SEMI-DILUTE RHEOLOGY OF PARTICLE SUSPENSIONS: DERIVATION OF DOI-TYPE MODELS

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Abstract. This work is devoted to the large-scale rheology of suspensions of non-Brownian inertialess rigid particles, possibly self-propelling, suspended in Stokes flow. Starting from a hydrodynamic model, we derive a semi-dilute mean-field description in form of a Doi-type model, which is given by a 'macroscopic' effective Stokes equation coupled with a 'microscopic' Vlasov equation for the statistical distribution of particle positions and orientations. This describes non-Newtonian effects as the viscosity in the effective Stokes equation depends on the local distribution of orientations via Einstein's formula. The main difficulty is the detailed analysis of multibody hydrodynamic interactions between the particles, which we perform by means of a cluster expansion combined with a multipole expansion in a suitable dilute regime.

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1. Introduction and main results

1.1. **General overview.** Suspensions of inertialess rigid particles in a Stokes flow are omnipresent both in natural phenomena and in practical applications, and they display reputedly complex rheological behaviors on large scales, including non-Newtonian effects, e.g. [9, 37]. Such behaviors are easily understood heuristically: First, we expect homogenization to hold on large scales, leading to a notion of effective viscosity for the suspension. This effective viscosity naturally depends on the local spatial arrangement of the particles on the microscale, which evolves itself with the fluid flow and can thus adapt in time to external forces. This leads to flow-induced microstructure, which can result in a nonlinear response to external forces, hence non-Newtonian effects on large scales. The complete understanding of such behaviors from a micro-macro perspective would require to couple homogenization with microstructure dynamics, which remains a completely open problem. A natural simplification is to focus on the dilute regime: in that case, particles are sparse and interact little, hence only reduced information on the microstructure should matter, for which a dynamical description might be simpler.

Previous work on particle suspensions have mainly been devoted to the following two preliminary questions:

— *Homogenization for a 'given' microstructure:*
Given instantaneous particle positions, the Stokes problem defining the fluid velocity can be approximated on large scales by an effective Stokes equation with some effective viscosity, say $\bar{\boldsymbol{B}}$. This is now well-understood in the framework of homogenization theory [14, 11] and we refer to [15] for a review.

— *Semi-dilute expansion of the effective viscosity:*
In the dilute regime, the effective viscosity $\bar{\boldsymbol{B}}$ can be expanded with respect to the particle volume fraction $\lambda \ll 1$. To first order, this expansion takes form of the celebrated Einstein formula

$$\bar{\boldsymbol{B}} = \mathrm{Id} + \lambda \bar{\boldsymbol{B}}_1 + O(\lambda^2), \tag{1.1}$$

where $\bar{\boldsymbol{B}}_1$ only depends on the single-particle distribution of shapes and orientations. The next-order correction further involves the statistical distribution of pairs of particles on the microscale. The expansion can be pursued to higher orders in form of a cluster expansion and is now well understood; see [12, 15] and references therein.

With these results at hand, it remains to couple homogenization with microstructure dynamics in the semi-dilute expansion. As Einstein's approximation (1.1) only involves the single-particle distribution, we can expect a mean-field description of the dynamics with accuracy $O(\lambda^2)$: it would take form of an effective Stokes equation with viscosity given by Einstein's approximation (1.1), coupled to a Vlasov equation for the single-particle distribution of positions and orientations; see indeed (1.15)–(1.16) below. This accounts for non-Newtonian effects due to the collective orientation of the particles and corresponds to the so-called Doi model first derived formally in [31, 24, 3].

The first rigorous results on the dilute dynamics [30, 25, 32] focused on the leading-order description, thus neglecting Einstein's $O(\lambda)$ correction to the effective viscosity (1.1): this leads to a simpler transport-Stokes system devoid of any non-Newtonian effect. In [29], Höfer and Schubert went one step further and managed to capture Einstein's correction in the effective Stokes equation, but their analysis was restricted to the case of spherical particles: orientations then play no role and no non-Newtonian effect is obtained. In the present work, we consider non-spherical particles, we manage to describe the mean-field distribution of their orientations, and we derive a Doi-type model, rigorously describing non-Newtonian effects for the first time in a micro-macro limit. We further extend our analysis to the case of active suspensions, including the effects of particle self-propulsion: this leads to an additional elastic stress in the effective Stokes equation as was indeed predicted in [38, 20, 36, 35, 6] and first derived in [18, 2] in the equilibrium setting.

Note that no mean-field description can hold beyond accuracy $O(\lambda^2)$ since higher-order corrections to the effective viscosity (1.1) would involve statistical information on the arrangement of the particles on the microscale, such as the microscopic two-particle distribution. Such information is beyond the scope of propagation of chaos and mean-field theory, and its dynamical description is left as an open problem; see Remark 4.2 for more detail.

1.2. **Hydrodynamic model for particle suspension.** We consider a system of N non-Brownian inertialess rigid particles, denoted by $I^n_{\varepsilon,N}$ for $1 \le n \le N$, of typical size $O(\varepsilon)$, possibly self-propelling, suspended in a Stokes flow. We start by introducing the precise model that we are going to study: we describe the set of rigid particles, then turn to their possible self-propulsion, before describing the underlying viscous solvent and the particle dynamics. We assume that the space dimension is $d > 2$ (the case $d = 2$ can be treated similarly, up to obvious modifications due to the logarithmic growth of the Stokeslet in the whole plane).

• *Elongated rigid particles.* Let $I^\circ \subset B$ be an axisymmetric connected closed set, which we take to be centered at $\int_{I^\circ} x\,dx = 0$ and to be of class C^2. We then consider N particles that are disjoint rigid copies $\{I^n_{\varepsilon,N}\}_{1\le n\le N}$ of the rescaled set εI°. More precisely, each particle $I^n_{\varepsilon,N}$ is characterized by its center $X^n_{\varepsilon,N} \in \mathbb{R}^d$ and by the direction $R^n_{\varepsilon,N} \in \mathbb{S}^{d-1}$ of its axis, in the sense of

$$I^n_{\varepsilon,N} := I_\varepsilon(X^n_{\varepsilon,N}, R^n_{\varepsilon,N}) := X^n_{\varepsilon,N} + \varepsilon I^\circ(R^n_{\varepsilon,N}),$$

where we have set $I^\circ(r) := \Theta(r) I^\circ$, where for a direction $r \in \mathbb{S}^{d-1}$ we denote by $\Theta(r) : \mathbb{R}^d \to \mathbb{R}^d$ the rotation that maps the axis of I° to r (the sign is fixed by choosing $r \mapsto \Theta(r)$ to be continuous). The set of all rigid particles is denoted by

$$\mathcal{I}_{\varepsilon,N} := \bigcup_{n=1}^N I^n_{\varepsilon,N}.$$

We also consider ε-neighborhoods $I^{n;+}_{\varepsilon,N} := I^n_{\varepsilon,N} + \varepsilon B$ and we assume that they are disjoint,

$$I^{n;+}_{\varepsilon,N} \cap I^{m;+}_{\varepsilon,N} = \varnothing, \qquad \text{for all } 1 \le n \ne m \le N, \tag{1.2}$$

which will be shown to be preserved along the dynamics in our regime of interest, cf. Proposition 1.1. The particle volume fraction is then

$$\lambda := |\mathcal{I}_{\varepsilon,N}| = N|\varepsilon I^\circ| = N\varepsilon^d |I^\circ|,$$

and we shall consider the macroscopic limit $N \uparrow \infty$, $\varepsilon \downarrow 0$, in the dilute regime $\lambda \ll 1$.

• *Particle activity.* We consider particles that may be active and propel themselves in the fluid (e.g. by consuming some underlying chemical energy, which is not included in the model for simplicity). By a balance of forces, self-propulsion must be described by a couple of forces of same intensity and opposite direction on each rigid particle and on the surrounding fluid:

— Each particle $I^n_{\varepsilon,N}$ is assumed to propel itself in the direction $R^n_{\varepsilon,N}$ of its own axis.

— The force of each particle on the surrounding fluid is typically exerted via a flagellar bundle, but the detail of the propulsion mechanism is not included in the model for simplicity: as e.g. in [5, Section 2.1] (see also [2]), we assume that the force exerted by particle $I^n_{\varepsilon,N}$ on the surrounding fluid can be effectively described by a force field

$$f^n_{\varepsilon,N} := f_\varepsilon(\cdot - X^n_{\varepsilon,N}, R^n_{\varepsilon,N}) \;:\; \mathbb{R}^d \setminus \mathcal{I}^n_{\varepsilon,N} \to \mathbb{R}^d.$$

where the function f_ε takes the form $f_\varepsilon(x,r) := \varepsilon^{-d} f(\frac{x}{\varepsilon}, r)$ for some bounded function $f : \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}^d$. We also assume that $f(\cdot, r)$ is axisymmetric around direction r, just like $I^\circ(r)$, for all $r \in \mathbb{S}^{d-1}$.

The balance of propulsion forces then takes form of the following assumption,

$$R^n_{\varepsilon,N} + \int_{\mathbb{R}^d \setminus \mathcal{I}^n_{\varepsilon,N}} f^n_{\varepsilon,N} = 0, \qquad \text{for all } 1 \le n \le N.$$

For notational convenience, we extend inside each particle

$$f^n_{\varepsilon,N}|_{\mathcal{I}^n_{\varepsilon,N}} := R^n_{\varepsilon,N} |I^n_{\varepsilon,N}|^{-1}, \tag{1.3}$$

or equivalently $f(\cdot,r)|_{I^\circ(r)} = r|I^\circ|^{-1}$, so the balance of forces reads

$$\int_{\mathbb{R}^d} f(\cdot,r) \,=\, 0, \qquad \text{for all } r. \tag{1.4}$$

We further assume that $f(\cdot,r)$ is compactly supported, say in $I^\circ(r)+B$, for all r, meaning that each particle propels itself only by acting on the surrounding fluid at bounded distance. By the separation assumption (1.2), we then note that the force fields $\{f^n_{\varepsilon,N}\}_{1\le n\le N}$ have pairwise disjoint supports.

In most previous work, e.g. [19, 20, 18], the action of a particle on the surrounding fluid was represented for simplicity by a point force, typically setting $f(x,r) := -\delta(x-\theta r)r$ for $x \notin I^\circ(r)$, for some parameter $\theta \in \mathbb{R}$ with $\theta r \notin I^\circ(r)$. This point-force model is a special case of ours (regularity issues for f play no important role), and the cases $\theta > 0$ and $\theta < 0$ then correspond to so-called puller and pusher particles, respectively. We refer e.g. to [39, Sections 2.1–2.2] for a review of other models for self-propulsion, such as squirmer models, which are different but which we believe could be treated analogously.

• *Inertialess particle dynamics in viscous solvent.* Particles are suspended in a homogeneous viscous fluid, which we assume to be described by the steady Stokes equation with unit viscosity. More precisely, given the set $\mathcal{I}_{\varepsilon,N}$ of particles at a given time, the fluid velocity $u_{\varepsilon,N}$ and pressure $p_{\varepsilon,N}$ satisfy the following Stokes equation in the fluid domain,

$$-\triangle u_{\varepsilon,N} + \nabla p_{\varepsilon,N} = h + \kappa \sum_{n=1}^N f^n_{\varepsilon,N}, \qquad \operatorname{div}(u_{\varepsilon,N}) = 0, \qquad \text{in } \mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}, \tag{1.5}$$

where h stands for some internal force in the fluid domain and where $\kappa \ge 0$ is the self-propulsion intensity. We assume for convenience $h \in W^{1,1}\cap W^{1,\infty}(\mathbb{R}^d)^d$. Next, we assume that the fluid flow satisfies no-slip conditions at particle boundaries, so we may implicitly extend the fluid velocity $u_{\varepsilon,N}$ inside the particles to coincide with the particle velocities. The rigidity of the particles then translates into a boundary condition,

$$\mathrm{D}(u_{\varepsilon,N}) = 0, \qquad \text{in } \mathcal{I}_{\varepsilon,N}, \tag{1.6}$$

where $\mathrm{D}(u) = \frac12(\nabla u + (\nabla u)')$ stands for the symmetric gradient. Equivalently, this means for all $1 \le n \le N$,

$$u_{\varepsilon,N} = V^n_{\varepsilon,N} + \Omega^n_{\varepsilon,N}(x - x^n_{\varepsilon,N}), \qquad \text{in } I^n_{\varepsilon,N}, \tag{1.7}$$

for some translational velocity $V^n_{\varepsilon,N} \in \mathbb{R}^d$ and angular velocity tensor $\Omega^n_{\varepsilon,N} \in \mathbb{R}^{d\times d}_{\text{skew}}$. As we neglect the inertia of the particles, Newton's equations of motion reduce to the balance of forces and torques, which take form of additional boundary conditions,

$$\begin{aligned} me + \kappa R^n_{\varepsilon,N} + \int_{\partial I^n_{\varepsilon,N}} \sigma(u_{\varepsilon,N},p_{\varepsilon,N})\nu \;&=\; 0,\\ \int_{\partial I^n_{\varepsilon,N}} (x - X^n_{\varepsilon,N}) \times \sigma(u_{\varepsilon,N},p_{\varepsilon,N})\nu \;&=\; 0, \qquad \text{for all } 1\le n\le N, \end{aligned} \tag{1.8}$$

where $\sigma(u,p) := 2\,\mathrm{D}(u) - p\,\mathrm{Id}$ is the Cauchy stress tensor and where $me \in \mathbb{R}^d$ stands for the buoyancy of the particles. (Henceforth, for $a,b \in \mathbb{R}^d$, we use the vectorial notation $a \times b \in \mathbb{R}^{d\times d}_{\text{skew}}$ with $(a\times b)_{\alpha\beta} := a_\alpha b_\beta - a_\beta b_\alpha$.) We choose the buoyancy me and the self-propulsion intensity κ as

$$me \;:=\; |I^n_{\varepsilon,N}|e \;=\; \tfrac{\lambda}{N}e, \qquad \kappa \;:=\; \tfrac{\kappa_0}{\varepsilon}|I^n_{\varepsilon,N}| \;=\; \kappa_0\tfrac{\lambda}{\varepsilon N},$$

for some fixed parameters $e \in \mathbb{R}^d$ and $\kappa_0 \geq 0$ with

$$|e| \leq 1, \qquad \kappa_0 \leq 1.$$

This choice corresponds to the scaling that leads to $O(\lambda)$ mean forces in the macroscopic limit. The case $e = 0$ then amounts to particles with neutral buoyancy, and the case $\kappa_0 = 0$ to passive particles. Note that our scaling for the buoyancy differs from previous work on the topic [25, 32, 29], where it was rather chosen to create a $O(1)$ mean force in the macroscopic limit; we are not able to consider such a stronger scaling in case of non-spherical particles due to singularity issues in the mean-field analysis of orientations.[1]

Summing up, given the set $\mathcal{I}_{\varepsilon,N}$ of particles at a given time, the instantaneous fluid velocity $u_{\varepsilon,N}$ is obtained as the unique weak solution in $\dot H^1(\mathbb{R}^d)^d$ of the Stokes problem (1.5)–(1.8), that is,

$$\begin{cases} -\triangle u_{\varepsilon,N} + \nabla p_{\varepsilon,N} = h + \kappa_0 \frac{\lambda}{\varepsilon N}\sum_{n=1}^N f^n_{\varepsilon,N}, & \text{in } \mathbb{R}^d \setminus \mathcal{I}^n_{\varepsilon,N},\\ \operatorname{div}(u_{\varepsilon,N}) = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}^n_{\varepsilon,N},\\ \mathrm{D}(u_{\varepsilon,N}) = 0, & \text{in } \mathcal{I}^n_{\varepsilon,N},\\ \frac{\lambda}{N} e + \kappa_0 \frac{\lambda}{\varepsilon N} R^n_{\varepsilon,N} + \int_{\partial I^n_{\varepsilon,N}} \sigma(u_{\varepsilon,N}, p_{\varepsilon,N})\nu = 0, & \text{for all } 1 \leq n \leq N,\\ \int_{\partial I^n_{\varepsilon,N}} (x - X^n_{\varepsilon,N}) \times \sigma(u_{\varepsilon,N}, p_{\varepsilon,N})\nu = 0, & \text{for all } 1 \leq n \leq N. \end{cases} \tag{1.9}$$

We recall that the weak formulation of this system takes on the following simple guise: for any test function $v \in \dot H^1(\mathbb{R}^d)^d$ that is incompressible, i.e. $\operatorname{div}(v) = 0$, and that is rigid inside particles, i.e. $\mathrm{D}(v) = 0$ in $\mathcal{I}_{\varepsilon,N}$, we have

$$2\int_{\mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}} \mathrm{D}(v) : \mathrm{D}(u_{\varepsilon,N}) = \int_{\mathbb{R}^d} v \cdot \Big(h \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}} + e \mathbb{1}_{\mathcal{I}_{\varepsilon,N}} + \kappa_0 \frac{\lambda}{\varepsilon N} \sum_n f^n_{\varepsilon,N} \Big). \tag{1.10}$$

Once the Stokes system (1.9) is solved for the instantaneous fluid velocity $u_{\varepsilon,N}$, particle positions and orientations can be updated according to

$$\partial_t X^n_{\varepsilon,N} = V^n_{\varepsilon,N}, \qquad \partial_t R^n_{\varepsilon,N} = \Omega^n_{\varepsilon,N} R^n_{\varepsilon,N}, \qquad \text{for all } 1 \leq n \leq N, \tag{1.11}$$

where $V^n_{\varepsilon,N}, \Omega^n_{\varepsilon,N}$ are given by (1.7), or alternatively

$$V^n_{\varepsilon,N} := \fint_{I^n_{\varepsilon,N}} u_{\varepsilon,N} \in \mathbb{R}^d, \qquad \Omega^n_{\varepsilon,N} := \fint_{I^n_{\varepsilon,N}} \nabla u_{\varepsilon,N} \in \mathbb{R}^{d\times d}_{\mathrm{skew}}. \tag{1.12}$$

In this way, the particles follow the fluid flow and interact with one another via the flow disturbance that they generate. The resulting dynamics is reputedly complex in view of the multibody, long-range, and singular nature of hydrodynamic interactions.

1.3. **Semi-dilute mean-field description.** We aim to investigate the collective macroscopic behavior of the fluid velocity $u_{\varepsilon,N}$ and of the particle empirical measures

$$\nu_{\varepsilon,N} := \sum_{n=1}^N \delta_{(X^n_{\varepsilon,N}, R^n_{\varepsilon,N})} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{S}^{d-1}), \tag{1.13}$$

$$\mu_{\varepsilon,N} := \sum_{n=1}^N \delta_{X^n_{\varepsilon,N}} \in \mathcal{P}(\mathbb{R}^d).$$

[1]In particular, while a drag force of order $O(\frac{1}{\lambda}\varepsilon^2)$ appeared in the Vlasov equation in [25, 32, 29] due to the buoyancy, this term would become $O(\varepsilon^2)$ in our scaling and is neglected in this work.

In the macroscopic limit $N \uparrow \infty$, $\varepsilon \downarrow 0$, in the dilute regime $\lambda \ll 1$, formal considerations lead to expect

$$(u_{\varepsilon,N}, \mu_{\varepsilon,N}) = (u_{\lambda,\varepsilon}, \mu_{\lambda,\varepsilon}) + O((\lambda+\varepsilon)^2), \qquad \nu_{\varepsilon,N} = \nu_{\lambda,\varepsilon} + O(\lambda+\varepsilon), \tag{1.14}$$

where $(u_{\lambda,\varepsilon}, \nu_{\lambda,\varepsilon}, \mu_{\lambda,\varepsilon})$ solves the following coupled system on $\mathbb{R}^d$: the macroscopic fluid velocity $u_{\lambda,\varepsilon}$ satisfies an effective Stokes equation,

$$\begin{cases} -\operatorname{div}\big[2\big(1+\lambda\langle\Sigma^\circ\nu_{\lambda,\varepsilon}\rangle\big)\mathrm{D}(u_{\lambda,\varepsilon})\big] + \nabla p_{\lambda,\varepsilon} \\ \qquad\qquad = (1-\lambda\mu_{\lambda,\varepsilon})h + \lambda\mu_{\lambda,\varepsilon}e + \kappa_0\lambda\beta_f \operatorname{div}\big[\langle(r\otimes r - \frac1d\,\mathrm{Id})\,\nu_{\lambda,\varepsilon}\rangle\big], \\ \operatorname{div}(u_{\lambda,\varepsilon}) = 0, \\ \mu_{\lambda,\varepsilon} = \langle\nu_{\lambda,\varepsilon}\rangle, \end{cases} \tag{1.15}$$

which is coupled to a Vlasov equation for the mean-field distribution $\nu_{\lambda,\varepsilon}$ of particle positions and orientations,

$$\begin{cases} \partial_t\nu_{\lambda,\varepsilon} + \operatorname{div}_x\big[\nu_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}+\kappa_0\varepsilon\alpha_f r)\big] + \operatorname{div}_r\big[\nu_{\lambda,\varepsilon}(\Omega^\circ\nabla u_{\lambda,\varepsilon})r\big] = 0, \\ \nu_{\lambda,\varepsilon}|_{t=0} = \nu^\circ. \end{cases} \tag{1.16}$$

Here, we use the short-hand notation $\langle g\rangle(x) := \int_{\mathbb{S}^{d-1}} g(x,r)\,d\sigma(r)$ for angular averaging, and the coefficient fields $r\mapsto\Sigma^\circ(r),\Omega^\circ(r)$ and constants α_f,β_f are defined as follows:

— *Passive effective viscosity:* The coefficient $1+\lambda\langle\Sigma^\circ\nu_{\lambda,\varepsilon}\rangle$ in the effective Stokes equation (1.15) corresponds to Einstein's formula for the dilute correction of the plain fluid viscosity due to the presence of rigid particles, see e.g. [12, Section 2] or [23]: for all $r\in\mathbb{S}^{d-1}$, the tensor $\Sigma^\circ(r)$ is the symmetric linear map on the set $\mathbb{R}^{d\times d}_{\mathrm{sym},0}$ of trace-free symmetric matrices, given by

$$E:\Sigma^\circ(r)E \;:=\; |I^\circ|^{-1}\int_{\mathbb{R}^d}|\mathrm{D}(u^\circ_{r,E})|^2, \qquad \text{for all } E\in\mathbb{R}^{d\times d}_{\mathrm{sym},0}, \tag{1.17}$$

where $u^\circ_{r,E}$ is the unique decaying solution of the single-particle problem

$$\begin{cases} -\triangle u^\circ_{r,E} + \nabla p^\circ_{r,E} = 0, & \text{in } \mathbb{R}^d\setminus I^\circ(r), \\ \operatorname{div}(u^\circ_{r,E}) = 0, & \text{in } \mathbb{R}^d\setminus I^\circ(r), \\ \mathrm{D}(u^\circ_{r,E}+Ex) = 0, & \text{in } I^\circ(r), \\ \int_{\partial I^\circ(r)}\sigma(u^\circ_{r,E},p^\circ_{r,E})\nu = 0, \\ \int_{\partial I^\circ(r)} x\times\sigma(u^\circ_{r,E},p^\circ_{r,E})\nu = 0, \end{cases} \tag{1.18}$$

which describes the flow disturbance generated by a strain rate E at a single rigid particle $I^\circ(r)$ oriented in direction r. In other words, $\Sigma^\circ(r)E$ measures the reaction of a particle oriented in direction r to a strain rate E, and it can equivalently be written as (half) the associated stresslet,

$$\Sigma^\circ(r)E \;=\; \tfrac12|I^\circ|^{-1}\int_{\partial I^\circ(r)}\sigma(u^\circ_{r,E}+Ex,p^\circ_{r,E})\nu\otimes^\circ_s x, \tag{1.19}$$

where $a\otimes^\circ_s b := \frac12(a\otimes b+b\otimes a)-\frac1d\,\mathrm{Id}(a\cdot b)$ is the trace-free symmetric tensor product. Note that the map $r\mapsto\Sigma^\circ(r)$ is easily checked to be smooth.

— *Passive particle rotation:* The local fluid deformation makes each particle rotate in a nontrivial way: in the dilute regime, we naturally define $\Omega^\circ(r)H$ as the angular velocity of a single particle oriented in direction r due to a local fluid deformation H. More precisely, $\Omega^\circ(r)$ is the linear map $\mathbb{R}_0^{d\times d}\to\mathbb{R}_{\text{skew}}^{d\times d}$ given by

$$\Omega^\circ(r)H\;:=\;\fint_{I^\circ(r)}\nabla u^\circ_{r,H^{\text{sym}}}+H,\qquad\text{for all }H\in\mathbb{R}_0^{d\times d},\tag{1.20}$$

where $u^\circ_{r,H^{\text{sym}}}$ is the solution of (1.18) with $E=H^{\text{sym}}$ the symmetric part of H. Note that the map $r\mapsto\Omega^\circ(r)$ is easily checked to be smooth.

— *Active elastic stress:* The particle self-propulsion generates an elastic stress in the Stokes equation (cf. last right-hand side term in (1.15)): in the dilute regime, it is naturally given in terms of the stresslet

$$\Sigma_f^\circ(r)\;:=\;\int_{\partial I^\circ(r)}\sigma(u^\circ_{r,f},p^\circ_{r,f})\nu\otimes_s^\circ x-\int_{\mathbb{R}^d}f(\cdot,r)\otimes^\circ x,\tag{1.21}$$

where $a\otimes^\circ b:=a\otimes b-\frac{1}{d}\operatorname{Id}(a\cdot b)$ is the trace-free tensor product and where $u^\circ_{r,f}$ is the unique decaying solution of the single-particle problem

$$\begin{cases}-\triangle u^\circ_{r,f}+\nabla p^\circ_{r,f}=f(\cdot,r), & \text{in }\mathbb{R}^d\setminus I^\circ(r),\\ \operatorname{div}(u^\circ_{r,f})=0, & \text{in }\mathbb{R}^d\setminus I^\circ(r),\\ \mathrm{D}(u^\circ_{r,f})=0, & \text{in }I^\circ(r),\\ r+\int_{\partial I^\circ(r)}\sigma(u^\circ_{r,f},p^\circ_{r,f})\nu=0, & \\ \int_{\partial I^\circ(r)}x\times\sigma(u^\circ_{r,f},p^\circ_{r,f})\nu=0, & \end{cases}\tag{1.22}$$

which describes the flow disturbance generated by the self-propulsion of a single particle oriented in direction r. Since the inclusion $I^\circ(r)$ and the propulsion force $f(\cdot,r)$ are both axisymmetric around direction r, this single-particle stresslet (1.21) can be written by symmetry as

$$\Sigma_f^\circ(r)\;=\;\beta_f\,r\otimes^\circ r,\qquad\text{for some }\beta_f\in\mathbb{R}.\tag{1.23}$$

The cases $\beta_f>0$ and $\beta_f<0$ correspond to so-called puller and pusher particles, respectively. In the macroscopic limit, the active elastic stress is then given by the angular average $\langle\Sigma_f^\circ\nu_{\lambda,\varepsilon}\rangle$, which appears as the last right-hand side term in the effective Stokes equation (1.15). It coincides with the expression predicted e.g. in [38, 20, 36, 35, 6] and first derived in [18, 2] in the equilibrium setting.

— *Swimming velocities:* The particle self-propulsion also generates a drag force on the particles, leading to effective swimming velocities: in the dilute regime, we naturally define $V_f^\circ(r)$ as the drag velocity of a single particle oriented in direction r due to its self-propulsion,

$$V_f^\circ(r)\;:=\;\fint_{I^\circ(r)}u^\circ_{r,f}\;\;\in\;\;\mathbb{R}^d.\tag{1.24}$$

Since the propulsion force $f(\cdot,r)$ is axisymmetric around r, this swimming velocity can be written by symmetry as

$$V_f^\circ(r)\;=\;\alpha_f\,r,\qquad\text{for some }\alpha_f>0.\tag{1.25}$$

The above kinetic model (1.15)–(1.16) is a variant of the so-called Doi model, which was first introduced in [31, 24, 3] for passive suspensions $\kappa_0=0$ (see also [8, 7, 9]), and which was adapted by Saintillan and Shelley [38, 36, 37] to active suspensions $\kappa_0\neq0$. The difference of the above with the standard form of the Doi model is twofold:

— Brownian effects are not considered in the present work. The inclusion of Brownian rotary effects on particle orientations was recently discussed in a simplified setting in [27], but the general mean-field description of Brownian suspensions with spatial diffusion remains a delicate open problem and is postponed to future work.

— While we consider particles with a given axisymmetric shape, the Doi model rather corresponds to the limit of very elongated particles as computed by slender-body theory, e.g. [3]. This amounts to replacing the effective coefficients $\Sigma^\circ, \Omega^\circ$ in (1.15)–(1.16) by

$$\begin{aligned} \Sigma^\circ(r) &\rightsquigarrow \alpha_1(r \otimes^\circ r) \otimes (r \otimes^\circ r), \\ \Omega^\circ(r)(\nabla u)r &\rightsquigarrow \big(\mathrm{Id} - r \otimes r\big)\big(\alpha_2\,\mathrm{D}(u) + \tfrac12 \nabla \times u\big)\, r, \end{aligned}$$

for some shape factors $\alpha_1, \alpha_2 > 0$.

Regardless of those differences, just as the usual Doi model, the kinetic model (1.15)–(1.16) describes the emergence of non-Newtonian effects due to mean-field particle orientations: orientations adapt collectively to the local fluid deformation and in turn modify the effective viscosity via Einstein's effective viscosity formula $1 + \lambda\langle \Sigma^\circ \nu_{\lambda,\varepsilon}\rangle$. We refer in particular to [22, 34] for a detailed study of properties of the Doi model.

1.4. **Main results.** We start by stating the asymptotically global well-posedness of the particle dynamics (1.9)–(1.12). It requires to control the evolution of the minimal interparticle distance and of the largest distance to the origin,

$$\mathrm{d}^{\min}_{\varepsilon,N} := \min_{1\le n\ne m\le N} |X^n_{\varepsilon,N} - X^m_{\varepsilon,N}|, \qquad \rho^{\max}_{\varepsilon,N} := \max_{1\le n\le N} |X^n_{\varepsilon,N}|.$$

The result is similar to corresponding statements in [25, 32], and a short proof is included in Section 3 for completeness, cf. Corollary 3.2.

Proposition 1.1 (Well-posedness of particle system)**.** *Assume that initial particle positions satisfy, for some $C_0 \ge 1$,*

$$\mathrm{d}^{\min}_{\varepsilon,N}(0) \ge \tfrac{1}{C_0} N^{-\frac1d}, \qquad \rho^{\max}_{\varepsilon,N}(0) \le C_0.$$

Given $T < \infty$, provided that $\lambda \log N \ll 1$ is small enough (depending on C_0, T, h), the particle dynamics (1.9)–(1.12) is well-posed up to time T and satisfies for all $0 \le t \le T$,

$$\mathrm{d}^{\min}_{\varepsilon,N}(t) \ge \tfrac{1}{C} N^{-\frac1d}, \qquad \rho^{\max}_{\varepsilon,N}(t) \le C,$$

for some $C \ge 1$ (depending on C_0, T, h). In particular, as $\varepsilon \simeq \lambda^{\frac1d} N^{-\frac1d} \ll N^{-\frac1d}$, this ensures that the disjointness condition (1.2) remains satisfied up to time T. ♢

We turn to the justification of the kinetic model (1.15)–(1.16) in the macroscopic limit. In fact, we only manage to derive it in general with accuracy $O((\lambda+\varepsilon)^2 + \kappa_0\varepsilon)$, which amounts to neglecting $O(\kappa_0\varepsilon)$ swimming velocities: it leads us to the simplified model

$$\begin{cases} -\mathrm{div}\big[2\big(1+\lambda\langle\Sigma^\circ\nu_\lambda\rangle\big)\mathrm{D}(u_\lambda)\big] + \nabla p_\lambda \\ \qquad\qquad = (1-\lambda\mu_\lambda)h + \lambda\mu_\lambda e + \kappa_0\lambda\beta_f\,\mathrm{div}\big[\langle (r\otimes r - \frac1d\,\mathrm{Id})\nu_\lambda\rangle\big], \\ \mathrm{div}(u_\lambda) = 0, \\ \mu_\lambda = \langle \nu_\lambda\rangle, \\ \partial_t\nu_\lambda + \mathrm{div}_x(\nu_\lambda u_\lambda) + \mathrm{div}_r(\nu_\lambda(\Omega^\circ\nabla u_\lambda)r) = 0, \\ \nu_\lambda|_{t=0} = \nu^\circ. \end{cases} \tag{1.26}$$

In order to further capture the $O(\kappa_0\varepsilon)$ swimming velocities in (1.15)–(1.16), we need to restrict to the monokinetic setting,

$$\nu_{\lambda,\varepsilon}(x,r) \,=\, \mu_{\lambda,\varepsilon}(x)\,\delta(r-r_{\lambda,\varepsilon}(x)),$$

and we will then derive the following monokinetic version of (1.15)–(1.16),

$$\left\{\begin{array}{l}-\operatorname{div}\big[2\big(1+\lambda\Sigma^\circ(r_{\lambda,\varepsilon})\mu_{\lambda,\varepsilon}\big)\mathrm{D}(u_{\lambda,\varepsilon})\big]+\nabla p_{\lambda,\varepsilon}\\ \qquad\qquad =(1-\lambda\mu_{\lambda,\varepsilon})h+\lambda\mu_{\lambda,\varepsilon}e+\kappa_0\lambda\beta_f\operatorname{div}\big[(r_{\lambda,\varepsilon}\otimes r_{\lambda,\varepsilon}-\frac1d\operatorname{Id})\mu_{\lambda,\varepsilon}\big],\\ \operatorname{div}(u_{\lambda,\varepsilon})=0,\\ \partial_t\mu_{\lambda,\varepsilon}+\operatorname{div}\big[\mu_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}+\kappa_0\varepsilon\alpha_f r_{\lambda,\varepsilon})\big]=0,\\ \partial_t r_{\lambda,\varepsilon}+(u_{\lambda,\varepsilon}+\kappa_0\varepsilon\alpha_f r_{\lambda,\varepsilon})\cdot\nabla r_{\lambda,\varepsilon}=(\Omega^\circ(r_{\lambda,\varepsilon})\nabla u_{\lambda,\varepsilon})r_{\lambda,\varepsilon},\\ (\mu_{\lambda,\varepsilon},r_{\lambda,\varepsilon})|_{t=0}=(\mu^\circ,r^\circ).\end{array}\right. \tag{1.27}$$

We start with the well-posedness of these macroscopic models in the smooth class. The proof is standard and is included in Section 6.1 for completeness.

Proposition 1.2 (Well-posedness of macroscopic models)**.**

(i) Given $T<\infty$, given $\gamma>1$ non-integer, and given an initial condition $\nu^\circ\in\mathcal P\cap W^{1,1}\cap W^{\gamma,\infty}(\mathbb R^d\times\mathbb S^{d-1})$, there is $\lambda_0>0$ (depending on T,h,γ,ν°) such that for all $0\le\lambda\le\lambda_0$ there is a unique solution (ν_λ,u_λ) of (1.26) up to time T with $\nu_\lambda\in\mathrm L^\infty([0,T];\mathcal P\cap W^{1,1}\cap W^{\gamma,\infty}(\mathbb R^d\times\mathbb S^{d-1}))$ and $u_\lambda\in\mathrm L^\infty([0,T];W^{\gamma+1,\infty}(\mathbb R^d)^d)$.

(ii) Given $T<\infty$, given $\gamma>1$ non-integer, and given initial conditions $\mu^\circ\in\mathcal P\cap W^{1,1}\cap W^{\gamma,\infty}(\mathbb R^d)$ and $r^\circ\in W^{\gamma,\infty}(\mathbb R^d;\mathbb S^{d-1})$, there are $\lambda_0,\varepsilon_0>0$ (depending on $T,h,\gamma,\mu^\circ,r^\circ$) such that for all $0\le\lambda\le\lambda_0$ and $0\le\varepsilon\le\varepsilon_0$ there is a unique solution $(\mu_{\lambda,\varepsilon},r_{\lambda,\varepsilon},u_{\lambda,\varepsilon})$ of (1.27) up to time T with $\mu_{\lambda,\varepsilon}\in\mathrm L^\infty([0,T];\mathcal P\cap W^{1,1}\cap W^{\gamma,\infty}(\mathbb R^d))$, $r_{\lambda,\varepsilon}\in\mathrm L^\infty([0,T];W^{\gamma,\infty}(\mathbb R^d;\mathbb S^{d-1}))$, and $u_{\lambda,\varepsilon}\in\mathrm L^\infty([0,T];W^{\gamma+1,\infty}(\mathbb R^d)^d)$. ◊

We can now state our main results, that is, the derivation of the Doi-type macroscopic models (1.26) or (1.27) from the particle dynamics (1.9)–(1.12), thus providing a rigorous version of (1.14). To our knowledge, this is the first time that non-Newtonian macroscopic models are rigorously derived from a hydrodynamic description of particle suspensions. It constitutes both a generalization of [29] to non-spherical and possibly active particles, and a generalization of [16, 18] to the particle dynamics. We start with the simplified model (1.26) without swimming velocities, which we derive with accuracy $O((\lambda+\varepsilon)^2+\kappa_0\varepsilon)$ (up to logarithmic corrections and initial well-preparedness).

Theorem 1.3 (Semi-dilute mean-field approximation)**.** *Assume that initial particle positions satisfy, for some $C_0\ge1$,*

$$\mathrm d^{\min}_{\varepsilon,N}(0)\,\ge\,\tfrac{1}{C_0}N^{-\frac1d},\qquad \rho^{\max}_{\varepsilon,N}(0)\,\le\,C_0,$$

and assume that the empirical measure (1.13) is initially close in the ∞-Wasserstein metric to some density $\nu^\circ\in\mathcal P\cap W^{1,1}\cap W^{\gamma,\infty}(\mathbb R^d\times\mathbb S^{d-1})$ for some $\gamma>1$,

$$W_\infty(\nu^\circ_{\varepsilon,N},\nu^\circ),W_\infty(\mu^\circ_{\varepsilon,N},\mu^\circ)\,\le\,C_0(\lambda\log N+\varepsilon),\qquad \mu^\circ:=\langle\nu^\circ\rangle.$$

Given $T<\infty$, assume that $\lambda\log N\ll1$ is small enough to ensure the well-posedness of the particle system up to time T as in Proposition 1.1, and let $0\le\lambda\le\lambda_0$ be small enough to ensure that (1.26) admits a unique solution $(\nu_\lambda,\mu_\lambda,u_\lambda)$ up to time T as in Proposition 1.2(i). Then we have for all $t\in[0,T]$,

$$W_\infty(\mu^t_{\varepsilon,N},\mu^t_\lambda)\,\lesssim\,(\lambda\log N+\varepsilon)^2+\kappa_0\varepsilon+W_\infty(\mu^\circ_{\varepsilon,N},\mu^\circ),$$

and in addition, for any boundary-layer thickness $\delta \in [\varepsilon, 1]$,

$$\begin{aligned}\|u_{\varepsilon,N}^t - u_\lambda^t\|_{\mathrm{L}^\infty(\mathbb{R}^d \setminus (\mathcal{I}_{\varepsilon,N}^t + \delta B))} &\lesssim (\lambda+\varepsilon)(\lambda\log N + \varepsilon) + (\lambda+\varepsilon)(\tfrac{\varepsilon}{\delta})^d,\\ W_\infty(\nu_{\varepsilon,N}^t, \nu_\lambda^t) &\lesssim \lambda \log N + \varepsilon,\end{aligned}$$

up to multiplicative constants depending on $C_0, T, h, \gamma, \nu^\circ$. ◊

We turn to derivation of the corresponding monokinetic version (1.27) of (1.9)–(1.12), including $O(\kappa_0\varepsilon)$ swimming forces, which we justify with improved accuracy $O((\lambda+\varepsilon)^2)$ (up to logarithmic corrections and initial well-preparedness).

Theorem 1.4 (Improved approximation in monokinetic regime)**.** *Assume that initial particle positions satisfy, for some* $C_0 \geq 1$,

$$\mathrm{d}_{\varepsilon,N}^{\min}(0) \geq \tfrac{1}{C_0} N^{-\frac{1}{d}}, \qquad \rho_{\varepsilon,N}^{\max}(0) \leq C_0,$$

and assume that the empirical measure (1.13) *is initially close in the* ∞*-Wasserstein metric to some monokinetic profile* $\nu^\circ(x,r) := \mu^\circ(x)\,\delta(r - r^\circ(x))$ *with* $\mu^\circ \in \mathcal{P} \cap W^{1,1} \cap W^{\gamma,\infty}(\mathbb{R}^d)$ *and* $r^\circ \in W^{\gamma,\infty}(\mathbb{R}^d; \mathbb{S}^{d-1})$ *for some* $\gamma > 1$,

$$W_\infty(\nu_{\varepsilon,N}^\circ, \nu^\circ) \leq C_0(\lambda \log N + \varepsilon).$$

Given $T < \infty$, *assume that* $\lambda \log N \ll 1$ *is small enough to ensure the well-posedness of the particle system up to time* T *as in Proposition 1.1, and let* $0 \leq \lambda \leq \lambda_0$ *and* $0 \leq \varepsilon \leq \varepsilon_0$ *be small enough to ensure that* (1.27) *admits a unique solution* $(\mu_{\lambda,\varepsilon}, r_{\lambda,\varepsilon}, u_{\lambda,\varepsilon})$ *up to time* T *as in Proposition 1.2. Then we have for all* $t \in [0,T]$,

$$W_\infty(\mu_{\varepsilon,N}^t, \mu_{\lambda,\varepsilon}^t) \lesssim (\lambda \log N + \varepsilon)^2 + W_\infty(\mu_{\varepsilon,N}^\circ, \mu^\circ),$$

and in addition, for any boundary-layer thickness $\delta \in [\varepsilon, 1]$,

$$\begin{aligned}\|u_{\varepsilon,N}^t - u_{\lambda,\varepsilon}^t\|_{\mathrm{L}^\infty(\mathbb{R}^d \setminus (\mathcal{I}_{\varepsilon,N}^t + \delta B))} &\lesssim (\lambda+\varepsilon)(\lambda\log N + \varepsilon) + (\lambda+\varepsilon)(\tfrac{\varepsilon}{\delta})^d,\\ W_\infty(\nu_{\varepsilon,N}^t, \nu_\lambda^t) &\lesssim \lambda \log N + \varepsilon,\end{aligned}$$

up to multiplicative constants depending on $C_0, T, h, \gamma, \mu^\circ, r^\circ$. ◊

As already emphasized, such mean-field descriptions cannot hold beyond the accuracy $O(\lambda^2)$. Indeed, the next-order $O(\lambda^2)$ correction to the approximate effective viscosity $1 + \lambda\langle \Sigma^\circ \nu_{\lambda,\varepsilon}\rangle$ in (1.26) or (1.27) should involve the statistical distribution of pairs of particles on the microscale, and such geometric information is beyond the scope of propagation of chaos and mean-field theory. We refer to Remark 4.2 below for further discussion.

Notation.

- In order to control the particle dynamics, on top of the minimal interparticle distance and the largest distance to the origin,
$$\mathrm{d}_{\varepsilon,N}^{\min} := \min_{1 \leq n \neq m \leq N} |X_{\varepsilon,N}^n - X_{\varepsilon,N}^m|, \qquad \rho_{\varepsilon,N}^{\max} := \max_{1 \leq n \leq N} |X_{\varepsilon,N}^n|. \tag{1.28}$$
we also consider the following quantities, as in [25], for any $\sigma \in [0,d]$,
$$\alpha_{\varepsilon,N}^\sigma := \max_{1 \leq n \leq N} \frac{1}{N} \sum_{m:m\neq n}^N |X_{\varepsilon,N}^n - X_{\varepsilon,N}^m|^{\sigma - d}. \tag{1.29}$$
- We denote by $\mathbb{R}_0^{d\times d}$, $\mathbb{R}_{\mathrm{sym},0}^{d\times d}$, and $\mathbb{R}_{\mathrm{skew}}^{d\times d}$ the set of trace-free matrices, of trace-free symmetric matrices, and of skew-symmetric matrices, respectively.

- For $a, b \in \mathbb{R}^d$, we denote by $a \otimes_s b := \frac{1}{2}(a \otimes b + b \otimes a)$ the symmetric tensor product, by $a \otimes^\circ b := a \otimes b - \frac{1}{d}\,\mathrm{Id}(a \cdot b)$ the trace-free tensor product, and by $a \otimes_s^\circ b := a \otimes_s b - \frac{1}{d}\,\mathrm{Id}(a \cdot b)$ the trace-free symmetric tensor product. We use the vectorial notation $a \times b \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ with $(a \times b)_{ij} := a_i b_j - a_j b_i$. For matrices A, B, we let $A : B := A_{ij} B_{ij}$, systematically using Einstein's summation convention on repeated indices. We also use the notation A^{sym} for the symmetric part of A, that is, $(A^{\mathrm{sym}})_{ij} := \frac{1}{2}(A_{ij} + A_{ji})$.
- For a vector field u and a matrix field T, we set $(\nabla u)_{ij} := \nabla_j u_i$, $\mathrm{D}(u) := (\nabla u)^{\mathrm{sym}}$, $(\nabla T)_{ijk} := \nabla_k T_{ij}$, $\mathrm{div}(T)_i := \nabla_j T_{ij}$. For a pressure field p, we denote the Cauchy stress tensor by $\sigma(u, p) := 2\mathrm{D}(u) - p\,\mathrm{Id}$. We define $(T * u)_i := T_{ij} * u_j$ as the convolution product of a vector field with a matrix kernel, and similarly for higher-order tensors.
- We use the short-hand notation $g \,\hat{*}\, \mu(x) := \int_{\mathbb{R}^d \setminus \{x\}} g(x - y)\, d\mu(y)$ for the diagonal-free convolution, which is equivalent to standard convolution if the measure μ is continuous.
- We let $\langle g \rangle(x) := \int_{\mathbb{S}^{d-1}} g(x, r)\, d\sigma(r)$ be the angular averaging of a function g on $\mathbb{R}^d \times \mathbb{S}^{d-1}$.
- We denote by $C \geq 1$ any constant that only depends on the dimension d, on the C^2 property of I°, and on the propulsion force f. We use the notation $\lesssim$ (resp. $\gtrsim$) for $\leq C\times$ (resp. $\geq \frac{1}{C}\times$) up to such a multiplicative constant C. We write $\ll$ (resp. $\gg$) for $\leq C\times$ (resp. $\geq C\times$) up to a sufficiently large multiplicative constant C. We add subscripts to indicate dependence on other parameters.
- The ball centered at x of radius r in $\mathbb{R}^d$ is denoted by $B_r(x)$, and we set $B := B_1(0)$.

2. Preliminary on Stokes analysis

In this section, we recall a series of preliminary results for the analysis of the steady Stokes equation with rigid inclusions. We start with the following standard lemma, showing how rigidity constraints can be viewed as creating source terms concentrated at particle boundaries in the Stokes equation; a short proof is included for convenience.

Lemma 2.1. *Given $h \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)^d$, if $u \in \dot H^1(\mathbb{R}^d)^d$ satisfies*

$$\begin{cases} -\triangle u + \nabla p = h, & \text{in } \mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}, \\ \mathrm{div}(u) = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}, \\ \mathrm{D}(u) = 0, & \text{in } \mathcal{I}_{\varepsilon,N}, \end{cases} \tag{2.1}$$

then the following relation holds in the weak sense in the whole space $\mathbb{R}^d$,

$$-\triangle u + \nabla(p \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}}) \,=\, h \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}} - \sum_{n=1}^N \delta_{\partial I^i_{\varepsilon,N}} \sigma(u,p)\nu. \qquad \Diamond$$

Proof. For a test function $v \in \dot H^1(\mathbb{R}^d)^d$, the incompressibility of u and the rigidity constraint in $\mathcal{I}_{\varepsilon,N}$ yield

$$\int_{\mathbb{R}^d} \nabla v : \nabla u \,=\, 2\int_{\mathbb{R}^d} \nabla v : \mathrm{D}(u) \,=\, 2\int_{\mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}} \nabla v : \mathrm{D}(u),$$

or equivalently, inserting the definition of the Cauchy stress tensor $\sigma(u,p) = 2\,\mathrm{D}(u) - p\,\mathrm{Id}$,

$$\int_{\mathbb{R}^d} \nabla v : \nabla u \,=\, \int_{\mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}} \nabla v : p\,\mathrm{Id} + \int_{\mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}} \nabla v : \sigma(u,p).$$

Integrating by parts in the last right-hand side term and noting that the Stokes equation in (2.1) yields $-\operatorname{div}(\sigma(u,p)) = h$ in $\mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}$, we deduce

$$\int_{\mathbb{R}^d} \nabla v : \nabla u \,=\, \int_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}} \nabla v : p\,\mathrm{Id} + \int_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}} v\cdot h - \sum_{n=1}^N \int_{\partial I^n_{\varepsilon,N}} v\cdot\sigma(u,p)\nu,$$

which is the conclusion. □

We state and prove the following basic trace estimate, which is used repeatedly to control force terms concentrated at particle boundaries.

Lemma 2.2 (Trace estimate)**.** *Given $1\le n\le N$, if $(u,p)\in H^1_{\mathrm{loc}}(\mathbb{R}^d)^d\times \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ satisfies*

$$\begin{cases} -\triangle u + \nabla p = 0, & \text{in } I^{n;+}_{\varepsilon,N}\setminus I^n_{\varepsilon,N},\\ \operatorname{div}(u) = 0, & \text{in } I^{n;+}_{\varepsilon,N}\setminus I^n_{\varepsilon,N},\\ \mathrm{D}(u) = 0, & \text{in } I^n_{\varepsilon,N},\\ \int_{\partial I^n_{\varepsilon,N}} \sigma(u,p)\nu = 0, & \\ \int_{\partial I^n_{\varepsilon,N}} (x-X^n_{\varepsilon,N})\times\sigma(u,p)\nu = 0, & \end{cases} \tag{2.2}$$

then we have for all $F\in H^1_{\mathrm{loc}}(\mathbb{R}^d)^d$ with $\operatorname{div}(F)=0$,

$$\Big|\int_{\partial I^n_{\varepsilon,N}} F\cdot\sigma(u,p)\nu\Big| \,\lesssim\, \|\mathrm{D}(F)\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N})}\|\mathrm{D}(u)\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N})},$$

where we recall $I^{n;+}_{\varepsilon,N} = I^n_{\varepsilon,N}+\varepsilon B$. ◊

Proof. The condition $\int_{\partial I^n_{\varepsilon,N}}\sigma(u,p)\nu = 0$ allows to rewrite

$$\int_{\partial I^n_{\varepsilon,N}} F\cdot\sigma(u,p)\nu \,=\, \int_{\partial I^n_{\varepsilon,N}}\Big(F - \fint_{I^{n;+}_{\varepsilon,N}} F\Big)\cdot\sigma(u,p)\nu.$$

Choosing a cut-off function χ_ε with

$$\chi_\varepsilon|_{I^n_{\varepsilon,N}} = 1, \qquad \operatorname{supp}\chi_\varepsilon\subset I^{n;+}_{\varepsilon,N}, \qquad |\nabla\chi_\varepsilon|\lesssim\varepsilon^{-1},$$

and noting that equation (2.2) yields $\operatorname{div}(\sigma(u,p))=0$ in the annulus $I^{n;+}_{\varepsilon,N}\setminus I^n_{\varepsilon,N}$, we find by integration by parts,

$$\int_{\partial I^n_{\varepsilon,N}} F\cdot\sigma(u,p)\nu = \int_{I^{n;+}_{\varepsilon,N}\setminus I^n_{\varepsilon,N}} \nabla\Big(\Big(F-\fint_{I^{n;+}_{\varepsilon,N}}F\Big)\chi_\varepsilon\Big) : \sigma(u,p).$$

Hence, by the Cauchy–Schwarz inequality followed by Poincaré's inequality, using properties of the cut-off function χ_ε,

$$\Big|\int_{\partial I^n_{\varepsilon,N}} F\cdot\sigma(u,p)\nu\Big| \,\lesssim\, \|\nabla F\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N})}\|\sigma(u,p)\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N}\setminus I^n_{\varepsilon,N})}.$$

For any $\Omega\in\mathbb{R}^{d\times d}_{\mathrm{skew}}$, as the last condition in (2.2) yields

$$\int_{\partial I^n_{\varepsilon,N}} \Omega(x-X^n_{\varepsilon,N})\cdot\sigma(u,p)\nu = 0$$

we can subtract $\Omega(x-X^n_{\varepsilon,N})$ to F in the above estimate, to the effect of

$$\Big|\int_{\partial I^n_{\varepsilon,N}} F\cdot\sigma(u,p)\nu\Big| \lesssim \|\nabla F-\Omega\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N})}\|\sigma(u,p)\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N}\setminus I^n_{\varepsilon,N})}.$$

Hence, taking the infimum over $\Omega\in\mathbb{R}^{d\times d}_{\mathrm{skew}}$ and appealing to Korn's inequality,

$$\Big|\int_{\partial I^n_{\varepsilon,N}} F\cdot\sigma(u,p)\nu\Big| \lesssim \|\mathrm{D}(F)\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N})}\|\sigma(u,p)\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N}\setminus I^n_{\varepsilon,N})}.$$

It remains to estimate the pressure field p in the right-hand side. As the incompressibility of F implies $\int_{\partial I^n_{\varepsilon,N}} F\cdot\nu=0$, any constant can be subtracted to the pressure p in the above, hence in particular

$$\Big|\int_{\partial I^n_{\varepsilon,N}} F\cdot\sigma(u,p)\nu\Big| \lesssim \|\mathrm{D}(F)\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N})}\bigg(\|\mathrm{D}(u)\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N})}+\Big\|p-\fint_{I^{n;+}_{\varepsilon,N}\setminus I^n_{\varepsilon,N}}p\Big\|_{\mathrm{L}^2(I^{n;+}_{\varepsilon,N}\setminus I^n_{\varepsilon,N})}\bigg).$$

Now appealing to a local pressure estimate for the steady Stokes equation, which follows from a standard argument based on the Bogovskii operator, e.g. [16, Lemma 3.3], the conclusion follows. □

Next, we recall the usual definition of the Stokeslet $\mathcal{G}$, that is the Green's function for the steady Stokes equation, and we recall its pointwise decay.

Lemma 2.3 (Pointwise Stokeslet estimates)**.** *For all $1\le i\le d$, we can define $\mathcal{G}_i\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)^d$ as the unique decaying distributional solution of*

$$-\triangle\mathcal{G}_i+\nabla P_i=e_i\delta_0,\qquad \operatorname{div}(\mathcal{G}_i)=0,\qquad \text{in } \mathbb{R}^d,$$

and we then set $\mathcal{G}_i=(\mathcal{G}_{ij})_{1\le j\le d}$ and $\mathcal{G}=(\mathcal{G}_{ij})_{1\le i,j\le d}$. This Green's function is explicitly given by

$$\mathcal{G}(x)\;=\;\tfrac{1}{2(d-2)|\partial B|}|x|^{2-d}\Big(\operatorname{Id}+(d-2)\tfrac{x}{|x|}\otimes\tfrac{x}{|x|}\Big),$$

hence it satisfies the following pointwise estimates,

$$|\mathcal{G}(x)|\lesssim|x|^{2-d},\qquad |\nabla\mathcal{G}(x)|\lesssim|x|^{1-d},\qquad |\nabla^2\mathcal{G}(x)|\lesssim|x|^{-d}.\qquad\Diamond$$

We also define a corresponding notion of Stokeslet for the steady Stokes problem with a single rigid inclusion and we state that it satisfies a similar pointwise decay. This is a particular case of the analysis in [12, Appendix A], where we further get a corresponding result for any finite family of well-separated rigid inclusions.

Lemma 2.4 (Pointwise Stokeslet estimates with rigid inclusions; [12])**.** *For all $1\le n\le N$, $y\in\mathbb{R}^d\setminus I^n_{\varepsilon,N}$, and $1\le i\le d$, we can define $\mathcal{G}^n_{\varepsilon,N;i}(\cdot,y)\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)^d$ as the unique decaying distributional solution of*

$$\begin{cases}-\triangle\mathcal{G}^n_{\varepsilon,N;i}(\cdot,y)+\nabla P^n_{\varepsilon,N;i}(\cdot,y)=e_i\delta_y, & \text{in } \mathbb{R}^d\setminus I^n_{\varepsilon,N},\\ \operatorname{div}(\mathcal{G}^n_{\varepsilon,N;i}(\cdot,y))=0, & \text{in } \mathbb{R}^d\setminus I^n_{\varepsilon,N},\\ \mathrm{D}(\mathcal{G}^n_{\varepsilon,N;i}(\cdot,y))=0, & \text{in } I^n_{\varepsilon,N},\\ \int_{\partial I^n_{\varepsilon,N}}\sigma(\mathcal{G}^n_{\varepsilon,N;i}(\cdot,y),P^n_{\varepsilon,N;i}(\cdot,y))\nu=0, & \\ \int_{\partial I^n_{\varepsilon,N}}(\cdot-X^i_{\varepsilon,N})\times\sigma(\mathcal{G}^n_{\varepsilon,N;i}(\cdot,y),P^n_{\varepsilon,N;i}(\cdot,y))\nu=0, & \end{cases}$$

and we then set $\mathcal{G}^n_{\varepsilon,N;i}=(\mathcal{G}^n_{\varepsilon,N;ij})_{1\le j\le d}$ *and* $\mathcal{G}^n_{\varepsilon,N}=(\mathcal{G}^n_{\varepsilon,N;ij})_{1\le i,j\le d}$*. This Green's function satisfies the following pointwise estimates,*

$$|\nabla_x\mathcal{G}^n_{\varepsilon,N}(x,y)|\lesssim|x-y|^{1-d},\qquad|\nabla_x\nabla_y\mathcal{G}^n_{\varepsilon,N}(x,y)|\lesssim|x-y|^{-d}.\qquad\Diamond$$

3. Lipschitz estimate on the fluid velocity

This section is devoted to the proof of the following a priori Lipschitz estimate on the fluid velocity in the dilute regime, which holds as long as particles are sufficiently well separated. In case of spherical passive particles, this was first established by Höfer in [25, Lemma 3.16], based on an expansion of the fluid velocity by means of the method of reflections. Note that the argument in that work was for spherical passive particles and indeed relied on the spherical shape of the particles, see e.g. [25, Lemma 3.10]. In the present contribution, we provide an alternative perturbative argument, which avoids the method of reflections and further allows to cover the case of non-spherical and active particles. The proof is displayed in Section 3.1.

Proposition 3.1 (Lipschitz estimate on fluid velocity)**.** *Recall the notation* (1.28)–(1.29)*, and assume that* $\mathrm{d}^{\min}_{\varepsilon,N}\ge4\varepsilon$ *and that* $\lambda\alpha^0_{\varepsilon,N}\ll1$ *is small enough. Then we have*

$$\|u_{\varepsilon,N}\|_{W^{1,\infty}(\mathbb{R}^d)}\lesssim_h 1.\qquad\Diamond$$

As a corollary, if particles are initially well separated, we can deduce that they remain so under the dynamics for asymptotically long times in a suitable dilute regime. This implies in particular Proposition 1.1. A proof is given in Section 3.2.

Corollary 3.2 (Well-posedness of particle model)**.** *Assume that initial particle positions satisfy, for some* $\theta_0\ge1$*,*

$$\mathrm{d}^{\min}_{\varepsilon,N}(0)\,\ge\,(\theta_0N)^{-\frac1d},\qquad\rho^{\max}_{\varepsilon,N}(0)\,\le\,\theta_0.$$

Given $\theta>\theta_0$*, further assuming that* $(\theta N)^{-\frac1d}\ge4\varepsilon$ *and that* $\lambda\theta\log(\theta N)\ll1$ *is small enough, there exists a maximal time* $T>C_h^{-1}\log(\theta/\theta_0)$ *such that the particle dynamics* (1.9)–(1.12) *is well-posed up to time* T *and satisfies for all* $t\in[0,T]$*,*

$$\mathrm{d}^{\min}_{\varepsilon,N}(t)\ge(\theta N)^{-\frac1d},\qquad\rho^{\max}_{\varepsilon,N}(t)\le\theta.\qquad\Diamond$$

3.1. Proof of Proposition 3.1.

We split the proof into four steps.

Step 1. Preliminary estimate on the fluid velocity away from rigid particles: proof that, for all $x\in\mathbb{R}^d$ with $\mathrm{dist}(x,\{X^n_{\varepsilon,N}\}_n)\ge\frac12\,\mathrm{d}^{\min}_{\varepsilon,N}$, we have

$$\begin{aligned}|u_{\varepsilon,N}(x)|&\lesssim C_h+|e|\lambda\alpha^2_{\varepsilon,N}+\kappa_0\lambda\alpha^1_{\varepsilon,N}\\&\qquad+\tfrac{\lambda}{N}\sum_{n=1}^N|x-X^n_{\varepsilon,N}|^{1-d}\Big(\fint_{I^n_{\varepsilon,N}+\varepsilon B}|\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12},\\|\nabla u_{\varepsilon,N}(x)|&\lesssim C_h+|e|\lambda\alpha^1_{\varepsilon,N}+\kappa_0\lambda\alpha^0_{\varepsilon,N}\\&\qquad+\tfrac{\lambda}{N}\sum_{n=1}^N|x-X^n_{\varepsilon,N}|^{-d}\Big(\fint_{I^n_{\varepsilon,N}+\varepsilon B}|\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12}.\end{aligned}\qquad(3.1)$$

From the Stokes problem (1.9), using Lemma 2.1, we find that the fluid velocity $u_{\varepsilon,N}$ satisfies the following equation in $\mathbb{R}^d$,

$$\begin{aligned}-\triangle u_{\varepsilon,N}+\nabla(p_{\varepsilon,N}\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}})\\=h\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}}+\kappa_0\tfrac{\lambda}{\varepsilon N}\sum_{n=1}^N f_{\varepsilon,N}^n\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}}-\sum_{n=1}^N\delta_{\partial I_{\varepsilon,N}^n}\sigma(u_{\varepsilon,N},p_{\varepsilon,N})\nu.\end{aligned}\tag{3.2}$$

In terms of the Stokeslet $\mathcal{G}$, cf. Lemma 2.3, noting that the assumption $\mathrm{d}_{\varepsilon,N}^{\min}\geq 4\varepsilon$ ensures that propulsion forces $\{f_{\varepsilon,N}^n\}_n$ have pairwise disjoint supports, we deduce

$$\begin{aligned}u_{\varepsilon,N}(x)\;=\;\int_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}}\mathcal{G}(x-\cdot)h+\kappa_0\tfrac{\lambda}{\varepsilon N}\sum_{n=1}^N\int_{I_{\varepsilon,N}^{n;+}\setminus I_{\varepsilon,N}^n}\mathcal{G}(x-\cdot)f_{\varepsilon,N}^n\\-\sum_{n=1}^N\int_{\partial I_{\varepsilon,N}^n}\mathcal{G}(x-\cdot)\,\sigma(u_{\varepsilon,N},p_{\varepsilon,N})\nu,\end{aligned}$$

hence, using boundary conditions for $u_{\varepsilon,N}$, cf. (1.9), and recalling (1.3),

$$\begin{aligned}u_{\varepsilon,N}(x)\;=\;\int_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}}\mathcal{G}(x-\cdot)h+\sum_{n=1}^N\int_{I_{\varepsilon,N}^n}\mathcal{G}(x-\cdot)e+\kappa_0\tfrac{\lambda}{\varepsilon N}\sum_{n=1}^N\int_{I_{\varepsilon,N}^{n;+}}\mathcal{G}(x-\cdot)f_{\varepsilon,N}^n\\-\sum_{n=1}^N\int_{\partial I_{\varepsilon,N}^n}\Big(\mathcal{G}(x-\cdot)-\fint_{I_{\varepsilon,N}^n}\mathcal{G}(x-\cdot)\Big)\sigma(u_{\varepsilon,N},p_{\varepsilon,N})\nu.\end{aligned}\tag{3.3}$$

Appealing to a trace estimate and to pointwise bounds on $\mathcal{G}$, cf. Lemmas 2.2 and 2.3, recalling the local balance condition (1.4) for propulsion forces, performing local integrals, and recalling $\varepsilon^d|I^\circ|=\frac{\lambda}{N}$, we find for all $x\in\mathbb{R}^d$ with $\operatorname{dist}(x,\mathcal{I}_{\varepsilon,N})\geq 2\varepsilon$,

$$\begin{aligned}|u_{\varepsilon,N}(x)|\;\lesssim\;\int_{\mathbb{R}^d}|x-\cdot|^{2-d}|h|+|e|\tfrac{\lambda}{N}\sum_{n=1}^N|x-X_{\varepsilon,N}^n|^{2-d}+\kappa_0\tfrac{\lambda}{N}\sum_{n=1}^N|x-X_{\varepsilon,N}^n|^{1-d}\\+\tfrac{\lambda}{N}\sum_{n=1}^N|x-X_{\varepsilon,N}^n|^{1-d}\Big(\fint_{I_{\varepsilon,N}^{n;+}}|\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12}.\end{aligned}$$

Similarly, first differentiating in space,

$$\begin{aligned}|\nabla u_{\varepsilon,N}(x)|\;\lesssim\;\int_{\mathbb{R}^d}|x-\cdot|^{1-d}|h|+|e|\tfrac{\lambda}{N}\sum_{n=1}^N|x-X_{\varepsilon,N}^n|^{1-d}+\kappa_0\tfrac{\lambda}{N}\sum_{n=1}^N|x-X_{\varepsilon,N}^n|^{-d}\\+\tfrac{\lambda}{N}\sum_{n=1}^N|x-X_{\varepsilon,N}^n|^{-d}\Big(\fint_{I_{\varepsilon,N}^{n;+}}|\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12}.\end{aligned}\tag{3.4}$$

Now note that, for all $x\in\mathbb{R}^d$ with $\operatorname{dist}(x,\{X_{\varepsilon,N}^n\}_n)\geq\frac12\mathrm{d}_{\varepsilon,N}^{\min}$, if $X_{\varepsilon,N}^p$ is a particle that is the closest to x, we can estimate for any $\sigma\in[0,d]$,

$$\tfrac1N\sum_{n=1}^N|x-X_{\varepsilon,N}^n|^{\sigma-d}\;\lesssim\;\tfrac1N\sum_{n:n\neq p}^N|X_{\varepsilon,N}^p-X_{\varepsilon,N}^n|^{\sigma-d}\;\leq\;\alpha_{\varepsilon,N}^\sigma,\tag{3.5}$$

where we recall that $\alpha^\sigma_{\varepsilon,N}$ is defined in (1.29). Using this bound in (3.4) and recalling the assumption $\mathrm{d}^{\min}_{\varepsilon,N}\ge 4\varepsilon$, the claim (3.1) follows.

Step 2. Preliminary estimate on the fluid velocity close to rigid particles: for all $x\in\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}$ with $|x-X^n_{\varepsilon,N}|\le\frac12\,\mathrm{d}^{\min}_{\varepsilon,N}$ for some $1\le n\le N$, we have

$$\begin{aligned}|u_{\varepsilon,N}(x)| &\lesssim C_h+|e|\lambda\alpha^2_{\varepsilon,N}+\kappa_0\lambda\alpha^1_{\varepsilon,N}\\ &\qquad+\tfrac{\lambda}{N}\sum_{m:m\ne n}^N|X^n_{\varepsilon,N}-X^m_{\varepsilon,N}|^{1-d}\Big(\fint_{I^{m;+}_{\varepsilon,N}}|\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12},\\ |\nabla u_{\varepsilon,N}(x)| &\lesssim C_h+|e|\lambda\alpha^1_{\varepsilon,N}+\kappa_0\lambda\alpha^0_{\varepsilon,N}\\ &\qquad+\tfrac{\lambda}{N}\sum_{m:m\ne n}^N|X^n_{\varepsilon,N}-X^m_{\varepsilon,N}|^{-d}\Big(\fint_{I^{m;+}_{\varepsilon,N}}|\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12}.\end{aligned}\tag{3.6}$$

Let $x\in\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}$ be fixed with $|x-X^n_{\varepsilon,N}|\le\frac12\,\mathrm{d}^{\min}_{\varepsilon,N}$ for some $1\le n\le N$. Note that we then have $\operatorname{dist}(x,\{X^m_{\varepsilon,N}\}_{m:m\ne n})\ge\frac12\,\mathrm{d}^{\min}_{\varepsilon,N}$. Testing equation (3.2) in $\mathbb{R}^d\setminus I^n_{\varepsilon,N}$ with the Stokeslet $\mathcal{G}^n_{\varepsilon,N}(\cdot,x)$ for the Stokes problem with a single rigid inclusion, cf. Lemma 2.4, we find, instead of (3.3),

$$\begin{aligned}u_{\varepsilon,N}(x) &= \int_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}}\mathcal{G}^n_{\varepsilon,N}(\cdot,x)h+\sum_{m=1}^N\int_{I^m_{\varepsilon,N}}\mathcal{G}^n_{\varepsilon,N}(\cdot,x)e+\kappa_0\tfrac{\lambda}{\varepsilon N}\sum_{m=1}^N\int_{I^{m;+}_{\varepsilon,N}}\mathcal{G}^n_{\varepsilon,N}(\cdot,x)f^m_{\varepsilon,N}\\ &\qquad-\sum_{m:m\ne n}^N\int_{\partial I^m_{\varepsilon,N}}\Big(\mathcal{G}^n_{\varepsilon,N}(\cdot,x)-\fint_{I^m_{\varepsilon,N}}\mathcal{G}^n_{\varepsilon,N}(\cdot,x)\Big)\sigma(u_{\varepsilon,N},p_{\varepsilon,N})\nu.\end{aligned}$$

Now appealing to a trace estimate and to the pointwise bounds on $\mathcal{G}^n_{\varepsilon,N}$, cf. Lemmas 2.2 and 2.4, and noting that $|x-X^m_{\varepsilon,N}|\gtrsim|X^n_{\varepsilon,N}-X^m_{\varepsilon,N}|$ for all m, the claim (3.6) follows similarly as in Step 2. The only difference is that we now need to treat diagonal contributions separately: for instance,

$$\begin{aligned}\Big|\sum_{m=1}^N\int_{I^m_{\varepsilon,N}}\mathcal{G}^n_{\varepsilon,N}(\cdot,x)e\Big| &\lesssim |e|\tfrac{\lambda}{N}\fint_{I^n_{\varepsilon,N}}|\cdot-x|^{2-d}+|e|\tfrac{\lambda}{N}\sum_{m:m\ne n}^N\fint_{I^m_{\varepsilon,N}}|\cdot-x|^{2-d}\\ &\lesssim |e|\tfrac{\lambda}{N}\langle x-X^n_{\varepsilon,N}\rangle_\varepsilon^{2-d}+|e|\tfrac{\lambda}{N}\sum_{m:m\ne n}^N|x-X^m_{\varepsilon,N}|^{2-d}\\ &\lesssim |e|(\varepsilon^2+\lambda\alpha^2_{\varepsilon,N}),\end{aligned}$$

where we use the short-hand notation $\langle x\rangle_\varepsilon:=(\varepsilon^2+|x|^2)^{1/2}$, and where the last inequality follows from (3.5) with $\operatorname{dist}(x,\{X^m_{\varepsilon,N}\}_{m:m\ne n})\ge\frac12\,\mathrm{d}^{\min}_{\varepsilon,N}$.

Step 3. Closed estimate on velocity gradients: proof that for all $1\le n\le N$,

$$\Big(\fint_{I^{n;+}_{\varepsilon,N}}|\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12}\lesssim C_h+|e|\lambda\alpha^1_{\varepsilon,N}.\tag{3.7}$$

This estimate is obtained by post-processing the results of the first two steps in the dilute regime $\lambda \ll 1$. From (3.6), we deduce in particular, for all $1 \le n \le N$,

$$\Big(\fint_{I^{n;+}_{\varepsilon,N}} |\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12} \lesssim C_h + |e|\lambda\alpha^1_{\varepsilon,N} + \kappa_0\lambda\alpha^0_{\varepsilon,N} + \tfrac{\lambda}{N}\sum_{m:m\ne n}^N |X^n_{\varepsilon,N} - X^m_{\varepsilon,N}|^{-d}\Big(\fint_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12}. \tag{3.8}$$

After summation, recalling (1.29), this leads us to

$$\sup_{1\le n\le N}\bigg[\tfrac1N\sum_{m:m\ne n}^N |X^n_{\varepsilon,N} - X^m_{\varepsilon,N}|^{-d}\Big(\fint_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12}\bigg] \lesssim \alpha^0_{\varepsilon,N}\big(C_h + |e|\lambda\alpha^1_{\varepsilon,N} + \kappa_0\lambda\alpha^0_{\varepsilon,N}\big) + \lambda\alpha^0_{\varepsilon,N}\sup_{1\le n\le N}\bigg[\tfrac1N\sum_{m:m\ne n}^N |X^n_{\varepsilon,N} - X^m_{\varepsilon,N}|^{-d}\Big(\fint_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12}\bigg].$$

Provided that $\lambda\alpha^0_{\varepsilon,N} \ll 1$ is small enough, the last right-hand side term can be absorbed, to the effect of

$$\sup_{1\le n\le N}\bigg[\tfrac1N\sum_{m:m\ne n}^N |X^n_{\varepsilon,N} - X^m_{\varepsilon,N}|^{-d}\Big(\fint_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12}\bigg] \lesssim \alpha^0_{\varepsilon,N}\big(C_h + |e|\lambda\alpha^1_{\varepsilon,N} + \kappa_0\lambda\alpha^0_{\varepsilon,N}\big).$$

Inserting this back into (3.8) and using $\kappa_0\lambda\alpha^0_{\varepsilon,N} \le 1$, the claim (3.7) follows.

Step 4. Conclusion.
Inserting (3.7) into (3.1) and (3.6), using again (3.5), using Hölder's inequality in form of

$$(\alpha^1_{\varepsilon,N})^2 \le \alpha^0_{\varepsilon,N}\alpha^2_{\varepsilon,N}, \qquad \alpha^1_{\varepsilon,N} \le (\alpha^0_{\varepsilon,N})^{\frac{d-1}{d}}, \qquad \alpha^2_{\varepsilon,N} \le (\alpha^1_{\varepsilon,N})^{\frac{d-2}{d-1}} \le (\alpha^0_{\varepsilon,N})^{\frac{d-2}{d}}, \tag{3.9}$$

and using $\lambda\alpha^0_{\varepsilon,N} \le 1$, we deduce for all $x \in \mathbb{R}^d \setminus \mathcal{I}_{\varepsilon,N}$,

$$|u_{\varepsilon,N}(x)| + |\nabla u_{\varepsilon,N}(x)| \lesssim_h 1,$$

which is the conclusion. □

3.2. **Proof of Corollary 3.2.** As long as particles are separated by a positive distance, the particle dynamics (1.9)–(1.12) is well-posed, see e.g. [25, Appendix A]. It remains to control interparticle distances. Given $\theta > \theta_0$, consider the maximal time

$$T^\theta_{\varepsilon,N} := \sup\Big\{t \ge 0 \;:\; \mathrm{d}^{\min}_{\varepsilon,N}(s) \ge (\theta N)^{-\frac1d},\ \rho^{\max}_{\varepsilon,N}(s) \le \theta \ \text{ for all } 0 \le s \le t\Big\}. \tag{3.10}$$

We split the proof into two steps.

Step 1. Preliminary estimate: for all $\sigma \in (0,d]$,

$$\begin{aligned} \alpha^\sigma_{\varepsilon,N} &\lesssim \sigma^{-1}(N^{\frac1d}\,\mathrm{d}^{\min}_{\varepsilon,N})^{-d}(\rho^{\max}_{\varepsilon,N})^\sigma, \\ \alpha^0_{\varepsilon,N} &\lesssim (N^{\frac1d}\,\mathrm{d}^{\min}_{\varepsilon,N})^{-d}\log\big(2 + \rho^{\max}_{\varepsilon,N} + (\mathrm{d}^{\min}_{\varepsilon,N})^{-1}\big). \end{aligned} \tag{3.11}$$

We start from

$$\tfrac{1}{N}\sum_{m:m\neq n}^{N}|X_{\varepsilon,N}^n-X_{\varepsilon,N}^m|^{\sigma-d}\lesssim\tfrac{1}{N}\sum_{m:m\neq n}^{N}\fint_{B(X_{\varepsilon,N}^m,\frac12\mathrm{d}_{\varepsilon,N}^{\min})}|\cdot-X_{\varepsilon,N}^n|^{\sigma-d}.$$

By definition, we have $\{X_{\varepsilon,N}^m\}_m\subset B(0,\rho_{\varepsilon,N}^{\max})$ and the balls $\{B(X_{\varepsilon,N}^m,\frac12\mathrm{d}_{\varepsilon,N}^{\min})\}_m$ are pairwise disjoint. For $\sigma\in(0,d]$, we then get

$$\begin{aligned}\tfrac{1}{N}\sum_{m:m\neq n}^{N}|X_{\varepsilon,N}^n-X_{\varepsilon,N}^m|^{\sigma-d}&\lesssim\tfrac{1}{N}(\mathrm{d}_{\varepsilon,N}^{\min})^{-d}\int_{B(0,2\rho_{\varepsilon,N}^{\max})}|\cdot|^{\sigma-d}\\&\lesssim\sigma^{-1}(N^{\frac1d}\mathrm{d}_{\varepsilon,N}^{\min})^{-d}(\rho_{\varepsilon,N}^{\max})^{\sigma},\end{aligned}$$

while for $\sigma=0$ we can bound

$$\begin{aligned}\tfrac{1}{N}\sum_{m:m\neq n}^{N}|X_{\varepsilon,N}^n-X_{\varepsilon,N}^m|^{-d}&\lesssim\tfrac{1}{N}(\mathrm{d}_{\varepsilon,N}^{\min})^{-d}\int_{\{y\in B(0,2\rho_{\varepsilon,N}^{\max}):|y|\ge\frac12\mathrm{d}_{\varepsilon,N}^{\min}\}}|y|^{-d}\,dy\\&\lesssim(N^{\frac1d}\mathrm{d}_{\varepsilon,N}^{\min})^{-d}\log\big(2+\rho_{\varepsilon,N}^{\max}+(\mathrm{d}_{\varepsilon,N}^{\min})^{-1}\big),\end{aligned}$$

and the claim (3.11) follows.

Step 2. Conclusion.
For $t\in[0,T_{\varepsilon,N}^\theta]$, we deduce from (3.11), for all $\sigma\in(0,d]$,

$$\alpha_{\varepsilon,N}^\sigma\lesssim\sigma^{-1}\theta^{1+\sigma},\qquad\alpha_{\varepsilon,N}^0\lesssim\theta\log(\theta N).$$

Provided that $(\theta N)^{-1/d}\ge4\varepsilon$ and that $\lambda\theta\log(\theta N)\ll1$ is small enough, we may then appeal to the Lipschitz bounds of Proposition 3.1, to the effect of

$$\tfrac{d}{dt}\mathrm{d}_{\varepsilon,N}^{\min}(t)\ge-C_h\,\mathrm{d}_{\varepsilon,N}^{\min}(t),\qquad\tfrac{d}{dt}\rho_{\varepsilon,N}^{\max}(t)\le C_h,\tag{3.12}$$

which entails for all $t\in[0,T_{\varepsilon,N}^\theta]$,

$$\mathrm{d}_{\varepsilon,N}^{\min}(t)\ge\mathrm{d}_{\varepsilon,N}^{\min}(0)e^{-tC_h},\qquad\rho_{\varepsilon,N}^{\max}(t)\le\rho_{\varepsilon,N}^{\max}(0)+tC_h.$$

By the initial assumptions, we deduce in particular $T_{\varepsilon,N}^\theta\ge C_h^{-1}\log(\theta/\theta_0)$, and the conclusion follows. ☐

4. Dilute expansion of particle velocities

This section is devoted to the following dilute expansion of particle velocities. In case of spherical passive particles, a corresponding expansion of translational velocities was already obtained in [29]. While previous contributions on the topic were based on the reflection method [30, 25, 32, 29], we take a different path inspired by our recent work with Gloria [12]: our dilute expansion is obtained instead by means of a cluster expansion combined with a monopole approximation. The main advantage of this method is to provide a systematic way to pursue the expansion to higher orders. The proof is split into several parts and is concluded by combining Lemmas 4.4 and 4.6 below.

Proposition 4.1 (Cluster expansion of particle velocities)**.** *Assume that* $\mathrm{d}_{\varepsilon,N}^{\min}\ge4\varepsilon$*, and that* $\lambda\alpha_{\varepsilon,N}^0\ll1$ *is small enough. Particle velocities* (1.12) *can then be expanded as follows, for all* $1\le n\le N$*,*

— *First-order expansion of translational velocities:*

$$|V^n_{\varepsilon,N} - (\mathcal{G} * h)(X^n_{\varepsilon,N})| \lesssim_h \lambda(\alpha^1_{\varepsilon,N} + 1) + \varepsilon;$$

— *Second-order expansion of translational velocities:*

$$\Big| V^n_{\varepsilon,N} - \big[\mathcal{G} \hat{*} \big((1-\lambda\mu_{\varepsilon,N})h + \lambda\mu_{\varepsilon,N} e\big)\big](X^n_{\varepsilon,N}) \\ - \lambda\big[\nabla\mathcal{G} \hat{*} \big(2\langle\Sigma^\circ\nu_{\varepsilon,N}\rangle \mathrm{D}(\mathcal{G} * h) + \kappa_0\langle\Sigma^\circ_f \nu_{\varepsilon,N}\rangle\big)\big](X^n_{\varepsilon,N}) - \kappa_0 \varepsilon V^\circ_f(R^n_{\varepsilon,N})\Big| \\ \lesssim_h (\lambda\alpha^1_{\varepsilon,N} + \varepsilon)\big(\lambda(\alpha^0_{\varepsilon,N} + 1) + \varepsilon\big);$$

— *First-order expansion of angular velocities:*

$$\big|\Omega^n_{\varepsilon,N} - \Omega^\circ(R^n_{\varepsilon,N})\nabla(\mathcal{G} * h)(X^n_{\varepsilon,N})\big| \lesssim_h \lambda(\alpha^0_{\varepsilon,N} + 1) + \varepsilon;$$

where we recall the short-hand notation $g \hat{*} \mu(x) = \int_{\mathbb{R}^d\setminus\{x\}} g(x-y)d\mu(y)$ *for diagonal-free convolutions, and where* $\Sigma^\circ, \Omega^\circ, \Sigma^\circ_f, V^\circ_f$ *are defined in (1.17)–(1.25). Moreover, away from the particles, we have the following expansion for the fluid velocity, for any* $\delta \in [\varepsilon, 1]$*,*

$$\Big\| u_{\varepsilon,N} - \mathcal{G} * \big((1-\lambda\mu_{\varepsilon,N})h + \lambda\mu_{\varepsilon,N} e\big) \\ - \lambda\nabla\mathcal{G} * \big(2\langle\Sigma^\circ\nu_{\varepsilon,N}\rangle \mathrm{D}(\mathcal{G} * h) + \kappa_0\langle\Sigma^\circ_f \nu_{\varepsilon,N}\rangle\big)\Big\|_{\mathrm{L}^\infty(\mathbb{R}^d\setminus(\mathcal{I}_{\varepsilon,N}+\delta B))} \\ \lesssim_h \big(\lambda\alpha^1_{\varepsilon,N} + \varepsilon\big)\big(\lambda(\alpha^0_{\varepsilon,N} + 1) + \varepsilon + (\tfrac{\varepsilon}{\delta})^d\big). \qquad \Diamond$$

Remark 4.2 (Higher orders and failure of mean-field theory)**.** These dilute expansions can be easily pursued to higher orders. For instance, the next-order expansion of the translational velocity $V^n_{\varepsilon,N}$ involves in particular the three-body contribution

$$\lambda^2\Big[\nabla\mathcal{G} \hat{*} \Big(2\langle\Sigma^\circ\nu_{\varepsilon,N}\rangle \mathrm{D}\big(\nabla\mathcal{G} \hat{*} (2\langle\Sigma^\circ\nu_{\varepsilon,N}\rangle \mathrm{D}(\mathcal{G} * h))\big)\Big)\Big](X^n_{\varepsilon,N}), \tag{4.1}$$

which physically describes the flow disturbance due to a stress difference at a particle boundary generated by the flow disturbance due to a stress difference at another particle boundary. Yet, although the expansion can be pursued to higher orders, a mean-field approximation cannot be justified to higher accuracy. Indeed, the above three-body contribution (4.1) involves particle interactions via the kernel $\nabla^2\mathcal{G}$, which is a Calderón–Zygmund kernel, with critical decay $|\nabla^2\mathcal{G}(x)| \simeq |x|^{-d}$. The criticality of this kernel is known to imply the failure of mean-field theory, see e.g. [33]: by scaling, any macroscopic limit should actually depend on the microscopic arrangement of the particles. This importance of the microscopic geometry was recently illustrated in [28]. More precisely, the next-order semi-dilute correction to the mean-field approximation would require to capture the statistical distribution of pairs of particles on the microscale — as was indeed anticipated in the introduction, in link with corrections to Einstein's formula (1.1), cf. [12, 17]. This leads us far beyond the scope of propagation of chaos and mean-field theory: even a formal description remains unclear and is left open for future work. (We emphasize that it is not about the two-particle *macroscopic* density, which commonly appears when describing corrections to mean field, e.g. [10]: instead, it is here about the distribution of pairs of particles *on the microscale*.) $\Diamond$

4.1. **Cluster expansion.** In view of (1.12), Proposition 4.1 amounts to establishing a dilute expansion for the fluid velocity $u_{\varepsilon,N}$. The latter is defined as the solution of the Stokes problem (1.9), which, in view of its weak formulation (1.10), is equivalent to setting

$$u_{\varepsilon,N} \;=\; \pi_{\varepsilon,N} v_{\varepsilon,N},$$

where $\pi_{\varepsilon,N}$ is the orthogonal projection

$$\pi_{\varepsilon,N} \;:\; \big\{u\in \dot H^1(\mathbb R^d)^d \,:\, \operatorname{div}(u)=0\big\} \;\twoheadrightarrow\; \big\{u\in \dot H^1(\mathbb R^d)^d \,:\, \operatorname{div}(u)=0,\ \operatorname{D}(u)|_{\mathcal I_{\varepsilon,N}}=0\big\},$$

and where $v_{\varepsilon,N}$ is the solution of

$$\begin{cases} -\triangle v_{\varepsilon,N}+\nabla q_{\varepsilon,N}=h\mathbb 1_{\mathbb R^d\setminus\mathcal I_{\varepsilon,N}}+e\mathbb 1_{\mathcal I_{\varepsilon,N}}+\kappa_0\frac{\lambda}{\varepsilon N}\sum_{n=1}^N f^n_{\varepsilon,N}, & \text{in } \mathbb R^d,\\ \operatorname{div}(v_{\varepsilon,N})=0, & \text{in } \mathbb R^d.\end{cases} \tag{4.2}$$

This decomposition is particularly useful as $v_{\varepsilon,N}$ depends linearly on the set of particles: this simplifies the hydodynamic problem to pairwise interactions between the particles, while all multibody effects are contained in the projection $\pi_{\varepsilon,N}$. The dilute expansion of $u_{\varepsilon,N}$ then amounts to expanding $\pi_{\varepsilon,N}$. For that purpose, rather than appealing to the method of reflections, as was done in previous work on the topic, e.g. [30, 25, 32, 26], we start from its cluster expansion as inspired by our work with Gloria [13, 12, 15].

In a nutshell, the cluster expansion of $\pi_{\varepsilon,N}$ amounts to decomposing multibody effects into a series of contributions involving subsets of particles of increasing cardinality. We start with some notation. For any index subset $J\subset\{1,\ldots,N\}$, we define the orthogonal projection

$$\pi^J_{\varepsilon,N} \;:\; \big\{u\in \dot H^1(\mathbb R^d)^d : \operatorname{div}(u)=0\big\} \;\twoheadrightarrow\; \big\{u\in \dot H^1(\mathbb R^d)^d : \operatorname{div}(u)=0,\ \operatorname{D}(u)|_{\cup_{n\in J}I^n_{\varepsilon,N}}=0\big\},$$

that is, the partial projection only taking into account particles with indices in J. Note that by definition we have in particular $\pi^{\varnothing}_{\varepsilon,N}=\operatorname{Id}$ and $\pi^{\{1,\ldots,N\}}_{\varepsilon,N}=\pi_{\varepsilon,N}$. Next, we consider differences of partial projections,

$$\delta^{\{n\}}\pi^J_{\varepsilon,N} \;:=\; \pi^{J\cup\{n\}}_{\varepsilon,N}-\pi^J_{\varepsilon,N},\qquad 1\le n\le N,$$

as well as higher-order differences,

$$\delta^K\pi^J_{\varepsilon,N} \;:=\; \sum_{I\subset K}(-1)^{\sharp(K\setminus I)}\pi^{J\cup I}_{\varepsilon,N},\qquad K\subset\{1,\ldots,N\}.$$

Note that by definition we have

$$\delta^{\varnothing}\pi^J_{\varepsilon,N}=\pi^J_{\varepsilon,N},\qquad\text{and}\qquad \delta^K\pi^J_{\varepsilon,N}=0\quad\text{whenever } K\cap J\ne\varnothing.$$

In these terms, the cluster expansion of $u_{\varepsilon,N}=\pi_{\varepsilon,N}v_{\varepsilon,N}$ takes on the following guise, where the kth term $u^{(k)}_{\varepsilon,N}$ describes the contribution of k-body interactions.

Lemma 4.3 (Cluster expansion)**.** *The following identity holds,*

$$u_{\varepsilon,N} \;=\; \sum_{k=0}^N u^{(k)}_{\varepsilon,N}, \tag{4.3}$$

where we have define $u^{(0)}_{\varepsilon,N}:=v_{\varepsilon,N}$ *and for all* $k\ge1$*,*

$$u^{(k)}_{\varepsilon,N} \;:=\; \sum_{1\le n_1<\ldots<n_k\le N}(\delta^{\{n_1,\ldots,n_k\}}\pi^{\varnothing}_{\varepsilon,N})v_{\varepsilon,N}. \tag{4.4}$$

◊

Proof. Identity (4.3) follows from the simple observation that

$$\begin{aligned}\sum_{k=0}^{N}\sum_{1\le n_1<\ldots<n_k\le N}\delta^{\{n_1,\ldots,n_k\}}\pi^{\varnothing}_{\varepsilon,N} &= \sum_{K\subset\{1,\ldots,N\}}\delta^{K}\pi^{\varnothing}_{\varepsilon,N}\\ &= \sum_{K\subset\{1,\ldots,N\}}\sum_{I\subset K}(-1)^{\sharp(K\setminus I)}\pi^{I}_{\varepsilon,N}\\ &= \sum_{I\subset\{1,\ldots,N\}}\pi^{I}_{\varepsilon,N}\sum_{k=0}^{N-\sharp I}(-1)^k\binom{N-\sharp I}{k}\\ &= \pi^{\{1,\ldots,N\}}_{\varepsilon,N} \;=\; \pi_{\varepsilon,N},\end{aligned}$$

where we used the fact that $\sum_{k=0}^{K}(-1)^k\binom{K}{k}=\mathbb{1}_{K=0}$. □

4.2. **Cluster expansion errors.** We turn to the accuracy of the cluster expansion (4.3) upon truncation in the dilute regime $\lambda\ll 1$. For the purpose of this work, we restrict ourselves to the second-order expansion, but higher orders can be dealt with analogously, cf. Remark 4.5.

Lemma 4.4. *Assume that $\mathrm{d}^{\min}_{\varepsilon,N}\ge 4\varepsilon$ and that $\lambda\alpha^0_{\varepsilon,N}\ll 1$ is small enough. Then we have*

$$\|u_{\varepsilon,N}-u^{(0)}_{\varepsilon,N}\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \;\lesssim_h\; \lambda\alpha^1_{\varepsilon,N}+\varepsilon, \tag{4.5}$$

$$\|u_{\varepsilon,N}-u^{(0)}_{\varepsilon,N}-u^{(1)}_{\varepsilon,N}\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \;\lesssim_h\; \lambda\alpha^0_{\varepsilon,N}(\lambda\alpha^1_{\varepsilon,N}+\varepsilon), \tag{4.6}$$

and moreover, for velocity gradients, for all $1\le n\le N$,

$$\big\|\nabla u_{\varepsilon,N}-\nabla\pi^{\{n\}}_{\varepsilon,N}u^{(0)}_{\varepsilon,N}\big\|_{\mathrm{L}^\infty(B(X^n_{\varepsilon,N},\frac12\mathrm{d}^{\min}_{\varepsilon,N}))} \lesssim_h \lambda\alpha^0_{\varepsilon,N}. \tag{4.7}$$

◊

Remark 4.5. The proof below can be immediately generalized to higher orders: we can show for all $1\le K<N$,

$$\Big\|u_{\varepsilon,N}-\sum_{k=0}^{K}u^{(k)}_{\varepsilon,N}\Big\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \lesssim_{h,K} (\lambda\alpha^0_{\varepsilon,N})^K(\lambda\alpha^1_{\varepsilon,N}+\varepsilon),$$

and moreover, for velocity gradients, for all $1\le n\le N$,

$$\Big\|\nabla u_{\varepsilon,N}-\nabla\pi^{\{n\}}_{\varepsilon,N}\sum_{k=0}^{K}u^{(k)}_{\varepsilon,N}\Big\|_{\mathrm{L}^\infty(B(X^n_{\varepsilon,N},\frac12\mathrm{d}^{\min}_{\varepsilon,N}))} \lesssim_{h,K} (\lambda\alpha^0_{\varepsilon,N})^{K+1}.$$

As this will not be used in this work, we omit the detail for shortness. ◊

Proof of Lemma 4.4. We split the proof into three steps, separately proving the different estimates in the statement.

Step 1. First-order expansion of $u_{\varepsilon,N}$: proof of (4.5).

Subtracting (3.2) from the defining equation for $u^{(0)}_{\varepsilon,N}=v_{\varepsilon,N}$, cf. (4.2), we get the following equation in $\mathbb{R}^d$,

$$-\triangle(u_{\varepsilon,N}-u^{(0)}_{\varepsilon,N})+\nabla p \;=\; -\sum_{n=1}^{N}\Big(e\mathbb{1}_{I^n_{\varepsilon,N}}+\kappa_0\tfrac{\lambda}{\varepsilon N}f^n_{\varepsilon,N}\mathbb{1}_{I^n_{\varepsilon,N}}+\delta_{\partial I^n_{\varepsilon,N}}\sigma(u_{\varepsilon,N},p_{\varepsilon,N})\nu\Big). \tag{4.8}$$

In terms of the Stokeslet $\mathcal{G}$, using the boundary conditions for $u_{\varepsilon,N}$, we deduce

$$(u_{\varepsilon,N}-u^{(0)}_{\varepsilon,N})(x)\,=\,-\sum_{n=1}^N\int_{\partial I^n_{\varepsilon,N}}\Big(\mathcal{G}(\cdot-x)-\fint_{I^n_{\varepsilon,N}}\mathcal{G}(\cdot-x)\Big)\sigma(u_{\varepsilon,N},p_{\varepsilon,N})\nu.$$

Appealing to a trace estimate and to pointwise bounds on $\mathcal{G}$, cf. Lemmas 2.2 and 2.3, and evalutating local integrals, this leads us to

$$|(u_{\varepsilon,N}-u^{(0)}_{\varepsilon,N})(x)|\,\lesssim\,\tfrac{\lambda}{N}\sum_{n=1}^N\langle x-X^n_{\varepsilon,N}\rangle_\varepsilon^{1-d}\Big(\fint_{I^{n;+}_{\varepsilon,N}}|\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12},$$

where we recall the short-hand notation $\langle x\rangle_\varepsilon=(\varepsilon^2+|x|^2)^{1/2}$. Now note that (3.5) yields for all $x\in\mathbb{R}^d$, after separating the diagonal contribution,

$$\tfrac{\lambda}{N}\sum_{n=1}^N\langle x-X^n_{\varepsilon,N}\rangle_\varepsilon^{1-d}\,\lesssim\,\lambda\alpha^1_{\varepsilon,N}+\varepsilon. \tag{4.9}$$

Inserting this into the above, and combining with the Lipschitz estimate of Proposition 3.1, we deduce for all $x\in\mathbb{R}^d$,

$$|(u_{\varepsilon,N}-u^{(0)}_{\varepsilon,N})(x)|\,\lesssim_h\,\lambda\alpha^1_{\varepsilon,N}+\varepsilon,$$

and the conclusion (4.5) follows.

Step 2. Second-order expansion of $u_{\varepsilon,N}$: proof of (4.6).
We use the short-hand notation $u^n_{\varepsilon,N}:=\pi^{\{n\}}_{\varepsilon,N}v_{\varepsilon,N}$ and we denote by $p^n_{\varepsilon,N}$ the associated pressure field in $\mathbb{R}^d\setminus I^n_{\varepsilon,N}$. Comparing (4.8) with the corresponding equation for

$$u^{(1)}_{\varepsilon,N}\,=\,\sum_{n=1}^N(u^n_{\varepsilon,N}-v_{\varepsilon,N}),$$

we obtain the following in $\mathbb{R}^d$,

$$-\triangle\big(u_{\varepsilon,N}-u^{(0)}_{\varepsilon,N}-u^{(1)}_{\varepsilon,N}\big)+\nabla p\,=\,-\sum_{n=1}^N\mathbb{1}_{\partial I^n_{\varepsilon,N}}\sigma\big(u_{\varepsilon,N}-u^n_{\varepsilon,N},p_{\varepsilon,N}-p^n_{\varepsilon,N}\big)\nu.$$

In terms of the Stokeslet $\mathcal{G}$, using boundary conditions, and appealing to a trace estimate and to pointwise bounds, we deduce as in Step 1,

$$\big|\big(u_{\varepsilon,N}-u^{(0)}_{\varepsilon,N}-u^{(1)}_{\varepsilon,N}\big)(x)\big|\,\lesssim\,\tfrac{\lambda}{N}\sum_{n=1}^N\langle x-X^n_{\varepsilon,N}\rangle_\varepsilon^{1-d}\Big(\fint_{I^{n;+}_{\varepsilon,N}}|\mathrm{D}(u_{\varepsilon,N}-u^n_{\varepsilon,N})|^2\Big)^{\frac12}. \tag{4.10}$$

It remains to estimate the last factor. For that purpose, given $1\le n\le N$, we start by noting that the difference $u_{\varepsilon,N}-u^n_{\varepsilon,N}$ satisfies the following equation in $\mathbb{R}^d\setminus I^n_{\varepsilon,N}$,

$$-\triangle\big(u_{\varepsilon,N}-u^n_{\varepsilon,N}\big)+\nabla p\,=\,-\sum_{m:m\ne n}^N\Big(e\mathbb{1}_{I^m_{\varepsilon,N}}+\kappa_0\tfrac{\lambda}{\varepsilon N}f^m_{\varepsilon,N}\mathbb{1}_{I^m_{\varepsilon,N}}+\delta_{\partial I^m_{\varepsilon,N}}\sigma(u_{\varepsilon,N},p_{\varepsilon,N})\nu\Big).$$

Testing this with the Stokeslet $\mathcal{G}^n_{\varepsilon,N}(\cdot,x)$ corresponding to the problem with a single rigid inclusion at $I^n_{\varepsilon,N}$, cf. Lemma 2.4, and using boundary conditions, we get in $\mathbb{R}^d\setminus I^n_{\varepsilon,N}$,

$$(u_{\varepsilon,N}-u^n_{\varepsilon,N})(x) \,=\, -\sum_{m:m\neq n}^N \int_{\partial I^m_{\varepsilon,N}} \Big(\mathcal{G}^n_{\varepsilon,N}(\cdot,x)-\fint_{I^m_{\varepsilon,N}}\mathcal{G}^n_{\varepsilon,N}(\cdot,x)\Big)\,\sigma(u_{\varepsilon,N},p_{\varepsilon,N})\nu, \quad (4.11)$$

hence, appealing to a trace estimate and to the pointwise bounds on $\mathcal{G}^n_{\varepsilon,N}$, cf. Lemmas 2.2 and 2.4,

$$\Big(\fint_{I^n_{\varepsilon,N}}|\mathrm{D}(u_{\varepsilon,N}-u^n_{\varepsilon,N})|^2\Big)^{\frac12}\,\lesssim\,\tfrac{\lambda}{N}\sum_{m:m\neq n}^N|X^n_{\varepsilon,N}-X^m_{\varepsilon,N}|^{-d}\Big(\fint_{I^{m;+}_{\varepsilon,N}}|\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12}.$$

Inserting this into (4.10), combining with the Lipschitz estimate of Proposition 3.1, and using (4.9) again, we get for all $x\in\mathbb{R}^d$,

$$\big|(u_{\varepsilon,N}-u^{(0)}_{\varepsilon,N}-u^{(1)}_{\varepsilon,N})(x)\big|\,\lesssim_h\,\lambda\alpha^0_{\varepsilon,N}(\lambda\alpha^1_{\varepsilon,N}+\varepsilon),$$

and the conclusion (4.6) follows.

Step 3. First-order expansion of $\nabla u_{\varepsilon,N}$: proof of (4.7).
Let $x\in\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}$ be fixed with $|x-X^n_{\varepsilon,N}|\le\frac12\mathrm{d}^{\min}_{\varepsilon,N}$ for some $1\le n\le N$. Starting from (4.11) and appealing again to a trace estimate and to the pointwise bounds on $\mathcal{G}^n_{\varepsilon,N}$, cf. Lemmas 2.2 and 2.4, we get

$$|\nabla(u_{\varepsilon,N}-u^n_{\varepsilon,N})(x)|\,\lesssim\,\tfrac{\lambda}{N}\sum_{m:m\neq n}^N|X^n_{\varepsilon,N}-X^m_{\varepsilon,N}|^{-d}\Big(\fint_{I^{m;+}_{\varepsilon,N}}|\mathrm{D}(u_{\varepsilon,N})|^2\Big)^{\frac12},$$

and the conclusion (4.7) then follows as above. □

4.3. **Analysis of cluster contributions.** In order to conclude the proof of Proposition 4.1 and to obtain the desired asymptotics for particle velocities (1.12), we build on the cluster estimates of Lemma 4.4 by further performing a multipole expansion of the cluster terms in the limit of small particles $\varepsilon\ll1$. For the purpose of this work, we restrict ourselves to the first two cluster terms and to their monopole approximation, but the description of higher-order cluster terms and their full multipole expansion could be pursued analogously. The conclusion of Proposition 4.1 directly follows by combining the following result with Lemma 4.4, further using (3.9) to slightly simplify the bounds.

Lemma 4.6 (Monopole approximation of cluster contributions)**.** *Assume that* $\mathrm{d}^{\min}_{\varepsilon,N}\ge4\varepsilon$ *and that* $\lambda\alpha^0_{\varepsilon,N}\ll1$ *is small enough. Then we have*

$$\Big|\fint_{I^n_{\varepsilon,N}}u^{(0)}_{\varepsilon,N}-(\mathcal{G}*h)(X^n_{\varepsilon,N})\Big|\,\lesssim_h\,\lambda(\alpha^2_{\varepsilon,N}+\kappa_0\alpha^1_{\varepsilon,N})+\varepsilon(\varepsilon+\kappa_0), \quad (4.12)$$

$$\Big|\fint_{I^n_{\varepsilon,N}}\nabla\pi^{\{n\}}_{\varepsilon,N}u^{(0)}_{\varepsilon,N}-\Omega^\circ(R^n_{\varepsilon,N})\nabla(\mathcal{G}*h)(X^n_{\varepsilon,N})\Big|\,\lesssim_h\,\lambda(\alpha^1_{\varepsilon,N}+\kappa_0\alpha^0_{\varepsilon,N})+\varepsilon, \quad (4.13)$$

and moreover,

$$\Big|\fint_{I^n_{\varepsilon,N}}(u^{(0)}_{\varepsilon,N}+u^{(1)}_{\varepsilon,N})-\big[\mathcal{G}\,\hat{*}\,\big((1-\lambda\mu_{\varepsilon,N})h+\lambda\mu_{\varepsilon,N}e\big)\big](X^n_{\varepsilon,N})$$

$$- \lambda\big[\nabla\mathcal{G} \hat{*} \big(2\langle\Sigma^\circ\nu_{\varepsilon,N}\rangle\mathrm{D}(\mathcal{G}*h)+\kappa_0\langle\Sigma_f^\circ\nu_{\varepsilon,N}\rangle\big)\big](X^n_{\varepsilon,N})-\kappa_0\varepsilon V_f^\circ(R^n_{\varepsilon,N})\Big|$$
$$\lesssim_h\ (\lambda\alpha^1_{\varepsilon,N}+\varepsilon)\big(\lambda(\alpha^2_{\varepsilon,N}+\alpha^1_{\varepsilon,N}+\alpha^0_{\varepsilon,N})+\varepsilon\big). \quad (4.14)$$

where we recall that $\Sigma^\circ,\Omega^\circ,\Sigma^\circ_f,V^\circ_f$ *are defined in* (1.17)–(1.25)*. Moreover, away from the particles, the fluid velocity satisfies for any* $\delta\in[\varepsilon,1]$*,*

$$\|u^{(0)}_{\varepsilon,N}-\mathcal{G}*h\|_{\mathrm{L}^\infty(\mathbb{R}^d\setminus(\mathcal{I}_{\varepsilon,N}+\delta B))}\lesssim_h\ \lambda(\alpha^2_{\varepsilon,N}+\kappa_0\alpha^1_{\varepsilon,N})+\delta(\delta+\kappa_0)(\tfrac{\varepsilon}{\delta})^d. \quad (4.15)$$

and

$$\Big\|\big(u^{(0)}_{\varepsilon,N}+u^{(1)}_{\varepsilon,N}\big)-\mathcal{G}*\big((1-\lambda\mu_{\varepsilon,N})h+\lambda\mu_{\varepsilon,N}e\big) \quad (4.16)$$
$$-\lambda\nabla\mathcal{G}*\big(2\langle\Sigma^\circ\nu_{\varepsilon,N}\rangle\mathrm{D}(\mathcal{G}*h)+\kappa_0\langle\Sigma^\circ_f\nu_{\varepsilon,N}\rangle\big)\Big\|_{\mathrm{L}^\infty(\mathbb{R}^d\setminus(\mathcal{I}_{\varepsilon,N}+\delta B))}$$
$$\lesssim_h\ (\lambda\alpha^1_{\varepsilon,N}+\varepsilon)\big(\lambda(\alpha^2_{\varepsilon,N}+\alpha^1_{\varepsilon,N}+\alpha^0_{\varepsilon,N})+\varepsilon+(\tfrac{\varepsilon}{\delta})^d\big). \quad \lozenge$$

Proof. We split the proof into five steps.

Step 1. First-order cluster contribution: proof of (4.12).
Recall that $u^{(0)}_{\varepsilon,N}=v_{\varepsilon,N}$ is defined by equation (4.2). In terms of the Stokeslet $\mathcal{G}$, it can be written as

$$\begin{aligned} v_{\varepsilon,N}(x) &= (\mathcal{G}*(h\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}}))(x)+\sum_{m=1}^N\bigg(\int_{I^m_{\varepsilon,N}}\mathcal{G}(x-\cdot)e+\kappa_0\tfrac{\lambda}{\varepsilon N}\int_{I^{m;+}_{\varepsilon,N}}\mathcal{G}(x-\cdot)f^m_{\varepsilon,N}\bigg)\\ &= (\mathcal{G}*h)(x)+\sum_{m=1}^N\int_{I^m_{\varepsilon,N}}\mathcal{G}(x-\cdot)(e-h)+\kappa_0\tfrac{\lambda}{\varepsilon N}\sum_{m=1}^N\int_{I^{m;+}_{\varepsilon,N}}\mathcal{G}(x-\cdot)f^m_{\varepsilon,N}. \quad (4.17)\end{aligned}$$

We average this over $x\in I^n_{\varepsilon,N}$ for some $1\le n\le N$ and it then remains to compute the local integrals. For the first right-hand side term, using a second-order Taylor expansion, the pointwise bounds on $\mathcal{G}$, cf. Lemma 2.3, and recalling $\int_{I^n_{\varepsilon,N}}(x-X^n_{\varepsilon,N})\,dx=0$, we find

$$\Big|\fint_{I^n_{\varepsilon,N}}\mathcal{G}*h-(\mathcal{G}*h)(X^n_{\varepsilon,N})\Big|\lesssim_h\ \varepsilon^2.$$

For the remaining terms, separating the diagonal contributions, and recalling the notation (1.29), the pointwise bounds on $\mathcal{G}$ directly yield

$$\begin{aligned}&\sum_{m=1}^N\fint_{I^n_{\varepsilon,N}}\Big|\int_{I^m_{\varepsilon,N}}\mathcal{G}(x-\cdot)(e-h)\Big|\,dx\\ &\quad\lesssim_h\ \varepsilon^d\sum_{m:m\ne n}|X^n_{\varepsilon,N}-X^m_{\varepsilon,N}|^{2-d}+\fint_{I^n_{\varepsilon,N}}\Big(\int_{I^n_{\varepsilon,N}}|x-\cdot|^{2-d}\Big)dx\\ &\quad\lesssim\ \lambda\alpha^2_{\varepsilon,N}+\varepsilon^2,\end{aligned}$$

and similarly, further recalling the definition $f^m_{\varepsilon,N}=\varepsilon^{-d}f(\frac1\varepsilon(\cdot-X^m_{\varepsilon,N}),R^m_{\varepsilon,N})$ and the balance of forces (1.4),

$$\tfrac{\lambda}{\varepsilon N}\sum_{m=1}^N\fint_{I^n_{\varepsilon,N}}\Big|\int_{I^{m;+}_{\varepsilon,N}}\mathcal{G}(x-\cdot)f^m_{\varepsilon,N}\Big|\,dx$$

$$\begin{aligned} &= \tfrac{\lambda}{\varepsilon N}\sum_{m=1}^{N} \fint_{I^n_{\varepsilon,N}} \Big| \int_{I^{m;+}_{\varepsilon,N}} \Big(\mathcal{G}(x-\cdot) - \fint_{I^{m;+}_{\varepsilon,N}} \mathcal{G}(x-\cdot)\Big) f^m_{\varepsilon,N} \Big| \, dx \\ &\lesssim \varepsilon^d \sum_{m:m\neq n}^{N} |X^n_{\varepsilon,N}-X^m_{\varepsilon,N}|^{1-d} + \fint_{I^n_{\varepsilon,N}} \Big(\int_{I^{n;+}_{\varepsilon,N}} |x-\cdot|^{1-d}\Big)\, dx \\ &\lesssim \lambda\alpha^1_{\varepsilon,N} + \varepsilon. \end{aligned}$$

Inserting these different estimates into (4.17), the conclusion (4.12) follows.

Step 2. Detailed estimates on the linear proxy $u^{(0)}_{\varepsilon,N} = v_{\varepsilon,N}$: for all $1 \le n \le N$, we have the following version of (3.7),

$$\Big(\fint_{I^{n;+}_{\varepsilon,N}} |\nabla v_{\varepsilon,N}|^2 \Big)^{\frac12} \lesssim_h 1, \tag{4.18}$$

and similarly for the next-order derivative,

$$\Big(\fint_{I^{n;+}_{\varepsilon,N}} |\nabla^2 v_{\varepsilon,N}|^2 \Big)^{\frac12} \lesssim_h 1 + \kappa_0 \tfrac{1}{\varepsilon}. \tag{4.19}$$

The first bound can be further refined in the following form, where we additionally capture the leading contribution: denoting by $v^n_{\varepsilon,N}$ the unique decaying solution of the following Stokes equation in $\mathbb{R}^d$,[2]

$$\begin{cases} -\triangle v^n_{\varepsilon,N} + \nabla q^n_{\varepsilon,N} = h\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}} + e\mathbb{1}_{\mathcal{I}_{\varepsilon,N}} + \kappa_0 \frac{\lambda}{\varepsilon N}\sum_{p:p\neq n} f^p_{\varepsilon,N}, & \text{in } \mathbb{R}^d, \\ \operatorname{div}(v^n_{\varepsilon,N}) = 0, & \text{in } \mathbb{R}^d, \end{cases} \tag{4.20}$$

we have

$$\|\nabla v^n_{\varepsilon,N} - \nabla(\mathcal{G}*h)(X^n_{\varepsilon,N})\|_{\mathrm{L}^\infty(I^{n;+}_{\varepsilon,N})} \lesssim_h \lambda(\alpha^1_{\varepsilon,N} + \kappa_0\alpha^0_{\varepsilon,N}) + \varepsilon. \tag{4.21}$$

The bounds (4.18) and (4.19) are easily obtained by similar estimates as in Step 1, recalling $\lambda\alpha^0_{\varepsilon,N} \le 1$, and we rather focus on the proof of (4.21). For that purpose, we start with the following identity for the solution of (4.20),

$$\begin{aligned} \nabla_\alpha v^n_{\varepsilon,N}(x) &= \nabla_\alpha(\mathcal{G}*h)(x) \\ &\quad + \sum_{m=1}^{N} \int_{I^m_{\varepsilon,N}} \nabla_\alpha \mathcal{G}(x-\cdot)(e-h) + \kappa_0\tfrac{\lambda}{\varepsilon N}\sum_{m:m\neq n}^{N} \int_{I^{m;+}_{\varepsilon,N}} \nabla_\alpha\mathcal{G}(x-\cdot) f^m_{\varepsilon,N}. \end{aligned}$$

Using again pointwise bounds on $\mathcal{G}$, we can estimate the last two right-hand side terms in $I^{n;+}_{\varepsilon,N}$ pointwise by $C_h(\lambda\alpha^1_{\varepsilon,N} + \varepsilon)$ and by $\kappa_0\lambda\alpha^0_{\varepsilon,N}$, respectively, and the claim (4.21) follows.

Step 3. Second-order cluster contribution: proof of (4.14).
By definition, cf. (4.4), we have

$$u^{(1)}_{\varepsilon,N} = \sum_{m=1}^{N} w^m_{\varepsilon,N}, \tag{4.22}$$

[2]This equation coincides with (4.2) up to removing the propulsion force of the n-th particle, which would create an additional $O(1)$ self-interaction term at $I^n_{\varepsilon,N}$.

in terms of $w^m_{\varepsilon,N} := u^m_{\varepsilon,N} - v_{\varepsilon,N}$, where we recall the short-hand notation $u^m_{\varepsilon,N} := \pi^{\{m\}} v_{\varepsilon,N}$. As in (4.8), we find the following equation for $w^m_{\varepsilon,N}$ in $\mathbb{R}^d$,

$$-\triangle w^m_{\varepsilon,N} + \nabla p \,=\, -\Big(e\mathbb{1}_{I^m_{\varepsilon,N}} + \kappa_0 \tfrac{\lambda}{\varepsilon N} f^m_{\varepsilon,N}\mathbb{1}_{I^m_{\varepsilon,N}} + \delta_{\partial I^m_{\varepsilon,N}}\sigma(u^m_{\varepsilon,N}, p^m_{\varepsilon,N})\nu\Big). \tag{4.23}$$

In terms of the Stokeslet $\mathcal{G}$, using boundary conditions for $u^m_{\varepsilon,N}$, we get for all $x \in \mathbb{R}^d$,

$$w^m_{\varepsilon,N}(x) \,=\, -\int_{\partial I^m_{\varepsilon,N}} \Big(\mathcal{G}(x-\cdot) - \fint_{I^m_{\varepsilon,N}} \mathcal{G}(x-\cdot)\Big)\sigma(u^m_{\varepsilon,N}, p^m_{\varepsilon,N})\nu. \tag{4.24}$$

Averaging this expression over $x \in I^n_{\varepsilon,N}$, summing over $m \neq n$, replacing $\mathcal{G}$ by its Taylor expansion, and appealing to a trace estimate and to pointwise bounds on $\mathcal{G}$, cf. Lemmas 2.2 and 2.3, we are led to

$$\begin{aligned}\bigg|\sum_{m:m\neq n}^N \fint_{I^n_{\varepsilon,N}} w^m_{\varepsilon,N} - \sum_{m:m\neq n}^N \nabla_\alpha \mathcal{G}(X^n_{\varepsilon,N} - X^m_{\varepsilon,N}) \int_{\partial I^m_{\varepsilon,N}} (x - X^m_{\varepsilon,N})_\alpha\, \sigma(u^m_{\varepsilon,N}, p^m_{\varepsilon,N})\nu\bigg| \\ \lesssim \tfrac{\varepsilon\lambda}{N} \sum_{m:m\neq n}^N |X^m_{\varepsilon,N} - X^n_{\varepsilon,N}|^{-d}\Big(\fint_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(u^m_{\varepsilon,N})|^2\Big)^{\frac12}.\end{aligned} \tag{4.25}$$

In order to estimate the last factor in the right-hand side, we appeal to an energy estimate: testing the equation (4.23) for $w^m_{\varepsilon,N}$ with $w^m_{\varepsilon,N}$ itself, and using boundary conditions for $w^m_{\varepsilon,N} = u^m_{\varepsilon,N} - v_{\varepsilon,N}$, we get the energy identity

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla w^m_{\varepsilon,N}|^2 \,&=\, -\int_{\partial I^m_{\varepsilon,N}} \Big(w^m_{\varepsilon,N} - \fint_{I^m_{\varepsilon,N}} w^m_{\varepsilon,N}\Big)\cdot \sigma(u^m_{\varepsilon,N}, p^m_{\varepsilon,N})\nu \\ &=\, \int_{\partial I^m_{\varepsilon,N}} \Big(v_{\varepsilon,N} - \fint_{I^m_{\varepsilon,N}} v_{\varepsilon,N}\Big)\cdot \sigma(u^m_{\varepsilon,N}, p^m_{\varepsilon,N})\nu,\end{aligned}$$

hence, by a trace estimate, cf. Lemma 2.2,

$$\int_{\mathbb{R}^d} |\mathrm{D}(w^m_{\varepsilon,N})|^2 \,\lesssim\, \Big(\int_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(v_{\varepsilon,N})|^2\Big)^{\frac12}\Big(\int_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(u^m_{\varepsilon,N})|^2\Big)^{\frac12},$$

which entails, by the triangle inequality, with $u^m_{\varepsilon,N} = w^m_{\varepsilon,N} + v_{\varepsilon,N}$,

$$\int_{\mathbb{R}^d} |\mathrm{D}(w^m_{\varepsilon,N})|^2 + \int_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(u^m_{\varepsilon,N})|^2 \,\lesssim\, \int_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(v_{\varepsilon,N})|^2. \tag{4.26}$$

Combining this with (4.18) and inserting into (4.25), we deduce

$$\begin{aligned}\bigg|\sum_{m:m\neq n}^N \fint_{I^n_{\varepsilon,N}} w^m_{\varepsilon,N} - \sum_{m:m\neq n}^N \nabla_\alpha \mathcal{G}(X^n_{\varepsilon,N} - X^m_{\varepsilon,N}) \int_{\partial I^m_{\varepsilon,N}} (x - X^m_{\varepsilon,N})_\alpha\, \sigma(u^m_{\varepsilon,N}, p^m_{\varepsilon,N})\nu\bigg| \\ \lesssim_h \varepsilon\lambda\alpha^0_{\varepsilon,N}.\end{aligned} \tag{4.27}$$

Recalling (4.22) and writing $w^n_{\varepsilon,N} = u^n_{\varepsilon,N} - u^{(0)}_{\varepsilon,N}$ for the diagonal term, we are then led to

$$\bigg|\fint_{I^n_{\varepsilon,N}} (u^{(0)}_{\varepsilon,N} + u^{(1)}_{\varepsilon,N}) - \fint_{I^n_{\varepsilon,N}} u^n_{\varepsilon,N} - \sum_{m:m\neq n}^N \nabla\mathcal{G}(X^n_{\varepsilon,N} - X^m_{\varepsilon,N})\, \Sigma^m_{\varepsilon,N}\bigg| \lesssim_h \varepsilon\lambda\alpha^0_{\varepsilon,N}, \tag{4.28}$$

where we have defined the stresslets

$$\Sigma^m_{\varepsilon,N} := \int_{\partial I^m_{\varepsilon,N}} \sigma(u^m_{\varepsilon,N}, p^m_{\varepsilon,N})\nu \otimes^\circ_s (x - X^m_{\varepsilon,N}), \qquad 1 \le m \le N. \tag{4.29}$$

Here, we recall that $\otimes^\circ_s$ stands for the trace-free symmetric tensor product: we have used both the vanishing torque condition for $u^m_{\varepsilon,N}$ and the incompressibility constraint $\operatorname{div}(\mathcal{G}) = 0$ to restrict $\Sigma^m_{\varepsilon,N}$ to its trace-free symmetric part in (4.28). It remains to evaluate the diagonal term and the stresslets in (4.28): we claim that for all $1 \le m \le N$,

$$\begin{aligned}
&\Big|\Sigma^m_{\varepsilon,N} - 2|I^m_{\varepsilon,N}|\Sigma^\circ(R^m_{\varepsilon,N})\,\mathrm{D}(\mathcal{G} * h)(X^m_{\varepsilon,N}) - \kappa_0|I^m_{\varepsilon,N}|\Sigma'_f(R^m_{\varepsilon,N})\Big| \\
&\qquad\qquad\qquad\qquad \lesssim_h \varepsilon^d\big(\lambda(\alpha^1_{\varepsilon,N} + \kappa_0\alpha^0_{\varepsilon,N}) + \varepsilon\big),
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
&\Big|\fint_{I^m_{\varepsilon,N}} u^m_{\varepsilon,N} - \big[\mathcal{G} \,\hat{*}\, \big((1-\lambda\mu_{\varepsilon,N})h + \lambda\mu_{\varepsilon,N}e\big)\big](X^m_{\varepsilon,N}) \\
&\qquad\qquad + \kappa_0\lambda[\nabla\mathcal{G} \,\hat{*}\, (\langle\nu_{\varepsilon,N}M_f\rangle)](X^m_{\varepsilon,N}) - \kappa_0\varepsilon V^\circ_f(R^m_{\varepsilon,N})\Big| \\
&\qquad\qquad\qquad\qquad \lesssim_h \varepsilon\big(\lambda(\alpha^2_{\varepsilon,N} + \alpha^1_{\varepsilon,N} + \alpha^0_{\varepsilon,N}) + \varepsilon\big),
\end{aligned} \tag{4.31}$$

where M_f stands for the first moment of the propulsion force,

$$M_f(r) := \int_{\mathbb{R}^d} f(\cdot, r) \otimes^\circ x, \tag{4.32}$$

and where we have set

$$\Sigma'_f(r) := \int_{\partial I^\circ(r)} \sigma(u^\circ_{r,f}, p^\circ_{r,f})\nu \otimes^\circ_s x, \tag{4.33}$$

recalling that $(u^\circ_{r,f}, p^\circ_{r,f})$ is the unique decaying solution of the single-particle problem (1.22). The proof of these two estimates (4.30)–(4.31) is split into the following three substeps. Inserting them into (4.28), noting that we recover $\Sigma'_f - M_f = \Sigma^\circ_f$ as defined in (1.21), and further using convolution notation, the conclusion (4.14) follows.

Substep 3.1. A suitable decomposition of $u^m_{\varepsilon,N}$.
Recalling that $u^m_{\varepsilon,N}$ satisfies the following single-particle problem,

$$\begin{cases}
-\triangle u^m_{\varepsilon,N} + \nabla p^m_{\varepsilon,N} = h\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}_{\varepsilon,N}} + e\mathbb{1}_{\mathcal{I}_{\varepsilon,N}\setminus I^m_{\varepsilon,N}} + \kappa_0\frac{\lambda}{\varepsilon N}\sum_{p=1}^N f^p_{\varepsilon,N}, & \text{in } \mathbb{R}^d\setminus I^m_{\varepsilon,N},\\
\operatorname{div}(u^m_{\varepsilon,N}) = 0, & \text{in } \mathbb{R}^d\setminus I^m_{\varepsilon,N},\\
\mathrm{D}(u^m_{\varepsilon,N}) = 0, & \text{in } I^m_{\varepsilon,N},\\
\frac{\lambda}{N}e + \kappa_0\frac{\lambda}{\varepsilon N}R^m_{\varepsilon,N} + \int_{\partial I^m_{\varepsilon,N}} \sigma(u^m_{\varepsilon,N}, p^m_{\varepsilon,N})\nu = 0, &\\
\int_{\partial I^m_{\varepsilon,N}} (x - X^m_{\varepsilon,N}) \times \sigma(u^m_{\varepsilon,N}, p^m_{\varepsilon,N})\nu = 0, &
\end{cases}$$

we may naturally decompose it as

$$u^m_{\varepsilon,N} = v^m_{\varepsilon,N} + w^m_{\varepsilon,N;0} + w^m_{\varepsilon,N;1} + e^m_{\varepsilon,N}, \tag{4.34}$$

where we recall that $v^m_{\varepsilon,N}$ is defined in (4.20), and where:

— $w^m_{\varepsilon,N;0}$ is the unique decaying solution of the single-particle problem

$$\begin{cases} -\triangle w^m_{\varepsilon,N;0} + \nabla q^m_{\varepsilon,N;0} = 0, & \text{in } \mathbb{R}^d \setminus I^m_{\varepsilon,N},\\ \operatorname{div}(w^m_{\varepsilon,N;0}) = 0, & \text{in } \mathbb{R}^d \setminus I^m_{\varepsilon,N},\\ \mathrm{D}(w^m_{\varepsilon,N;0}) + H^m_{\varepsilon,N} = 0, & \text{in } I^m_{\varepsilon,N},\\ \int_{\partial I^m_{\varepsilon,N}} \sigma(w^m_{\varepsilon,N;0}, q^m_{\varepsilon,N;0})\nu = 0, \\ \int_{\partial I^m_{\varepsilon,N}} (x - X^m_{\varepsilon,N}) \times \sigma(w^m_{\varepsilon,N;0}, q^m_{\varepsilon,N;0})\nu = 0, \end{cases} \tag{4.35}$$

where the strain rate $H^m_{\varepsilon,N}$ is chosen as

$$H^m_{\varepsilon,N} := \mathrm{D}(\mathcal{G} * h)(X^m_{\varepsilon,N}); \tag{4.36}$$

— $w^m_{\varepsilon,N;1}$ is the unique decaying solution of the single-particle problem

$$\begin{cases} -\triangle w^m_{\varepsilon,N;1} + \nabla q^m_{\varepsilon,N;1} = \kappa_0 \frac{\lambda}{\varepsilon N} f^m_{\varepsilon,N}, & \text{in } \mathbb{R}^d \setminus I^m_{\varepsilon,N},\\ \operatorname{div}(w^m_{\varepsilon,N;1}) = 0, & \text{in } \mathbb{R}^d \setminus I^m_{\varepsilon,N},\\ \mathrm{D}(w^m_{\varepsilon,N;1}) = 0, & \text{in } I^m_{\varepsilon,N},\\ \kappa_0 \frac{\lambda}{\varepsilon N} R^m_{\varepsilon,N} + \int_{\partial I^m_{\varepsilon,N}} \sigma(w^m_{\varepsilon,N;1}, q^m_{\varepsilon,N;1})\nu = 0, \\ \int_{\partial I^m_{\varepsilon,N}} (x - X^m_{\varepsilon,N}) \times \sigma(w^m_{\varepsilon,N;1}, q^m_{\varepsilon,N;1})\nu = 0. \end{cases} \tag{4.37}$$

Using Lemma 2.1, a direct computation then entails that the remainder

$$e^m_{\varepsilon,N} := u^m_{\varepsilon,N} - v^m_{\varepsilon,N} - w^m_{\varepsilon,N;0} - w^m_{\varepsilon,N;1}$$

satisfies the following equation in $\mathbb{R}^d$,

$$-\triangle e^m_{\varepsilon,N} + \nabla p = -\Big(e \mathbb{1}_{I^m_{\varepsilon,N}} + \delta_{\partial I^m_{\varepsilon,N}} \sigma\big(u^m_{\varepsilon,N} - w^m_{\varepsilon,N;0} - H^m_{\varepsilon,N}x - w^m_{\varepsilon,N;1}, q^m_{\varepsilon,N} - q^m_{\varepsilon,N;0} - q^m_{\varepsilon,N;1}\big)\nu\Big).$$

Testing this equation with $e^m_{\varepsilon,N}$ itself, and using boundary conditions, we get

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla e^m_{\varepsilon,N}|^2 &= -\int_{\partial I^m_{\varepsilon,N}} \Big(e^m_{\varepsilon,N} - \fint_{I^m_{\varepsilon,N}} e^m_{\varepsilon,N}\Big) \\ &\qquad \cdot \sigma\big(u^m_{\varepsilon,N} - w^m_{\varepsilon,N;0} - H^m_{\varepsilon,N}x - w^m_{\varepsilon,N;1}, q^m_{\varepsilon,N} - q^m_{\varepsilon,N;0} - q^m_{\varepsilon,N;1}\big)\nu \\ &= \int_{\partial I^m_{\varepsilon,N}} \Big((v^m_{\varepsilon,N} - H^m_{\varepsilon,N}x) - \fint_{I^m_{\varepsilon,N}} (v^m_{\varepsilon,N} - H^m_{\varepsilon,N}x)\Big) \\ &\qquad \cdot \sigma\big(u^m_{\varepsilon,N} - w^m_{\varepsilon,N;0} - H^m_{\varepsilon,N}x - w^m_{\varepsilon,N;1}, q^m_{\varepsilon,N} - q^m_{\varepsilon,N;0} - q^m_{\varepsilon,N;1}\big)\nu, \end{aligned}$$

hence, by a trace estimate, cf. Lemma 2.2,

$$\int_{\mathbb{R}^d} |\nabla e^m_{\varepsilon,N}|^2 \lesssim \Big(\int_{I^m_{\varepsilon,N}} |\mathrm{D}(v^m_{\varepsilon,N}) - H^m_{\varepsilon,N}|^2\Big)^{\frac12} \Big(\int_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(u^m_{\varepsilon,N} - w^m_{\varepsilon,N;0} - w^m_{\varepsilon,N;1}) - H^m_{\varepsilon,N}|^2\Big)^{\frac12}.$$

By the triangle inequality, reconstructing $e^m_{\varepsilon,N}$ in the last factor, we are led to

$$\int_{\mathbb{R}^d} |\nabla e^m_{\varepsilon,N}|^2 \lesssim \int_{I^m_{\varepsilon,N}} |\mathrm{D}(v^m_{\varepsilon,N}) - H^m_{\varepsilon,N}|^2,$$

and thus, appealing to (4.21) and recalling the choice (4.36),

$$\int_{\mathbb{R}^d} |\nabla e^m_{\varepsilon,N}|^2 \lesssim_h \varepsilon^d \big(\lambda(\alpha^1_{\varepsilon,N} + \kappa_0 \alpha^0_{\varepsilon,N}) + \varepsilon\big)^2. \tag{4.38}$$

Substep 3.2. Proof of (4.30).
Inserting the above decomposition (4.34) for $u^m_{\varepsilon,N}$ into the definition (4.29) of the stresslet, and using the remainder estimate (4.38) together with a trace estimate, cf. Lemma 2.2, we get

$$\Big|\Sigma^m_{\varepsilon,N} - \int_{\partial I^m_{\varepsilon,N}} \sigma(v^m_{\varepsilon,N}, q^m_{\varepsilon,N})\nu \otimes^\circ_s (x - X^m_{\varepsilon,N}) - \int_{\partial I^m_{\varepsilon,N}} \sigma(w^m_{\varepsilon,N;0}, q^m_{\varepsilon,N;0})\nu \otimes^\circ_s (x - X^m_{\varepsilon,N})$$
$$- \int_{\partial I^m_{\varepsilon,N}} \sigma(w^m_{\varepsilon,N;1}, q^m_{\varepsilon,N;1})\nu \otimes^\circ_s (x - X^m_{\varepsilon,N})\Big|$$
$$\lesssim \varepsilon^d \Big(\fint_{I^{m;+}_{\varepsilon,N}} |\mathrm{D}(e^m_{\varepsilon,N})|^2 \Big)^{\frac12} \lesssim_h \varepsilon^d \big(\lambda(\alpha^1_{\varepsilon,N} + \kappa_0 \alpha^0_{\varepsilon,N}) + \varepsilon\big). \quad (4.39)$$

It remains to evaluate the different terms in the left-hand side. First, we compute

$$\int_{\partial I^m_{\varepsilon,N}} \sigma(v^m_{\varepsilon,N}, q^m_{\varepsilon,N})\nu \otimes^\circ_s (x - X^m_{\varepsilon,N}) = \int_{I^m_{\varepsilon,N}} 2\mathrm{D}(v^m_{\varepsilon,N}),$$

and thus, by (4.21),

$$\Big| \int_{\partial I^m_{\varepsilon,N}} \sigma(v^m_{\varepsilon,N}, q^m_{\varepsilon,N})\nu \otimes^\circ_s (x - X^m_{\varepsilon,N}) - 2|I^m_{\varepsilon,N}|\, \mathrm{D}(\mathcal{G} * h)(X^m_{\varepsilon,N})\Big|$$
$$\lesssim_h \varepsilon^d \big(\lambda(\alpha^1_{\varepsilon,N} + \kappa_0 \alpha^0_{\varepsilon,N}) + \varepsilon\big).$$

Next, as $w^m_{\varepsilon,N;0}$ and $w^m_{\varepsilon,N;1}$ satisfy the single-particle problems (4.35)–(4.37), which can be compared to (1.18) and (1.22), we simply find by scaling

$$\int_{\partial I^m_{\varepsilon,N}} \sigma(w^m_{\varepsilon,N;0}, q^m_{\varepsilon,N;0})\nu \otimes^\circ_s (x - X^m_{\varepsilon,N}) = 2|I^m_{\varepsilon,N}|\big(\Sigma^\circ(R^m_{\varepsilon,N})H^m_{\varepsilon,N} - H^m_{\varepsilon,N}\big),$$
$$\int_{\partial I^m_{\varepsilon,N}} \sigma(w^m_{\varepsilon,N;1}, q^m_{\varepsilon,N;1})\nu \otimes^\circ_s (x - X^m_{\varepsilon,N}) = \kappa_0 |I^m_{\varepsilon,N}| \Sigma'_f(R^m_{\varepsilon,N}),$$

where we recall the definition of Σ°, Σ'_f in (1.19) and (4.33). Inserting these different computations into (4.39), and recalling $H^m_{\varepsilon,N} = \mathrm{D}(\mathcal{G} * h)(X^m_{\varepsilon,N})$, cf. (4.36), the claim (4.30) follows.

Substep 3.3. Proof of (4.31).
Using again the decomposition (4.34) for $u^m_{\varepsilon,N}$, and noting that, by the Sobolev embedding, the remainder estimate (4.38) yields

$$\Big| \fint_{I^m_{\varepsilon,N}} e^m_{\varepsilon,N} \Big| \lesssim \varepsilon^{1-\frac d2} \Big(\int_{I^m_{\varepsilon,N}} |e^m_{\varepsilon,N}|^{\frac{2d}{d-2}} \Big)^{\frac{d-2}{2d}}$$
$$\lesssim \varepsilon^{1-\frac d2} \Big(\int_{I^m_{\varepsilon,N}} |\nabla e^m_{\varepsilon,N}|^2 \Big)^{\frac12}$$
$$\lesssim_h \varepsilon\big(\lambda(\alpha^1_{\varepsilon,N} + \kappa_0 \alpha^0_{\varepsilon,N}) + \varepsilon\big),$$

we deduce

$$\Big| \fint_{I^m_{\varepsilon,N}} u^m_{\varepsilon,N} - \fint_{I^m_{\varepsilon,N}} v^m_{\varepsilon,N} - \fint_{I^m_{\varepsilon,N}} w^m_{\varepsilon,N;0} - \fint_{I^m_{\varepsilon,N}} w^m_{\varepsilon,N;1} \Big| \lesssim_h \varepsilon\big(\lambda(\alpha^1_{\varepsilon,N} + \kappa_0 \alpha^0_{\varepsilon,N}) + \varepsilon\big). \quad (4.40)$$

It remains to evaluate the different terms in the left-hand side. First, recalling equation (4.20) for $v_{\varepsilon,N}^m$, we can represent, in terms of the Stokeslet $\mathcal{G}$,

$$v_{\varepsilon,N}^m(x) \;=\; (\mathcal{G}*h)(x)+\sum_{p=1}^N\int_{I_{\varepsilon,N}^p}\mathcal{G}(x-\cdot)(e-h)+\kappa_0\tfrac{\lambda}{\varepsilon N}\sum_{p:p\neq m}^N\int_{I_{\varepsilon,N}^{p;+}}\mathcal{G}(x-\cdot)f_{\varepsilon,N}^p. \quad (4.41)$$

Averaging this over $x\in I_{\varepsilon,N}^m$, using a Taylor expansion and the pointwise bounds on $\mathcal{G}$, cf. Lemma 2.3, and recalling $f_{\varepsilon,N}^m=\varepsilon^{-d}f(\frac1\varepsilon(\cdot-X_{\varepsilon,N}^m),R_{\varepsilon,N}^m)$ and the balance of forces (1.4), we find for the off-diagonal contributions,

$$\Big|\fint_{I_{\varepsilon,N}^m}\mathcal{G}*h-(\mathcal{G}*h)(X_{\varepsilon,N}^m)\Big|\;\lesssim_h\;\varepsilon^2,$$

$$\Big|\sum_{p:p\neq m}^N\fint_{I_{\varepsilon,N}^m}\Big(\int_{I_{\varepsilon,N}^p}\mathcal{G}(x-\cdot)(e-h)\Big)\,dx-\tfrac{\lambda}{N}\sum_{p:p\neq m}^N\mathcal{G}(X_{\varepsilon,N}^m-X_{\varepsilon,N}^p)\big(e-h(X_{\varepsilon,N}^p)\big)\Big|$$
$$\lesssim_h\;\varepsilon\lambda(\alpha_{\varepsilon,N}^2+\alpha_{\varepsilon,N}^1),$$

$$\Big|\tfrac{\lambda}{\varepsilon N}\sum_{p:p\neq m}^N\fint_{I_{\varepsilon,N}^m}\Big(\int_{I_{\varepsilon,N}^{p;+}}\mathcal{G}(x-\cdot)f_{\varepsilon,N}^p\Big)+\tfrac{\lambda}{N}\sum_{p:p\neq m}^N\nabla\mathcal{G}(X_{\varepsilon,N}^m-X_{\varepsilon,N}^p)M_f(R_{\varepsilon,N}^p)\Big|\;\lesssim\;\varepsilon\lambda\alpha_{\varepsilon,N}^0,$$

in terms of the first moment M_f of the propulsion force, cf. (4.32). In addition, for the diagonal contribution in the second right-hand side term of (4.41), we can estimate

$$\fint_{I_{\varepsilon,N}^m}\Big|\int_{I_{\varepsilon,N}^m}\mathcal{G}(x-\cdot)(e-h)\Big|\;\lesssim_h\;\fint_{I_{\varepsilon,N}^m}\Big(\int_{I_{\varepsilon,N}^m}|x-\cdot|^{2-d}\Big)\;\lesssim\;\varepsilon^2.$$

Inserting these different computations into (4.41), and using convolution notation, we get

$$\Big|\fint_{I_{\varepsilon,N}^m}v_{\varepsilon,N}^m-\big[\mathcal{G}\,\hat{*}\,\big((1-\lambda\mu_{\varepsilon,N})h+\lambda\mu_{\varepsilon,N}e\big)\big](X_{\varepsilon,N}^m)+\kappa_0\lambda\big[\nabla\mathcal{G}*(\langle\nu_{\varepsilon,N}M_f\rangle)\big](X_{\varepsilon,N}^m)\Big|$$
$$\lesssim_h\;\varepsilon\big(\lambda(\alpha_{\varepsilon,N}^2+\alpha_{\varepsilon,N}^1+\kappa\alpha_{\varepsilon,N}^0)+\varepsilon\big). \quad (4.42)$$

We turn to the analysis of the last two left-hand side contributions in (4.40). As $w_{\varepsilon,N;0}^m$ solves the single-particle problem (4.35), we note that it actually satisfies the Dirichlet condition $w_{\varepsilon,N;0}^m(x)=-H_{\varepsilon,N}^m(x-X_{\varepsilon,N}^m)$ in $I_{\varepsilon,N}^m$, hence

$$\fint_{I_{\varepsilon,N}^m}w_{\varepsilon,N;0}^m\;=\;0.$$

As $w_{\varepsilon,N;1}^m$ satisfies the single-particle problem (4.37), which can be compared to (1.22), we find by scaling

$$\fint_{I_{\varepsilon,N}^m}w_{\varepsilon,N;1}^m\;=\;\kappa_0\varepsilon V_f^\circ(R_{\varepsilon,N}^m),$$

where we recall the definition (1.24) of V_f°. Inserting these identities into (4.40), together with (4.42), recalling $H_{\varepsilon,N}^m=\mathrm{D}(\mathcal{G}*h)(X_{\varepsilon,N}^m)$, cf. (4.36), and using convolution notation, the claim (4.31) follows.

Step 4. Angular velocities: proof of (4.13).
Recalling the short-hand notation $u^n_{\varepsilon,N} = \pi^{\{n\}}_{\varepsilon,N} u^{(0)}_{\varepsilon,N}$, we start again with the decomposition (4.34): using the remainder estimate (4.38), we get

$$\Big| \fint_{I^n_{\varepsilon,N}} \nabla u^n_{\varepsilon,N} - \fint_{I^n_{\varepsilon,N}} \nabla v^n_{\varepsilon,N} - \fint_{I^n_{\varepsilon,N}} \nabla w^n_{\varepsilon,N;0} - \fint_{I^n_{\varepsilon,N}} \nabla w^n_{\varepsilon,N;1} \Big| \lesssim_h \lambda(\alpha^1_{\varepsilon,N} + \kappa_0 \alpha^0_{\varepsilon,N}) + \varepsilon. \quad (4.43)$$

It remains to evaluate the different terms in the left-hand side. First, we get from (4.21),

$$\Big| \fint_{I^n_{\varepsilon,N}} \nabla v^n_{\varepsilon,N} - \nabla(\mathcal{G} * h)(X^n_{\varepsilon,N}) \Big| \lesssim_h \lambda(\alpha^1_{\varepsilon,N} + \kappa_0 \alpha^0_{\varepsilon,N}) + \varepsilon.$$

Next, as $w^n_{\varepsilon,N;0}$ satisfies the single-particle problems (4.35), which can be compared to (1.18), we find by scaling

$$\begin{aligned} \fint_{I^n_{\varepsilon,N}} \nabla w^n_{\varepsilon,N;0} &= \fint_{I^\circ(R^n_{\varepsilon,N})} \nabla u^\circ_{R^n_{\varepsilon,N}, H^n_{\varepsilon,N}} \\ &= \Omega^\circ(R^n_{\varepsilon,N}) \nabla(\mathcal{G} * h)(X^n_{\varepsilon,N}) - \nabla(\mathcal{G} * h)(X^n_{\varepsilon,N}), \end{aligned}$$

where we recall the definition (1.20) of Ω° and the choice (4.36) of $H^n_{\varepsilon,N}$ as the symmetric part of $\nabla(\mathcal{G} * h)(X^n_{\varepsilon,N})$. Finally, as $w^n_{\varepsilon,N;1}$ satisfies the single-particle problem (4.37) with rigidity constraint $\mathrm{D}(w^n_{\varepsilon,N;1}) = 0$ in $I^n_{\varepsilon,N}$, and as the inclusion $I^m_{\varepsilon,N}$ and the propulsion force $f^m_{\varepsilon,N}$ are both axisymmetric in the direction $R^m_{\varepsilon,N}$, centered at $X^m_{\varepsilon,N}$, we find by symmetry

$$\fint_{I^n_{\varepsilon,N}} \nabla w^n_{\varepsilon,N;1} = 0.$$

Inserting these different computations into (4.43), the conclusion (4.13) follows.

Step 5. Fluid velocity away from the particles: proof of (4.15)–(4.16).
Let the boundary-layer thickness $\delta \in [\varepsilon, 1]$ be fixed. The proof of (4.15)–(4.16) is slightly simpler than that of (4.12)–(4.14) as self-interaction terms can be ignored away from the particles. We start with (4.15). Recalling the representation (4.17) for $u^{(0)}_{\varepsilon,N} = v_{\varepsilon,N}$, and using the pointwise bounds on $\mathcal{G}$, cf. Lemma 2.3, we get for all $x \in \mathbb{R}^d \setminus (\mathcal{I}_{\varepsilon,N} + \delta B)$,

$$\|u^{(0)}_{\varepsilon,N} - \mathcal{G} * h\|_{\mathrm{L}^\infty(\mathbb{R}^d \setminus (\mathcal{I}_{\varepsilon,N} + \delta B))} \lesssim_h \tfrac{\lambda}{N} \sum_{m=1}^N \Big(|x - X^m_{\varepsilon,N}|^{2-d} + \kappa_0 |x - X^m_{\varepsilon,N}|^{1-d} \Big).$$

To estimate the right-hand side, we use (3.5) in the following modified form away from the particles: for all $x \in \mathbb{R}^d \setminus (\mathcal{I}_{\varepsilon,N} + \delta B)$, we can estimate for any $\sigma \in [0, d]$, after separating the diagonal contribution as in (4.9),

$$\tfrac{1}{N} \sum_{m=1}^N |x - X^m_{\varepsilon,N}|^{\sigma - d} \lesssim \alpha^\sigma_{\varepsilon,N} + \tfrac{1}{N} \delta^{\sigma - d} \simeq \alpha^\sigma_{\varepsilon,N} + \tfrac{1}{\lambda} \delta^\sigma (\tfrac{\varepsilon}{\delta})^d. \quad (4.44)$$

Using this to estimate the above right-hand side, we get the conclusion (4.15).

We turn to the proof of (4.16) and we start with an improved estimate on $u^{(0)}_{\varepsilon,N} = v_{\varepsilon,N}$. Replacing $\mathcal{G}$ by its Taylor expansion in the representation (4.17) for the latter, we find for all $x \in \mathbb{R}^d \setminus (\mathcal{I}_{\varepsilon,N} + \delta B)$,

$$\Big|u^{(0)}_{\varepsilon,N}(x) - \big[\mathcal{G} * \big((1-\lambda\mu_{\varepsilon,N})h + \lambda\mu_{\varepsilon,N}e\big)\big](x) + \kappa_0\lambda\big[\nabla\mathcal{G} * (\langle M_f\nu_{\varepsilon,N}\rangle)\big](x)\Big| \\ \lesssim_h \varepsilon\tfrac{\lambda}{N}\sum_{m=1}^N \big(|x - X^m_{\varepsilon,N}|^{2-d} + |x - X^m_{\varepsilon,N}|^{1-d} + \kappa_0|x - X^m_{\varepsilon,N}|^{-d}\big),$$

and thus, using (4.44) again to estimate the right-hand side,

$$\Big\|u^{(0)}_{\varepsilon,N} - \mathcal{G} * \big((1-\lambda\mu_{\varepsilon,N})h + \lambda\mu_{\varepsilon,N}e\big) + \kappa_0\lambda\nabla\mathcal{G} * (\langle M_f\nu_{\varepsilon,N}\rangle)\Big\|_{\mathrm{L}^\infty(\mathbb{R}^d\setminus(\mathcal{I}_{\varepsilon,N}+\delta B))} \\ \lesssim_h \varepsilon\big(\lambda(\alpha^2_{\varepsilon,N} + \alpha^1_{\varepsilon,N} + \kappa_0\alpha^0_{\varepsilon,N}) + (\delta+\kappa_0)(\tfrac{\varepsilon}{\delta})^d\big). \quad (4.45)$$

We turn to the corresponding analysis of $u^{(1)}_{\varepsilon,N}$. As in (4.22)–(4.24), we can write

$$u^{(1)}_{\varepsilon,N} = -\sum_{m=1}^N \int_{\partial I^m_{\varepsilon,N}} \Big(\mathcal{G}(x-\cdot) - \fint_{I^m_{\varepsilon,N}} \mathcal{G}(x-\cdot)\Big)\sigma(u^m_{\varepsilon,N}, p^m_{\varepsilon,N})\nu. \quad (4.46)$$

Replacing $\mathcal{G}$ by its Taylor expansion, we deduce for all $x \in \mathbb{R}^d \setminus (\mathcal{I}_{\varepsilon,N} + \delta B)$,

$$\Big|u^{(1)}_{\varepsilon,N}(x) - \sum_{m=1}^N \nabla\mathcal{G}(x - X^m)\Sigma^m_{\varepsilon,N}\Big| \lesssim \varepsilon\tfrac{\lambda}{N}\sum_{m=1}^N |x - X^m_{\varepsilon,N}|^{-d}\Big(\fint_{I^m_{\varepsilon,N}} |\mathrm{D}(u^m_{\varepsilon,N})|^2\Big)^{\frac12},$$

in terms of the stresslets $\{\Sigma^m_{\varepsilon,N}\}_m$ defined in (4.29). Appealing to (4.26) in combination with (4.18) to estimate the last factor, and using (4.44) again, we get

$$\Big\|u^{(1)}_{\varepsilon,N} - \sum_{m=1}^N \nabla\mathcal{G}(\cdot - X^m)\Sigma^m_{\varepsilon,N}\Big\|_{\mathrm{L}^\infty(\mathbb{R}^d\setminus(\mathcal{I}_{\varepsilon,N}+\delta B))} \lesssim_h \varepsilon\big(\lambda\alpha^0_{\varepsilon,N} + (\tfrac{\varepsilon}{\delta})^d\big).$$

Now inserting the approximation (4.30) for the stresslets, using (4.44) again, using convolution notation, and noting that $(\frac{\varepsilon}{\delta})^d\delta \le \varepsilon$, we deduce

$$\Big\|u^{(1)}_{\varepsilon,N} - \lambda\nabla\mathcal{G} * \big(2\langle\Sigma^\circ\nu_{\varepsilon,N}\rangle\mathrm{D}(\mathcal{G}*h) + \kappa_0\langle\Sigma'_f\nu_{\varepsilon,N}\rangle\big)\Big\|_{\mathrm{L}^\infty(\mathbb{R}^d\setminus(\mathcal{I}_{\varepsilon,N}+\delta B))} \\ \lesssim_h \big(\lambda\alpha^1_{\varepsilon,N} + \varepsilon\big)\big(\lambda(\alpha^1_{\varepsilon,N} + \alpha^0_{\varepsilon,N}) + \varepsilon + (\tfrac{\varepsilon}{\delta})^d\big).$$

Combining this with (4.45) and recalling $\Sigma^\circ_f = \Sigma'_f - M_f$, we get the conclusion (4.16). □

5. Mean-field approximation

Given the dilute expansion of particle velocities in Proposition 4.1, we now appeal to a mean-field argument to derive a macroscopic equation for the particle density. For that purpose, as in [25, 32, 29], we use the ∞-Wasserstein method developed by Hauray and Jabin in [21] (see also [4]), combined with a refinement found in [29]. This method can generally be applied whenever particle interactions are less singular than Coulomb forces at short distances, which is indeed the case in the present setting as hydrodynamic interactions are given to leading order by the Stokeslet. We start with the following mean-field approximation with accuracy $O((\lambda+\varepsilon)^2 + \kappa_0\varepsilon)$, neglecting swimming velocities.

Proposition 5.1. *Assume that there is a solution* $\tilde\nu_\lambda\in C([0,T];\mathcal P\cap\mathrm L^\infty(\mathbb R^d\times\mathbb S^{d-1}))$ *of the following transport equation up to some time* $T>0$,

$$\left\{\begin{array}{l}\partial_t\tilde\nu_\lambda+{\operatorname{div}}_x(\tilde\nu_\lambda\tilde u_\lambda)+{\operatorname{div}}_r(\tilde\nu_\lambda\tilde\Omega_\lambda r)=0,\\ \tilde\nu_\lambda|_{t=0}=\nu^\circ,\end{array}\right.\tag{5.1}$$

where the translational and angular velocity fields $\tilde u_\lambda$ *and* $\tilde\Omega_\lambda$ *are given by*

$$\begin{aligned}\tilde u_\lambda(x)&:=\big[\mathcal G*\big((1-\lambda\tilde\mu_\lambda)h+\lambda\tilde\mu_\lambda e\big)\big](x)\\&\qquad+\lambda\big[\nabla\mathcal G*\big(2\langle\Sigma^\circ\tilde\nu_\lambda\rangle\,\mathrm D(\mathcal G*h)+\kappa_0\langle\Sigma^\circ_f\tilde\nu_\lambda\rangle\big)\big](x),\\ \tilde\Omega_\lambda(x,r)&:=\Omega^\circ(r)\nabla(\mathcal G*h)(x),\end{aligned}\tag{5.2}$$

in terms of the spatial density $\tilde\mu_\lambda:=\langle\tilde\nu_\lambda\rangle$. *Denote by* $W^\infty_{\varepsilon,N},Z^\infty_{\varepsilon,N}$ *the* ∞*-Wasserstein distances of the empirical measures* $\mu_{\varepsilon,N},\nu_{\varepsilon,N}$ *to their mean-field approximations* $\tilde\mu_\lambda,\tilde\nu_\lambda$,

$$W^\infty_{\varepsilon,N}(t):=W_\infty(\mu^t_{\varepsilon,N},\tilde\mu^t_\lambda),\qquad Z^\infty_{\varepsilon,N}(t):=W_\infty(\nu^t_{\varepsilon,N},\tilde\nu^t_\lambda)\vee W_\infty(\mu^t_{\varepsilon,N},\tilde\mu^t_\lambda).\tag{5.3}$$

Assume that initial particle positions satisfy, for some $\theta_0\ge1$,

$$\mathrm d^{\min}_{\varepsilon,N}(0)\ge(\theta_0N)^{-\frac1d},\qquad\rho^{\max}_{\varepsilon,N}(0)\le\theta_0,\qquad Z^\infty_{\varepsilon,N}(0)\le\theta_0.$$

Given $\theta>\theta_0$, *further assume that* $(\theta N)^{-\frac1d}\ge4\varepsilon$ *and that* $\lambda\theta\log(2+\theta N)\ll1$ *is small enough, and assume that the maximal time* $T^\theta_{\varepsilon,N}$ *in Corollary 3.2 satisfies* $T^\theta_{\varepsilon,N}\ge T$. *Then we have for all* $t\in[0,T]$,

$$Z^\infty_{\varepsilon,N}(t)\le\mathcal C_{\theta,h,\mu}(t)\big(Z^\infty_{\varepsilon,N}(0)+\lambda\log N+\varepsilon\big),\tag{5.4}$$

$$\begin{aligned}W^\infty_{\varepsilon,N}(t)\le\mathcal C_{\theta,h,\mu}(t)\Big(&W^\infty_{\varepsilon,N}(0)+(\lambda+\varepsilon)(\lambda\log N+\varepsilon)+\kappa_0\varepsilon\\&+\lambda\big(Z^\infty_{\varepsilon,N}(0)+\lambda\log N+\varepsilon\big)\log\big(2+\tfrac{1}{Z^\infty_{\varepsilon,N}(0)+\lambda\log N+\varepsilon}\big)\Big),\end{aligned}\tag{5.5}$$

and for any $\delta\in[\varepsilon,1]$,

$$\|u^t_{\varepsilon,N}-\tilde u^t_\lambda\|_{\mathrm L^\infty(\mathbb R^d\setminus(\mathcal I_{\varepsilon,N}+\delta B))}\le\mathcal C_{\theta,h,\mu}(t)\Big((\lambda+\varepsilon)\big(\lambda\log N+\varepsilon+(\tfrac\varepsilon\delta)^d\big)+\lambda Z^\infty_{\varepsilon,N}(0)\Big),\tag{5.6}$$

where $\mathcal C_{\theta,h,\mu}(t)$ *stands for any constant that further depends on* θ,h,t, *and on an upper bound on* $\max_{0\le s\le t}\|\tilde\mu^s_\lambda\|_{\mathrm L^\infty(\mathbb R^d)}$. ◊

Proof. For all $t\in[0,T]$, denote by $\hat X^t_{\varepsilon,N}:\mathbb R^d\to\mathbb R^d$ an optimal transport map such that

$$(\hat X^t_{\varepsilon,N})_*\tilde\mu^t_\lambda=\mu^t_{\varepsilon,N},\qquad W^\infty_{\varepsilon,N}(t)=\operatorname*{sup\,ess}_{x;\tilde\mu^t_\lambda}|x-\hat X^t_{\varepsilon,N}(x)|,\tag{5.7}$$

and also denote by $(\tilde X^t_{\varepsilon,N},\tilde R^t_{\varepsilon,N}):\mathbb R^d\times\mathbb S^{d-1}\to\mathbb R^d\times\mathbb S^{d-1}$ an optimal transport map such that

$$\begin{aligned}(\tilde X^t_{\varepsilon,N},\tilde R^t_{\varepsilon,N})_*\tilde\nu^t_\lambda&=\nu^t_{\varepsilon,N},\\ W_\infty(\nu^t_{\varepsilon,N},\tilde\nu^t_\lambda)&=\operatorname*{sup\,ess}_{x,r;\tilde\nu^t_\lambda}\big|(x,r)-(\tilde X^t_{\varepsilon,N}(x,r),\tilde R^t_{\varepsilon,N}(x,r))\big|.\end{aligned}\tag{5.8}$$

Here, we use the notation $\operatorname{sup\,ess}_{x;\mu}g(x)$ for the essential supremum of a function g with respect to a measure μ. We split the proof into three steps.

Step 1. First-order control on $Z^\infty_{\varepsilon,N}$: proof that

$$\tfrac{d^+}{dt}Z^\infty_{\varepsilon,N}\lesssim_h Z^\infty_{\varepsilon,N}+\lambda\big(\alpha^0_{\varepsilon,N}+\|\tilde\mu_\lambda\|_{\mathrm L^1\cap\mathrm L^\infty(\mathbb R^d)}\big)+\varepsilon,\tag{5.9}$$

where $\frac{d^+}{dt}$ stands for the right-derivative.

Recalling (5.3), we focus on the bound on $W_\infty(\nu_{\varepsilon,N},\tilde\nu_\lambda)$, while the argument is analogous for $W_\infty(\mu_{\varepsilon,N},\tilde\mu_\lambda)$. In view of the particle dynamics (1.11) and of the macroscopic transport equation (5.1), we can estimate the time-derivative of the ∞-Wasserstein distance (5.8) by using characteristics similarly as in [4], to the effect of

$$\begin{aligned}\tfrac{d^+}{dt}W_\infty(\nu_{\varepsilon,N},\tilde\nu_\lambda)\ \le\ \operatorname*{sup\,ess}_{x,r;\tilde\nu_\lambda}\Big(&|\tilde u_\lambda(x)-V_{\varepsilon,N}(\tilde X_{\varepsilon,N}(x,r))|\\&+\big|\tilde\Omega_\lambda(x,r)r-\Omega_{\varepsilon,N}(\tilde X_{\varepsilon,N}(x,r))\tilde R_{\varepsilon,N}(x,r)\big|\Big),\end{aligned}$$

where we have defined $V_{\varepsilon,N},\Omega_{\varepsilon,N}$ by setting $V_{\varepsilon,N}(X^n_{\varepsilon,N})=V^n_{\varepsilon,N}$ and $\Omega_{\varepsilon,N}(X^n_{\varepsilon,N})=\Omega^n_{\varepsilon,N}$ for all $1\le n\le N$. Inserting the first-order expansions of particle translational and angular velocities stated in Proposition 4.1, as well as the definition (5.2) of limiting velocities, we deduce

$$\begin{aligned}\tfrac{d^+}{dt}W_\infty(\nu_{\varepsilon,N},\tilde\nu_\lambda)\ \lesssim_h\ &\operatorname*{sup\,ess}_{x,r;\tilde\nu_\lambda}\big|(\mathcal G*h)(x)-(\mathcal G*h)(\tilde X_{\varepsilon,N}(x,r))\big|\\&+\operatorname*{sup\,ess}_{x,r;\tilde\nu_\lambda}\Big|\big(\Omega^\circ(r)\nabla(\mathcal G*h)(x)\big)r-\big(\Omega^\circ(\tilde R_{\varepsilon,N}(x,r))\nabla(\mathcal G*h)(\tilde X_{\varepsilon,N}(x,r))\big)\tilde R_{\varepsilon,N}(x,r)\Big|\\&+\lambda\big(\alpha^0_{\varepsilon,N}+\|\tilde\mu_\lambda\|_{\mathrm L^1\cap\mathrm L^\infty(\mathbb R^d)}\big)+\varepsilon.\end{aligned}$$

Using that for $\tilde\nu_\lambda$-almost all x,r we have

$$|x-\tilde X_{\varepsilon,N}(x,r)|,|r-\tilde R_{\varepsilon,N}(x,r)|\ \le\ W_\infty(\nu_{\varepsilon,N},\tilde\nu_\lambda),$$

and recalling that Ω° is smooth, the claim (5.9) follows.

Step 2. Second-order control on $W^\infty_{\varepsilon,N}$: proof that

$$\begin{aligned}\tfrac{d^+}{dt}W^\infty_{\varepsilon,N}\ \lesssim_h\ &W^\infty_{\varepsilon,N}+(\lambda\alpha^1_{\varepsilon,N}+\varepsilon)\big(\lambda(\alpha^0_{\varepsilon,N}+1)+\varepsilon\big)+\kappa_0\varepsilon\\&+\lambda\Big(Z^\infty_{\varepsilon,N}\log\big(2+\tfrac{1}{Z^\infty_{\varepsilon,N}}\big)+(W^\infty_{\varepsilon,N})^2+(W^\infty_{\varepsilon,N})^2(N^{\frac1d}\,\mathrm d^{\min}_{\varepsilon,N})^{2-d}\\&\qquad+Z^\infty_{\varepsilon,N}(N^{\frac1d}\,\mathrm d^{\min}_{\varepsilon,N})^{1-d}\Big)\|\tilde\mu_\lambda\|_{\mathrm L^1\cap\mathrm L^\infty(\mathbb R^d)}.\end{aligned}\tag{5.10}$$

Averaging the limiting transport equation (5.1) over r, we find

$$\partial_t\tilde\mu_\lambda+\operatorname{div}_x(\tilde\mu_\lambda\tilde u_\lambda)=0.$$

Comparing this transport equation with the particle dynamics (1.11), we can again estimate the time-derivative of the ∞-Wasserstein distance (5.7) by using characteristics as in [4],

$$\tfrac{d^+}{dt}W^\infty_{\varepsilon,N}\ \le\ \operatorname*{sup\,ess}_{x;\tilde\mu_\lambda}|\tilde u_\lambda(x)-V_{\varepsilon,N}(\hat X_{\varepsilon,N}(x))|.$$

Inserting the second-order expansion of particle velocities stated in Proposition 4.1, as well as the definition (5.2) of limiting velocities, we deduce

$$\begin{aligned}\tfrac{d^+}{dt}W^\infty_{\varepsilon,N}\ \lesssim_h\ &\operatorname*{sup\,ess}_{x;\tilde\mu_\lambda}\big|(\mathcal G*h)(x)-(\mathcal G*h)(\hat X_{\varepsilon,N}(x))\big|\\&+\lambda\operatorname*{sup\,ess}_{x;\tilde\mu_\lambda}\Big|\big[\mathcal G*(\tilde\mu_\lambda(e-h))\big](x)-\big[\mathcal G\,\hat*\,(\mu_{\varepsilon,N}(e-h))\big](\hat X_{\varepsilon,N}(x))\Big|\end{aligned}$$

$$+ \lambda \operatorname*{sup\,ess}_{x;\tilde\mu_\lambda} \Big| \big[\nabla\mathcal{G} * (\langle \Sigma_f^\circ \tilde\nu_\lambda\rangle)\big](x) - \big[\nabla\mathcal{G} \hat{*} (\langle \Sigma_f^\circ \nu_{\varepsilon,N}\rangle)\big](\hat X_{\varepsilon,N}(x))\Big|$$

$$+ \lambda \operatorname*{sup\,ess}_{x;\tilde\mu_\lambda} \Big| \big[\nabla\mathcal{G} * (\langle \Sigma^\circ \tilde\nu_\lambda\rangle\, \mathrm{D}(\mathcal{G}*h))\big](x) - \big[\nabla\mathcal{G} \hat{*} (\langle \Sigma^\circ \nu_{\varepsilon,N}\rangle\, \mathrm{D}(\mathcal{G}*h))\big](\hat X_{\varepsilon,N}(x))\Big|$$

$$+ (\lambda\alpha^1_{\varepsilon,N}+\varepsilon)\big(\lambda(\alpha^0_{\varepsilon,N}+1)+\varepsilon\big)+\kappa_0\varepsilon. \quad (5.11)$$

We analyze the different right-hand side terms separately. First, as in Step 1, the first term is bounded by $C_h W^\infty_{\varepsilon,N}$. Next, the second term can be decomposed as

$$\Big|\big[\mathcal{G}*(\tilde\mu_\lambda(e-h))\big](x) - \big[\mathcal{G}\hat{*}(\mu_{\varepsilon,N}(e-h))\big](\hat X_{\varepsilon,N}(x))\Big|$$
$$\le \Big| \int_{\mathbb{R}^d} \mathcal{G}(x-y)\,\tilde\mu_\lambda(y)\,dy - \int_{\mathbb{R}^d\setminus\{\hat X_{\varepsilon,N}(x)\}} \mathcal{G}(\hat X_{\varepsilon,N}(x)-y)\,\mu_{\varepsilon,N}(y)\,dy\Big|$$
$$+ \Big| \int_{\mathbb{R}^d} \mathcal{G}(x-y)\,h(y)\,\tilde\mu_\lambda(y)\,dy - \int_{\mathbb{R}^d\setminus\{\hat X_{\varepsilon,N}(x)\}} \mathcal{G}(\hat X_{\varepsilon,N}(x)-y)\,h(y)\,\mu_{\varepsilon,N}(y)\,dy\Big|,$$

and thus, recalling $(\hat X_{\varepsilon,N})_*\tilde\mu_\lambda = \mu_{\varepsilon,N}$,

$$\Big|\big(\mathcal{G}*(\tilde\mu_\lambda(e-h))\big)(x) - \big(\mathcal{G}\hat{*}(\mu_{\varepsilon,N}(e-h))\big)(\hat X_{\varepsilon,N}(x))\Big|$$
$$\le \int_{\mathbb{R}^d} \Big|\mathcal{G}(x-y) - (\mathbb{1}_{\neq}\mathcal{G})(\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y))\Big|\,\tilde\mu_\lambda(y)\,dy$$
$$+ \int_{\mathbb{R}^d} \Big|\mathcal{G}(x-y)\,h(y) - (\mathbb{1}_{\neq}\mathcal{G})(\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y))\,h(\hat X_{\varepsilon,N}(y))\Big|\,\tilde\mu_\lambda(y)\,dy,$$

where we use the short-hand notation $(\mathbb{1}_{\neq}\mathcal{G})(x-y) := \mathbb{1}_{x\neq y}\mathcal{G}(x-y)$. The last term can be simplified by using $|h(y)-h(\hat X_{\varepsilon,N}(y))| \lesssim_h W^\infty_{\varepsilon,N}$,

$$\Big|\big[\mathcal{G}*(\tilde\mu_\lambda(e-h))\big](x) - \big[\mathcal{G}\hat{*}(\mu_{\varepsilon,N}(e-h))\big](\hat X_{\varepsilon,N}(x))\Big|$$
$$\lesssim_h W^\infty_{\varepsilon,N}\|\tilde\mu_\lambda\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)} + \int_{\mathbb{R}^d} \Big|\mathcal{G}(x-y) - (\mathbb{1}_{\neq}\mathcal{G})(\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y))\Big|\,\tilde\mu_\lambda(y)\,dy. \quad (5.12)$$

We split the remaining integral into two parts, distinguishing between the contributions of $|x-y|\ge 4W^\infty_{\varepsilon,N}$ and $|x-y|<4W^\infty_{\varepsilon,N}$. On the one hand, appealing to pointwise bounds on the Stokeslet, cf. Lemma 2.3, in form of

$$|\mathcal{G}(x)-\mathcal{G}(x')| \lesssim \tfrac{|x-x'|}{(|x|\wedge|x'|)^{d-1}},$$

and noting that the condition $|x-y|\ge 4W^\infty_{\varepsilon,N}$ entails

$$|\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y)| \ge |x-y| - 2W^\infty_{\varepsilon,N} \ge \tfrac12|x-y|,$$

we find

$$\int_{y:|x-y|\ge 4W^\infty_{\varepsilon,N}} \Big|\mathcal{G}(x-y) - (\mathbb{1}_{\neq}\mathcal{G})(\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y))\Big|\,\tilde\mu_\lambda(y)\,dy$$
$$= \int_{y:|x-y|\ge 4W^\infty_{\varepsilon,N}} \Big|\mathcal{G}(x-y) - \mathcal{G}(\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y))\Big|\,\tilde\mu_\lambda(y)\,dy$$
$$\lesssim \int_{y:|x-y|\ge 4W^\infty_{\varepsilon,N}} \tfrac{|x-\hat X_{\varepsilon,N}(x)|+|y-\hat X_{\varepsilon,N}(y)|}{(|x-y|\wedge|\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y)|)^{d-1}}\,\tilde\mu_\lambda(y)\,dy$$

$$\lesssim W^\infty_{\varepsilon,N}\int_{y:|x-y|\ge 4W^\infty_{\varepsilon,N}}|x-y|^{1-d}\,\tilde\mu_\lambda(y)\,dy$$
$$\lesssim W^\infty_{\varepsilon,N}\|\tilde\mu_\lambda\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)}. \tag{5.13}$$

On the other hand, using pointwise bounds on the Stokeslet, cf. Lemma 2.3, we get

$$\int_{y:|x-y|<4W^\infty_{\varepsilon,N}}\Big|\mathcal{G}(x-y)-(\mathbb{1}_{\neq}\mathcal{G})(\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y))\Big|\,\tilde\mu_\lambda(y)\,dy$$
$$\lesssim \int_{y:|x-y|<4W^\infty_{\varepsilon,N}}|x-y|^{2-d}\,\tilde\mu_\lambda(y)\,dy$$
$$+\int_{y:|x-y|<4W^\infty_{\varepsilon,N}}|\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y)|^{2-d}\,\mathbb{1}_{\hat X_{\varepsilon,N}(x)\neq\hat X_{\varepsilon,N}(y)}\,\tilde\mu_\lambda(y)\,dy,$$

where the first right-hand side term is bounded by $(W^\infty_{\varepsilon,N})^2\|\tilde\mu_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^d)}$. In the usual form of the method developed by Hauray and Jabin [21, 4], the second right-hand side term would be estimated by $(W^\infty_{\varepsilon,N})^d(\mathrm{d}^{\min}_{\varepsilon,N})^{2-d}$. This would however yield a quite disappointing estimate in our setting as we only control $W^\infty_{\varepsilon,N}$ at best up to an $O(\lambda^2)$ error in the dilute regime. Instead, we use an idea by Höfer and Schubert [29]: as shown in [29, Lemma 3.1], there holds for any $\sigma\in(0,d)$ and $\ell\ge W^\infty_{\varepsilon,N}$,

$$\tfrac1N\sum_{m:m\neq n}|X^n_{\varepsilon,N}-X^m_{\varepsilon,N}|^{\sigma-d}\,\mathbb{1}_{|X^n_{\varepsilon,N}-X^m_{\varepsilon,N}|\le\ell}\lesssim_\sigma \|\tilde\mu_\lambda\|^{\frac{\sigma}{d}}_{\mathrm{L}^\infty(\mathbb{R}^d)}\ell^\sigma(N^{\frac1d}\,\mathrm{d}^{\min}_{\varepsilon,N})^{\sigma-d}. \tag{5.14}$$

Noting that the condition $|x-y|<4W^\infty_{\varepsilon,N}$ implies $|\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y)|\le 6W^\infty_{\varepsilon,N}$, and recalling $(\hat X_{\varepsilon,N})_*\tilde\mu_\lambda=\mu_{\varepsilon,N}$, we deduce

$$\int_{y:|x-y|<4W^\infty_{\varepsilon,N}}|\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y)|^{2-d}\,\mathbb{1}_{\hat X_{\varepsilon,N}(x)\neq\hat X_{\varepsilon,N}(y)}\,\tilde\mu_\lambda(y)\,dy$$
$$\le\ \tfrac1N\sum_{m:X^m_{\varepsilon,N}\neq\hat X_{\varepsilon,N}(x)}|\hat X_{\varepsilon,N}(x)-X^m_{\varepsilon,N}|^{2-d}\,\mathbb{1}_{|\hat X_{\varepsilon,N}(x)-X^m_{\varepsilon,N}|\le 6W^\infty_{\varepsilon,N}}$$
$$\lesssim\ \|\tilde\mu_\lambda\|^{\frac2d}_{\mathrm{L}^\infty(\mathbb{R}^d)}(W^\infty_{\varepsilon,N})^2(N^{\frac1d}\,\mathrm{d}^{\min}_{\varepsilon,N})^{2-d},$$

so the above becomes

$$\int_{y:|x-y|<4W^\infty_{\varepsilon,N}}\Big|\mathcal{G}(x-y)-(\mathbb{1}_{\neq}\mathcal{G})(\hat X_{\varepsilon,N}(x)-\hat X_{\varepsilon,N}(y))\Big|\,\tilde\mu_\lambda(y)\,dy$$
$$\lesssim (W^\infty_{\varepsilon,N})^2\Big(\|\tilde\mu_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^d)}+\|\tilde\mu_\lambda\|^{\frac2d}_{\mathrm{L}^\infty(\mathbb{R}^d)}(N^{\frac1d}\,\mathrm{d}^{\min}_{\varepsilon,N})^{2-d}\Big).$$

Inserting this into (5.12), together with (5.13), we obtain

$$\Big|\big[\mathcal{G}*(\tilde\mu_\lambda(e-h))\big](x)-\big[\mathcal{G}\,\hat*\,(\mu_{\varepsilon,N}(e-h))\big](\hat X_{\varepsilon,N}(x))\Big|$$
$$\lesssim_h\ \Big(W^\infty_{\varepsilon,N}+(W^\infty_{\varepsilon,N})^2+(W^\infty_{\varepsilon,N})^2(N^{\frac1d}\,\mathrm{d}^{\min}_{\varepsilon,N})^{2-d}\Big)\|\tilde\mu_\lambda\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)} \tag{5.15}$$

We turn to the third right-hand side term in (5.11). Using the transport plan $(\tilde X_{\varepsilon,N},\tilde R_{\varepsilon,N})$ given by (5.8), with $(\tilde X_{\varepsilon,N},\tilde R_{\varepsilon,N})_*\tilde\nu_\lambda=\nu_{\varepsilon,N}$, and recalling that Σ°_f is smooth (it is actually quadratic, cf. (1.23)), we can estimate

$$
\begin{aligned}
&\Big|\big[\nabla\mathcal{G}*(\langle\Sigma_f^\circ\tilde\nu_\lambda\rangle)\big](x)-\big[\nabla\mathcal{G}\,\hat{*}\,(\langle\Sigma_f^\circ\nu_{\varepsilon,N}\rangle)\big](\hat X_{\varepsilon,N}(x))\Big|\\
&\quad\le\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\Big|\nabla\mathcal{G}(x-y)\Sigma_f^\circ(r)\\
&\qquad\qquad-(\mathbb{1}_{\neq}\nabla\mathcal{G})\big(\hat X_{\varepsilon,N}(x)-\tilde X_{\varepsilon,N}(y,r)\big)\Sigma_f^\circ(\tilde R_{\varepsilon,N}(y,r))\Big|\,\tilde\nu_\lambda(y,r)\,dyd\sigma(r)\\
&\quad\lesssim Z_{\varepsilon,N}^\infty\|\tilde\mu_\lambda\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)}\\
&\qquad+\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\Big|\nabla\mathcal{G}(x-y)-(\mathbb{1}_{\neq}\nabla\mathcal{G})(\hat X_{\varepsilon,N}(x)-\tilde X_{\varepsilon,N}(y,r))\Big|\,\tilde\nu_\lambda(y,r)\,dyd\sigma(r).
\end{aligned}\tag{5.16}
$$

Now arguing similarly as above, we split the remaining integral into two parts, distinguishing between the contributions of $|x-y|\ge 4Z_{\varepsilon,N}^\infty$ and $|x-y|<4Z_{\varepsilon,N}^\infty$. On the one hand, using pointwise bounds on the Stokeslet, cf. Lemma 2.3, in form of

$$
|\nabla\mathcal{G}(x)-\nabla\mathcal{G}(x')|\lesssim\tfrac{|x-x'|}{(|x|\wedge|x'|)^d},
$$

and noting that the condition $|x-y|\ge 4Z_{\varepsilon,N}^\infty$ entails $|\hat X_{\varepsilon,N}(x)-\tilde X_{\varepsilon,N}(y,r)|\ge\frac12|x-y|$, we find

$$
\begin{aligned}
&\int_{y,r:|x-y|\ge 4Z_{\varepsilon,N}^\infty}\Big|\nabla\mathcal{G}(x-y)-(\mathbb{1}_{\neq}\nabla\mathcal{G})(\hat X_{\varepsilon,N}(x)-\tilde X_{\varepsilon,N}(y,r))\Big|\,\tilde\nu_\lambda(y,r)\,dyd\sigma(r)\\
&\quad\lesssim Z_{\varepsilon,N}^\infty\int_{y:|x-y|\ge 4Z_{\varepsilon,N}^\infty}|x-y|^{-d}\,\tilde\mu_\lambda(y)\,dy\\
&\quad\lesssim Z_{\varepsilon,N}^\infty\log\big(2+\tfrac{1}{Z_{\varepsilon,N}^\infty}\big)\|\tilde\mu_\lambda\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)}.
\end{aligned}
$$

On the other hand, using pointwise bounds on the Stokeslet, as well as (5.14), we get

$$
\begin{aligned}
&\int_{y,r:|x-y|\le 4Z_{\varepsilon,N}^\infty}\Big|\nabla\mathcal{G}(x-y)-(\mathbb{1}_{\neq}\nabla\mathcal{G})(\hat X_{\varepsilon,N}(x)-\tilde X_{\varepsilon,N}(y,r))\Big|\,\tilde\nu_\lambda(y,r)\,dyd\sigma(r)\\
&\qquad\qquad\lesssim Z_{\varepsilon,N}^\infty\Big(\|\tilde\mu_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^d)}+\|\tilde\mu_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^d)}^{\frac1d}(N^{\frac1d}\,\mathrm{d}_{\varepsilon,N}^{\min})^{1-d}\Big).
\end{aligned}
$$

Inserting these two estimates into (5.16), we deduce

$$
\begin{aligned}
&\Big|\big[\nabla\mathcal{G}*(\langle\Sigma_f^\circ\tilde\nu_\lambda\rangle)\big](x)-\big[\nabla\mathcal{G}\,\hat{*}\,(\langle\Sigma_f^\circ\nu_{\varepsilon,N}\rangle)\big](\hat X_{\varepsilon,N}(x))\Big|\\
&\qquad\lesssim\Big(Z_{\varepsilon,N}^\infty\log\big(2+\tfrac{1}{Z_{\varepsilon,N}^\infty}\big)+Z_{\varepsilon,N}^\infty(N^{\frac1d}\,\mathrm{d}_{\varepsilon,N}^{\min})^{1-d}\Big)\|\tilde\mu_\lambda\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)}.
\end{aligned}\tag{5.17}
$$

Finally, regarding the fourth right-hand side term in (5.11), a similar argument yields

$$
\begin{aligned}
&\Big|\big[\nabla\mathcal{G}*(\langle\Sigma^\circ\tilde\nu_\lambda\rangle\,\mathrm{D}(\mathcal{G}*h))\big](x)-\big[\nabla\mathcal{G}\,\hat{*}\,(\langle\Sigma^\circ\nu_{\varepsilon,N}\rangle\,\mathrm{D}(\mathcal{G}*h))\big](\hat X_{\varepsilon,N}(x))\Big|\\
&\qquad\lesssim_h\Big(Z_{\varepsilon,N}^\infty\log\big(2+\tfrac{1}{Z_{\varepsilon,N}^\infty}\big)+Z_{\varepsilon,N}^\infty(N^{\frac1d}\,\mathrm{d}_{\varepsilon,N}^{\min})^{1-d}\Big)\|\tilde\mu_\lambda\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)}.
\end{aligned}
$$

Inserting this into (5.11), together with (5.15), and (5.17), the claim (5.10) follows.

Step 3. Conclusion.
We use the notation $\mathcal{C}_{h,\mu}(t)$ for any constant that further depends on h,t, and on an upper

bound on $\|\tilde\mu_\lambda^s\|_{\mathrm{L}^\infty([0,t];\mathrm{L}^\infty(\mathbb{R}^d))}$. Recall the assumption $T^\theta_{\varepsilon,N}\ge T$, where $T^\theta_{\varepsilon,N}$ is the maximal time in Corollary 3.2,

$$T^\theta_{\varepsilon,N}:=\sup\Big\{t\ge0~:~\mathrm{d}^{\min}_{\varepsilon,N}(s)\ge(\theta N)^{-\frac1d},~\rho^{\max}_{\varepsilon,N}(s)\le\theta~~\text{for all }0\le s\le t\Big\}.$$

By the Gronwall inequality, the result (5.9) of Step 1 yields

$$Z^\infty_{\varepsilon,N}(t)\lesssim_{h,t}Z^\infty_{\varepsilon,N}(0)+\lambda\big(\alpha^0_{\varepsilon,N}+1+\|\tilde\mu_\lambda\|_{\mathrm{L}^\infty([0,t];\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d))}\big)+\varepsilon,$$

and thus, using (3.11), for all $t\in[0,T]$,

$$Z^\infty_{\varepsilon,N}(t)~\le~\mathcal{C}_{h,\mu}(t)\big(Z^\infty_{\varepsilon,N}(0)+\lambda\theta\log(\theta N)+\varepsilon\big),\tag{5.18}$$

which already proves (5.4).

We turn to the proof of (5.5). Using (5.18) to control the last $O(\lambda)$ term in the result (5.10) of Step 2, and using again (3.11), we get for all $t\in[0,T]$,

$$\begin{aligned}\tfrac{d^+}{dt}W^\infty_{\varepsilon,N}~\lesssim_h~&W^\infty_{\varepsilon,N}+(\lambda\theta^2+\varepsilon)\big(\lambda\theta\log(\theta N)+\varepsilon\big)+\kappa_0\varepsilon\\&+\lambda\mathcal{C}_{h,\mu}\Big(\big(Z^\infty_{\varepsilon,N}(0)+\lambda\theta\log(\theta N)+\varepsilon\big)\log\big(2+\tfrac{1}{Z^\infty_{\varepsilon,N}(0)+\lambda\log N+\varepsilon}\big)\\&\qquad+\theta(Z^\infty_{\varepsilon,N}(0))^2+\theta Z^\infty_{\varepsilon,N}(0)+\theta\varepsilon+\lambda\theta^2\log(\theta N)\Big).\end{aligned}$$

By the Gronwall inequality, this yields (5.5).

It remains to prove (5.6). We start with the expansion of the fluid velocity away from the particles as stated in Proposition 4.1: using (3.11), replacing $\mu_{\varepsilon,N},\nu_{\varepsilon,N}$ by $\tilde\mu_\lambda,\tilde\nu_\lambda$ up to errors estimated in terms of ∞-Wasserstein distances, and recognizing the definition (5.2) of $\tilde u_\lambda$, we get for any $\delta\in[\varepsilon,1]$,

$$\|u_{\varepsilon,N}-\tilde u_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^d\setminus(\mathcal{I}_{\varepsilon,N}+\delta B))}\lesssim_h(\lambda\theta^2+\varepsilon)\big(\lambda\theta\log(\theta N)+\varepsilon+(\tfrac{\varepsilon}{\delta})^d\big)+\lambda\big(Z^\infty_{\varepsilon,N}+(Z^\infty_{\varepsilon,N})^2\big).$$

Combined with (5.18), this yields the conclusion (5.6). ☐

Next, we turn to a corresponding mean-field approximation in the monokinetic regime, in which case we manage to further capture the effects of $O(\kappa_0\varepsilon)$ swimming forces. We then get an approximation with accuracy $O((\lambda+\varepsilon)^2)$ (up to logarithmic corrections and up to initial well-preparedness).

Proposition 5.2. *Assume that there is a solution $\tilde\mu_{\lambda,\varepsilon}\in C([0,T];\mathcal{P}\cap\mathrm{L}^\infty(\mathbb{R}^d))$ and $\tilde r_{\varepsilon,\lambda}\in C([0,T];W^{1,\infty}(\mathbb{R}^d;\mathbb{S}^{d-1}))$ of the following equations up to some time $T>0$,*

$$\left\{\begin{array}{l}\partial_t\tilde\mu_{\lambda,\varepsilon}+\operatorname{div}[\tilde\mu_{\lambda,\varepsilon}(\tilde u_{\lambda,\varepsilon}+\kappa_0\varepsilon V_f^\circ(\tilde r_{\lambda,\varepsilon}))]=0,\\ \partial_t\tilde r_{\lambda,\varepsilon}+(\tilde u_{\lambda,\varepsilon}+\kappa_0\varepsilon V_f^\circ(\tilde r_{\lambda,\varepsilon}))\cdot\nabla\tilde r_{\lambda,\varepsilon}=\tilde\Omega_{\lambda,\varepsilon}(\cdot,\tilde r_{\lambda,\varepsilon})\tilde r_{\lambda,\varepsilon},\\ (\tilde\mu_{\lambda,\varepsilon},\tilde r_{\lambda,\varepsilon})|_{t=0}=(\mu^\circ,r^\circ),\end{array}\right.\tag{5.19}$$

where the translational and angular velocity fields $\tilde u_{\lambda,\varepsilon}$ and $\tilde\Omega_{\lambda,\varepsilon}$ are given by

$$\begin{aligned}\tilde u_{\lambda,\varepsilon}(x)~&:=~\big[\mathcal{G}*\big((1-\lambda\tilde\mu_{\lambda,\varepsilon})h+\lambda\tilde\mu_{\lambda,\varepsilon}e\big)\big](x)\\&\qquad+\lambda\big[\nabla\mathcal{G}*\big(\tilde\mu_{\lambda,\varepsilon}(2\Sigma^\circ(\tilde r_{\lambda,\varepsilon})\mathrm{D}(\mathcal{G}*h)+\kappa_0\Sigma_f^\circ(\tilde r_{\lambda,\varepsilon}))\big)\big](x),\\ \tilde\Omega_{\lambda,\varepsilon}(x,r)~&:=~\Omega^\circ(r)\nabla(\mathcal{G}*h)(x).\end{aligned}\tag{5.20}$$

Denote by $W^\infty_{\varepsilon,N},Z^\infty_{\varepsilon,N}$ the ∞-Wasserstein distances of the empirical measures $\mu_{\varepsilon,N}$ and $\nu_{\varepsilon,N}$ to their mean-field approximations $\tilde\mu_{\lambda,\varepsilon}$ and $\tilde\nu_{\lambda,\varepsilon}(x,r):=\tilde\mu_{\lambda,\varepsilon}(x)\,\delta(r-\tilde r_{\lambda,\varepsilon}(x))$,

$$W^\infty_{\varepsilon,N}(t)~:=~W_\infty(\mu^t_{\varepsilon,N},\tilde\mu^t_{\lambda,\varepsilon})~\le~Z^\infty_{\varepsilon,N}(t)~:=~W_\infty(\nu^t_{\varepsilon,N},\tilde\nu^t_{\lambda,\varepsilon}).\tag{5.21}$$

Assume that initial particle positions satisfy, for some $\theta_0 \geq 1$,

$$\mathrm{d}^{\min}_{\varepsilon,N}(0) \,\geq\, (\theta_0 N)^{-\frac1d}, \qquad \rho^{\max}_{\varepsilon,N}(0) \,\leq\, \theta_0, \qquad Z^\infty_{\varepsilon,N}(0) \leq \theta_0.$$

Given $\theta > \theta_0$, further assume that $(\theta N)^{-\frac1d} \geq 4\varepsilon$ and that $\lambda\theta\log(2+\theta N) \ll 1$ is small enough, and assume that the maximal time $T^\theta_{\varepsilon,N}$ in Corollary 3.2 satisfies $T^\theta_{\varepsilon,N} \geq T$. Then, in the above terms, we have for all $t \in [0,T]$,

$$Z^\infty_{\varepsilon,N}(t) \;\leq\; \mathcal{C}_{\theta,h,\mu,r}(t)\big(Z^\infty_{\varepsilon,N}(0) + \lambda\log N + \varepsilon\big), \tag{5.22}$$

$$\begin{aligned} W^\infty_{\varepsilon,N}(t) \;\leq\; \mathcal{C}_{\theta,h,\mu,r}(t)\Big(&W^\infty_{\varepsilon,N}(0) + \kappa_0\varepsilon Z^\infty_{\varepsilon,N}(0) + (\lambda+\varepsilon)(\lambda\log N + \varepsilon)\\ &+\lambda\big(Z^\infty_{\varepsilon,N}(0)+\lambda\log N+\varepsilon\big)\log\big(2+\tfrac{1}{Z^\infty_{\varepsilon,N}(0)+\lambda\log N+\varepsilon}\big)\Big), \end{aligned} \tag{5.23}$$

and for any $\delta \in [\varepsilon,1]$,

$$\|u_{\varepsilon,N} - \tilde u_{\lambda,\varepsilon}\|_{\mathrm{L}^\infty(\mathbb{R}^d\setminus(\mathcal{I}_{\varepsilon,N}+\delta B))} \,\leq\, \mathcal{C}_{\theta,h,\mu,r}(t)\Big((\lambda+\varepsilon)\big(\lambda\log N + \varepsilon + (\tfrac{\varepsilon}{\delta})^d\big) + \lambda Z^\infty_{\varepsilon,N}(0)\Big),$$

where $\mathcal{C}_{\theta,h,\mu,r}(t)$ stands for any constant that further depends on θ, h, t, and on an upper bound on $\max_{0\leq s\leq t}\|(\tilde\mu^s_{\lambda,\varepsilon}, \nabla\tilde r^s_{\lambda,\varepsilon})\|_{\mathrm{L}^\infty(\mathbb{R}^d)}$. ◊

Proof. The proof follows that of Proposition 5.1 and we only indicate the main changes. For all $t\in[0,T]$, denote by $\hat X^t_{\varepsilon,N} : \mathbb{R}^d \to \mathbb{R}^d$ an optimal transport map such that

$$(\hat X^t_{\varepsilon,N})_*\tilde\mu^t_{\lambda,\varepsilon} \,=\, \mu^t_{\varepsilon,N}, \qquad W^\infty_{\varepsilon,N}(t) \,=\, \sup_{x;\tilde\mu^t_{\lambda,\varepsilon}}\mathrm{ess}\,|x - \hat X^t_{\varepsilon,N}(x)|.$$

Also note that the definition (5.21) of $Z^\infty_{\varepsilon,N}$ can be written as

$$Z^\infty_{\varepsilon,N}(t) \,:=\, \inf_X\Big[\sup_{x;\tilde\mu^t_{\lambda,\varepsilon}}\mathrm{ess}\,\Big(|x - X^t(x)| + \big|\tilde r^t_{\lambda,\varepsilon}(x) - R_{\varepsilon,N}(X^t(x))\big|\Big)\Big], \tag{5.24}$$

where the infimum runs over all measurable maps $X:\mathbb{R}^d\to\mathbb{R}^d$ with $X_*\tilde\mu^t_{\lambda,\varepsilon} = \mu^t_{\varepsilon,N}$, and where the map $R_{\varepsilon,N}$ is defined by setting $R_{\varepsilon,N}(X^n_{\varepsilon,N}) := R^n_{\varepsilon,N}$ for all $1\leq n\leq N$. We denote by $\tilde X^t_{\varepsilon,N}:\mathbb{R}^d\to\mathbb{R}^d$ an optimal transport map for this problem (5.24).

With this notation, we now turn to the control of $Z^\infty_{\varepsilon,N}$. Using characteristics, its time-derivative is estimated as follows,

$$\begin{aligned} \tfrac{d^+}{dt}Z^\infty_{\varepsilon,N} \,\leq\, \sup_{x;\tilde\mu_{\lambda,\varepsilon}}\mathrm{ess}\,\Big(&\big|\tilde u_{\lambda,\varepsilon}(x) + \kappa_0\varepsilon V^\circ_f(\tilde r_{\lambda,\varepsilon}(x)) - V_{\varepsilon,N}(\tilde X_{\varepsilon,N}(x))\big|\\ &+\big|\partial_t\tilde r_{\lambda,\varepsilon}(x) + (\tilde u_{\lambda,\varepsilon}(x) + \kappa_0\varepsilon V^\circ_f(\tilde r_{\lambda,\varepsilon}(x)))\cdot\nabla\tilde r_{\lambda,\varepsilon} - \Omega_{\varepsilon,N}(\tilde X_{\varepsilon,N}(x))R_{\varepsilon,N}(\tilde X_{\varepsilon,N}(x))\big|\Big) \end{aligned}$$

where, just as for $R_{\varepsilon,N}$, we have defined $V_{\varepsilon,N}$ and $\Omega_{\varepsilon,N}$ by setting $V_{\varepsilon,N}(X^n_{\varepsilon,N}) = V^n_{\varepsilon,N}$ and $\Omega_{\varepsilon,N}(X^n_{\varepsilon,N}) = \Omega^n_{\varepsilon,N}$ for all $1\leq n\leq N$. Inserting the first-order expansions of particle translational and angular velocities stated in Proposition 4.1, as well as the definition (5.20) of limiting velocities, we find

$$\begin{aligned} \tfrac{d^+}{dt}Z^\infty_{\varepsilon,N} \,\lesssim_h\, &\sup_{x;\tilde\mu_{\lambda,\varepsilon}}\mathrm{ess}\,\Big(\big|(\mathcal{G}*h)(x) - (\mathcal{G}*h)(\tilde X_{\varepsilon,N}(x))\big|\\ &+\big|\big(\Omega^\circ(\tilde r_{\lambda,\varepsilon}(x))\nabla(\mathcal{G}*h)(x)\big)\tilde r_{\lambda,\varepsilon}(x) - \big(\Omega^\circ(R_{\varepsilon,N}(\tilde X_{\varepsilon,N}(x)))\nabla(\mathcal{G}*h)(\tilde X_{\varepsilon,N}(x))\big)R_{\varepsilon,N}(\tilde X_{\varepsilon,N}(x))\big|\Big)\\ &+\lambda\big(\alpha^0_{\varepsilon,N} + \|\tilde\mu_{\lambda,\varepsilon}\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)}\big) + \varepsilon, \end{aligned}$$

and thus, by definition of $Z^\infty_{\varepsilon,N}$, cf. (5.24), as Ω° is smooth,

$$\tfrac{d^+}{dt} Z^\infty_{\varepsilon,N} \lesssim_h Z^\infty_{\varepsilon,N} + \lambda\big(\alpha^0_{\varepsilon,N} + \|\tilde\mu_{\lambda,\varepsilon}\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)}\big) + \varepsilon. \tag{5.25}$$

We turn to the corresponding control on $W^\infty_{\varepsilon,N}$. Arguing again by means of characteristics, we find

$$\tfrac{d^+}{dt} W^\infty_{\varepsilon,N} \le \operatorname*{sup\,ess}_{x;\tilde\mu_{\lambda,\varepsilon}} \big|\tilde u_{\lambda,\varepsilon}(x) + \kappa_0\varepsilon V^\circ_f(\tilde r_{\lambda,\varepsilon}(x)) - V_{\varepsilon,N}(\hat X_{\varepsilon,N}(x))\big|,$$

and thus, as above, by Proposition 4.1,

$$\tfrac{d^+}{dt} W^\infty_{\varepsilon,N} \lesssim_h W^\infty_{\varepsilon,N} + \lambda\big(\alpha^0_{\varepsilon,N} + \|\tilde\mu_{\lambda,\varepsilon}\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)}\big) + \varepsilon. \tag{5.26}$$

In order to get a more precise control on $W^\infty_{\varepsilon,N}$, we rather insert the second-order expansion of particle velocities stated in Proposition 4.1, to the effect of

$$\begin{aligned}\tfrac{d^+}{dt} \tilde W^\infty_{\varepsilon,N} \lesssim_h \operatorname*{sup\,ess}_{x;\tilde\mu_{\lambda,\varepsilon}} \Big(&\big|(\mathcal{G}*h)(x) - (\mathcal{G}*h)(\hat X_{\varepsilon,N}(x))\big| \\ &+ \kappa_0\varepsilon\big|V^\circ_f(\tilde r_{\lambda,\varepsilon}(x)) - V^\circ_f(R_{\varepsilon,N}(\hat X_{\varepsilon,N}(x)))\big| \\ &+ \lambda\big|\big[\mathcal{G}*\big(\tilde\mu_{\lambda,\varepsilon}(e-h)\big)\big](x) - \big[\mathcal{G}\,\hat{*}\,\big(\mu_{\varepsilon,N}(e-h)\big)\big](\hat X_{\varepsilon,N}(x))\big| \\ &+ \lambda\big|\big[\nabla\mathcal{G}*\big(\tilde\mu_{\lambda,\varepsilon}\Sigma^\circ_f(\tilde r_{\lambda,\varepsilon})\big)\big](x) - \big[\nabla\mathcal{G}\,\hat{*}\,\langle\Sigma^\circ_f\nu_{\varepsilon,N}\rangle\big)\big](\hat X_{\varepsilon,N}(x))\big| \\ &+ \lambda\big|\big[\nabla\mathcal{G}*\big(\tilde\mu_{\lambda,\varepsilon}\Sigma^\circ(\tilde r_{\lambda,\varepsilon})\mathrm{D}(\mathcal{G}*h)\big)\big](x) - \big[\nabla\mathcal{G}\,\hat{*}\,\big(\langle\Sigma^\circ\nu_{\varepsilon,N}\rangle\mathrm{D}(\mathcal{G}*h)\big)\big](\hat X_{\varepsilon,N}(x))\big|\Big) \\ &+ (\lambda\alpha^1_{\varepsilon,N}+\varepsilon)\big(\lambda(\alpha^0_{\varepsilon,N}+1)+\varepsilon\big).\end{aligned} \tag{5.27}$$

To estimate the right-hand side, we note that the Lipschitz continuity of $\tilde r_{\lambda,\varepsilon}$ yields

$$\operatorname*{sup\,ess}_{x;\tilde\mu_{\lambda,\varepsilon}} |\tilde r_{\lambda,\varepsilon}(x) - R_{\varepsilon,N}(\hat X_{\varepsilon,N}(x))| \le \big(1 + \|\nabla\tilde r_{\lambda,\varepsilon}\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\big) Z^\infty_{\varepsilon,N}.$$

The first two right-hand side terms in (5.27) are then bounded by

$$C_h W^\infty_{\varepsilon,N} + \kappa_0\varepsilon\big(1 + \|\nabla\tilde r_{\lambda,\varepsilon}\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\big) Z^\infty_{\varepsilon,N},$$

while the remaining three terms in bracket can be estimated similarly as in Step 2 of the proof of Proposition 5.1 above. This leads us to

$$\begin{aligned}\tfrac{d^+}{dt} W^\infty_{\varepsilon,N} \lesssim_h\; & W^\infty_{\varepsilon,N} + \kappa_0\varepsilon\big(1 + \|\nabla\tilde r_{\lambda,\varepsilon}\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\big) Z^\infty_{\varepsilon,N} + (\lambda\alpha^1_{\varepsilon,N}+\varepsilon)\big(\lambda(\alpha^0_{\varepsilon,N}+1)+\varepsilon\big) \\ &+ \lambda\Big(Z^\infty_{\varepsilon,N}\log\big(2 + \tfrac{1}{Z^\infty_{\varepsilon,N}}\big) + (W^\infty_{\varepsilon,N})^2 + (W^\infty_{\varepsilon,N})^2(N^{\frac1d}\,\mathrm{d}^{\min}_{\varepsilon,N})^{2-d} \\ &\qquad + Z^\infty_{\varepsilon,N}(N^{\frac1d}\,\mathrm{d}^{\min}_{\varepsilon,N})^{1-d}\Big)\|\tilde\mu_{\lambda,\varepsilon}\|_{\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^d)}.\end{aligned}$$

Combining this with (5.25) and (5.26), and appealing to the Gronwall inequality, the conclusion easily follows as in Step 3 of the proof of Proposition 5.1. We skip the detail for shortness. ☐

6. Conclusion: proof of main results

It remains to ensure the well-posedness of the macroscopic models (1.26) and (1.27) as stated in Proposition 1.2, and to conclude the proof of Theorems 1.3 and 1.4.

6.1. **Proof of Proposition 1.2.** We focus on the proof of Proposition 1.2(i) for the well-posedness of (1.26), while the argument is similar for (1.27). Let some non-integer $\gamma > 1$ be fixed. We proceed by an iteration argument. For $n = 0$, let $\nu_0 := \nu^\circ$ and $u_0 := 0$. Next, for all $n \geq 0$, we iteratively define ν_{n+1} as the solution of the linear transport equation

$$\begin{cases} \partial_t \nu_{n+1} + \operatorname{div}_x(\nu_{n+1} u_n) + \operatorname{div}_r(\nu_{n+1}(\Omega^\circ \nabla u_n) r) = 0, \\ \nu_{n+1}|_{t=0} = \nu^\circ, \end{cases} \tag{6.1}$$

and ∇u_{n+1} as the solution of the linear Stokes equation

$$\begin{cases} -\triangle u_{n+1} + \nabla p_{n+1} = (1-\lambda\mu_n) h + \lambda \mu_n e + \lambda \operatorname{div}\big[2\langle \Sigma^\circ \nu_n\rangle \mathrm{D}(u_n) + \kappa_0 \langle \Sigma_f^\circ \nu_n\rangle\big], \\ \operatorname{div}(u_{n+1}) = 0, \end{cases}$$

which means, in terms of the Stokeslet $\mathcal{G}$,

$$u_{n+1} \;=\; \mathcal{G} * \big((1-\lambda\mu_n)h + \lambda\mu_n e\big) + \lambda \nabla \mathcal{G} * \big(2\langle \Sigma^\circ \nu_n\rangle\, \mathrm{D}(u_n) + \kappa_0 \langle \Sigma_f^\circ \nu_n\rangle\big). \tag{6.2}$$

For all $n \geq 0$, if we have $u_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W^{\gamma+1,\infty}(\mathbb{R}^d)^d)$, recalling that Ω° is smooth and that initially $\nu^\circ \in \mathcal{P} \cap W^{1,1} \cap W^{\gamma,\infty}(\mathbb{R}^d \times \mathbb{S}^{d-1})$, the standard theory of transport equations in Hölder spaces (e.g. [1, Theorem 3.14]) ensures that (6.1) admits a unique weak solution $\nu_{n+1} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap W^{1,1} \cap W^{\gamma,\infty}(\mathbb{R}^d \times \mathbb{S}^{d-1}))$ with

$$\|\nu^t_{n+1}\|_{W^{1,1}\cap W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})} \leq \|\nu^\circ\|_{W^{1,1}\cap W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})} \exp\Big(C_\gamma t + C_\gamma \int_0^t \|u_n\|_{W^{\gamma+1,\infty}(\mathbb{R}^d)}\Big). \tag{6.3}$$

Moreover, if $\nu_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap W^{\gamma,\infty}(\mathbb{R}^d \times \mathbb{S}^{d-1}))$ and $u_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W^{\gamma+1,\infty}(\mathbb{R}^d)^d)$, recalling that $\Sigma^\circ, \Sigma_f^\circ$ are smooth and using the standard theory for the Stokes equation in Hölder spaces, we find that equation (6.2) yields $u_{n+1} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W^{\gamma+1,\infty}(\mathbb{R}^d)^d)$ with

$$\|u_{n+1}\|_{W^{\gamma+1,\infty}(\mathbb{R}^d)} \;\lesssim_{h,\gamma}\; 1 + \lambda \|\nu_n\|_{\mathrm{L}^1\cap W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})} \big(1 + \|u_n\|_{W^{\gamma+1,\infty}(\mathbb{R}^d)}\big). \tag{6.4}$$

By induction, this proves that we can indeed construct unique global weak solutions for the scheme (6.1)–(6.2) with $\nu_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap W^{1,1} \cap W^{\gamma,\infty}(\mathbb{R}^d \times \mathbb{S}^{d-1}))$ and $u_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W^{\gamma+1,\infty}(\mathbb{R}^d)^d)$ for all $n \geq 0$. From the above a priori estimates (6.3)–(6.4), absorbing the nonlinearity for small λ, we conclude the following: given $T > 0$, provided that $\lambda \ll_{T,h,\gamma,\nu^\circ} 1$ is small enough, we have for all $n \geq 0$ and $t \in [0,T]$,

$$\begin{aligned} \|\nu^t_n\|_{W^{1,1}\cap W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})} &\leq e^{C_{h,\gamma} t} \|\nu^\circ\|_{W^{1,1}\cap W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})}, \\ \|u^t_n\|_{W^{\gamma+1,\infty}(\mathbb{R}^d)} &\leq C_{h,\gamma}. \end{aligned} \tag{6.5}$$

Further appealing to the Aubin lemma, we may then extract a subsequence of $(\nu_n, u_n)_n$ that converges strongly to some limit (ν, u) in $C([0,T]; \mathrm{L}^1(\mathbb{R}^d \times \mathbb{S}^{d-1})) \times C([0,T]; W^{1,\infty}(\mathbb{R}^d)^d)$. Passing to the limit in the iterative scheme (6.1)–(6.2), we find that the limit (ν, u) precisely satisfies the Vlasov–Stokes system (1.26), and we may also pass to the limit in the a priori estimates (6.5).

It remains to establish the uniqueness of the solution of (1.26). Let $(\nu, u), (\nu_*, u_*)$ be two solutions in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T]; \mathcal{P} \cap W^{1,1} \cap W^{\gamma,\infty}(\mathbb{R}^d \times \mathbb{S}^{d-1})) \times \mathrm{L}^\infty_{\mathrm{loc}}([0,T]; W^{\gamma+1,\infty}(\mathbb{R}^d)^d)$ with the same initial data $\nu|_{t=0} = \nu_*|_{t=0} = \nu^\circ$. The equation for the difference $\nu - \nu_*$ can be written as

$$\partial_t(\nu-\nu_*)+\mathrm{div}_x\big((\nu-\nu_*)u\big)+\mathrm{div}_r\big((\nu-\nu_*)(\Omega^\circ\nabla u)r\big)$$
$$= -\mathrm{div}_x\big(\nu_*(u-u_*)\big)-\mathrm{div}_r\big(\nu_*(\Omega^\circ\nabla(u-u_*))r\big),$$

and the equation for $u-u_*$ as

$$u-u_* \;=\; \lambda\mathcal{G}*\big((\mu-\mu_*)(e-h)\big)+\lambda\nabla\mathcal{G}*\big(2\langle\Sigma^\circ\nu\rangle\,\mathrm{D}(u-u_*)\big)$$
$$+\lambda\nabla\mathcal{G}*\big(2\langle\Sigma^\circ(\nu-\nu_*)\rangle\,\mathrm{D}(u_*)+\kappa_0\langle\Sigma_f^\circ(\nu-\nu_*)\rangle\big).$$

Similar a priori estimates as in (6.3)–(6.4) then yield for all $t\in[0,T]$,

$$\|\nu^t-\nu_*^t\|_{\mathrm{L}^1\cap W^{\gamma-1,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})}\;\lesssim_s\;\exp\Big(C_\gamma t+C_\gamma\int_0^t\|u\|_{W^{\gamma,\infty}(\mathbb{R}^d)}\Big)\times\int_0^t\|\nu_*\|_{W^{1,1}\cap W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})}\|u-u_*\|_{W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})},\tag{6.6}$$
$$\|u-u_*\|_{W^{\gamma,\infty}(\mathbb{R}^d)}\;\lesssim_{h,\gamma}\;\lambda\|\nu\|_{\mathrm{L}^1\cap W^{\gamma-1,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})}\|u-u_*\|_{W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})}+\lambda\big(1+\|u_*\|_{W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})}\big)\|\nu-\nu_*\|_{\mathrm{L}^1\cap W^{\gamma-1,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

Using a priori bounds on (ν,u) and (ν_*,u_*) and recalling the choice $\lambda\ll_{T,h,\gamma,\nu^\circ}1$ to absorb the nonlinearity in the estimate for $u-u_*$, the conclusion $(\nu,u)=(\nu_*,u_*)$ follows from the Gronwall inequality. □

6.2. **Proof of Theorems 1.3 and 1.4.** We focus on the proof of Theorem 1.3, which is a simple post-processing of Proposition 5.1. Theorem 1.4 is similarly obtained by post-processing of Proposition 5.2 and we skip the corresponding detail for shortness.

First note that a simpler version of the proof of Proposition 1.2(i) also yields the following well-posedness result for (5.1)–(5.2): given $T>0$, non-integer $\gamma>1$, and given an initial condition $\nu^\circ\in\mathcal{P}\cap W^{1,1}\cap W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})$, there is $\lambda_0>0$ (depending on T,h,γ,ν°) such that for all $0\le\lambda\le\lambda_0$ there is a unique solution $(\tilde\nu_\lambda,\tilde u_\lambda)$ of (5.1)–(5.2) up to time T with $\tilde\nu_\lambda\in C([0,T];\mathcal{P}\cap W^{1,1}\cap W^{\gamma,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1}))$ and $\tilde u_\lambda\in C([0,T];W^{\gamma+1,\infty}(\mathbb{R}^d)^d)$.

In order to deduce Theorem 1.3 from Proposition 5.1, it remains to compare $(\tilde\nu_\lambda,\tilde u_\lambda)$ to the solution (ν_λ,u_λ) of (1.26): we shall prove for all $t\in[0,T]$,

$$\|u_\lambda^t-\tilde u_\lambda^t\|_{W^{\gamma,\infty}(\mathbb{R}^d)}\;\lesssim_{t,h,\gamma}\;\lambda^2,\tag{6.7}$$
$$W_\infty(\mu_\lambda^t,\tilde\mu_\lambda^t)\;\lesssim_{t,h}\;\lambda^2,\tag{6.8}$$
$$W_\infty(\nu_\lambda^t,\tilde\nu_\lambda^t)\;\lesssim_{t,h}\;\lambda.\tag{6.9}$$

First, taking the difference between the equations for ν_λ and $\tilde\nu_\lambda$, we find

$$\partial_t(\nu_\lambda-\tilde\nu_\lambda)+\mathrm{div}_x\big((\nu_\lambda-\tilde\nu_\lambda)u_\lambda\big)+\mathrm{div}_r\big((\nu_\lambda-\tilde\nu_\lambda)(\Omega^\circ\nabla u_\lambda)r\big)$$
$$= -\mathrm{div}_x\big(\tilde\nu_\lambda(u_\lambda-\tilde u_\lambda)\big)-\mathrm{div}_r\big(\tilde\nu_\lambda(\Omega^\circ\nabla(u_\lambda-\mathcal{G}*h))r\big),$$

from which we can deduce the following estimate, similarly as in (6.6), further using a priori bounds on $(\nu_\lambda,u_\lambda),(\tilde\nu_\lambda,\tilde u_\lambda)$, for all $t\in[0,T]$,

$$\|\nu_\lambda^t-\tilde\nu_\lambda^t\|_{\mathrm{L}^1\cap W^{\gamma-1,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})}\;\lesssim_{t,h,\gamma,\nu^\circ}\;\int_0^t\Big(\|u_\lambda-\tilde u_\lambda\|_{W^{\gamma-1,\infty}(\mathbb{R}^d)}+\|u_\lambda-\mathcal{G}*h\|_{W^{\gamma,\infty}(\mathbb{R}^d)}\Big).$$

Decomposing $u_\lambda - \mathcal{G}*h = (u_\lambda - \tilde u_\lambda) + (\tilde u_\lambda - \mathcal{G}*h)$ and using (5.2) to estimate the second piece, this yields for all $t\in[0,T]$,

$$\|\nu_\lambda^t - \tilde\nu_\lambda^t\|_{\mathrm{L}^1\cap W^{\gamma-1,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})} \lesssim_{t,h,\gamma,\nu^\circ} \lambda + \int_0^t \|u_\lambda - \tilde u_\lambda\|_{W^{\gamma,\infty}(\mathbb{R}^d)}. \tag{6.10}$$

Next, comparing the Stokes equation for u_λ in (1.26) with the definition (5.2) of $\tilde u_\lambda$,

$$\begin{aligned} u_\lambda - \tilde u_\lambda \;=\; & \lambda\mathcal{G}*\big((\mu_\lambda - \tilde\mu_\lambda)(e-h)\big) + \lambda\nabla\mathcal{G}*\big(2\langle\Sigma^\circ\tilde\nu_\lambda\rangle\,\mathrm{D}(u_\lambda - \tilde u_\lambda)\big)\\ &+ \lambda\nabla\mathcal{G}*\big(2\langle\Sigma^\circ(\nu_\lambda-\tilde\nu_\lambda)\rangle\,\mathrm{D}(u_\lambda) + \kappa_0\langle\Sigma_f^\circ(\nu_\lambda - \tilde\nu_\lambda)\rangle\big), \end{aligned} \tag{6.11}$$

which implies, using again a priori bounds on $(\nu_\lambda,u_\lambda),(\tilde\nu_\lambda,\tilde u_\lambda)$, for all $t\in[0,T]$,

$$\|u_\lambda^t - \tilde u_\lambda^t\|_{W^{\gamma,\infty}(\mathbb{R}^d)} \lesssim_{t,h,\gamma,\nu^\circ} \lambda\|\nu_\lambda^t - \tilde\nu_\lambda^t\|_{\mathrm{L}^1\cap W^{\gamma-1,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})} + \lambda\|u_\lambda^t - \tilde u_\lambda^t\|_{W^{\gamma,\infty}(\mathbb{R}^d)},$$

and thus, for $\lambda \ll_{T,h,\gamma,\nu^\circ} 1$ small enough,

$$\|u_\lambda^t - \tilde u_\lambda^t\|_{W^{\gamma,\infty}(\mathbb{R}^d)} \lesssim_{t,h,\gamma,\nu^\circ} \lambda\|\nu_\lambda^t - \tilde\nu_\lambda^t\|_{\mathrm{L}^1\cap W^{\gamma-1,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

Combining this with (6.10) and appealing to the Gronwall inequality, we deduce

$$\begin{aligned} \|\nu_\lambda^t - \tilde\nu_\lambda^t\|_{W^{\gamma-1,\infty}(\mathbb{R}^d\times\mathbb{S}^{d-1})} &\lesssim_{t,h,\gamma,\nu^\circ} \lambda,\\ \|u_\lambda^t - \tilde u_\lambda^t\|_{W^{\gamma,\infty}(\mathbb{R}^d)} &\lesssim_{t,h,\gamma,\nu^\circ} \lambda^2. \end{aligned} \tag{6.12}$$

The last estimate is already (6.7). We turn to proof of (6.8) and (6.9), that is, corresponding estimates for the ∞-Wasserstein distances between ν_λ, $\mu_\lambda = \langle\nu_\lambda\rangle$ and $\tilde\nu_\lambda$, $\tilde\mu_\lambda = \langle\tilde\nu_\lambda\rangle$. For all $t\in[0,T]$, denote by $\hat X_\lambda^t:\mathbb{R}^d\to\mathbb{R}^d$ an optimal transport map such that

$$(\hat X_\lambda^t)_*\mu_\lambda^t \;=\; \tilde\mu_\lambda^t, \qquad W_\infty(\mu_\lambda^t,\tilde\mu_\lambda^t) \;=\; \sup_{x;\mu_\lambda^t}\operatorname{ess}|x - \hat X_\lambda^t(x)|,$$

and denote by $(\tilde X_\lambda^t,\tilde R_\lambda^t):\mathbb{R}^d\times\mathbb{S}^{d-1}\to\mathbb{R}^d\times\mathbb{S}^{d-1}$ an optimal transport map such that

$$(\tilde X_\lambda^t,\tilde R_\lambda^t)_*\nu_\lambda^t \;=\; \tilde\nu_\lambda^t, \qquad W_\infty(\nu_\lambda^t,\tilde\nu_\lambda^t) \;=\; \sup_{x,r;\nu_\lambda^t}\operatorname{ess}\big|(x,r) - (\tilde X_\lambda^t(x,r),\tilde R_\lambda^t(x,r))\big|.$$

Comparing the transport equations for μ_λ and $\tilde\mu_\lambda$, which are obtained by integrating out equations (1.26) and (5.1) with respect to r,

$$\partial_t\mu_\lambda + \operatorname{div}(u_\lambda\mu_\lambda) = 0, \qquad \partial_t\tilde\mu_\lambda + \operatorname{div}(\tilde u_\lambda\tilde\mu_\lambda) = 0,$$

and using characteristics, we can estimate

$$\tfrac{d^+}{dt}W_\infty(\mu_\lambda,\tilde\mu_\lambda) \;\le\; \sup_{x;\mu_\lambda}\operatorname{ess}|u_\lambda(x) - \tilde u_\lambda(\hat X_\lambda(x))|,$$

and thus, using the a priori Lipschitz bound on $\tilde u_\lambda$,

$$\tfrac{d^+}{dt}W_\infty(\mu_\lambda,\tilde\mu_\lambda) \lesssim_{t,h,\gamma,\nu^\circ} W_\infty(\mu_\lambda,\tilde\mu_\lambda) + \|u_\lambda - \tilde u_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^d)}.$$

Combining this with (6.12) and appealing to the Gronwall inequality, the claim (6.8) follows. Next, comparing the transport equations for ν_λ and $\tilde\nu_\lambda$, again using characteristics,

$$\begin{aligned} &\tfrac{d^+}{dt}W_\infty(\nu_\lambda,\tilde\nu_\lambda)\\ &\le \sup_{x,r;\nu_\lambda}\operatorname{ess}\Big(\big|u_\lambda(x) - \tilde u_\lambda(\tilde X_\lambda(x,r))\big| + \big|(\Omega^\circ(r)\nabla u_\lambda(x))r - \tilde\Omega_\lambda(\tilde X_\lambda(x,r),\tilde R_\lambda(x,r))\tilde R_\lambda(x,r)\big|\Big). \end{aligned}$$

Inserting the definition (5.2) of $\tilde\Omega_\lambda$, and using again the a priori Lipschitz bound on $\tilde u_\lambda$ and the smoothness of Ω°, we are led to

$$\tfrac{d^+}{dt}W_\infty(\nu_\lambda,\tilde\nu_\lambda)\lesssim_{t,h} W_\infty(\nu_\lambda,\tilde\nu_\lambda)+\|u_\lambda-\tilde u_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^d)}+\|\nabla(u_\lambda-\mathcal{G}*h)\|_{\mathrm{L}^\infty(\mathbb{R}^d)}.$$

Decomposing again $u_\lambda-\mathcal{G}*h=(u_\lambda-\tilde u_\lambda)+(\tilde u_\lambda-\mathcal{G}*h)$, using (5.2) to estimate the second piece, and then combining the result with (6.12) and appealing to the Gronwall inequality, the claim (6.9) follows. This ends the proof of (6.7)–(6.9). The conclusion of Theorem 1.3 can then be deduced from Proposition 5.1, replacing $(\tilde\nu_\lambda,\tilde u_\lambda)$ by (ν_λ,u_λ). □

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