A DUALITY METHOD FOR MEAN-FIELD LIMITS WITH SINGULAR INTERACTIONS

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Abstract. We introduce a new approach to justify mean-field limits for first- and second-order particle systems with singular interactions. It is based on a duality approach combined with the analysis of linearized dual correlations, and it allows to cover for the first time arbitrary square-integrable interaction forces at possibly vanishing temperature. In case of first-order systems, it allows to recover in particular the mean-field limit to the 2d Euler and Navier–Stokes equations. We postpone to a forthcoming work the development of quantitative estimates and the extension to more singular interactions.

Contents

1. Introduction

Consider the classical Newton dynamics for N indistinguishable point-particles with pairwise interactions. Letting $d \geq 1$ be the space dimension, we denote by $X_{i,N} \in \Omega$ and $V_{i,N} \in \mathbb{R}^d$ the positions and velocities of the particles, labeled by $1 \leq i \leq N$, where the space domain $\Omega \subset \mathbb{R}^d$ stands either for the whole space $\mathbb{R}^d$ or for the periodic torus $\mathbb{T}^d$. The evolution of the particle system is given by the following ODEs,

$$\begin{cases} \dfrac{d}{dt} X_{i,N} = V_{i,N}, \\ \dfrac{d}{dt} V_{i,N} = \dfrac{1}{N-1} \displaystyle\sum_{j:j\neq i}^{N} K(X_{i,N} - X_{j,N}), \end{cases}$$

where $K : \Omega \to \mathbb{R}^d$ is an interaction force kernel with $K \in L^1_{loc}(\Omega;\mathbb{R}^d)$ and with the action-reaction condition $K(x-y) = -K(y-x)$, and where the mean-field scaling is considered.[1] As the upcoming results apply identically to stochastic models, we rather consider more generally the following system of SDEs including the effects of Brownian forces,

$$\begin{cases} dX_{i,N} = V_{i,N}dt, \\ dV_{i,N} = \dfrac{1}{N-1}\displaystyle\sum_{j:j\neq i}^N K(X_{i,N}-X_{j,N})\,dt + \sqrt{2\alpha}\,dB_{i,N}, \end{cases} \tag{1}$$

where $\{B_{i,N}\}_{1\le i\le N}$ are N independent Brownian motions and where the temperature $0 \le \alpha < \infty$ is a fixed parameter. We take α independent of N for simplicity, but easy extensions would allow to have $\alpha = \alpha_N$ with for example $\alpha_N \to 0$.

Taking a statistical perspective, we consider a probability density F_N on the N-particle phase space $\mathbb{D}^N := (\Omega\times\mathbb{R}^d)^N$, and Newton's equations (1) are then equivalent to the following Liouville equation,

$$\partial_t F_N + \sum_{i=1}^N \Big(v_i\cdot\nabla_{x_i}F_N + \frac{1}{N-1}\sum_{j:j\neq i}^N K(x_i-x_j)\cdot\nabla_{v_i}F_N\Big) = \alpha\sum_{i=1}^N \Delta_{v_i}F_N. \tag{2}$$

The exchangeability of the particles amounts to assuming that F_N is symmetric with respect to its different entries

$$z_i = (x_i,v_i) \in \mathbb{D} := \Omega\times\mathbb{R}^d, \quad \text{for } 1\le i\le N.$$

More precisely, we shall assume that at initial time $t=0$ particles are f°-chaotic in the sense of

$$F_N|_{t=0} \;=\; (f^\circ)^{\otimes N}, \tag{3}$$

for some probability density $f^\circ \in \mathcal{P}(\mathbb{D})\cap L^\infty(\mathbb{D})$. This exact chaoticity assumption could be partly relaxed in our argument, but we do not pursue in that direction here.

In the macroscopic limit $N\uparrow\infty$, we aim at an averaged description of the system, describing the evolution of a typical particle as given by

[1] We use the prefactor $\frac{1}{N-1}$ instead of the usual $\frac{1}{N}$ for mere convenience, but it of course does not change anything in the sequel.

first marginal

$$F_{N,1}(z) := \int_{\mathbb{D}^{N-1}} F_N(z, z_2, \ldots, z_N)\, dz_2 \ldots dz_N.$$

As is well known, formally neglecting particle correlations, we are led to expect that $F_{N,1}$ remains close to a solution $f \in L^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{D}) \cap L^\infty(\mathbb{D}))$ of the following mean-field Vlasov equation,

$$\partial_t f + v \cdot \nabla_x f + (K * f) \cdot \nabla_v f = \alpha \Delta_v f, \qquad f|_{t=0} = f^\circ, \tag{4}$$

where $K * f(x) := \int_{\mathbb{D}} K(x - x') f(x', v')\, dx' dv'$. More generally, the following propagation of chaos is expected: for all $k \geq 0$, the kth marginal

$$F_{N,k}(z_1, \ldots, z_k) := \int_{\mathbb{D}^{N-k}} F_N(z_1, \ldots, z_N)\, dz_{k+1} \ldots dz_N \tag{5}$$

is expected to remain close to the tensor product $f^{\otimes k}$ of the mean-field Vlasov solution.

In the present contribution, we introduce a new dual approach to justify the mean-field limit, which allows to cover for the first time arbitrary square-integrable interaction forces at possibly vanishing temperature. We postpone to a forthcoming work the development of quantitative estimates and the extension to even more singular interactions taking advantage of the explicit hierarchy in Lemma 14 in Appendix B. The result holds for arbitrary 'weak duality solutions' of the Liouville equation in the sense introduced in Appendix A: this notion of solution does not require the existence of renormalized solutions and can be checked to exist whenever $K \in L^1_{loc}(\Omega; \mathbb{R}^d)$.

Theorem 1. *Let $0 \leq \alpha < \infty$, let $K \in L^2_{loc}(\Omega; \mathbb{R}^d)$, and assume for convenience $K \in L^\infty_{loc}(|x| > 1)$. Consider a global weak duality solution $F_N \in L^\infty_{loc}(\mathbb{R}^+; L^1(\mathbb{D}^N) \cap L^\infty(\mathbb{D}^N))$ of the Liouville equation (2), in the sense of Appendix A, with f°-chaotic initial data (3) for some density $f^\circ \in \mathcal{P}(\mathbb{D}) \cap L^\infty(\mathbb{D})$. Let $f \in L^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{D}) \cap L^\infty(\mathbb{D}))$ be a bounded weak solution of the Vlasov equation (4) with initial data f°, and assume that for some $T > 0$ it has bounded Fisher information*

$$\int_0^T \Big(\int_{\mathbb{D}} |\nabla_v \log f|^2 f \Big)^{\frac{1}{2}} < \infty.$$

Then, the following propagation of chaos holds: for all $k \geq 0$, the kth marginal $F_{N,k}$, given by (5), converges to $f^{\otimes k}$ as $N \uparrow \infty$ in the sense of distributions on $[0, T] \times \mathbb{D}^k$.

The method can also be adapted to first-order dynamics: the corresponding result that we obtain in this way takes on the following guise.

As the adaptation is straightforward, we omit the detail for shortness. Compared to previous work on mean field for first-order dynamics, it allows to treat for the first time quite singular interactions that have no specific energy structure. As explained at the end of the statement, by symmetry, the local squared integrability of the interaction force can be slightly relaxed up to assuming more regularity for the mean-field solution, so that this result covers in particular the well-known case of the 2d Euler and Navier–Stokes equations.

Theorem 2. *Let $0 \le \alpha < \infty$, let $K \in L^2_{loc}(\Omega;\mathbb{R}^d)$ with $\mathrm{div}(K) \in L^2_{loc}(\Omega)$, and assume for convenience $K \in W^{1,\infty}(|x|>1)$. Consider a global weak duality solution $F_N \in L^\infty_{loc}(\mathbb{R}^+; L^1(\Omega^N) \cap L^\infty(\Omega^N))$, in the sense of Appendix A, of the Liouville equation*

$$\partial_t F_N + \frac{1}{N-1}\sum_{i\neq j}^N \mathrm{div}_{x_i}\big(K(x_i-x_j)F_N\big) = \alpha \sum_{i=1}^N \Delta_{x_i} F_N, \tag{6}$$

with f°-chaotic initial data $F_N|_{t=0} = (f^\circ)^{\otimes N}$ for some density $f^\circ \in \mathcal{P}(\Omega)\cap L^\infty(\Omega)$. Let $f \in L^\infty(\mathbb{R}^+; \mathcal{P}(\Omega)\cap L^\infty(\Omega))$ be a bounded weak solution of the McKean–Vlasov equation

$$\partial_t f + \mathrm{div}\big((K*f)f\big) = \alpha\Delta f, \qquad f|_{t=0} = f^\circ,$$

and assume that for some $T>0$ it has controlled Fisher information

$$\int_0^T \Big(\int_\Omega |\nabla \log f|^2 f\Big)^{\frac12} < \infty.$$

Then, the following propagation of chaos holds: for all $k \ge 0$, the kth marginal $F_{N,k}$ of F_N converges to $f^{\otimes k}$ as $N\uparrow\infty$ in the sense of distributions on $[0,T]\times\mathbb{D}^k$.

In addition, in this result, the condition $K \in L^2_{loc}(\Omega;\mathbb{R}^d)$ can be partly relaxed up to assuming more regularity for f: more precisely, it can be replaced by the following weaker condition,

$$\int_0^T \Big(\int_{\Omega^2} \big|K(x-y)\cdot\big(\nabla\log f(x) - \nabla \log f(y)\big)\big|^2 f(x)f(y)\,dxdy\Big)^{\frac12} < \infty,$$

which holds for instance whenever the kernel satisfies $|x|K \in L^2_{loc}(\mathbb{R}^d)$ and assuming $\nabla \log f \in L^1(0,T;W^{1,\infty}(\Omega))$.

Duality methods have been first developed in [17] around renormalized solutions of the 2d Euler equations, following an earlier idea of [18] for the transport equation, and they have been further developed later on to construct Eulerian and Lagrangian solutions to the continuity and Euler equations with L^1 vorticity [16] and to show that smooth approximation is not a selection principle for the transport equation with rough vector field [15]. To be complete, we can cite for instance [48] and references therein (for instance [6]) for definitions of dual solutions related to transport type equations.

To the authors' knowledge, duality methods have however never been used for mean-field limit purposes. In order to study mean-field limits by duality methods, dual marginals of observables and their correlation structure will be considered and analyzed. In a nutshell, we shall derive a hierarchy of equations on linearized dual correlations and show that the limiting hierarchy has a unique solution: assuming that dual correlations vanish initially in the macroscopic limit, we then deduce by uniqueness that this property propagates over time, which can be used to conclude the desired mean-field limit result. Note that correlation functions associated with the joint N-particle density F_N, as well as their evolution equations, were first used in the framework of kinetic theory in [27, 41, 50, 2, 42] in form of so-called cumulant expansions. Linearized correlations close to an initial equilibrium have been used in particular in [2, 3, 4] for problems related to dilute gases of hard spheres, and they have also been strongly used in recent papers such as [20, 22] and more recently in [21] for mean-field limit problems. Yet, the perspective followed here is different as we study correlations of the *dual* equations far from any notion of equilibrium.

Comparison to previous results. In recent years, mean-field limit problems with singular interactions have been largely investigated for specific kernels. At vanishing temperature, the mean-field limit for *first-order systems* was classically obtained for example in [28, 29] or [52, 53] for 2d Euler, and it was remarkably extended in [54] to essentially any Riesz interaction kernel by means of modulated energy techniques. For the corresponding first-order setting with positive temperature, we refer in particular to [23, 44, 39] for the mean-field limit to 2d Navier–Stokes, to [9, 10, 24, 14] for singular attractive kernels, and to [49] for multiplicative noise. Uniform-in-time propagation of chaos was even recently obtained in [31, 51, 13].

In the case of *second-order systems*, on the contrary, much less is known. The mean-field limit was classically obtained in [7, 19] for Lipschitz kernels K, see also [55]. In dimension $d = 1$, the mean-field limit to the Vlasov–Poisson–Fokker–Planck system was derived in [30, 35]. In dimension $d \geq 2$, the only results for unbounded interaction kernels were obtained in [33, 34], but those are valid only for vanishing temperature and for mildly singular interaction kernels with $|K(x)| \lesssim |x|^{-\gamma}$ and $|\nabla K(x)| \lesssim |x|^{-\gamma-1}$ for some $\gamma < 1$. In [38], the mean-field limit was derived for any $K \in L^\infty(\mathbb{R}^d)$ without needing control on ∇K.

We also mention [11], where the mean-field limit is derived for bounded but discontinuous interactions based on so-called vision cones.

The case of singular interaction kernels K with N-dependent truncation is much better understood: given $|K(x)| \lesssim |x|^{-\gamma}$, one can consider the mean-field limit problem with K replaced instead by a truncated kernel $|K_N(x)| \lesssim |x+\varepsilon_N|^{-\gamma}$ with some regularizing parameter $\varepsilon_N \to 0$ as $N \to \infty$. This was classically considered for example in [25, 26, 56, 57], and we refer to the more recent works [46, 47] where the conditions on the regularization parameter ε_N are further weakened. The mean-field limit for such truncated kernels is also studied in case of positive temperature, see for example [12, 36].

In the special case of the Cucker–Smale flocking model, it is possible to take advantage of some dispersion properties of the dynamics in order to prove the mean-field limit result for some range of singular interaction kernels; we refer for instance to [43].

Recently, new hierarchical approaches have been introduced to bound marginals for particle systems with appropriate non-degenerate diffusion. Using relative entropy, [45] was the first to derive the optimal quantitative estimates for the convergence of the marginals to the limiting tensorized solution (obtaining optimal rates $O(1/N)$ instead of $O(1/\sqrt{N})$ for marginals, as first obtained for smooth interactions in [20]). While formulated for first-order systems, the method of [45], as noted by the author, also applies to second-order systems with diffusion in velocity. It takes advantage of the regularization provided by the diffusion to avoid "losing" a derivative in the hierarchy estimates. The use of the relative entropy however imposes that the interaction kernel belongs to an exponential Orlicz space. More recently, a novel hierarchical approach has been developed in [8], starting from the BBGKY hierarchy and allowing to justify the mean-field limit of interacting particle systems with positive temperature with interaction kernel deriving from a potential, leading to the first ever derivation of the mean-field limit to the Vlasov–Poisson–Fokker–Planck system in 2d, as well as to a partial result in 3d.

Here we propose the first method allowing to consider possible vanishing temperature parameter covering first- and second-order systems with a singular kernel K in $L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$. Unfortunately, in its present form, our method does not allow to consider more general singular kernels for instance to justify the mean-field limit to the Vlasov–Poisson–Fokker–Planck equation, even in $2d$. We postpone to a forthcoming work the development of quantitative estimates and the extension to more singular interactions.

2. A dual approach to mean field

Henceforth, we let the space domain Ω be either the whole space $\mathbb{R}^d$ or the periodic torus $\mathbb{T}^d$ in dimension $d \geq 1$, we let $0 \leq \alpha < \infty$ be fixed, we consider an interaction force kernel $K \in L^1_{loc}(\Omega;\mathbb{R}^d) \cap L^\infty(|x| > 1)$ with the action-reaction condition $K(x-y) = -K(y-x)$, and we consider an initial density $f^\circ \in \mathcal{P}(\mathbb{D}) \cap L^\infty(\mathbb{D})$ on phase space $\mathbb{D} := \Omega \times \mathbb{R}^d$. Our approach starts with the following observation, which provides some sort of a dual reformulation for mean-field limit questions. Note that the notion of weak duality solutions for the Liouville equation as introduced in Appendix A is precisely designed for the present result to hold, while the rest of the argument will hold for an arbitrary weak solution of the backward Liouville equation.

Proposition 3. *Let F_N be a global weak duality solution of the Liouville equation* (2) *in the sense of Appendix A, with f°-chaotic initial data* (3)*. Let $f \in L^\infty_{loc}(\mathbb{R}^+;\mathcal{P}(\mathbb{D}) \cap L^\infty(\mathbb{D}))$ be a weak solution of the corresponding mean-field Vlasov equation* (4) *with initial data f°, and assume $\nabla_v f \in L^1_{loc}(\mathbb{R}^+; L^1(\mathbb{D}))$. Given $k \geq 1$, $T > 0$, and $\psi \in C^\infty_c(\mathbb{D})$, further consider a bounded weak solution $\Phi_N \in L^\infty([0,T] \times \mathbb{D}^N)$ of the corresponding backward Liouville equation*

$$\partial_t \Phi_N + \sum_{i=1}^N \Big(v_i \cdot \nabla_{x_i} \Phi_N + \frac{1}{N-1} \sum_{j:j\neq i}^N K(x_i - x_j) \cdot \nabla_{v_i} \Phi_N \Big) = -\alpha \sum_{i=1}^N \Delta_{v_i} \Phi_N, \tag{7}$$

with final condition

$$\Phi_N(z_1,\ldots,z_N)|_{t=T} = \binom{N}{k}^{-1} \sum_{1 \leq i_1 < \ldots < i_k \leq N} \psi(z_{i_1}) \ldots \psi(z_{i_k}), \tag{8}$$

for which F_N is a global weak duality solution in the sense of Appendix A. Then there holds

$$\int_{\mathbb{D}^k} \psi^{\otimes k} F_{N,k}(T) \xrightarrow{N\uparrow\infty} \Big(\int_{\mathbb{D}} \psi f(T) \Big)^k \tag{9}$$

if and only if we have

$$N \int_0^T \Big(\int_{\mathbb{D}^N} V_f(z_1,z_2)\, \Phi_N\, f^{\otimes N} \Big)\, dt \xrightarrow{N\uparrow\infty} 0, \tag{10}$$

where we have defined

$$V_f(z_i,z_j) := \big(K(x_i - x_j) - K * f(x_i) \big) \cdot (\nabla_{v_i} \log f)(z_i). \tag{11}$$

Proof. We start by comparing the equations for Φ_N and F_N: as they are taken to be in duality in the sense of Appendix A, we get

$$\int_{\mathbb{D}^N} \Phi_N(T)F_N(T) \,=\, \int_{\mathbb{D}^N} \Phi_N(0)F_N(0),$$

and thus, inserting the final condition (8) for Φ_N and the f°-chaotic initial data (3) for F_N,

$$\int_{\mathbb{D}^k} \psi^{\otimes k} F_{N,k}(T) \,=\, \int_{\mathbb{D}^N} \Phi_N(0)\,(f^\circ)^{\otimes N}. \tag{12}$$

The right-hand side can be further decomposed as

$$\int_{\mathbb{D}^N} \Phi_N(0)(f^\circ)^{\otimes N} = \int_{\mathbb{D}^N} \Phi_N(T) f(T)^{\otimes N} - \int_0^T \Big(\partial_t \int_{\mathbb{D}^N} \Phi_N(t) f(t)^{\otimes N}\Big) dt,$$

and thus, recalling again the final condition (8) and using the equations for Φ_N and for f,

$$\begin{aligned}\int_{\mathbb{D}^N} \Phi_N(0)(f^\circ)^{\otimes N} \,&=\, \int_{\mathbb{D}^k} \psi^{\otimes k} f(T)^{\otimes k} \\ &\qquad - \frac{1}{N-1}\sum_{i\neq j}^N \int_0^T \Big(\int_{\mathbb{D}^N} V_f(z_i,z_j)\,\Phi_N\, f^{\otimes N}\Big) dt,\end{aligned}$$

where V_f is defined in the statement. This identity is easily justified by an approximation argument for bounded weak solutions Φ_N and f, provided that $\nabla_v f \in L^1([0,T]\times\mathbb{D})$. Note that this assumption precisely ensures that the last right-hand side term makes sense in this identity. Combining this with (12), and recalling that Φ_N is a symmetric function in its N variables, this concludes the proof. $\square$

3. Dual linearized correlations

Given $k \geq 1$, $T > 0$, and $\psi \in C_c^\infty(\mathbb{D})$, and given a bounded weak solution $\Phi_N \in L^\infty([0,T]\times\mathbb{D}^N)$ of the backward Liouville equation (7)–(8), we aim to prove the dual convergence property (10). For that purpose, we analyze correlations of the dual solution Φ_N with respect to the mean-field density $f^{\otimes N}$, somehow showing that the special structure of the final condition (8) is approximately preserved. We first define the marginals $\{M_{N,n}\}_{0\leq n\leq N}$ of Φ_N with respect to $f^{\otimes N}$: for all $0 \leq n \leq N$, we let

$$\begin{aligned}&M_{N,n}(z_1,\ldots,z_n) \\ &\qquad := \int_{(\mathbb{R}^d)^{N-n}} \Phi_N(z_1,\ldots,z_N)\, f^{\otimes N-n}(z_{n+1},\ldots,z_N)\, dz_{n+1}\ldots dz_N.\end{aligned} \tag{13}$$

As Φ_N is a bounded symmetric function in its N variables, the marginal $M_{N,n}$ is also bounded and symmetric in its n variables. Next, corresponding linearized correlation functions $\{C_{N,n}\}_{0\le n\le N}$ (also sometimes called cumulants) are defined to satisfy the following cluster expansion,

$$\Phi_N(z_1,\ldots,z_N) \;=\; \sum_{n=0}^{N}\sum_{\sigma\in P_n^N} C_{N,n}(z_\sigma), \tag{14}$$

where P_n^N denotes the set of all subsets of $[N] := \{1,\ldots,N\}$ with n elements, and where for an index subset $\sigma = \{i_1,\ldots,i_k\}$ we write $z_\sigma := (z_{i_1},\ldots,z_{i_k})$. Further requiring $C_{N,n}$ to be a symmetric function in its n variables and to satisfy

$$\int_{\mathbb{D}} C_{N,n}(z_1,\ldots,z_n)\, f(z_j)\, dz_j = 0, \qquad \text{for all } 1\le j\le n, \tag{15}$$

the above relation (14) uniquely defines the linearized dual correlations $\{C_{N,n}\}_{0\le n\le N}$.

Equivalently, the cluster expansion (14) can be inverted and correlations can be defined explicitly in terms of marginals: for all $0\le n\le N$,

$$C_{N,n}(z_1,\ldots,z_n) \;=\; \sum_{k=0}^{n}(-1)^{n-k}\sum_{\sigma\in P_k^n} M_{N,k}(z_\sigma). \tag{16}$$

Note that, restricting the cluster expansion (14) to k particles, we have for all $0\le k\le N$,

$$M_{N,k}(z_1,\ldots,z_k) = \sum_{l=0}^{k}\sum_{\tau\in P_l^k} C_{N,l}(z_\tau). \tag{17}$$

In terms of these correlation functions, the quantity in (10) that we aim to estimate takes the form

$$N\int_0^T \Big(\int_{\mathbb{D}^N} V_f(z_1,z_2)\,\Phi_N\, f^{\otimes N}\Big)dt = N\int_0^T \Big(\int_{\mathbb{D}^2} V_f\, C_{N,2}\, f^{\otimes 2}\Big)dt, \tag{18}$$

where we have used the fact that the definition (11) of V_f satisfies

$$\int_{\mathbb{D}} V_f(z_1,z_2)\, f(z_j)\, dz_j = 0 \qquad \text{for } j=1,2. \tag{19}$$

This identity (18) leads us to the following straightforward consequence of Proposition 3; we formulate it here in the L^2 setting of Theorem 1, but it can be adapted to an arbitrary L^p setting.

Proposition 4. *Further assume that $K \in L^2_{loc}(\Omega;\mathbb{R}^d)$ and that f satisfies*

$$\int_0^T \Big(\int_{\mathbb{D}} |\nabla_v \log f|^2 f\Big)^{\frac12} < \infty.$$

If we can show

$$NC_{N,2} \overset{*}{\rightharpoonup} 0 \quad \text{weakly-* in } L^\infty(0,T;L^2(\mathbb{D}^2,f^{\otimes 2})), \tag{20}$$

then we have $\int_{\mathbb{D}^k} \psi^{\otimes k} F_{N,k}(T) \to (\int_{\mathbb{D}} \psi f(T))^k$ in the setting of Proposition 3.

Proof. The assumptions on K and f ensure that V_f belongs to $L^1(0,T;$ $L^2(\mathbb{D}^2,f^{\otimes 2}))$. The convergence property (20) would then precisely ensure that (18) tends to 0 as $N \uparrow \infty$. By Proposition 3, this implies (9) and the conclusion follows. ☐

This indicates the importance of dual correlations. In order to prove the desired convergence (20), we start with the following a priori estimates on correlations, which are inspired from [2, Proposition 4.2] and simply follow from symmetry considerations. In particular, for $n = 2$, this proves the (weighted) L^2 boundedness of $NC_{N,2}$.

Lemma 5. *For all $0 \le n \le N$, we have uniformly on $[0,T]$,*

$$\Big(\int_{\mathbb{D}^n} |C_{N,n}|^2 f^{\otimes n}\Big)^{\frac12} \le \binom{N}{n}^{-\frac12} \|\psi\|_{L^\infty(\mathbb{D})}.$$

Proof. From the cluster expansion (14) and the orthogonality property (15) of cumulants, we find

$$\int_{\mathbb{D}^N} |\Phi_N|^2 f^{\otimes N} = \sum_{n=0}^N \binom{N}{n} \int_{\mathbb{D}^n} |C_{N,n}|^2 f^{\otimes n}.$$

The left-hand side is bounded by $\|\Phi_N\|_{L^\infty(\mathbb{D}^N)}$, which is controlled on $[0,T]$ by the final value $\|\Phi_N(T)\|_{L^\infty(\mathbb{D}^N)}$ for bounded weak solutions of the backward Liouville equation, cf. Definition 10. Recalling the final condition (8), we deduce on $[0,T]$,

$$\int_{\mathbb{D}^N} |\Phi_N|^2 f^{\otimes N} \le \|\Phi_N\|^2_{L^\infty(\mathbb{D}^N)} \le \|\Phi_N(T)\|^2_{L^\infty(\mathbb{D}^N)} \le \|\psi\|^2_{L^\infty(\mathbb{D})},$$

and the conclusion follows. ☐

For $n = 2$, the above estimate only yields the boundedness of $NC_{N,2}$, which is certainly not enough to show the desired convergence property (20). Yet, by weak compactness, it at least provides the following convergence result that will be used later on.

Lemma 6. *Up to extraction of a subsequence as $N\uparrow\infty$, we have the following weak convergences for rescaled correlations, for all $n\ge 0$,*

$$N^{\frac{n}{2}}C_{N,n} \overset{*}{\rightharpoonup} n!^{\frac12}\bar C_n \quad \text{in } L^\infty(0,T;L^2(\mathbb{D}^n,f^{\otimes n})), \tag{21}$$

for some $\bar C_n\in L^\infty(0,T;L^2(\mathbb{D}^n,f^{\otimes n}))$. In addition,

$$\sup_{0\le t\le T}\sup_{n\ge 0}\int_{\mathbb{D}^n}|\bar C_n(t)|^2 f(t)^{\otimes n} \,\le\, \|\psi\|^2_{L^\infty(\mathbb{D})}. \tag{22}$$

Proof. Consider the rescaled correlation

$$\bar C_{N,n} := \binom{N}{n}^{\frac12} C_{N,n}, \qquad 0\le n\le N.$$

For all $n\ge 0$, Lemma 5 implies that $\bar C_{N,n}$ is bounded as $N\uparrow\infty$ in $L^\infty(0,T;L^2(\mathbb{D}^n,f^{\otimes n}))$. Up to extraction of a subsequence, we thus have $\bar C_{N,n}\overset{*}{\rightharpoonup}\bar C_n$ for some $\bar C_n$ in that space, which is equivalent to (21). In addition, the a priori estimates on the extracted limit follow from Lemma 5 by weak lower semicontinuity. ☐

In these terms, the desired convergence (20) is equivalent to $\bar C_2=0$. To prove this, a finer analysis of dual correlations is required. Arguing as in [20], following characteristics, it is in fact possible to check

$$C_{N,n}(t)=O(e^{Ct}N^{-n}) \tag{23}$$

provided that the force kernel K is smooth, which then shows that the a priori estimates of Lemma 5 are strongly suboptimal (at least for $t=O(1)$, cf. [21]): in particular, we could deduce in that case $\bar C_n=0$ for all $n\ge 1$. However, in case of a singular kernel K, as considered here, the analysis is much more delicate to handle.

We shall proceed by examining the hierarchy of equations satisfied by correlation functions $\{C_{N,n}\}_n$. For that purpose, we first note that by definition for all t the correlation $C_{N,n}(t)$ belongs to the following subspace of $L^2(\mathbb{D}^n,f(t)^{\otimes n})$,

$$\begin{aligned} H_n(t) := \Big\{ & g\in L^2(\mathbb{D}^n,f(t)^{\otimes n}) \,:\, g \text{ is symmetric in its } n \text{ variables,}\\ & \text{and } \int_{\mathbb{D}} g(z_1,\ldots,z_n)\,f(t,z_j)\,dz_j=0 \text{ for all } 1\le j\le n\Big\}.\end{aligned}$$

The cluster expansion (17) then yields the following relation between marginals and correlations: for all $1\le n\le N$ and $g_n\in H_n(t)$,

$$\int_{\mathbb{D}^n} g_n\,M_{N,n}(t)\,f(t)^{\otimes n} \,=\, \int_{\mathbb{D}^n} g_n\,C_{N,n}(t)\,f(t)^{\otimes n}, \tag{24}$$

which shows that $C_{N,n}(t)$ is in fact the orthogonal projection of $M_{N,n}(t)$ onto $H_n(t)$ for the L^2 space weighted by $f(t)^{\otimes n}$. In this spirit, instead of deriving a complete equation for $C_{N,n}$, it is enough to derive its weak formulation on $L^\infty([0,T];H_n)$, that is, to derive an equation up to a remainder term $R_{N,n}(t) \in H_n(t)^\perp$. For the sake of completeness, we include in Appendix B the complete equations with explicit expressions for remainders, which requires more involved combinatorial computations but will be useful for future purposes.

Proposition 7. *Assume $\nabla_v f \in L^1([0,T]\times\mathbb{D})$. For all $0 \le n \le N$, we have in the distributional sense on $[0,T]\times\mathbb{D}^n$,*

(25)
$$\begin{aligned}
&\partial_t C_{N,n} + \sum_{i=1}^n v_i\cdot\nabla_{x_i} C_{N,n} + \alpha\sum_{i=1}^n \Delta_{v_i} C_{N,n}\\
&-\frac{N-n}{N-1}\sum_{j=1}^n\int_{\mathbb{D}} V_f(z_*,z_j)\,C_{N,n}(z_{[n]\setminus\{j\}},z_*)\,f(z_*)\,dz_*\\
&+\frac{N-n}{N-1}\sum_{i=1}^n (K*f)(x_i)\cdot\nabla_{v_i}C_{N,n} + \frac{1}{N-1}\sum_{i\ne j}K(x_i-x_j)\cdot\nabla_{v_i}C_{N,n}\\
&-\frac{(N-n)(N-n-1)}{N-1}\int_{\mathbb{D}^2} V_f(z_*,z_*')\,C_{N,n+2}(z_{[n]},z_*,z_*')\,f(z_*)f(z_*')\,dz_*dz_*'\\
&+\frac{N-n}{N-1}\sum_{i=1}^n\nabla_{v_i}\cdot\int_{\mathbb{D}}\big(K(x_i-x_*)\,-K*f(x_i)\big)\,C_{n+1}(z_{[n]},z_*)\,f(z_*)\,dz_*\\
&-\frac{N-n}{N-1}\sum_{j=1}^n\int_{\mathbb{D}}V_f(z_*,z_j)\,C_{N,n+1}(z_{[n]},z_*)\,f(z_*)\,dz_*\\
&+\frac{1}{N-1}\sum_{i\ne j}K(x_i-x_j)\cdot\nabla_{v_i}C_{N,n-1}(z_{[n]\setminus\{j\}}) \quad=\quad R_{N,n},
\end{aligned}$$

for some remainder term $R_{N,n}\in W^{-2,1}_{loc}([0,T]\times\mathbb{D}^n)$ that is orthogonal to H_n in the following weak sense,

$$\int_0^T\int_{\mathbb{D}^n} h_n R_{N,n} \,=\, 0 \quad \textit{for all } h_n\in C_c^\infty([0,T]\times\mathbb{D}^n)$$
$$\textit{such that } \int_{\mathbb{D}} h_n(t,z_{[n]})\,dz_j = 0 \textit{ a.e. for all } 1\le j\le n.$$

Proof. Note that the assumption $\nabla_v f \in L^1([0,T]\times\mathbb{D})$ precisely ensures that $z\mapsto\int_{\mathbb{D}}|V_f(\cdot,z)|f$ is locally integrable. We start by deriving the BBGKY-type hierarchy of equations satisfied by dual marginals: by definition (13), combining the equation for Φ_N and the mean-field equation for f, we find for all $0 \le n \le N$,

$$
\begin{aligned}
(26)\quad \partial_t M_{N,n} + \sum_{i=1}^n v_i \cdot \nabla_{x_i} M_{N,n} + \alpha \sum_{i=1}^n \Delta_{v_i} M_{N,n}& \\
= -\frac{1}{N-1} \sum_{i\neq j}^n K(x_i - x_j) \cdot \nabla_{v_i} M_{N,n}& \\
+ \frac{N-n}{N-1} \sum_{j=1}^n \int_{\mathbb{D}} V_f(z_*, z_j)\, M_{N,n+1}(z_{[n]}, z_*)\, f(z_*)\, dz_*& \\
- \frac{N-n}{N-1} \sum_{i=1}^n \int_{\mathbb{D}} K(x_i - x_*) \cdot \nabla_{v_i} M_{N,n+1}(z_{[n]}, z_*)\, f(z_*)\, dz_*& \\
+ \frac{(N-n)(N-n-1)}{N-1}& \\
\times \int_{\mathbb{D}^2} V_f(z_*, z'_*)\, M_{N,n+2}(z_{[n]}, z_*, z'_*)\, f(z_*) f(z'_*)\, dz_* dz'_*,&
\end{aligned}
$$

where we have set $M_{N,N+1}, M_{N,N+2} = 0$ for notational simplicity. We now turn to corresponding equations for correlations. For that purpose, we appeal to identity (24). More precisely, by an approximation argument, setting $h_n = g_n f^{\otimes n}$ and recalling $M_{N,n}, C_{N,n} \in L^\infty([0,T] \times \mathbb{D}^n)$, we note that (24) can be upgraded as follows: for all $1 \le n \le N$, $t \in [0,T]$, and $p_n \in L^1(\mathbb{D}^n)$, we have

$$
\begin{aligned}
(27)\quad \int_{\mathbb{D}^n} p_n M_{N,n}(t) \;=\; \int_{\mathbb{D}^n} p_n C_{N,n}(t)& \\
\text{provided that } \int_{\mathbb{D}} p_n(z_{[n]})\, dz_j = 0 \quad \text{a.e. for all } 1 \le j \le n.&
\end{aligned}
$$

(Note that, by symmetry of both $M_{N,n}$ and $C_{N,n}$, the test function p_n does indeed not need to be taken symmetric.) Now consider a smooth test function $h_n \in C_c^\infty([0,T] \times \mathbb{D}^n)$ that satisfies

$$
(28)\qquad \int_{\mathbb{D}} h_n(t, z_{[n]})\, dz_j = 0 \quad \text{a.e. for all } 1 \le j \le n.
$$

In the weak sense on $[0,T]$, we can decompose

$$
\begin{aligned}
\frac{d}{dt}\Big(\int_{\mathbb{D}^n} h_n C_{N,n} - \int_{\mathbb{D}^n} h_n M_{N,n} \Big) = \Big(\int_{\mathbb{D}^n} h_n \partial_t C_{N,n} - \int_{\mathbb{D}^n} h_n \partial_t M_{N,n} \Big)& \\
+ \Big(\int_{\mathbb{D}^n} C_{N,n} \partial_t h_n - \int_{\mathbb{D}^n} M_{N,n} \partial_t h_n \Big).&
\end{aligned}
$$

By (27) and by the choice (28) of h_n, both the left-hand side and the second right-hand side term vanish, hence

$$\int_{\mathbb{D}^n} h_n \partial_t C_{N,n} \;=\; \int_{\mathbb{D}^n} h_n \partial_t M_{N,n}.$$

Inserting equation (26), we get

$$\int_{\mathbb{D}^n} h_n \partial_t C_{N,n} \\ = T_1 + T_2 + \frac{N-n}{N-1}(T_3+T_4) + \frac{(N-n)(N-n-1)}{N-1} T_5, \tag{29}$$

where we have set for abbreviation

$$T_1 \;:=\; \int_{\mathbb{D}^n} M_{N,n}\Big(\sum_{i=1}^n v_i\cdot\nabla_{x_i} + \frac{1}{N-1}\sum_{i\neq j}^n K(x_i-x_j)\cdot\nabla_{v_i}\Big)h_n,$$

$$T_2 \;:=\; -\alpha\sum_{i=1}^n \int_{\mathbb{D}^n} M_{N,n}\,\Delta_{v_i} h_n,$$

$$T_3 \;:=\; \sum_{j=1}^n \int_{\mathbb{D}^{n+1}} V_f(z_{n+1},z_j)\,h_n(z_{[n]})M_{N,n+1}(z_{[n+1]})f(z_{n+1})\,dz_{[n+1]},$$

$$T_4 \;:=\; \sum_{i=1}^n \int_{\mathbb{D}^{n+1}} K(x_i-x_{n+1})\cdot\nabla_{v_i} h_n(z_{[n]}) \\ \times M_{N,n+1}(z_{[n+1]})\,f(z_{n+1})\,dz_{[n+1]},$$

$$T_5 \;:=\; \int_{\mathbb{D}^{n+2}} V_f(z_{n+1},z_{n+2})h_n(z_{[n]})M_{N,n+2}(z_{[n+2]})\,f(z_{n+1})f(z_{n+2})\,dz_{[n+2]}.$$

In order to derive the desired equation for $C_{N,n}$, it remains to express these five quantities $\{T_j\}_{1\le j\le 5}$ in terms of correlations instead of marginals. For the first two terms T_1 and T_2, noting that the choice (28) of h_n yields

$$\int_{\mathbb{D}}\Big(\sum_{i=1}^n v_i\cdot\nabla_{x_i}h_n + \frac{1}{N-1}\sum_{i\neq j}^n K(x_i-x_j)\cdot\nabla_{v_i}h_n\Big)\,dz_j \;=\; 0,$$

$$\int_{\mathbb{D}}\Big(\sum_{i=1}^n \Delta_{v_i}h_n\Big)\,dz_j \;=\; 0, \qquad \text{for all } 1\le j\le n,$$

we immediately get from (27),

$$T_1 \;=\; \int_{\mathbb{D}^n} C_{N,n}\Big(\sum_{i=1}^n v_i\cdot\nabla_{x_i} + \frac{1}{N-1}\sum_{i\neq j}^n K(x_i-x_j)\cdot\nabla_{v_i}\Big)h_n,$$

$$T_2 \;=\; -\alpha\sum_{i=1}^n\int_{\mathbb{D}^n} C_{N,n}\,\Delta_{v_i}h_n.$$

We turn to the third term T_3 and decompose it as

$$T_3 \;=\; \int_{\mathbb{D}^{n+1}} L_{n+1}M_{N,n+1} \\ +\sum_{j=1}^n\int_{\mathbb{D}^{n+1}} W_n^j(z_{[n+1]\setminus\{j\}})\,M_{N,n+1}(z_{[n+1]})\,f(z_j)\,dz_{[n+1]},$$

in terms of

$$L_{n+1}(z_{[n+1]}) \;:=\; \sum_{j=1}^n\Big(V_f(z_{n+1},z_j)\,h_n(z_{[n]})f(z_{n+1}) \\ -W_n^j(z_{[n+1]\setminus\{j\}})f(z_j)\Big),$$

$$W_n^j(z_{[n+1]\setminus\{j\}}) \;:=\; f(z_{n+1})\int_{\mathbb{D}} V_f(z_{n+1},z_j)\,h_n(z_{[n]})\,dz_j.$$

For all $1\le j\le n$, by definition of marginals, we have

$$\int_{\mathbb{D}^{n+1}} W_n^j(z_{[n+1]\setminus\{j\}})\,M_{N,n+1}(z_{[n+1]})\,f(z_j)\,dz_{[n+1]} \;=\; \int_{\mathbb{D}^n} W_n^jM_{N,n}.$$

Recalling the properties (19) of V_f, we note that $\int_{\mathbb{D}} L_{n+1}(z_{[n+1]})\,dz_l=0$ for all $1\le l\le n{+}1$ and $\int_{\mathbb{D}} W_n^j(z_{[n]})\,dz_l=0$ for all $1\le l\le n$. Appealing again to (27), we are led to

$$T_3 \;=\; \int_{\mathbb{D}^{n+1}} L_{n+1}C_{N,n+1}+\sum_{j=1}^n\int_{\mathbb{D}^n} W_n^jC_{N,n},$$

that is, by definition of L_{n+1} and W_n^j,

$$T_3 \;=\; \sum_{j=1}^n\int_{\mathbb{D}^n} h_n(z_{[n]})\bigg(\int_{\mathbb{D}} V_f(z_*,z_j)\,C_{N,n+1}(z_{[n]},z_*)\,f(z_*)\,dz_* \\ +\int_{\mathbb{D}} V_f(z_*,z_j)\,C_{N,n}(z_{[n]\setminus\{j\}},z_*)\,f(z_*)\,dz_*\bigg)dz_{[n]}.$$

We turn to the fourth term T_4. Adding and subtracting $K*f(x_i)$ to the interaction kernel $K(x_i-x_{n+1})$, and recalling the definition of marginals, we can decompose this term T_4 as

$$T_4 \;=\; \sum_{i=1}^n\int_{\mathbb{D}^{n+1}} \big(K(x_i-x_{n+1})-K*f(x_i)\big)\cdot\nabla_{v_i}h_n(z_{[n]}) \\ \times M_{N,n+1}(z_{[n+1]})\,f(z_{n+1})\,dz_{[n+1]}$$

$$+\sum_{i=1}^n \int_{\mathbb{D}^n} (K*f)(x_i)\cdot\nabla_{v_i}h_n(z_{[n]})\,M_{N,n}(z_{[n]})\,dz_{[n]}.$$

We are now again in position to appeal to (27), leading us to

$$\begin{aligned}T_4 \,=\, \sum_{i=1}^n \int_{\mathbb{D}^{n+1}} \big(K(x_i-x_{n+1})-K*f(x_i)\big)\cdot\nabla_{v_i}h_n(z_{[n]})\\ \times C_{N,n+1}(z_{[n+1]})\,f(z_{n+1})\,dz_{[n+1]}\\ +\sum_{i=1}^n \int_{\mathbb{D}^n} (K*f)(x_i)\cdot\nabla_{v_i}h_n(z_{[n]})\,C_{N,n}(z_{[n]})\,dz_{[n]}.\end{aligned}$$

Finally, for the last term T_5, by the choice (28) of h_n and the properties (19) of V_f, we can directly appeal to (27) to find

$$T_5 \,=\, \int_{\mathbb{D}^{n+2}} V_f(z_{n+1},z_{n+2})h_n(z_{[n]})C_{N,n+2}(z_{[n+2]})\,f(z_{n+1})f(z_{n+2})\,dz_{[n+2]}.$$

Collecting the above identities for the different terms $\{T_l\}_{1\le l\le 5}$, combining them together into (29), and letting $R_{N,n}$ be defined as the left-hand side in (25), we obtain

$$\int_{\mathbb{D}^n} h_n R_{N,n} = 0.$$

As this holds for any test function h_n satisfying (28), the conclusion follows. By definition of $R_{N,n}$ as the left-hand side in (25), given that correlation functions are uniformly bounded as $\Phi_N\in L^\infty([0,T]\times\mathbb{D}^N)$, we note that $R_{N,n}$ belongs to $W^{-2,1}([0,T]\times\mathbb{D}^n)$ as stated. ☐

Using Lemma 6 to pass to the limit in the above hierarchy of equations for correlations, we are led to the following limit system. Note that the remainder term $R_{N,n}$ orthogonal to H_n vanishes in the limit $N\uparrow\infty$ as the obtained limit system (30) is seen to naturally propagate the structure of H_n.

Proposition 8. *Let $K\in L^2_{loc}(\Omega;\mathbb{R}^d)$, assume that f satisfies*

$$\int_0^T \Big(\int_{\mathbb{D}} |\nabla_v \log f|^2 f\Big)^{\frac12} < \infty,$$

and denote by $\{\bar C_n\}_n$ the limit rescaled correlations extracted in Lemma 6, with $\bar C_n\in L^\infty(0,T;L^2(\mathbb{D}^n,f^{\otimes n}))$. For all $n\ge 0$, we have in the distributional sense on $[0,T]\times\mathbb{D}^n$,

$$(30)\quad \partial_t\bar C_n+\sum_{i=1}^n v_i\cdot\nabla_{x_i}\bar C_n+\sum_{i=1}^n (K*f)(x_i)\cdot\nabla_{v_i}\bar C_n+\alpha\sum_{i=1}^n \Delta_{v_i}\bar C_n$$

$$= \sum_{j=1}^{n} \int_{\mathbb{D}} V_f(z_*, z_j)\, \bar{C}_n(z_{[n]\setminus\{j\}}, z_*)\, f(z_*)\, dz_*$$
$$+ \sqrt{n+1}\,\sqrt{n+2} \int_{\mathbb{D}^2} V_f(z_*, z'_*)\, \bar{C}_{n+2}(z_{[n]}, z_*, z'_*)\, f(z_*) f(z'_*)\, dz_* dz'_*,$$

with final conditions

$$\bar{C}_n|_{t=T} \;=\; \begin{cases} (\int_{\mathbb{D}} \psi f(T))^k, & \textit{for } n = 0, \\ 0, & \textit{for } n \geq 1 \end{cases} \tag{31}$$

Proof. As assumptions on K and f ensure $V_f \in L^1(0,T; L^2_{loc}(\mathbb{D}^2, f^{\otimes 2}))$, we can pass to the limit in the weak formulation of equations for correlations given in Proposition 7 along the limit extracted in Lemma 6. We deduce that this limit $\{\bar{C}_n\}_n$ satisfies the following hierarchy: for all $n \geq 0$, we have in the distributional sense on $[0,T] \times \mathbb{D}^n$,

$$\partial_t \bar{C}_n + \sum_{i=1}^{n} v_i \cdot \nabla_{x_i} \bar{C}_n + \sum_{i=1}^{n} (K * f)(x_i) \cdot \nabla_{v_i} \bar{C}_n + \alpha \sum_{i=1}^{n} \Delta_{v_i} \bar{C}_n \tag{32}$$
$$- \sum_{j=1}^{n} \int_{\mathbb{D}} V_f(z_*, z_j)\, \bar{C}_n(z_{[n]\setminus\{j\}}, z_*)\, f(z_*)\, dz_*$$
$$- \sqrt{n+1}\,\sqrt{n+2} \int_{\mathbb{D}^2} V_f(z_*, z'_*)\, \bar{C}_{n+2}(z_{[n]}, z_*, z'_*)\, f(z_*) f(z'_*)\, dz_* dz'_* \;=\; \bar{R}_n,$$

for some remainder term $\bar{R}_n \in W^{-2,1}_{loc}([0,T] \times \mathbb{D}^n)$ that is again orthogonal to H_n in the following weak sense,

$$\int_0^T \int_{\mathbb{D}^n} h_n \bar{R}_n \;=\; 0 \quad \text{for all } h_n \in C_c^\infty([0,T] \times \mathbb{D}^n) \tag{33}$$
$$\text{such that } \int_{\mathbb{D}} h_n(t, z_{[n]})\, dz_j = 0 \text{ a.e. for all } 1 \leq j \leq n.$$

To obtain the prefactor in the last left-hand side term in (32), we have simply noted that

$$\frac{(N-n)(N-n-1)}{N-1} \binom{N}{n}^{\frac{1}{2}} \binom{N}{n+2}^{-\frac{1}{2}} \;\to\; \sqrt{n+1}\,\sqrt{n+2}.$$

From here, we split the proof into two steps.

Step 1: proof of (30).
In order to establish equation (30), it remains to show that the remainder term in (32) actually vanishes, $\bar{R}_n = 0$. Up to an approximation argument, we may assume for convenience that f further satisfies

$\int_0^T\int_{\mathbb{D}}|v|f<\infty$, and we shall show in that setting

$$\int_0^T\int_{\mathbb{D}^n} g_n\bar R_n f^{\otimes n}\,=\,0,\qquad\text{for all } g_n\in C_c^\infty([0,T]\times\mathbb{D}^n). \tag{34}$$

With the additional assumption on f, we first emphasize that the product $\int_0^T\int_{\mathbb{D}^n} g_n\bar R_n f^{\otimes n}$ makes perfect sense for all $g_n\in C_b^2([0,T]\times\mathbb{D}^n)$: indeed, for $\bar R_n$ defined as the left-hand side in (32), testing with $g_nf^{\otimes n}$, using the mean-field equation for f, and recognizing in the first line of (32) the dual of the mean-field operator, we obtain up to an approximation argument, for all $g_n\in C_b^2([0,T]\times\mathbb{D}^n)$,

$$\begin{aligned}
\int_0^T\int_{\mathbb{D}^n} g_n\bar R_n f^{\otimes n}\,=\,&\int_{\mathbb{D}^n}(g_n\bar C_nf^{\otimes n})(T)-\int_{\mathbb{D}^n}(g_n\bar C_nf^{\otimes n})(0)\\
-\int_0^T\int_{\mathbb{D}^n}\Big(\partial_tg_n+&\sum_{i=1}^n v_i\cdot\nabla_{x_i}g_n+\sum_{i=1}^n(K*f)(x_i)\cdot\nabla_{v_i}g_n-\alpha\sum_{i=1}^n\Delta_{v_i}g_n\Big)\bar C_nf^{\otimes n}\\
&+2\alpha\int_0^T\int_{\mathbb{D}^n}\bar C_n\sum_{i=1}^n\nabla_{v_i}g_n\cdot\nabla_{v_i}f^{\otimes n}\\
-\sum_{j=1}^n\int_0^T\int_{\mathbb{D}^{n+1}}&V_f(z_{n+1},z_j)\,g_n(z_{[n]})\,\bar C_n(z_{[n+1]\setminus\{j\}})\,f^{\otimes n+1}(z_{[n+1]})\,dz_{[n+1]}\\
&-\sqrt{n+1}\,\sqrt{n+2}\int_0^T\int_{\mathbb{D}^{n+2}}V_f(z_{n+1},z_{n+2})\,g_n(z_{[n]})\\
&\qquad\times\bar C_{n+2}(z_{[n+2]})\,f^{\otimes n+2}(z_{[n+2]})\,dz_{[n+2]},
\end{aligned}$$

where all the terms make sense by the assumptions on f and K and by the uniform boundedness of correlation functions. In particular, from this identity, recalling that just like $\bar C_{N,n}$ the limit correlation $\bar C_n$ satisfies $\int_{\mathbb{D}}\bar C_n(z_{[n]})\,f(z_j)\,dz_j\,=\,0$ for all $1\le j\le n$, and using the cancellation properties (19) for V_f, we find in the distributional sense

$$\int_{\mathbb{D}}\bar R_n(z_{[n]})\,f(z_j)\,dz_j=0,\qquad\text{for all } 1\le j\le n. \tag{35}$$

Now, given a fixed test function $g_n\in C_c^\infty([0,T]\times\mathbb{D}^n)$ that is symmetric in its n variables, we can define its marginals, for $1\le l\le n$,

$$g_{n,l}(z_1,\ldots,z_l):=\int_{\mathbb{D}^{n-l}}g_n(z_1,\ldots,z_n)\,f^{\otimes(n-l)}(z_{l+1},\ldots,z_n)\,dz_{l+1}\ldots dz_n,$$

as well as the corresponding correlations

$$c_{n,l}(z_1,\ldots,z_l):=\sum_{j=0}^l(-1)^{l-n}\sum_{\sigma\in P_j^l}g_{n,l}(z_\sigma).$$

As in (17), we note that these correlations satisfy the following cluster expansion,

$$g_n(z_1,\ldots,z_n) = \sum_{l=0}^{n} \sum_{\sigma\in P_l^n} c_{n,l}(z_\sigma).$$

From this decomposition, the cancellation properties (35) for $\bar{R}_n$ lead us to

$$\int_0^T \int_{\mathbb{D}^n} g_n \bar{R}_n f^{\otimes n} \;=\; \int_0^T \int_{\mathbb{D}^n} c_{n,n} \bar{R}_n f^{\otimes n}.$$

Now by definition we find that $c_{n,n}$ satisfies $\int_{\mathbb{D}} c_{n,n}(t, z_{[n]})\, f(z_j)\, dz_j = 0$ a.e. for all $1 \le j \le n$, and the orthogonality property (33) then yields the claim (34).

Step 2: proof of final condition (31).
From the final condition (8), we find for marginals

$$\begin{aligned} (36)\quad & M_{N,n}(T, z_1,\ldots,z_n) \\ &= \binom{N}{k}^{-1} \sum_{l=0}^{k\wedge n} \binom{N-n}{k-l} \Big(\int_{\mathbb{D}} \psi f(T)\Big)^{k-l} \sum_{\sigma\in P_l^n} \psi^{\otimes l}(z_\sigma). \end{aligned}$$

Recalling (27), given $p_n \in C_c^\infty(\mathbb{D}^n)$ with $\int_{\mathbb{D}} p_n(z_{[n]})\, dz_j = 0$ for all j, we find

$$\begin{aligned} \int_{\mathbb{D}^n} p_n C_{N,n}(T) \;&=\; \int_{\mathbb{D}^n} p_n M_{N,n}(T) \\ &=\; 1_{k\ge n} \binom{N}{k}^{-1} \binom{N-n}{k-n} \Big(\int_{\mathbb{D}} \psi f(T)\Big)^{k-n} \int_{\mathbb{D}^n} p_n \psi^{\otimes n}. \end{aligned}$$

After rescaling, letting $N \uparrow \infty$ along the limit extracted in Lemma 6, we get

$$\bar{C}_{N,0}(T) \;=\; \Big(\int_{\mathbb{D}} \psi f(T)\Big)^k, \qquad \int_{\mathbb{D}^n} p_n \bar{C}_{N,n}(T) \;=\; 0, \quad \text{for all } n \ge 1,$$

and the claim (31) follows. □

Next, we show that the limiting hierarchy (30) for correlations allows for a unique solution.

Lemma 9. *Let $K \in L^2_{loc}(\Omega;\mathbb{R}^d)$ and assume that f satisfies*

$$\int_0^T \Big(\int_{\mathbb{D}} |\nabla_v \log f|^2 f\Big)^{\frac{1}{2}} < \infty.$$

If $\{C_n\}_n$ satisfies the limiting hierarchy (30) *in the distributional sense on $[0,T]$ with vanishing final condition $C_n|_{t=T}=0$ for all n and with the a priori estimate*

$$\sup_{0\le t\le T}\sup_{n\ge 0}\int_{\mathbb{D}^n}|C_n(t)|^2 f(t)^{\otimes n}\,<\,\infty, \tag{37}$$

then we have $C_n\equiv 0$ for all n.

Proof. Let $\{C_n\}_n$ be as in the statement. As it satisfies the hierarchy (30) in the weak sense, we can compute

$$\begin{aligned}\frac12\frac{d}{dt}\int|C_n|^2f^{\otimes n}\,&=\,\alpha\sum_{i=1}^n\int|\nabla_{v_i}C_n|^2f^{\otimes n}\\
&+n\int V_f(z,z')\,C_n(z_{[n-1]},z)\,C_n(z_{[n-1]},z')\,f^{\otimes(n+1)}(z_{[n-1]},z,z')\,dz_{[n-1]}dzdz'\\
&+(n+2)\int V_f(z_{n+1},z_{n+2})\,C_n(z_{[n]})\,C_{n+2}(z_{[n+2]})\,f^{\otimes(n+2)}(z_{[n+2]})\,dz_{[n+2]},\end{aligned}$$

and thus, by the Cauchy–Schwarz inequality,

$$\begin{aligned}&\frac{d}{dt}\Big(\int|C_n|^2f^{\otimes n}\Big)^{\frac12}\\
&\qquad\ge\,-n\Lambda_f\Big(\int|C_n|^2f^{\otimes n}\Big)^{\frac12}-(n+2)\Lambda_f\Big(\int|C_{n+2}|^2f^{\otimes(n+2)}\Big)^{\frac12},\end{aligned}$$

where we have set $\Lambda_f:=(\int_{\mathbb{D}^2}|V_f|^2f^{\otimes 2})^{\frac12}$ for abbreviation. In terms of the generating function

$$Z(t,r)\,:=\,\sum_{n=0}^\infty r^n\Big(\int|C_n|^2f^{\otimes n}\Big)^{\frac12},$$

which is well-defined for all $t\in[0,T]$ and $r\in[0,1)$ by the a priori estimate (37), we get

$$\partial_tZ(t,r)\,\ge\,-\Lambda_f(t)\Big(r\partial_rZ(t,r)+\tfrac1r\partial_rZ(t,r)\Big)\,\ge\,-2\Lambda_f(t)\tfrac1r\partial_rZ(t,r).$$

Solving this differential inequality by the method of characteristics, with the final condition $Z(T,\cdot)=0$, we deduce for all $r\in[0,1)$ and $T_0\in[0,T]$, provided that $r^2+4\int_{T_0}^T\Lambda_f<1$,

$$0\,\le\,Z(T_0,r)\,\le\,Z\Big(T,\big(r^2+4{\textstyle\int_{T_0}^T}\Lambda_f\big)^{\frac12}\Big)\,=\,0.$$

Recall that the assumptions on K and f ensure $\int_0^T\Lambda_f<\infty$, so that we can find $T_0<T$ with $\int_{T_0}^T\Lambda_f<\frac18$. For this choice of T_0, the above yields $Z(t,r)=0$ for all $t\in[T_0,T]$ and $r\in[0,\frac12]$, which means $C_n(t)=0$

for all n and $t \in [T_0, T]$. Iterating this argument over successive time intervals, we conclude $C_n(t) = 0$ for all n and $t \in [0, T]$. $\square$

Combining the above different observations, we are now in position to conclude the proof of Theorem 1.

Proof of Theorem 1. By Proposition 8, the limit rescaled correlations $\{\bar{C}_n\}_n$ extracted in Lemma 6 satisfy the limit hierarchy (30) as well as the a priori estimates (22). Now, using the cancellation properties (19) for V_f, a trivial solution of the hierarchy (30) with the same final condition (31) is actually given by the constant

$$C_n(t) \;:=\; \bar{C}_n|_{t=T} \;=\; \begin{cases} (\int_{\mathbb{D}} \psi f(T))^k, & \text{for } n = 0, \\ 0, & \text{for } n \geq 1. \end{cases}$$

The uniqueness result of Lemma 9 then entails that the extracted limit $\{\bar{C}_n\}_n$ is necessarily equal to this trivial solution. In particular, this means $\bar{C}_2 \equiv 0$, thus proving the convergence (20) independently of extractions. By Proposition 4, this entails $\int_{\mathbb{D}^k} \psi^{\otimes k} F_{N,k}(T) \to (\int_{\mathbb{D}} \psi f(T))^k$. By the arbitrariness of $k \geq 1$ and $\psi \in C_c^\infty(\mathbb{D})$, this concludes the proof. $\square$

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Appendix A. Weak duality solutions

In this appendix, we introduce a new notion of "weak duality solutions" for the Liouville equation, which is the relevant one for our duality approach to mean field. In a nutshell, weak duality solutions are defined to be in duality with *some*, but possibly not all, bounded weak solutions of the backward equation with given final condition. We shall see below that weak duality solutions always exist whenever $K \in L^1_{loc}(\Omega; \mathbb{R}^d)$ and $F_N^\circ \in L^1 \cap L^p(\mathbb{D}^N)$ for some $p > 1$, cf. Proposition 11. We emphasize that weak duality solutions do not need to be actual weak solutions of the Liouville equation, as in particular the product $K(x_i - x_j)F_N$ might not be defined. In addition, bounded

[2]Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

renormalized solutions for the backward equation are not required to exist either, so that in particular no uniqueness is guaranteed in general. To be complete, we can cite for instance [48] and references citedtherein (for intance [6]) for dual solutions definitions related to transport type equations. Of course, weak duality solutions coincide with renormalized solutions whenever the latter exist: for $\alpha = 0$, this is known to be the case for instance if $K \in BV_{loc}(\Omega;\mathbb{R}^d)$, cf. [5], or if $K \in L^1_{loc}(\Omega;\mathbb{R}^d) \cap BV_{loc}(\Omega \setminus \{0\};\mathbb{R}^d)$ takes the form $K = -\nabla V$ with $V(x) \geq -C(1+|x|^2)$, cf. [32], and we refer to [40] for the case $\alpha > 0$.

Definition 10. *Let $K \in L^1_{loc}(\Omega;\mathbb{R}^d)$.*

(i) *Given $T > 0$ and $\Phi_N^T \in L^\infty(\mathbb{D}^N)$, we consider the following backward Liouville equation on $(-\infty, T]$,*

$$\partial_t \Phi_N + \sum_{i=1}^N \Big(v_i \cdot \nabla_{x_i} \Phi_N + \frac{1}{N-1} \sum_{j:j\neq i}^N K(x_i - x_j) \cdot \nabla_{v_i} \Phi_N \Big) = -\alpha \sum_{i=1}^N \Delta_{v_i} \Phi_N, \tag{38}$$

with final data $\Phi_N|_{t=T} = \Phi_N^T$. We say that $\Phi_N \in L^\infty(-\infty, T; L^\infty(\mathbb{D}^N))$ is a global bounded weak solution *of this backward Liouville equation with final data Φ_N^T if it belongs to $\mathcal{C}_{loc}(-\infty, T; w^*L^\infty(\mathbb{D}^N))$ with $\|\Phi_N\|_{L^\infty((-\infty,\,T]\times\mathbb{D}^N)} \leq \|\Phi_N^T\|_{L^\infty(\mathbb{D}^N)}$, and if it satisfies* (38) *in the weak sense: for all $G \in \mathcal{C}_c^\infty((-\infty, T] \times \mathbb{D}^N)$,*

$$\begin{aligned} &\int_{-\infty}^T \int_{\mathbb{D}^N} (\partial_t G) \Phi_N - \int_{\mathbb{D}^N} G(T)\, \Phi_N^T \\ &+ \sum_{i=1}^N \int_{-\infty}^T \int_{\mathbb{D}^N} \Big(v_i \cdot \nabla_{x_i} G + \frac{1}{N-1} \sum_{j:j\neq i} K(x_i - x_j) \cdot \nabla_{v_i} G \Big) \Phi_N \\ &\qquad\qquad = \alpha \sum_{i=1}^N \int_{-\infty}^T \int_{\mathbb{D}^N} (\Delta_{v_i} G) \Phi_N. \end{aligned}$$

(ii) *Given $F_N^\circ \in L^1(\mathbb{D}^N)$, we say that $F_N \in L^\infty_{loc}(\mathbb{R}^+; L^1(\mathbb{D}^N))$ is a* **global weak duality solution** *of the Liouville equation* (2) *with initial data F_N° if it belongs to $\mathcal{C}_{loc}(\mathbb{R}^+; wL^1_{loc}(\mathbb{D}^N))$ and if, for any $T > 0$ and any $\Phi_N^T \in L^\infty(\mathbb{D}^N)$ that tends to 0 at infinity, there exists a global bounded weak solution $\Phi_N \in L^\infty(-\infty, T; L^\infty(\mathbb{D}^N))$ of the backward Liouville equation* (38) *on $(-\infty, T]$ with final data Φ_N^T, in the sense of* (i) *above, such that we have the duality formula*

$$\int_{\mathbb{D}^N} \Phi_N^T F_N(T) = \int_{\mathbb{D}^N} \Phi_N(0) F_N^\circ.$$

(Note that both sides of this formula make pointwise sense thanks to the time continuity of F_N and Φ_N.)

As the following result shows, global bounded weak duality solutions always exist whenever $K \in L^1_{loc}(\Omega;\mathbb{R}^d)$ and $F_N^\circ \in L^1(\mathbb{D}^N)\cap L^p(\mathbb{D}^N)$ for some $p>1$.

Proposition 11. *Let $K \in L^1_{loc}(\Omega;\mathbb{R}^d)$ and $F_N^\circ \in L^1(\mathbb{D}^N)\cap L^p(\mathbb{D}^N)$ for some $1<p\le\infty$. Then there exists a global weak duality solution $F_N \in L^\infty(\mathbb{R}^+;L^1(\mathbb{D}^N)\cap L^p(\mathbb{D}^N))$ of the Liouville equation* (2) *with initial data F_N°. If in addition $K\in L^{p'}_{loc}(\Omega;\mathbb{R}^d)$ with $1/p+1/p'=1$, then there exists such a global weak duality solution that further satisfies* (2) *in the weak sense.*

Proof. We proceed by approximation: let $K_\epsilon \in \mathcal{C}_b^\infty(\Omega;\mathbb{R}^d)$ and $F_{N,\epsilon}^\circ \in L^1(\mathbb{D}^N)\cap L^\infty(\mathbb{D}^N)$ such that $K_\epsilon \to K$ in $L^1_{loc}(\Omega;\mathbb{R}^d)$ and $F_{N,\epsilon}^\circ \to F_N^\circ$ in $L^1(\mathbb{D}^N)\cap L^p(\mathbb{D}^N)$ as $\epsilon\to 0$. For fixed ϵ, by regularity of K_ϵ and uniform boundedness of $F_{N,\epsilon}^\circ$, it is well-known that there exists a unique global weak solution $F_{N,\epsilon}\in L^\infty(\mathbb{R}^+;L^1(\mathbb{D}^N)\cap L^\infty(\mathbb{D}^N))$ of the Liouville equation (2) with kernel K replaced by K_ϵ and with initial data F_N° replaced by $F_{N,\epsilon}^\circ$, cf. e.g. [18], and it satisfies the a priori estimate

$$\|F_{N,\epsilon}\|_{L^\infty(\mathbb{R}^+;L^1(\mathbb{D}^N)\cap L^p(\mathbb{D}^N))} \le \|F_{N,\epsilon}^\circ\|_{L^1(\mathbb{D}^N)\cap L^p(\mathbb{D}^N)}. \tag{39}$$

By weak compactness, up to a subsequence as $\epsilon\downarrow 0$, we deduce that $F_{N,\epsilon}$ converges weakly-* to some F_N in $L^\infty(\mathbb{R}^+;L^1(\mathbb{D}^N)\cap L^p(\mathbb{D}^N))$. In the particular case when we have in addition $K\in L^{p'}_{loc}(\Omega;\mathbb{R}^d)$, we can pass to the limit in the weak formulation of the equation for $F_{N,\epsilon}$ and conclude that F_N actually satisfies the weak formulation of the Liouville equation (2) with initial data F_N°.

We turn to the duality property. Given $T>0$ and given $\Phi_N^T\in L^\infty(\mathbb{D}^N)$, by regularity of K_ϵ, there exists a unique global weak solution $\Phi_{N,\epsilon}\in L^\infty(-\infty,T;L^\infty(\mathbb{D}^N))$ of the backward Liouville equation (38) on $(-\infty,T]$ with kernel K replaced by K_ϵ and with final data Φ_N^T, and it satisfies the a priori estimate

$$\|\Phi_{N,\epsilon}\|_{L^\infty(-\infty,T;L^\infty(\mathbb{D}^N))} \le \|\Phi_N^T\|_{L^\infty(\mathbb{D}^N)}. \tag{40}$$

By weak compactness, up to a subsequence as $\epsilon\downarrow 0$, we deduce that $\Phi_{N,\epsilon}$ converges weakly-* to some Φ_N in $L^\infty(-\infty,T;L^\infty(\mathbb{D}^N))$ and that the latter satisfies the weak formulation of the backward Liouville equation (38) with final data Φ_N^T. From (40) and from the equation for $\Phi_{N,\epsilon}$, we find that the time derivatives $(\partial_t\Phi_{N,\epsilon})_\epsilon$ are bounded in $L^\infty(-\infty,T;W^{-2,1}_{loc}(\mathbb{D}^N))$. By the Aubin–Lions–Simon lemma, this entails that $(\Phi_{N,\epsilon})_\epsilon$ is precompact e.g. in $\mathcal{C}_{loc}(-\infty,T;W^{-1,\infty}_{loc}(\mathbb{D}^N))$. As $W^{-1,\infty}=$

$(W^{1,1})^*$ and as $W^{1,1}(\mathbb{D}^N)$ is dense in $L^1(\mathbb{D}^N)$, we conclude that $(\Phi_{N,\epsilon})_\epsilon$ is actually precompact in $\mathcal{C}_{loc}(-\infty, T; w^*L^\infty(\mathbb{D}^N))$. This proves the desired time continuity of the limit Φ_N, which is thus by definition a global bounded weak solution of the backward Liouville equation.

For fixed ϵ, by regularity of K_ϵ, the bounded weak solutions $F_{N,\epsilon}$ and $\Phi_{N,\epsilon}$ can be shown to satisfy the following duality formula, cf. [18, Section II.4], for all $t \geq 0$,

$$\int_{\mathbb{D}^N} \Phi_N^T F_{N,\epsilon}(t) \;=\; \int_{\mathbb{D}^N} \Phi_{N,\epsilon}(T-t) F_{N,\epsilon}^\circ. \tag{41}$$

Recalling that $(\Phi_{N,\epsilon})_\epsilon$ is precompact in $C_{loc}(-\infty, T; w^*L^\infty(\mathbb{D}^N))$, that $(F_{N,\epsilon}^\circ)_\epsilon$ converges strongly to F_N° in $L^1(\mathbb{D}^N)$, that $(F_{N,\epsilon})_\epsilon$ converges weakly-* to F_N in $L^\infty(\mathbb{R}^+; L^1(\mathbb{D}^N) \cap L^p(\mathbb{D}^N))$, and that the choice of $\Phi_N^T \in L^\infty(\mathbb{D}^N)$ is arbitrary, this identity entails that $(F_{N,\epsilon})_\epsilon$ is precompact in $C_{loc}(\mathbb{R}^+; wL^p(\mathbb{D}^N))$. We may now pass to the pointwise limit in (41) and conclude that F_N is a weak duality solution. $\square$

We emphasize that a weak duality solution F_N in the above sense may not remain a probability density, even if F_N^0 was one. If $\mathbb{D}^N$ is unbounded, the proof does not ensure that we do not lose mass at infinity in finite time: F_N is only tested against Φ_N, which vanishes at infinity, so that we cannot take $\Phi_N = 1$. This issue can correspond to a possible blow-up in the original many-particle system. With the assumptions of the proposition above, there is indeed no reason to expect strong solutions to exist globally in time for the trajectories of the particles. Of course if trajectories go to infinity in finite time with a positive probability, then the corresponding configurations have to vanish from F_N leading to a loss of mass.

We can also introduce a corresponding notion of weak duality solutions for the Liouville equation (6) associated to first-order systems as considered in Theorem 2. We state below a corresponding existence result; details are omitted for shortness. Note that the assumption $\operatorname{div} K \in L^1_{loc}(\Omega)$ is needed to make sense of the weak formulation of the backward equation. Again, weak duality solutions coincide with renormalized solutions when the latter exist: for $\alpha = 0$, this is known to be the case for instance if $K \in BV_{loc}(\Omega; \mathbb{R}^d)$ with $\operatorname{div} K \in L^\infty_{loc}(\Omega)$, cf. [1], and we refer to [40] for the case $\alpha > 0$.

Definition 12. *Let $K \in L^1_{loc}(\Omega;\mathbb{R}^d)$ with $\operatorname{div} K \in L^1_{loc}(\Omega)$.*

(i) *Given $T > 0$ and $\Phi_N^T \in L^\infty(\Omega^N)$, we consider the following backward Liouville equation on $(-\infty, T]$,*

$$\partial_t \Phi_N + \frac{1}{N-1}\sum_{i\neq j}^N K(x_i - x_j)\cdot \nabla_{x_i}\Phi_N = -\alpha \sum_{i=1}^N \Delta_{x_i}\Phi_N, \tag{42}$$

with final data $\Phi_N|_{t=T} = \Phi_N^T$. We say that $\Phi_N \in L^\infty(-\infty, T; L^\infty(\Omega^N))$ is a global bounded weak solution *of this backward equation with final data Φ_N^T if it belongs to $\mathcal{C}_{loc}(-\infty, T; w^*L^\infty(\Omega^N))$ with $\|\Phi_N\|_{L^\infty((-\infty,\,T]\times\Omega^N)} \le \|\Phi_N^T\|_{L^\infty(\Omega^N)}$, and if it satisfies* (42) *in the weak sense.*

(ii) *Given $F_N^\circ \in L^1(\Omega^N)$, we say that $F_N \in L^\infty_{loc}(\mathbb{R}^+; L^1(\Omega^N))$ is a* **global weak duality solution** *of the Liouville equation* (6) *with initial data F_N° if it belongs to $\mathcal{C}_{loc}(\mathbb{R}^+; wL^1_{loc}(\Omega^N))$ and if, for any $T > 0$ and any $\Phi_N^T \in L^\infty(\Omega^N)$ that converges to* 0 *at infinity, there exists a global bounded weak solution $\Phi_N \in L^\infty(-\infty, T; L^\infty(\Omega^N))$ of the backward Liouville equation* (42) *on $(-\infty, T]$ with final data Φ_N^T, in the sense of* (i) *above, such that we have the duality formula*

$$\int_{\Omega^N} \Phi_N^T F_N(T) = \int_{\Omega^N} \Phi_N(0) F_N^\circ.$$

Proposition 13. *Let $K \in L^1_{loc}(\Omega;\mathbb{R}^d)$ with $\operatorname{div} K \in L^1_{loc}(\Omega)$, and let $F_N^\circ \in L^1(\Omega^N)\cap L^p(\Omega^N)$ for some $p > 1$. Then there exists a global weak duality solution $F_N \in L^\infty(\mathbb{R}^+; L^1(\Omega^N)\cap L^p(\Omega^N))$ of the Liouville equation* (6) *with initial data F_N°.*

Appendix B. Explicit hierarchy

In this appendix, we derive the full hierarchy of equations for linearized dual correlations $\{C_{N,n}\}_{0\le n\le N}$ without focusing on their weak formulation on H_n: in other words, we derive the expression for the remainder $R_{N,n}$ in Proposition 7. For that purpose, we start from the equations (26) for marginals, we appeal to (16) and (17) to transform them into equations for correlations, and we work out the combinatorics. Note that from equation (43) below it is now straightforward to recover the result of Proposition 8 after rescaling and passing to the limit.

Lemma 14. *For all $0 \le n \le N$, we have in the distributional sense on $[0,T]\times\mathbb{D}^n$,*

$$\partial_t C_{N,n} + \sum_{i=1}^n v_i\cdot\nabla_{x_i} C_{N,n} + \alpha\sum_{i=1}^n \Delta_{v_i} C_{N,n} \tag{43}$$

$$= \frac{1}{N-1} S_N^{n,+} C_{N,n-1} + S_N^{n,\circ} C_{N,n} + S_N^{n,-} C_{N,n+1} + N S_N^{n,=} C_{N,n+2},$$

where we have set $C_{N,-1}, C_{N,N+1}, C_{N,N+2} \equiv 0$ *for notational simplicity, and where we have defined the following operators*

$$\begin{aligned}
S_N^{n,+} C_{N,n-1} :=\ & \sum_{i\neq j}^{n} (K * f)(x_i) \cdot \nabla_{v_i} C_{N,n-1}(z_{[n]\setminus\{j\}}) \\
& - \sum_{i\neq j}^{n} \int_{\mathbb{D}} V_f(z_*, z_j)\, C_{N,n-1}(z_{[n]\setminus\{i,j\}}, z_*)\, f(z_*)\, dz_* \\
& - \sum_{i\neq j}^{n} K(x_i - x_j) \cdot \nabla_{v_i} C_{N,n-1}(z_{[n]\setminus\{j\}}), \\
S_N^{n,\circ} C_{N,n} :=\ & - \frac{N-n}{N-1} \sum_{i=1}^{n} (K * f)(x_i) \cdot \nabla_{v_i} C_{N,n} \\
& + \frac{N-n}{N-1} \sum_{j=1}^{n} \int_{\mathbb{D}} V_f(z_*, z_j)\, C_{N,n}(z_{[n]\setminus\{j\}}, z_*)\, f(z_*)\, dz_* \\
& - \frac{1}{N-1} \sum_{i\neq j}^{n} K(x_i - x_j) \cdot \nabla_{v_i} C_{N,n}(z_{[n]}) \\
& + \frac{1}{N-1} \sum_{i\neq j}^{n} \int_{\mathbb{D}} K(x_i - x_*) \cdot \nabla_{v_i} C_{N,n}(z_{[n]\setminus\{j\}}, z_*)\, f(z_*)\, dz_* \\
& - \frac{1}{N-1} \sum_{i\neq j}^{n} \int_{\mathbb{D}} V_f(z_*, z_j)\, C_{N,n}(z_{[n]\setminus\{i\}}, z_*)\, f(z_*)\, dz_* \\
& + \frac{1}{N-1} \sum_{i\neq j}^{n} \int_{\mathbb{D}^2} V_f(z_*, z_*')\, C_{N,n}(z_{[n]\setminus\{i,j\}}, z_*, z_*')\, f(z_*) f(z_*')\, dz_* dz_*', \\
S_N^{n,-} C_{N,n+1} :=\ & \frac{N-n}{N-1} \sum_{j=1}^{n} \int_{\mathbb{D}} V_f(z_*, z_j)\, C_{N,n+1}(z_{[n]}, z_*)\, f(z_*)\, dz_* \\
& - \frac{N-n}{N-1} \sum_{i=1}^{n} \int_{\mathbb{D}} K(x_i - x_*) \cdot \nabla_{v_i} C_{N,n+1}(z_{[n]}, z_*)\, f(z_*)\, dz_* \\
& - 2\frac{N-n}{N-1} \sum_{i=1}^{n} \int_{\mathbb{D}^2} V_f(z_*, z_*')\, C_{N,n+1}(z_{[n]\setminus\{i\}}, z_*, z_*')\, f(z_*) f(z_*')\, dz_* dz_*', \\
S_N^{n,=} C_{N,n+2} :=\ & \frac{(N-n)(N-n-1)}{N(N-1)} \\
& \qquad \times \int_{\mathbb{D}^2} V_f(z_*, z_*')\, C_{N,n+2}(z_{[n]}, z_*, z_*')\, f(z_*) f(z_*')\, dz_* dz_*'.
\end{aligned}$$

Moreover, the final condition (8) *for correlations at time $t = T$ takes the form*

$$C_{N,n}(T, z_1, \ldots, z_n) \\ = 1_{n\le k} \binom{N}{n}^{-1} \binom{k}{n} \sum_{l=0}^{n} (-1)^{n+l} \sum_{\tau\in P_l^n} \Big(\int_{\mathbb{D}} \psi f\Big)^{k-l} \psi^{\otimes l}(z_\tau). \tag{44}$$

Proof. We first recall the BBGKY-type hierarchy (26) of equations satisfied by dual marginals. In order to derive corresponding equations for correlations, we start from (16), writing for all $0 \le n \le N$,

$$\partial_t C_{N,n} = \sum_{k=0}^{n} (-1)^{n-k} \sum_{\sigma\in P_k^n} \partial_t M_{N,k}(z_\sigma).$$

Inserting the equations (26) for marginals, we are led to

$$\partial_t C_{N,n} + \sum_{i=1}^{n} v_i \cdot \nabla_{x_i} C_{N,n} + \alpha \sum_{i=1}^{n} \Delta_{v_i} C_{N,n} = T_1 + T_2 + T_3 + T_4,$$

where we have set

$$T_i := \sum_{k=0}^{n} (-1)^{n-k} \sum_{\sigma\in P_k^n} T'_{i,k}(z_\sigma),$$

in terms of

$$\begin{aligned}
T'_{1,k}(z_\sigma) &:= -\frac{1}{N-1} \sum_{\substack{i,j\in\sigma\\ i\neq j}} K(x_i - x_j) \cdot \nabla_{v_i} M_{N,k}(z_\sigma),\\
T'_{2,k}(z_\sigma) &:= \frac{N-k}{N-1} \sum_{j\in\sigma} \int_{\mathbb{D}} V_f(z_*, z_j) M_{N,k+1}(z_\sigma, z_*)\, f(z_*)\, dz_*,\\
T'_{3,k}(z_\sigma) &:= -\frac{N-k}{N-1} \sum_{i\in\sigma} \int_{\mathbb{D}} K(x_i - x_*) \cdot \nabla_{v_i} M_{N,k+1}(z_\sigma, z_*)\, f(z_*)\, dz_*,\\
T'_{4,k}(z_\sigma) &:= \frac{(N-k)(N-k-1)}{N-1}\\
&\qquad \times \int_{\mathbb{D}^2} V_f(z_*, z'_*)\, M_{N,k+2}(z_\sigma, z_*, z'_*)\, f(z_*) f(z'_*)\, dz_* dz'_*.
\end{aligned}$$

We consider the different contributions $\{T_i\}_{1\le i\le 4}$ separately and use the cluster expansion (14) to express each of them in terms of correlations instead of marginals. First, for T_1, reorganizing the sums, we find

$$T_1 = -\frac{1}{N-1} \sum_{k=0}^{n} (-1)^{n-k} \sum_{\sigma\in P_k^n} \sum_{\substack{i,j\in\sigma\\ i\neq j}} K(x_i - x_j) \cdot \nabla_{v_i} M_{N,k}(z_\sigma)$$

$$
\begin{aligned}
&= -\frac{1}{N-1}\sum_{k=0}^{n}(-1)^{n-k}\sum_{\sigma\in P_k^n}\sum_{\substack{i,j\in\sigma\\ i\neq j}} K(x_i-x_j)\cdot\nabla_{v_i}\sum_{l=0}^{k}\sum_{\tau\in P_l^\sigma} C_{N,l}(z_\tau)\\
&= -\frac{1}{N-1}\sum_{i\neq j}^{n} K(x_i-x_j)\cdot\nabla_{v_i}\sum_{l=0}^{n}\sum_{\tau\in P_l^n} C_{N,l}(z_\tau)\\
&\qquad\qquad\times\sum_{k=l}^{n}(-1)^{n-k}\sharp\big\{\sigma\in P_k^n:\sigma\supset\tau\cup\{i,j\}\big\},
\end{aligned}
$$

and thus, distinguishing the cases whether $j\in\tau$ or $j\notin\tau$, noting that the contribution vanishes if $i\notin\tau$, and computing the cardinalities,

$$
\begin{aligned}
T_1 \;=\; &-\frac{1}{N-1}\sum_{i\neq j}^{n} K(x_i-x_j)\cdot\nabla_{v_i}\\
&\times\bigg(\sum_{l=0}^{n}\sum_{\substack{\tau\in P_l^n\\ \tau\ni j}} C_{N,l}(z_\tau)\sum_{k=l}^{n}(-1)^{n-k}\binom{n-l}{k-l}\\
&\qquad+\sum_{l=0}^{n}\sum_{\substack{\tau\in P_l^n\\ \tau\not\ni j}} C_{N,l}(z_\tau)\sum_{k=l}^{n}(-1)^{n-k}\binom{n-l-1}{k-l-1}\bigg).
\end{aligned}
$$

Now using the binomial identity

$$
\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k} \;=\; 1_{p=0}, \tag{45}
$$

we deduce

$$
T_1 \;=\; -\frac{1}{N-1}\sum_{i\neq j}^{n} K(x_i-x_j)\cdot\nabla_{v_i}\Big(C_{N,n}(z_{[n]})+C_{N,n-1}(z_{[n]\setminus\{j\}})\Big).
$$

Next, for the second term T_2, using that $\int_{\mathbb{R}^d\times\mathbb{R}^d} V_f(z_*,z)\,f(z_*)\,dz_*=0$, we can similarly write

$$
\begin{aligned}
T_2 \;&=\; \sum_{k=0}^{n}(-1)^{n-k}\sum_{\sigma\in P_k^n}\frac{N-k}{N-1}\sum_{j\in\sigma}\int_{\mathbb{D}} V_f(z_*,z_j)M_{N,k+1}(z_\sigma,z_*)\,f(z_*)\,dz_*\\
&=\; \sum_{k=0}^{n}(-1)^{n-k}\sum_{\sigma\in P_k^n}\frac{N-k}{N-1}\sum_{j\in\sigma}\int_{\mathbb{D}} V_f(z_*,z_j)\\
&\qquad\qquad\times\sum_{l=0}^{k}\sum_{\tau\in P_l^\sigma} C_{N,l+1}(z_\tau,z_*)\,f(z_*)\,dz_*
\end{aligned}
$$

$$= \sum_{j=1}^{n}\sum_{l=0}^{n}\sum_{\tau\in P_l^n}\int_{\mathbb{D}} V_f(z_*,z_j)C_{N,l+1}(z_\tau,z_*)\,f(z_*)\,dz_* \times\sum_{k=l}^{n}(-1)^{n-k}\frac{N-k}{N-1}\sharp\{\sigma\in P_k^n:\sigma\supset\tau\cup\{j\}\},$$

and thus, distinguishing the cases whether $j\in\tau$ or $j\notin\tau$, computing the cardinalities, and using again the binomial identity (45), now in form of

$$\sum_{k=l}^{n}(-1)^{n-k}\frac{N-k}{N-1}\binom{n-l}{k-l} = 1_{l=n}\frac{N-n}{N-1}-1_{l=n-1}\frac{1}{N-1},$$

we obtain

$$\begin{aligned} T_2 &= \frac{N-n}{N-1}\sum_{j=1}^{n}\int_{\mathbb{D}} V_f(z_*,z_j)C_{N,n+1}(z_{[n]},z_*)\,f(z_*)\,dz_* \\ &- \frac{1}{N-1}\sum_{i\neq j}^{n}\int_{\mathbb{D}} V_f(z_*,z_j)C_{N,n}(z_{[n]\setminus\{i\}},z_*)\,f(z_*)\,dz_* \\ &+ \frac{N-n}{N-1}\sum_{j=1}^{n}\int_{\mathbb{D}} V_f(z_*,z_j)C_{N,n}(z_{[n]\setminus\{j\}},z_*)\,f(z_*)\,dz_* \\ &- \frac{1}{N-1}\sum_{i\neq j}^{n}\int_{\mathbb{D}} V_f(z_*,z_j)C_{N,n-1}(z_{[n]\setminus\{i,j\}},z_*)\,f(z_*)\,dz_*. \end{aligned}$$

Similar computations for T_3 and T_4 easily yield

$$\begin{aligned} T_3 &= -\frac{N-n}{N-1}\sum_{i=1}^{n}\int_{\mathbb{D}} K(x_i-x_*)\cdot\nabla_{v_i}C_{N,n+1}(z_{[n]},z_*)\,f(z_*)\,dz_* \\ &\quad +\frac{1}{N-1}\sum_{i\neq j}^{n}\int_{\mathbb{D}} K(x_i-x_*)\cdot\nabla_{v_i}C_{N,n}(z_{[n]\setminus\{j\}},z_*)\,f(z_*)\,dz_* \\ &\quad -\frac{N-n}{N-1}\sum_{i=1}^{n} K*f(x_i)\cdot\nabla_{v_i}C_{N,n}(z_{[n]}) \\ &\quad +\frac{1}{N-1}\sum_{i\neq j}^{n} K*f(x_i)\cdot\nabla_{v_i}C_{N,n-1}(z_{[n]\setminus\{j\}}), \end{aligned}$$

and

$$T_4 = \frac{1}{N-1}\sum_{i\neq j}^{n}\int_{\mathbb{D}^2} V_f(z_*,z_*')\,C_{N,n}(z_{[n]\setminus\{i,j\}},z_*,z_*')\,f(z_*)f(z_*')\,dz_*dz_*'$$

$$
\begin{aligned}
&- \quad 2\frac{N-n}{N-1}\sum_{i=1}^{n}\int_{\mathbb{D}^2} V_f(z_*,z_*')\, C_{N,n+1}(z_{[n]\setminus\{i\}},z_*,z_*')\, f(z_*)f(z_*')\, dz_*dz_*' \\
&+ \quad \frac{(N-n)(N-n-1)}{N-1} \\
&\qquad\qquad \times \int_{\mathbb{D}^2} V_f(z_*,z_*')\, C_{N,n+2}(z_{[n]},z_*,z_*')\, f(z_*)f(z_*')\, dz_*dz_*'.
\end{aligned}
$$

Combining these different computations, we are precisely led to the claimed equations (43) for correlation functions.

It remains to derive the final condition (44) for correlations. We recall that from (8) we find (36) as the final condition for marginals. Inserting this into the definition (16) of correlation functions, and using the combinatorial identity[3]

$$
\sum_{r=l}^{n}(-1)^{n-r}\binom{N-r}{k-l}\binom{n-l}{r-l} \;=\; (-1)^{n+l}\binom{N-n}{k-n},
$$

the claim (44) follows after straightforward computations. □

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