HOMOGENIZATION OF ACTIVE SUSPENSIONS AND REDUCTION OF EFFECTIVE VISCOSITY

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Abstract. We consider a suspension of active rigid particles (swimmers) in a steady Stokes flow, using for simplicity a steady-state model where particles are distributed according to a stationary ergodic random process, and we study its homogenization in the macroscopic limit. A key point in the model is that swimmers are allowed to adapt their propulsion to the surrounding fluid deformation: swimming forces are not prescribed a priori, but are rather obtained through the retroaction of the fluid. Qualitative homogenization of this nonlinear model requires an unusual proof that crucially relies on a semi-quantitative two-scale analysis. Thanks to the introduction of new correctors that accurately capture spatial oscillations created by swimming forces, we identify the contribution of the activity to the effective viscosity. In agreement with the physics literature, an analysis in the dilute regime shows that the activity of the particles can either increase or decrease the effective viscosity (depending on the swimming mechanism), in a way that strongly differs from the well-known effect of passive suspensions.

Contents

1. Introduction and main results

1.1. **General overview.** This work is devoted to the large-scale rheology of suspensions of active particles in viscous fluids, where *active* particles are devices that can propel themselves in the fluid (in a direction that can depend on the surrounding fluid flow itself). Our main purpose is to establish rigorously, in a simple (yet somewhat realistic) physical model, that the presence of active particles in a fluid can strongly reduce its viscosity.

Important examples include suspensions of bacteria [36], micro-algae [37], nanomotors [29], etc. Compared to passive systems, active suspensions exhibit a particularly rich phenomenology, with the experimental observation of pattern formations [1] and unsteady whirls and jets [28]. Due to this complexity, the response to an external forcing can defy intuition, with rheological measurements displaying in some settings a transition to a superfluid-like behavior [27].

Our contribution is threefold: we formulate a suitable steady-state model for active suspensions based on the physics literature, we prove homogenization for this model and identify the effective rheology on large scales, and we finally analyze the latter in the dilute regime — then recovering standard explicit results of the physics literature.

As reviewed in Section 1.2 below, Einstein's celebrated viscosity formula [12] shows that the presence of *passive* rigid particles can only *increase* the plain fluid viscosity on large scales as those particles hinder the fluid flow. The case of active suspensions is radically different: the activity of micro-swimmers may overcome viscous dissipation and reduce the effective viscosity. As first pointed out in [24], this is related to the emergence of a state of collective orientation of the particles under a given external force: the preferred orientation happens to determine the increase or decrease in viscosity and was predicted to depend on the type of swimming mechanism. In 2009, Sokolov and Aranson [35] observed experimentally a striking reduction of the effective viscosity of a suspension of *Bacillus subtilis* in a dilute regime. In contrast, for motile microalgae such as *Chlamydomonas reinhardtii*, a drastic increase in the viscosity was observed experimentally in [31] — drastic even compared to suspensions with the same amount of dead cells! This confirmed that the *swimming mechanism* plays a key role in the modification of the viscosity on large scales and subsequent studies have classified these into two categories:

— "pusher" particles, such as *Bacillus subtilis*, move into the fluid via an extensile motion, e.g. using a flagella, and tend to decrease the effective viscosity;
— "puller" particles, such as *Chlamydomonas reinhardtii*, are contractile swimmers, and their activity can increase the effective viscosity.

Those experimental results have attracted much attention in the physics community over the past decade and several models have been introduced to explain the effect of the swimming mechanism on the effective viscosity. We refer in particular to the heuristic analysis in [22, 23, 32], where the preferred orientation of the particles is formally computed depending on the external shear flow, and is shown to affect the effective viscosity in the dilute regime, in agreement with experiments. In these works, the swimming mechanism is usually modeled for simplicity by point dipole forces at particle locations. We refer to [30, 33, 34] and references therein for the physical background.

From a modeling point of view, a key question in the study of suspensions is how to describe the positions and orientations of the suspended particles and how they are coupled to the fluid flow. There are two natural ways to proceed:

(I) Dynamical model: Particles follow the fluid flow and propel themselves, so their positions and orientations evolve in time accordingly. The geometry of the array of particles can then adapt dynamically to external forces, leading to a nonlinear response (non-Newtonian effects).

(II) Nonlinear steady-state model: The statistics of the positions are assumed to be given, as well as the statistics of the orientations, but the latter are allowed to depend on the surrounding fluid flow. The problem is then reduced to understanding the retroaction of the fluid on the self-propulsion of the particles as their orientations adapt to the surrounding fluid deformation.

From the rigorous mathematical perspective, the literature is particularly scarce, with the exception of the recent work [19] by Girodroux-Lavigne. In this work, the author considers a dilute steady-state model, consisting of a steady Stokes fluid with a dilute suspension of active particles for which swimming forces are fully prescribed in advance (hence, are part of the data). This reduces the analysis to a combination of homogenization (due to the presence of rigid particles, via a dilute analysis) with averaging of the active part (in form of a source term). This can be viewed as a very first step towards both situations (I) and (II): it remains to be combined with the retroaction of the fluid, which can be modeled either by the particle dynamics (I), or by the dependence of particle orientations on the surrounding fluid in a nonlinear steady-state model (II). Rigorous results in a realistic dynamical setting (I) seem out of reach at the moment beyond dilute regimes [3], and the present contribution is rather devoted to the steady-state setting (II) in a general non-dilute regime.

The sequel of this introduction is organized as follows: In Section 1.2, we recall the steady-state model for a steady Stokes fluid with a suspension of passive rigid particles, and we state the associated homogenization result previously obtained by the last two authors. In Section 1.3, we introduce a corresponding model for active suspensions in a steady Stokes flow, as inspired by the review articles [33, 34]. In Section 1.4, we relate this model to the physics literature and show how it can be used to recover standard results of the physics literature on a heuristic level. Section 1.5 is dedicated to the main results of this paper: the rigorous homogenization of this nonlinear model, and the analysis of the effective rheology in the dilute regime. Last, in Section 1.6, based on these results, we investigate the contribution of active particles to the effective viscosity, and rigorously establish that a significant reduction can take place in the case of pushers.

1.2. **Reminder on passive suspensions.** Given an underlying probability space $(\Omega, \mathbb{P})$ (with expectation $\mathbb{E}[\cdot]$), let $\{x_n\}_n$ be a random point process on the ambient space $\mathbb{R}^d$, consider an associated collection of random shapes $\{I_n^\circ\}_n$, where each I_n° is a connected random open subset of the unit ball B centered at the origin (in the sense that $\int_{I_n^\circ} y\,dy = 0$), and then define the corresponding inclusions

$$I_n := x_n + I_n^\circ.$$

Note that random shapes are not required to be independent of the point process $\{x_n\}_n$. We then consider the random set

$$\mathcal{I} := \bigcup_n I_n,$$

which we assume to satisfy the following for some $\vartheta > 0$.

Hypothesis 1.1 (Particle suspension).

(a) Stationarity and ergodicity: *The random set $\mathcal{I}$ is stationary and ergodic.*

(b) Uniform C^2 regularity: *The random shapes $\{I_n^\circ\}_n$ satisfy interior and exterior ball conditions with radius ϑ almost surely.*

(c) Uniform hardcore condition: *For some $\ell \geq \vartheta$, there holds $(I_n + \ell B) \cap (I_m + \ell B) = \varnothing$ almost surely for all $n \neq m$. We let ℓ be the largest such value, that is, half the interparticle distance*

$$\ell := \tfrac{1}{2} \inf_{n \neq m} \operatorname{dist}(I_n, I_m). \tag{1.1}$$

$\Diamond$

Now consider a tank, described as a bounded Lipschitz domain $U \subset \mathbb{R}^d$, and assume that it is filled with a steady Stokes fluid with a suspension of particles of size ε, described as the ε-rescaling of $\mathcal{I}$. More precisely, we only consider particles that are included in U and remove those close to the boundary: define $\mathcal{N}_\varepsilon(U)$ as the set of indices n such that $\varepsilon(I_n + \ell B) \subset U$, and set

$$\mathcal{I}_\varepsilon(U) := \bigcup_{n \in \mathcal{N}_\varepsilon(U)} \varepsilon I_n.$$

We write u_ε for the fluid velocity, P_ε for the corresponding pressure. We assume Dirichlet conditions $u_\varepsilon = 0$ on ∂U, and we extend the fluid velocity inside particles with the rigidity constraint

$$\mathrm{D}(u_\varepsilon) := \tfrac{1}{2}(\nabla u_\varepsilon + (\nabla u_\varepsilon)^T) = 0, \qquad \text{in } \mathcal{I}_\varepsilon(U).$$

Recall the definition of the Cauchy stress tensor

$$\sigma(u_\varepsilon, P_\varepsilon) := 2\,\mathrm{D}(u_\varepsilon) - P_\varepsilon \,\mathrm{Id},$$

where Id denotes the identity matrix. Given an internal force h, the fluid velocity $u_\varepsilon \in H^1_0(U)^d$ and the associated pressure $P_\varepsilon \in \mathrm{L}^2(U \setminus \mathcal{I}_\varepsilon(U))$ are then given as the solutions of the Stokes system

$$\begin{cases} -\triangle u_\varepsilon + \nabla P_\varepsilon = h & \text{in } U \setminus \mathcal{I}_\varepsilon(U), \\ \operatorname{div}(u_\varepsilon) = 0, & \text{in } U \setminus \mathcal{I}_\varepsilon(U), \\ \mathrm{D}(u_\varepsilon) = 0, & \text{in } \mathcal{I}_\varepsilon(U), \\ \int_{\varepsilon \partial I_n} \sigma(u_\varepsilon, P_\varepsilon)\nu = 0, & \forall n \in \mathcal{N}_\varepsilon(U), \\ \int_{\varepsilon \partial I_n} \Theta(x - \varepsilon x_n) \cdot \sigma(u_\varepsilon, P_\varepsilon)\nu = 0, & \forall n \in \mathcal{N}_\varepsilon(U),\ \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{1.2}$$

where ν stands for the outward normal to ∂I_n, where $\mathbb{M}^{\mathrm{skew}}$ is the set of skew-symmetric matrices, and where we assume the additional anchoring condition

$$\int_{U \setminus \mathcal{I}_\varepsilon(U)} P_\varepsilon = 0,$$

which we shall abbreviate as choosing $P_\varepsilon \in \mathrm{L}^2(U \setminus \mathcal{I}_\varepsilon(U))/\mathbb{R}$. The homogenization of the Stokes system (1.2) was the object of [2, 4, 8], where the last two authors proved that $(u_\varepsilon, P_\varepsilon \mathbb{1}_{U \setminus \mathcal{I}_\varepsilon(U)})$ converges almost surely weakly in $H^1_0(U) \times \mathrm{L}^2(U)/\mathbb{R}$ to the unique solution $(\bar u, \bar P)$ of the homogenized Stokes system

$$\begin{cases} -\operatorname{div}(2\bar{\boldsymbol{B}}_{\mathrm{pas}}\,\mathrm{D}(\bar u)) + \nabla \bar P = (1-\lambda) f & \text{in } U, \\ \operatorname{div}(\bar u) = 0, & \text{in } U, \end{cases} \tag{1.3}$$

where λ stands for the particle volume fraction

$$\lambda := \mathbb{E}\left[\mathbb{1}_{\mathcal{I}}\right], \tag{1.4}$$

and where the effective viscosity $\bar{\boldsymbol{B}}_{\mathrm{pas}}$ is a symmetric linear map on the set $\mathbb{M}_0^{\mathrm{sym}}$ of symmetric trace-free matrices. We recall that the latter satisfies $E : \bar{\boldsymbol{B}}_{\mathrm{pas}} E > |E|^2$ for all $E \in \mathbb{M}_0^{\mathrm{sym}}$ as soon as $\lambda > 0$, meaning that the presence of (passive) rigid particles always increases the effective viscosity. We refer to [9] for a review of the topic.

In view of the quantitative homogenization results that we shall need later, we occasionally make quantitative ergodicity assumptions in form of the validity of the following multiscale variance inequality introduced by the last two authors in [6, 7]. This assumption holds for instance for hardcore Poisson point process with exponentially decaying π.

Hypothesis 1.2 (Quantitative mixing assumption)**.** *There exists a non-increasing weight function $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ with superalgebraic decay (that is, $\pi(t) \leq C_p \langle t\rangle^{-p}$ for all $p < \infty$) such that the random set $\mathcal{I}$ satisfies, for all $\sigma(\mathcal{I})$-measurable random variables $Y(\mathcal{I})$,*

$$\operatorname{Var}[Y(\mathcal{I})] \leq \mathbb{E}\bigg[\int_0^\infty \int_{\mathbb{R}^d} \Big(\partial^{\mathrm{osc}}_{\mathcal{I},B_t(x)} Y(\mathcal{I})\Big)^2 dx\, \langle t\rangle^{-d} \pi(t)\, dt\bigg],$$

where the "oscillation" ∂^{osc} of the random variable $Y(\mathcal{I})$ is defined by

$$\begin{aligned}\partial^{\mathrm{osc}}_{\mathcal{I},B_t(x)} Y(\mathcal{I}) := \operatorname{sup\,ess}\Big\{Y(\mathcal{I}') \,:\, \mathcal{I}' \cap (\mathbb{R}^d \setminus B_t(x)) = \mathcal{I} \cap (\mathbb{R}^d \setminus B_t(x))\Big\} \\ - \operatorname{inf\,ess}\Big\{Y(\mathcal{I}') \,:\, \mathcal{I}' \cap (\mathbb{R}^d \setminus B_t(x)) = \mathcal{I} \cap (\mathbb{R}^d \setminus B_t(x))\Big\}. \qquad \Diamond\end{aligned}$$

1.3. **Hydrodynamic model for active suspensions.** As opposed to *passive* particles, *active* particles propel themselves by applying a force on the surrounding fluid. In a steady-state perspective, we assume that we are given a random ensemble of particle positions and swimming directions, and we aim to evaluate the associated large-scale rheology. Swimming directions should not be taken as uniformly distributed, but should depend on the surrounding fluid deformation, which leads to a nontrivial interaction between the fluid flow and particles' swimming forces. More precisely, our model is based on the following assumption: if the fluid is locally deformed, then the distribution of orientations depends on some local average of the symmetrized velocity gradient of the surrounding fluid around each particle. Although this steady-state perspective is certainly simplistic, our model does not prescribe the retroaction of the fluid on the particles a priori, but leaves it as part of the problem. We start by modeling the swimming mechanism for a single particle, before combining it with (1.2) into a model for the whole active suspension.

1.3.1. *Single-particle swimming mechanism.* Let us place ourselves at the scale of an isolated particle I, and denote by u the fluid velocity outside I. Given a nonnegative smooth kernel χ with unit mass $\int_{\mathbb{R}^d} \chi = 1$, the locally averaged fluid deformation felt by the particle is taken of the form

$$E_I(u) := \fint_I \chi * \mathrm{D}(u). \tag{1.5}$$

The precise choice of this operator does not matter in our analysis, provided that it is a compact operator applied to a restriction of $\mathrm{D}(u)$ around I. Given a value $E_I(u) = E$ of this averaged fluid deformation, the particle adapts its random swimming direction: we denote by $\bar f(E) \in \mathbb{R}^d$ the resulting propulsion force and by $\tilde f(E) \in \mathbb{M}^{\mathrm{skew}}$ the resulting torque on the particle. By the action-reaction principle, this force and torque must result from an action of the particle on the surrounding fluid. The detail of this action depends on the details of the swimming mechanism (flagella, cilia, etc.). The force field exerted by the particle on the fluid is denoted by $f(E) = f(\cdot, E)$, depending on the deformation E, and is taken to be supported in the immediate neighborhood $(I + B) \setminus I$ of the particle.

By the action-reaction principle for forces and torques, we must have

$$\begin{aligned} \bar f(E)+\int_{(I+B)\setminus I} f(E) &= 0, \\ \Theta:\tilde f(E)+\int_{(I+B)\setminus I}\Theta x\cdot f(E) &= 0, \qquad \forall\Theta\in\mathbb{M}^{\mathrm{skew}}, \end{aligned} \tag{1.6}$$

since the barycenter of particle I is $\int_I y\,dy=0$. The relation (1.6) reads as a local neutrality condition that actually entails that swimming forces act as dipoles in the fluid equations. We emphasize that this is fundamentally different from the sedimentation problem studied in [11], for which the force on particles originates from gravity and is not compensated locally by opposite forces in the surrounding fluid — the backflow is then uniform and leads to more important large-scale effects.

We further make the following assumptions on the regularity of the swimming force with respect to the fluid deformation.

Hypothesis 1.3 (Swimming mechanism)**.** *The random force field defines almost surely a smooth map $\mathbb{M}_0^{\mathrm{sym}}\to\mathrm{L}^\infty((I+B)\setminus I)^d:E\mapsto f(E)$ such that, for all $k\ge1$,*

$$\begin{aligned} \|f(E)\|_{\mathrm{L}^\infty((I+B)\setminus I)} &\le C\langle E\rangle, \\ \|\partial^k f(E)\|_{\mathrm{L}^\infty((I+B)\setminus I)} &\le C_k. \end{aligned}$$

◊

In view of quantitative homogenization results, we shall occasionally need to further assume that for large strain rates E the swimming direction becomes a deterministic function of E. This technical assumption is physically reasonable.

Hypothesis 1.4 (Swimming in large strain rate)**.** *There exist a deterministic direction field $f^\infty:\mathbb{M}_0^{\mathrm{sym}}\to\mathbb{R}^d$, a random strength field $\mathcal{S}\in\mathrm{L}^\infty((I+B)\setminus I)$, and an exponent $\gamma>0$, such that for all $E\in\mathbb{M}_0^{\mathrm{sym}}$ and $k\ge1$ we have almost surely*

$$\begin{aligned} \|f(E)-\mathcal{S}f^\infty(E)\|_{\mathrm{L}^\infty((I+B)\setminus I)} &\le C\langle E\rangle^{1-\gamma}, \\ \|\partial^k f(E)-\mathcal{S}\partial^k f^\infty(E)\|_{\mathrm{L}^\infty((I+B)\setminus I)} &\le C_k\langle E\rangle^{-\gamma}. \end{aligned}$$

◊

1.3.2. *Resulting system for many particles.* Before including the above single-particle swimming mechanism into the passive suspension model (1.2), we start by making an assumption on the joint law of particles' swimming forces.

Hypothesis 1.5 (Joint swimming forces)**.** *Let $\{f_n(E)\}_n$ be a sequence of random maps, such that f_n satisfies Hypothesis 1.3 with $I=I_n$ for all n, and such that $\mathcal{I}$ and $\sum_n f_n(E)$ are jointly stationary for all E.* ◊

In order to include these swimming forces into the model (1.2) for a suspension of small particles $\{\varepsilon I_n\}_n$, they need to be properly rescaled. The natural scaling happens to be $O(\frac1\varepsilon)$, which is indeed the only scaling giving rise to a nontrivial and finite contribution in the macroscopic limit $\varepsilon\downarrow0$. We add a coupling parameter κ, which stands for the activity strength and will need to be chosen small enough to perform the analysis. In this ε-rescaling, the kernel χ defining the local averaged fluid deformations felt by the particles (1.5) should naturally be replaced by $\chi_\varepsilon:=\varepsilon^{-d}\chi(\frac{\cdot}{\varepsilon})$. This however leads to important difficulties due to the highly oscillatory local behavior of the fluid flow. Instead, we need to replace it by $\chi_\delta:=\delta^{-d}\chi(\frac{\cdot}{\delta})$, for some intermediate averaging scale $\varepsilon\ll\delta\ll1$.

The resulting hydrodynamic model takes on the following guise,

$$\left\{\begin{array}{ll} -\triangle u_\varepsilon+\nabla P_\varepsilon & \\ \qquad =h+\frac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)}f_{n,\varepsilon}(\fint_{\varepsilon I_n}\chi_\delta*\mathrm{D}(u_\varepsilon)), & \text{in } U\setminus\mathcal{I}_\varepsilon(U),\\ \operatorname{div}(u_\varepsilon)=0, & \text{in } U\setminus\mathcal{I}_\varepsilon(U),\\ u_\varepsilon=0, & \text{on } \partial U,\\ \mathrm{D}(u_\varepsilon)=0, & \text{in } \mathcal{I}_\varepsilon(U),\\ \int_{\varepsilon\partial I_n}\sigma(u_\varepsilon,P_\varepsilon)\nu+\frac{\kappa}{\varepsilon}\bar f_n(\fint_{\varepsilon I_n}\chi_\delta*\mathrm{D}(u_\varepsilon))=0, & \forall n\in\mathcal{N}_\varepsilon(U),\\ \int_{\varepsilon\partial I_n}\Theta(\cdot-\varepsilon x_n)\cdot\sigma(u_\varepsilon,P_\varepsilon)\nu & \\ \qquad\qquad +\frac{\kappa}{\varepsilon}\Theta:\tilde f_n(\fint_{\varepsilon I_n}\chi_\delta*\mathrm{D}(u_\varepsilon))=0, & \forall n\in\mathcal{N}_\varepsilon(U),\,\Theta\in\mathbb{M}^{\mathrm{skew}}, \end{array}\right. \tag{1.7}$$

where we have set

$$f_{n,\varepsilon}(E)\;:=\;f_{n,\varepsilon}(\cdot,E)\;:=\;\varepsilon^{-d}f_n\big(\tfrac{\cdot}{\varepsilon},E\big).$$

As above, the pressure is anchored via $\int_{U\setminus\mathcal{I}_\varepsilon(U)}P_\varepsilon=0$.

In what follows, it will be convenient to use an equivalent formulation of swimming forces. While each force field f_n is supported in the particle neighborhood $(I_n+B)\setminus I_n$ in the fluid domain, we may naturally extend f_n inside the particle domain I_n to match its propulsion force and torque. More precisely, we can uniquely define f_n in I_n as an affine function such that

$$\bar f_n(E)\;=\;\int_{I_n}f_n(E),\qquad \tilde f_n(E)\;=\;\int_{I_n}f_n(E)\otimes(x-x_n). \tag{1.8}$$

In terms of these extensions, the neutrality condition (1.6) takes the simpler form

$$\begin{aligned} \int_{I_n+B}f_n(E)\;&=\;0, \qquad\qquad\qquad\qquad (1.9)\\ \int_{I_n+B}\Theta(x-x_n)\cdot f_n(E)\;&=\;0,\qquad \forall\Theta\in\mathbb{M}^{\mathrm{skew}}, \end{aligned}$$

and the system (1.7) then becomes

$$\left\{\begin{array}{ll} -\triangle u_\varepsilon+\nabla P_\varepsilon & \\ \qquad =h+\frac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)}f_{n,\varepsilon}(\fint_{\varepsilon I_n}\chi_\delta*\mathrm{D}(u_\varepsilon)), & \text{in } U\setminus\mathcal{I}_\varepsilon(U),\\ \operatorname{div}(u_\varepsilon)=0, & \text{in } U\setminus\mathcal{I}_\varepsilon(U),\\ u_\varepsilon=0, & \text{on } \partial U,\\ \mathrm{D}(u_\varepsilon)=0, & \text{in } \mathcal{I}_\varepsilon(U),\\ \int_{\varepsilon\partial I_n}\sigma(u_\varepsilon,P_\varepsilon)\nu+\frac{\kappa}{\varepsilon}\int_{\varepsilon I_n}f_{n,\varepsilon}(\fint_{\varepsilon I_n}\chi_\delta*\mathrm{D}(u_\varepsilon))=0, & \forall n\in\mathcal{N}_\varepsilon(U),\\ \int_{\varepsilon\partial I_n}\Theta(x-\varepsilon x_n)\cdot\sigma(u_\varepsilon,P_\varepsilon)\nu & \\ \quad +\frac{\kappa}{\varepsilon}\int_{\varepsilon I_n}\Theta(x-\varepsilon x_n)\cdot f_{n,\varepsilon}(\fint_{\varepsilon I_n}\chi_\delta*\mathrm{D}(u_\varepsilon))=0, & \forall n\in\mathcal{N}_\varepsilon(U),\,\Theta\in\mathbb{M}^{\mathrm{skew}}. \end{array}\right. \tag{1.10}$$

With this reformulation, we may readily check that the solution u_ε can be viewed as the orthogonal projection on $\{u\in H^1_0(U)^d:\mathrm{D}(u)|_{\mathcal{I}_\varepsilon(U)}=0,\,\operatorname{div}(u)=0\}$ of the solution of

$$\left\{\begin{array}{ll} -\triangle v_\varepsilon+\nabla R_\varepsilon=h+\frac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)}f_{n,\varepsilon}(\fint_{\varepsilon I_n}\chi_\delta*\mathrm{D}(u_\varepsilon)), & \text{in } U,\\ \operatorname{div}(v_\varepsilon)=0, & \text{in } U,\\ v_\varepsilon=0, & \text{on } \partial U. \end{array}\right. \tag{1.11}$$

This observation is not used in the sequel.

1.4. **Heuristics and relation to the physics literature.** In the physics literature, one usually considers a forcing term in form of an imposed strain rate $E \in \mathbb{M}_0^{\mathrm{sym}}$ at infinity — in which case it is natural to replace (1.5) by E itself. The velocity field of the suspension on microscopic scale is then given by $u_E + Ex$, where u_E is a suitable solution of the following infinite-volume problem,

$$\left\{\begin{array}{ll} -\triangle u_E + \nabla P_E = \kappa \sum_n f_n(E), & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \operatorname{div}(u_E) = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \mathrm{D}(u_E + Ex) = 0, & \text{in } \mathcal{I}, \\ \int_{\partial I_n} \sigma(u_E + Ex, P_E)\nu + \kappa \int_{I_n} f_n(E) = 0, & \forall n, \\ \int_{\partial I_n} \Theta(x - x_n) \cdot \sigma(u_E + Ex, P_E)\nu & \\ \qquad\qquad + \kappa \int_{I_n} \Theta(x - x_n) \cdot f_n(E) = 0, & \forall n,\ \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{array}\right. \tag{1.12}$$

In these terms, the effective viscosity $\bar{\boldsymbol{B}}_{\mathrm{tot}}(E)$ of the suspension in direction E is obtained as the associated ensemble-averaged stress. Splitting the contributions of the stress in the fluid domain and in the particles, and taking into account swimming forces, we get for all $E' \in \mathbb{M}_0^{\mathrm{sym}}$,

$$\begin{aligned} E' : 2\bar{\boldsymbol{B}}_{\mathrm{tot}}(E) \,=\, & E' : \mathbb{E}\big[\sigma(u_E + Ex, P_E)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big] \\ & + E' : \mathbb{E}\big[\sigma_E \mathbb{1}_{\mathcal{I}}\big] - \kappa \mathbb{E}\bigg[\sum_n E'(x - x_n) \cdot f_n(E)\bigg], \end{aligned} \tag{1.13}$$

where σ_E stands for the stress inside the particles, which we shall define in the proof of Lemma 2.5 below via the extension problem

$$\left\{\begin{array}{ll} -\operatorname{div}(\sigma_E) = \kappa f_n(E), & \text{in } I_n, \\ \sigma_E \nu = \sigma(u_E + Ex, P_E)\nu, & \text{on } \partial I_n. \end{array}\right. \tag{1.14}$$

Noting that rigidity constraints $\mathrm{D}(u_E + Ex) = 0$ in $\mathcal{I}$ yield

$$\sigma(u_E + Ex, P_E)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}} \,=\, 2\,\mathrm{D}(u_E + Ex) - P_E \,\mathrm{Id}\, \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}},$$

and recalling that $\mathbb{E}\left[\mathrm{D}(u_E)\right] = 0$ (as the average of a gradient), the first contribution in (1.13) takes the form

$$E' : \mathbb{E}\big[\sigma(u_E + Ex, P_E)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big] \,=\, 2E' : E.$$

Next, using (1.14), integrating by parts, and using (1.8), and the skew-symmetry of $\tilde{f}_n(E)$, one can reformulate the second contribution in (1.13) as the ensemble-averaged stresslet on the particles

$$\begin{aligned} \mathbb{E}\big[\sigma_E \mathbb{1}_{\mathcal{I}}\big] \,&=\, \mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|} \int_{I_n} \sigma_E\bigg] \\ &=\, \mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|} \int_{\partial I_n} \sigma(u_E + Ex, P_E)\nu \otimes_s (x - x_n)\bigg]. \end{aligned}$$

The effective viscosity (1.13) thus takes the form

$$\begin{aligned} & E' : 2\bar{\boldsymbol{B}}_{\mathrm{tot}}(E) \,=\, 2E' : E \\ & + \mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|} \int_{\partial I_n} E'(x - x_n) \cdot \sigma(u_E + Ex, P_E)\nu\bigg] - \kappa \mathbb{E}\bigg[\sum_n E'(x - x_n) \cdot f_n(E)\bigg]. \end{aligned} \tag{1.15}$$

For convenience, we shall distinguish the passive from the active contributions in this expression, and we decompose the solution u_E as $u_E = \psi_E + \kappa\phi_E$ in terms of the so-called passive and active correctors ψ_E and ϕ_E, defined as suitable solutions of

$$\begin{cases} -\triangle\psi_E + \nabla\Sigma_E = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \operatorname{div}(\psi_E) = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \mathrm{D}(\psi_E) + E = 0, & \text{in } \mathcal{I}, \\ \int_{\partial I_n} \sigma(Ex + \psi_E, \Sigma_E)\nu = 0, & \forall n, \\ \int_{\partial I_n} \Theta(x - x_n) \cdot \sigma(Ex + \psi_E, \Sigma_E)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}},\ \forall n. \end{cases}$$

and

$$\begin{cases} -\triangle\phi_E + \nabla\Pi_E = \sum_n f_n(E), & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \operatorname{div}(\phi_E) = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \mathrm{D}(\phi_E) = 0, & \text{in } \mathcal{I}, \\ \int_{\partial I_n} \sigma(\phi_E, \Pi_E)\nu + \int_{I_n} f_n(E) = 0, & \forall n, \\ \int_{\partial I_n} \Theta(x - x_n) \cdot \sigma(\phi_E, \Pi_E)\nu + \int_{I_n} \Theta(x - x_n) \cdot f_n(E) = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}},\ \forall n. \end{cases}$$

Precise definitions of these correctors are postponed to Section 2. In these terms, the effective viscosity takes the form

$$\bar{\boldsymbol{B}}_{\mathrm{tot}}(E) \;=\; \bar{\boldsymbol{B}}_{\mathrm{pas}}E + \kappa\bar{\boldsymbol{B}}_{\mathrm{act}}(E), \tag{1.16}$$

where the passive and active contributions are given by

$$\begin{aligned} E' : 2\bar{\boldsymbol{B}}_{\mathrm{pas}}E \;&:=\; 2E' : E + \mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{\partial I_n} E'(x - x_n) \cdot \sigma(\psi_E + Ex, \Sigma_E)\nu\bigg], \\ E' : 2\bar{\boldsymbol{B}}_{\mathrm{act}}(E) \;&:=\; \mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{\partial I_n} E'(x - x_n) \cdot \sigma(\phi_E, \Pi_E)\nu\bigg] \\ &\qquad -\mathbb{E}\bigg[\sum_n E'(x - x_n) \cdot f_n(E)\bigg]. \end{aligned}$$

Using equations for correctors, these expressions are equivalently given by

$$\begin{aligned} E : 2\bar{\boldsymbol{B}}_{\mathrm{pas}}E \;&=\; \mathbb{E}\big[2|\mathrm{D}(\psi_E) + E|^2\big], \\ E' : 2\bar{\boldsymbol{B}}_{\mathrm{act}}(E) \;&=\; -\mathbb{E}\bigg[\sum_n (\psi_{E'} + E'(x - x_n)) \cdot f_n(E)\bigg]. \end{aligned} \tag{1.17}$$

As one could have expected, the active contribution $E' : \bar{\boldsymbol{B}}_{\mathrm{act}}(E)$ coincides with the averaged swimming force along the passive corrector in direction E'. Note in particular that the active corrector ϕ_E *does not appear* in that formulation. The first main contribution of the present article is to properly justify these effective viscosity formulas using homogenization theory, cf. Theorem 1.7.

While these general formulas are difficult to analyze in practice without resorting to numerical simulations, it is a classical problem in the physics community to derive simpler approximate formulas in the dilute regime, which are easier to interpret and provide a useful grasp at the physical behavior of suspensions. This is made possible by replacing correctors by explicit solutions of single-particle problems: we refer to Theorem 1.9 and Section 1.6 below for justification of such dilute approximations based on the methods introduced by the last two authors in [5].

1.5. **Main results: well-posedness, homogenization, and dilute regime.** We turn to the statement of our main results and we start with the well-posedness of the hydrodynamic model (1.7). It requires either the coupling constant κ to be small enough or the interparticle distance ℓ to be large enough. Note that condition (1.18) below is nearly almost optimal in general: the same condition with $\eta = 0$ is required to ensure the perturbative well-posedness of the homogenized equation (1.20), see first paragraph of Section 3.2.

Proposition 1.6 (Well-posedness of the hydrodynamic model)**.** *Let Hypotheses 1.1, 1.3, and 1.5 hold, and assume $\varepsilon\ell \le \delta$. Provided for some $\eta > 0$ we have that*

$$\kappa\ell^{\eta-d} \ll 1 \tag{1.18}$$

is small enough (only depending on η, the dimension d, the domain U, and the unscaled kernel χ), the system (1.7) is well-posed almost surely for any $h \in \mathrm{L}^2(U)^d$: there exists a unique almost sure weak solution $(u_\varepsilon, P_\varepsilon) \in \mathrm{L}^2(\Omega; H^1_0(U)^d \times \mathrm{L}^2(U \setminus \mathcal{I}_\varepsilon(U))/\mathbb{R})$ and it satisfies almost surely

$$\int_U |\nabla u_\varepsilon|^2 + \int_{U\setminus\mathcal{I}_\varepsilon(U)} (P_\varepsilon)^2 \lesssim_\eta (1+\kappa^2)\Big(\kappa^2\ell^{-d} + \int_{U\setminus\mathcal{I}_\varepsilon(U)} |h|^2\Big). \tag{1.19}$$

$\Diamond$

We now state the homogenization result for this model in the macroscopic limit $\varepsilon \downarrow 0$, in the simplified situation when the mesoscopic averaging scale δ is fixed (see the proof for the associated corrector result).

Theorem 1.7 (Homogenization at fixed $\delta > 0$)**.** *Let Hypotheses 1.1, 1.3, and 1.5 hold, as well as the smallness condition (1.18) to ensure well-posedness. For any $h \in W^{1,\infty}(U)^d$, as $\varepsilon \downarrow 0$ with $\delta > 0$ fixed, the almost sure weak solution $(u_\varepsilon, P_\varepsilon)$ of (1.7) satisfies almost surely*

$$\begin{array}{rcll} u_\varepsilon & \rightharpoonup & \bar u_\delta, & \text{in } H^1_0(U), \\ P_\varepsilon \mathbb{1}_{U\setminus\mathcal{I}_\varepsilon(U)} & \rightharpoonup & (1-\lambda)\bar P_\delta + (1-\lambda)\bar{\boldsymbol{b}} : \mathrm{D}(\bar u_\delta) & \\ & & \quad +(1-\lambda)\kappa\big(\bar{\boldsymbol{c}}(\chi_\delta * \mathrm{D}(\bar u_\delta)) - \fint_U \bar{\boldsymbol{c}}(\chi_\delta * \mathrm{D}(\bar u_\delta))\big), & \text{in } \mathrm{L}^2(U), \end{array}$$

where $(\bar u_\delta, \bar P_\delta) \in H^1_0(U)^d \times \mathrm{L}^2(U)/\mathbb{R}$ is the unique solution of the well-posed macroscopic system

$$\left\{\begin{array}{ll} -\mathrm{div}(2\bar{\boldsymbol{B}}_{\mathrm{pas}}\,\mathrm{D}(\bar u_\delta)) - \mathrm{div}(2\kappa\bar{\boldsymbol{B}}_{\mathrm{act}}(\chi_\delta * \mathrm{D}(\bar u_\delta))) + \nabla\bar P_\delta \,=\, (1-\lambda)h, & \text{in } U, \\ \mathrm{div}(\bar u_\delta) = 0, & \text{in } U, \\ \bar u_\delta = 0, & \text{on } \partial U, \end{array}\right. \tag{1.20}$$

where $\lambda := \mathbb{E}[\mathbb{1}_\mathcal{I}]$ is the particle volume fraction, and where the effective tensors $\bar{\boldsymbol{B}}_{\mathrm{pas}}$, $\bar{\boldsymbol{B}}_{\mathrm{act}}$, $\bar{\boldsymbol{b}}$, $\bar{\boldsymbol{c}}$ are defined as follows, in terms of the correctors (ψ, Σ) and (ϕ, Π) given in (2.1) and (2.4) below,

- *The* passive effective viscosity *$\bar{\boldsymbol{B}}_{\mathrm{pas}}$ is a positive definite symmetric linear map on the space of symmetric trace-free matrices $\mathbb{M}_0^{\mathrm{sym}}$: together with the associated symmetric trace-free matrix $\bar{\boldsymbol{b}} \in \mathbb{M}_0^{\mathrm{sym}}$, it is defined for all $E \in \mathbb{M}_0^{\mathrm{sym}}$ by*

$$2(\bar{\boldsymbol{B}}_{\mathrm{pas}} - \mathrm{Id})E + (\bar{\boldsymbol{b}} : E)\,\mathrm{Id} \,:=\, \mathbb{E}\Big[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{\partial I_n} \sigma(\psi_E + Ex, \Sigma_E)\nu \otimes_s (x - x_n)\Big],$$

or equivalently,

$$E:2\bar{\boldsymbol{B}}_{\rm pas}E \;:=\; \mathbb{E}\big[2|\mathrm{D}(\psi_E)+E|^2\big], \tag{1.21}$$

$$\bar{\boldsymbol{b}}:E \;:=\; \tfrac{1}{d}\,\mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{\partial I_n}(x-x_n)\cdot\sigma(\psi_E+Ex,\Sigma_E)\nu\bigg]. \tag{1.22}$$

- *The* active effective viscosity $\bar{\boldsymbol{B}}_{\rm act}$ *is given by*

$$\bar{\boldsymbol{B}}_{\rm act} \;:=\; \bar{\boldsymbol{C}}+\tfrac{1}{2}\bar{\boldsymbol{F}}, \tag{1.23}$$

where the map $\bar{\boldsymbol{C}}:\mathbb{M}_0^{\rm sym}\to\mathbb{M}_0^{\rm sym}$*, together with the associated map* $\bar{\boldsymbol{c}}:\mathbb{M}_0^{\rm sym}\to\mathbb{R}$*, is defined for all* $E\in\mathbb{M}_0^{\rm sym}$ *by*

$$2\bar{\boldsymbol{C}}(E)+\bar{\boldsymbol{c}}(E)\,\mathrm{Id} \;:=\; \mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{\partial I_n}\sigma(\phi_E,\Pi_E)\nu\otimes_s(x-x_n)\bigg], \tag{1.24}$$

and where the map $\bar{\boldsymbol{F}}:\mathbb{M}_0^{\rm sym}\to\mathbb{M}_0^{\rm sym}$ *is defined for all* $E,E'\in\mathbb{M}_0^{\rm sym}$ *by*

$$E':\bar{\boldsymbol{F}}(E) \;:=\; -\mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{I_n+B}E'(x-x_n)\cdot f_n(E)\bigg], \tag{1.25}$$

or equivalently for all $E,E'\in\mathbb{M}_0^{\rm sym}$*,*

$$E':2\bar{\boldsymbol{B}}_{\rm act}(E) \;=\; -\mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{I_n+B}(\psi_{E'}+E'(x-x_n))\cdot f_n(E)\bigg], \tag{1.26}$$

$$\bar{\boldsymbol{c}}(E) \;=\; \tfrac{1}{d}\mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\Big(\int_{\partial I_n}(x-x_n)\cdot\sigma(\phi_E,\Pi_E)\nu\Big)\bigg]. \qquad\Diamond$$

Next, we combine the above (nonlocal) homogenization limit with the (local) limit $\delta\downarrow 0$. In order to get beyond a purely diagonal regime, cf. (3.55) below, we need to appeal to the quantitative homogenization techniques developed in [10] by the last two authors in the context of the Stokes equation with rigid inclusions. This requires a quantitative mixing assumption such as Hypothesis 1.2 (which holds in particular for hardcore Poisson point processes). Although from the modeling viewpoint the choice $\delta\sim\varepsilon$ could be more natural, it is not accessible to our analysis, and we are restricted to (1.27) below.

Theorem 1.8 (Homogenization as $\delta\downarrow 0$)**.** *Let Hypotheses 1.1, 1.3, and 1.5 hold, as well as the smallness condition* (1.18) *to ensure well-posedness, and further let the quantitative mixing Hypothesis 1.2 and the technical Hypothesis 1.4 hold. Then, for any* $h\in W^{1,\infty}(U)^d$*, as* $\varepsilon,\delta\downarrow 0$*, in the regime*

$$\delta^{-s}\varepsilon \;\to\; 0, \qquad \text{for some } s>d+1, \tag{1.27}$$

the almost sure weak solution $(u_\varepsilon,P_\varepsilon)$ *of* (1.7) *satisfies almost surely*

$$\begin{array}{rcll} u_\varepsilon & \rightharpoonup & \bar u, & \text{in } H^1_0(U)^d,\\ P_\varepsilon\mathbb{1}_{U\setminus\mathcal{I}_\varepsilon(U)} & \rightharpoonup & (1-\lambda)\bar P+(1-\lambda)\bar{\boldsymbol{b}}:\mathrm{D}(\bar u) & \\ && \quad +(1-\lambda)\kappa\big(\bar{\boldsymbol{c}}(\mathrm{D}(\bar u))-\fint_U\bar{\boldsymbol{c}}(\mathrm{D}(\bar u))\big), & \text{in } \mathrm{L}^2(U), \end{array}$$

where $(\bar u,\bar P)\in H^1_0(U)^d\times\mathrm{L}^2(U)/\mathbb{R}$ *is the unique solution of*

$$\left\{\begin{array}{ll} -\mathrm{div}(2\bar{\boldsymbol{B}}_{\rm tot}(\mathrm{D}(\bar u)))+\nabla\bar P \;=\; (1-\lambda)h, & \text{in } U,\\ \mathrm{div}(\bar u)=0, & \text{in } U,\\ \bar u=0, & \text{on } \partial U. \end{array}\right. \tag{1.28}$$

in terms of the total effective viscosity

$$\bar{\boldsymbol{B}}_{\rm tot}(E) \,:=\, \bar{\boldsymbol{B}}_{\rm pas}E + \kappa\bar{\boldsymbol{B}}_{\rm act}(E),$$

where we recall that $\lambda, \bar{\boldsymbol{B}}_{\rm pas}, \bar{\boldsymbol{B}}_{\rm act}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{c}}$ *are defined in Theorem 1.7 above.* ◊

The above shows that the effective stress-strain constitutive relation $E \mapsto 2\bar{\boldsymbol{B}}_{\rm pas}E$ is replaced by the nonlinear (non-Newtonian) relation $E \mapsto 2\bar{\boldsymbol{B}}_{\rm pas}E + 2\bar{\boldsymbol{B}}_{\rm act}(E)$ due to the effect of particle activity.

Our last result concerns the analysis of the latter in the dilute regime, and we establish the active counterpart to Einstein's effective viscosity formula. Before stating the result, we need to recall some notation from [5]: Denote by $Q_r(x) = x + [-\frac{r}{2}, \frac{r}{2})^d$ the cube of sidelength r centered at x, and set $Q(x) = Q_1(x)$, $Q_r = Q_r(0)$, and $Q = Q_1(0)$. The intensity of the point process $\{x_n\}_n$ is

$$\lambda_1 \,:=\, \mathbb{E}\big[\sharp\{n : x_n \in Q\}\big],$$

and we further define the two- and three-point intensities as

$$\begin{aligned}
\lambda_2 \,&:=\, \ell^{-2d} \sup_{x\in\mathbb{R}^d} \mathbb{E}\Big[\sharp\big\{(n,m) : n \neq m,\ x_n \in Q_\ell,\ x_m \in Q_\ell(x)\big\}\Big],\\
\lambda_3 \,&:=\, \ell^{-3d} \sup_{x_1,x_2\in\mathbb{R}^d} \mathbb{E}\Big[\sharp\big\{(n_0,n_1,n_2) :\ n_0,n_1,n_2 \text{ pairwise distinct},\\
&\qquad\qquad x_{n_0} \in Q_\ell,\ x_{n_1} \in Q_\ell(x_1),\ x_{n_2} \in Q_\ell(x_2)\big\}\Big],
\end{aligned}$$

where we recall that ℓ stands for (half) the interparticle distance, cf. (1.1). Note that by definition the intensity can be compared to the particle volume fraction λ, cf. (1.4), and we have for $k = 2, 3$,

$$\lambda_1 \simeq \lambda \lesssim \ell^{-d} \qquad \text{and} \qquad \lambda_k \lesssim \ell^{-kd}.$$

We further recall that the two- and three-point densities g_2, g_3 of the point process are defined by the following relations, for all $\zeta_2 \in C^\infty_c((\mathbb{R}^d)^2)$ and $\zeta_3 \in C^\infty_c((\mathbb{R}^d)^3)$,

$$\begin{aligned}
\iint_{(\mathbb{R}^d)^2} \zeta_2(x,y)\, g_2(x,y)\, dxdy \,&=\, \mathbb{E}\Big[\sum_{n\neq m} \zeta_2(x_n,x_m)\Big], \qquad (1.29)\\
\iint_{(\mathbb{R}^d)^3} \zeta_3(x,y,z)\, g_3(x,y,z)\, dxdydz \,&=\, \mathbb{E}\Big[\sum_{\substack{n_0,n_1,n_2\\ \text{distinct}}} \zeta_3(x_{n_0},x_{n_1},x_{n_2})\Big].
\end{aligned}$$

In these terms, the above definitions of the two- and three-point intensities are equivalently written as

$$\begin{aligned}
\lambda_2 \,&=\, \sup_{x\in\mathbb{R}^d} \fint_{Q_\ell} \fint_{Q_\ell(x)} g_2(y,z)\, dydz, \qquad (1.30)\\
\lambda_3 \,&=\, \sup_{x_1,x_2\in\mathbb{R}^d} \fint_{Q_\ell} \fint_{Q_\ell(x_1)} \fint_{Q_\ell(x_2)} g_3(x,y,z)\, dxdydz.
\end{aligned}$$

With this notation at hand, we may now state the following result on the first-order dilute expansion of the effective viscosity. The expansion of the passive contribution $\bar{\boldsymbol{B}}_{\rm pas}$ was already established in [5], and we extend it here to the active setting.

Theorem 1.9 (Dilute expansion of the effective viscosity)**.** *Let Hypotheses 1.1, 1.3, and 1.5 hold, and further assume* *(a)* Independence condition: *The random shapes and swimming forces* $\{I_n^\circ, f_n\}_n$ *are iid copies of a given random open subset* I° *and of a random map* f°*, independently of the point process* $\{x_n\}_n$.

(b) Decay of correlations: *The point process* $\{x_n\}_n$ *is strongly mixing, and the two- and three-point correlation functions*

$$\begin{aligned} h_2(x,y) &:= g_2(x,y) - \lambda_1^2, \\ h_3(x,y,z) &:= g_3(x,y,z) - \lambda_1^3 - \lambda_1(h_2(x,y) + h_2(x,z) + h_2(y,z)), \end{aligned}$$

have algebraic decay: there exists $\gamma > 0$ *such that for all* x, y, z*,*

$$|h_2(x,y)| \lesssim \langle x-y\rangle^{-\gamma}, \qquad |h_3(x,y,z)| \lesssim \langle x-y\rangle^{-\gamma} \wedge \langle x-z\rangle^{-\gamma} \wedge \langle y-z\rangle^{-\gamma}. \tag{1.31}$$

Then, we have

$$\begin{aligned} \Big|\bar{\boldsymbol{B}}_{\mathrm{tot}}(E) - \big(E + \lambda_1 \bar{\boldsymbol{B}}_{\mathrm{pas}}^{(1)} E + \kappa\lambda_1 \bar{\boldsymbol{B}}_{\mathrm{act}}^{(1)}(E)\big)\Big| \\ \lesssim \langle E\rangle \Big(\lambda_2|\log\lambda_1| + (\lambda_1\lambda_2 + \lambda_1\lambda_3|\log\lambda_1|)^{\frac12}\Big), \end{aligned} \tag{1.32}$$

where we have set

$$\begin{aligned} E' : 2\bar{\boldsymbol{B}}_{\mathrm{pas}}^{(1)} E &:= \mathbb{E}\bigg[\int_{\partial I^\circ} E'x \cdot \sigma(\psi_E^\circ + Ex, \Sigma_E^\circ)\nu\bigg], \\ E' : 2\bar{\boldsymbol{B}}_{\mathrm{act}}^{(1)}(E) &:= -\mathbb{E}\bigg[\int_{2B} (\psi_{E'}^\circ + E'x) \cdot f^\circ(E)\bigg], \end{aligned}$$

in terms of the solution $(\psi_E^\circ, \Sigma_E^\circ)$ *of the single-particle problem*

$$\begin{cases} -\triangle\psi_E^\circ + \nabla\Sigma_E^\circ = 0, & \text{in } \mathbb{R}^d\setminus I^\circ, \\ \operatorname{div}(\psi_E^\circ) = 0, & \text{in } \mathbb{R}^d\setminus I^\circ, \\ \mathrm{D}(\psi_E^\circ + Ex) = 0, & \text{in } I^\circ, \\ \int_{\partial I^\circ} \sigma(\psi_E^\circ + Ex, \Sigma_E^\circ)\nu = 0, & \\ \int_{\partial I^\circ} \Theta x \cdot \sigma(\psi_E^\circ + Ex, \Sigma_E^\circ)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{1.33}$$

In particular, in case of spherical particles $I^\circ = B$*, these expressions take the explicit forms*

$$\begin{aligned} E' : 2\bar{\boldsymbol{B}}_{\mathrm{pas}}^{(1)} E &:= (d+2)|B| E' : E, \\ E' : 2\bar{\boldsymbol{B}}_{\mathrm{act}}^{(1)}(E) &:= -\int_{\mathbb{R}^d\setminus B} \big(1 - \tfrac{1}{|x|^{d+2}}\big) E'x \cdot \mathbb{E}\left[f^\circ(E)\right] \\ &\qquad + \tfrac{d+2}{2} \int_{\mathbb{R}^d\setminus B} \big(1 - \tfrac{1}{|x|^2}\big) \tfrac{(x\cdot E'x)x}{|x|^{d+2}} \cdot \mathbb{E}\left[f^\circ(E)\right]. \qquad \Diamond \end{aligned} \tag{1.34}$$

Note that the error bound in (1.32) *is* $\ll \lambda_1$ *provided that* $\lambda_2|\log\lambda_1| \ll \lambda_1$*, and is in particular bounded by* $\ell^{-3d/2} \ll \ell^{-d}$ *in the regime* $\ell \gg 1$*.*

Alternatively, arguing similarly as for the reformulation (1.26) of (1.24)–(1.25), the dilute active effective viscosity can be written as

$$E' : 2\bar{\boldsymbol{B}}_{\mathrm{act}}^{(1)}(E) = E' : \mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\bigg(\int_{\partial I_n} \sigma(\phi_E^n, \Pi_E^n)\nu \otimes_s (x-x_n) - \int_{I_n+B} f_n(E) \otimes_s (x-x_n)\bigg)\bigg].$$

In particular, if $f_n(E)$ is rotationally symmetric around the direction $\bar f_n(E)$, we find by symmetry,

$$2\bar{\boldsymbol{B}}^{(1)}_{\mathrm{act}}(E) \;=\; \lambda_1\mathbb{E}\Big[\alpha(E)\Big(\tfrac{\bar f_n(E)\otimes\bar f_n(E)}{|\bar f_n(E)|^2}-\tfrac1d\,\mathrm{Id}\Big)\Big], \tag{1.35}$$

for some random prefactor $\alpha(E)\in\mathbb{R}$, the sign of which is actually of critical interest and depends on the shape of the swimming mechanism.

1.6. **Viscosity reduction.** The main motivation of this work is to introduce a nontrivial (and hopefully somewhat realistic) model for active suspensions and rigorously establish a reduction of the viscosity of the plain fluid due to the activity of the particles. In the dilute regime, the above formulas provide a rigorous contribution to this celebrated topic. Although the question can be reduced to understanding the sign of $\alpha(E)$ in (1.35), we take a shorter path here based on (1.34). To make computations explicit, we restrict ourselves to the following simplified model for the swimming mechanism, see e.g. [19, 22],

$$f^\circ(E) \;:=\; \bar f(E)(|B|^{-1}\mathbb{1}_B-\delta_{x(E)}),$$

where the force exerted by the particle on the surrounding fluid is reduced to a Dirac force at a surrounding point $x(E)\in\mathbb{R}^d\setminus B$. Consider a shear deformation $E=\frac s2(e_1\otimes e_2+e_2\otimes e_1)$ for some $s\in\mathbb{R}$. In these terms, the formula (1.34) reads

$$\begin{aligned}E:\bar{\boldsymbol{B}}^{(1)}_{\mathrm{act}}(E) \;=\; \tfrac s2\big(1-\tfrac{1}{|x(E)|^{d+2}}\big)\big(x(E)_1\bar f(E)_2+x(E)_2\bar f(E)_1\big)&\\ -\,s\tfrac{d+2}{2}\big(1-\tfrac{1}{|x(E)|^2}\big)\tfrac{x(E)_1x(E)_2}{|x(E)|^{d+2}}\,x(E)\cdot\bar f(E).&\end{aligned}$$

For a spherical particle with its swimming device viewed as a rigid elongated particle, the motion in shear flow has been well-studied on the formal level: in the limit of a strong angular diffusion, a standard heuristic computation shows that the preferred orientation of the particle is $e:=\frac{1}{\sqrt2}(e_1+e_2)$ or its opposite; see e.g. [14, Section V.8]. This leads us to choosing

$$\bar f(E)=\pm|\bar f(E)|e\qquad\text{and}\qquad x(E)=\pm\gamma|x(E)|e,$$

where $\gamma=1$ in case when the Dirac swimming force is ahead of the particle ("puller" particle) and $\gamma=-1$ in case when it is behind the particle ("pusher" particle). Hence, we get

$$E:\bar{\boldsymbol{B}}^{(1)}_{\mathrm{act}}(E) \;=\; \gamma\tfrac s2|x(E)||\bar f(E)|\Big(1-\tfrac{d+2}{2}\tfrac{1}{|x(E)|^d}+\tfrac d2\tfrac{1}{|x(E)|^{d+2}}\Big),$$

which is negative (resp. positive) in case of a pusher (resp. puller). This is in full agreement with well-known experiments and predictions of [35, 36, 27].

To conclude on the extent of the possible viscosity reduction, we come back to Theorem 1.9 in case $\ell\gg1$. As stated, the error bound in the result is then of order $O(\ell^{-3d/2})$,

$$\bar{\boldsymbol{B}}_{\mathrm{tot}}(E)=\big(1+\tfrac{d+2}{2}|B|\lambda_1\big)E+\kappa\lambda_1\bar{\boldsymbol{B}}^{(1)}_{\mathrm{act}}(E)+O(\ell^{-3d/2}). \tag{1.36}$$

If $\bar{\boldsymbol{B}}^{(1)}_{\mathrm{act}}(E)<0$, we infer that the total effective viscosity is smaller than the viscosity of the plain fluid,

$$E:\bar{\boldsymbol{B}}_{\mathrm{tot}}(E)\;<\;|E|^2,$$

provided that the activity is strong enough $\kappa\gg1$ in such a way that the active contribution exceeds the passive one. Moreover, as $\lambda_1=O(\ell^{-d})$, we see that the viscosity reduction could become of order 1 if $\kappa=O(\ell^d)$. This drastic reduction of viscosity is however prohibited by the (only nearly optimal) condition (1.18), which is used to ensure the well-posedness of the microscopic model. In some sense, the above analysis can be compared to the enhancement of elastostriction by active charges analyzed in [13].

Outline of the article. The article is organized as follows. Section 2 is dedicated to the introduction of correctors, the key quantities in homogenization. The proofs of the homogenization results of Theorems 1.7 and 1.8 are the object of Section 3, while the dilute analysis and the proof of Theorem 1.9 are postponed to Section 4.

Notation.

— For vector fields u, u', and matrix fields T, T', we set $(\nabla u)_{ij} = \nabla_j u_i$, $\operatorname{div}(T)_i = \nabla_j T_{ij}$, $T : T' = T_{ij}T'_{ij}$, $(u \otimes u')_{ij} = u_i u'_j$, $(T^s)_{ij} = \frac{1}{2}(T_{ij} + T_{ji})$, $\mathrm{D}(u) = (\nabla u)^s$, $(u \otimes_s u') = (u \otimes u')^s$. For a 3-tensor field S, the matrix $\operatorname{div}(S)$ is defined by $\operatorname{div}(S)_{ij} = \nabla_k S_{ijk}$. For a matrix E and a vector field u, we write $\partial_E u = E : \nabla u$. We systematically use Einstein's summation convention on repeated indices.

— For a vector field u and a scalar field P, we recall the notation $\sigma(u, P) = 2\,\mathrm{D}(u) - P\,\mathrm{Id}$ for the Cauchy stress tensor.

— We denote by $C \geq 1$ any constant that only depends on dimension d, on the constant ϑ in Hypothesis 1.1, and on the reference domain U. We use the notation $\lesssim$ (resp. $\gtrsim$) for $\leq C\times$(resp. $\geq \frac{1}{C}\times$) up to such a multiplicative constant C. We write $\ll$ (resp. $\gg$) for $\leq C\times$ (resp. $\geq C\times$) up to a sufficiently large multiplicative constant C. We add subscripts to C, $\lesssim$, $\gtrsim$ in order to indicate dependence on other parameters.

— We write $\mathbb{M}_0 \subset \mathbb{R}^{d\times d}$ for the subset of trace-free matrices, $\mathbb{M}_0^{\mathrm{sym}}$ for the subset of symmetric trace-free matrices, and $\mathbb{M}^{\mathrm{skew}}$ for the subset of skew-symmetric matrices.

— The ball centered at x of radius r in $\mathbb{R}^d$ is denoted by $B_r(x)$, and we set $B(x) = B_1(x)$, $B_r = B_r(0)$ and $B = B_1(0)$.

— We use the standard notation $\langle a \rangle := (1 + |a|^2)^{1/2}$.

2. Corrector problems

We start by recalling the relevant correctors for the passive suspension problem as introduced in [2, 4, 8, 10], and then we define new correctors for the active problem.

2.1. Passive corrector problems.

2.1. Passive corrector problems. Correctors for passive suspensions were first defined in [8] and are key to the definition of the associated effective viscosity $\bar{\boldsymbol{B}}$, cf. (1.3).

Lemma 2.1 (Passive correctors [8]). *Let Hypothesis 1.1 hold. For all $E \in \mathbb{M}_0^{\mathrm{sym}}$, there exist a unique random field $\psi_E \in \mathrm{L}^2(\Omega; H^1_{\mathrm{loc}}(\mathbb{R}^d)^d)$ and a unique pressure field $\Sigma_E \in \mathrm{L}^2(\Omega; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \mathcal{I}))$ such that:*

— *almost surely, realizations of ψ_E and Σ_E satisfy*

$$\begin{cases} -\triangle \psi_E + \nabla \Sigma_E = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \operatorname{div}(\psi_E) = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \mathrm{D}(\psi_E + Ex) = 0, & \text{in } \mathcal{I}, \\ \int_{\partial I_n} \sigma(\psi_E + Ex, \Sigma_E)\nu = 0, & \forall n, \\ \int_{\partial I_n} \Theta(x - \varepsilon x_n) \cdot \sigma(\psi_E + Ex, \Sigma_E)\nu = 0, & \forall n,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{2.1}$$

— *the corrector gradient* $\nabla\psi_E$ *and the pressure* $\Sigma_E \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}$ *are stationary, with*

$$\mathbb{E}\big[\nabla\psi_E\big]=0,\quad \mathbb{E}\big[\Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big]=0,$$
$$\mathbb{E}\big[|\nabla\psi_E|^2\big]+\mathbb{E}\big[\Sigma_E^2\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big]\lesssim\lambda|E|^2,$$

and with the anchoring condition $\int_B\psi_E=0$.

In addition, the following properties hold.

(i) Ergodic theorem: *almost surely,*

$$\begin{array}{rcll}(\nabla\psi_E)(\frac{\cdot}{\varepsilon}) & \rightharpoonup & \mathbb{E}\left[\nabla\psi_E\right]=0, & \\ (\Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\frac{\cdot}{\varepsilon}) & \rightharpoonup & \mathbb{E}\big[\Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big]=0, & \text{weakly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ as } \varepsilon\downarrow 0.\end{array}$$

(ii) Sublinearity: *almost surely, for all* $q<\frac{2d}{d-2}$,

$$\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\to 0\quad\text{strongly in } \mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^d)^d \text{ as } \varepsilon\downarrow 0. \qquad \Diamond$$

As in [2, 10], in view of quantitative estimates, we further need to define an associated extended flux J and a flux corrector Υ. More precisely, J is a solenoidal extension of the natural flux $\sigma(\psi_E+Ex,\Sigma_E)$ outside the particles, and Υ is the associated vector potential in the Coulomb gauge.

Lemma 2.2 (Passive flux correctors [2, 10])**.** *Let Hypothesis 1.1 hold. For all* $E\in\mathbb{M}_0^{\mathrm{sym}}$, *there exists a stationary random 2-tensor field* $J_E=(J_{E;ij})_{1\le i,j\le d}$ *with finite second moment such that, almost surely,*

$$\begin{array}{rcl}J_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}} & = & \sigma(\psi_E+Ex,\Sigma_E)\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\\ \operatorname{div}(J_E) & = & 0,\end{array}$$

and for all n,

$$\|J_E\|_{\mathrm{L}^2(I_n)}\lesssim\|\sigma(\psi_E+Ex,\Sigma_E)\|_{\mathrm{L}^2((I_n+B)\setminus I_n)}. \tag{2.2}$$

Moreover, there exists a unique random 3-tensor field $\Upsilon_E=(\Upsilon_{E;ijk})_{1\le i,j,k\le d}$ *such that:*

— *for all* i,j,k, *almost surely, realizations of* $\Upsilon_{E;ijk}$ *belong to* $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ *and satisfy*

$$-\Delta\Upsilon_{E;ijk}\;=\;\partial_j J_{E;ik}-\partial_k J_{E;ij}; \tag{2.3}$$

— *the random field* $\nabla\Upsilon_E$ *is stationary, has vanishing expectation, has finite second moment, and satisfies the anchoring condition* $\fint_B\Upsilon_E=0$.

In addition, the following properties hold.

(i) Skew-symmetry: *almost surely,* $\Upsilon_{E;ijk}=-\Upsilon_{E;ikj}$ *for all* i,j,k.

(ii) Vector potential: *almost surely, for all* i,

$$\operatorname{div}(\Upsilon_{E;i})=J_{E;i}-\mathbb{E}[J_{E;i}],$$

where we have set $\Upsilon_{E;i}=(\Upsilon_{E;ijk})_{1\le j,k\le d}$ *and* $J_{E;i}=(J_{E;ij})_{1\le j\le d}$.

(iii) Ergodic theorem: *almost surely,*

$$(\nabla\Upsilon_E)(\tfrac{\cdot}{\varepsilon})\rightharpoonup\mathbb{E}\left[\nabla\Upsilon_E\right]=0\text{weakly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ as } \varepsilon\downarrow 0.$$

(iv) Sublinearity: *almost surely, for all* $q<\frac{2d}{d-2}$,

$$\varepsilon\Upsilon_E(\tfrac{\cdot}{\varepsilon})\rightharpoonup 0\quad\text{strongly in } \mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^d)^d \text{ as } \varepsilon\downarrow 0.$$

(v) Effective constants: *the expectation of J_E takes the form*

$$\mathbb{E}\left[J_E\right] = 2\bar{\boldsymbol{B}}_{\mathrm{pas}}E + (\bar{\boldsymbol{b}} : E)\,\mathrm{Id},$$

in terms of the effective constants $\bar{\boldsymbol{B}}_{\mathrm{pas}}$ and $\bar{\boldsymbol{b}}$ defined in (1.21) and (1.22). $\Diamond$

2.2. **Active corrector problems.** We turn to the definition of suitable correctors for the active suspension problem. These new correctors characterize the contribution of swimming forces of the particles in a uniform fluid velocity gradient $E \in \mathbb{M}_0^{\mathrm{sym}}$.

Lemma 2.3. *Let Hypotheses 1.1, 1.3, and 1.5 hold. For all $E \in \mathbb{M}_0^{\mathrm{sym}}$, there exist a unique random field $\phi_E \in \mathrm{L}^2(\Omega; H^1_{\mathrm{loc}}(\mathbb{R}^d)^d)$ and a unique pressure field $\Pi_E \in \mathrm{L}^2(\Omega; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \mathcal{I}))$ such that:*

— *almost surely, realizations of ϕ_E and Π_E satisfy*

$$\begin{cases} -\triangle \phi_E + \nabla \Pi_E = \sum_n f_n(E), & \text{in } \mathbb{R}^d \setminus \mathcal{I},\\ \mathrm{div}(\phi_E) = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I},\\ \mathrm{D}(\phi_E) = 0, & \text{in } \mathcal{I},\\ \int_{\partial I_n} \sigma(\phi_E, \Pi_E)\nu + \bar{f}_n(E) = 0, & \forall n,\\ \int_{\partial I_n} \Theta(x - x_n) \cdot \sigma(\phi_E, \Pi_E)\nu + \Theta : \tilde{f}_n(E) = 0, & \forall n,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{2.4}$$

— *the corrector gradient $\nabla\phi_E$ and the pressure $\Pi_E \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}$ are stationary, with*

$$\begin{gathered} \mathbb{E}\big[\nabla \phi_E\big] = 0, \quad \mathbb{E}\big[\Pi_E \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big] = 0,\\ \mathbb{E}\big[|\nabla \phi_E|^2\big] + \mathbb{E}\big[\Pi_E^2 \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big] \lesssim \lambda \langle E \rangle^2, \end{gathered} \tag{2.5}$$

and with the anchoring condition $\int_B \phi_E = 0$.

In addition, the following properties hold.

(i) Ergodic theorem: *almost surely,*

$$\begin{aligned} (\nabla \phi_E)(\tfrac{\cdot}{\varepsilon}) &\rightharpoonup \mathbb{E}\left[\nabla \phi_E\right] = 0,\\ (\Pi_E \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon}) &\rightharpoonup \mathbb{E}\big[\Pi_E \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big] = 0, \qquad \text{weakly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ as } \varepsilon \downarrow 0. \end{aligned}$$

(ii) Sublinearity: *almost surely, for all $q < \frac{2d}{d-2}$,*

$$\varepsilon \phi_E(\tfrac{\cdot}{\varepsilon}) \to 0 \qquad \text{strongly in } \mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^d) \text{ as } \varepsilon \downarrow 0.$$ $\Diamond$

Proof. The argument is similar to that in [8, Proposition 2.1] for Lemma 2.1 above, and we only briefly show the needed adaptations: we describe the structure of equation (2.4), following the first step of the proof of [8, Proposition 2.1], while the rest of the proof in [8] is then easily repeated and is skipped here for brevity. More precisely, we only show the following: if ϕ_E is a solution of (2.4) with $\nabla\phi_E, \Pi_E$ stationary with finite second moments, then it satisfies for all stationary fields $v \in \mathrm{L}^2(\Omega; H^1_{\mathrm{loc}}(\mathbb{R}^d)^d)$ with $\mathrm{div}(v) = 0$ and $\mathrm{D}(v)|_{\mathcal{I}} = 0$,

$$\mathbb{E}\left[\nabla v : \nabla \phi_E\right] = -\mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|} \int_{I_n+B} v \cdot f_n(E)\bigg], \tag{2.6}$$

where, by the Cauchy–Schwarz and the Poincaré inequalities, together with the hardcore assumption and the property $\int_{I_n+B} f_n(E) = 0$ (cf. (1.9)), the right-hand side is bounded by

$$\bigg|\mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|} \int_{I_n+B} v \cdot f_n(E)\bigg]\bigg| \lesssim \mathbb{E}\big[|\nabla v|^2\big]^{\frac12} \bigg(\lambda\, \mathbb{E}\bigg[\int_{I_\circ+B} |f_\circ(E)|^2\bigg]\bigg)^{\frac12}.$$

To prove this claim, we start by noting that the hardcore assumption allows to construct almost surely for all $R>0$ a cut-off function η_R such that

$$\eta_R|_{B_R}=1, \qquad \eta_R|_{\mathbb{R}^d\setminus B_{R+5}}=0, \qquad |\nabla \eta_R|\lesssim 1,$$

and such that η_R is constant in I_n+B for all n. As ϕ_E is divergence-free, testing equation (2.4) with $\eta_R v$ and integrating by parts, we find

$$\begin{aligned}2\int_{\mathbb{R}^d\setminus\mathcal{I}} \mathrm{D}(\eta_R v):\mathrm{D}(\phi_E)-\int_{\mathbb{R}^d\setminus\mathcal{I}}\nabla\eta_R\cdot v\Pi_E\\ =\sum_n\int_{(I_n+B)\setminus I_n}\eta_R v\cdot f_n(E)-\sum_n\int_{\partial I_n}\eta_R v\cdot\sigma(\phi_E,\Pi_E)\nu.\end{aligned}\tag{2.7}$$

Since $\mathrm{D}(\phi_E)|_{\mathcal{I}}=0$ and $\nabla\eta_R|_{\mathcal{I}}=0$, the left-hand side of (2.7) writes

$$2\int_{\mathbb{R}^d\setminus\mathcal{I}} \mathrm{D}(\eta_R v):\mathrm{D}(\phi_E)-\int_{\mathbb{R}^d\setminus\mathcal{I}}\nabla\eta_R\cdot v\Pi_E=\int_{\mathbb{R}^d}\nabla(\eta_R v):\nabla\phi_E-\int_{\mathbb{R}^d}\nabla\eta_R\cdot v\Pi_E.$$

As η_R is constant in I_n+B and $\mathrm{D}(v)=0$ in I_n, we can rewrite the last right-hand side term as

$$\begin{aligned}\int_{\partial I_n}\eta_R v\cdot\sigma(\phi_E,\Pi_E)\nu \;=\; \eta_R(x_n)\bigg(\Big(\fint_{I_n} v\Big)\cdot\int_{\partial I_n}\sigma(\phi_E,\Pi_E)\nu\\ +\Big(\fint_{I_n}(\nabla v)^{\mathrm{skew}}\Big):\int_{\partial I_n}\sigma(\phi_E,\Pi_E)\nu\otimes(x-x_n)\bigg),\end{aligned}$$

and thus, in view of the boundary conditions for (ϕ_E,Π_E) in (2.4), using the notation (1.8),

$$\begin{aligned}\int_{\partial I_n}\eta_R v\cdot\sigma(\phi_E,\Pi_E)\nu &= -\eta_R(x_n)\bigg(\Big(\fint_{I_n} v\Big)\cdot\bar f_n(E)+\Big(\fint_{I_n}\nabla v\Big):\tilde f_n(E)\bigg)\\ &= -\int_{I_n}\eta_R v\cdot f_n(E).\end{aligned}$$

Inserting this into (2.7), we get

$$\int_{\mathbb{R}^d}\nabla(\eta_R v):\nabla\phi_E-\int_{\mathbb{R}^d}\nabla\eta_R\cdot v\Pi_E=\sum_n\int_{I_n+B}\eta_R v\cdot f_n(E).\tag{2.8}$$

Expanding the gradient in the left-hand side, passing to the limit $R\uparrow\infty$, and using the stationarity of $\nabla\phi_E,\Pi_E$ and of v, the claim (2.6) follows. From there, we may then refer to the proof in [8, Proposition 2.1]. ☐

Next, as in Lemma 2.2, we further need to define an associated extended flux and a flux corrector for the active suspension problem. The difficulty, however, is that even in the fluid domain $\mathbb{R}^d\setminus\mathcal{I}$ the flux $\sigma(\phi_E,\Pi_E)$ is not divergence-free. It thus needs to be first suitably compensated and we are led to defining the following auxiliary corrector γ_E. The proof of the upcoming lemma (which is a simplified version of Lemma 2.3) is straightforward and skipped for brevity.

Lemma 2.4. *Let Hypotheses 1.1, 1.3, and 1.5 hold. For all* $E\in\mathbb{M}_0^{\mathrm{sym}}$*, there exists a unique random field* $\gamma_E\in\mathrm{L}^2(\Omega;H^1_{\mathrm{loc}}(\mathbb{R}^d)^d)$ *such that:*

— *almost surely, the realizations of γ_E satisfy*

$$-\triangle\gamma_E = \sum_n f_n(E); \tag{2.9}$$

— *the gradient field $\nabla\gamma_E$ is stationary with*

$$\mathbb{E}\big[\nabla\gamma_E\big] = 0, \qquad \mathbb{E}\big[|\nabla\gamma_E|^2\big] \lesssim \lambda\langle E\rangle^2,$$

and with the anchoring condition $\int_B \gamma_E = 0$.

In addition, the following properties hold.

(i) Ergodic theorem: *almost surely,*

$$(\nabla\gamma_E)(\tfrac{\cdot}{\varepsilon}) \ \rightharpoonup\ \mathbb{E}\left[\nabla\gamma_E\right] = 0 \qquad \text{weakly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)^d \text{ as } \varepsilon\downarrow 0.$$

(ii) Sublinearity: *almost surely, for all $q < \frac{2d}{d-2}$,*

$$\varepsilon\gamma_E(\tfrac{\cdot}{\varepsilon}) \ \to\ 0 \qquad \text{strongly in } \mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^d)^d \text{ as } \varepsilon\downarrow 0. \qquad \Diamond$$

Using the above-defined γ_E to compensate the divergence of the flux $\sigma(\phi_E,\Pi_E)$ in the fluid domain, and extending it similarly as in Lemma 2.2 inside the particles, we are now in the position to define a flux K_E, a divergence-free compensated flux L_E, and the associated flux corrector θ_E.

Lemma 2.5. *Under Hypotheses 1.1, 1.3, and 1.5, there exists a stationary random symmetric 2-tensor field $K_E = (K_{E;ij})_{1\le i,j\le d}$ with finite second moment*

$$\mathbb{E}\big[|K_E|^2\big] \lesssim \lambda\langle E\rangle^2, \tag{2.10}$$

such that, almost surely,

$$\begin{aligned} K_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}} &= \sigma(\phi_E,\Pi_E)\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\\ \operatorname{div}(K_E) &= -\sum_n f_n(E). \end{aligned} \tag{2.11}$$

Moreover, the expectation of K_E takes the form

$$\mathbb{E}\left[K_E\right] \;=\; 2\bar{\boldsymbol{C}}(E) + \bar{\mathbf{c}}(E)\,\mathrm{Id}, \tag{2.12}$$

in terms of the effective tensors $\bar{\boldsymbol{C}}(E)$ and $\bar{\mathbf{c}}(E)$ defined in (1.24). Next, for the divergence-free compensated flux

$$L_E \;:=\; K_E - \mathbb{E}\left[K_E\right] - \nabla\gamma_E, \tag{2.13}$$

there exists a unique random 3-tensor field $\theta_E = (\theta_{E;ijk})_{1\le i,j,k\le d}$ such that:

— *for all i,j,k, almost surely, realizations of $\theta_{E;ijk}$ belong to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and satisfy*

$$-\triangle\theta_{E;ijk} \;=\; \partial_j L_{E;ik} - \partial_k L_{E;ij}; \tag{2.14}$$

— *the random field $\nabla\theta_E$ is stationary, has vanishing expectation, has finite second moment, and satisfies the anchoring condition $\fint_B \theta_E = 0$.*

In addition, the following properties hold.

(i) Skew-symmetry: *almost surely, $\theta_{E;ijk} = -\theta_{E;ikj}$ for all i,j,k.*

(ii) Vector potential: *almost surely, for all i,*

$$\operatorname{div}(\theta_{E;i}) = L_{E;i},$$

where we have set $\theta_{E;i} = (\theta_{E;ijk})_{1\le j,k\le d}$ and $L_{E;i} = (L_{E;ij})_{1\le j\le d}$.

(iii) Ergodic theorem: *almost surely,*

$$(\nabla\theta_E)(\tfrac{\cdot}{\varepsilon}) \quad\rightharpoonup\quad \mathbb{E}\left[\nabla\theta_E\right]=0 \qquad \textit{weakly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)^d \textit{ as } \varepsilon\downarrow 0.$$

(iv) Sublinearity: *almost surely, for all* $q<\frac{2d}{d-2}$,

$$\varepsilon\theta_E(\tfrac{\cdot}{\varepsilon}) \;\to\; 0 \qquad \textit{strongly in } \mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^d)^d \textit{ as } \varepsilon\downarrow 0. \qquad \Diamond$$

Proof. We split the proof into three main steps.

Step 1. Construction and properties of the extended flux K_E.
For all $g\in C^\infty_c(\mathbb{R}^d)$, equation (2.4) yields by integration by parts,

$$\int_{\mathbb{R}^d\setminus\mathcal{I}}\nabla g:\sigma(\phi_E,\Pi_E) \;=\; \sum_n\int_{(I_n+B)\setminus I_n} g\cdot f_n(E)-\sum_n\int_{\partial I_n} g\cdot\sigma(\phi_E,\Pi_E)\nu,$$

and thus, using boundary conditions for (ϕ_E,Π_E) in (2.4) and recalling the notation (1.8),

$$\int_{\mathbb{R}^d\setminus\mathcal{I}}\nabla g:\sigma(\phi_E,\Pi_E)+\sum_n\int_{\partial I_n}\Big(g-\Big(\fint_{I_n} g\Big)-\Big(\fint_{I_n}(\nabla g)^{\mathrm{skew}}\Big)(x-x_n)\Big)\cdot\sigma(\phi_E,\Pi_E)\nu$$
$$=\sum_n\int_{I_n+B} g\cdot f_n(E). \qquad (2.15)$$

For all n, we may then consider the following Neumann problem,

$$\begin{cases} -\triangle\phi_E^n+\nabla\Pi_E^n=f_n(E), & \text{in } I_n,\\ \operatorname{div}(\phi_E^n)=0, & \text{in } I_n,\\ \sigma(\phi_E^n,\Pi_E^n)\nu=\sigma(\phi_E,\Pi_E)\nu, & \text{on } \partial I_n.\end{cases} \qquad (2.16)$$

Note that this only defines ϕ_E^n up to a rigid motion, which is fixed by choosing ϕ_E^n with $\fint_{I_n}\phi_E^n=0$ and $\fint_{I_n}\nabla\phi_E^n\in\mathbb{M}_0^{\mathrm{sym}}$. Assuming that this Neumann problem (2.16) is well-posed, and setting

$$\begin{aligned}\tilde q_E &:= \mathrm{D}(\phi_E)+\sum_n\mathrm{D}(\phi_E^n)\mathbb{1}_{I_n},\\ \tilde\Pi_E &:= \Pi_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}+\sum_n\Pi_E^n\mathbb{1}_{I_n},\end{aligned} \qquad (2.17)$$

we easily deduce that the extended flux

$$K_E := 2\tilde q_E-\tilde\Pi_E\,\mathrm{Id} \qquad (2.18)$$

satisfies the desired relations (2.11) (recall that $\mathrm{D}(\phi_E)|_{\mathcal{I}}=0$). It remains to check the well-posedness of this Neumann problem and to establish the bound (2.10). Note that this well-posedness, together with the uniqueness in our construction of the pressures $\{\Pi_E^n\}_n$ below, ensures that $\tilde q_E$ and $\tilde\Pi_E$ are stationary. We split the proof into three further substeps.

Substep 1.1. Well-posedness of the Neumann problem (2.16) for ϕ_E^n.
The weak formulation of (2.16) reads as follows: for all divergence-free fields $v\in H^1(I_n)^d$,

$$2\int_{I_n}\mathrm{D}(v):\mathrm{D}(\phi_E^n)=\mathcal{L}_E(v), \qquad (2.19)$$

where $\mathcal{L}_E$ stands for the linear form

$$\mathcal{L}_E(v) := \int_{I_n} v\cdot f_n(E) + \int_{\partial I_n} v\cdot\sigma(\phi_E,\Pi_E)\nu.$$

Using the boundary conditions for (ϕ_E,Π_E) in (2.4) and recalling the notation (1.8), the latter can be reformulated as

$$\mathcal{L}_E(v) = \int_{\partial I_n}\Big(v-\Big(\fint_{I_n} v\Big)-\Big(\fint_{I_n}(\nabla v)^{\mathrm{skew}}\Big)(x-x_n)\Big)\cdot\sigma(\phi_E,\Pi_E)\nu. \tag{2.20}$$

We shall show that (2.19) is well-posed in the following Hilbert subspace of $H^1(I_n)^d$,

$$\mathcal{T} := \Big\{v\in H^1(I_n)^d \,:\, \operatorname{div}(v)=0,\ \fint_{I_n} v=0,\ \text{and}\ \fint_{I_n}\nabla v\in\mathbb{M}_0^{\mathrm{sym}}\Big\}.$$

First note that Korn's inequality yields for all $v\in\mathcal{T}$,

$$\|\nabla v\|_{\mathrm{L}^2(I_n)} \lesssim \|\mathrm{D}(v)\|_{\mathrm{L}^2(I_n)},$$

which entails that the bilinear form $(v,\psi)\mapsto 2\int_{I_n}\mathrm{D}(v):\mathrm{D}(\psi)$ is continuous and coercive on $\mathcal{T}\times\mathcal{T}$. By the Lax–Milgram theorem, in order to prove the well-posedness of (2.19), it remains to show that the linear form $\mathcal{L}_E$ is continuous on $\mathcal{T}$ as well.

In order to deal with the Neumann condition, we consider an extension map

$$T_n \,:\, \big\{v\in H^1(I_n)^d:\operatorname{div}(v)=0\big\} \to \big\{v\in H_0^1(I_n+B)^d:\operatorname{div}(v)=0\big\},$$

such that $T_nv|_{I_n}=v|_{I_n}$ and

$$\|\nabla T_nv\|_{\mathrm{L}^2(I_n+B)} \lesssim \|\nabla v\|_{\mathrm{L}^2(I_n)}. \tag{2.21}$$

Smuggling in T_n in (2.20), integrating by parts, and inserting equation (2.4), the linear form $\mathcal{L}_E$ can be rewritten as

$$\begin{aligned}\mathcal{L}_E(v) &= -\int_{(I_n+B)\setminus I_n}\operatorname{div}\Big[\sigma(\phi_E,\Pi_E)\,T_n\Big(v-\Big(\fint_{I_n}v\Big)-\Big(\fint_{I_n}(\nabla v)^{\mathrm{skew}}\Big)(x-x_n)\Big)\Big]\\ &= \int_{(I_n+B)\setminus I_n} T_n\Big(v-\Big(\fint_{I_n}v\Big)-\Big(\fint_{I_n}(\nabla v)^{\mathrm{skew}}\Big)(x-x_n)\Big)\cdot f_n(E)\\ &\quad -2\int_{(I_n+B)\setminus I_n}\mathrm{D}(\phi_E):\mathrm{D}\Big[T_n\Big(v-\Big(\fint_{I_n}v\Big)-\Big(\fint_{I_n}(\nabla v)^{\mathrm{skew}}\Big)(x-x_n)\Big)\Big].\end{aligned}$$

Hence, in view of (2.21) and of Korn's inequality,

$$|\mathcal{L}_E(v)| \lesssim \|\mathrm{D}(v)\|_{\mathrm{L}^2(I_n)}\Big(\|\mathrm{D}(\phi_E)\|_{\mathrm{L}^2(I_n+B)}+\|f_n(E)\|_{\mathrm{L}^2(I_n+B)}\Big),$$

which proves the continuity of $\mathcal{L}_E$ on $\mathcal{T}$.

By the Lax–Milgram theorem, we deduce that there exists a unique solution $\phi_E^n\in\mathcal{T}$ of (2.19), and that it satisfies

$$\|\mathrm{D}(\phi_E^n)\|_{\mathrm{L}^2(I_n)} \lesssim \|\mathrm{D}(\phi_E)\|_{\mathrm{L}^2(I_n+B)}+\|f_n(E)\|_{\mathrm{L}^2(I_n+B)}.$$

By Korn's inequality, this further yields

$$\|\nabla\phi_E^n\|_{\mathrm{L}^2(I_n)} \lesssim \|\mathrm{D}(\phi_E)\|_{\mathrm{L}^2(I_n+B)}+\|f_n(E)\|_{\mathrm{L}^2(I_n+B)}. \tag{2.22}$$

Substep 1.2. Construction of the pressure.
As (2.17) reads $\tilde q_E = \mathrm{D}(\phi_E) + \mathrm{D}(\phi_E^n)\mathbb{1}_{I_n}$ in $I_n + B$, combining equation (2.4) for ϕ_E and equation (2.19) for ϕ_E^n, we find for all $v \in C_c^\infty(I_n + B)^d$ with $\operatorname{div}(v) = 0$,

$$2\int_{\mathbb{R}^d} \mathrm{D}(v) : \tilde q_E = \int_{\mathbb{R}^d} v \cdot f_n(E).$$

Appealing e.g. to [26, Proposition 12.10], we deduce that there exists an associated pressure field $\Pi_E^n \in \mathrm{L}^2_{\mathrm{loc}}(I_n + B)$, which is unique up to an additive constant, such that for all test functions $v \in C_c^\infty(I_n + B)^d$,

$$\int_{\mathbb{R}^d} \mathrm{D}(v) : (2\tilde q_E - \Pi_E^n \,\mathrm{Id}) = \int_{\mathbb{R}^d} v \cdot f_n(E). \tag{2.23}$$

Since for all $v \in C_c^\infty((I_n + B) \setminus I_n)^d$ we get

$$\int_{\mathbb{R}^d} \mathrm{D}(v) : \sigma(\phi_E, \Pi_E^n) \;=\; \int_{\mathbb{R}^d} \mathrm{D}(v) : (2\tilde q_E - \Pi_E^n \,\mathrm{Id}) \;=\; \int_{\mathbb{R}^d} v \cdot f_n(E),$$

and comparing with equation (2.4), we deduce that Π_E^n can be chosen uniquely to coincide with Π_E on $(I_n + B) \setminus I_n$.

It remains to estimate the above-constructed pressure. Using that Π_E^n coincides with Π_E in $(I_n + B) \setminus I_n$, we can split

$$\begin{aligned}\int_{I_n+B} \Pi_E^n \;&=\; \int_{(I_n+B)\setminus I_n} \Pi_E + \int_{I_n} \Pi_E^n \\ &=\; \int_{(I_n+B)\setminus I_n} \Pi_E + \int_{I_n} \Big(\Pi_E^n - \fint_{I_n+B} \Pi_E^n\Big) + |I_n| \fint_{I_n+B} \Pi_E,\end{aligned}$$

to the effect that

$$(|I_n + B| - |I_n|)\Big|\fint_{I_n+B} \Pi_E^n\Big| \;\le\; \Big|\int_{(I_n+B)\setminus I_n} \Pi_E\Big| + \Big|\int_{I_n} \Big(\Pi_E^n - \fint_{I_n+B} \Pi_E^n\Big)\Big|.$$

Hence,

$$\begin{aligned}\|\Pi_E^n\|_{\mathrm{L}^2(I_n)} \;&\lesssim\; \Big\|\Pi_E^n - \fint_{I_n+B} \Pi_E^n\Big\|_{\mathrm{L}^2(I_n+B)} + \Big|\fint_{I_n+B} \Pi_E^n\Big| \\ &\lesssim\; \Big\|\Pi_E^n - \fint_{I_n+B} \Pi_E^n\Big\|_{\mathrm{L}^2(I_n+B)} + \Big|\fint_{(I_n+B)\setminus I_n} \Pi_E\Big|.\end{aligned}$$

Starting from (2.23), a standard argument based on the Bogovskii operator yields

$$\Big\|\Pi_E^n - \fint_{I_n+B} \Pi_E^n\Big\|_{\mathrm{L}^2(I_n+B)} \lesssim \|\tilde q_E\|_{\mathrm{L}^2(I_n+B)},$$

so that the above becomes

$$\|\Pi_E^n\|_{\mathrm{L}^2(I_n)} \lesssim \|\tilde q_E\|_{\mathrm{L}^2(I_n+B)} + \|\Pi_E\|_{\mathrm{L}^2((I_n+B)\setminus I_n)}.$$

Combining this with (2.22), we get

$$\|\nabla \phi_E^n\|_{\mathrm{L}^2(I_n)} + \|\Pi_E^n\|_{\mathrm{L}^2(I_n)} \lesssim \|\sigma(\phi_E, \Pi_E)\|_{\mathrm{L}^2((I_n+B)\setminus I_n)} + \|f_n(E)\|_{\mathrm{L}^2(I_n+B)}. \tag{2.24}$$

Substep 1.3. Proof of (2.10).
By definition (2.18), the bound (2.24) yields for all n,

$$\begin{aligned} \|K_E\|_{\mathrm{L}^2(I_n)} &\lesssim \|\sigma(\phi_E^n,\Pi_E^n)\|_{\mathrm{L}^2(I_n)} \\ &\lesssim \|\sigma(\phi_E,\Pi_E)\|_{\mathrm{L}^2((I_n+B)\setminus I_n)} + \|f_n(E)\|_{\mathrm{L}^2(I_n+B)}. \end{aligned} \tag{2.25}$$

Hence, for all $R>0$,

$$\|K_E\|^2_{\mathrm{L}^2(B_R)} \lesssim \|\sigma(\phi_E,\Pi_E)\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\|^2_{\mathrm{L}^2(B_{R+5})} + \sum_{n:I_n\cap B_R\neq\varnothing}\int_{I_n+B}|f_n(E)|^2,$$

and thus, by stationarity, letting $R\uparrow\infty$,

$$\|K_E\|^2_{\mathrm{L}^2(\Omega)} \lesssim \|\sigma(\phi_E,\Pi_E)\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\|^2_{\mathrm{L}^2(\Omega)} + \lambda\,\mathbb{E}\bigg[\int_{I_\circ+B}|f_\circ(E)|^2\bigg].$$

Combined with (2.5), this yields (2.10).

Step 2. Formula for $\mathbb{E}[K_E]$ and definition of $\bar{\boldsymbol{C}}(E)$ and $\bar{\boldsymbol{c}}(E)$.
We split the proof into two further substeps, separately proving formula (2.12) for $\mathbb{E}[K_E]$ and establishing the alternative formulas (1.26) for $\bar{\boldsymbol{C}}(E)$ and $\bar{\boldsymbol{c}}(E)$.

Substep 2.1. Proof of (2.12).
The hardcore assumption allows to construct almost surely for all $R>0$ a cut-off function η_R such that

$$\eta_R|_{B_R}=1,\qquad \eta_R|_{\mathbb{R}^d\setminus B_{R+5}}=0,\qquad |\nabla\eta_R|\lesssim 1,$$

and such that η_R is constant in I_n+B for all n. By definition of K_E and (ϕ_E,Π_E), we have

$$\mathbb{E}\big[K_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big] = \mathbb{E}\big[\sigma(\phi_E,\Pi_E)\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big] = 2\mathbb{E}\big[\mathrm{D}(\phi_E)\big] - \mathbb{E}\big[\Pi_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big]\,\mathrm{Id} = 0,$$

and the ergodic theorem then yields almost surely,

$$\mathbb{E}[K_E] = \mathbb{E}\big[K_E\mathbb{1}_{\mathcal{I}}\big] = \lim_{R\uparrow\infty}|B_R|^{-1}\int_{\mathcal{I}}\eta_R\,K_E. \tag{2.26}$$

By definition of K_E and the choice of η_R, integration by parts and the equation (2.16) for (ϕ_E^n,Σ_E^n) yield

$$\begin{aligned} \int_{\mathcal{I}}\eta_R\,K_E &= \sum_n\eta_R(x_n)\int_{I_n}\sigma(\phi_E^n,\Pi_E^n) \\ &= \sum_n\eta_R(x_n)\int_{\partial I_n}\sigma(\phi_E^n,\Pi_E^n)\nu\otimes(x-x_n) \\ &\quad+\sum_n\eta_R(x_n)\int_{I_n}f_n(E)\otimes(x-x_n). \end{aligned}$$

By the boundary conditions for (ϕ_E^n,Π_E^n) and recalling the notation (1.8), we deduce

$$\int_{\mathcal{I}}\eta_R\,K_E = \int_{\mathbb{R}^d}\eta_R\sum_n\tfrac{\mathbb{1}_{I_n}}{|I_n|}\Big(\tilde f_n(E)+\int_{\partial I_n}\sigma(\phi_E,\Pi_E)\nu\otimes(x-x_n)\Big).$$

Letting $R\uparrow\infty$, the ergodic theorem then entails

$$\mathbb{E}[K_E] = \mathbb{E}\bigg[\sum_n\tfrac{\mathbb{1}_{I_n}}{|I_n|}\Big(\tilde f_n(E)+\int_{\partial I_n}\sigma(\phi_E,\Pi_E)\nu\otimes(\cdot-x_n)\Big)\bigg].$$

Since K_E is symmetric, taking the symmetric part of this identity yields (2.12) in combination with the skew-symmetry of $\tilde f_n(E)$ and the definition (1.24) of $\bar{\boldsymbol{C}}(E)$ and $\bar{\boldsymbol{c}}(E)$.

Step 2.2. Proof of (1.26).
We start from (1.24), projected in some direction $E' \in \mathbb{M}_0^{\rm sym}$,

$$2E' : \bar{\boldsymbol{C}}(E) \,=\, \mathbb{E}\bigg[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{\partial I_n} E'(x-x_n)\cdot\sigma(\phi_E,\Pi_E)\nu\bigg],$$

which we reformulate using the ergodic theorem as

$$2E' : \bar{\boldsymbol{C}}(E) \,=\, \lim_{R\uparrow\infty} |B_R|^{-1}\sum_n \int_{\partial I_n} \eta_R E'(x-x_n)\cdot\sigma(\phi_E,\Pi_E)\nu, \tag{2.27}$$

with η_R as above. We turn to a suitable reformulation of the right-hand side. Adding and subtracting the passive corrector $\psi_{E'}$, we can write

$$\begin{aligned}\sum_n \int_{\partial I_n} &\eta_R E'(x-x_n)\cdot\sigma(\phi_E,\Pi_E)\nu\\ &= \sum_n \int_{\partial I_n} \eta_R(\psi_{E'}+E'(x-x_n))\cdot\sigma(\phi_E,\Pi_E)\nu - \sum_n\int_{\partial I_n}\eta_R\psi_{E'}\cdot\sigma(\phi_E,\Pi_E)\nu. \end{aligned}\tag{2.28}$$

For the first right-hand side term, we use that $\psi_{E'}+E'(x-x_n)$ is a rigid motion in I_n, cf. (2.1), we appeal to boundary conditions for (ϕ_E,Π_E) in (2.4), and we recall the notation (1.8), which leads us to

$$\sum_n \int_{\partial I_n} \eta_R(\psi_{E'}+E'(x-x_n))\cdot\sigma(\phi_E,\Pi_E)\nu \,=\, -\sum_n\int_{I_n}\eta_R(\psi_{E'}+E'(x-x_n))\cdot f_n(E).$$

In order to reformulate the second right-hand side term of (2.28), we appeal to the weak formulation of equation (2.4) for ϕ_E: testing this equation with $\eta_R\psi_{E'}$ yields, as in (2.7),

$$\begin{aligned}\int_{\mathbb{R}^d}\nabla(\eta_R\psi_{E'}):\nabla\phi_E - \int_{\mathbb{R}^d}\nabla\eta_R\cdot\psi_{E'}\Pi_E \,=\, &\sum_n\int_{(I_n+B)\setminus I_n}\eta_R\psi_{E'}\cdot f_n(E)\\ &-\sum_n\int_{\partial I_n}\eta_R\psi_{E'}\cdot\sigma(\phi_E,\Pi_E)\nu.\end{aligned}$$

Hence, (2.28) turns into

$$\begin{aligned}\sum_n \int_{\partial I_n} \eta_R E'(x-x_n)\cdot\sigma(\phi_E,\Pi_E)\nu \,=\, &-\sum_n\int_{I_n}\eta_R(\psi_{E'}+E'(x-x_n))\cdot f_n(E)\\ -\int_{\mathbb{R}^d}\nabla\eta_R\cdot\psi_{E'}\Pi_E &+\int_{\mathbb{R}^d}\nabla(\eta_R\psi_{E'}):\nabla\phi_E - \sum_n\int_{(I_n+B)\setminus I_n}\eta_R\psi_{E'}\cdot f_n(E),\end{aligned}$$

and thus, after reorganizing the terms, using that the skew-symmetry of $\tilde f_n(E)$ in (1.8) yields $\int_{I_n}E'(x-x_n)\cdot f_n(E)=0$,

$$\begin{aligned}\sum_n \int_{\partial I_n} \eta_R E'(x-x_n)\cdot\sigma(\phi_E,\Pi_E)\nu \,=\, &\int_{\mathbb{R}^d}\nabla(\eta_R\psi_{E'}):\nabla\phi_E\\ &-\int_{\mathbb{R}^d}\nabla\eta_R\cdot\psi_{E'}\Pi_E - \sum_n\int_{I_n+B}\eta_R\psi_{E'}\cdot f_n(E).\end{aligned}$$

Alternatively, expanding the first right-hand side term,

$$\sum_n \int_{\partial I_n} \eta_R E'(x-x_n)\cdot\sigma(\phi_E,\Pi_E)\nu \;=\; \int_{\mathbb{R}^d} \nabla\psi_{E'} : \nabla(\eta_R\phi_E) - \int_{\mathbb{R}^d} \nabla\eta_R\cdot\psi_{E'}\Pi_E \\ + \int_{\mathbb{R}^d} (\psi_{E'}\otimes\nabla\eta_R):\nabla\phi_E - \int_{\mathbb{R}^d}(\phi_E\otimes\nabla\eta_R):\nabla\psi_{E'} - \sum_n \int_{I_n+B} \eta_R\psi_{E'}\cdot f_n(E).$$

Now testing equation (2.1) for $\psi_{E'}$ with $\eta_R\phi_E$, and using that $\eta_R\phi_E$ is rigid in $\mathcal{I}$, we find that the first right-hand side term only yields a term involving Σ_E

$$\sum_n \int_{\partial I_n} \eta_R E'(x-x_n)\cdot\sigma(\phi_E,\Pi_E)\nu \;=\; -\int_{\mathbb{R}^d} \nabla\eta_R\cdot(\psi_{E'}\Pi_E+\phi_E\Sigma_{E'}) \\ + \int_{\mathbb{R}^d} (\psi_{E'}\otimes\nabla\eta_R):\nabla\phi_E - \int_{\mathbb{R}^d}(\phi_E\otimes\nabla\eta_R):\nabla\psi_{E'} - \sum_n \int_{I_n+B} \eta_R\psi_{E'}\cdot f_n(E).$$

Using the sublinearity of $\psi_{E'}$ and ϕ_E and the stationarity of $\nabla\psi_{E'},\nabla\phi_E,\Pi_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}$, cf. Lemmas 2.1 and 2.3, we can now pass to the limit $R\uparrow\infty$ in this identity, and we obtain

$$\lim_{R\uparrow\infty}|B_R|^{-1}\sum_n\int_{\partial I_n}\eta_R E'(x-x_n)\cdot\sigma(\phi_E,\Pi_E)\nu \;=\; -\mathbb{E}\Big[\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{I_n+B}\psi_{E'}\cdot f_n(E)\Big].$$

Inserting this into (2.27), the conclusion (1.26) follows. Finally, the formula for $\bar{\boldsymbol{c}}(E)$ simply follows by taking the trace in (1.24).

Step 3. Construction of θ.
This construction is standard: as L_E is stationary, it follows from stationary calculus, e.g. [26, Chapter 7], that there is a unique solution $\theta_E=(\theta_{E;ijk})_{ijk}\in\mathrm{L}^2(\Omega;H^1_{\mathrm{loc}}(\mathbb{R}^d))$ of

$$-\triangle\theta_{E;ijk} \;=\; \partial_j L_{E;ik}-\partial_k L_{E;ij}, \tag{2.29}$$

such that $\nabla\theta_E$ is stationary, has vanishing expectation, has finite second moment,

$$\|\nabla\theta_E\|_{\mathrm{L}^2(\Omega)}\lesssim\|L_E\|_{\mathrm{L}^2(\Omega)}\lesssim\|K_E\|_{\mathrm{L}^2(\Omega)}+\|\nabla\gamma_E\|_{\mathrm{L}^2(\Omega)},$$

and satisfies the anchoring condition $\fint_B\theta_E=0$. By uniqueness, the skew-symmetry of θ_E follows from the skew-symmetry of the right-hand side of (2.29) with respect to indices j,k. The fact that θ_E is a vector potential for L_E follows from this defining equation as e.g. in [20, Section 3.1]. Finally, the sublinearity of θ_E is a standard property for random fields with stationary gradient and vanishing expectation; see e.g. [26, Chapter 7]. ☐

In view of the internal contribution to the effective viscosity, cf. (1.25), we also define the following stationary field,

$$F_E \;:=\; -\sum_n\tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{I_n+B}f_n(E)\otimes(x-x_n), \tag{2.30}$$

which is such that

$$\bar{\boldsymbol{F}}_E \;=\; \mathbb{E}\left[F_E\right]. \tag{2.31}$$

When considering two-scale expansions, since correctors $\phi_E,\Pi_E,\gamma_E,\theta_E$, fluxes K_E,L_E, and F_E are nonlinear with respect to E, we shall need to consider derivatives of these objects with respect to E. We introduce a general notation for linearized quantities.

Definition 2.6. *Given $E,E',E''\in\mathbb{M}_0^{\mathrm{sym}}$, we use the following notation for directional derivatives of swimming forces $f_n(E)$ in directions E',E'',*

$$\begin{aligned}\partial_{E'}f_n(E) &:= \lim_{h\to0}\tfrac1h\big(f_n(E+hE')-f_n(E)\big),\\ \partial_{E',E''}f_n(E) &:= \lim_{h\to0}\tfrac1h\big(\partial_{E'}f_n(E+hE'')-\partial_{E'}f_n(E)\big).\end{aligned}$$

— First linearized correctors: *We define $(\mathfrak{d}\phi_{E,E'},\mathfrak{d}\Pi_{E,E'})$ as in Lemma 2.3 with the swimming forces $f_n(E)$ replaced by $\partial_{E'}f_n(E)$ (and similarly for $\mathfrak{d}F_{E,E'}$). We then define $\mathfrak{d}\gamma_{E,E'}$ as in Lemma 2.4 and $\mathfrak{d}K_{E,E'},\mathfrak{d}L_{E,E'},\mathfrak{d}\theta_{E,E'}$ as in Lemma 2.5 with $f_n(E)$ replaced by $\partial_{E'}f_n(E)$ and with (ϕ_E,Π_E) replaced by $(\mathfrak{d}\phi_{E,E'},\mathfrak{d}\Pi_{E,E'})$.*

— Second linearized correctors: *We define $(\mathfrak{d}^2\phi_{E,E',E''},\mathfrak{d}^2\Pi_{E,E',E''})$ as in Lemma 2.3 with swimming forces $f_n(E)$ replaced by $\partial_{E',E''}f_n(E)$ (and similarly for $\mathfrak{d}^2F_{E,E',E''}$). We then define $\mathfrak{d}^2\gamma_{E,E',E''}$ as in Lemma 2.4 and $\mathfrak{d}^2K_{E,E',E''},\mathfrak{d}^2L_{E,E',E''},\mathfrak{d}^2\theta_{E,E',E''}$ as in Lemma 2.5 with $f_n(E)$ replaced by $\partial_{E',E''}f_n(E)$ and with (ϕ_E,Π_E) replaced by $(\mathfrak{d}^2\phi_{E,E',E''},\mathfrak{d}^2\Pi_{E,E',E''})$.* $\Diamond$

Finally, we state that the effective maps $\bar{\boldsymbol{C}},\bar{\boldsymbol{c}},\bar{\boldsymbol{F}}$ are smooth, with derivatives given in terms of linearized correctors. The proof, based on Hypothesis 1.3, is straightforward and left to the reader.

Lemma 2.7. *The effective maps $\bar{\boldsymbol{C}},\bar{\boldsymbol{c}},\bar{\boldsymbol{F}}$ are smooth and*

$$|\bar{\boldsymbol{C}}(E)|,|\bar{\boldsymbol{c}}(E)|,|\bar{\boldsymbol{F}}(E)|\lesssim\lambda\langle E\rangle,\qquad |\partial^k\bar{\boldsymbol{C}}(E)|,|\partial^k\bar{\boldsymbol{c}}(E)|,|\partial^k\bar{\boldsymbol{F}}(E)|\lesssim\lambda. \tag{2.32}$$

Moreover, we have for all $E,F\in\mathbb{M}_0^{\mathrm{sym}}$,

$$2\partial_F\bar{\boldsymbol{C}}(E)+\partial_F\bar{\boldsymbol{c}}(E)\,\mathrm{Id}\;=\;\mathbb{E}\bigg[\sum_n\tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{\partial I_n}\sigma(\mathfrak{d}\phi_{E,F},\mathfrak{d}\Pi_{E,F})\nu\otimes_s(x-x_n)\bigg].$$

In particular, in view of (2.12), this yields $\partial_F\mathbb{E}\,[K_E]=\mathbb{E}\,[\mathfrak{d}K_{E,F}]$. $\Diamond$

3. Proof of the homogenization result

This section is devoted to the proof of Theorem 1.7. We quickly establish the well-posedness result of Proposition 1.6 before turning to the analysis of the limit $\varepsilon\downarrow0$. For any map V and domain D, we henceforth use the short-hand notation $V_{\upharpoonright D}:=\fint_D V$.

3.1. Well-posedness of hydrodynamic model.

This section is devoted to the proof of Proposition 1.6. We start by recalling the following standard computation (see e.g. [8]), which we already partly encountered when proving existence of correctors.

Lemma 3.1 (e.g. [8])**.** *If vector fields u,h and a scalar field P satisfy the following relations,*

$$\begin{cases}-\triangle u+\nabla P=h, & \text{in }\mathbb{R}^d\setminus\mathcal{I},\\ \operatorname{div}(u)=0, & \text{in }\mathbb{R}^d\setminus\mathcal{I},\\ \mathrm{D}(u)=0, & \text{in }\mathcal{I},\end{cases}$$

then the following holds in $\mathbb{R}^d$

$$-\triangle u+\nabla(P\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})=h\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}-\sum_n\delta_{\partial I_n}\sigma(u,P)\nu. \tag{3.1}$$

The same is true if $\mathbb{R}^d$ and $\mathcal{I}$ are replaced by U and $\mathcal{I}_\varepsilon(U)$, respectively. $\Diamond$

We turn to the proof of Proposition 1.6 and proceed by a fixed-point argument. Let $\varepsilon>0$ be fixed. Given $v\in W^{1,1}(U)^d$, define $T_\varepsilon(v)\in H^1_0(U)^d$ as the unique solution of the following linear problem, with associated pressure $P_\varepsilon(v)\in \mathrm{L}^2(U\setminus\mathcal{I}_\varepsilon(U))$,

$$\begin{cases} -\triangle T_\varepsilon(v)+\nabla P_\varepsilon(v) & \\ \qquad = h+\frac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)} f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(v)_{|\varepsilon I_n}), & \text{in } U\setminus\mathcal{I}_\varepsilon(U),\\ \operatorname{div}(T_\varepsilon(v))=0, & \text{in } U\setminus\mathcal{I}_\varepsilon(U),\\ \mathrm{D}(T_\varepsilon(v))=0, & \text{in } \mathcal{I}_\varepsilon(U),\\ \int_{\varepsilon\partial I_n}\sigma(T_\varepsilon(v),P_\varepsilon(v))\nu+\frac{\kappa}{\varepsilon}\bar f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(v)_{|\varepsilon I_n})=0, & \forall n\in\mathcal{N}_\varepsilon(U),\\ \int_{\varepsilon\partial I_n}\Theta(x-\varepsilon x_n)\cdot\sigma(T_\varepsilon(v),P_\varepsilon(v))\nu & \\ \qquad\qquad +\kappa\,\Theta:\tilde f_n(\chi_\delta*\mathrm{D}(v)_{|\varepsilon I_n})=0, & \forall\Theta\in\mathbb{M}^{\mathrm{skew}},\forall n\in\mathcal{N}_\varepsilon(U).\end{cases}\tag{3.2}$$

We split the proof into three steps. We start by proving the result under the stronger smallness condition $\kappa\ell^{-d/2}\ll 1$, before relaxing it to (1.18).

Step 1. Suboptimal contraction estimate: for all $v,w\in H^1_0(U)^d$ we have

$$\int_U|\nabla(T_\varepsilon(v)-T_\varepsilon(w))|^2\lesssim_\chi(\kappa\ell^{-\frac d2})^2\int_U|\nabla(v-w)|^2.\tag{3.3}$$

This proves that T_ε is a contraction on $H^1_0(U)^d$ provided that $\kappa\ell^{-\frac d2}\ll 1$ is small enough. Under this condition, we deduce the well-posedness of the hydrodynamic model (1.7) in $H^1_0(U)^d$.

We turn to the proof of (3.3). For abbreviation, we set $T_\varepsilon(v,w):=T_\varepsilon(v)-T_\varepsilon(w)$ and $P_\varepsilon(v,w)=P_\varepsilon(v)-P_\varepsilon(w)$. Testing the equations for $T_\varepsilon(v)$ and $T_\varepsilon(w)$ with $T_\varepsilon(v,w)$ in form of (3.1), we find

$$\begin{aligned}\int_U|\nabla T_\varepsilon(v,w)|^2 &= -\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon\partial I_n}T_\varepsilon(v,w)\cdot\sigma(T_\varepsilon(v,w),P_\varepsilon(v,w))\nu\\ &\quad+\frac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon(I_n+B)\setminus\varepsilon I_n}T_\varepsilon(v,w)\cdot\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(v)_{|\varepsilon I_n})-f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(w)_{|\varepsilon I_n})\Big),\end{aligned}$$

and thus, using the rigidity of $T_\varepsilon(v,w)$ in εI_n and using boundary conditions, recalling the notation (1.8),

$$\int_U|\nabla T_\varepsilon(v,w)|^2=\frac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon(I_n+B)}T_\varepsilon(v,w)\cdot\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(v)_{|\varepsilon I_n})-f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(w)_{|\varepsilon I_n})\Big).$$

By the neutrality condition (1.9), appealing to Poincaré's inequality, using the hardcore assumption, and recalling that $f_{n,\varepsilon}$ is Lipschitz, we get

$$\int_U|\nabla T_\varepsilon(v,w)|^2\lesssim\kappa\Big(\int_U|\nabla T_\varepsilon(v,w)|^2\Big)^{\frac12}\Big(\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon I_n}|\chi_\delta*\mathrm{D}(v-w)|^2\Big)^{\frac12}.$$

By Jensen's inequality, with $\int_{\mathbb{R}^d}\chi_\delta=1$, the second factor is bounded by

$$\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon I_n}|\chi_\delta*\mathrm{D}(v-w)|^2\lesssim\int_U\Big(\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon I_n}\chi_\delta(x-y)\,dx\Big)|\mathrm{D}(v-w)(y)|^2\,dy.$$

The expression into brackets can be estimated as follows,

$$\begin{aligned}\sup_{y\in\mathbb{R}^d}\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon I_n}\chi_\delta(x-y)\,dx &\le \sup_{y\in\mathbb{R}^d}\sum_n\int_{\frac{\varepsilon}{\delta}I_n}\chi(x-y)\,dx\\ &\lesssim (\tfrac{\varepsilon}{\delta})^d\sup_{y\in\mathbb{R}^d}\sum_n\sup_{y+\frac{\varepsilon}{\delta}I_n}\chi\\ &\le (\tfrac{\varepsilon}{\delta})^d\sup_{y\in\mathbb{R}^d}\sum_n\sup_{\frac{\varepsilon}{\delta}B_\ell(y+x_n)}\chi\\ &\lesssim \ell^{-d}\sup_{y\in\mathbb{R}^d}\sum_n\int_{\frac{\varepsilon}{\delta}B_\ell(y+x_n)}\Big(\sup_{B_{2\ell\frac{\varepsilon}{\delta}}(z)}\chi\Big)\,dz,\end{aligned}$$

and thus, by the hardcore assumption, provided $\varepsilon\ell\le\delta$,

$$\sup_{y\in\mathbb{R}^d}\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon I_n}\chi_\delta(x-y)\,dx\,\lesssim\,\ell^{-d}\int_{\mathbb{R}^d}\Big(\sup_{B_2(z)}\chi\Big)\,dz.$$

Inserting this into the above, we get

$$\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon I_n}|\chi_\delta*\mathrm{D}(v-w)|^2\,\lesssim_\chi\,\ell^{-d}\int_U|\nabla(v-w)|^2, \tag{3.4}$$

and thus,

$$\int_U|\nabla T_\varepsilon(v,w)|^2\,\lesssim_\chi\,\kappa\ell^{-\frac{d}{2}}\Big(\int_U|\nabla T_\varepsilon(v,w)|^2\Big)^{\frac12}\Big(\int_U|\nabla(v-w)|^2\Big)^{\frac12},$$

that is, (3.3).

Step 2. Improved contraction estimate: given $1<p\le2$ and given $\ell\gg_p 1$ large enough, we have for all $v,w\in W_0^{1,p}(U)^d$,

$$\|\nabla(T_\varepsilon(v)-T_\varepsilon(w))\|_{\mathrm{L}^p(U)}\,\lesssim_{p,\chi}\,\kappa\ell^{-\frac{d}{p}}\|\nabla(v-w)\|_{\mathrm{L}^p(U)}.$$

This proves that T_ε is a contraction on $W_0^{1,p}(U)^d$ provided that $\kappa\ell^{-\frac{d}{p}}\ll1$ is small enough, which then implies the well-posedness of the hydrodynamic model (1.7) in $W_0^{1,p}(U)^d$.

We appeal to the dilute deterministic L^p regularity theory developed by Höfer in [25]. Given $1<p\le2$, provided that $\ell\gg_p1$ is large enough (depending on p and dimension d), it allows us to deduce almost surely,

$$\|\nabla(T_\varepsilon(v)-T_\varepsilon(w))\|_{\mathrm{L}^p(U)}\,\lesssim_p\,\kappa\Big\|\sum_{n\in\mathcal{N}_\varepsilon(U)}\big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(v)_{\nmid\varepsilon I_n})-f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(w)_{\nmid\varepsilon I_n})\big)\Big\|_{\mathrm{L}^p(U)},$$

and thus, using properties of $\{f_{n,\varepsilon}\}_n$,

$$\begin{aligned}&\|\nabla(T_\varepsilon(v)-T_\varepsilon(w))\|_{\mathrm{L}^p(U)}\\ &\lesssim_p\ \kappa\Big(\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon(I_n+B)}\big|f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(v)_{\nmid\varepsilon I_n})-f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(w)_{\nmid\varepsilon I_n})\big|^p\Big)^{\frac1p}\\ &\lesssim\ \kappa\Big(\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon I_n}|\chi_\delta*\mathrm{D}(v-w)|^p\Big)^{\frac1p}.\end{aligned}$$

Repeating the argument in (3.4), this proves the claim.

Step 3. Conclusion.
Testing equation (3.2) with its solution $T_\varepsilon(v)$ itself, using boundary conditions and recalling the notation (1.8), we find

$$\int_U |\nabla T_\varepsilon(v)|^2 = \int_{U\setminus\mathcal{I}_\varepsilon(U)} h\cdot T_\varepsilon(v) + \tfrac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon(I_n+B)} T_\varepsilon(v)\cdot f_{n,\varepsilon}(\chi_\delta * \mathrm{D}(v)_{|\varepsilon I_n}),$$

and thus, by the neutrality condition (1.9), appealing to Poincaré's inequality and recalling that $f_{n,\varepsilon}$ is Lipschitz, arguing exactly as in Step 1,

$$\int_U |\nabla T_\varepsilon(v)|^2 \lesssim \kappa^2\ell^{-d} + \int_{U\setminus\mathcal{I}_\varepsilon(U)} |h|^2 + \kappa^2 \sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon I_n} |\chi_\delta * \mathrm{D}(v)|^2.$$

Now using the hardcore condition and Young's inequality, we deduce for all $1<p\le 2$,

$$\int_U |\nabla T_\varepsilon(v)|^2 \lesssim \kappa^2\ell^{-d} + \int_{U\setminus\mathcal{I}_\varepsilon(U)} |h|^2 + \kappa^2\|\chi_\delta\|^2_{\mathrm{L}^1\cap\mathrm{L}^2(U)}\|\nabla v\|^2_{\mathrm{L}^p(U)}.$$

This proves that T_ε embeds $W^{1,p}_0(U)^d$ into $H^1_0(U)^d$, so that the solution $u_\varepsilon = T_\varepsilon(u_\varepsilon)$ in $W^{1,p}_0(U)^d$ constructed in Step 2 actually belongs to $H^1_0(U)^d$, and the a priori estimate (1.19) follows by iteration. ☐

3.2. **Qualitative homogenization at fixed δ.** This section is devoted to the proof of Theorem 1.7. Before turning to the proof, we argue for the well-posedness of the homogenized equation (1.20). Using $|\bar{\boldsymbol{C}}(E)|, |\bar{\boldsymbol{F}}(E)| \lesssim \lambda\langle E\rangle$, cf. (2.32), a perturbative argument as in the proof of Proposition 1.6 yields the well-posedness of (1.20) with $(\bar u_\delta, \bar P_\delta) \in H^1_0(U)^d\times \mathrm{L}^2(U)/\mathbb{R}$ provided that $\kappa\lambda \ll 1$ is small enough. As $\lambda \lesssim \ell^{-d}$, this holds in particular under the smallness condition (1.18).

Interestingly, our proof of qualitative homogenization relies on a semi-quantitative two-scale analysis and we do not believe that there exists a simpler and purely qualitative proof. In terms of a suitable limiting profile $\bar u_\varepsilon$ (mildly depending on ε and to be identified at the end of the proof), we consider the following two-scale expansions for the solution $(u_\varepsilon, P_\varepsilon)$ of the hydrodynamic model (1.7),

$$\begin{aligned} u_\varepsilon &\rightsquigarrow \bar u_\varepsilon + \varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\partial_E \bar u_\varepsilon + \varepsilon\kappa\phi_{\chi_\delta * \mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon}), \\ P_\varepsilon \mathbb{1}_{\mathbb{R}^d\setminus\varepsilon\mathcal{I}} &\rightsquigarrow \bar P_\varepsilon + \bar{\boldsymbol{b}} : \mathrm{D}(\bar u_\varepsilon) + \kappa\bar{\boldsymbol{c}}(\chi_\delta * \mathrm{D}(u_\varepsilon)) \\ &\qquad + (\Sigma_E \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\partial_E\bar u_\varepsilon + (\kappa\Pi_{\chi_\delta * \mathrm{D}(u_\varepsilon)(x)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{x}{\varepsilon}), \end{aligned} \tag{3.5}$$

where u_ε is implicitely extended by 0 on $\mathbb{R}^d\setminus U$ to ensure that $\chi_\delta * \mathrm{D}(u_\varepsilon)$ is well-defined, and where we implicitly sum over E in an orthonormal basis of $\mathbb{M}_0^{\mathrm{sym}}$. We start with two comments on the form of these two-scale expansions:

— The most unusual feature is that correctors ϕ, Π are evaluated at a background fluid deformation $\chi_\delta * \mathrm{D}(u_\varepsilon)$ depending on the microscopic solution u_ε. This non-standard choice is taken as an intermediate step and happens to be providential in our proof, where the limit of this background deformation $\chi_\delta * \mathrm{D}(u_\varepsilon) \sim \chi_\delta * \mathrm{D}(\bar u_\varepsilon)$ can only be identified at the very end. To our knowledge, the necessity of such a two-step homogenization argument is new to the literature and gives the present problem an interest of its own.

— The choice of the non-oscillating part $\bar P_\varepsilon + \bar{\boldsymbol{b}} : \mathrm{D}(\bar u_\varepsilon) + \kappa\bar{\boldsymbol{c}}(\chi_\delta * \mathrm{D}(u_\varepsilon))$ in the two-scale expansion of the pressure is dictated by the proof and is similar to the case of passive suspensions [8]: the pressure for (1.7) does not converge to the pressure of the homogenized problem in its naïve form.

As $\chi_\delta * \mathrm{D}(u_\varepsilon)$ is smooth, the maps $(x,y) \mapsto (\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)(x)})(y), (\Pi_{\chi_\delta*\mathrm{D}(u_\varepsilon)(x)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(y)$ are Carathéodory functions, which ensures that $x \mapsto \phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)(x)}(\frac{x}{\varepsilon}), (\Pi_{\chi_\delta*\mathrm{D}(u_\varepsilon)(x)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\frac{x}{\varepsilon})$ in (3.5) are measurable. When differentiating such composed functions, some care is needed in the notation. Given a smooth field $V : \mathbb{R}^d \to \mathbb{M}_0^{\mathrm{sym}}$, we denote by ϕ_V the function of two variables $(x,y) \mapsto \phi_{V(x)}(y)$, and we use the short-hand notation $\phi_V(\frac{x}{\varepsilon}) := \phi_{V(x)}(\frac{x}{\varepsilon})$. We use the following notation for the derivative of ϕ_V with V frozen,

$$(\nabla\phi_{|V})(\tfrac{x}{\varepsilon}) \,:=\, \nabla_y\phi_{V(x)}|_{y=\frac{x}{\varepsilon}},$$

so that the total derivative is then given by

$$\nabla(\varepsilon\phi_V)(\tfrac{x}{\varepsilon}) \,=\, (\nabla\phi_{|V})(\tfrac{x}{\varepsilon}) + \varepsilon\mathfrak{d}\phi_{V(x),E}(\tfrac{x}{\varepsilon}) \otimes \nabla(E : V(x)), \tag{3.6}$$

where the linearized corrector $\mathfrak{d}\phi$ is given in Definition 2.6 and where we implicitly sum over E in an orthonormal basis of $\mathbb{M}_0^{\mathrm{sym}}$. In addition, if V is a smooth random field, we use the following notation for the expectation of ϕ_V with V frozen: for any random field a,

$$\mathbb{E}[a(x)\phi_{|V}(\tfrac{x}{\varepsilon})] \,:=\, \mathbb{E}\left[a(x)\phi_E(\tfrac{x}{\varepsilon})\right]|_{E=V(x)}.$$

The same notation is used for other correctors and fluxes.

As usual, correctors are not capturing the relevant behavior close to the boundary of the domain U: in particular, the two-scale expansion in (3.5) does not vanish at the boundary. To circumvent this, we proceed by truncating correctors in a neighborhood of the boundary. We set for abbreviation

$$\partial_r U := \{x \in U : \mathrm{dist}(x, \partial U) < r\},$$

and, given $r_\varepsilon \geq 4\varepsilon$ (which we shall optimize later on), we choose a smooth cut-off function $\eta_\varepsilon \in C_c^\infty(U; [0,1])$ such that

$$\eta_\varepsilon|_{U\setminus\partial_{2r_\varepsilon}U} = 1, \qquad \eta_\varepsilon|_{\partial_{r_\varepsilon}U} = 0, \qquad |\nabla\eta_\varepsilon| \lesssim \tfrac{1}{r_\varepsilon},$$

and such that η_ε is constant inside each of the fattened particles $\{\varepsilon(I_n + \frac12 B)\}_n$. Note that by definition the set $\mathcal{I}_\varepsilon(U)$ coincides with $\varepsilon\mathcal{I}$ on the support of η_ε,

$$\eta_\varepsilon\mathbb{1}_{\mathcal{I}_\varepsilon(U)} = \eta_\varepsilon\mathbb{1}_{\varepsilon\mathcal{I}}. \tag{3.7}$$

In these terms, truncating the two-scale expansions (3.5), we are led to considering the following truncated two-scale expansion errors,

$$w_\varepsilon \,:=\, u_\varepsilon - u_\varepsilon^{2s}, \qquad Q_\varepsilon \,:=\, P_\varepsilon\mathbb{1}_{U\setminus\mathcal{I}_\varepsilon(U)} - P_\varepsilon^{2s},$$

where the two-scale expansion $(u_\varepsilon^{2s}, P_\varepsilon^{2s})$ of $(u_\varepsilon, P_\varepsilon)$ is given by

$$\begin{aligned} u_\varepsilon^{2s}(x) \;&:=\; \bar u_\varepsilon(x) + \varepsilon\eta_\varepsilon(x)\psi_E(\tfrac{x}{\varepsilon})\partial_E\bar u_\varepsilon(x) + \varepsilon\kappa\eta_\varepsilon(x)\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{x}{\varepsilon}), \\ P_\varepsilon^{2s}(x) \;&:=\; -P_\varepsilon^*(x) + \bar P_\varepsilon(x) + \eta_\varepsilon(x)\bar{\boldsymbol{b}} : \mathrm{D}(\bar u_\varepsilon)(x) + \kappa\eta_\varepsilon(x)\bar{\boldsymbol{c}}(\chi_\delta * \mathrm{D}(u_\varepsilon)(x)) \\ &\qquad + \eta_\varepsilon(x)(\Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{x}{\varepsilon})(\partial_E\bar u_\varepsilon)(x) + \kappa\eta_\varepsilon(x)(\Pi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{x}{\varepsilon}), \end{aligned} \tag{3.8}$$

where we have further added a locally constant pressure field

$$P_\varepsilon^* \,:=\, P_\varepsilon'\mathbb{1}_{U\setminus\mathcal{I}_\varepsilon(U)} + \sum_{n\in\mathcal{N}_\varepsilon(U)} P_{\varepsilon,n}''\mathbb{1}_{\varepsilon I_n}, \tag{3.9}$$

in terms of some constants P'_ε and $\{P''_{\varepsilon,n}\}_n$ to be suitably chosen later on, cf. (3.44), and where the limiting profile $(\bar u_\varepsilon, \bar P_\varepsilon) \in H^1_0(U)^d \times \mathrm{L}^2(U)/\mathbb{R}$ is chosen as the unique solution of

$$-\operatorname{div}(2\bar{\boldsymbol{B}}_{\mathrm{pas}}\,\mathrm{D}(\bar u_\varepsilon)) + \nabla \bar P_\varepsilon \;=\; (1-\lambda)h + \operatorname{div}(2\kappa\bar{\boldsymbol{B}}_{\mathrm{act}}(\chi_\delta * \mathrm{D}(u_\varepsilon))). \tag{3.10}$$

Again, this equation is viewed as a convenient intermediate step towards the relevant homogenized equation (1.20): as in the two-scale expansion (3.8), the background fluid deformation $\chi_\delta * \mathrm{D}(u_\varepsilon)$ is expressed here in terms of the microscopic solution u_ε itself and its limit will only be identified at the very end of the proof. We split the proof into three main steps.

Step 1. Equation for the two-scale expansion error $(w_\varepsilon, Q_\varepsilon)$: in the weak sense in U,

$$\begin{aligned}
-\triangle w_\varepsilon + \nabla Q_\varepsilon \;=\; &-\operatorname{div}\Big((J_E\mathbb{1}_{\mathcal I})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon + \kappa(K_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathcal I})(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\Big)\\
&-\sum_{n\in\mathcal N_\varepsilon(U)}\Big(\delta_{\varepsilon\partial I_n}\sigma(u_\varepsilon, P_\varepsilon + P'_\varepsilon - P''_{\varepsilon,n})\nu + \tfrac{\kappa}{\varepsilon}f_{n,\varepsilon}\big(\chi_\delta * \mathrm{D}(u_\varepsilon)_{|\varepsilon I_n}\big)\mathbb{1}_{\varepsilon I_n}\Big)\\
-\tfrac{\kappa}{\varepsilon}\sum_{n\in\mathcal N_\varepsilon(U)}&\Big(\eta_\varepsilon f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon)) - f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{|\varepsilon I_n}\big)\Big) - \kappa\mathfrak{d}F_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\\
&+(1-\eta_\varepsilon)(\lambda - \mathbb{1}_{\mathcal I_\varepsilon(U)})h - \kappa\bar{\boldsymbol F}(\chi_\delta*\mathrm{D}(u_\varepsilon))\nabla\eta_\varepsilon\\
&-\operatorname{div}\Big((1-\eta_\varepsilon)\big(2(\bar{\boldsymbol B}_{\mathrm{pas}} - \mathrm{Id})\,\mathrm{D}(\bar u_\varepsilon) + 2\kappa\bar{\boldsymbol B}_{\mathrm{act}}(\chi_\delta*\mathrm{D}(u_\varepsilon))\big)\Big)\\
+\varepsilon\operatorname{div}\Big(&\big(2\psi_E\otimes_s - \mathrm{Id}\otimes\psi_E - \Upsilon_E\big)(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon) - \eta_\varepsilon h\otimes(\triangle^{-1}\nabla\mathbb{1}_{\mathcal I})(\tfrac{\cdot}{\varepsilon})\\
&+\kappa\big(2\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\otimes_s - \mathrm{Id}\otimes\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)} - \theta_{\chi_\delta*\mathrm{D}(u_\varepsilon)} + \gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\nabla\eta_\varepsilon\\
&+\kappa\big(2\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes_s - \mathrm{Id}\otimes\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E} - \mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\\
&\qquad+\mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\Big)\\
&+\kappa\varepsilon\nabla_i\Big((\triangle^{-1}\nabla_i\mathfrak{d}F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\Big)\\
+\varepsilon(\triangle^{-1}\nabla_j\mathbb{1}_{\mathcal I})(\tfrac{\cdot}{\varepsilon})\nabla_j(\eta_\varepsilon h) &- \kappa\varepsilon\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\triangle\eta_\varepsilon - \kappa\varepsilon\mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon}) : \eta_\varepsilon\triangle(\chi_\delta*\partial_E u_\varepsilon)\\
&-\kappa\varepsilon(\triangle^{-1}\nabla_i\mathfrak{d}F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E})(\tfrac{\cdot}{\varepsilon})\nabla_i(\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon))\\
&+2\kappa\varepsilon\big(\mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E} - \mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon}) : \nabla(\chi_\delta*\partial_E u_\varepsilon)\otimes\nabla\eta_\varepsilon\\
&+\kappa\varepsilon\big(\mathfrak{d}^2\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'} - \mathfrak{d}^2\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'}\otimes\mathrm{Id} - \triangle^{-1}\nabla\mathfrak{d}^2F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'})\big)(\tfrac{\cdot}{\varepsilon})\\
&\qquad : \eta_\varepsilon\nabla(\chi_\delta*\partial_{E'}u_\varepsilon)\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon).
\end{aligned}\tag{3.11}$$

We split the proof into six substeps.

Substep 1.1. Reformulation of the equation for u_ε:

$$\begin{aligned}
-\triangle u_\varepsilon + \nabla(P_\varepsilon\mathbb{1}_{U\setminus\mathcal I_\varepsilon(U)} + P^*_\varepsilon) \;=\; &h\mathbb{1}_{U\setminus\mathcal I_\varepsilon(U)} + \tfrac{\kappa}{\varepsilon}\sum_{n\in\mathcal N_\varepsilon(U)} f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{|\varepsilon I_n}\big)\\
&-\sum_{n\in\mathcal N_\varepsilon(U)}\Big(\delta_{\varepsilon\partial I_n}\sigma(u_\varepsilon, P_\varepsilon + P'_\varepsilon - P''_{\varepsilon,n})\nu + \tfrac{\kappa}{\varepsilon}f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{|\varepsilon I_n}\big)\mathbb{1}_{\varepsilon I_n}\Big).
\end{aligned}\tag{3.12}$$

Starting from equation (1.7) in form of (3.1), we find

$$-\triangle u_\varepsilon + \nabla(P_\varepsilon\mathbb{1}_{U\setminus\mathcal I_\varepsilon(U)})$$

$$= h\mathbb{1}_{U\setminus\mathcal{I}_\varepsilon(U)} + \tfrac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)} f_{n,\varepsilon}\big(\chi_\delta * \mathrm{D}(u_\varepsilon)_{|\varepsilon I_n}\big)\mathbb{1}_{\varepsilon(I_n+B)\setminus\varepsilon I_n} - \sum_{n\in\mathcal{N}_\varepsilon(U)} \delta_{\varepsilon\partial I_n}\sigma(u_\varepsilon,P_\varepsilon)\nu.$$

Adding and subtracting the contribution of swimming forces on the particles, this becomes

$$\begin{aligned} -\triangle u_\varepsilon + \nabla(P_\varepsilon\mathbb{1}_{U\setminus\mathcal{I}_\varepsilon(U)}) \;=\; & h\mathbb{1}_{U\setminus\mathcal{I}_\varepsilon(U)} + \tfrac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)} f_{n,\varepsilon}\big(\chi_\delta * \mathrm{D}(u_\varepsilon)_{|\varepsilon I_n}\big) \\ & - \sum_{n\in\mathcal{N}_\varepsilon(U)}\Big(\delta_{\varepsilon\partial I_n}\sigma(u_\varepsilon,P_\varepsilon)\nu + \tfrac{\kappa}{\varepsilon}f_{n,\varepsilon}\big(\chi_\delta * \mathrm{D}(u_\varepsilon)_{|\varepsilon I_n}\big)\mathbb{1}_{\varepsilon I_n}\Big). \end{aligned}$$

Adding ∇P_ε^* to both sides, cf. (3.9), the claim (3.12) follows.

Substep 1.2. Equation for the two-scale expansion for the two-scale expansion error u_ε^{2s}:

$$\begin{aligned} -\triangle u_\varepsilon^{2s} + \nabla(P_\varepsilon^{2s}+P_\varepsilon^*) \;=\; & \nabla\Big(\bar P_\varepsilon + \eta_\varepsilon\bar{\boldsymbol{b}} : \mathrm{D}(\bar u_\varepsilon) + \kappa\eta_\varepsilon\bar{\boldsymbol{c}}(\chi_\delta * \mathrm{D}(u_\varepsilon))\Big) \\ & - T_\varepsilon^1 - \kappa T_\varepsilon^2 - \operatorname{div}\big(2(1-\eta_\varepsilon)\,\mathrm{D}(\bar u_\varepsilon)\big) \\ & - \varepsilon\operatorname{div}\Big(2\psi_E(\tfrac{\cdot}{\varepsilon})\otimes_s\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon) + 2\kappa\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\otimes_s\nabla\eta_\varepsilon \\ & \qquad + 2\kappa\boldsymbol{\partial}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E_\alpha}(\tfrac{\cdot}{\varepsilon})\otimes_s\eta_\varepsilon\nabla(\chi_\delta*\partial_{E_\alpha}u_\varepsilon)\Big) \\ & + \varepsilon\nabla\Big(\psi_E(\tfrac{\cdot}{\varepsilon})\cdot\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon) + \kappa\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\cdot\nabla\eta_\varepsilon \\ & \qquad + \kappa\boldsymbol{\partial}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E_\alpha}(\tfrac{\cdot}{\varepsilon})\cdot\eta_\varepsilon\nabla(\chi_\delta*\partial_{E_\alpha}u_\varepsilon)\Big), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} T_\varepsilon^1 &:= \operatorname{div}\Big(\big(2(\mathrm{D}(\psi_E)+E) - \Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\,\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\partial_E\bar u_\varepsilon\Big), \\ T_\varepsilon^2 &:= \operatorname{div}\Big(\big(2\,\mathrm{D}(\phi_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}) - \Pi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\,\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\Big) \end{aligned}$$

are two terms that we shall further reformulate in the upcoming substeps.

For the two-scale expansion error $(u_\varepsilon^{2s},P_\varepsilon^{2s})$ defined in (3.8), we have

$$\begin{aligned} -\triangle u_\varepsilon^{2s} + \nabla(P_\varepsilon^{2s}+P_\varepsilon^*) \;=\; & -\triangle\bar u_\varepsilon + \nabla\Big(\bar P_\varepsilon + \eta_\varepsilon\bar{\boldsymbol{b}} : \mathrm{D}(\bar u_\varepsilon) + \kappa\eta_\varepsilon\bar{\boldsymbol{c}}(\chi_\delta*\mathrm{D}(u_\varepsilon))\Big) \\ & - \triangle\Big(\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon + \varepsilon\kappa\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\Big) \\ & + \nabla\Big((\Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon + \kappa(\Pi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\Big). \end{aligned} \tag{3.14}$$

It remains to reformulate the penultimate right-hand side term. By the identity

$$\triangle h = \operatorname{div}(2\,\mathrm{D}(h)) - \nabla\operatorname{div}(h),$$

we can write

$$\begin{aligned} \triangle\big(\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon\big) &= \operatorname{div}\big(2\,\mathrm{D}(\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon)\big) - \nabla\operatorname{div}\big(\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon\big), \\ \triangle\big(\varepsilon\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\big) &= \operatorname{div}\big(2\,\mathrm{D}(\varepsilon\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon)\big) - \nabla\operatorname{div}\big(\varepsilon\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\big). \end{aligned}$$

First, as $\operatorname{div}(\psi_E)=0$, we find

$$\begin{aligned} \mathrm{D}(\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon) &= \mathrm{D}(\psi_E)(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon + \varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\otimes_s\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon), \\ \operatorname{div}(\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon) &= \varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\cdot\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon). \end{aligned} \tag{3.15}$$

Second, as $\operatorname{div}(\phi_E)=0$, further recalling (3.6), we find

$$\begin{aligned}\mathrm{D}(\varepsilon\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon) &= \mathrm{D}(\phi_{|\chi_\delta*\mathrm{D}(u_\varepsilon)})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon+\varepsilon\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\otimes_s\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\\ &\qquad+\varepsilon\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\otimes_s\nabla\eta_\varepsilon,\\ \operatorname{div}(\varepsilon\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon) &= \varepsilon\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\cdot\nabla\eta_\varepsilon\\ &\qquad+\varepsilon\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\cdot\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon),\end{aligned}\tag{3.16}$$

where we recall that we implicitly sum over E in an orthonormal basis of $\mathbb{M}_0^{\mathrm{sym}}$, and where the linearized corrector $\mathfrak{d}\phi$ is given in Definition 2.6. In these terms, the penultimate right-hand side term in (3.14) takes the form

$$\begin{aligned}&\triangle\Big(\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon+\varepsilon\kappa\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\Big)\\ &\quad= \operatorname{div}\Big(2\,\mathrm{D}(\psi_E)(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon+2\kappa\,\mathrm{D}(\phi_{|\chi_\delta*\mathrm{D}(u_\varepsilon)})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon+2\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\otimes_s\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon)\\ &\qquad\qquad+2\varepsilon\kappa\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\otimes_s\nabla\eta_\varepsilon+2\varepsilon\kappa\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E_\alpha}(\tfrac{\cdot}{\varepsilon})\otimes_s\eta_\varepsilon\nabla(\chi_\delta*\partial_{E_\alpha}u_\varepsilon)\Big)\\ &\quad\quad-\nabla\Big(\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\cdot\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon)+\varepsilon\kappa\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\cdot\nabla\eta_\varepsilon\\ &\qquad\qquad\qquad+\varepsilon\kappa\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E_\alpha}(\tfrac{\cdot}{\varepsilon})\cdot\eta_\varepsilon\nabla(\chi_\delta*\partial_{E_\alpha}u_\varepsilon)\Big).\end{aligned}$$

Inserting this identity into (3.14), using that $\operatorname{div}(\bar u_\varepsilon)=0$, and decomposing

$$\triangle\bar u_\varepsilon=\operatorname{div}(2\,\mathrm{D}(\bar u_\varepsilon))=\operatorname{div}\big(2(1-\eta_\varepsilon)\,\mathrm{D}(\bar u_\varepsilon)\big)+\operatorname{div}(2E\eta_\varepsilon\partial_E\bar u_\varepsilon),$$

the claim (3.13) follows.

Substep 1.3. Proof of

$$\begin{aligned}T_\varepsilon^1&=\operatorname{div}\Big(\big(2(\mathrm{D}(\psi_E)+E)-\Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\,\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\partial_E\bar u_\varepsilon\Big)\\ &=\operatorname{div}\big(2\eta_\varepsilon\bar{\boldsymbol{B}}_{\mathrm{pas}}\,\mathrm{D}(\bar u_\varepsilon)\big)+\nabla\big(\eta_\varepsilon\bar{\boldsymbol{b}}:\mathrm{D}(\bar u_\varepsilon)\big)-\varepsilon\operatorname{div}\big(\Upsilon_E(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon)\big)\\ &\qquad-\operatorname{div}\big((J_E\mathbb{1}_\mathcal{I})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon\big).\end{aligned}\tag{3.17}$$

In terms of the extended flux J_E, cf. Lemma 2.2, we have

$$\operatorname{div}\Big(\big(2(\mathrm{D}(\psi_E)+E)-\Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\,\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\partial_E\bar u_\varepsilon\Big)=\operatorname{div}\big((J_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\partial_E\bar u_\varepsilon\big).$$

Recalling that $\operatorname{div}(J_E)=0$, we can decompose

$$\begin{aligned}&\operatorname{div}\Big(\big(2(\mathrm{D}(\psi_E)+E)-\Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\,\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\partial_E\bar u_\varepsilon\Big)\\ &\qquad\qquad=J_E(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon)-\operatorname{div}\big((J_E\mathbb{1}_\mathcal{I})(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\partial_E\bar u_\varepsilon\big).\end{aligned}$$

As Lemma 2.2 further yields $J_E-\mathbb{E}[J_E]=\operatorname{div}(\Upsilon_E)$ and $\mathbb{E}[J_E]=2\bar{\boldsymbol{B}}_{\mathrm{pas}}E+(\bar{\boldsymbol{b}}:E)\,\mathrm{Id}$, where the flux corrector Υ_E is skew-symmetric in its last two indices, the claim (3.17) follows. For completeness, we recall the standard argument based on skew-symmetry of Υ that leads to this identity: for any smooth scalar field ζ, we have

$$\begin{aligned}(J_E-\mathbb{E}[J_E])\nabla\zeta&=\operatorname{div}(\Upsilon_E)\nabla\zeta\\ &=e_i(\nabla_k\Upsilon_{E;ijk})\nabla_j\zeta\\ &=e_i\nabla_k\big(\underbrace{\Upsilon_{E;ijk}}_{=-\Upsilon_{E;ikj}}\nabla_j\zeta\big)-\underbrace{e_i\Upsilon_{E;ijk}\nabla^2_{jk}\zeta}_{=0}\end{aligned}$$

$$= \quad -\mathrm{div}(\Upsilon_E \nabla\zeta). \tag{3.18}$$

Substep 1.4. Proof of

$$\begin{aligned}
T^2_\varepsilon = \mathrm{div}\Big(\big(2\,\mathrm{D}(\phi_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}) - \Pi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\,\mathrm{Id}\,\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\Big)&\\
= \mathrm{div}(2\eta_\varepsilon\bar{\boldsymbol{C}}(\chi_\delta*\mathrm{D}(u_\varepsilon))) + \nabla(\eta_\varepsilon\bar{\boldsymbol{c}}(\chi_\delta*\mathrm{D}(u_\varepsilon)))&\\
-\tfrac{1}{\varepsilon}\eta_\varepsilon\sum_{n\in\mathcal{N}_\varepsilon(U)} f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon)) - \mathrm{div}\big((K_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\big)&\\
-\varepsilon\mathrm{div}\Big(\big(\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon)} - \gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\otimes\mathrm{Id}\,\big)(\tfrac{\cdot}{\varepsilon})\nabla\eta_\varepsilon\Big)&\\
-\varepsilon\mathrm{div}\Big(\big(\mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E} - \mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\,\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\Big)&\\
-\varepsilon\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\triangle\eta_\varepsilon - \varepsilon\mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon}) : \eta_\varepsilon\triangle(\chi_\delta*\partial_E u_\varepsilon)&\\
+2\varepsilon\big(\mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E} - \mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\,\big)(\tfrac{\cdot}{\varepsilon}) : \nabla(\chi_\delta*\partial_E u_\varepsilon)\otimes\nabla\eta_\varepsilon&\\
+\varepsilon\big(\mathfrak{d}^2\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'} - \mathfrak{d}^2\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'}\otimes\mathrm{Id}\,\big)(\tfrac{\cdot}{\varepsilon})&\\
: \eta_\varepsilon\nabla(\chi_\delta*\partial_{E'}u_\varepsilon)\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon).&
\end{aligned}\tag{3.19}$$

In terms of the extended flux K_E, cf. Lemma 2.5, we have

$$\mathrm{div}\Big(\big(2\,\mathrm{D}(\phi_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}) - \Pi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\,\mathrm{Id}\,\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\Big) \;=\; \mathrm{div}\big((K_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\big).$$

As Lemma 2.5 further yields $\mathbb{E}[K_E] = 2\bar{\boldsymbol{C}}(E) + \bar{\boldsymbol{c}}(E)\,\mathrm{Id}$ and $\mathrm{div}(K_E) = -\sum_n f_n(E)$, and appealing to (3.6) and (3.7), we find

$$\begin{aligned}
&\mathrm{div}\Big(\big(2\,\mathrm{D}(\phi_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}) - \Pi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\,\mathrm{Id}\,\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\Big)\\
&\quad= \mathrm{div}(2\eta_\varepsilon\bar{\boldsymbol{C}}(\chi_\delta*\mathrm{D}(u_\varepsilon))) + \nabla(\eta_\varepsilon\bar{\boldsymbol{c}}(\chi_\delta*\mathrm{D}(u_\varepsilon))) - \tfrac{1}{\varepsilon}\eta_\varepsilon\sum_{n\in\mathcal{N}_\varepsilon(U)} f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon))\\
&\qquad-\mathrm{div}\big((K_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\big) + \big(K_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon}) - \mathbb{E}\left[K_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}\right]\big)\nabla\eta_\varepsilon\\
&\qquad\quad+\big(\mathfrak{d}K_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon}) - \mathbb{E}\left[\mathfrak{d}K_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\right]\big)\,\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon).
\end{aligned}\tag{3.20}$$

It remains to reformulate the last two right-hand side terms. As Lemma 2.5 yields

$$K_E - \mathbb{E}[K_E] \;=\; \mathrm{div}(\theta_E) + \nabla\gamma_E,$$

we get

$$\big(K_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon}) - \mathbb{E}\left[K_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}\right]\big)\nabla\eta_\varepsilon \;=\; \big(\mathrm{div}(\theta_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}) + \nabla\gamma_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}\big)(\tfrac{\cdot}{\varepsilon})\nabla\eta_\varepsilon,$$

and thus, by Leibniz' rule, using (3.6) and the skew-symmetry of θ (whence the minus sign of the first right-hand side term, cf. (3.18)),

$$\begin{aligned}
&\big(K_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon}) - \mathbb{E}\left[K_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}\right]\big)\nabla\eta_\varepsilon\\
&\quad= -\varepsilon\mathrm{div}\Big(\big(\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon)} - \gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\otimes\mathrm{Id}\,\big)(\tfrac{\cdot}{\varepsilon})\nabla\eta_\varepsilon\Big) - \varepsilon\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\triangle\eta_\varepsilon\\
&\qquad\quad+\varepsilon\big(\mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E} - \mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\,\big)(\tfrac{\cdot}{\varepsilon}) : \nabla(\chi_\delta*\partial_E u_\varepsilon)\otimes\nabla\eta_\varepsilon.
\end{aligned}$$

Similarly, with the notation of Definition 2.6, we have

$$\big(\mathfrak{d}K_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon}) - \mathbb{E}\left[\mathfrak{d}K_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E}\right]\big)\,\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)$$

$$= \Big(\operatorname{div}(\mathfrak{d}\theta_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E}) + \nabla\mathfrak{d}\gamma_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E}\Big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon),$$

and thus, by Leibniz' rule, using (3.6) and the skew-symmetry of $\mathfrak{d}\theta$,

$$\begin{aligned}
&\Big(\mathfrak{d}K_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon}) - \mathbb{E}\left[\mathfrak{d}K_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E}\right]\Big)\,\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\\
&\quad= -\varepsilon\operatorname{div}\Big(\big(\mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E} - \mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\Big)\\
&\qquad+\varepsilon\big(\mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E} - \mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon}) : \nabla(\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon))\\
&\quad+\varepsilon\big(\mathfrak{d}^2\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'} - \mathfrak{d}^2\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon}) : \eta_\varepsilon\nabla(\chi_\delta*\partial_{E'}u_\varepsilon)\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon).
\end{aligned}$$

Inserting this into (3.20), the claim (3.19) follows.

Substep 1.5. In order to reconstruct $\bar{\boldsymbol{B}}_{\mathrm{act}}(\chi_\delta*\mathrm{D}(u_\varepsilon))$, we shall need the following identity:

$$\begin{aligned}
\operatorname{div}(\eta_\varepsilon\bar{\boldsymbol{F}}(\chi_\delta*\mathrm{D}(u_\varepsilon))) \;&=\; \mathfrak{d}F_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon) + \bar{\boldsymbol{F}}(\chi_\delta*\mathrm{D}(u_\varepsilon))\nabla\eta_\varepsilon\\
&\quad-\varepsilon\nabla_i\Big((\triangle^{-1}\nabla_i\mathfrak{d}F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\Big)\\
&\quad+\varepsilon(\triangle^{-1}\nabla_i\mathfrak{d}F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E})(\tfrac{\cdot}{\varepsilon})\nabla_i(\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon))\\
&\quad+\varepsilon(\triangle^{-1}\nabla_i\mathfrak{d}^2F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\nabla_i(\chi_\delta*\partial_{E'}(u_\varepsilon)).
\end{aligned}\tag{3.21}$$

As $\mathbb{E}\left[F_E\right] = \bar{\boldsymbol{F}}(E)$, cf. (2.31), and using the notation in Definition 2.6, we can decompose

$$\begin{aligned}
\mathfrak{d}F_{E,E'} \;&=\; \partial_{E'}\bar{\boldsymbol{F}}(E) + \big(\mathfrak{d}F_{E,E'} - \mathbb{E}\left[\mathfrak{d}F_{E,E'}\right]\big)\\
&=\; \partial_{E'}\bar{\boldsymbol{F}}(E) + \nabla_i\triangle^{-1}\nabla_i\mathfrak{d}F_{E,E'}.
\end{aligned}$$

Using this together with Leibniz' rule, we find

$$\begin{aligned}
\mathfrak{d}F_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon) \;&=\; \partial_E\bar{\boldsymbol{F}}(\chi_\delta*\mathrm{D}(u_\varepsilon))\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\\
&\quad+\varepsilon\nabla_i\Big((\triangle^{-1}\nabla_i\mathfrak{d}F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E})(\tfrac{\cdot}{\varepsilon})\Big)\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\\
&\quad-\varepsilon(\triangle^{-1}\nabla_i\mathfrak{d}^2F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\nabla_i(\chi_\delta*\partial_{E'}(u_\varepsilon)).
\end{aligned}$$

Further reformulating the first right-hand side term, noting that

$$\partial_E\bar{\boldsymbol{F}}(\chi_\delta*\mathrm{D}(u_\varepsilon))\nabla(\chi_\delta*\partial_E u_\varepsilon) \;=\; \operatorname{div}(\bar{\boldsymbol{F}}(\chi_\delta*\mathrm{D}(u_\varepsilon))),$$

the claim (3.21) follows.

Substep 1.6. Proof of (3.11).
Subtracting (3.12) and (3.13), inserting identities (3.17), (3.19), and (3.21), using equation (3.10) for $\bar{u}_\varepsilon$, and decomposing

$$\begin{aligned}
h\mathbb{1}_{U\setminus\mathcal{I}_\varepsilon(U)} - (1-\lambda)h \;&=\; (\lambda - \mathbb{1}_{\mathcal{I}_\varepsilon(U)})(1-\eta_\varepsilon)h + (\lambda - \mathbb{1}_{\varepsilon\mathcal{I}})\eta_\varepsilon h\\
&=\; (\lambda - \mathbb{1}_{\mathcal{I}_\varepsilon(U)})(1-\eta_\varepsilon)h\\
&\qquad-\varepsilon\nabla_j\Big((\triangle^{-1}\nabla_j\mathbb{1}_{\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon h\Big) + \varepsilon(\triangle^{-1}\nabla_j\mathbb{1}_{\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\nabla_j(\eta_\varepsilon h),
\end{aligned}$$

the claim (3.11) follows after straightforward simplifications.

In the rest of the proof, for notational convenience, we do not make explicit the dependence of estimates wrt κ and ℓ.

Step 2. Energy estimate for (3.11): for all $K \geq 1$,

$$\int_U |\nabla w_\varepsilon|^2 \lesssim \tfrac{1}{K}\int_U (Q_\varepsilon)^2 + \Big(1+\|(h,\nabla \bar u_\varepsilon)\|_{W^{1,\infty}(U)}^2 + \|\chi_\delta * \mathrm{D}(u_\varepsilon)\|_{W^{2,\infty}(U)}^6\Big)$$
$$\times\bigg(\int_{\partial_{3r_\varepsilon}U} |(1,\nabla\psi,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\nabla\phi,\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^2$$
$$+\varepsilon^2 K\int_U (1+r_\varepsilon^{-2}\mathbb{1}_{\partial_{3r_\varepsilon}U})|(1,\psi,\nabla\psi,\Upsilon,\phi,\theta,\gamma,\mathfrak{d}\phi,\nabla\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\gamma,\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,$$
$$\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)(\tfrac{\cdot}{\varepsilon})|^2\bigg), \qquad (3.22)$$

where we use the following notation for correctors,

$$|\psi| := \sup_E |E|^{-1}|\psi_E|, \qquad (3.23)$$

and similarly for $|\Upsilon|$. For the active corrector ϕ, which depends nonlinearly on the direction E, as well as for θ,γ, we rather set

$$|\phi| := \sup_{|E|\le C_\delta(h)} \langle E\rangle^{-1}|\phi_E|, \qquad (3.24)$$

where the deterministic constant $C_\delta(h)>0$ is chosen such that almost surely

$$\|\chi_\delta * \mathrm{D}(u_\varepsilon)\|_{\mathrm{L}^\infty(U)} \le \|\chi_\delta\|_{\mathrm{L}^2(\mathbb{R}^d)}\|\nabla u_\varepsilon\|_{\mathrm{L}^2(U)} \le C_\delta(h).$$

Such a constant can be chosen in view of (1.19). For the linearized correctors $\mathfrak{d}\phi$ and $\mathfrak{d}^2\phi$, as well as for $\mathfrak{d}\theta,\mathfrak{d}\gamma,\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F$, we similarly set

$$|\mathfrak{d}\phi| := \sup_{|E|\le C_\delta(h)}\sup_{E'} |E'|^{-1}|\mathfrak{d}\phi_{E,E'}|,$$
$$|\mathfrak{d}^2\phi| := \sup_{|E|\le C_\delta(h)}\sup_{E',E''} |E'|^{-1}|E''|^{-1}|\mathfrak{d}^2\phi_{E,E',E''}|.$$

By Poincaré's inequality in form of

$$\sup_{\varepsilon(I_n+B)}\Big|\nabla\bar u_\varepsilon - \fint_{\varepsilon(I_n+B)}\nabla\bar u_\varepsilon\Big| \lesssim \varepsilon\|\nabla^2\bar u_\varepsilon\|_{\mathrm{L}^\infty(U)},$$
$$\sup_{\varepsilon(I_n+B)}\Big|\chi_\delta*\mathrm{D}(u_\varepsilon) - \fint_{\varepsilon(I_n+B)}\chi_\delta*\mathrm{D}(u_\varepsilon)\Big| \lesssim \varepsilon\|\chi_\delta*\mathrm{D}(u_\varepsilon)\|_{W^{1,\infty}(U)},$$

the claim (3.22) follows from

$$\int_U|\nabla w_\varepsilon|^2 \lesssim \tfrac{1}{K}\int_U(Q_\varepsilon)^2$$
$$+\int_{\partial_{3r_\varepsilon}U}|(1,\nabla\psi,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\nabla\phi,\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^2\big(1+|h|^2+|\nabla\bar u_\varepsilon|^2+[\nabla\bar u_\varepsilon]_{4\varepsilon}^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)$$
$$+\varepsilon^2 r_\varepsilon^{-2}K\int_{\partial_{3r_\varepsilon}U}|(\psi,\Upsilon,\phi,\theta,\gamma,\mathfrak{d}\phi)(\tfrac{\cdot}{\varepsilon})|^2\big(1+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)$$
$$+\varepsilon^2K\int_U|(1,\psi,\Upsilon,\phi,\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\gamma,\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)(\tfrac{\cdot}{\varepsilon})|^2$$
$$\times\Big(|\langle\nabla\rangle h|^2+|\nabla^2\bar u_\varepsilon|^2+|\nabla\langle\nabla\rangle\chi_\delta*\mathrm{D}(u_\varepsilon)|^2+|\nabla\chi_\delta*\mathrm{D}(u_\varepsilon)|^4$$
$$+|\nabla(\chi_\delta*\partial_E u_\varepsilon)|^2\big(1+[\chi_\delta*\mathrm{D}(u_\varepsilon)]_{4\varepsilon}^4\big)\Big)$$

$$+K\sum_n\int_{\varepsilon(I_n+B)}|(\psi,\nabla\psi)(\tfrac{\cdot}{\varepsilon})|^2\Big|\nabla\bar u_\varepsilon-\fint_{\varepsilon(I_n+B)}\nabla\bar u_\varepsilon\Big|^2$$
$$+K\sum_n\int_{\varepsilon(I_n+B)}|(\mathfrak{d}\phi,\nabla\mathfrak{d}\phi)(\tfrac{\cdot}{\varepsilon})|^2\Big|\chi_\delta*\mathrm{D}(u_\varepsilon)-\fint_{\varepsilon(I_n+B)}\chi_\delta*\mathrm{D}(u_\varepsilon)\Big|^2, \quad (3.25)$$

where we use the short-hand notation $[g]_{4\varepsilon}(x):=(\fint_{B_{4\varepsilon}(x)}|g|^2)^{\frac12}$. We split the proof of (3.25) into seven substeps.

Substep 2.1. Preliminary.
In order to obtain (3.25), we may wish to test equation (3.11) with w_ε itself. However, w_ε is not rigid inside particles, which prevents us from taking advantage of the boundary conditions. To circumvent this issue, we make use of the following truncation maps $T_0^\varepsilon,T_1^\varepsilon$: for all $g\in C_b^\infty(\mathbb{R}^d)^d$,

$$\begin{aligned}
T_0^\varepsilon[g] &:= (1-\rho^\varepsilon)g+\sum_n\rho_n^\varepsilon\Big(\fint_{\varepsilon(I_n+B)}g\Big),\\
T_1^\varepsilon[g] &:= (1-\rho^\varepsilon)g+\sum_n\rho_n^\varepsilon\bigg(\Big(\fint_{\varepsilon(I_n+B)}g\Big)+(\cdot-\varepsilon x_n)_j\Big(\fint_{\varepsilon(I_n+B)}\partial_jg\Big)\bigg),
\end{aligned}$$

where for all n we have chosen a cut-off function $\rho_n^\varepsilon\in C_c^\infty(\mathbb{R}^d;[0,1])$ with

$$\rho_n^\varepsilon|_{\varepsilon(I_n+\frac14B)}=1,\qquad \rho_n^\varepsilon|_{\mathbb{R}^d\setminus\varepsilon(I_n+\frac12B)}=0,\qquad |\nabla\rho_n^\varepsilon|\lesssim\tfrac1\varepsilon,$$

and where we have set for abbreviation $\rho^\varepsilon:=\sum_n\rho_n^\varepsilon$. In these terms, we shall test (3.11) with the following modification of the two-scale expansion error w_ε, cf. (3.8),

$$\tilde w_\varepsilon := u_\varepsilon-T_1^\varepsilon[\bar u_\varepsilon]-\varepsilon\eta_\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})T_0^\varepsilon[\partial_E\bar u_\varepsilon]-\varepsilon\kappa\eta_\varepsilon\phi_{T_0^\varepsilon[\chi_\delta*\mathrm{D}(u_\varepsilon)]}(\tfrac{\cdot}{\varepsilon}).$$

Testing equation (3.11) with $\tilde w_\varepsilon$, we obtain

$$I_0^\varepsilon = I_1^\varepsilon+I_2^\varepsilon+I_3^\varepsilon+I_4^\varepsilon+I_5^\varepsilon, \quad (3.26)$$

in terms of

$$\begin{aligned}
I_0^\varepsilon &:= \int_U\nabla\tilde w_\varepsilon:\nabla w_\varepsilon-\int_UQ_\varepsilon\,\mathrm{div}(\tilde w_\varepsilon),\\
I_1^\varepsilon &:= \int_U\eta_\varepsilon\nabla\tilde w_\varepsilon:\Big((J_E\mathbb{1}_{\mathcal I})(\tfrac{\cdot}{\varepsilon})\partial_E\bar u_\varepsilon+\kappa(K_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathcal I})(\tfrac{\cdot}{\varepsilon})\Big)\\
I_2^\varepsilon &:= -\sum_{n\in\mathcal N_\varepsilon(U)}\bigg(\int_{\varepsilon\partial I_n}\tilde w_\varepsilon\cdot\sigma(u_\varepsilon,P_\varepsilon+P'_\varepsilon-P''_{\varepsilon,n})\nu+\tfrac\kappa\varepsilon\int_{\varepsilon I_n}\tilde w_\varepsilon\cdot f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\restriction\varepsilon I_n}\big)\bigg)\\
I_3^\varepsilon &:= -\tfrac\kappa\varepsilon\sum_{n\in\mathcal N_\varepsilon(U)}\int_{\varepsilon(I_n+B)}\eta_\varepsilon\tilde w_\varepsilon\cdot\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon))-f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\restriction\varepsilon I_n}\big)\Big)\\
&\quad -\kappa\int_U\eta_\varepsilon\tilde w_\varepsilon\otimes\nabla(\chi_\delta*\partial_Eu_\varepsilon):\mathfrak{d}F_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon}),\\
I_4^\varepsilon &:= \int_U(1-\eta_\varepsilon)(\lambda-\mathbb{1}_{\mathcal I_\varepsilon(U)})\tilde w_\varepsilon\cdot h+\tfrac\kappa\varepsilon\sum_{n\in\mathcal N_\varepsilon(U)}\int_{\varepsilon(I_n+B)}(1-\eta_\varepsilon)\tilde w_\varepsilon\cdot f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\restriction\varepsilon I_n}\big)\\
&\quad +2\int_U(1-\eta_\varepsilon)\,\mathrm{D}(\tilde w_\varepsilon):\big((\bar{\boldsymbol B}_{\mathrm{pas}}-\mathrm{Id})\,\mathrm{D}(\bar u_\varepsilon)+\kappa\bar{\boldsymbol B}_{\mathrm{act}}(\chi_\delta*\mathrm{D}(u_\varepsilon))\big)
\end{aligned}$$

$$-\kappa\int_U \tilde w_\varepsilon\otimes\nabla\eta_\varepsilon:\bar{\boldsymbol F}(\chi_\delta*\mathrm{D}(u_\varepsilon))$$

and

$$\begin{aligned}
I_5^\varepsilon \;:=\; & -\varepsilon\int_U \nabla\tilde w_\varepsilon:\Big(\big(2\psi_E\otimes_s-\mathrm{Id}\otimes\psi_E-\Upsilon_E\big)(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon)-\eta_\varepsilon h\otimes(\triangle^{-1}\nabla\mathbb{1}_{\mathcal I})(\tfrac{\cdot}{\varepsilon})\\
&\qquad+\kappa\big(2\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\otimes_s-\mathrm{Id}\otimes\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}-\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon)}+\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\nabla\eta_\varepsilon\\
&\qquad+\kappa\big(2\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes_s-\mathrm{Id}\otimes\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}-\mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\\
&\qquad\qquad+\mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\Big)\\
&-\kappa\varepsilon\int_U \eta_\varepsilon\nabla_i\tilde w_\varepsilon\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon):(\triangle^{-1}\nabla_i\mathfrak{d}F_{|\chi_\delta*\mathrm{D}(u_\varepsilon)})(\tfrac{\cdot}{\varepsilon})\\
&+\varepsilon\int_U \tilde w_\varepsilon\cdot\Big((\triangle^{-1}\nabla_j\mathbb{1}_{\mathcal I})(\tfrac{\cdot}{\varepsilon})\nabla_j(\eta_\varepsilon h)-\kappa\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\triangle\eta_\varepsilon\\
&\qquad-\kappa\mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon}):\eta_\varepsilon\triangle(\chi_\delta*\partial_E u_\varepsilon)\\
&\qquad-\kappa(\triangle^{-1}\nabla_i\mathfrak{d}F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E})(\tfrac{\cdot}{\varepsilon})\nabla_i(\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon))\\
&\qquad+2\kappa\big(\mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}-\mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon}):\nabla(\chi_\delta*\partial_E u_\varepsilon)\otimes\nabla\eta_\varepsilon\\
&\qquad+\kappa\big(\mathfrak{d}^2\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'}-\mathfrak{d}^2\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'}\otimes\mathrm{Id}-\triangle^{-1}\nabla_i\mathfrak{d}^2F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'}\big)(\tfrac{\cdot}{\varepsilon})\\
&\qquad\qquad:\eta_\varepsilon\nabla(\chi_\delta*\partial_{E'}u_\varepsilon)\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon)\Big).
\end{aligned}$$

We analyze the different terms separately in the upcoming six substeps.

Substep 2.2. Proof that for all $K\ge1$,

$$\begin{aligned}
I_0^\varepsilon\ \ge\ &(1-\tfrac{1}{2K})\int_U|\nabla w_\varepsilon|^2-\tfrac1K\int_U(Q_\varepsilon)^2-K\int_U|\nabla(\tilde w_\varepsilon-w_\varepsilon)|^2\\
&-\varepsilon^2r_\varepsilon^{-2}CK\int_{\partial_{2r_\varepsilon}U}|(\psi,\phi,\mathfrak{d}\phi)(\tfrac{\cdot}{\varepsilon})|^2\big(1+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\\
&-\varepsilon^2CK\int_U|(\psi,\phi,\mathfrak{d}\phi)(\tfrac{\cdot}{\varepsilon})|^2\big(|\nabla^2\bar u_\varepsilon|^2+|\nabla\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big).
\end{aligned}\tag{3.27}$$

Adding and subtracting w_ε to $\tilde w_\varepsilon$, we find from Young's inequality, for all $K\ge1$,

$$I_0^\varepsilon\ \ge\ (1-\tfrac{1}{2K})\int_U|\nabla w_\varepsilon|^2-\tfrac1K\int_U(Q_\varepsilon)^2-K\int_U|\nabla(\tilde w_\varepsilon-w_\varepsilon)|^2-K\int_U|\mathrm{div}(w_\varepsilon)|^2.$$

Since $\mathrm{div}(u_\varepsilon)=\mathrm{div}(\bar u_\varepsilon)=0$, we get from (3.15) and (3.16),

$$\begin{aligned}
\mathrm{div}(w_\varepsilon)\ =\ &-\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\cdot\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon)-\varepsilon\kappa\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\cdot\nabla\eta_\varepsilon\\
&-\varepsilon\kappa\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\cdot\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon),
\end{aligned}\tag{3.28}$$

and the claim (3.27) follows using the properties of η_ε.

Substep 2.3. Proof that

$$\begin{aligned}
I_1^\varepsilon\ \lesssim\ &\Big(\int_{\partial_{2r_\varepsilon}U}\big(1+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\Big)^{\frac12}\\
&\times\Big(\int_{\partial_{3r_\varepsilon}U}|(1,\nabla\psi,\Sigma\mathbb{1}_{\mathbb R^d\setminus\mathcal I},\nabla\phi,\Pi\mathbb{1}_{\mathbb R^d\setminus\mathcal I})(\tfrac{\cdot}{\varepsilon})|^2[\nabla\bar u_\varepsilon]_{4\varepsilon}^2\Big)^{\frac12}.
\end{aligned}\tag{3.29}$$

First note that for all n we have in εI_n, since $\mathrm{D}(\psi_E)|_{I_n} = -E$,

$$\begin{aligned}\mathrm{D}(\tilde w_\varepsilon)|_{\varepsilon I_n} &= -\Big(\mathrm{D}(T_1^\varepsilon[\bar u_\varepsilon]) + \eta_\varepsilon\, \mathrm{D}(\psi_E)(\tfrac{\cdot}{\varepsilon})T_0^\varepsilon[\partial_E \bar u_\varepsilon]\Big)\Big|_{\varepsilon I_n}\\ &= -(1-\eta_\varepsilon)|_{\varepsilon I_n} \fint_{\varepsilon(I_n+B)} \mathrm{D}(\bar u_\varepsilon). \qquad (3.30)\end{aligned}$$

As by construction J and K are symmetric matrix fields, we may replace $\nabla \tilde w_\varepsilon$ by $\mathrm{D}(\tilde w_\varepsilon)$ in the definition of I_1^ε, and we thus find

$$I_1^\varepsilon = -\sum_{n\in\mathcal N_\varepsilon(U)} (\eta_\varepsilon(1-\eta_\varepsilon))(\varepsilon x_n) \fint_{\varepsilon(I_n+B)} \mathrm{D}(\bar u_\varepsilon) : \int_{\varepsilon I_n} \Big(J_E(\tfrac{\cdot}{\varepsilon})\partial_E \bar u_\varepsilon + \kappa(K_{\chi_\delta * \mathrm{D}(u_\varepsilon)})(\tfrac{\cdot}{\varepsilon})\Big).$$

By the hardcore assumption, using the properties of η_ε, and using (2.2), (2.25), and the Lipschitz continuity of $E \mapsto f_n(E)$ (cf. Hypothesis 1.3), the claim (3.29) follows.

Substep 2.4. Proof that

$$\begin{aligned} I_2^\varepsilon \lesssim\; &\Big(\int_U |\nabla w_\varepsilon|^2\Big)^{\frac12}\Big(\int_{\partial_{3r_\varepsilon} U} |\nabla \bar u_\varepsilon|^2\Big)^{\frac12}\\ &+ \Big(\int_{\partial_{3r_\varepsilon} U} |\nabla \bar u_\varepsilon|^2\Big)^{\frac12}\Big(\int_{\partial_{3r_\varepsilon} U} |(1,\nabla\psi,\nabla\phi)(\tfrac{\cdot}{\varepsilon})|^2\big(1+|\nabla\bar u_\varepsilon|^2 + |\chi_\delta * \mathrm{D}(u_\varepsilon)|^2\big)\Big)^{\frac12}\\ &+ \varepsilon\Big(\int_{\partial_{3r_\varepsilon} U} |\nabla \bar u_\varepsilon|^2\Big)^{\frac12}\Big(r_\varepsilon^{-2}\int_{\partial_{3r_\varepsilon} U} |(\psi,\phi)(\tfrac{\cdot}{\varepsilon})|^2\big(1+|\nabla\bar u_\varepsilon|^2 + |\chi_\delta * \mathrm{D}(u_\varepsilon)|^2\big)\Big)^{\frac12}\\ &+ \varepsilon\Big(\int_{\partial_{3r_\varepsilon} U} |\nabla \bar u_\varepsilon|^2\Big)^{\frac12}\Big(\int_{\partial_{3r_\varepsilon} U} |(\psi,\mathfrak{d}\phi)(\tfrac{\cdot}{\varepsilon})|^2\big(|\nabla^2\bar u_\varepsilon|^2 + |\nabla\chi_\delta * \mathrm{D}(u_\varepsilon)|^2\big)\Big)^{\frac12}. \qquad (3.31)\end{aligned}$$

Appealing to the boundary conditions in (1.7), recalling the notation (1.8), and using $\int_{\varepsilon\partial I_n}\nu = 0$ and $\int_{\varepsilon\partial I_n}\nu\otimes(x-\varepsilon x_n) = |\varepsilon I_n|\,\mathrm{Id}$, we have

$$I_2^\varepsilon = -\sum_{n\in\mathcal N_\varepsilon(U)} \Big(\fint_{\varepsilon I_n} \mathrm{D}(\tilde w_\varepsilon)\Big) : \int_{\varepsilon\partial I_n} \sigma(u_\varepsilon, P_\varepsilon + P_\varepsilon' - P_{\varepsilon,n}'')\nu\otimes(x-\varepsilon x_n),$$

and thus, using (3.30) again, and noting that the identities $\int_{\varepsilon\partial I_n}\nu\otimes(\cdot-\varepsilon x_n) = |\varepsilon I_n|\,\mathrm{Id}$ and $\operatorname{div}(\bar u_\varepsilon)=0$ allow to remove any constant from the pressure field P_ε in this expression,

$$I_2^\varepsilon = \sum_{n\in\mathcal N_\varepsilon(U)} (1-\eta_\varepsilon)(\varepsilon x_n)\Big(\fint_{\varepsilon(I_n+B)} \mathrm{D}(\bar u_\varepsilon)\Big) : \int_{\varepsilon\partial I_n} \sigma(u_\varepsilon, P_\varepsilon - P_{\varepsilon,n}''')\nu\otimes(x-\varepsilon x_n), \qquad (3.32)$$

where we have chosen $P_{\varepsilon,n}''' := \fint_{\varepsilon(I_n+B)\setminus\varepsilon I_n} P_\varepsilon$. In order to estimate the right-hand side, we shall turn surface integrals into volume integrals, proceeding as for the construction of K_E in Lemma 2.5. More precisely, for all n, we consider the following Neumann problem,

$$\begin{cases} -\triangle u_\varepsilon^n + \nabla P_\varepsilon^n = f_{n,\varepsilon}\big(\chi_\delta * \mathrm{D}(u_\varepsilon)_{|\varepsilon I_n}\big), & \text{in } \varepsilon I_n,\\ \operatorname{div}(u_\varepsilon^n) = 0, & \text{in } \varepsilon I_n,\\ \sigma(u_\varepsilon^n, P_\varepsilon^n)\nu = \sigma(u_\varepsilon, P_\varepsilon - P_{\varepsilon,n}''')\nu, & \text{on } \varepsilon\partial I_n.\end{cases}$$

As for (2.16), we can show that there is a unique solution $u_\varepsilon^n \in H_0^1(\varepsilon I_n)^d$ with $\fint_{\varepsilon I_n} u_\varepsilon^n = 0$ and $\fint_{\varepsilon I_n}\nabla u_\varepsilon^n \in \mathbb M_0^{\mathrm{sym}}$, and a unique pressure $P_\varepsilon^n \in \mathrm{L}^2(\varepsilon I_n)/\mathbb R$, such that

$$\|(\nabla u_\varepsilon^n, P_\varepsilon^n)\|_{\mathrm{L}^2(\varepsilon I_n)} \lesssim \|\mathrm{D}(u_\varepsilon)\|_{\mathrm{L}^2(\varepsilon(I_n+B))} + \big\|f_{n,\varepsilon}\big(\chi_\delta * \mathrm{D}(u_\varepsilon)_{|\varepsilon I_n}\big)\big\|_{\mathrm{L}^2(\varepsilon(I_n+B))}. \qquad (3.33)$$

In these terms, we can reformulate the surface integral in (3.32) as follows,

$$\begin{aligned}\int_{\varepsilon\partial I_n}\sigma(u_\varepsilon,P_\varepsilon)\nu\otimes(x-\varepsilon x_n)\,&=\,\int_{\varepsilon I_n}\nabla_j\Big(\sigma(u_\varepsilon^n,P_\varepsilon^n)e_j\otimes(x-\varepsilon x_n)\Big)\\&=\,-\int_{\varepsilon I_n}f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon I_n}\big)\otimes(x-\varepsilon x_n)+\int_{\varepsilon I_n}\sigma(u_\varepsilon^n,P_\varepsilon^n),\end{aligned}$$

and thus, in view of (3.33),

$$\begin{aligned}&\bigg|\int_{\varepsilon\partial I_n}\sigma(u_\varepsilon,P_\varepsilon)\nu\otimes(x-\varepsilon x_n)\bigg|\\&\qquad\lesssim\,|\varepsilon I_n|^{\frac12}\Big(|\varepsilon I_n|^{\frac12}+\|\mathrm{D}(u_\varepsilon)\|_{\mathrm{L}^2(\varepsilon(I_n+B))}+\|\chi_\delta*\mathrm{D}(u_\varepsilon)\|_{\mathrm{L}^2(\varepsilon I_n)}\Big).\end{aligned}$$

Inserting this into (3.32) and using the properties of η_ε, we are led to

$$I_2^\varepsilon\,\lesssim\,\Big(\int_{\partial_{3r_\varepsilon}U}|\nabla\bar u_\varepsilon|^2\Big)^{\frac12}\Big(\int_{\partial_{3r_\varepsilon}U}\big(1+|\nabla u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\Big)^{\frac12}.$$

Inserting then the two-scale expansion to replace the norm of ∇u_ε in the right-hand side,

$$u_\varepsilon\,=\,w_\varepsilon+\bar u_\varepsilon+\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\partial_E\bar u_\varepsilon+\varepsilon\kappa\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon,$$

the claim (3.31) follows.

Substep 2.5. Proof that

$$\begin{aligned}|I_3^\varepsilon|\,\lesssim\,\varepsilon\Big(\int_U|\nabla\tilde w_\varepsilon|^2\Big)^{\frac12}\Big(\int_U&|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^4+|\nabla^2(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\\&+|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\big(1+[\chi_\delta*\mathrm{D}(u_\varepsilon)]_{4\varepsilon}^4\big)\Big)^{\frac12}.\end{aligned}\tag{3.34}$$

We start by decomposing

$$\begin{aligned}I_3^\varepsilon\,=\,&-\tfrac{\kappa}{\varepsilon}\sum_{n\in\mathcal N_\varepsilon(U)}\int_{\varepsilon(I_n+B)}\Big(\eta_\varepsilon\tilde w_\varepsilon-\fint_{\varepsilon I_n}\eta_\varepsilon\tilde w_\varepsilon\Big)\cdot\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon))-f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon I_n}\big)\Big)\\&-\tfrac{\kappa}{\varepsilon}\sum_{n\in\mathcal N_\varepsilon(U)}\Big(\fint_{\varepsilon I_n}\eta_\varepsilon\tilde w_\varepsilon\Big)\cdot\int_{\varepsilon(I_n+B)}\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon))-f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon I_n}\big)\Big)\\&-\kappa\int_U\eta_\varepsilon\tilde w_\varepsilon\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon):\mathfrak{d}F_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon}).\end{aligned}\tag{3.35}$$

We now estimate the first right-hand side term. Using the Lipschitz regularity of $f_{n,\varepsilon}$, applying Poincaré's inequality, and using the hardcore assumption, we get

$$\begin{aligned}&\bigg|\tfrac1\varepsilon\sum_{n\in\mathcal N_\varepsilon(U)}\int_{\varepsilon(I_n+B)}\Big(\eta_\varepsilon\tilde w_\varepsilon-\fint_{\varepsilon I_n}\eta_\varepsilon\tilde w_\varepsilon\Big)\cdot\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon))-f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon I_n}\big)\Big)\bigg|\\&\qquad\lesssim\,\varepsilon\Big(\int_U|\nabla(\eta_\varepsilon\tilde w_\varepsilon)|^2\Big)^{\frac12}\Big(\int_U|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\Big)^{\frac12}.\end{aligned}$$

We further appeal to the following version of Poincaré's inequality for $\tilde{w}_\varepsilon \in H_0^1(U)^d$ in $\partial_{2r_\varepsilon}U$ (this will be used several times in the proof): by the properties of η_ε,

$$\int_U |\nabla \eta_\varepsilon|^2 \tilde{w}_\varepsilon^2 \lesssim r_\varepsilon^{-2} \int_{\partial_{2r_\varepsilon}U} \tilde{w}_\varepsilon^2 \lesssim \int_U |\nabla \tilde{w}_\varepsilon|^2. \tag{3.36}$$

The above then becomes

$$\Big|\tfrac{1}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)} \int_{\varepsilon(I_n+B)} \Big(\eta_\varepsilon\tilde{w}_\varepsilon - \fint_{\varepsilon I_n}\eta_\varepsilon\tilde{w}_\varepsilon\Big)\cdot\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon)) - f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon I_n}\big)\Big)\Big| \\ \lesssim \varepsilon\Big(\int_U|\nabla\tilde{w}_\varepsilon|^2\Big)^{\frac12}\Big(\int_U|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\Big)^{\frac12}. \tag{3.37}$$

We turn to the analysis of the last two terms in (3.35). By a Taylor expansion (of the form $|u(a+b)-u(a)-cu'(a)|\lesssim b^2\sup|u''|+|b-c||u'(a)|$), using the C^2 regularity of $f_{n,\varepsilon}$, we can estimate

$$\Big|\int_{\varepsilon(I_n+B)}\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon)) - f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon I_n}\big)\Big) \\ -\Big(\fint_{\varepsilon I_n}\nabla_i(\chi_\delta*\partial_E u_\varepsilon)\Big)\int_{\varepsilon(I_n+B)}(x-\varepsilon x_n)_i\;\partial_E f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon I_n})\Big| \\ \lesssim \varepsilon^2\int_{\varepsilon(I_n+B)}|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2+\varepsilon^2\int_{\varepsilon(I_n+B)}|\nabla^2(\chi_\delta*\mathrm{D}(u_\varepsilon))|.$$

Summing over n and recognizing the definition of $\mathfrak{d}F$, cf. (2.30),

$$\mathfrak{d}F_{E,E'} := -\sum_n \tfrac{\mathbb{1}_{I_n}}{|I_n|}\int_{I_n+B}\partial_{E'}f_n(E)\otimes(x-x_n),$$

we are led to

$$\Big|\tfrac{1}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)}\Big(\fint_{\varepsilon I_n}\eta_\varepsilon\tilde{w}_\varepsilon\Big)\cdot\int_{\varepsilon(I_n+B)}\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon)) - f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon I_n}\big)\Big) \\ +\int_U\eta_\varepsilon\tilde{w}_\varepsilon\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon):\mathfrak{d}F_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\Big| \\ \lesssim \varepsilon\Big(\int_U|\eta_\varepsilon\tilde{w}_\varepsilon|^2\Big)^{\frac12}\Big(\int_U|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^4+|\nabla^2(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\Big)^{\frac12} \\ +\varepsilon\Big(\int_U|\nabla(\eta_\varepsilon\tilde{w}_\varepsilon)|^2\Big)^{\frac12}\Big(\int_U|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\big(1+[\chi_\delta*\mathrm{D}(u_\varepsilon)]_{4\varepsilon}^4\big)\Big)^{\frac12}.$$

Further appealing to Poincaré's inequality and to (3.36), this becomes

$$\Big|\tfrac{1}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)}\Big(\fint_{\varepsilon I_n}\eta_\varepsilon\tilde{w}_\varepsilon\Big)\cdot\int_{\varepsilon(I_n+B)}\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon)) - f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon I_n}\big)\Big) \\ +\int_U\eta_\varepsilon\tilde{w}_\varepsilon\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon):\mathfrak{d}F_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\Big| \\ \lesssim \varepsilon\Big(\int_U|\nabla\tilde{w}_\varepsilon|^2\Big)^{\frac12}\Big(\int_U|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^4+|\nabla^2(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2$$

$$+|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\big(1+[\chi_\delta*\mathrm{D}(u_\varepsilon)]_{4\varepsilon}^4\big)\Big)^{\frac12}.$$

Combining this with (3.35) and (3.37), the claim (3.34) follows.

Substep 2.6. Proof that

$$I_4^\varepsilon \;\lesssim\; \Big(\int_U|\nabla\tilde w_\varepsilon|^2\Big)^{\frac12}\Big(\int_{\partial_{3r_\varepsilon}U}\big(1+|h|^2+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\Big)^{\frac12}, \tag{3.38}$$

$$\begin{aligned}
I_5^\varepsilon \;\lesssim\;& \varepsilon\Big(\int_U|\nabla\tilde w_\varepsilon|^2\Big)^{\frac12}\Big(r_\varepsilon^{-2}\int_{\partial_{2r_\varepsilon}U}|(\psi,\Upsilon,\phi,\theta,\gamma)|^2\big(1+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\Big)^{\frac12}\\
&+\varepsilon\Big(\int_U|\nabla\tilde w_\varepsilon|^2\Big)^{\frac12}\Big(\int_U|(\psi,\Upsilon,\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal I},\triangle^{-1}\nabla\mathfrak{d}F)|^2\\
&\qquad\qquad\times\big(|\langle\nabla\rangle h|^2+|\nabla^2\bar u_\varepsilon|^2+|\nabla\langle\nabla\rangle\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\Big)^{\frac12}\\
&+\varepsilon\Big(\int_U|\nabla\tilde w_\varepsilon|^2\Big)^{\frac12}\Big(\int_U|(\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathfrak{d}^2F)(\tfrac{\cdot}{\varepsilon})|^2|\nabla\chi_\delta*\mathrm{D}(u_\varepsilon)|^4\Big)^{\frac12}.
\end{aligned} \tag{3.39}$$

Using properties of η_ε, the neutrality condition (1.9), and Poincaré's inequality, a direct estimate yields

$$I_4^\varepsilon\lesssim\Big(\int_{\partial_{3r_\varepsilon}U}\big(1+|h|^2+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\Big)^{\frac12}\Big(\int_U|\nabla\tilde w_\varepsilon|^2+\int_U|\nabla\eta_\varepsilon|^2|\tilde w_\varepsilon|^2\Big)^{\frac12},$$

and the claimed estimate (3.38) follows by applying (3.36) again. The bound (3.39) on I_5^ε is obtained by similar straightforward computations.

Substep 2.7. Proof of (3.25).
Starting from (3.26), combining estimates (3.27), (3.29), (3.31), (3.34), (3.38), and (3.39), adding and subtracting w_ε to $\tilde w_\varepsilon$ in the right-hand side, and applying Young's inequality, we get after straightforward simplifications, for all $K\ge1$,

$$\begin{aligned}
\int_U|\nabla w_\varepsilon|^2 \;\lesssim\;& \tfrac1K\int_U(Q_\varepsilon)^2+K\int_U|\nabla(\tilde w_\varepsilon-w_\varepsilon)|^2\\
&+\int_{\partial_{3r_\varepsilon}U}|(1,\nabla\psi,\Sigma\mathbb{1}_{\mathbb R^d\setminus\mathcal I},\nabla\phi,\Pi\mathbb{1}_{\mathbb R^d\setminus\mathcal I})(\tfrac{\cdot}{\varepsilon})|^2\big(1+|h|^2+|\nabla\bar u_\varepsilon|^2+[\nabla\bar u_\varepsilon]_{4\varepsilon}^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\\
&+\varepsilon^2r_\varepsilon^{-2}K\int_{\partial_{3r_\varepsilon}U}|(\psi,\Upsilon,\phi,\theta,\gamma,\mathfrak{d}\phi)(\tfrac{\cdot}{\varepsilon})|^2\big(1+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\\
&+\varepsilon^2K\int_U|(1,\psi,\Upsilon,\phi,\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\gamma,\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal I},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)(\tfrac{\cdot}{\varepsilon})|^2\\
&\qquad\times\Big(|\langle\nabla\rangle h|^2+|\nabla^2\bar u_\varepsilon|^2+|\nabla\langle\nabla\rangle\chi_\delta*\mathrm{D}(u_\varepsilon)|^2+|\nabla\chi_\delta*\mathrm{D}(u_\varepsilon)|^4\\
&\qquad\qquad+|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\big(1+[\chi_\delta*\mathrm{D}(u_\varepsilon)]_{4\varepsilon}^4\big)\Big).
\end{aligned} \tag{3.40}$$

It remains to evaluate the norm of $\nabla(\tilde w_\varepsilon-w_\varepsilon)$. By definition of $\tilde w_\varepsilon$, we have

$$\begin{aligned}
\nabla(\tilde w_\varepsilon-w_\varepsilon) \;=\;& \nabla(\bar u_\varepsilon-T_1^\varepsilon[\bar u_\varepsilon])+\nabla\big(\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon(\partial_E\bar u_\varepsilon-T_0^\varepsilon[\partial_E\bar u_\varepsilon])\big)\\
&+\nabla\big(\varepsilon\kappa\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon-\varepsilon\kappa\phi_{T_0^\varepsilon[\chi_\delta*\mathrm{D}(u_\varepsilon)]}(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\big),
\end{aligned}$$

and thus, inserting the definition of the truncation operators $T_0^\varepsilon, T_1^\varepsilon$, and noting that η_ε is constant in the support of the cut-off functions $\{\rho_\varepsilon^n\}_n$ by definition,

$$\begin{aligned}\nabla(\tilde w_\varepsilon - w_\varepsilon) &= \sum_n \rho_n^\varepsilon\Big(\nabla\bar u_\varepsilon - \fint_{\varepsilon(I_n+B)}\nabla\bar u_\varepsilon\Big) + \sum_n \nabla\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\rho_n^\varepsilon\Big(\partial_E\bar u_\varepsilon - \fint_{\varepsilon(I_n+B)}\partial_E\bar u_\varepsilon\Big)\\
&\quad + \kappa\sum_n \big(\nabla\phi_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon}) - \nabla\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon(I_n+B)}}(\tfrac{\cdot}{\varepsilon})\big)\eta_\varepsilon\rho_n^\varepsilon\\
&\quad + \sum_n \varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\rho_n^\varepsilon\nabla\partial_E\bar u_\varepsilon + \sum_n \varepsilon\kappa\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\otimes\eta_\varepsilon\rho_n^\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\\
&\quad + \sum_n\Big(\bar u_\varepsilon - \Big(\fint_{\varepsilon(I_n+B)}\bar u_\varepsilon\Big) - \Big(\fint_{\varepsilon(I_n+B)}\nabla\bar u_\varepsilon\Big)(x-\varepsilon x_n)\Big)\otimes\nabla\rho_n^\varepsilon\\
&\quad + \sum_n \varepsilon\psi_E(\tfrac{\cdot}{\varepsilon})\otimes\eta_\varepsilon\nabla\rho_n^\varepsilon\Big(\partial_E\bar u_\varepsilon - \fint_{\varepsilon(I_n+B)}\partial_E\bar u_\varepsilon\Big)\\
&\quad + \sum_n \varepsilon\kappa\big(\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon}) - \phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon(I_n+B)}}(\tfrac{\cdot}{\varepsilon})\big)\otimes\eta_\varepsilon\nabla\rho_n^\varepsilon.\end{aligned}$$

Using that

$$\begin{aligned}\int_{\varepsilon(I_n+B)} &\big|\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon}) - \phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon(I_n+B)}}(\tfrac{\cdot}{\varepsilon})\big|^2\\
&\lesssim \varepsilon^2\int_{\varepsilon(I_n+B)}|\mathfrak{d}\phi(\tfrac{\cdot}{\varepsilon})|^2\Big|\chi_\delta*\mathrm{D}(u_\varepsilon) - \fint_{\varepsilon(I_n+B)}\chi_\delta*\mathrm{D}(u_\varepsilon)\Big|^2,\end{aligned}$$

and

$$\begin{aligned}\int_{\varepsilon(I_n+B)} &\big|\nabla\phi_{|\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon}) - \nabla\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)_{\nmid\varepsilon(I_n+B)}}(\tfrac{\cdot}{\varepsilon})\big|^2\\
&\lesssim \varepsilon^2\int_{\varepsilon(I_n+B)}|(\nabla\mathfrak{d}\phi)(\tfrac{\cdot}{\varepsilon})|^2\Big|\chi_\delta*\mathrm{D}(u_\varepsilon) - \fint_{\varepsilon(I_n+B)}\chi_\delta*\mathrm{D}(u_\varepsilon)\Big|^2,\end{aligned}$$

a direct computation then leads us to

$$\begin{aligned}\int_U|\nabla(\tilde w_\varepsilon - w_\varepsilon)|^2 &\lesssim \varepsilon^2\int_U|(1,\psi)(\tfrac{\cdot}{\varepsilon})|^2|\nabla^2\bar u_\varepsilon|^2 + \varepsilon^2\int_U|\mathfrak{d}\phi(\tfrac{\cdot}{\varepsilon})|^2|\nabla\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\\
&\quad + \sum_n\int_{\varepsilon(I_n+B)}|(\psi,\nabla\psi)(\tfrac{\cdot}{\varepsilon})|^2\Big|\nabla\bar u_\varepsilon - \fint_{\varepsilon(I_n+B)}\nabla\bar u_\varepsilon\Big|^2\\
&\quad + \sum_n\int_{\varepsilon(I_n+B)}|(\mathfrak{d}\phi,\nabla\mathfrak{d}\phi)(\tfrac{\cdot}{\varepsilon})|^2\Big|\chi_\delta*\mathrm{D}(u_\varepsilon) - \fint_{\varepsilon(I_n+B)}\chi_\delta*\mathrm{D}(u_\varepsilon)\Big|^2. \qquad (3.41)\end{aligned}$$

Inserting this into (3.40), the conclusion (3.25) follows.

Step 3. Pressure estimate for (3.11):

$$\begin{aligned}\int_U(Q_\varepsilon)^2 &\lesssim \int_U|\nabla w_\varepsilon|^2 + r_\varepsilon\Big(1 + \|(h,\nabla\bar u_\varepsilon,\chi_\delta*\mathrm{D}(u_\varepsilon))\|^2_{\mathrm{L}^\infty(U)}\Big)\\
&\quad + \varepsilon^2\Big(1 + \|(h,\nabla\bar u_\varepsilon,\bar P_\varepsilon)\|^2_{W^{1,\infty}(U)} + \|\chi_\delta*\mathrm{D}(u_\varepsilon)\|^6_{W^{2,\infty}(U)}\Big)\\
&\times\int_U\big(1 + r_\varepsilon^{-2}\mathbb{1}_{\partial_{3r_\varepsilon}U}\big)\big|\big(1,\psi,\Upsilon,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\phi,\theta,\gamma,\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\mathfrak{d}\gamma,\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\end{aligned}$$

$$\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F\big)(\tfrac{\cdot}{\varepsilon})\Big|^2, \quad (3.42)$$

which follows from

$$\begin{aligned}\int_U (Q_\varepsilon)^2 \lesssim\ & \int_U |\nabla w_\varepsilon|^2 + \int_{\partial_{3r_\varepsilon}U}\big(1+|h|^2+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\\
&+\varepsilon^2 r_\varepsilon^{-2}\int_{\partial_{3r_\varepsilon}U}|(\psi,\Upsilon,\phi,\theta,\gamma,\mathfrak{d}\theta,\mathfrak{d}\gamma)(\tfrac{\cdot}{\varepsilon})|^2\big(1+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\\
&+\varepsilon^2\int_U|(1,\psi,\Upsilon,\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\gamma,\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)(\tfrac{\cdot}{\varepsilon})|^2\\
&\qquad\times\Big(|\langle\nabla\rangle h|^2+|\nabla^2\bar u_\varepsilon|^2+|\nabla\bar P_\varepsilon|^2+|\nabla\langle\nabla\rangle\chi_\delta*\mathrm{D}(u_\varepsilon)|^2+|\nabla\chi_\delta*\mathrm{D}(u_\varepsilon)|^4\\
&\qquad\qquad+|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\big(1+[\chi_\delta*\mathrm{D}(u_\varepsilon)]_{4\varepsilon}^4\big)\Big)\\
&+\sum_n\int_{\varepsilon(I_n+B)}|(\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^2\Big|\nabla\bar u_\varepsilon-\fint_{\varepsilon(I_n+B)}\nabla\bar u_\varepsilon\Big|^2\\
&+\sum_n\int_{\varepsilon(I_n+B)}|(\mathfrak{d}\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^2\Big|\chi_\delta*\mathrm{D}(u_\varepsilon)-\fint_{\varepsilon(I_n+B)}\chi_\delta*\mathrm{D}(u_\varepsilon)\Big|^2, \quad (3.43)\end{aligned}$$

after taking uniform norms of $h,\bar u_\varepsilon,\chi_\delta*\mathrm{D}(u_\varepsilon)$ and using Poincaré's inequality in the last two summands.

Similarly as in Step 2, we shall appeal to a truncated version of Q_ε,

$$\begin{aligned}\tilde Q_\varepsilon :=\ & P_\varepsilon\mathbb{1}_{\mathbb{R}^d\setminus\varepsilon\mathcal{I}}+P_\varepsilon^*-T_0^\varepsilon[\bar P_\varepsilon]-\eta_\varepsilon\bar{\boldsymbol{b}}:T_0^\varepsilon[\mathrm{D}(\bar u_\varepsilon)]-\kappa\eta_\varepsilon\bar{\boldsymbol{c}}(T_0^\varepsilon[\chi_\delta*\mathrm{D}(u_\varepsilon)])\\
&-(\Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon T_0^\varepsilon[\partial_E\bar u_\varepsilon]-\kappa(\Pi_{T_0^\varepsilon[\chi_\delta*\mathrm{D}(u_\varepsilon)]}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon,\end{aligned}$$

where we recall that P_ε^* stands for some locally constant pressure field, cf. (3.9),

$$P_\varepsilon^* := P_\varepsilon'\mathbb{1}_{U\setminus\mathcal{I}_\varepsilon(U)}+\sum_{n\in\mathcal{N}_\varepsilon(U)}P_{\varepsilon,n}''\mathbb{1}_{\varepsilon I_n},$$

and where we now choose the constants P_ε' and $\{P_{\varepsilon,n}''\}_n$ in such a way that

$$\tilde Q_\varepsilon|_{\mathcal{I}_\varepsilon(U)}=0 \qquad\text{and}\qquad \int_U\tilde Q_\varepsilon=0. \quad (3.44)$$

Using the Bogovskii operator as in [4], we can construct a vector field $S_\varepsilon\in H_0^1(U)^d$ such that $S_\varepsilon|_{\varepsilon I_n}$ is a constant for all $n\in\mathcal{N}_\varepsilon(U)$ and such that

$$\begin{aligned}\operatorname{div}(S_\varepsilon) &= -\tilde Q_\varepsilon \qquad\text{in } U,\\
\int_U|\nabla S_\varepsilon|^2 &\lesssim \int_U|\tilde Q_\varepsilon|^2. \end{aligned}\quad (3.45)$$

Testing equation (3.11) with S_ε, using the property that $S_\varepsilon|_{\varepsilon I_n}$ is a constant for all $n\in\mathcal{N}_\varepsilon(U)$, and using the boundary conditions in (1.10), we find

$$\begin{aligned}\int_U\tilde Q_\varepsilon Q_\varepsilon =\ & -\int_U\nabla S_\varepsilon:\nabla w_\varepsilon-\kappa\int_U\eta_\varepsilon S_\varepsilon\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon):\mathfrak{d}F_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon})\\
&-\tfrac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon(I_n+B)}\eta_\varepsilon S_\varepsilon\cdot\Big(f_{n,\varepsilon}(\chi_\delta*\mathrm{D}(u_\varepsilon))-f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{\upharpoonright\varepsilon I_n}\big)\Big)\end{aligned}$$

$$
\begin{aligned}
&+\int_U (1-\eta_\varepsilon)(\lambda-\mathbb{1}_{\mathcal{I}_\varepsilon(U)})S_\varepsilon\cdot h-\kappa\int_U S_\varepsilon\otimes\nabla\eta_\varepsilon:\bar{\boldsymbol{F}}(\chi_\delta*\mathrm{D}(u_\varepsilon))\\
&+\tfrac{\kappa}{\varepsilon}\sum_{n\in\mathcal{N}_\varepsilon(U)}\int_{\varepsilon(I_n+B)}(1-\eta_\varepsilon)S_\varepsilon\cdot f_{n,\varepsilon}\big(\chi_\delta*\mathrm{D}(u_\varepsilon)_{|\varepsilon I_n}\big)\\
&+\int_U(1-\eta_\varepsilon)\nabla S_\varepsilon:\Big(2(\bar{\boldsymbol{B}}_{\mathrm{pas}}-\mathrm{Id})\,\mathrm{D}(\bar u_\varepsilon)+2\kappa\bar{\boldsymbol{C}}(\chi_\delta*\mathrm{D}(u_\varepsilon))+\kappa\bar{\boldsymbol{F}}(\chi_\delta*\mathrm{D}(u_\varepsilon))\Big)\\
&-\varepsilon\int_U\nabla S_\varepsilon:\Big(\big(2\psi_E\otimes_s-\mathrm{Id}\otimes\psi_E-\Upsilon_E\big)(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\varepsilon\partial_E\bar u_\varepsilon)-\eta_\varepsilon h\otimes(\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\\
&\qquad\qquad+\kappa\big(2\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\otimes_s-\mathrm{Id}\otimes\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}-\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon)}+\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\nabla\eta_\varepsilon\\
&\qquad\qquad+\kappa\big(2\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes_s-\mathrm{Id}\otimes\mathfrak{d}\phi_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}-\mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\\
&\qquad\qquad\qquad\qquad\qquad+\mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon)\Big)\\
&-\varepsilon\int_U\eta_\varepsilon\nabla_i S_\varepsilon\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon):(\triangle^{-1}\nabla_i\mathfrak{d}F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E})(\tfrac{\cdot}{\varepsilon})\\
&+\varepsilon\int_U S_\varepsilon\cdot\Big((\triangle^{-1}\nabla_j\mathbb{1}_{\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\nabla_j(\eta_\varepsilon h)-\kappa\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon)}(\tfrac{\cdot}{\varepsilon})\triangle\eta_\varepsilon\\
&\qquad\qquad-\kappa\mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}(\tfrac{\cdot}{\varepsilon}):\eta_\varepsilon\triangle(\chi_\delta*\partial_E u_\varepsilon)\\
&\qquad\qquad-\kappa(\triangle^{-1}\nabla_i\mathfrak{d}F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E})(\tfrac{\cdot}{\varepsilon})\nabla_i(\eta_\varepsilon\nabla(\chi_\delta*\partial_E u_\varepsilon))\\
&\qquad\qquad+2\kappa\big(\mathfrak{d}\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}-\mathfrak{d}\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E}\otimes\mathrm{Id}\big)(\tfrac{\cdot}{\varepsilon}):\nabla(\chi_\delta*\partial_E u_\varepsilon)\otimes\nabla\eta_\varepsilon\\
&\qquad\qquad+\kappa\big(\mathfrak{d}^2\theta_{\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'}-\mathfrak{d}^2\gamma_{\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'}\otimes\mathrm{Id}-\triangle^{-1}\nabla\mathfrak{d}^2F_{|\chi_\delta*\mathrm{D}(u_\varepsilon),E,E'}\big)(\tfrac{\cdot}{\varepsilon})\\
&\qquad\qquad\qquad\qquad\qquad:\eta_\varepsilon\nabla(\chi_\delta*\partial_{E'}u_\varepsilon)\otimes\nabla(\chi_\delta*\partial_E u_\varepsilon)\Big).
\end{aligned}
$$

Adding and subtracting Q_ε to $\tilde Q_\varepsilon$ in the left-hand side, and proceeding as in Step 2 to estimate the different contributions, we deduce

$$
\begin{aligned}
\int_U(Q_\varepsilon)^2\;&\lesssim\;\int_U(\tilde Q_\varepsilon-Q_\varepsilon)^2+\Big(\int_U|\nabla S_\varepsilon|^2\Big)^{\frac12}\Big(\int_U|\nabla w_\varepsilon|^2\Big)^{\frac12}\\
&+\Big(\int_U|\nabla S_\varepsilon|^2\Big)^{\frac12}\Big(\int_{\partial_{3r_\varepsilon}U}(1+|h|^2+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2)\Big)^{\frac12}\\
&+\varepsilon\Big(\int_U|\nabla S_\varepsilon|^2\Big)^{\frac12}\Big(r_\varepsilon^{-2}\int_{\partial_{3r_\varepsilon}U}|(\psi,\Upsilon,\phi,\theta,\gamma,\mathfrak{d}\theta,\mathfrak{d}\gamma)|^2\big(1+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\Big)^{\frac12}\\
&+\varepsilon\Big(\int_U|\nabla S_\varepsilon|^2\Big)^{\frac12}\Big(\int_U|(1,\psi,\Upsilon,\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\gamma,\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)|^2\\
&\qquad\qquad\times\Big(|\langle\nabla\rangle h|^2+|\nabla^2\bar u_\varepsilon|^2+|\nabla\langle\nabla\rangle\chi_\delta*\mathrm{D}(u_\varepsilon)|^2+|\nabla\chi_\delta*\mathrm{D}(u_\varepsilon)|^4\\
&\qquad\qquad\qquad\qquad+|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\big(1+[\chi_\delta*\mathrm{D}(u_\varepsilon)]_{4\varepsilon}^4\big)\Big)\Big)^{\frac12}.
\end{aligned}
$$

Hence, by (3.45) and Young's inequality,

$$
\int_U(Q_\varepsilon)^2\;\lesssim\;\int_U(\tilde Q_\varepsilon-Q_\varepsilon)^2+\int_U|\nabla w_\varepsilon|^2+\int_{\partial_{3r_\varepsilon}U}(1+|h|^2+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2)
$$

$$
\begin{aligned}
&+\varepsilon^2 r_\varepsilon^{-2}\int_{\partial_{3r_\varepsilon}U}|(\psi,\Upsilon,\phi,\theta,\gamma,\mathfrak{d}\theta,\mathfrak{d}\gamma)|^2\big(1+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\\
&+\varepsilon^2\int_U|(1,\psi,\Upsilon,\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\gamma,\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)|^2\\
&\qquad\times\Big(|\langle\nabla\rangle h|^2+|\nabla^2\bar u_\varepsilon|^2+|\nabla\langle\nabla\rangle\chi_\delta*\mathrm{D}(u_\varepsilon)|^2+|\nabla\chi_\delta*\mathrm{D}(u_\varepsilon)|^4\\
&\qquad\qquad+|\nabla(\chi_\delta*\mathrm{D}(u_\varepsilon))|^2\big(1+[\chi_\delta*\mathrm{D}(u_\varepsilon)]_{4\varepsilon}^4\big)\Big).
\end{aligned}\tag{3.46}
$$

It remains to estimate the norm of $\tilde Q_\varepsilon-Q_\varepsilon$ in the right-hand side. By definition of $\tilde Q_\varepsilon$, we have

$$
\begin{aligned}
\tilde Q_\varepsilon-Q_\varepsilon\;=\;&\bar P_\varepsilon-T_0^\varepsilon[\bar P_\varepsilon]+\eta_\varepsilon\bar{\boldsymbol b}:\big(\mathrm{D}(\bar u_\varepsilon)-T_0^\varepsilon[\mathrm{D}(\bar u_\varepsilon)]\big)\\
&+\kappa\eta_\varepsilon\big(\bar{\boldsymbol c}(\chi_\delta*\mathrm{D}(u_\varepsilon))-\bar{\boldsymbol c}(T_0^\varepsilon[\chi_\delta*\mathrm{D}\,u_\varepsilon)])\big)+(\Sigma_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})\eta_\varepsilon(\partial_E\bar u_\varepsilon-T_0^\varepsilon[\partial_E\bar u_\varepsilon])\\
&+\kappa\big(\Pi_{\chi_\delta*\mathrm{D}(u_\varepsilon)}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}-\Pi_{T_0^\varepsilon[\chi_\delta*\mathrm{D}(u_\varepsilon)]}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big)(\tfrac{\cdot}{\varepsilon})\,\eta_\varepsilon,
\end{aligned}
$$

and thus, using (2.32) and proceeding as for (3.41), we get

$$
\begin{aligned}
\int_U(\tilde Q_\varepsilon-Q_\varepsilon)^2\;\lesssim\;&\varepsilon^2\int_U\big(|\nabla^2\bar u_\varepsilon|^2+|\nabla\bar P_\varepsilon|^2+|\nabla\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\\
&+\sum_n\int_{\varepsilon(I_n+B)}|(\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^2\Big|\nabla\bar u_\varepsilon-\fint_{\varepsilon(I_n+B)}\nabla\bar u_\varepsilon\Big|^2\\
&+\sum_n\int_{\varepsilon(I_n+B)}|(\mathfrak{d}\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^2\Big|\chi_\delta*\mathrm{D}(u_\varepsilon)-\fint_{\varepsilon(I_n+B)}\chi_\delta*\mathrm{D}(u_\varepsilon)\Big|^2.
\end{aligned}\tag{3.47}
$$

Inserting this into (3.46), the conclusion (3.43) follows.

Step 4. Conclusion.
Choosing $K\ge1$ large enough to absorb part of the pressure into the left-hand side, the combination of (3.22) and (3.42) yields

$$
\begin{aligned}
\int_U|\nabla w_\varepsilon|^2+\int_U(Q_\varepsilon)^2\;\lesssim\;&\Big(1+\|(h,\nabla\bar u_\varepsilon,\bar P_\varepsilon)\|_{W^{1,\infty}(U)}^2+\|\chi_\delta*\mathrm{D}(u_\varepsilon)\|_{W^{2,\infty}(U)}^6\Big)\\
&\times\bigg(\int_{\partial_{3r_\varepsilon}U}|(1,\nabla\psi,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\nabla\phi,\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^2\\
&+\varepsilon^2\int_U\big(1+r_\varepsilon^{-2}\mathbb{1}_{\partial_{3r_\varepsilon}U}\big)|(1,\psi,\nabla\psi,\Upsilon,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\phi,\theta,\gamma,\mathfrak{d}\phi,\nabla\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\mathfrak{d}\gamma,\\
&\qquad\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)(\tfrac{\cdot}{\varepsilon})|^2\bigg).
\end{aligned}\tag{3.48}
$$

By Proposition 1.6, we have $\|\nabla u_\varepsilon\|_{\mathrm{L}^2(U)}\lesssim1+\|h\|_{\mathrm{L}^2(U)}$, and thus for all $s>0$,

$$
\|\chi_\delta*\mathrm{D}(u_\varepsilon)\|_{W^{1+s,\infty}(U)}\;\lesssim\;\|\chi_\delta\|_{H^{1+s}(\mathbb{R}^d)}(1+\|h\|_{\mathrm{L}^2(U)}).
$$

By the regularity theory for the Stokes equation, using (2.32), we then deduce that the solution $(\bar u_\varepsilon,\bar P_\varepsilon)$ of (3.10) satisfies for all $s>0$,

$$
\begin{aligned}
\|(\nabla\bar u_\varepsilon,\bar P_\varepsilon)\|_{W^{1,\infty}(U)}\;&\lesssim\;\|h\|_{W^{s,\infty}(U)}+\|\bar{\boldsymbol B}_{\mathrm{act}}(\chi_\delta*\mathrm{D}(u_\varepsilon))\|_{W^{1+s,\infty}(U)}\\
&\lesssim\;\|h\|_{W^{s,\infty}(U)}+\|\chi_\delta\|_{H^{1+s}(\mathbb{R}^d)}(1+\|h\|_{\mathrm{L}^2(U)}).
\end{aligned}\tag{3.49}
$$

Inserting these estimates into (3.48), we get

$$\begin{aligned}\int_U |\nabla w_\varepsilon|^2 + \int_U (Q_\varepsilon)^2 \\ \lesssim_{\chi\delta} \ (1+\|h\|_{W^{1,\infty}(U)}^6)\bigg(\int_{\partial_{3r_\varepsilon}U} \big|\big(1,\nabla\psi,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\nabla\phi,\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big)(\tfrac{\cdot}{\varepsilon})\big|^2 \\ +\varepsilon^2\int_U \big(1+r_\varepsilon^{-2}\mathbb{1}_{\partial_{3r_\varepsilon}U}\big)\big|(1,\psi,\nabla\psi,\Upsilon,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\phi,\theta,\gamma,\mathfrak{d}\phi,\nabla\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\mathfrak{d}\gamma, \\ \mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)(\tfrac{\cdot}{\varepsilon})\big|^2\bigg). \qquad (3.50)\end{aligned}$$

By the ergodic theorem and by the sublinearity of correctors, cf. Lemmas 2.1, 2.2, 2.3, 2.4, and 2.5, we have for any fixed $r>0$, almost surely,

$$\lim_{\varepsilon\downarrow 0}\int_{\partial_{3r}U} \big|\big(1,\nabla\psi,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\nabla\phi,\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}\big)(\tfrac{\cdot}{\varepsilon})\big|^2 \ \lesssim\ Cr, \qquad (3.51)$$

and

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\varepsilon^2\int_U \big(1+r^{-2}\mathbb{1}_{\partial_{3r}U}\big)\big|(1,\psi,\nabla\psi,\Upsilon,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\phi,\theta,\gamma,\mathfrak{d}\phi,\nabla\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\mathfrak{d}\gamma, \\ \mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)(\tfrac{\cdot}{\varepsilon})\big|^2 \ =\ 0. \qquad (3.52)\end{aligned}$$

Some care is however needed to prove these convergence results as we take suprema in the notation (3.23)–(3.24): while the linear dependence of $\nabla\psi_E$ on E makes the suprema $|\nabla\psi| = \sup_E |E|^{-1}|\nabla\psi_E|$ trivial, the same is not true for $\nabla\phi_E$. In that case, we use the Sobolev embedding in form of

$$|(\nabla\phi)(\tfrac{\cdot}{\varepsilon})|^2 \le \sup_{|E|\le C_\delta(h)} |\nabla\phi_E(\tfrac{\cdot}{\varepsilon})|^2 \lesssim \sum_{l=0}^k \int_{|E|\le C_\delta(h)} |(\nabla\mathfrak{d}^l\phi_E)(\tfrac{\cdot}{\varepsilon})|^2\,dE, \qquad (3.53)$$

for some $k>\frac12\dim\mathbb{M}_0^{\mathrm{sym}}$, and thus

$$\int_{\partial_{3r}U} |(\nabla\phi)(\tfrac{\cdot}{\varepsilon})|^2 \lesssim \sum_{l=0}^k \int_{|E|\le C_\delta(h)} \Big(\int_{\partial_{3r}U} |(\nabla\mathfrak{d}^l\phi_E)(\tfrac{\cdot}{\varepsilon})|^2\Big)\,dE,$$

to which the ergodic theorem for (linearized) correctors $\nabla\mathfrak{d}^l\phi$ in Lemma 2.3 can now be applied, leading to the claim (3.51). Similarly, writing

$$\begin{aligned}\varepsilon^2\int_U \big(1+r^{-2}\mathbb{1}_{\partial_{3r}U}\big)|\phi(\tfrac{\cdot}{\varepsilon})|^2 \ &\le\ \varepsilon^2\int_U \big(1+r^{-2}\mathbb{1}_{\partial_{3r}U}\big)\sup_{|E|\le C_\delta(h)} |\phi_E(\tfrac{\cdot}{\varepsilon})|^2 \\ &\lesssim\ \sum_{l=0}^k \int_{|E|\le C_\delta(h)} \Big(\varepsilon^2\int_U \big(1+r^{-2}\mathbb{1}_{\partial_{3r}U}\big)|\mathfrak{d}^l\phi_E(\tfrac{\cdot}{\varepsilon})|^2\Big)\,dE,\end{aligned}$$

the claim (3.52) indeed follows from the sublinearity of the (linearized) correctors $\mathfrak{d}^l\phi$ in Lemma 2.3.

Next, inserting (3.51)–(3.52) into (3.50) and appealing to a diagonalization argument, we conclude that there exists a (random) sequence $r_\varepsilon\downarrow 0$ such that for this choice we have, almost surely,

$$\lim_{\varepsilon\downarrow 0}\Big(\int_U |\nabla w_\varepsilon|^2+\int_U (Q_\varepsilon)^2\Big)\,=\,0,$$

that is, $w_\varepsilon \to 0$ in $H^1_0(U)$ and $Q_\varepsilon \to 0$ in $\mathrm{L}^2(U)$. On the other hand, note that a priori estimates (1.19) and (3.49) entail that up to an extraction we have $u_\varepsilon \rightharpoonup u_0$ and $\bar u_\varepsilon \rightharpoonup \bar u_0$ in $H^1_0(U)$, for some $u_0, \bar u_0 \in H^1_0(U)^d$. Passing to the weak limit in equation (3.10) along this subsequence, we find that $\bar u_0$ satisfies

$$-\operatorname{div}(2\bar{\boldsymbol B}_{\mathrm{pas}}\,\mathrm{D}(\bar u_0))+\nabla \bar P_0=(1-\lambda)h+\operatorname{div}(2\kappa\bar{\boldsymbol B}_{\mathrm{act}}(\chi_\delta*\mathrm{D}(u_0))). \tag{3.54}$$

Now, by definition of the two-scale expansion error w_ε, cf. (3.8), together with the sublinearity of correctors, cf. Lemmas 2.1 and 2.3, the convergence $w_\varepsilon\to 0$ in $H^1_0(U)$ implies $u_\varepsilon-\bar u_\varepsilon\to 0$ in $\mathrm{L}^2(U)$, and thus $u_0=\bar u_0$. From (3.54), we deduce that $\bar u := u_0=\bar u_0$ actually satisfies the homogenized equation (1.20). In view of the well-posedness for the latter, we conclude $u_\varepsilon \rightharpoonup \bar u$ in $H^1_0(U)$ independently of extractions.

We turn to the convergence of the pressure field. Recall that we have shown $Q_\varepsilon\to 0$ in $\mathrm{L}^2(U)$. The a priori estimate (3.49) ensures $\bar P_\varepsilon \rightharpoonup \bar P$ in $\mathrm{L}^2(U)$, where $\bar P$ is the unique pressure field in $\mathrm{L}^2(U)/\mathbb{R}$ for the homogenized equation (3.54). By definition of Q_ε, cf. (3.8), together with the ergodic theorem for corrector pressures, cf. Lemmas 2.1 and 2.3, and with the choice (3.44) of P'_ε, the convergence of the pressure follows. ☐

3.3. **Quantitative homogenization and limit $\delta\downarrow 0$.** This section is devoted to the proof of Theorem 1.8. As the above proof is semi-quantitative, one can infer convergence rates provided that quantitative mixing assumptions such as Hypothesis 1.2 are further made on the statistical ensemble of inclusions. Quantitative rates then allow in particular to let the parameter δ tend to 0 in a nontrivial regime. We split the proof into three steps.

Step 1. Convergence of the homogenized equation (1.20) as $\delta\downarrow 0$.
Writing equation (1.20) as

$$-\mathrm{div}(2\bar{\boldsymbol B}_{\mathrm{pas}}\,\mathrm{D}(\bar u_\delta))+\nabla\bar P_\delta=(1-\lambda)h+\operatorname{div}(2\kappa\bar{\boldsymbol B}_{\mathrm{act}}(\chi_\delta*\mathrm{D}(\bar u_\delta))),$$

and appealing to the regularity theory for the Stokes equation, the unique solution $(\bar u_\delta,\bar P_\delta)\in H^1_0(U)^d\times\mathrm{L}^2(U)/\mathbb{R}$ satisfies for all $0<\eta<1$,

$$\|(\nabla\bar u_\delta,\bar P_\delta)\|_{W^{1-\eta,\infty}(U)}\,\lesssim_\eta\,\|h\|_{\mathrm{L}^\infty(U)}+\kappa\|\bar{\boldsymbol B}_{\mathrm{act}}(\chi_\delta*\mathrm{D}(\bar u_\delta))\|_{W^{1-\eta,\infty}(U)}.$$

Using (2.32), we deduce

$$\begin{aligned}\|(\nabla\bar u_\delta,\bar P_\delta)\|_{W^{1-\eta,\infty}(U)}\;&\lesssim_\eta\;\|h\|_{\mathrm{L}^\infty(U)}+\kappa\lambda\|\chi_\delta*\mathrm{D}(\bar u_\delta)\|_{W^{1-\eta,\infty}(U)}\\&\lesssim_\eta\;\|h\|_{\mathrm{L}^\infty(U)}+\kappa\lambda\|\nabla\bar u_\delta\|_{W^{1-\eta,\infty}(U)}.\end{aligned}$$

The smallness condition (1.18) yields $\kappa\lambda\lesssim\kappa\ell^{-d}\ll 1$, and we thus infer

$$\|(\nabla\bar u_\delta,\bar P_\delta)\|_{W^{1-\eta,\infty}(U)}\,\lesssim_\eta\,\|h\|_{\mathrm{L}^\infty(U)}.$$

Up to an extraction, this implies $(\nabla\bar u_\delta,\bar P_\delta)\to(\nabla\bar u_0,\bar P_0)$ in $\mathrm{L}^\infty(U)$, for some limit $(\bar u_0,\bar P_0)\in H^1_0(U)^d\times\mathrm{L}^2(U)/\mathbb{R}$. Passing to the limit in equation (1.20), we find that $(\bar u_0,\bar P_0)$ satisfies equation (1.28). Provided that $\kappa\lambda\ll 1$ is small enough, which is ensured by the smallness condition (1.18), the well-posedness of (1.28) follows from the same argument as for (1.20). We conclude that $(\bar u_0,\bar P_0)=(\bar u,\bar P)$ is the unique solution of (1.28) and that $\bar u_\delta\to\bar u$ in $W^{1,\infty}(U)$ and $\bar P_\delta\to\bar P$ in $\mathrm{L}^\infty(U)$.

Combining this with Theorem 1.7, by a diagonalization argument, we deduce that there is a (random) sequence $\delta_\varepsilon^\circ \downarrow 0$ such that, for any sequence $0 < \delta_\varepsilon \le \delta_\varepsilon^\circ$, the solution $(u_\varepsilon, P_\varepsilon)$ of (1.10) with $\delta = \delta_\varepsilon$ satisfies, as $\varepsilon \downarrow 0$,

$$\begin{array}{rcll} u_\varepsilon & \rightharpoonup & \bar u, & \text{in } H_0^1(U)^d, \\ P_\varepsilon \mathbb{1}_{U \setminus \mathcal{I}_\varepsilon(U)} & \rightharpoonup & (1-\lambda)\bar P + (1-\lambda)\bar{\boldsymbol{b}} : \mathrm{D}(\bar u) & \\ & & \qquad +(1-\lambda)\kappa\big(\bar{\boldsymbol{c}}(\mathrm{D}(\bar u)) - \fint_U \bar{\boldsymbol{c}}(\mathrm{D}(\bar u))\big), & \text{in } \mathrm{L}^2(U). \end{array} \tag{3.55}$$

To improve on such a diagonal result, we need to prove a quantitative version of Theorem 1.7 and capture the precise dependence on δ. This is the purpose of the next two steps.

Step 2. Corrector estimates.
As we proved in [10] for passive correctors, under a quantitative mixing assumption such as Hypothesis 1.2, we have for all $s < \infty$,

$$\begin{gathered} \mathbb{E}\Big[|(\nabla\psi, \Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}, \nabla\Upsilon)(x)|^s\Big]^{\frac1s} \lesssim_s 1, \\ \mathbb{E}\big[|(\psi, \Upsilon)(x)|^s\big]^{\frac1s} \lesssim_s \mu_d(x) := \left\{ \begin{array}{lll} \log^{\frac12}(2+|x|) & : & d=2, \\ 1 & : & d>2. \end{array} \right. \end{gathered} \tag{3.56}$$

which optimally quantifies the sublinearity of passive correctors. The method in [10] applies mutadis mutandis to active correctors, and yields the following: for all $E, E', E'' \in \mathbb{M}_0^{\mathrm{sym}}$ and $s < \infty$, we get

$$\begin{aligned} &\mathbb{E}\Big[|(\nabla\phi_E, \Pi_E\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}, \nabla\theta_E, \nabla\mathfrak{d}\phi_E, \nabla\mathfrak{d}\theta_{E,E'}, \nabla\mathfrak{d}^2\phi_{E,E',E''}, \nabla\mathfrak{d}^2\theta_{E,E',E''})(x)|^s\Big]^{\frac1s} \lesssim_{s,E,E',E''} 1, \\ &\mathbb{E}\Big[|(\phi_E, \gamma_E, \theta_E, \mathfrak{d}\phi_{E,E'}, \mathfrak{d}\gamma_{E,E'}, \mathfrak{d}\theta_{E,E'}, \\ &\qquad\qquad \mathfrak{d}^2\phi_{E,E',E''}, \mathfrak{d}^2\gamma_{E,E',E''}, \mathfrak{d}^2\theta_{E,E',E''})(x)|^s\Big]^{\frac1s} \lesssim_{s,E,E',E''} \mu_d(x). \end{aligned} \tag{3.57}$$

Yet, for our purposes, we further need corresponding estimates on suprema such as

$$|\nabla\phi| = \sup_{|E|\le C_\delta(h)} \langle E\rangle^{-1}|\nabla\phi_E|,$$

which is not trivial due to the nonlinear dependence on E. As we aim at capturing the best dependence on δ, we cannot appeal to brutal Sobolev estimates as in (3.53). Instead, we shall take advantage of the above moment estimates (3.57) together with Hypothesis 1.4. More precisely, we decompose

$$|\nabla\phi| \le \sup_E \langle E\rangle^{-1}|\nabla\phi_E^\infty| + \sup_E \langle E\rangle^{-1}|\nabla(\phi_E - \phi_E^\infty)|, \tag{3.58}$$

where we compare ϕ_E to the random field ϕ_E^∞ that is defined via the same corrector problem (2.4) & (2.5) with the swimming forces $\{f_n(E)\}_n$ replaced by their large-E approximations $\{f^\infty(E)\xi_n\}_n$, cf. Hypothesis 1.4. On the one hand, as $f^\infty(E)$ is a deterministic function of E, the supremum of $\nabla\phi_E^\infty$ over E becomes trivial and moment estimates can be established in the following form, for all $s < \infty$,

$$\mathbb{E}\Big[\big(\sup_E \langle E\rangle^{-1}|\nabla\phi_E^\infty|\big)^s\Big]^{\frac1s} \lesssim_s 1.$$

On the other hand, by the Sobolev embedding, we can bound for all $s<\infty$, provided $s>\dim\mathbb{M}_0^{\rm sym}$,

$$\sup_E \langle E\rangle^{-1}|\nabla(\phi_E-\phi_E^\infty)| \lesssim \Big(\int_{\mathbb{M}_0^{\rm sym}}\langle E\rangle^{-s}\big(|\nabla(\phi_E-\phi_E^\infty)|+|\nabla(\mathfrak{d}\phi_E-\mathfrak{d}\phi_E^\infty)|\big)^s\,dE\Big)^{\frac1s}$$

and thus, using the proof of moment estimates (3.57) together with Hypothesis 1.4 in form of

$$\mathbb{E}\Big[\big(|\nabla(\phi_E-\phi_E^\infty)|+|\nabla(\mathfrak{d}\phi_E-\mathfrak{d}\phi_E^\infty)|\big)^s\Big]^{\frac1s}\lesssim_s \langle E\rangle^{1-\gamma},$$

we deduce for all $s<\infty$ with $s>(1\vee\frac1\gamma)\dim\mathbb{M}_0^{\rm sym}$,

$$\mathbb{E}\Big[\big(\sup_E\langle E\rangle^{-1}|\nabla(\phi_E-\phi_E^\infty)|\big)^s\Big]^{\frac1s}\lesssim\Big(\int_{\mathbb{M}_0^{\rm sym}}\langle E\rangle^{-\gamma s}\,dE\Big)^{\frac1s}\lesssim_s 1.$$

Combining these bounds with (3.58), we deduce for all $s<\infty$,

$$\mathbb{E}\left[|\nabla\phi|^s\right]^{\frac1s}\lesssim_s 1,$$

uniformly with respect to $\delta>0$. This string of arguments allows us to post-process (3.57) into

$$\begin{aligned}\mathbb{E}\Big[|(\nabla\phi,\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\nabla\theta,\nabla\mathfrak{d}\phi,\nabla\mathfrak{d}\theta,\nabla\mathfrak{d}^2\phi,\nabla\mathfrak{d}^2\theta)(x)|^s\Big]^{\frac1s}&\lesssim_s 1,\\ \mathbb{E}\Big[|(\phi,\gamma,\theta,\mathfrak{d}\phi,\mathfrak{d}\gamma,\mathfrak{d}\theta,\mathfrak{d}^2\phi,\mathfrak{d}^2\gamma,\mathfrak{d}^2\theta)(x)|^s\Big]^{\frac1s}&\lesssim_s \mu_d(x).\end{aligned}$$

Step 3. Conclusion.
In order to capture the best dependence on δ, we need a version of (3.48) where the dependence on $\bar u_\varepsilon,\chi_\delta*\mathrm{D}(u_\varepsilon)$ does not deteriorate in terms of uniform norms. Rather combining (3.25) and (3.43), and choosing $K\ge1$ large enough, we get

$$\begin{aligned}&\int_U|\nabla w_\varepsilon|^2+\int_U(Q_\varepsilon)^2\\ &\lesssim\int_{\partial_{3r_\varepsilon}U}|(1,\nabla\psi,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\nabla\phi,\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^2\big(1+|h|^2+|\nabla\bar u_\varepsilon|^2+[\nabla\bar u_\varepsilon]_{4\varepsilon}^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\\ &\qquad+\varepsilon^2r_\varepsilon^{-2}\int_{\partial_{3r_\varepsilon}U}|(\psi,\Upsilon,\phi,\theta,\gamma,\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\gamma)(\tfrac{\cdot}{\varepsilon})|^2\big(1+|\nabla\bar u_\varepsilon|^2+|\chi_\delta*\mathrm{D}(u_\varepsilon)|^2\big)\\ &\qquad+\varepsilon^2\int_U|(1,\psi,\Upsilon,\phi,\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\gamma,\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)(\tfrac{\cdot}{\varepsilon})|^2\\ &\qquad\qquad\times\Big(|\langle\nabla\rangle h|^2+|\nabla^2\bar u_\varepsilon|^2+|\nabla\bar P_\varepsilon|^2+|\nabla\langle\nabla\rangle\chi_\delta*\mathrm{D}(u_\varepsilon)|^2+|\nabla\chi_\delta*\mathrm{D}(u_\varepsilon)|^4\\ &\qquad\qquad\qquad+|\nabla(\chi_\delta*D(u_\varepsilon))|^2\big(1+[\chi_\delta*\mathrm{D}(u_\varepsilon)]_{4\varepsilon}^4\big)\Big)\\ &\qquad+\sum_n\int_{\varepsilon(I_n+B)}|(\psi,\nabla\psi,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^2\Big|\nabla\bar u_\varepsilon-\fint_{\varepsilon(I_n+B)}\nabla\bar u_\varepsilon\Big|^2\\ &\qquad+\sum_n\int_{\varepsilon(I_n+B)}|(\mathfrak{d}\phi,\nabla\mathfrak{d}\phi,\mathfrak{d}\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^2\Big|\chi_\delta*\mathrm{D}(u_\varepsilon)-\fint_{\varepsilon(I_n+B)}\chi_\delta*\mathrm{D}(u_\varepsilon)\Big|^2.\end{aligned}$$

By Proposition 1.6, we have $\|\nabla u_\varepsilon\|_{\mathrm{L}^2(U)} \lesssim 1 + \|h\|_{\mathrm{L}^2(U)}$ and thus, for all $k \ge 0$ and $s \ge 2$,

$$\begin{aligned}\|\chi_\delta * \mathrm{D}(u_\varepsilon)\|_{W^{k,s}(U)} &\lesssim \delta^{-k-d(\frac12-\frac1s)}\big(1+\|h\|_{\mathrm{L}^2(U)}\big),\\ \|\chi_\delta * \mathrm{D}(u_\varepsilon)\|_{W^{k,s}(\partial_{r_\varepsilon}U)} &\lesssim \delta^{-k-\frac d2}\big(\delta^d\wedge r_\varepsilon\big)^{\frac1s}\big(1+\|h\|_{\mathrm{L}^2(U)}\big).\end{aligned}$$

By the regularity theory for the Stokes equation, using (2.32), we then deduce that the solution $(\bar u_\varepsilon, \bar P_\varepsilon)$ of (3.10) satisfies for all $s \ge 2$ and $\eta > 0$,

$$\begin{aligned}\|(\nabla\bar u_\varepsilon, \bar P_\varepsilon)\|_{W^{1,s}(U)} &\lesssim \delta^{-1-d(\frac12-\frac1s)}\big(1+\|h\|_{\mathrm{L}^{2\vee s}(U)}\big),\\ \|(\nabla\bar u_\varepsilon, \bar P_\varepsilon)\|_{\mathrm{L}^{s}(\partial_{3r_\varepsilon}U)} &\lesssim \delta^{-\frac d2}\big(\delta^d\wedge(r_\varepsilon\delta^{-\eta})\big)^{\frac1s}\big(1+\|h\|_{\mathrm{L}^\infty(U)}\big).\end{aligned}$$

Inserting these estimates into the above, we get for all $s \ge 1$,

$$\begin{aligned}&\int_U |\nabla w_\varepsilon|^2 + \int_U (Q_\varepsilon)^2\\ &\lesssim \Big(r_\varepsilon + (r_\varepsilon\delta^{-d-\eta})\wedge(r_\varepsilon\delta^{-d})^{\frac1s}\Big)\Big(\fint_{\partial_{3r_\varepsilon}U}|(1,\nabla\psi,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\nabla\phi,\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}})(\tfrac{\cdot}{\varepsilon})|^{2s}\Big)^{\frac1s}\\ &\quad+\varepsilon^2 r_\varepsilon^{-2}\Big(r_\varepsilon + (r_\varepsilon\delta^{-d-\eta})\wedge(r_\varepsilon\delta^{-d})^{\frac1s}\Big)\Big(\fint_{\partial_{3r_\varepsilon}U}|(\psi,\Upsilon,\phi,\theta,\gamma,\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\gamma)(\tfrac{\cdot}{\varepsilon})|^{2s}\Big)^{\frac1s}\\ &\qquad+\varepsilon^2\delta^{-2-d(2+\frac1s)}\Big(\int_U|(1,\psi,\Upsilon,\nabla\psi,\Sigma\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\phi,\mathfrak{d}\phi,\nabla\mathfrak{d}\phi,\mathfrak{d}\theta,\mathfrak{d}\Pi\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}},\mathfrak{d}\gamma,\\ &\qquad\qquad\mathfrak{d}^2\theta,\mathfrak{d}^2\gamma,\triangle^{-1}\nabla\mathbb{1}_{\mathcal{I}},\triangle^{-1}\nabla\mathfrak{d}F,\triangle^{-1}\nabla\mathfrak{d}^2F)(\tfrac{\cdot}{\varepsilon})|^{2s}\Big)^{\frac1s},\end{aligned}$$

where the multiplicative constant depends on the $W^{1,\infty}(U)$ norm of h. Hence, taking the expectation and using the corrector estimates of Step 2, we get for all $s < \infty$,

$$\begin{aligned}&\mathbb{E}\Big[\Big(\int_U|\nabla w_\varepsilon|^2\Big)^s\Big]^{\frac1s} + \mathbb{E}\Big[\Big(\int_U(Q_\varepsilon)^2\Big)^s\Big]^{\frac1s}\\ &\qquad\lesssim_h \big(1+\varepsilon^2\mu_d(\tfrac1\varepsilon)^2 r_\varepsilon^{-2}\big)\Big(r_\varepsilon + (r_\varepsilon\delta^{-d-\eta})\wedge(r_\varepsilon\delta^{-d})^{\frac1s}\Big) + \varepsilon^2\mu_d(\tfrac1\varepsilon)^2\delta^{-2-d(2+\frac1s)}\end{aligned}$$

Choosing $r_\varepsilon := \varepsilon\mu_d(\frac1\varepsilon)$, this becomes

$$\begin{aligned}&\mathbb{E}\Big[\Big(\int_U|\nabla w_\varepsilon|^2\Big)^s\Big]^{\frac1s} + \mathbb{E}\Big[\Big(\int_U(Q_\varepsilon)^2\Big)^s\Big]^{\frac1s}\\ &\qquad\lesssim_h \varepsilon\mu_d(\tfrac1\varepsilon) + \varepsilon\mu_d(\tfrac1\varepsilon)\delta^{-d-\eta} + (\varepsilon\mu_d(\tfrac1\varepsilon))^2\delta^{-2-d(2+\frac1s)}.\end{aligned}$$

As $\varepsilon,\delta \downarrow 0$ in the regime (1.27), we thus get $w_\varepsilon \to 0$ in $\mathrm{L}^s(\Omega; H^1(U))$ and $Q_\varepsilon \to 0$ in $\mathrm{L}^s(\Omega;\mathrm{L}^2(U))$. Arguing as for Theorem 1.7, and further using the result of Step 1, the conclusion follows. ∎

4. Dilute expansion of the effective viscosity

This section is devoted to the proof of Theorem 1.9, that is, the first-order dilute expansion of the effective viscosity $\bar{\boldsymbol B}_{\mathrm{tot}}$. We recall that Einstein's formula for the passive contribution was already established in [5] (see also [16, 15, 18, 17]), in form of

$$\big|\bar{\boldsymbol B}_{\mathrm{pas}} - \mathrm{Id} - \lambda_1\bar{\boldsymbol B}^{(1)}_{\mathrm{pas}}\big| \lesssim \lambda_2|\log\lambda_1|,$$

and it remains to prove

$$\big|\bar{\boldsymbol{B}}_{\mathrm{act}}(E)-\lambda_1\bar{\boldsymbol{B}}^{(1)}_{\mathrm{act}}(E)\big| \lesssim \langle E\rangle\Big(\lambda_2|\log\lambda_1|+(\lambda_1\lambda_2+\lambda_1\lambda_3|\log\lambda_1|)^{\frac12}\Big). \tag{4.1}$$

We split the proof into four steps.

Step 1. Periodic approximation.
Define a periodized version $\mathcal{P}_L$ of the point process $\mathcal{P}=\{x_n\}_n$ on the cube $Q_L:=[-\frac{L}{2},\frac{L}{2})^d$,

$$\mathcal{P}_L:=\{x_n:n\in N_L\},\qquad N_L:=\{n:x_n\in Q_{L-4}\},$$

and consider the corresponding random set

$$\mathcal{I}_L:=\bigcup_{n\in N_L}I_n,\qquad I_n:=x_n+I_n^\circ.$$

For notational convenience, we choose an enumeration $\mathcal{P}_L:=\{x_{n,L}\}_n$ and we set $I_{n,L}:=x_{n,L}+I^\circ_{n,L}$. By definition, under Hypothesis 1.1, for all L, the periodized random set $\mathcal{I}_L+L\mathbb{Z}^d$ satisfies the same regularity and hardcore conditions as in Hypothesis 1.1. Moreover, we emphasize the stabilization property $\mathcal{P}_L|_{Q_{L-4}}=\mathcal{P}|_{Q_{L-4}}$. Next, we define $\psi_{E;L}\in\mathrm{L}^2(\Omega;H^1_{\mathrm{per}}(Q_L)^d)$ as the unique almost sure solution of the periodic version of (2.1),

$$\begin{cases}-\triangle\psi_{E;L}+\nabla\Sigma_{E;L}=0, & \text{in } Q_L\setminus\mathcal{I}_L,\\ \operatorname{div}(\psi_{E;L})=0, & \text{in } Q_L\setminus\mathcal{I}_L,\\ \mathrm{D}(\psi_{E;L}+Ex)=0, & \text{in } \mathcal{I}_L,\\ \int_{\partial I_{n,L}}\sigma(\psi_{E;L}+Ex,\Sigma_{E;L})\nu=0, & \forall n,\\ \int_{\partial I_{n,L}}\Theta(x-\varepsilon x_{n,L})\cdot\sigma(\psi_{E;L}+Ex,\Sigma_{E;L})\nu=0, & \forall n,\ \forall\Theta\in\mathbb{M}^{\mathrm{skew}}.\end{cases} \tag{4.2}$$

It is easily checked that the map $\bar{\boldsymbol{B}}_{\mathrm{act}}$ defined in (1.26) can be reformulated as

$$\begin{aligned}E':2\bar{\boldsymbol{B}}_{\mathrm{act}}(E)&=\lim_{L\uparrow\infty}E':2\bar{\boldsymbol{B}}_{\mathrm{act};L}(E),\\ E':2\bar{\boldsymbol{B}}_{\mathrm{act};L}(E)&:=-\mathbb{E}\bigg[L^{-d}\sum_{n\in Q_L}\int_{I_{n,L}+B}(\psi_{E';L}+E'(x-x_{n,L}))\cdot f_{n,L}(E)\bigg].\end{aligned} \tag{4.3}$$

We start by decomposing

$$E':2\bar{\boldsymbol{B}}_{\mathrm{act};L}(E)\,=\,E':\bar{\boldsymbol{B}}^{(1)}_{\mathrm{act};L}(E)+E':R_{1;L}(E)+E':R_{2;L}(E), \tag{4.4}$$

where we have set

$$E':\bar{\boldsymbol{B}}^{(1)}_{\mathrm{act};L}(E)\,:=\,-L^{-d}\mathbb{E}\bigg[\sum_n\int_{I_{n,L}+B}(\psi^n_{E';L}+E'(x-x_{n,L}))\cdot f_{n,L}(E)\bigg],$$

and where the remainders $R_{1;L}(E),R_{2;L}(E)$ are given by

$$\begin{aligned}E':R_{1;L}(E)&:=&-L^{-d}\mathbb{E}\bigg[\sum_{n\neq m}\int_{I_{n,L}+B}\psi^m_{E';L}\cdot f_{n,L}(E)\bigg],\\ E':R_{2;L}(E)&:=&-L^{-d}\mathbb{E}\bigg[\sum_n\int_{I_{n,L}+B}\Big(\psi_{E';L}-\sum_m\psi^m_{E';L}\Big)\cdot f_{n,L}(E)\bigg],\end{aligned}$$

in terms of the solution $\psi^n_{E';L}$ of the single-particle periodized problem

$$\begin{cases} -\triangle\psi^n_{E;L}+\nabla\Sigma^n_{E;L}=0, & \text{in } Q_L\setminus I_{n,L},\\ \operatorname{div}(\psi^n_{E;L})=0, & \text{in } Q_L\setminus I_{n,L},\\ \mathrm{D}(\psi^n_{E;L}+Ex)=0, & \text{in } I_{n,L},\\ \fint_{\partial I_{n,L}}\sigma(\psi^n_{E;L}+Ex,\Sigma^n_{E;L})\nu=0, & \forall n,\\ \fint_{\partial I_{n,L}}\Theta(x-\varepsilon x_{n,L})\cdot\sigma(\psi^n_{E;L}+Ex,\Sigma^n_{E;L})\nu=0, & \forall n,\ \forall\Theta\in\mathbb{M}^{\mathrm{skew}}.\end{cases}\tag{4.5}$$

Step 2. Proof that

$$|R_{1;L}|\lesssim\lambda_2\big(|\log\lambda_1|+\tfrac{\log L}{L}\big).\tag{4.6}$$

As by assumption the point process is independent of particles' shapes and swimming forces, we can write, in terms of the two-point intensity g_2, cf. (1.29),

$$E':R_{1;L}(E)\;=\;-L^{-d}\iint_{Q_{L-4}\times Q_{L-4}}\mathbb{E}\Big[\int_{Q_L}\psi^\circ_{E';L}(\cdot+x-y)\cdot\tilde f^\circ(E)\Big]\,g_2(x,y)\,dxdy,$$

where $\tilde f^\circ$ is an iid copy of f°, hence independent of $\psi^\circ_{E';L}$. Noting that the periodicity of $\psi^\circ_{E';L}$ yields

$$\int_{Q_L}\Big(\int_{Q_L}\psi^\circ_{E';L}(\cdot+x-y)\cdot\tilde f^\circ(E)\Big)\,dx=0,\tag{4.7}$$

we can replace the two-point density g_2 by the correlation function $h_2=g_2-\lambda_1^2$, to the effect of

$$\begin{aligned}E':R_{1;L}(E)\;=\;&-L^{-d}\iint_{Q_L\times Q_{L-4}}\mathbb{E}\Big[\int_{Q_L}\psi^\circ_{E';L}(\cdot+x-y)\cdot\tilde f^\circ(E)\Big]\,h_2(x,y)\,dxdy\\&+L^{-d}\iint_{(Q_L\setminus Q_{L-4})\times Q_{L-4}}\mathbb{E}\Big[\int_{Q_L}\psi^\circ_{E';L}(\cdot+x-y)\cdot\tilde f^\circ(E)\Big]\,g_2(x,y)\,dxdy.\end{aligned}$$

The neutrality condition (1.9) entails

$$\Big|\int_{Q_L}\psi^\circ_{E';L}(\cdot+x-y)\cdot\tilde f^\circ(E)\Big|\lesssim\langle E\rangle\Big(\int_{2B}|\nabla\psi^\circ_{E';L}(\cdot+x-y)|^2\Big)^{\frac12},$$

and thus, using standard decay estimates, see e.g. [5, Lemma 4.2],

$$\Big|\int_{Q_L}\psi^\circ_{E';L}(\cdot+x-y)\cdot\tilde f^\circ(E)\Big|\lesssim\langle E\rangle\langle x-y\rangle^{-d}.$$

The above then becomes

$$\begin{aligned}|R_{1;L}|\lesssim L^{-d}&\iint_{Q_L\times Q_L}\langle x-y\rangle^{-d}\,|h_2(x,y)|\,dxdy\\&+L^{-d}\iint_{(Q_L\setminus Q_{L-4})\times Q_L}\langle x-y\rangle^{-d}\,g_2(x,y)\,dxdy.\end{aligned}$$

The definition of two-point intensity (1.30) and the decay of correlations (1.31) yield

$$\begin{aligned}|h_2(x',y')|&\lesssim(\lambda_1^2+g_2(x,y))\wedge\langle x-y\rangle^{-\gamma},\\ g_2(x,y)&=g_2(x,y)\mathbb{1}_{|x-y|\ge2\ell},\\ \fint_{B_\ell(x)\times B_\ell(y)}&g_2(x',y')\,dx'dy'\lesssim\lambda_2,\end{aligned}$$

and the claim (4.6) easily follows.

Step 3. Proof that
$$|R_{2;L}| \lesssim (\lambda_1)^{\frac12}(\lambda_2+\lambda_3|\log\lambda_1|)^{\frac12}. \tag{4.8}$$
We start by decomposing
$$\begin{aligned} E':R_{2;L}(E) \,=\, -L^{-d}\mathbb{E}\Big[\sum_n\int_{(I_{n,L}+B)\setminus I_{n,L}}\Big(\psi_{E';L}-\sum_m\psi^m_{E';L}\Big)\cdot f_{n,L}(E)\Big] \\ +L^{-d}\mathbb{E}\Big[\sum_n\int_{\partial I_{n,L}}\Big(\psi_{E';L}-\sum_m\psi^m_{E';L}\Big)\cdot\sigma(\phi_{E;L},\Pi_{E;L})\nu\Big] \\ -L^{-d}\mathbb{E}\Big[\sum_n\int_{I_{n,L}+B}\Big(\psi_{E';L}-\sum_m\psi^m_{E';L}\Big)\cdot\Big(f_{n,L}(E)\mathbb{1}_{I_{n,L}}+\delta_{\partial I_{n,L}}\sigma(\phi_{E;L},\Pi_{E;L})\nu\Big)\Big]. \end{aligned} \tag{4.9}$$
The first two terms can be recovered in the weak formulation of the equation for $\phi_{E;L}$ (periodized version of (2.4) as in (4.2)), when tested with $\psi_{E';L}-\sum_m\psi^m_{E';L}$,
$$\begin{aligned} -L^{-d}\mathbb{E}\Big[\sum_n\int_{(I_{n,L}+B)\setminus I_{n,L}}\Big(\psi_{E';L}-\sum_m\psi^m_{E';L}\Big)\cdot f_{n,L}(E)\Big] \\ +L^{-d}\mathbb{E}\Big[\sum_n\int_{\partial I_{n,L}}\Big(\psi_{E';L}-\sum_m\psi^m_{E';L}\Big)\cdot\sigma(\phi_{E;L},\Pi_{E;L})\nu\Big] \\ =\,-\int_{Q_L}\nabla\phi_{E;L}:\nabla\Big(\psi_{E';L}-\sum_m\psi^m_{E';L}\Big), \end{aligned}$$
and thus, similarly testing the equation for $\psi_{E';L}-\sum_m\psi^m_{E';L}$ with $\phi_{E;L}$, using boundary conditions and the rigidity condition for $\phi_{E;L}$ in $\mathcal{I}_L$,
$$\begin{aligned} -L^{-d}\mathbb{E}\Big[\sum_n\int_{(I_{n,L}+B)\setminus I_{n,L}}\Big(\psi_{E';L}-\sum_m\psi^m_{E';L}\Big)\cdot f_{n,L}(E)\Big] \\ +L^{-d}\mathbb{E}\Big[\sum_n\int_{\partial I_{n,L}}\Big(\psi_{E';L}-\sum_m\psi^m_{E';L}\Big)\cdot\sigma(\phi_{E;L},\Pi_{E;L})\nu\Big] \,=\, 0. \end{aligned}$$
Next, using boundary conditions for $\phi_{E;L}$, noting that $\psi_{E';L}-\psi^n_{E';L}$ is rigid in $I_{n,L}$, the last term in (4.9) can be rewritten as
$$\begin{aligned} -L^{-d}\mathbb{E}\Big[\sum_n\int_{I_{n,L}+B}\Big(\psi_{E';L}-\sum_m\psi^m_{E';L}\Big)\cdot\Big(f_{n,L}(E)\mathbb{1}_{I_{n,L}}+\delta_{\partial I_{n,L}}\sigma(\phi_{E;L},\Pi_{E;L})\nu\Big)\Big] \\ =\,L^{-d}\mathbb{E}\Big[\sum_{n\ne m}\int_{I_{n,L}+B}\psi^m_{E';L}\cdot\Big(f_{n,L}(E)\mathbb{1}_{I_{n,L}}+\delta_{\partial I_{n,L}}\sigma(\phi_{E;L},\Pi_{E;L})\nu\Big)\Big]. \end{aligned}$$
Inserting these identities into (4.9), we get
$$E':R_{2;L}(E) \,=\, L^{-d}\mathbb{E}\Big[\sum_{n\ne m}\int_{I_{n,L}+B}\psi^m_{E';L}\cdot\Big(f_{n,L}(E)\mathbb{1}_{I_{n,L}}+\delta_{\partial I_{n,L}}\sigma(\phi_{E;L},\Pi_{E;L})\nu\Big)\Big].$$
Using boundary conditions for $\phi_{E;L}$ to replace $\psi^m_{E';L}$ by $\psi^m_{E';L}-\fint_{I_{n,L}}\psi^m_{E';L}$, using the Poincaré inequality, and a trace estimate, this can be estimated as

$$|E' : R_{2;L}(E)| \lesssim \mathbb{E}\bigg[L^{-d}\sum_n \int_{I_{n,L}+B}\Big|\sum_{m:m\neq n}\nabla\psi^m_{E';L}\Big|^2\bigg]^{\frac12} \\ \times \mathbb{E}\bigg[L^{-d}\sum_n \int_{I_{n,L}+B}|f_{n,L}(E)|^2 + L^{-d}\sum_n\int_{I_{n,L}+B}|\nabla\phi_{E;L}|^2\bigg]^{\frac12}.$$

Taking advantage of explicit renormalizations as in [5, Section 4.4], the first factor can easily be estimated by

$$\mathbb{E}\bigg[L^{-d}\sum_n \int_{I_{n,L}+B}\Big|\sum_{m:m\neq n}\nabla\psi^m_{E';L}\Big|^2\bigg] \lesssim (\lambda_2+\lambda_3|\log\lambda_1|)|E'|^2.$$

Further noting that

$$\mathbb{E}\bigg[L^{-d}\sum_n\int_{I_{n,L}+B}|f_{n,L}(E)|^2\bigg] \lesssim \lambda_1\int_{2B}|f_\circ(E)|^2 \lesssim \lambda_1\langle E\rangle^2, \tag{4.10}$$

and using the hardcore condition, we deduce

$$|R_{2;L}(E)| \lesssim (\lambda_2+\lambda_3|\log\lambda_1|)^{\frac12}\bigg(\lambda_1\langle E\rangle^2+\mathbb{E}\Big[\fint_{Q_L}|\nabla\phi_{E;L}|^2\Big]\bigg)^{\frac12}. \tag{4.11}$$

It remains to estimate the last integral: starting from the energy identity for $\phi_{E;L}$,

$$\int_{Q_L}|\nabla\phi_{E;L}|^2 = \sum_n\int_{I_{n,L}+B}\phi_{E;L}\cdot f_{n,L}(E),$$

and using boundary conditions and the Poincaré inequality to estimate the right-hand side, we find

$$\int_{Q_L}|\nabla\phi_{E;L}|^2 \lesssim \bigg(\sum_n\int_{I_{n,L}+B}|f_{n,L}(E)|^2\bigg)^{\frac12}\bigg(\sum_n\int_{I_{n,L}+B}|\nabla\phi_{E;L}|^2\bigg)^{\frac12}.$$

Hence, using the hardcore condition, absorbing the last factor, taking the expectation, and combining with (4.10), we obtain

$$\mathbb{E}\Big[\fint_{Q_L}|\nabla\phi_{E;L}|^2\Big] \lesssim \mathbb{E}\bigg[L^{-d}\sum_n\int_{I_{n,L}+B}|f_{n,L}(E)|^2\bigg] \lesssim \lambda_1\langle E\rangle^2.$$

Inserting this into (4.11), the claim (4.8) follows.

Step 4. Conclusion.
In view of Steps 1, 2 and 3, it remains to examine the limit of the main term $\bar{\boldsymbol{B}}^{(1)}_{\mathrm{act};L}$ in (4.4). As by assumption the point process $\{x_n\}$ is independent of particles' shapes and swimming forces, we can write

$$E' : \bar{\boldsymbol{B}}^{(1)}_{\mathrm{act};L}(E) \;=\; -\lambda_1 L^{-d}|Q_{L-4}|\mathbb{E}\bigg[\int_{2B}(\psi^\circ_{E';L}+E'x)\cdot f^\circ(E)\bigg],$$

and thus, as $L\uparrow\infty$,

$$\lim_{L\uparrow\infty} E' : \bar{\boldsymbol{B}}^{(1)}_{\mathrm{act};L}(E) \;=\; -\lambda_1\mathbb{E}\bigg[\int_{2B}(\psi^\circ_{E'}+E'x)\cdot f^\circ(E)\bigg],$$

in terms of the solution $\psi^\circ_{E'}$ of the whole-space single-particle problem (1.33). In case of spherical particles, $I^\circ = B$, the latter is explicitly solvable, cf. [21, Section 2.1.3],

$$\psi^\circ_{E'}(x) = \begin{cases} -E'x & : \quad |x| \le 1, \\ -\frac{d+2}{2}\frac{(x\cdot E'x)x}{|x|^{d+2}}\big(1-\frac{1}{|x|^2}\big) - \frac{E'x}{|x|^{d+2}} & : \quad |x|>1, \end{cases}$$

and the conclusion follows. ☐

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