Functions with constant mean on similar countable subsets of $\mathbb{R}^2$

Cédric De Groote and Mitia Duerinckx

Abstract

We prove the following generalization of a problem proposed at the 70th William Lowell Putnam Mathematical Competition. Given a nonempty finite set E of n points in $\mathbb{R}^2$ and a function $f : \mathbb{R}^2 \to \mathbb{R}^d$ such that the arithmetic mean of the values of f at the n points of every image of E by a direct similarity is equal to a constant, then f is constant on $\mathbb{R}^2$. This result is extended to nonempty countable sets, and its validity is discussed in a more general context.

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function that satisfies $f(a) + f(b) + f(c) + f(d) = 0$ whenever a, b, c, and d are the four vertices of a square, then f is the null function. This problem, proposed at the 70th William Lowell Putnam Mathematical Competition [1], can be solved by a very simple geometric argument, which can easily be adapted to all regular n-gons for a given $n \geq 3$ in $\mathbb{R}^2$. We prove here a more general result.

Two nonempty subsets A and B of $\mathbb{R}^2$ are said to be directly similar (denoted by $A \sim B$) if there exists a direct similarity σ of $\mathbb{R}^2$ such that $\sigma(A) = B$.

Theorem 1. *If $E = \{p_1, \ldots, p_n\}$ is a nonempty finite set of n points in $\mathbb{R}^2$ and if $f : \mathbb{R}^2 \to \mathbb{R}^d$ is a function such that the arithmetic mean of the values of f at the points of every set $E' \sim E$ is equal to $c \in \mathbb{R}^d$, then $f(x) = c$ for all $x \in \mathbb{R}^2$.*

Proof. Since the case $n = 1$ is trivial, we may assume that $n \geq 2$. In order to prove that $f(x) = c$ for every point $x \in \mathbb{R}^2$, it suffices to prove that $f(p_1) = c$. Indeed, if E' is the image of E under the translation mapping p_1 onto x, the orbits of E and E' in the group of all direct similarities of $\mathbb{R}^2$ are exactly the same, and so there is no loss of generality in assuming that $x = p_1$.

Let o be a point of $\mathbb{R}^2$ such that $o \notin E$. For every $j = 2, \ldots, n$, let σ_j be a direct similarity of $\mathbb{R}^2$ fixing o such that $\sigma_j(p_1) = p_j$, and let σ_1 be the identity, so that we can write $E = \{\sigma_1(p_1), \sigma_2(p_1), \ldots, \sigma_n(p_1)\}$. Every σ_j is the product of a rotation with a homothecy, both with center o.

If τ is the translation mapping o onto p_1, we define, for each $j = 2, \ldots, n$,

$$\begin{aligned} & F_j = \{\sigma_1^\tau(p_j), \sigma_2^\tau(p_j), \ldots, \sigma_n^\tau(p_j)\} \\ \text{and} \quad & G_j = \{\sigma_j^\tau(p_1), \sigma_j^\tau(p_2), \ldots, \sigma_j^\tau(p_n)\}. \end{aligned}$$

Note that σ_j^τ, *i.e.* σ_j conjugated by τ, is a direct similarity fixing p_1.

Clearly $G_j \sim E$ for all $j = 2, \ldots, n$, because $G_j = \sigma_j^\tau(E)$. Moreover, $F_j \sim E$ since

$$F_j = \tau(\{\sigma_1(\tau^{-1}(p_j)), \ldots, \sigma_n(\tau^{-1}(p_j))\}) = \tau(\{\sigma_1(\sigma_j^*(p_1)), \ldots, \sigma_n(\sigma_j^*(p_1))\})$$

(where σ_j^* is the direct similarity of $\mathbb{R}^2$ fixing o and mapping p_1 onto $\tau^{-1}(p_j)$), and so, by using the commutativity of the group of all direct similarities of $\mathbb{R}^2$ fixing o,

$$F_j = (\tau \circ \sigma_j^*)(\{\sigma_1(p_1), \ldots, \sigma_n(p_1)\}) \sim E.$$

By definition of f, we have, for every $j = 2, \ldots, n$,

$$\frac{1}{n}\Big[f(p_j) + \sum_{k=2}^{n} f(\sigma_k^\tau(p_j))\Big] = c \qquad \text{(because } F_j \sim E\text{)}, \tag{1}$$

$$\frac{1}{n}\Big[f(p_1) + \sum_{k=2}^{n} f(\sigma_j^\tau(p_k))\Big] = c \qquad \text{(because } G_j \sim E\text{)}, \tag{2}$$

and moreover

$$\frac{1}{n}\Big[\sum_{k=1}^{n} f(p_k)\Big] = c. \tag{3}$$

By adding equality (3) to the sum of the $n-1$ equalities (2) and subtracting the $n-1$ equalities (1), we get

$$\frac{1}{n}\left[\sum_{k=1}^{n} f(p_k)\right] + \sum_{j=2}^{n} \frac{1}{n}\left[f(p_1) + \sum_{k=2}^{n} f(\sigma_j^\tau(p_k))\right] - \sum_{j=2}^{n} \frac{1}{n}\left[f(p_j) + \sum_{k=2}^{n} f(\sigma_k^\tau(p_j))\right]$$
$$= c + (n-1)c - (n-1)c,$$

which reduces to

$$\frac{1}{n} f(p_1) + \frac{n-1}{n} f(p_1) = c,$$

and so $f(p_1) = c$. □

It is tempting to try to extend this result to weighted averages, E being a nonempty ordered collection of n points. However, the fact that the weights are not necessarily equal breaks the symmetry between the points and our method of proof no longer works.

Note that Theorem 1 can be naturally extended to nonempty countable sets as follows:

Theorem 2. *If $E = \{p_i | i \in \mathbb{N}\} \subset \mathbb{R}^2$ is a nonempty countable set and if $f : \mathbb{R}^2 \to \mathbb{R}^d$ is a function such that, for all $E' = \{q_i | i \in \mathbb{N}\} \sim E$, the series $\sum_{i=0}^{\infty} f(q_i)$ is absolutely convergent and equal to 0, then $f(x) = 0$ for all $x \in \mathbb{R}^2$.*

This can be proven by exactly the same argument as the one given above, since the absolute convergence of the series $\sum_{i=0}^{\infty} f(q_i)$ allows us to rearrange its terms freely.

The above theorems motivate many further questions. What can be said if we consider, rather than direct similarities, another subgroup of the group $AGL(2, \mathbb{R})$ of all affine transformations of $\mathbb{R}^2$, or if we replace $\mathbb{R}^2$ by $\mathbb{R}^p$ or even by K^p for any field K? More precisely, for any subgroup G of $AGL(p, K)$, if $E = \{p_1, \ldots, p_n\}$ is a nonempty finite set of n points in K^p and if $f : K^p \to \mathbb{R}^d$ is a function such that the arithmetic mean of the values of f at the points of every set $E' = g(E)$ (where $g \in G$) is equal to $c \in \mathbb{R}^d$, does this imply that f is a constant function?

It is straightforward to see that the above proof can be directly adapted to give a positive answer to this general question only if G contains a transitive subgroup H such that the stabilizer of any point is transitive and abelian; in other words, G has to contain a 2-transitive subgroup H such that the stabilizer of the origin is abelian. Such subgroups H of $AGL(p, K)$ can be determined in general [2]: H exists if and only if K admits a field extension K' of degree p, and then H is isomorphic to $AGL(1, K')$. In particular, when $K = \mathbb{R}$, no such subgroup exists if $p \geq 3$ (as a consequence of Hurwitz's theorem), whereas H is the subgroup of direct similarities for $p = 2$. When $K = \mathbb{Q}$, such subgroups H exist for every dimension p.

However, when there is no such subgroup H in G, our argument cannot be extended and the problem is open.

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References

[1] The 70th William Lowell Putnam Mathematical Competition, *Amer. Math. Monthly* **117** (2010) 714-721.

[2] F. Buekenhout, private communication, 2011.

Department of Mathematics, Université Libre de Bruxelles, Boulevard du Triomphe, B-1050 Brussels, Belgium
mitia.duerinckx@ulb.ac.be

Department of Mathematics, Université Libre de Bruxelles, Boulevard du Triomphe, B-1050 Brussels, Belgium
cedric.de.groote@ulb.ac.be