg\| \int_{\mathbb{X}} \big|\sigma_0^T(\partial_\mu U)(t-s,\mu^N_s)\big|^2 \mu^N_s - \int_{\mathbb{X}} \big|\sigma_0^T(\partial_\mu U)(t-s,m(s,\mu^N_0))\big|^2 m(s,\mu^N_0)\bigg\|_{\mathrm{L}^2(\Omega_B)}&\\ \lesssim_{W,\beta,\lambda,\phi,a,p_0}\, N^{-\frac12} e^{-\lambda(t-s)} \int_{\mathbb{X}} \langle z\rangle^{p_0} \mu^N_0(\mathrm{d}z)&.\end{aligned}$$

Inserting this into (7.18), and using the short-hand notation

$$\kappa_{s,t}(\phi,\mu) \,:=\, \int_{\mathbb{X}} \big|\sigma_0^T(\partial_\mu U)(t-s,m(s,\mu))\big|^2 m(s,\mu),$$

we deduce for all $\theta \in \mathbb{R}$, $0 \le s \le t$, and $\lambda \in [0, \frac18 p_0\lambda_0)$,

$$\begin{aligned}\bigg|\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}_B\big[g(\theta + D^N_{s,t}(\phi))\big] - \tfrac12 \kappa_{s,t}(\mu^N_0)\,\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\mathbb{E}_B\big[g(\theta + D^N_{s,t}(\phi))\big]\bigg|&\\ \lesssim_{W,\beta,\lambda,\phi,a,p_0}\, N^{-\frac12} e^{-\lambda(t-s)} \|g''\|_{\mathrm{L}^\infty(\mathbb{R})} \int_{\mathbb{X}} \langle z\rangle^{p_0} \mu^N_0(\mathrm{d}z)&.\end{aligned}$$

In order to solve this approximate heat equation for the map $(s,\theta) \mapsto \mathbb{E}_B[g(\theta + D^N_{s,t}(\phi))]$ on $[0,t] \times \mathbb{R}$, we can appeal for instance to the Feynman–Kac formula with $D^N_{0,t}(\phi) = 0$. Equivalently, this amounts to integrating the above estimate with the associated heat kernel. It leads us to deduce for all $\theta \in \mathbb{R}$ and $0 \le s \le t$,

$$\bigg|\mathbb{E}_B\big[g(\theta + D^N_{s,t}(\phi))\big] - \mathbb{E}_{\mathcal{N}}\Big[g\Big(\theta + \mathcal{N}\sqrt{\textstyle\int_0^s \kappa_{u,t}(\phi,\mu^N_0)\,\mathrm{d}u}\Big)\Big]\bigg| \,\lesssim_{W,\beta,\phi,a,p_0}\, N^{-\frac12} \|g''\|_{\mathrm{L}^\infty(\mathbb{R})} \int_{\mathbb{X}} \langle z\rangle^{p_0} \mu^N_0(\mathrm{d}z).$$

In particular, setting $\theta = 0$ and $s = t$, recalling $D^N_{t,t}(\phi) = D^N_t(\phi)$, taking the expectation with respect to initial data, and recalling $\mathbb{E}_\circ[\mu^N_0] = \mu_\circ$ and the moment assumption for $\mu_\circ$, we get

$$\bigg|\mathbb{E}\big[g(D^N_t(\phi))\big] - \mathbb{E}_\circ\mathbb{E}_{\mathcal{N}}\Big[g\Big(\mathcal{N}\sqrt{\textstyle\int_0^t \kappa_{u,t}(\phi,\mu^N_0)\,\mathrm{d}u}\Big)\Big]\bigg| \,\lesssim_{W,\beta,\phi,a,\mu_\circ}\, N^{-\frac12} \|g''\|_{\mathrm{L}^\infty(\mathbb{R})}.$$

Using the notation (7.3) and recalling that the definition of $m^{(1)}$ in Lemma 4.6 amounts to

$$m^{(1)}(t,\mu_\circ,y) = U_{t,0}[\delta_y - \mu_\circ],$$

we recognize the definition (7.15) of σ_t^D,

$$\begin{aligned}\int_0^t \kappa_{u,t}(\phi,\mu)\,\mathrm{d}u &= \int_0^t \Big(\int_{\mathbb{X}} \big|\sigma_0^T(\partial_\mu U)(t-s,m(s,\mu))\big|^2 m(s,\mu)\Big)\,\mathrm{d}s\\ &= \int_0^t \Big(\int_{\mathbb{X}} \big|\sigma_0^T\nabla U_{t,s}^*[\phi])\big|^2 m(s,\mu)\Big)\,\mathrm{d}s\\ &= \sigma_t^D(\phi,\mu)^2, \end{aligned}\tag{7.19}$$

and the claim (7.17) follows.

Step 2. Proof that for all $g\in C_b^2(\mathbb{R})$ and $t\ge 0$,

$$\Big|\mathbb{E}_\circ\mathbb{E}_{\mathcal{N}}\Big[g\big(\sigma_t^D(\phi,\mu_0^N)\mathcal{N}\big)\Big]-\mathbb{E}_{\mathcal{N}}\Big[g\big(\sigma_t^D(\phi,\mu_\circ)\mathcal{N}\big)\Big]\Big| \lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-\frac12}\|g''\|_{\mathrm{L}^\infty(\mathbb{R})}. \tag{7.20}$$

Combining this with the result (7.17) of Step 1, and recalling the definition (1.18) of the second-order Zolotarev metric, this will conclude the proof of (7.14). Set for shortness $\sigma_t^N:=\sigma_t^D(\phi,\mu_0^N)$ and $\sigma_t:=\sigma_t^D(\phi,\mu_\circ)$. We can decompose

$$\mathbb{E}_{\mathcal{N}}\big[g(\sigma_t^N\mathcal{N})\big]-\mathbb{E}_{\mathcal{N}}\big[g(\sigma_t\mathcal{N})\big] = (\sigma_t^N-\sigma_t)\int_0^1 \mathbb{E}_{\mathcal{N}}\Big[\mathcal{N}g'\Big(\big(\sigma_t+\theta(\sigma_t^N-\sigma_t)\big)\mathcal{N}\Big)\Big]\,\mathrm{d}\theta,$$

and a Gaussian integration by parts then yields

$$\mathbb{E}_{\mathcal{N}}\big[g(\sigma_t^N\mathcal{N})\big]-\mathbb{E}_{\mathcal{N}}\big[g(\sigma_t\mathcal{N})\big] = (\sigma_t^N-\sigma_t)\int_0^1 \big(\sigma_t+\theta(\sigma_t^N-\sigma_t)\big)\,\mathbb{E}_{\mathcal{N}}\Big[g''\Big(\big(\sigma_t+\theta(\sigma_t^N-\sigma_t)\big)\mathcal{N}\Big)\Big]\,\mathrm{d}\theta.$$

Hence,

$$\Big|\mathbb{E}_{\mathcal{N}}\big[g(\sigma_t^N\mathcal{N})\big]-\mathbb{E}_{\mathcal{N}}\big[g(\sigma_t\mathcal{N})\big]\Big| \le \|g''\|_{\mathrm{L}^\infty(\mathbb{R})}|\sigma_t^N-\sigma_t|\big(|\sigma_t|+|\sigma_t^N-\sigma_t|\big).$$

Taking the expectation with respect to initial data, the claim (7.20) would follow provided that we could show for all $t\ge 0$,

$$\begin{aligned}|\sigma_t^D(\phi,\mu_\circ)| &\lesssim_{W,\beta,\phi,a,\mu_\circ} 1, &(7.21)\\ \mathbb{E}_\circ\big[|\sigma_t^D(\phi,\mu_0^N)-\sigma_t^D(\phi,\mu_\circ)|^2\big]^{\frac12} &\lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-\frac12}. &(7.22)\end{aligned}$$

For that purpose, we first recall that by (7.19) we can write

$$\sigma_t^D(\phi,\mu)^2 = \int_0^t \Big(\int_{\mathbb{X}} \big|\sigma_0^T(\partial_\mu U)(t-s,m(s,\mu))\big|^2 m(s,\mu)\Big)\,\mathrm{d}s, \tag{7.23}$$

with the short-hand notation $U(t,\mu):=\int_{\mathbb{X}}\phi\, m(t,\mu)$. Applying Lemma 4.6 to estimate the linear derivative, and combining it with the moment bounds of Lemma 4.7, the claim (7.21) follows after straightforward computations. We turn to the proof of (7.22). By the triangle inequality, we can decompose

$$\mathbb{E}_\circ\big[|\sigma_t^D(\phi,\mu_0^N)-\sigma_t^D(\phi,\mu_\circ)|^2\big]^{\frac12} \le \big|\mathbb{E}_\circ[\sigma_t^D(\phi,\mu_0^N)]-\sigma_t^D(\phi,\mu_\circ)\big| + \mathrm{Var}_\circ[\sigma_t^D(\phi,\mu_0^N)]^{\frac12},$$

and we estimate both terms separately. On the one hand, starting again from (7.23), appealing to Lemma 2.3, using Lemma 4.6 to estimate the multiple linear derivatives, and combining it with the moment bounds of Lemma 4.7 and with the moment assumption for $\mu_\circ$, we easily get

$$\big|\mathbb{E}_\circ[\sigma_t^D(\phi,\mu_0^N)]-\sigma_t^D(\phi,\mu_\circ)\big| \lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-1}.$$

On the other hand, using the variance inequality (2.13) for Glauber calculus, and appealing to (2.20) to bound Glauber derivatives in terms of linear derivatives, we find

$$\begin{aligned}\mathrm{Var}_\circ\left[\sigma_t^D(\phi,\mu_0^N)\right] &\le \sum_{j=1}^N \mathbb{E}_\circ\left[|D_j^\circ \sigma_t^D(\phi,\mu_0^N)|^2\right]\\ &\lesssim N^{-1}\,\mathbb{E}_\circ\bigg[\Big|\int_0^1\int_{\mathbb{X}}\int_{\mathbb{X}} \frac{\delta\sigma_t^D}{\delta\mu}\Big(\phi,\mu_0^N+\tfrac{1-s}{N}(\delta_z-\delta_{Z_0^{1,N}}),y\Big)\,(\delta_{Z_0^{1,N}}-\delta_z)(dy)\,\mu_\circ(dz)\,ds\Big|^2\bigg].\end{aligned}$$

Further using Lemma 4.6 to estimate the multiple linear derivatives, and combining it with the moment bounds of Lemma 4.7 and with the moment assumption for $\mu_\circ$, we deduce

$$\mathrm{Var}_\circ[\sigma_t^D(\phi,\mu_0^N)] \lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-1},$$

and the claim (7.22) follows. ☐

7.5. **Proof of Theorem 1.3.** Let κ_0,λ_0 be as in Theorem 3.1 and let $\kappa\in[0,\kappa_0]$ be fixed. Let also $g\in C_b^2(\mathbb{R})$ be momentarily fixed with $g'(0)=0$ and $\|g''\|_{\mathrm{L}^\infty(\mathbb{R})}=1$. By Lemma 7.1, we find

$$\Big|\mathbb{E}\big[g(S_t^N(\phi))\big]-\mathbb{E}\big[g\big(C_t^N(\phi)+D_t^N(\phi)\big)\big]\Big| \lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-\frac12}.$$

Next, appealing to (7.16) in Lemma 7.3 for the asymptotic normality of $D_t^N(\phi)$ given $C_t^N(\phi)$, we deduce

$$\Big|\mathbb{E}\big[g(S_t^N(\phi))\big]-\mathbb{E}_\circ\mathbb{E}_{\mathcal{N}}\big[g\big(C_t^N(\phi)+\sigma_t^D(\phi,\mu_\circ)\mathcal{N}\big)\big]\Big| \lesssim_{W,\beta,\phi,a,\mu_\circ} N^{-\frac12}, \tag{7.24}$$

where $\mathcal{N}$ stands for a standard normal random variable taken independent both of initial data and of Brownian forces. It remains to combine this with our analysis of fluctuations of $C_t^N(\phi)$. Appealing to the asymptotic normality of $C_t^N(\phi)$ as stated in Lemma 7.2, and testing that result in Zolotarev metric with the function $\mathbb{E}_{\mathcal{N}}[g(\cdot+\sigma_t^D(\phi,\mu_\circ)\mathcal{N})]\in C_b^2(\mathbb{R})$, we deduce for all $\lambda\in[0,\frac12 p_0\lambda_0)$,

$$\begin{aligned}&\Big|\mathbb{E}\big[g(S_t^N(\phi))\big]-\mathbb{E}_{\mathcal{N}}\mathbb{E}_{\mathcal{N}'}\big[g\big(\sigma_t^C(\phi,\mu_\circ)\mathcal{N}'+\sigma_t^D(\phi,\mu_\circ)\mathcal{N}\big)\big]\Big|\\ &\qquad\lesssim_{W,\beta,\lambda,\phi,a,\mu_\circ} N^{-\frac12}\bigg(1+e^{-\lambda t}\Big(\sigma_t^C(\phi,\mu_\circ)+(N^{-\frac13}e^{-\lambda t})^{\frac12}\Big)^{-1}\bigg),\end{aligned}$$

where $\mathcal{N}'$ stands for another standard normal random variable taken independent of initial data, of Brownian forces, and of $\mathcal{N}$. Taking the supremum over g, and noting that $\sigma_t^C(\phi,\mu_\circ)\mathcal{N}'+\sigma_t^D(\phi,\mu_\circ)\mathcal{N}$ has the same distribution as $\sigma_t(\phi,\mu_\circ)\mathcal{N}$ with total variance

$$\sigma_t(\phi,\mu_\circ)^2 := \sigma_t^C(\phi,\mu_\circ)^2+\sigma_t^D(\phi,\mu_\circ)^2,$$

we conclude

$$\begin{aligned}\mathrm{d}_2\Big(S_t^N(\phi)\,,\,\sigma_t(\phi,\mu_\circ)\mathcal{N}\Big) &\lesssim_{W,\beta,\lambda,\phi,a,\mu_\circ} N^{-\frac12}\bigg(1+e^{-\lambda t}\Big(\sigma_t^C(\phi,\mu_\circ)+(N^{-\frac13}e^{-\lambda t})^{\frac12}\Big)^{-1}\bigg)\\ &\le N^{-\frac12}+N^{-\frac13}e^{-\frac12\lambda t}.\end{aligned}$$

Noting that the total variance $\sigma_t(\phi,\mu_\circ)$ coincides with the variance predicted by the Gaussian Dean–Kawasaki equation, cf. (7.5), the conclusion follows. ☐

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