ON NONLINEAR SCHRÖDINGER EQUATIONS WITH RANDOM INITIAL DATA

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Abstract. This note is concerned with the global well-posedness of nonlinear Schrödinger equations in the continuum with spatially homogeneous random initial data.

1. Introduction

Motivated by weak turbulence theory, e.g. [10], we consider nonlinear Schrödinger equations with spatially homogeneous statistical ensembles of initial data. As a prototypical example, we study the defocusing cubic equation in $\mathbb{R}^d$,

$$i\partial_t u = -\triangle u + |u|^2 u, \qquad u|_{t=0} = u^\circ. \tag{1.1}$$

A statistical ensemble of initial data amounts to considering an initial condition that is a realization $u^\circ(\cdot,\omega)$ of a random field $u^\circ : \mathbb{R}^d \times \Omega \to \mathbb{C}$ on some probability space $(\Omega, \mathbb{P})$. The spatial homogeneity condition then implies that initial mass and energy diverge:[1]

$$\mathbb{E}\left[\int_{\mathbb{R}^d} |u^\circ|^2\right] = \infty, \qquad \mathbb{E}\left[\int_{\mathbb{R}^d} |\nabla u^\circ|^2 + \tfrac{1}{2}|u^\circ|^4\right] = \infty.$$

This divergence is a key aspect at the very core of weak turbulence: Strichartz' estimates are not applicable in this infinite-energy setting, which is thus in sharp contrast with the finite-energy phenomenology and scattering results [7].

The present note is concerned with the global well-posedness of (1.1) in this infinite-energy setting. The main difficulty is related to the lack of a uniform bound on the propagation speed: mass that is initially spread out might move together and blow up. This contrasts with the case of the nonlinear wave equation, as well as of the *discrete* nonlinear Schrödinger equation, for which there is an (approximate) finite propagation speed and global well-posedness follows, see [3, Propositions 1–3]. As explained in Examples 2.3 below, periodic and quasi-periodic initial data can in fact be viewed as particular instances of the spatially homogeneous random setting. While the periodic case is well understood [2], the almost periodic case remains largely open and we refer to recent work by Oh [12, 11] on the topic. In the general random setting, the problem seems to have only been considered very recently by Dodson, Soffer, and Spencer [3], who established local well-posedness in the real analytic category. If the nonlinearity $|u|^2u$ in (1.1) is replaced by a regularized version $|\phi * u|^2(\phi * u)$ for some smooth decaying kernel ϕ, then the problem is strongly reduced and global well-posedness is obtained in [3] in the C^k category.

[1] Under an additional ergodicity assumption, it further entails almost surely

$$\int_{\mathbb{R}^d} |u^\circ(\cdot,\omega)|^2 = \infty, \qquad \int_{\mathbb{R}^d} |\nabla u^\circ(\cdot,\omega)|^2 + \tfrac{1}{2}|u^\circ(\cdot,\omega)|^4 = \infty.$$

Our main result in this note states the global well-posedness of the nonlinear Schrödinger equation (1.1) in the spatially homogeneous energy space provided that a tiny dissipation is added. This tiny dissipation is physically relevant in the context of weak turbulence, e.g. [10], and the constructed solution is controlled uniformly with respect to this dissipation. The definition of a meaningful vanishing-dissipation limit remains an open problem (beyond the local-in-time real analytic framework of [3]). Precise definitions of spatial homogeneity and of the functional space $\mathbb{X}$ below are postponed to the next section.

Theorem 1. *Let $1 \le d < 4$. Given a probability space $(\Omega, \mathbb{P})$, let $\mathbb{X}$ be the Banach space of spatially homogeneous jointly measurable random fields $v : \mathbb{R}^d \times \Omega \to \mathbb{C}$ with*

$$\|v\|_{\mathbb{X}} := \mathbb{E}\big[|\nabla v|^2\big]^{\frac12} + \mathbb{E}\big[|v|^4\big]^{\frac14} < \infty.$$

For all $\varepsilon > 0$ and $u^\circ \in \mathbb{X}$, there exists a unique global weak solution $u_\varepsilon \in \mathrm{L}^\infty(\mathbb{R}^+; \mathbb{X})$ to the equation

$$(-\varepsilon + i)\partial_t u_\varepsilon = -\triangle u_\varepsilon + |u_\varepsilon|^2 u_\varepsilon, \qquad u_\varepsilon|_{t=0} = u^\circ, \tag{1.2}$$

in the sense that Duhamel's formula holds almost everywhere,

$$u_\varepsilon^t = e^{\frac{t}{\varepsilon - i}\triangle} u^\circ - \tfrac{1}{\varepsilon - i} \int_0^t e^{\frac{t-s}{\varepsilon-i}\triangle}(|u_\varepsilon^s|^2 u_\varepsilon^s)\, ds, \qquad t \ge 0.$$

In addition, it satisfies the following dissipation estimates: for all $t \ge 0$,

$$\mathbb{E}\left[|u_\varepsilon^t|^2\right] \le \mathbb{E}\left[|u_\varepsilon^t|^2\right] + \tfrac{2\varepsilon}{1+\varepsilon^2} \int_0^t \mathbb{E}\left[|\nabla u_\varepsilon^s|^2 + |u_\varepsilon^s|^4\right] ds \;=\; \mathbb{E}\left[|u^\circ|^2\right],$$

$$\mathbb{E}\big[|\nabla u_\varepsilon^t|^2 + \tfrac12 |u_\varepsilon^t|^4\big] \;\le\; \mathbb{E}\big[|\nabla u^\circ|^2 + \tfrac12 |u^\circ|^4\big]. \qquad \Diamond$$

Notation.

- We denote by $C \ge 1$ any constant that only depends on the space dimension d. We use the notation $\lesssim$ (resp. $\gtrsim$) for $\le C\times$ (resp. $\ge \frac{1}{C}\times$) up to such a multiplicative constant C. We write $\simeq$ when both $\lesssim$ and $\gtrsim$ hold. We add subscripts to $C, \lesssim, \gtrsim, \simeq$ to indicate dependence on other parameters.
- The ball centered at x and of radius r in $\mathbb{R}^d$ is denoted by $B_r(x)$, and we write for abbreviation $B(x) := B_1(x)$ and $B_r := B_r(0)$.

2. Statistical spatial homogeneity

2.1. **Definition and examples.** Given a reference probability space $(\Omega, \mathbb{P})$, we recall the notion of statistical spatial homogeneity for random fields.

Definition 2.1. A *random field* on $\mathbb{R}^d$ is a map $v : \mathbb{R}^d \times \Omega \to \mathbb{C}$ such that for all $x \in \mathbb{R}^d$ the function $v(x, \cdot) : \Omega \to \mathbb{C}$ is measurable. It is said to be *(statistically) spatially homogeneous* if its finite-dimensional law is shift-invariant, that is, if for any finite set $E \subset \mathbb{R}^d$ the law of $\{v(x+y, \cdot)\}_{x \in E}$ does not depend on the shift $y \in \mathbb{R}^d$. In addition, it is said to be *jointly measurable* if the map $v : \mathbb{R}^d \times \Omega \to \mathbb{C}$ is jointly measurable. We denote by $\mathrm{L}^0_{\mathrm{hom}}(\mathbb{R}^d \times \Omega)$ the set of spatially homogeneous jointly measurable random fields. $\Diamond$

Note that the joint measurability condition ensures that realizations $v(\cdot, \omega)$ are almost surely measurable functions on $\mathbb{R}^d$ and can thus be taken as meaningful initial data in (1.1) or (1.2). The following observation by von Neumann [14] gives an alternative characterization of joint measurability in this context, which can be viewed as a stochastic version of Lusin's theorem (see also [8, Section 7.1]).

Lemma 2.2 (Joint measurability; [14, 8])**.** *A spatially homogeneous random field v is jointly measurable if and only if it is stochastically continuous, that is, if for all $\delta > 0$ it satisfies $\mathbb{P}[|v(x,\cdot) - v(y,\cdot)| > \delta] \to 0$ as $|x-y| \to 0$.* ♢

Examples 2.3. Important examples of spatially homogeneous random fields are found among Gaussian fields, and we also explain how periodic and almost periodic settings can be viewed as particular instances of this random framework (see also [13, p.846]).

(a) *Gaussian fields:* A gauge-invariant Gaussian random field v is a family $\{v(x,\cdot)\}_{x\in\mathbb{R}^d}$ of complex-valued Gaussian random variables such that v and $e^{i\theta}v$ have the same finite-dimensional law for all $\theta \in [0, 2\pi)$. Equivalently, this means for all x, y,

$$\mathbb{E}\left[v(x,\cdot)\right] = 0, \qquad \mathrm{Cov}\left[v(x,\cdot); v(y,\cdot)\right] = 0,$$

and we denote by $c(x,y) := \mathrm{Cov}\left[\overline{v(x,\cdot)}; v(y,\cdot)\right]$ the covariance function. This random field v is spatially homogeneous if and only if c is of the form $c(x,y) = c_0(x-y)$ for some function $c_0 : \mathbb{R}^d \to \mathbb{C}$. Note that c_0 is necessarily a positive definite bounded function. In addition, the field v is stochastically continuous, hence jointly measurable by Lemma 2.2, if and only if c_0 is continuous at the origin.

(b) *Periodic setting:* Given a 1-periodic measurable function $v_{\mathrm{per}} : \mathbb{R}^d \to \mathbb{C}$, we choose the probability space $(\Omega, \mathbb{P})$ as the periodic cell $[0,1)^d$ endowed with Lebesgue's measure, and we define an associated random field $v : \mathbb{R}^d \times \Omega \to \mathbb{C}$ by $v(x,\omega) := v_{\mathrm{per}}(x+\omega)$. The latter is clearly spatially homogeneous and jointly measurable, and for $\omega = 0$ we recover $v(x,0) = v_{\mathrm{per}}(x)$.

(c) *Almost periodic setting:* Denote by $\mathfrak{B}(\mathbb{R}^d)$ the Bohr compactification of the additive group $(\mathbb{R}^d, +)$ and let $\mathfrak{b} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ the associated continuous homomorphism, see e.g. [9]. By definition, given an almost periodic function $v_{\mathrm{ap}} : \mathbb{R}^d \to \mathbb{C}$, there exists a continuous function $V_{\mathrm{ap}} : \mathfrak{B}(\mathbb{R}^d) \to \mathbb{C}$ such that $v_{\mathrm{ap}} = V_{\mathrm{ap}} \circ \mathfrak{b}$. We choose the probability space $(\Omega, \mathbb{P})$ as $\mathfrak{B}(\mathbb{R}^d)$ endowed with its normalized Haar measure, and we define a random field $v : \mathbb{R}^d \times \Omega \to \mathbb{C}$ by $v(x,\omega) := V_{\mathrm{ap}}(\mathfrak{b}(x) + \omega)$, where we use the notation '+' for the group law on $\mathfrak{B}(\mathbb{R}^d)$. This random field is clearly spatially homogeneous and jointly measurable, and for $\omega = 0$ we recover $v(x,0) = v_{\mathrm{ap}}(x)$. ♢

2.2. **Functional setting.** In this section, we define more carefully the functional space $\mathbb{X}$ used in the statement of Theorem 1. For $1 \le q < \infty$, we denote by $\mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d \times \Omega)$ the Banach space of spatially homogeneous jointly measurable random fields $v \in \mathrm{L}^0_{\mathrm{hom}}(\mathbb{R}^d \times \Omega)$ such that the following norm is finite,

$$\|v\|_{\mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)} := \|v\|_{\mathrm{L}^q([0,1)^d\times\Omega)}.$$

By spatial homogeneity, see Definition 2.1, this is in fact equivalent to

$$\|v\|_{\mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)} = \|v(x,\cdot)\|_{\mathrm{L}^q(\Omega)} \qquad \text{for any } x \in \mathbb{R}^d.$$

As $\mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d \times \Omega)$ is invariant under spatial translations and as its elements are almost surely locally L^q-integrable, the spatial gradient ∇ can be defined on $\mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d \times \Omega)$ and its domain is denoted by $W^{1,q}_{\mathrm{hom}}(\mathbb{R}^d \times \Omega)$. More generally, for all $s \ge 0$, we define $W^{s,q}_{\mathrm{hom}}(\mathbb{R}^d \times \Omega)$ as the Banach space of random fields $v \in \mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d \times \Omega)$ such that the following norm is finite,

$$\|v\|_{W^{s,q}_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)} := \|(1-\triangle)^{\frac{s}{2}} v\|_{\mathrm{L}^q([0,1)^d\times\Omega)},$$

and for $q=2$ we use the usual notation $H^s_{\mathrm{hom}}(\mathbb{R}^d\times\Omega):=W^{s,2}_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$. In these terms, the space $\mathbb{X}$ in Theorem 1 coincides with $H^1_{\mathrm{hom}}\cap\mathrm{L}^4_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$.

Next, we give an alternative description of these spaces and we explain how equations (1.1) or (1.2) in this spatially homogeneous random setting are equivalent to abstract equations on the probability space; this construction is standard for corrector equations in stochastic homogenization theory, see e.g. [13, Section 2] and [8, Section 7.1]. Let $u^\circ\in\mathrm{L}^0_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$ be a reference random field. Since we consider equations with realizations of u° as initial data, we can henceforth assume that the probability space $(\Omega,\mathbb{P})$ is endowed with the σ-algebra $\sigma(u^\circ)$ generated by u°.[2] Translations $u^\circ(\cdot,\omega)\mapsto u^\circ(\cdot+x,\omega)$ then induce a unique multiplicative linear action $T=\{T_x\}_{x\in\mathbb{R}^d}$ of the additive group $(\mathbb{R}^d,+)$ on the algebra of random variables.

Lemma 2.4 (Properties of T)**.**

(i) For $1\le q\le\infty$, the maps T_x's are isometries on $\mathrm{L}^q(\Omega)$.

(ii) For $1\le q<\infty$, the action T is a C_0-group of isometries on $\mathrm{L}^q(\Omega)$. ◊

Proof. Item (i) follows from the fact that u° is spatially homogeneous. We turn to the proof of (ii). Given $q<\infty$, it remains to check that $\|T_xX-X\|_{\mathrm{L}^q(\Omega)}\to0$ as $|x|\to0$ for all $X\in\mathrm{L}^q(\Omega)$. By a truncation argument, it suffices to argue for $X\in\mathrm{L}^\infty(\Omega)$. By the joint measurability of u°, the map $(x,\omega)\mapsto(T_xX)(\omega)$ is also jointly measurable, hence stochastically continuous by Lemma 2.2. Writing for any $\delta>0$,

$$\|T_xX-X\|^q_{\mathrm{L}^q(\Omega)}\;\le\;\delta^q+(2\|X\|_{\mathrm{L}^\infty(\Omega)})^q\,\mathbb{P}\left[|T_xX-X|>\delta\right],$$

the conclusion follows from stochastic continuity. □

In terms of T, we can define the extension $X^\sharp\in\mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$ of any random variable $X\in\mathrm{L}^q(\Omega)$, and the restriction $v^\flat\in\mathrm{L}^q(\Omega)$ of any random field $v\in\mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$, via

$$X^\sharp(x,\omega):=(T_xX)(\omega),\qquad v^\flat(\omega)=v(0,\omega),$$

and we note that $(X^\sharp)^\flat=X$ and $(v^\flat)^\sharp=v$, thus yielding a canonical isomorphism

$$\mathrm{L}^q(\Omega)\;\cong\;\mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d\times\Omega). \tag{2.1}$$

For $1\le q<\infty$, as T is a C_0-group of isometries on $\mathrm{L}^q(\Omega)$, cf. Lemma 2.4(ii), we can define the *T-gradient* $\nabla^\flat$ as the generator of this group. It is a densely defined operator on $\mathrm{L}^q(\Omega)$, its domain is denoted by $W^{1,q}(\Omega)$, and it is skew-adjoint on $\mathrm{L}^2(\Omega)$. Alternatively, this operator $\nabla^\flat$ can be reinterpreted via the isomorphism (2.1):

$$W^{1,q}(\Omega)\cong W^{1,q}_{\mathrm{hom}}(\mathbb{R}^d\times\Omega),\qquad(\nabla^\flat X)^\sharp=\nabla X^\sharp\qquad\text{for all }X\in W^{1,q}(\Omega).$$

We also define the corresponding T-Laplacian $-\triangle^\flat:=-\nabla^\flat\cdot\nabla^\flat$ on $\mathrm{L}^q(\Omega)$, which is non-negative and essentially self-adjoint on $\mathrm{L}^2(\Omega)$. For all $s\ge0$, we denote by $W^{s,q}(\Omega)$ the Banach space of random variables $X\in\mathrm{L}^q(\Omega)$ such that $X^\sharp\in W^{s,q}_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$.

The above construction entails that spatially homogeneous solutions of (1.2) are equivalent to solutions of a corresponding abstract equation on the probability space. Note that expressions like $e^{\frac{t}{\varepsilon-i}\triangle}v$ in (2.2) below make sense almost surely for $v\in\mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$ since the kernel of $e^{\frac{t}{\varepsilon-i}\triangle}$ has Gaussian decay while realizations of v have subexponential

[2] That is, the σ-algebra generated by all sets of the form $\{\omega\in\Omega:u^\circ(x_1,\omega)\in A_1,\ldots,u^\circ(x_n,\omega)\in A_n\}$ with $n\ge1$, $x_1,\ldots,x_n\in\mathbb{R}^d$, and Borel subsets $A_1,\ldots A_n\subset\mathbb{C}$.

growth almost surely, cf. (3.13). In the periodic case, in view of Example 2.3(b), $\nabla^\flat$ is the periodic gradient and this result amounts to reducing (1.2) to the corresponding equation on the periodic cell $\Omega = [0,1)^d$.

Lemma 2.5. *Given $T>0$, the following two properties are equivalent:*

- *A random field $u_\varepsilon \in \mathrm{L}^\infty([0,T]; H^1_{\mathrm{hom}} \cap \mathrm{L}^4_{\mathrm{hom}}(\mathbb{R}^d\times\Omega))$ is a weak solution of (1.2) in the sense that Duhamel's formula holds almost everywhere,*
$$u^t_\varepsilon \,=\, e^{\frac{t}{\varepsilon-i}\triangle} u^\circ - \tfrac{1}{\varepsilon-i}\int_0^t e^{\frac{t-s}{\varepsilon-i}\triangle}(|u^s_\varepsilon|^2 u^s_\varepsilon)\,ds, \qquad 0\le t\le T. \tag{2.2}$$
- *We have $u^t_\varepsilon = (U^t_\varepsilon)^\sharp$ where $U_\varepsilon \in \mathrm{L}^\infty([0,T]; H^1\cap\mathrm{L}^4(\Omega))$ is a weak solution of the following abstract equation on the probability space,*
$$(-\varepsilon+i)\partial_t U_\varepsilon = -\triangle^\flat U_\varepsilon + |U_\varepsilon|^2 U_\varepsilon, \qquad U_\varepsilon|_{t=0} = (u^\circ)^\flat, \tag{2.3}$$
in the sense that Duhamel's formula holds almost everywhere,
$$U^t_\varepsilon \,=\, e^{\frac{t}{\varepsilon-i}\triangle^\flat}(u^\circ)^\flat - \tfrac{1}{\varepsilon-i}\int_0^t e^{\frac{t-s}{\varepsilon-i}\triangle^\flat}(|U^s_\varepsilon|^2 U^s_\varepsilon)\,ds, \qquad 0\le t\le T. \qquad \Diamond$$

2.3. **Lack of functional tools.** In contrast with the periodic case, the T-Laplacian $-\triangle^\flat$ on $\mathrm{L}^2(\Omega)$ typically has absolutely continuous spectrum and no spectral gap above 0. As shown in [4], the spectrum can actually be arbitrary and depends on the structure of the underlying probability space; we focus here for simplicity on the Gaussian setting.

Lemma 2.6 (Spectrum of T-Laplacian; [4])**.** *Assume that $u^\circ \in \mathrm{L}^2_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$ is Gaussian in the sense of Example 2.3(a) and that its covariance function c_0 has an absolutely continuous Fourier transform. Then, the spectrum of $-\triangle^\flat$ on $\mathrm{L}^2(\Omega)$ is $[0,\infty)$ and is made of a simple eigenvalue at 0 embedded in absolutely continuous spectrum.* $\Diamond$

In particular, this entails that Poincaré's inequality and compact Rellich embeddings do not hold on $H^1(\Omega)$. In addition, we show that Sobolev embeddings also fail and that the parabolic semigroup $\{e^{t\triangle^\flat}\}_{t\ge0}$ yields no improved integrability. Heuristically, this lack of functional tools is related to the fact that the T-gradient $\nabla^\flat$ only contains information on a finite set of directions, while Ω is typically an infinite product space. This constitutes a key difficulty for the analysis of nonlinear equations such as (1.2) in this setting.

Lemma 2.7 (Lack of functional tools)**.** *Assume that $u^\circ \in \mathrm{L}^2_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$ is Gaussian in the sense of Example 2.3(a) and that its covariance function c_0 is integrable. Then,*

(i) Poincaré's inequality: $\|X-\mathbb{E}[X]\|_{\mathrm{L}^2(\Omega)} \le C\|\nabla^\flat X\|_{\mathrm{L}^2(\Omega)}$ *does not hold on $H^1(\Omega)$.*

(ii) Compact Rellich embeddings: *The space $H^m(\Omega)$ is not compactly embedded in $\mathrm{L}^2(\Omega)$ for any $m>0$.*

(iii) Sobolev embeddings: $\|X\|_{\mathrm{L}^q(\Omega)} \le C\|X\|_{H^m(\Omega)}$ *does not hold on $H^m(\Omega)$ for any $m\ge0$ and $q>2$.*

(iv) Parabolic improvement of integrability: *Given $z\in\mathbb{C}$ with $\Re z\ge 0$, the inequality $\|e^{z\triangle^\flat}X\|_{\mathrm{L}^q(\Omega)} \le C\|X\|_{\mathrm{L}^2(\Omega)}$ does not hold on $\mathrm{L}^q(\Omega)$ for any $q>2$.* $\Diamond$

Proof. We start with items (i) and (ii). They can both be viewed as consequences of Lemma 2.6, but we rather provide a quick direct proof. Given a real-valued test function $\xi\in C^\infty_c(\mathbb{R}^d)$, consider the Gaussian random variables
$$X_n(\omega) \,:=\, n^{-\frac{d}{2}}\int_{\mathbb{R}^d} \xi(\tfrac{y}{n})\,u^\circ(y,\omega)\,dy, \qquad n\ge1.$$

A direct computation yields as $n\uparrow\infty$,

$$\|X_n\|^2_{\mathrm{L}^2(\Omega)}\to R:=\|\xi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\int_{\mathbb{R}^d}c_0,\qquad n^2\|\nabla^\flat X_n\|^2_{\mathrm{L}^2(\Omega)}\to\|\nabla\xi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\int_{\mathbb{R}^d}c_0,\tag{2.4}$$

which contradicts the validity of Poincaré's inequality, hence proves (i). Since the sequence $(X_n)_n$ is bounded in $H^m(\Omega)$ for any $m\ge 0$, and since it converges weakly but not strongly to 0 in $\mathrm{L}^2(\Omega)$, item (ii) follows.

We turn to the proof of item (iii). A simple application of Wick's theorem yields for all integers $r\ge1$,

$$\|X_n\|^{2r}_{\mathrm{L}^{2r}(\Omega)}=r!\|X_n\|^{2r}_{\mathrm{L}^2(\Omega)}.$$

Extending this to all $r\ge1$, and combining with (2.4), we find

$$\|X_n\|^{2r}_{\mathrm{L}^{2r}(\Omega)}\xrightarrow{n\uparrow\infty}R^r\Gamma(r+1).\tag{2.5}$$

Also note that (2.4) yields $\|\nabla^m X_n\|^{2r}_{\mathrm{L}^{2r}(\Omega)}\to0$ for all $r,m\ge1$. Hence, given $q\ge2$ and $m\ge0$, we deduce for all integers $p\ge1$,

$$\frac{\|X_n^p\|_{\mathrm{L}^q(\Omega)}}{\|X_n^p\|_{H^m(\Omega)}}\xrightarrow{n\uparrow\infty}\frac{\Gamma(qp+1)^{1/q}}{\Gamma(2p+1)^{1/2}}.$$

For $q>2$, the right-hand side blows up exponentially as $p\uparrow\infty$, thus contradicting the validity of Sobolev's inequality.

It remains to prove item (iv) and we start with the easy case when $\Re z=0$. The essential self-adjointness of the T-Laplacian $-\triangle^\flat$ on $\mathrm{L}^2(\Omega)$ entails that $e^{-z\triangle^\flat}$ is unitary on $\mathrm{L}^2(\Omega)$ in that case, hence the inequality $\|e^{z\triangle^\flat}X\|_{\mathrm{L}^q(\Omega)}\le C\|X\|_{\mathrm{L}^2(\Omega)}$ for all $X\in\mathrm{L}^2(\Omega)$ would in fact imply $\|X\|_{\mathrm{L}^q(\Omega)}\le C\|X\|_{\mathrm{L}^2(\Omega)}$ for all $X\in\mathrm{L}^2(\Omega)$, a contradiction.

We turn to the proof of (iv) for $\Re z>0$. As the operator $e^{z\triangle}$ has kernel $K_z(x):=(4\pi z)^{-\frac d2}e^{-\frac1{4z}|x|^2}$, and as realizations of the Gaussian random field $X_n^\sharp$ have subexponential growth almost surely, we can write in view of (2.1),

$$e^{z\triangle^\flat}X_n^p=\int_{\mathbb{R}^d}K_z(x)\,X_n^\sharp(x,\cdot)^p\,dx.$$

Appealing to Wick's theorem as above, using (2.4) in form of

$$\mathbb{E}\left[X_n^\sharp(x,\cdot)\overline{X_n^\sharp(y,\cdot)}\right]\xrightarrow{n\uparrow\infty}R,$$

and noting that $\int_{\mathbb{R}^d}K_z=1$, we find for all $r\ge1$ and all integers $p\ge1$,

$$\|e^{z\triangle^\flat}X_n^p\|^{2r}_{\mathrm{L}^{2r}(\Omega)}\xrightarrow{n\uparrow\infty}R^{pr}\Gamma(pr+1).$$

Hence, combined with (2.5), given $q\ge2$, we deduce for all integers $p\ge1$,

$$\frac{\|e^{z\triangle^\flat}X_n^p\|_{\mathrm{L}^q(\Omega)}}{\|X_n^p\|_{\mathrm{L}^2(\Omega)}}\xrightarrow{n\uparrow\infty}\frac{\Gamma(\frac12qp+1)^{1/q}}{\Gamma(p+1)^{1/2}}.$$

For $q>2$, the right-hand side blows up exponentially as $p\uparrow\infty$, and the conclusion follows. □

3. Well-posedness for (1.2) with random data

We turn to the proof of Theorem 1, where we recall $\mathbb{X} = H^1_{\mathrm{hom}} \cap \mathrm{L}^4_{\mathrm{hom}}(\mathbb{R}^d \times \Omega)$ with the above notation. In order to overcome the lack of functional tools to study the nonlinear equation (1.2) in this setting, cf. Lemma 2.7, we rather focus on almost sure realizations in local Sobolev spaces, for which standard tools are available. Dissipation is crucial to compensate for the lack of finite propagation speed and allows to prove well-posedness in local Sobolev spaces with polynomial growth, which is then post-processed into a well-posedness result in the spatially homogeneous random setting.

Proof of Theorem 1. We start by introducing local Sobolev spaces with polynomial growth, which are natural spaces to control almost sure realizations of random fields. Given $\ell \geq 0$, we consider the uniformly localized L^q and H^1 norms with ℓ-growth,

$$\begin{aligned}\|v\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)} &:= \sup_{x_0\in\mathbb{R}^d} \Big(\langle x_0\rangle^{-q\ell} \fint_{B(x_0)} |v|^q \Big)^{\frac1q},\\ \|v\|_{H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)} &:= \sup_{x_0\in\mathbb{R}^d} \Big(\langle x_0\rangle^{-2\ell} \fint_{B(x_0)} (|v|^2+|\nabla v|^2) \Big)^{\frac12},\end{aligned}$$

and we denote by $\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)$ and $H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$ the corresponding subspaces of $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$. We also consider the uniformly localized energy functional with ℓ-growth,

$$\mathcal{E}_{\mathrm{uloc},\ell}(v) := \sup_{x_0\in\mathbb{R}^d} \Big(\langle x_0\rangle^{-2\ell} \fint_{B(x_0)} \big(|\nabla v|^2 + \tfrac12 |v|^4\big)\Big).$$

As shown below, cf. (3.13), given $\ell > \frac{d}{q}$, realizations of a random field $v \in \mathrm{L}^q_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$ belong almost surely to $\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)$. With this in mind, we start by studying the nonlinear Schrödinger equation (1.2) in $H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$, and next we exploit uniqueness to construct a unique dynamics in $H^1_{\mathrm{hom}} \cap \mathrm{L}^4_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$. Note that $H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$ embeds in $\mathrm{L}^4_{\mathrm{uloc},\ell}(\mathbb{R}^d)$ by the Sobolev embedding (with $d \leq 4$), while on the contrary $H^1_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$ does in general not embed in $\mathrm{L}^4_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$, cf. Lemma 2.7(iii). The proof is split into two main steps.

Step 1. Global well-posedness in $H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$.
For $d < 4$, given $\varepsilon > 0$ and $\ell \geq 0$, we show that for all $v^\circ \in H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$ the equation

$$(-\varepsilon + i)\partial_t v_\varepsilon = -\triangle v_\varepsilon + |v_\varepsilon|^2 v_\varepsilon, \qquad v_\varepsilon|_{t=0} = v^\circ$$

admits a unique global weak (Duhamel) solution v_ε in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d))$. We split the proof into four further substeps.

Substep 1.1. Equivalent definition of $\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)$.
In terms of the exponential cut-off $\chi(x) := e^{-|x|}$, setting for abbreviation $\chi_{x_0} := \chi(\cdot - x_0)$, we show that for all q, ℓ we have

$$\|v\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)} \simeq_{q,\ell} \sup_{x_0\in\mathbb{R}^d} \Big(\langle x_0\rangle^{-q\ell} \int_{\mathbb{R}^d} \chi_{x_0} |v|^q \Big)^{\frac1q}. \tag{3.1}$$

Indeed, as $\chi_{x_0} \gtrsim 1$ on $B(x_0)$, the left-hand side in (3.1) is clearly bounded above by the right-hand side. The converse inequality follows from the following,

$$\begin{aligned}\Big(\langle x_0\rangle^{-q\ell}\int_{\mathbb{R}^d}\chi_{x_0}|v|^q\Big)^{\frac1q} &\le \Big(\langle x_0\rangle^{-q\ell}\sum_{z\in\mathbb{Z}^d}\int_{B(x_0+z)}\chi_{x_0}|v|^q\Big)^{\frac1q}\\ &\lesssim \Big(\langle x_0\rangle^{-q\ell}\sum_{z\in\mathbb{Z}^d}e^{-|z|}\langle x_0+z\rangle^{q\ell}\Big)^{\frac1q}\|v\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)}\\ &\lesssim \|v\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)}.\end{aligned}$$

Substep 1.2. Localized parabolic estimates.
Given $\varepsilon>0$, $\ell\ge0$, and $1\le p\le q\le\infty$ with $\frac d2(\frac1p-\frac1q)<1$, we show for all $g\in\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)$ and $t\ge0$,

$$\|e^{\frac{t}{\varepsilon-i}\triangle}g\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)} \lesssim_{p,q,\ell,\varepsilon} t^{-\frac d2(\frac1p-\frac1q)}e^{C_\varepsilon t}\|g\|_{\mathrm{L}^p_{\mathrm{uloc},\ell}(\mathbb{R}^d)}, \tag{3.2}$$

and in addition, provided $\frac d2(\frac1p-\frac1q)<\frac12$,

$$\|\nabla e^{\frac{t}{\varepsilon-i}\triangle}g\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)} \lesssim_{p,q,\ell,\varepsilon} t^{-\frac d2(\frac1p-\frac1q)-\frac12}e^{C_\varepsilon t}\|g\|_{\mathrm{L}^p_{\mathrm{uloc},\ell}(\mathbb{R}^d)}. \tag{3.3}$$

Multiplying the parabolic evolution $\{e^{\frac{t}{\varepsilon-i}\triangle}g\}_{t\ge0}$ with the exponential cut-off χ_{x_0}, and using Duhamel's formula, we easily find

$$\begin{aligned}\chi_{x_0}(e^{\frac{t}{\varepsilon-i}\triangle}g) &= e^{\frac{t}{\varepsilon-i}\triangle}(\chi_{x_0}g)\\ &+\tfrac{1}{\varepsilon-i}\int_0^t e^{\frac{t-s}{\varepsilon-i}\triangle}\big((\triangle\chi_{x_0})(e^{\frac{s}{\varepsilon-i}\triangle}g)\big)\,ds-\tfrac{2}{\varepsilon-i}\int_0^t\nabla\cdot e^{\frac{t-s}{\varepsilon-i}\triangle}\big((\nabla\chi_{x_0})(e^{\frac{s}{\varepsilon-i}\triangle}g)\big)\,ds.\end{aligned} \tag{3.4}$$

Note that the parabolic semigroup $\{e^{\frac{t}{\varepsilon-i}\triangle}\}_{t\ge0}$ has kernel $K^t_\varepsilon(x):=(\frac{\varepsilon-i}{4\pi t})^{\frac d2}e^{-\frac{\varepsilon-i}{4t}|x|^2}$, which implies by Young's convolution inequality, for all $g\in C^\infty_c(\mathbb{R}^d)$ and $1\le p\le q\le\infty$, letting r be such that $\frac1p+\frac1r=\frac1q+1$,

$$\|e^{\frac{t}{\varepsilon-i}\triangle}g\|_{\mathrm{L}^q(\mathbb{R}^d)} = \|K^t_\varepsilon*g\|_{\mathrm{L}^q(\mathbb{R}^d)} \le \|K^t_\varepsilon\|_{\mathrm{L}^r(\mathbb{R}^d)}\|g\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_{p,q,\varepsilon} t^{-\frac d2(\frac1p-\frac1q)}\|g\|_{\mathrm{L}^p(\mathbb{R}^d)}, \tag{3.5}$$

and similarly,

$$\|\nabla e^{\frac{t}{\varepsilon-i}\triangle}g\|_{\mathrm{L}^q(\mathbb{R}^d)} \lesssim_{p,q,\varepsilon} t^{-\frac d2(\frac1p-\frac1q)-\frac12}\|g\|_{\mathrm{L}^p(\mathbb{R}^d)}. \tag{3.6}$$

For $1\le p\le q\le\infty$, taking the L^q norm in (3.4) and using these parabolic estimates, we obtain

$$\begin{aligned}\|\chi_{x_0}(e^{\frac{t}{\varepsilon-i}\triangle}g)\|_{\mathrm{L}^q(\mathbb{R}^d)} &\le C_{p,q,\varepsilon}\,t^{-\frac d2(\frac1p-\frac1q)}\|\chi_{x_0}g\|_{\mathrm{L}^p(\mathbb{R}^d)}\\ &\qquad+C_\varepsilon\int_0^t\big(1+(t-s)^{-\frac12}\big)\|\chi_{x_0}(e^{\frac{s}{\varepsilon-i}\triangle}g)\|_{\mathrm{L}^q(\mathbb{R}^d)}\,ds,\end{aligned}$$

and the claim (3.2) easily follows from Grönwall's inequality together with (3.1). Using (3.6) instead of (3.5), the claim (3.3) is obtained in a similar way.

Substep 1.3. Local well-posedness in $\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)$.
Given $\varepsilon>0$, $\ell\ge0$, and $3\le q\le\infty$ with $q>d$, we show that for all $v^\circ\in\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)$ there exists $T>0$ such that the equation

$$(-\varepsilon+i)\partial_t v_\varepsilon = -\triangle v_\varepsilon+|v_\varepsilon|^2v_\varepsilon, \qquad v_\varepsilon|_{t=0}=v^\circ$$

admits a unique weak (Duhamel) solution v_ε in $\mathrm{L}^\infty([0,T];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))$. In case $q>\frac{6d}{d+2}$, we further have $v_\varepsilon\in\mathrm{L}^\infty([0,T];H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d))$ provided $v^\circ\in H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$.

To prove well-posedness in $\mathrm{L}^\infty([0,T];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))$, we argue by a Picard fixed-point argument: for $T>0$ we define an operator $\Phi_{T,\varepsilon}(\cdot;v^\circ)$ on $\mathrm{L}^\infty([0,T];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))$ by

$$(\Phi_{T,\varepsilon}(v;v^\circ))^t\;:=\;e^{\frac{t}{\varepsilon-i}\triangle}v^\circ-\tfrac{1}{\varepsilon-i}\int_0^t e^{\frac{t-s}{\varepsilon-i}\triangle}(|v^s|^2v^s)\,ds,\qquad 0\le t\le T,\tag{3.7}$$

and it suffices to show that for all $v,w\in\mathrm{L}^\infty([0,T];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))$,

$$\begin{aligned}&\|\Phi_{T,\varepsilon}(v;v^\circ)\|_{\mathrm{L}^\infty([0,T];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))}\\&\qquad\lesssim_{q,\ell,\varepsilon}\;e^{C_\varepsilon T}\Big(\|v^\circ\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)}+\|v\|^3_{\mathrm{L}^\infty([0,T];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))}\Big),\end{aligned}\tag{3.8}$$

$$\begin{aligned}&\|\Phi_{T,\varepsilon}(v;v^\circ)-\Phi_{T,\varepsilon}(w;v^\circ)\|_{\mathrm{L}^\infty([0,T];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))}\\&\qquad\lesssim_{q,\ell,\varepsilon}\;T^{1-\frac{d}{q}}e^{C_\varepsilon T}\|(v,w)\|^2_{\mathrm{L}^\infty([0,T];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))}\|v-w\|_{\mathrm{L}^\infty([0,T];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))}.\end{aligned}\tag{3.9}$$

It remains to prove these two estimates. Applying the localized parabolic estimate (3.2) with exponents $\frac{q}{3}\le q$, we find for all $3\le q\le\infty$ with $q>d$,

$$\begin{aligned}\|(\Phi_{T,\varepsilon}(v;v^\circ))^t\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)}\;&\lesssim_{q,\ell,\varepsilon}\;e^{C_\varepsilon t}\Big(\|v^\circ\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)}+\int_0^t(t-s)^{-\frac{d}{q}}\big\||v^s|^2v^s\big\|_{\mathrm{L}^{q/3}_{\mathrm{uloc},\ell}(\mathbb{R}^d)}\,ds\Big)\\&\lesssim_{q,\ell,\varepsilon}\;e^{C_\varepsilon t}\Big(\|v^\circ\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)}+t^{1-\frac{d}{q}}\|v\|^3_{\mathrm{L}^\infty([0,t];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))}\Big),\end{aligned}$$

which proves (3.8). Similarly, we find

$$\begin{aligned}&\|(\Phi_{T,\varepsilon}(v;v^\circ))^t-(\Phi_{T,\varepsilon}(w;v^\circ))^t\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)}\\&\qquad\qquad\lesssim_{q,\ell,\varepsilon}\;e^{C_\varepsilon t}\int_0^t(t-s)^{-\frac{d}{q}}\big\||v^s|^2v^s-|w^s|^2w^s\big\|_{\mathrm{L}^{q/3}_{\mathrm{uloc},\ell}(\mathbb{R}^d)}ds,\end{aligned}$$

and thus, further using Hölder's inequality in form of

$$\big\||v^s|^2v^s-|w^s|^2w^s\big\|_{\mathrm{L}^{q/3}_{\mathrm{uloc},\ell}(\mathbb{R}^d)}\;\lesssim\;\|(v^s,w^s)\|^2_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)}\|v^s-w^s\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)},$$

we deduce

$$\begin{aligned}&\|(\Phi_{T,\varepsilon}(v;v^\circ))^t-(\Phi_{T,\varepsilon}(w;v^\circ))^t\|_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)}\\&\qquad\qquad\lesssim_{q,\ell,\varepsilon}\;t^{1-\frac{d}{q}}e^{C_\varepsilon t}\|(v,w)\|^2_{\mathrm{L}^\infty([0,t];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))}\|v-w\|_{\mathrm{L}^\infty([0,t];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))},\end{aligned}$$

which proves (3.9).

It remains to show that for $q>\frac{6d}{d+2}$ this local weak (Duhamel) solution $v_\varepsilon=\Phi_{T,\varepsilon}(v_\varepsilon;v^\circ)$ in $\mathrm{L}^\infty([0,T];\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))$ also belongs to $\mathrm{L}^\infty([0,T];H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d))$ provided $v^\circ\in H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$. Without loss of generality we choose $q\le 6$. As the condition on q ensures $\frac{d}{2}(\frac{3}{q}-\frac12)<\frac12$, we appeal to the localized parabolic estimate (3.3) with exponents $\frac{q}{3}\le 2$, to the effect of

$$\begin{aligned}\|\nabla v^t_\varepsilon\|_{\mathrm{L}^2_{\mathrm{uloc},\ell}(\mathbb{R}^d)}\;&\lesssim_{q,\ell,\varepsilon}\;e^{C_\varepsilon t}\Big(\|\nabla v^\circ\|_{\mathrm{L}^2_{\mathrm{uloc},\ell}(\mathbb{R}^d)}+\int_0^t(t-s)^{-\frac{d}{2}(\frac{3}{q}-\frac12)-\frac12}\|v^s\|^3_{\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d)}ds\Big)\\&\lesssim_{q,\ell,\varepsilon}\;e^{C_\varepsilon t}\Big(\|\nabla v^\circ\|_{\mathrm{L}^2_{\mathrm{uloc},\ell}(\mathbb{R}^d)}+\|v\|^3_{\mathrm{L}^\infty(\mathbb{R}^+;\mathrm{L}^q_{\mathrm{uloc},\ell}(\mathbb{R}^d))}\Big),\end{aligned}$$

hence $v_\varepsilon^t \in \mathrm{L}^\infty([0,T];H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d))$.

Substep 1.4. Conclusion: global well-posedness in $H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$.
We argue that it suffices to prove the following localized energy estimate: for all $T>0$ and $v^\circ \in H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$, if $v_\varepsilon \in \mathrm{L}^\infty([0,T];H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d))$ is a weak (Duhamel) solution of

$$(-\varepsilon+i)\partial_t v_\varepsilon = -\triangle v_\varepsilon + |v_\varepsilon|^2 v_\varepsilon, \qquad v_\varepsilon|_{t=0} = v^\circ, \tag{3.10}$$

then we have for all $0 \le t \le T$,

$$\mathcal{E}_{\mathrm{uloc},\ell}(v_\varepsilon^t) \lesssim_\ell e^{\frac{1}{2\varepsilon}t}\,\mathcal{E}_{\mathrm{uloc},\ell}(v^\circ). \tag{3.11}$$

Since by definition

$$\mathcal{E}_{\mathrm{uloc},\ell}(g) \simeq \|g\|^2_{H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)} + \|g\|^4_{\mathrm{L}^4_{\mathrm{uloc},\ell/2}(\mathbb{R}^d)},$$

we can naturally combine this energy estimate (3.11) together with the local well-posedness result of Step 3 with $q=4$. As the restriction $q > \frac{6d}{d+2}$ reduces in that case to $d<4$, and as $H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$ embeds in $\mathrm{L}^4_{\mathrm{uloc},\ell}(\mathbb{R}^d)$ by the Sobolev embedding for $d<4$, this leads us to the claimed global well-posedness result.

It remains to prove the energy estimate (3.11). If the solution v_ε of (3.10) was smooth, then, using the standard notation $\langle a,b\rangle := \Re(\bar a b)$ for the scalar product in $\mathbb{C}$, we could compute by equation (3.10) and Young's inequality,

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^d} \chi_{x_0}\big(|\nabla v_\varepsilon|^2 + \tfrac12 |v_\varepsilon|^4\big) &= 2\int_{\mathbb{R}^d} \chi_{x_0}\big(\langle \nabla v_\varepsilon, \nabla \partial_t v_\varepsilon\rangle + \langle |v_\varepsilon|^2 v_\varepsilon, \partial_t v_\varepsilon\rangle\big)\\
&= 2\int_{\mathbb{R}^d} \chi_{x_0}\langle -\triangle v_\varepsilon + |v_\varepsilon|^2 v_\varepsilon, \partial_t v_\varepsilon\rangle - 2\int_{\mathbb{R}^d} \nabla\chi_{x_0}\cdot\langle \nabla v_\varepsilon, \partial_t v_\varepsilon\rangle\\
&= -2\varepsilon \int_{\mathbb{R}^d} \chi_{x_0}|\partial_t v_\varepsilon|^2 - 2\int_{\mathbb{R}^d} \nabla\chi_{x_0}\cdot\langle \nabla v_\varepsilon, \partial_t v_\varepsilon\rangle\\
&\le \tfrac{1}{2\varepsilon}\int_{\mathbb{R}^d} \chi_{x_0}|\nabla v_\varepsilon|^2,
\end{aligned}$$

hence, by Grönwall's inequality,

$$\int_{\mathbb{R}^d} \chi_{x_0}\big(|\nabla v_\varepsilon^t|^2 + \tfrac12 |v_\varepsilon^t|^4\big) \le e^{\frac{1}{2\varepsilon}t}\int_{\mathbb{R}^d} \chi_{x_0}\big(|\nabla v^\circ|^2 + \tfrac12 |v^\circ|^4\big). \tag{3.12}$$

This estimate can be justified in our non-smooth setting by an approximation procedure as e.g. in [5, 6], and the claim (3.11) then follows from (3.1).

Step 2. Global well-posedness in $H^1_{\mathrm{hom}} \cap \mathrm{L}^4_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$.
Let $d<4$. Given $u^\circ \in H^1_{\mathrm{hom}} \cap \mathrm{L}^4_{\mathrm{hom}}(\mathbb{R}^d\times\Omega)$, we prove the existence of a unique almost sure global weak (Duhamel) solution u_ε in $\mathrm{L}^\infty(\mathbb{R}^+;H^1_{\mathrm{hom}}\cap\mathrm{L}^4_{\mathrm{hom}}(\mathbb{R}^d\times\Omega))$ of equation (1.2). We split the proof into four further substeps.

Substep 2.1. Existence and uniqueness for realizations.
Let $\ell > \frac{d}{2}$ be fixed. The localized energy of a realization $u^\circ(\cdot,\omega)$ is trivially bounded by

$$\mathcal{E}_{\mathrm{uloc},\ell}(u^\circ(\cdot,\omega)) \lesssim M_{u^\circ}(\omega) := \sum_{z\in\mathbb{Z}^d}\langle z\rangle^{-2\ell} \fint_{B(z)} \big(|\nabla u^\circ(\cdot,\omega)|^2 + \tfrac12 |u^\circ(\cdot,\omega)|^4\big).$$

As u° belongs to $H^1_{\mathrm{hom}} \cap \mathrm{L}^4_{\mathrm{hom}}(\mathbb{R}^d \times \Omega)$, the choice $\ell > \frac{d}{2}$ ensures $\mathbb{E}[M_{u^\circ}] < \infty$. This implies that there is a subset $\Omega_0 \subset \Omega$ with maximal probability such that

$$\mathcal{E}_{\mathrm{uloc},\ell}(u^\circ(\cdot,\omega)) \lesssim M_{u^\circ}(\omega) < \infty \qquad \text{for all } \omega \in \Omega_0, \tag{3.13}$$

hence $u^\circ(\cdot,\omega) \in H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$. Therefore, in view of Step 1, for all $\omega \in \Omega_0$, there exists a unique weak (Duhamel) solution $u_\varepsilon(\cdot,\omega) \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d))$ of

$$(-\varepsilon + i)\partial_t u_\varepsilon(\cdot,\omega) = -\triangle u_\varepsilon(\cdot,\omega) + |u_\varepsilon(\cdot,\omega)|^2 u_\varepsilon(\cdot,\omega), \qquad u_\varepsilon(\cdot,\omega)|_{t=0} = u^\circ(\cdot,\omega), \tag{3.14}$$

and it satisfies the following energy estimate for all $t \geq 0$,

$$\mathcal{E}_{\mathrm{uloc},\ell}(u^t_\varepsilon(\cdot,\omega)) \lesssim_\ell e^{\frac{1}{2\varepsilon}t}\mathcal{E}_{\mathrm{uloc},\ell}(u^\circ(\cdot,\omega)) \lesssim e^{\frac{1}{2\varepsilon}t} M_{u^\circ}(\omega). \tag{3.15}$$

Substep 2.2. Measurability.
For all T we show that the above-defined map $\Omega_0 \to \mathrm{L}^\infty([0,T]; H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)) : \omega \mapsto u_\varepsilon(\cdot,\omega)$ is Bochner measurable.

On the one hand, by Fubini's theorem, the joint measurability of u° on $\Omega \times \mathbb{R}^d$ ensures that the map $\Omega_0 \to H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d) : \omega \mapsto u^\circ(\cdot,\omega)$ is weakly measurable, hence also Bochner measurable by Pettis' theorem [1, Lemma 11.37] as $H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)$ is a separable Banach space. On the other hand, arguing again as in Substep 1.3, we note that the solution operator $H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d) \to \mathrm{L}^\infty([0,T]; H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)) : u^\circ(\cdot,\omega) \mapsto u_\varepsilon(\cdot,\omega)$ is locally Lipschitz continuous. The Bochner measurability of u_ε follows by composition.

Substep 2.3. Spatial homogeneity.
For all T, we show that u_ε belongs to $\mathrm{L}^\infty([0,T]; H^1_{\mathrm{hom}} \cap \mathrm{L}^4_{\mathrm{hom}}(\mathbb{R}^d \times \Omega))$ up to modification on null sets.

For all $\omega \in \Omega_0$ and $x \in \mathbb{R}^d$, since by definition $(T_x u^\circ)(\cdot,\omega) = u^\circ(\cdot + x,\omega)$, cf. Section 2.2, the uniqueness of a weak (Duhamel) solution entails for almost all t,y,

$$(T_x u^t_\varepsilon)(y,\omega) = u^t_\varepsilon(y+x,\omega). \tag{3.16}$$

In other words, u_ε satisfies an "almost everywhere" version of spatial homogeneity, and it remains to modify it on null sets to make it spatially homogeneous in the sense of Definition 2.1. By the measurability statement of Substep 2.2 and by the bound (3.15) with $M_{u^\circ} \in \mathrm{L}^1(\Omega)$, we have $u_\varepsilon \in \mathrm{L}^2(\Omega_0; \mathrm{L}^\infty([0,T]; H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)))$. Up to modification on null sets, we deduce $u_\varepsilon \in \mathrm{L}^\infty([0,T]; \mathrm{L}^2(\Omega; H^1_{\mathrm{uloc},\ell}(\mathbb{R}^d)))$, and we then define for all $\delta > 0$,

$$U^t_{\varepsilon,\delta}(\omega) := \fint_{B_\delta} u^t_\varepsilon(\cdot,\omega).$$

By definition and by (3.15), the family $(U_{\varepsilon,\delta})_{\delta>0}$ is bounded in $\mathrm{L}^\infty([0,T]; H^1 \cap \mathrm{L}^4(\Omega))$. Up to an extraction, $U_{\varepsilon,\delta}$ converges weakly to some U_ε in that space as $\delta \downarrow 0$. For all x, this implies that $T_x U_{\varepsilon,\delta}$ converges weakly to $T_x U_\varepsilon$. Now by (3.16), we find $T_x U^t_{\varepsilon,\delta}(\omega) = \fint_{B_\delta(x)} u^t_\varepsilon(\cdot,\omega)$ for almost all t,x,ω. Passing to the limit yields $T_x U^t_\varepsilon(\omega) = u^t_\varepsilon(x,\omega)$ for almost all t,x,ω, and the claim follows.

Substep 2.4. Conclusion.
It remains to check the dissipation estimates. We start with the dissipation of the mass.

Let $\chi_R(x) := \chi(\frac{1}{R}x)$. If the solution u_ε was smooth, equation (1.2) would allow to compute, almost surely,

$$\begin{aligned}\partial_t \int_{\mathbb{R}^d} \chi_R |u_\varepsilon(\cdot,\omega)|^2 &= 2\int_{\mathbb{R}^d} \chi_R \langle u_\varepsilon(\cdot,\omega), \partial_t u_\varepsilon(\cdot,\omega)\rangle \\ &= -2\int_{\mathbb{R}^d} \chi_R \big\langle u_\varepsilon(\cdot,\omega), \tfrac{1}{\varepsilon-i}\big(-\triangle u_\varepsilon(\cdot,\omega) + |u_\varepsilon(\cdot,\omega)|^2 u_\varepsilon(\cdot,\omega)\big)\big\rangle \\ &= -\tfrac{2\varepsilon}{\varepsilon^2+1}\int_{\mathbb{R}^d} \chi_R\big(|\nabla u_\varepsilon(\cdot,\omega)|^2 + |u_\varepsilon(\cdot,\omega)|^4\big) - 2\int_{\mathbb{R}^d} \nabla\chi_R \cdot \big\langle u_\varepsilon(\cdot,\omega), \tfrac{1}{\varepsilon-i}\nabla u_\varepsilon(\cdot,\omega)\big\rangle,\end{aligned}$$

hence, by integration, with $|\nabla\chi_R| \le \frac{1}{R}\chi_R$,

$$\begin{aligned}\bigg|\int_{\mathbb{R}^d} \chi_R |u_\varepsilon^t(\cdot,\omega)|^2 - \int_{\mathbb{R}^d} \chi_R |u^\circ(\cdot,\omega)|^2 + \tfrac{2\varepsilon}{\varepsilon^2+1}\int_0^t \int_{\mathbb{R}^d} \chi_R\big(|\nabla u_\varepsilon(\cdot,\omega)|^2 + |u_\varepsilon(\cdot,\omega)|^4\big)\bigg| \\ \le \tfrac{1}{R}\int_0^t \int_{\mathbb{R}^d} \chi_R\big(|u_\varepsilon(\cdot,\omega)|^2 + |\nabla u_\varepsilon(\cdot,\omega)|^2\big).\end{aligned}$$

Up to convolving u_ε with a smooth kernel and passing to the limit, this estimate is easily justified in our non-smooth setting. Now taking the expectation, using the spatial homogeneity, and letting $R \uparrow \infty$, the claimed mass dissipation identity follows.

It remains to prove the corresponding energy dissipation estimate. Repeating the argument for (3.12), but replacing the exponential cut-off χ_{x_0} by χ_R, we get

$$\int_{\mathbb{R}^d} \chi_R\big(|\nabla u_\varepsilon^t(\cdot,\omega)|^2 + \tfrac12 |u_\varepsilon^t(\cdot,\omega)|^4\big) \le e^{\frac{1}{2\varepsilon R^2}t}\int_{\mathbb{R}^d} \chi_R\big(|\nabla u^\circ(\cdot,\omega)|^2 + \tfrac12 |u^\circ(\cdot,\omega)|^4\big).$$

Now taking the expectation, using the spatial homogeneity, and letting $R \uparrow \infty$, the claimed energy dissipation estimate follows. □

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