HYDRODYNAMIC LIMIT OF MULTISCALE VISCOELASTIC MODELS FOR RIGID PARTICLE SUSPENSIONS

MITIA DUERINCKX, LUCAS ERTZBISCHOFF,
ALEXANDRE GIRODROUX-LAVIGNE, RICHARD M. HÖFER

Abstract. We study the multiscale viscoelastic Doi model for suspensions of Brownian rigid rod-like particles, as well as its generalization by Saintillan and Shelley for self-propelled particles. We consider the regime of a small Weissenberg number, which corresponds to a fast rotational diffusion compared to the fluid velocity gradient, and we analyze the resulting hydrodynamic approximation. More precisely, we show the asymptotic validity of macroscopic nonlinear viscoelastic models, in form of so-called ordered fluid models, as an expansion in the Weissenberg number. The result holds for zero Reynolds number in 3D and for arbitrary Reynolds number in 2D. Along the way, we establish several new well-posedness and regularity results for nonlinear fluid models, which may be of independent interest.

Contents

1. Introduction

1.1. **General overview.** Suspensions of rigid particles in Stokesian fluids are ubiquitous both in nature and in applications and are known to typically exhibit non-Newtonian behaviors. These systems can be described on different scales: either by macroscopic non-Newtonian fluid models, or by so-called multiscale kinetic models, or else on the microscopic scale as suspended particles moving in a fluid flow. Let us briefly describe these different levels of physical description:

- *Macroscopic non-Newtonian fluid models:*
  An incompressible fluid flow is generally described by the Navier–Stokes equations,

$$\left\{\begin{array}{l}\rho_{\mathrm{fl}}(\partial_t + u\cdot\nabla)u - \operatorname{div}(\sigma) + \nabla p \,=\, h,\\ \operatorname{div}(u) = 0,\end{array}\right. \tag{1.1}$$

where $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ is the fluid velocity field, where the source term h accounts for internal forces, where $\rho_{\rm fl}$ stands for the fluid density (constant, say), and where σ is the deviatoric fluid stress (a trace-free symmetric matrix field). For Newtonian fluids, the stress is linear in the strain rate, that is,

$$\sigma = 2\mu\,\mathrm{D}(u), \tag{1.2}$$

where the strain rate $\mathrm{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is defined as the symmetric gradient of the fluid velocity, and where the viscosity $\mu \geq 0$ is a constant. In contrast, non-Newtonian fluids are characterized by more complex constitutive laws relating the stress σ to the strain rate $\mathrm{D}(u)$, describing various possible types of nonlinear and memory effects. In *macroscopic* non-Newtonian models, these laws are assumed to take some explicit form and are usually fitted phenomenologically to experimental rheological measurements. In Reiner–Rivlin and in generalized Newtonian fluid models, the stress σ is simply taken to be a local function of $\mathrm{D}(u)$, meaning that only nonlinear effects are retained. To further describe memory effects of non-Newtonian fluids, such as viscoelastic properties, more realistic models rather relate σ and $\mathrm{D}(u)$ via integral or differential equations, which aim to take into account the dependence of the stress on the fluid deformation history. Such models that are frame invariant are generically called *simple fluids*, of which the celebrated Oldroyd–B and FENE–P models are particular cases. In the fast relaxation limit, simple fluid models reduce to the hierarchy of so-called ordered fluid models. We refer to Section 3 for details.

- *Multiscale kinetic models for suspensions:*
  So-called *multiscale* or *micro-macro* models go one step away from the pure phenomenology, towards a microscopically more accurate description of particle suspensions. More precisely, the macroscopic fluid equation (1.1) gets coupled via the stress σ to a kinetic equation describing the evolution of the suspended solid phase. The latter is modeled by a particle density function $f(t, x, n) \in \mathbb{R}^+$ at time t, where x is the position of particles and n is their 'state': for instance, $n \in \mathbb{R}^d$ may describe the relative position of endpoints of elongated particles (hence $n \in \mathbb{S}^{d-1}$ in case of rigid suspended particles as will be considered in this work). The evolution of the particle density f is then modeled by a Fokker–Planck equation describing transport with the fluid and diffusive effects. Finally, the coupling to the macroscopic fluid equation is expressed through an explicit constitutive law $\sigma = \sigma(f, \mathrm{D}(u))$, which is typically derived from formal microscopic considerations in dilute regime. Such models describe how the microscopic state of the particles adapts collectively to local fluid deformations and how the macroscopic fluid flow gets itself effectively impacted. Popular models include the kinetic FENE and Hookean dumbbell models for dilute suspensions of flexible polymers, the so-called Doi model for suspensions of Brownian rigid rod-like particles, and the Doi–Saintillan–Shelley for corresponding active particles. We refer to Section 2 for details.
- *Microscopic models:*
  At the particle scale, we can formulate a fully detailed hydrodynamic model describing the motion of suspended particles in the Stokesian background fluid flow. It takes form of equation (1.1) restricted to the fluid domain, with Newtonian constitutive law (1.2), coupled with Newton's equations of motion for the particles. We refer for instance to [HS23] for a discussion of this complex dynamics.

From the modeling perspective, while microscopic models are certainly impractical due to the huge number of particles in real-life systems, one can argue that multiscale kinetic models are more satisfactory than macroscopic fluid models as they retain some information from the fluid-particle coupling and therefore better reveal mechanisms leading to non-Newtonian behavior. Yet, macroscopic models appeal through their simpler description: in particular, they are much more accessible for numerical simulations and proner to comparison with experimental rheological measurements.

Most previous works on particle suspensions have aimed either to study properties of macroscopic non-Newtonian fluid models or multiscale kinetic models, or else to derive multiscale kinetic models rigorously from microscopic particle dynamics, which has indeed attracted considerable interest in recent years; see Section 1.3 below for references. In the present contribution, we propose to fill the gap in the micro-macro understanding of non-Newtonian effects of particle suspensions by further studying the derivation of explicit macroscopic fluid models from multiscale kinetic models in suitable regimes.

In some exceptional cases, the kinetic equation describing the state of the particles in micro-macro models can be integrated out and leads to a closed equation for the stress σ in terms of the strain rate $\mathrm{D}(u)$. This is for instance the case for the kinetic Hookean dumbbell model, which is well-known to be formally equivalent to the Oldroyd–B macroscopic fluid model (see e.g. the recent rigorous analysis in [DS23]). In general, however, no exact macroscopic closure is available and we can only hope for perturbative closures to be valid in suitable asymptotic regimes.

In fact, in the regime of weak non-Newtonian effects, a fairly large class of non-Newtonian macroscopic fluid models is believed to be well approximated by a special family of models, called *ordered fluid* models. More precisely, the latter are expected to be good approximations for all viscoelastic fluids in the fading memory regime, that is, in the regime when the elastic time-dependent effects due to suspended particles in the fluid have an inherent relaxation timescale that is much shorter than the overall timescale of the fluid flow. This ratio of timescales is the so-called Weissenberg number Wi. In other words, ordered fluid models arise formally as expansions of any viscoelastic fluid model at small Wi, which is sometimes referred to as the retarded motion expansion. A first-order fluid is a Newtonian fluid, a second-order fluid is a non-Newtonian fluid where effects of order $O(\mathrm{Wi}^2)$ are neglected, a third-order fluid amounts to neglecting effects of order $O(\mathrm{Wi}^3)$, etc.

In the present work, we focus on the so-called Doi model, which is a multiscale kinetic model describing suspensions of Brownian rigid rod-like particles, and we further consider its generalization by Saintillan and Shelley for active (self-propelled) particles. We rigorously analyze the hydrodynamic limit of these models in the small-Wi expansion, which was extensively studied on a formal level in the physics literature in the 1970s, see [HL72; Bre74], and we confirm the asymptotic validity of ordered fluid models in this setting. As a prerequisite, the justification of the asymptotic expansion requires a careful study both of the Doi–Saintillan–Shelley model and of ordered fluid models: in particular, we establish some new well-posedness and regularity results for these nonlinear viscoelastic fluid models, which we believe are nontrivial and of independent interest. Moreover, as the Doi–Saintillan–Shelley model can itself be derived from a microscopic hydrodynamic model (at least formally), see Section 1.3, our derivation of macroscopic ordered fluid models comes together with explicit expressions for rheological parameters in terms of microscopic characteristics of the underlying particle suspension.

1.2. **Informal statement of main results.** We start from the following dimensionless Doi–Saintillan–Shelley model describing suspensions of (active or passive) Brownian rigid rod-like particles in a Stokesian fluid,

$$\begin{cases} \mathrm{Re}\,\big(\partial_t u_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon\big) - \Delta u_\varepsilon + \nabla p_\varepsilon = h + \frac{1}{\varepsilon}\mathrm{div}(\sigma_1[f_\varepsilon]) + \mathrm{div}(\sigma_2[f_\varepsilon,\nabla u_\varepsilon]),\\ \partial_t f_\varepsilon + \mathrm{div}_x\big((u_\varepsilon + U_0 n)f_\varepsilon\big) + \mathrm{div}_n\big(\pi_n^\perp(\nabla u_\varepsilon)n f_\varepsilon\big) = \frac{1}{\mathrm{Pe}}\Delta_x f_\varepsilon + \frac{1}{\varepsilon}\Delta_n f_\varepsilon,\\ \mathrm{div}(u_\varepsilon) = 0, \end{cases} \tag{1.3}$$

in terms of the elastic and viscous stresses

$$\sigma_1[f] \;:=\; \lambda\theta \int_{\mathbb{S}^{d-1}} \big(n\otimes n - \tfrac{1}{d}\,\mathrm{Id}\,\big) f(\cdot,n)\,dn, \tag{1.4}$$

$$\sigma_2[f,\nabla u] \;:=\; \lambda \int_{\mathbb{S}^{d-1}} (n\otimes n)(\nabla u)(n\otimes n)\,f\,dn, \tag{1.5}$$

where $\varepsilon := \mathrm{Wi} > 0$ stands for the Weissenberg number, $\mathrm{Re} \ge 0$ is the Reynolds number, $\mathrm{Pe} > 0$ is the so-called Péclet number, and where the source term h is taken to be smooth and accounts for internal forces. The state variable n in the kinetic Fokker–Planck equation describes the orientation of rigid particles on the unit sphere, hence $n \in \mathbb{S}^{d-1}$: we write Δ_n and div_n for the Laplace–Beltrami operator and the divergence on the sphere $\mathbb{S}^{d-1}$, and we also use the short-hand notation $\pi_n^\perp := \mathrm{Id} - n\otimes n$ for the orthogonal projection onto $n^\perp$. The above system is introduced in detail in Section 2, where in particular the constants $U_0, \lambda, \theta \in \mathbb{R}$ are further described. The Doi model for passive suspensions is recovered for the special choice $U_0 = 0$ and $\theta = 6$. We set this model for simplicity in a finite box $\mathbb{T}^d = [0,1)^d$ with periodic boundary conditions, and consider space dimension $d = 2$ or 3.

Note that we take into account a non-vanishing spatial diffusion $O(\frac{1}{\mathrm{Pe}})$ in (1.3), which differs from the setting usually considered in applications where $\mathrm{Pe} = \infty$: a nontrivial spatial diffusion $\mathrm{Pe} < \infty$ is actually needed in the present work for technical well-posedness reasons. Due to this spatial diffusion, the structure of ordered fluid equations needs to be slightly adapted, leading to the nonstandard definition of Rivlin–Ericksen tensors in (1.8) below; see Section 3 for details. This is reminiscent of the version of the Oldroyd–B model with stress diffusion that is often considered both for analytical and numerical studies [RT21].

Our main result is the following asymptotic validity of second-order fluid models. We refer to Theorem 4.2 in Section 4 for a more detailed statement, including the precise well-preparedness requirement for kinetic initial data, as well as the explicit expression for the effective coefficients $\mu, \nu, \gamma_1, \gamma_2$ and their rheological interpretation. New results on the well-posedness of the kinetic model (1.3) and of the second-order fluid model (1.8) are postponed to Sections 2 and 3, respectively.

Theorem 1.1 (Informal statement of the main result). *Consider either the Stokes case* $\mathrm{Re} = 0$ *with* $d \le 3$, *or the Navier–Stokes case* $\mathrm{Re} = 1$ *with* $d = 2$. *Given an initial particle density* $f_\varepsilon^\circ \in C^\infty \cap \mathcal{P}(\mathbb{T}^d \times \mathbb{S}^{d-1})$ *that is well-prepared in a sense that will be clarified later (see Assumption 4.1), and given also an initial fluid velocity* $u^\circ \in C^\infty(\mathbb{T}^d)^d$ *with* $\mathrm{div}(u^\circ) = 0$ *in the Navier–Stokes case, consider a weak global solution* $(u_\varepsilon, f_\varepsilon)$ *of the Cauchy problem for the Doi–Saintillan–Shelley model* (1.3)*. For all* $T > 0$*, provided that*

$$\varepsilon \ll 1 \qquad \textit{and} \qquad \lambda\theta(1+\mathrm{Pe}) \ll 1 \tag{1.6}$$

are small enough, the fluid velocity u_ε *satisfies*

$$\begin{aligned} \|\nabla(u_\varepsilon - \bar u_\varepsilon)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d)^{d^2})} &\lesssim \varepsilon^2, \qquad \textit{if}\ \ \mathrm{Re} = 0,\\ \|u_\varepsilon - \bar u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^d)^d)} + \|\nabla(u_\varepsilon - \bar u_\varepsilon)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d)^{d^2})} &\lesssim \varepsilon^2, \qquad \textit{if}\ \ \mathrm{Re} = 1, \end{aligned}$$

and the particles' spatial density $\rho_\varepsilon := \fint_{\mathbb{S}^{d-1}} f_\varepsilon(\cdot,n)\,\mathrm{d}n$ *satisfies*

$$\|\rho_\varepsilon - \bar\rho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \|\nabla(\rho_\varepsilon - \bar\rho_\varepsilon)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d)^d)} \lesssim \varepsilon^2,$$

where $(\bar u_\varepsilon, \bar\rho_\varepsilon)$ *solves the following (non-standard) second-order fluid equation*

$$\begin{cases} \mathrm{Re}(\partial_t + \bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon - \operatorname{div}(\bar\sigma_\varepsilon) + \nabla\bar p_\varepsilon \;=\; h + O(\varepsilon^2), \\ (\partial_t + \bar u_\varepsilon\cdot\nabla)\bar\rho_\varepsilon - (\frac{1}{\mathrm{Pe}} + \varepsilon\nu)\Delta\bar\rho_\varepsilon = O(\varepsilon^2), \\ \bar\sigma_\varepsilon = (1+\mu\bar\rho_\varepsilon)A_1(\bar u_\varepsilon) + \varepsilon\gamma_1 A_2'(\bar u_\varepsilon,\bar\rho_\varepsilon) + \varepsilon\gamma_2\bar\rho_\varepsilon A_1(\bar u_\varepsilon)^2, \\ \operatorname{div}(\bar u_\varepsilon) = 0, \end{cases} \tag{1.7}$$

for some explicit coefficients $\mu,\nu,\gamma_1,\gamma_2\in\mathbb{R}$*, where* $A_1(u) := 2\,\mathrm{D}(u) = (\nabla u)^T + \nabla u$ *is the strain rate and where* A_2' *is a (non-standard) inhomogeneous, diffusive version of the second Rivlin-Ericksen tensor, defined by*

$$A_2'(\rho,u) \;:=\; \big(\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u\cdot\nabla\big)\big(\rho A_1(u)\big) + \rho\big((\nabla u)^T A_1(u) + A_1(u)(\nabla u)\big). \tag{1.8}$$

◊

Remark 1.2. A few comments are in order:

(a) The well-preparedness assumption for the initial kinetic data will be clarified later on in Assumption 4.1. Informally, it amounts to assuming that initial data are locally at equilibrium with respect to the dynamics of orientations and are perturbatively compatible with the formal ε-expansion. It allows to avoid initial boundary layers.

(b) The existence of global weak solutions for the Doi–Saintillan–Shelley model (1.3), which is assumed to hold in the above statement, is indeed proven in Section 2.2 below. We further establish a new weak-strong uniqueness principle for this system, which implies some stability that is at the very heart of the above result.

(c) Although the second-order fluid model (1.7) is well-known to be ill-posed whenever $\gamma_1 < 0$ (which is indeed the case for relevant effective parameters), we can define several well-posed notions of approximate solutions that only satisfy (1.7) up to higher-order $O(\varepsilon^2)$ errors. This discussion is postponed to Section 3, where we present two approaches to fix this issue: First, we introduce the notion of approximate *hierarchical solutions*, which naturally appear as low-Wi expansions; see Propositions 3.2 and 3.4. Second, by means of a Boussinesq-type perturbative rearrangement, we additionally provide a reformulation of second-order fluid equations in terms of a closed well-posed system; see Proposition 3.3.

(d) The explicit expression for the effective second-order fluid coefficients $\mu,\nu,\gamma_1,\gamma_2$ is postponed to Section 4, where we further describe the rheological properties of the obtained macroscopic fluid model. The expressions for the coefficients agree with those computed in [HL72; Bre74] in the case of passive suspensions. Moreover, they qualitatively match experimental data and formal predictions on active suspensions.

(e) A similar result could be obtained with the same approach when starting from the co-rotational kinetic FENE model for elastic polymers (see e.g. [LM07] for a review of this system). For conciseness, we do not repeat our analysis in that setting and leave the adaptation to the reader. ◊

1.3. **Previous results.** We briefly review previous rigorous results related to the multiscale description of particle suspensions and related systems.

Derivation of the Doi–Saintillan–Shelley model. The systematic theoretical study of the effective rheology of suspensions has been initiated by Einstein [Ein06], who found that passive non-Brownian spherical rigid particles effectively increase the fluid viscosity by $\frac{5}{2}\phi\mu$, where ϕ is the volume fraction of the particles and μ is the viscosity of the solvent. Jeffery [Jef22] studied the analogous problem for ellipsoidal particles and found an increase of the viscosity depending locally on orientations of the particles. By slender-body theory, in the limit of very elongated particles, Jeffery's viscous stress exactly takes the form of σ_2 in (1.3); see e.g. [Bre74; KK13]. Starting in the 1930s, there is a vast literature in physics on the rheology of suspensions of non-spherical rigid particles, see e.g. [Kuh32; Eis33; Pet38; Bur38], but this early work was restricted to specific fluid flows like simple shear, and Brownian effects were neglected. Brownian particles were first considered in [Sim40; KK45; RK50; Sai51], by means of different models and justifications, finally leading to the additional elastic stress σ_1 in (1.3). These models were largely reviewed in [LH71; HL72; Bre74]: in particular, the multiscale kinetic model (1.3) in the passive case ($U_0 = 0$, $\theta = 6$) then entered textbooks such as [DE88; Gra18] and became known in the mathematical literature as the Doi model. The extension to active suspensions has been proposed by Saintillan and Shelley [SS08]: by coarse-graining force dipoles exerted by the particles on the fluid, they derived a further contribution of the elastic stress σ_1 due to particles' activity; see also [HABK08; Hai+09; Sai10b; PRB16; DVY19].

On the mathematical side, the derivation of the viscous stress σ_2 from microscopic models has received considerable attention in the last years. When the fluid is modeled by the Stokes equations and the particle distribution is given, the effective increase of the fluid viscosity is by now well understood [HM12; NS20; GVH20; HW20; DG23b; GV21; GVH21; DG21; GVM22; Due22; DG23a]; see [DG22] for a review. In a similar setting, for active suspensions, the elastic stress σ_1 has been derived in [GL22; BDG22]. Yet, non-Newtonian effects originate from the retroaction of the fluid on the particles, that is, from coupling the particle density to the fluid flow: beyond the derivation of σ_1 and σ_2 in the static setting, the derivation of kinetic equations for the particle density in the time-dependent setting is of key interest. First steps in that direction have been undertaken in [HS21; HMS24; Due23]. Finally, regarding the derivation of the passive part of the elastic stress σ_1 for Brownian particles, we refer to [HLM23], where the authors start from a simplified microscopic model where the particle dynamics is given by Brownian motion and not coupled to the fluid.

In a different direction, we also mention recent work [HS10; AO22; CZDGV22], where the (linear) stability and mixing properties of the Doi–Saintillan–Shelley model have been investigated (neglecting however the viscous stress σ_2).

Macroscopic rheology and formal closures. Most of the above-mentioned works in the physics literature on the derivation of the Doi model do not stop at multiscale models but also aim at macroscopic non-Newtonian models, as well as at explicit calculations of stresses in specific flows such as simple shear. In particular, they typically give formulas for the shear viscosity in simple shear flow for very large or very small Weissenberg number Wi. Normal-stress differences (see (3.6) below) have also been computed at small Wi by Giesekus [Gie62], showing that the elastic stress σ_1 does not contribute to the second normal-stress difference but that the viscous stress does. This was in contradiction with Weissenberg's original conjecture that all real-world fluids must have vanishing second normal-stress difference, a conjecture that was later falsified also through experiments. Hinch and Leal [LH71; HL72] systematically computed expansions for the stress in simple shear flow both at small and at large Wi. They also noticed that their findings at small Wi agree with second-order fluid models, but they did not investigate whether this holds in more general fluid flows.

Although no exact macroscopic closure is available for the Doi model in the general non-perturbative setting, formal approximate closures have been studied for example in [DE88, Chapter 8.7]. A particular instance is the Oldoyd–B model, which is well-known to be remarkably an exact macroscopic closure for the kinetic Hookean dumbbell model; this formal connection was made rigorous in [DS23] (in the stress-diffusive case). For the Doi and Doi–Saintillan–Shelley models, an exact closure is not available, thus raising the question whether macroscopic closures are at least asymptotically valid in some scaling regimes: for instance, we will see that the validity of the Oldoyd–B closure already fails at order $O(\mathrm{Wi})$ for the Doi model in the small-Wi regime (see Remark 3.1). The situation is very similar for kinetic FENE models for elastic polymers: no exact macroscopic closure holds, but the so-called FENE–P model is still a very popular formal approximate closure (see e.g. [BDJ80], where the name FENE–P is attributed in reference to an earlier paper of Peterlin [Pet66]).

*Small-*Wi *regime and hydrodynamic limits.* At small Wi, the Doi–Saintillan–Shelley system (1.3) undergoes a strong diffusion in orientation (cf. factor $\frac{1}{\varepsilon} = \frac{1}{\mathrm{Wi}}$ in front of Δ_n): to leading order, the orientations of the particles relax instantaneously to the steady state, which corresponds to isotropic orientations. While this leading order amounts to a trivial Newtonian behavior, next-order $O(\mathrm{Wi})$ corrections encode nontrivial non-Newtonian effects where the stress starts to depend on the local fluid deformation and its history. The formal perturbative expansion in powers of Wi is comparable to the Hilbert expansion method in the Boltzmann theory [Hil12; Caf80; Gol05; SR09], where one looks for a solution of the Boltzmann equation as a formal power series in terms of the Knudsen number $\mathrm{Kn} \ll 1$ and where the leading-order approximation simply leads to compressible Euler equations. From the closely related Chapman–Enskog asymptotics expansion, one can (formally) obtain compressible Navier–Stokes equations as a $O(\mathrm{Kn})$ correction to the compressible Euler system, up to $O(\mathrm{Kn}^2)$ errors (see e.g. [Gol05, Section 5.2] or [SR09, Section 2.2.2]). In a similar way, for the Doi model, second-order fluid equations are obtained in this work as a $O(\mathrm{Wi})$ correction to the Stokes equations, up to $O(\mathrm{Wi}^2)$ errors.

Our results can be compared with corresponding results for the Doi–Onsager model for liquid crystals, which indeed shares some similarities with the Doi model that we consider in this work. In [EZ06; WZZ15], the macroscopic Ericksen–Leslie system is derived from the Doi–Onsager model by means of a Hilbert expansion. Note however that in that case the leading term in the expansion already yields a non-trivial system, so that higher-order corrections are not investigated in [EZ06; WZZ15]. We also mention recent related work on hydrodynamic limits for alignment models [DM08; DFMA17; DFMAT19; DFL22] and for flocking models [KMT15; KV15; FK18], as well as a preliminary result for kinetic FENE models for elastic polymers [LPD02].

Note that hydrodynamic limits have been investigated also for various other kinetic models for particle suspensions in different settings: for instance, in the context of sedimentation for small inertial particles, let us mention the inertialess limits studied in [Jab00; Höf18; HKM23; Ert23], as well as the high-diffusion limit in velocity investigated in [GJV04a; GJV04b; MV08; SY20].

1.4. **Structure of the article.** The article is split into five main sections, in addition to three appendices containing proofs of secondary results:

- In Section 2, we give a detailed account of the Doi–Saintillan–Shelley model: we describe the non-dimensionalization leading to its form (1.3), and we state a new well-posedness result for this system, see Proposition 2.1, which we prove in Appendix A.
- Section 3 provides an introduction to ordered fluid models, starting with their basic non-Newtonian rheological properties. We then discuss the ill-posedness of second-order fluid equations for the relevant range of coefficients, and we present two approaches to fix this issue: we introduce *hierarchical solutions* in Proposition 3.2 and we define a Boussinesq-type perturbative rearrangement in Proposition 3.3. As we allow the suspended particle density to be spatially inhomogeneous in general, and as we include a non-vanishing spatial diffusion $O(\frac{1}{\mathrm{Pe}})$ in the model for technical reasons, we actually derive ordered fluid models of the form (1.7), which slightly differ from their standard version: we motivate and introduce these nonstandard models in Section 3.4 and give the corresponding definition of hierarchical solutions in Proposition 3.4. The proofs of Propositions 3.2, 3.3 and 3.4 are postponed to Appendix B.
- In Section 4, we provide a more detailed formulation of our main result on the derivation of second-ordered fluid equations from the Doi–Saintillan–Shelley theory at small Weissenberg number, see Theorem 4.2. We further comment on the derivation of higher-order fluid models, for which details are postponed to Appendix C.
- In Section 5 we prove our main result, that is, the rigorous ε-expansion of the solution $(u_\varepsilon, f_\varepsilon)$ of the Doi–Saintillan–Shelley system (1.3).

Notations. We summarize the main notations that we use in this work:

- We denote by $C \geq 1$ any constant that only depends on the dimension d and possibly on other controlled quantities to be specified. We use the notation $\lesssim$ for $\leq C\times$ up to such a multiplicative constant C. We write $\ll$ (resp. $\gg$) for $\leq C\times$ (resp. $\geq C\times$) up to a sufficiently large multiplicative constant C. When needed, we add subscripts to indicate dependence on other parameters.
- For a vector field u and a matrix field S, we set $(\nabla u)_{ij} := \nabla_j u_i$, $S^T_{ij} := S_{ji}$, $\mathrm{D}(u) := \frac{1}{2}(\nabla u + (\nabla u)^T)$, and $\mathrm{div}(S)_i := \nabla_j S_{ij}$ (we systematically use Einstein's summation convention on repeated indices).
- We denote by $\mathrm{d}n$ the (not normalized) Lebesgue measure on the $(d-1)$-dimensional unit sphere $\mathbb{S}^{d-1}$, and we denote its area by $\omega_d := |\mathbb{S}_{d-1}|$. Differential operators with a subscript n (such as div_n and Δ_n) refer to differential operators on $\mathbb{S}^{d-1}$, endowed with the natural Riemannian metric.
- For $n \in \mathbb{S}^{d-1}$, we denote by $\pi_n^\perp := \mathrm{Id} - n \otimes n$ the orthogonal projection on $n^\perp$.
- We let $\langle g \rangle(x) := \fint_{\mathbb{S}^{d-1}} g(x,n)\,\mathrm{d}n := \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} g(x,n)\,\mathrm{d}n$ be the angular averaging of a function g on $\mathbb{T}^d \times \mathbb{S}^{d-1}$. We also use the short-hand notation $P_1^\perp g := g - \langle g \rangle$.
- We denote by $H^k(\mathbb{T}^d)$ (resp. $H^k(\mathbb{T}^d \times \mathbb{S}^{d-1})$) the standard L^2 Sobolev spaces for functions depending on $x \in \mathbb{T}^d$ (resp. $(x,n) \in \mathbb{T}^d \times \mathbb{S}^{d-1}$), and we use the notation $\|\cdot\|_{H^k_x}$ (resp. $\|\cdot\|_{H^k_{x,n}}$) for the corresponding norms. For time-dependent functions, given a Banach space X and $t > 0$, the norm of $\mathrm{L}^p(0,t;X)$ is denoted by $\|\cdot\|_{\mathrm{L}^p_t X}$.
- The space of probability measures on $\mathbb{T}^d$ (resp. on $\mathbb{T}^d \times \mathbb{S}^{d-1}$) is denoted by $\mathcal{P}(\mathbb{T}^d)$ (resp. by $\mathcal{P}(\mathbb{T}^d \times \mathbb{S}^{d-1})$).

2. Doi–Saintillan–Shelley theory

For simplicity, we focus in this work on systems in a finite box $\mathbb{T}^d = [0,1)^d$ with periodic boundary conditions. In terms of the fluid velocity field $u\colon \mathbb{R}^+ \times \mathbb{T}^d \to \mathbb{R}^d$ and of the particles' position and orientation density function $f\colon \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{S}^{d-1} \to \mathbb{R}^+$, we consider the so-called Doi–Saintillan–Shelley kinetic model for a suspension of very elongated, rigid, active particles in an incompressible viscous fluid flow (see [DE88, Chapter 8] and [Sai18]),

$$\begin{cases} \rho_{\rm fl}(\partial_t + u\cdot\nabla)u - \mu_{\rm fl}\Delta u + \nabla p \;=\; h + {\rm div}(\sigma_1[f]) + {\rm div}(\sigma_2[f,\nabla u]), \\ \partial_t f + {\rm div}_x\big((u+V_0 n)f\big) + {\rm div}_n\big(\pi_n^\perp(\nabla u)nf\big) \\ \qquad\qquad\qquad\qquad = D_{\rm tr}{\rm div}_x\big(({\rm Id}+n\otimes n)\nabla_x f\big) + D_{\rm ro}\Delta_n f, \\ {\rm div}(u) = 0, \end{cases} \tag{2.1}$$

where elastic and viscous stresses are respectively given by

$$\begin{aligned} \sigma_1[f] \;&:=\; \big(3k_B\Theta + \alpha\mu_{\rm fl}|V_0|\ell^2\big)\int_{\mathbb{S}^{d-1}}\big(n\otimes n - \tfrac1d\,{\rm Id}\,\big)f(\cdot,n)\,{\rm d}n, \\ \sigma_2[f,\nabla u] \;&:=\; \tfrac12\zeta_{\rm ro}\int_{\mathbb{S}^{d-1}}(n\otimes n) : {\rm D}(u)(n\otimes n)\,f(\cdot,n)\,{\rm d}n, \end{aligned} \tag{2.2}$$

where h is an internal force, and where the different parameters $\rho_{\rm fl}, \mu_{\rm fl}, D_{\rm tr}, D_{\rm ro}, \zeta_{\rm ro}, V_0, \ell$ are all assumed to be constant. In this kinetic description, $\rho_{\rm fl}$ stands for the fluid density, $\mu_{\rm fl}$ for the solvent viscosity, $D_{\rm tr}$ and $D_{\rm ro}$ for the translational and rotational diffusion coefficients, $\zeta_{\rm ro}$ for the rotational resistance coefficient, V_0 for the self-propulsion speed of the particles, and ℓ for the length of the particles. The fluid flow is coupled to the kinetic equation for the particle density via the additional stresses σ_1 (elastic stress) and σ_2 (viscous stress), which make the fluid equations non-Newtonian. We briefly describe the structure and physical origin of these contributions (see Section 1.3 for references):

— The viscous stress σ_2 arises from the rigidity of suspended particles in the fluid flow: it is formally understood by homogenization of the solid phase, viewed as inclusions with infinite shear viscosity in the fluid. The above expression (2.2) is an approximation for very elongated particles: for general axisymmetric particles, the viscous stress involves additional terms depending on the precise shape of the particles (see e.g. [HL72] for spheroids), but slender body theory indeed shows that it reduces to the above form in the limit of very elongated particles (see e.g. [KK13, Section 3.4]).

— The elastic stress σ_1 contains a passive and an active contribution. The passive part, proportional to the the Boltzmann constant k_B and to the absolute temperature Θ, is created by the random torques that are responsible for the rotational Brownian motion of the particles. These torques indeed create stresses due to the rigidity and anisotropy of the particles. On the other hand, the active part arises directly from the swimming mechanism, which is encoded in the parameter $\alpha \in \mathbb{R}$: a so-called puller particle corresponds to $\alpha > 0$, and a pusher particle corresponds to $\alpha < 0$.

For $V_0 = 0$, the model (2.1) reduces to the classical Doi model for passive Brownian particles [DE88]. Finally, for very elongated particles in 3D, we also note that the translational and rotational diffusion and resistance coefficients are asymptotically given as follows (see [Dho96, Section 5.15]),

$$\begin{aligned} D_{\rm ro} &= \tfrac{k_B\Theta}{\zeta_{\rm ro}}+D_{\rm act}, &\qquad \zeta_{\rm ro} &= \tfrac{\pi\mu_{\rm fl}\ell^3}{3\log(\ell/a)},\\ D_{\rm tr} &= \tfrac{k_B\Theta}{\zeta_{\rm tr}}, &\qquad \zeta_{\rm tr} &= \tfrac{2\pi\mu_{\rm fl}\ell}{\log(\ell/a)}, \end{aligned} \tag{2.3}$$

where a is the width of the particles and where $D_{\rm act}$ is some possible active contribution to the rotational diffusion (tumbling). Compared to the above model (2.1), we henceforth make two minor simplifications:

— While the translational diffusion in (2.1) is proportional to $\mathrm{Id}+n\otimes n$, hence is twice as strong in the direction of particle orientation as in the orthogonal directions, we choose to neglect this $O(1)$ difference and assume that the diffusion is isotropic. This choice is for simplicity and does not change anything in the analysis of the model.

— It has been observed in the seminal work [BB72] that, for the example of E. coli bacteria and related microswimmers, the contribution of thermal rotation and active tumbling is of the same order, and we therefore set $D_{\rm act}=0$ for simplicity.

2.1. **Non-dimensionalization and relevant regimes.** We non-dimensionalize the above model (2.1) (after the two above-described minor simplifications), in terms of the typical speed u_0 of the fluid, the typical macroscopic length scale L, and the typical number N of particles in a cube of side length L. We further rescale time according to the time scale of the fluid flow, that is, $T=L/u_0$. More precisely, we define

$$\hat u(t,x):=\tfrac{1}{u_0}u(Tt,Lx),\qquad \hat f(t,x):=\tfrac{L^d}{N}f(Tt,Lx),\qquad \hat h(t,x):=\tfrac{L^2}{\mu u_0}h(Tt,Lx).$$

This leads to the following dimensionless model, dropping the hats for simplicity,

$$\begin{cases} \mathrm{Re}(\partial_t+u\cdot\nabla)u-\Delta u+\nabla p \,=\, h+\mathrm{div}(\sigma_1[f])+\mathrm{div}(\sigma_2[f,\nabla u]),\\ \partial_t f+\mathrm{div}_x\big((u+U_0n)f\big)+\mathrm{div}_n\big(\pi_n^\perp(\nabla u)nf\big) \,=\, \tfrac{1}{\mathrm{Pe}}\Delta_x f+\tfrac{1}{\mathrm{Wi}}\Delta_n f,\\ \mathrm{div}(u)=0, \end{cases}$$

where the dimensionless counterparts of the additional stresses σ_1,σ_2 now take the form

$$\begin{aligned} \sigma_1[f] &= (6\tfrac{\lambda}{\mathrm{Wi}}+\gamma)\int_{\mathbb S^{d-1}}\big(n\otimes n-\tfrac1d\,\mathrm{Id}\,\big)f(\cdot,n)\,dn,\\ \sigma_2[f,\nabla u] &= \lambda\int_{\mathbb S^{d-1}}(n\otimes n)(\nabla u)(n\otimes n)\,f(\cdot,n)\,dn. \end{aligned}$$

Here, $\mathrm{Re},\mathrm{Pe},\mathrm{Wi}>0$ stand for the so-called Reynolds, Péclet, and Weissenberg numbers, $\lambda>0$ depends only on the shape and number density of the particles, and $\gamma\in\mathbb R$ accounts for the active contribution to the stress. More precisely, these parameters are given by

$$\begin{aligned} \mathrm{Re} &= \tfrac{\rho_{\rm fl}u_0L}{\mu_{\rm fl}}, &\qquad \lambda &= \tfrac{\zeta_{\rm ro}N}{2\mu_{\rm fl}L^d},\\ \mathrm{Pe} &= \tfrac{u_0L\zeta_{\rm tr}}{k_B\Theta}, &\qquad \mathrm{Wi} &= \tfrac{u_0\zeta_{\rm ro}}{k_B\Theta L},\\ \gamma &= \alpha N\tfrac{|U_0|\ell^2}{L^{d-1}}, &\qquad U_0 &= \tfrac{V_0}{u_0}, \end{aligned}$$

and we briefly comment on their range and interpretation:

— The Weissenberg number Wi is the ratio between convection and relaxation time scales. For the kinetic viscoelastic models under consideration, it coincides with the rotational Péclet number. For elongated particles in 3D, as $\zeta_{\rm ro}$ is proportional to the cube of the particle length, cf. (2.3), the regime when Wi is of order $O(1)$ is very narrow, and lies under standard flow conditions at particle lengths of around 10 microns. In this work, we are interested in the derivation of hydrodynamic approximations in case of very small $\mathrm{Wi} \ll 1$: this means that we have applications in mind where the particles have a length of a few microns and below, which is in particular the case for many types of bacteria. For notational convenience, we rename the Weissenberg number as

$$\varepsilon := \mathrm{Wi} \ll 1.$$

— The (translational) Péclet number Pe is typically much larger than its rotational counterpart Wi since[1]

$$\frac{\mathrm{Pe}}{\mathrm{Wi}} = \frac{\zeta_{\mathrm{tr}} L^2}{\zeta_{\mathrm{ro}}} = \frac{6L^2}{\ell^2} \gg 1.$$

In fact, from the application-oriented perspective, it makes sense to consider $\mathrm{Pe} \gg 1$. However, due to well-posedness and stability issues, our analysis crucially relies on keeping Pe not too large, and we shall generally assume $\mathrm{Pe} \simeq 1$. We can actually allow for Pe to slightly diverge, but more slowly than the inverse of the volume fraction of the particles, cf. (1.6).

— The shape parameter λ is typically quite small as it is proportional to the volume fraction NL^{-d} of the particles.

— The prefactor in the viscous stress σ_1 reads

$$6\frac{\lambda}{\mathrm{Wi}} + \gamma = \frac{\lambda}{\mathrm{Wi}}\Big(6 + 2\alpha \frac{|V_0|\ell^2 \mu_{\mathrm{fl}}}{k_B \Theta}\Big),$$

where the term $\frac{|V_0|\ell^2 \mu_{\mathrm{fl}}}{k_B \Theta}$ is of order $1-10$ for typical microswimmers like E. coli bacteria [BB72]. We shall set for abbreviation

$$\theta := 6 + 2\alpha \frac{|V_0|\ell^2 \mu_{\mathrm{fl}}}{k_B \Theta}. \tag{2.4}$$

Note in particular that for passive particles $V_0 = 0$ we find $\theta = 6$.

— The self-propulsion speed V_0 of active particles is typically around 10 micron per second. This is so slow with respect to typical shear flows considered in experimental settings that the ratio $U_0 = \frac{V_0}{u_0}$ is typically tiny. However, extremely low shear rates leading to $u_0 \sim V_0$ can possibly also be achieved, as for instance in the experiments reported by [Lóp+15]. For that reason, we choose not to neglect U_0 in the equations and to keep track of its effects.

In conclusion, we are led to consider the following system,

$$\left\{\begin{array}{l}
\mathrm{Re}(\partial_t + u_\varepsilon \cdot \nabla) u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon = h + \frac{1}{\varepsilon}\mathrm{div}(\sigma_1[f_\varepsilon]) + \mathrm{div}(\sigma_2[f_\varepsilon, \nabla u_\varepsilon]),\\
\partial_t f_\varepsilon + \mathrm{div}_x\big((u_\varepsilon + U_0 n) f_\varepsilon\big) + \mathrm{div}_n\big(\pi_n^\perp(\nabla u_\varepsilon) n f_\varepsilon\big) = \frac{1}{\mathrm{Pe}}\Delta_x f_\varepsilon + \frac{1}{\varepsilon}\Delta_n f_\varepsilon,\\
\mathrm{div}(u_\varepsilon) = 0,\\
\int_{\mathbb{T}^d} u_\varepsilon = 0 \quad \text{if } \mathrm{Re} = 0,\\
\sigma_1[f] = \lambda\theta \int_{\mathbb{S}^{d-1}} \big(n \otimes n - \frac{1}{d}\,\mathrm{Id}\big) f(\cdot, n)\,\mathrm{d}n,\\
\sigma_2[f, \nabla u] = \lambda \int_{\mathbb{S}^{d-1}} (n \otimes n)(\nabla u)(n \otimes n)\, f(\cdot, n)\,\mathrm{d}n,
\end{array}\right. \tag{2.5}$$

which is complemented by initial conditions

$$\left\{\begin{array}{l}
f_\varepsilon|_{t=0} = f_\varepsilon^\circ,\\
u_\varepsilon|_{t=0} = u^\circ \quad \text{if } \mathrm{Re} \neq 0,
\end{array}\right. \tag{2.6}$$

[1]The same holds with an additional logarithmic correction in 2D.

and we consider the asymptotic limit $\varepsilon \downarrow 0$ in the regime with $\lambda \leq 1$ and with $\lambda\theta \ll 1$ small enough. Regarding the Reynolds number, we focus on $\mathrm{Re} \in \{0,1\}$ for simplicity. Due to classical regularity issues for the Navier–Stokes equations, we actually limit ourselves to the Stokes case $\mathrm{Re} = 0$ for the 3D model, but we also study the Navier–Stokes case $\mathrm{Re} = 1$ in 2D. Note that the 3D Navier–Stokes case is much more complicated and is not discussed in this work as the well-posedness of the system is still open in that case.

2.2. **Well-posedness of the Doi–Saintillan–Shelley model.** The system (2.5) is an instance of the more general class of Fokker–Planck–Navier–Stokes systems, but we emphasize two main peculiarities:

— We include in (2.5) the contribution of the viscous stress σ_2, which arises from the rigidity of underlying suspended particles on the microscale and effectively modifies the solvent viscosity.

— We also include the effect of particle swimming via U_0. This creates local changes in the spatial density $\rho_\varepsilon = \fint_{\mathbb{S}^{d-1}} f_\varepsilon(\cdot, n)\,dn$, which no longer remains constant in general.

In contrast, most previous works have focused on the corresponding model without viscous stress σ_2 and without particle swimming $U_0 = 0$. In that simplified setting, for the 3D Stokes case, the existence of global entropy solutions was proven in [OT08], and the global well-posedness of smooth solutions was proven in [CS10] (without translational diffusion, $\mathrm{Pe} = \infty$). In the Navier–Stokes case, corresponding well-posedness results were obtained for instance in [CM08]. The model including the viscous stress $\sigma_2[f_\varepsilon, \nabla u_\varepsilon]$ but without particle swimming $U_0 = 0$ was first studied in [LM07], where the existence of global weak solutions was proven for the Navier–Stokes case in 2D and 3D (without translational diffusion, $\mathrm{Pe} = \infty$). We also refer to [La19] for the global well-posedness of smooth solutions in the 2D Navier–Stokes case. Particle swimming $U_0 \neq 0$ was first considered in [CL13], where the authors studied the corresponding model without viscous stress σ_2 and proved the global existence of weak entropy solutions both for the Stokes and Navier–Stokes cases in 2D and 3D, as well as the existence of energy solutions for the Stokes case and their uniqueness in 2D.

Building on similar ideas, we can actually establish the existence of global energy solutions for the full model (2.5) in the 2D and 3D Stokes cases, as well as in the 2D Navier–Stokes case, and we further obtain a weak-strong uniqueness principle. To our knowledge, this is surprisingly the first result where both the viscous stress and the swimming forces are included at the same time. The proof is postponed to Appendix A.

Proposition 2.1. *Consider either the Stokes case* $\mathrm{Re} = 0$ *with* $d \leq 3$*, or the Navier–Stokes case* $\mathrm{Re} = 1$ *with* $d = 2$*. Given* $\varepsilon > 0$*, given* $h \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{T}^d))$*, given an initial condition* $f^\circ_\varepsilon \in \mathrm{L}^2 \cap \mathcal{P}(\mathbb{T}^d \times \mathbb{S}^{d-1})$*, and given also* $u^\circ \in \mathrm{L}^2(\mathbb{T}^d)^d$ *with* $\mathrm{div}(u^\circ) = 0$ *in the Navier–Stokes case, the Cauchy problem* (2.5)*–*(2.6) *admits a global weak solution* $(u_\varepsilon, f_\varepsilon)$ *with:*

(i) in the Stokes case $\mathrm{Re} = 0$*,* $d \leq 3$*,*

$$u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{T}^d)^d),$$

$$f_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2 \cap \mathcal{P}(\mathbb{T}^d \times \mathbb{S}^{d-1})) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{T}^d \times \mathbb{S}^{d-1}));$$

(ii) in the Navier–Stokes case $\mathrm{Re} = 1$*,* $d = 2$*,*

$$u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{T}^2)^2) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{T}^2)^2),$$

$$f_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2 \cap \mathcal{P}(\mathbb{T}^2 \times \mathbb{S}^1)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{T}^2 \times \mathbb{S}^1)).$$

In both cases, a weak-strong uniqueness principle further holds: if $(u_\varepsilon, f_\varepsilon)$ *and* $(u'_\varepsilon, f'_\varepsilon)$ *are two such global weak solutions with identical initial conditions, and if* $(u'_\varepsilon, f'_\varepsilon)$ *has the following additional regularity,*

$$\begin{aligned} u'_\varepsilon &\in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; W^{1,\infty}(\mathbb{T}^d)^d), \\ f'_\varepsilon &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{S}^{d-1})), \end{aligned}$$

then we have $(u_\varepsilon, f_\varepsilon) = (u'_\varepsilon, f'_\varepsilon)$. ◊

3. Ordered fluid models

Various non-Newtonian fluid models have been considered in the literature, taking into account nonlinear and memory effects in different ways on the macroscopic scale. We focus here on so-called ordered fluid models, which will indeed be shown to naturally appear as hydrodynamic approximations of the Doi–Saintillan–Shelley theory. We start by defining such models and reviewing their non-Newtonian properties; for more details on modelling aspects and applications, we refer to [DR95; Böh87; BAH87; Jos13; PTMD13]. Next, we develop a perturbative well-posedness theory and we introduce (non-standard) inhomogeneous, diffusive versions of these models.

3.1. Standard ordered fluid models.

Ordered fluid equations are of the form

$$\begin{cases} \mathrm{Re}(\partial_t + u\cdot\nabla)u - \mathrm{div}(\sigma) + \nabla p = h, \\ \mathrm{div}(u) = 0, \end{cases} \tag{3.1}$$

where σ is a function of the Rivlin–Ericksen tensors $\{A_k(u)\}_k$ defined iteratively through

$$\begin{aligned} A_1(u) &:= 2\mathrm{D}(u) = \nabla u + (\nabla u)^T, \\ A_{n+1}(u) &:= (\partial_t + u\cdot\nabla)A_n(u) + (\nabla u)^T A_n(u) + A_n(u)\nabla u, \qquad n \geq 1. \end{aligned} \tag{3.2}$$

In other words, $A_{n+1}(u)$ is the so-called lower-convected derivative of $A_n(u)$, which ensures frame indifference of the equations.[2] While first-order fluids are of the form (3.1) with constitutive law $\sigma = \eta_0 A_1(u)$, thus coinciding with standard Newtonian fluids, second-order fluids are of the form (3.1) with

$$\sigma := \eta_0 A_1(u) + \alpha_1 A_2(u) + \alpha_2 A_1(u)^2, \tag{3.3}$$

see e.g. [Böh87, Eq. (8.48)], and third-order fluids amount to

$$\begin{aligned} \sigma := \eta_0 A_1(u) &+ \alpha_1 A_2(u) + \alpha_2 A_1(u)^2 \\ &+ \beta_1 A_3(u) + \beta_2\big(A_1(u)A_2(u) + A_2(u)A_1(u)\big) + \beta_3 A_1(u)\mathrm{tr}\big(A_1(u)^2\big), \end{aligned} \tag{3.4}$$

[2] In the literature, *upper*-convected instead of lower-convected derivatives are sometimes used to define ordered fluids (see e.g. [BCAH87, Section 6]). This is merely a choice of convention, as both lead to equivalent fluid equations (although the value and interpretation of parameters of course depends on the chosen convention). Indeed, the upper-convected derivative of $A_n(u)$ can be rewritten in terms of the lower-convected derivative as

$$(\partial_t + u\cdot\nabla)A_n(u) - A_n(u)(\nabla u)^T - (\nabla u)A_n(u) = A_{n+1}(u) - \big(A_1(u)A_n(u) + A_n(u)A_1(u)\big).$$

for some coefficients $\eta_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$.[3] Since third- and higher-order fluid models are fairly complicated and involve a lot of terms, the most widely used ordered fluid model for viscoelastic fluids is the second-order model (3.3).

We briefly recall the basic rheological properties of these ordered fluid models depending on the different parameter values; we focus here on the 3D setting. The coefficient $\eta_0 > 0$ in (3.3) and (3.4) is the zero-shear viscosity, as in the first-order (Newtonian) model, while other parameters account for various non-Newtonian behaviors.

— *Shear-dependent viscosity:* In a simple shear flow $u_0(t,x) = \kappa x_2 e_1$ with shear rate $\kappa > 0$, the shear-dependent viscosity is defined as

$$\eta(\kappa) := \frac{\sigma_{12}}{\kappa}. \tag{3.5}$$

Noting that

$$A_1(u_0) = \kappa \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1(u_0)^2 = \kappa^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2(u_0) = \kappa^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we compute for the second-order fluid that the shear-dependent viscosity simply coincides with the zero-shear viscosity, $\eta(\kappa) = \eta_0$. For the third-order model, in contrast, we find a nontrivial shear-dependent relation,

$$\eta(\kappa) = \eta_0 + 2\kappa^2(\beta_2 + \beta_3).$$

Most real-life viscoelastic fluids, and in particular passive dilute suspensions, happen to be shear-thinning, meaning that the map $\kappa \mapsto \eta(\kappa)$ is decreasing: this holds for the third-order fluid model provided that the coefficients satisfy $\beta_2 + \beta_3 < 0$.

— *Normal-stress differences:* In a simple shear flow $u_0(t,x) = \kappa x_2 e_1$ with shear rate $\kappa \in \mathbb{R}$, non-Newtonian fluids typically display non-zero normal stresses. This is responsible for a number of phenomena, of which the rod-climbing effect is the best known; see e.g. [Jos13, Chapter 17] and [BAH87, Chapter 6]. Normal stress coefficients are defined as[4]

$$\nu_{10} := \frac{\sigma_{11} - \sigma_{22}}{\kappa^2}, \qquad \nu_{20} := \frac{\sigma_{22} - \sigma_{33}}{\kappa^2}, \tag{3.6}$$

and thus, for second- and third-order models,

$$\nu_{10} = -2\alpha_1, \qquad \nu_{20} = 2\alpha_1 + \alpha_2. \tag{3.7}$$

In other words, α_1, α_2 are related to normal stress coefficients via

$$\alpha_1 = -\tfrac{1}{2}\nu_{10}, \qquad \alpha_2 = \nu_{10} + \nu_{20}. \tag{3.8}$$

For most real-life viscoelastic fluids, and in particular for polymer solutions, it is found experimentally that $\nu_{10} > 0$, $\nu_{20} < 0$, and that $|\nu_{20}|$ is considerably smaller than $|\nu_{10}|$ (by a factor of around 10, see e.g. [Böh87, Section 2.2] and [PTMD13, Section2.2]), which means in particular, in terms of second-order fluid coefficients,

$$\alpha_1 < 0, \qquad \alpha_2 > 0. \tag{3.9}$$

[3] In arbitrary dimension d, a further term $\beta_4 A_1(u)^3$ should be included in general in the stress σ of third-order fluids, but it is redundant in dimensions $d \leq 3$ as we then have $B^3 = \frac{1}{2} B \,\mathrm{tr}(B^2) + \frac{1}{3}\mathrm{tr}(B^3)$ by the Cayley–Hamilton theorem for any symmetric trace-free matrix B.

[4] Beware of different sign conventions for the normal stress coefficients. We follow here the definition of [Böh87, Chapter 2.2].

— *Elongational viscosity:* The apparent viscosity of a non-Newtonian fluid may be completely different in an elongational flow. Given a uniaxial elongational flow $u_0(t,x) = \kappa(x_1e_1 - \frac{1}{2}(x_2e_2 + x_3e_3))$ in the direction e_1, the elongational viscosity is defined as

$$\eta_E := \frac{\sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33})}{\kappa}. \tag{3.10}$$

Noting that

$$A_1(u_0) = \kappa \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad A_2(u_0) = A_1(u_0)^2 = \kappa^2 \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.11}$$

we compute for the second-order model,

$$\eta_E = 3\eta_0 + 3\kappa(\alpha_1 + \alpha_2).$$

For real-life viscoelastic fluids, it is typically observed that the elongational viscosity increases with the strain rate (so-called strain-thickening behavior), which holds provided that coefficients satisfy $\alpha_1 + \alpha_2 > 0$.

— *Retardation phase shift in oscillatory flow:* In a simple shear flow $u_0(t,x) = \kappa(t)x_2e_1$ with oscillatory shear rate $\kappa(t) = \sin t$, we compute

$$A_2(u_0) = \dot\kappa \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \kappa^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which gives rise to a phase shift for σ_{12} in the second-order model, in form of

$$\sigma_{12} = \eta_0\kappa + \alpha_1\dot\kappa = (\eta_0^2 + \alpha_1^2)^{\frac{1}{2}} \sin\big(t + \arctan(\tfrac{\alpha_1}{\eta_0})\big).$$

This constitutes another typical (time-dependent) non-Newtonian feature, in link with the dependence of the stress on the flow history.

Remark 3.1 (Connection to Oldroyd–B model)**.** The Oldroyd–B model is a special case of a simple fluid model, which is particularly popular as a formal exact closure of the kinetic Hookean dumbbell model. It is characterized by the following constitutive equation for the stress tensor,

$$\sigma = 2\eta_s\,\mathrm{D}(u) + \eta_p\tau, \qquad \tau + \mathrm{Wi}\big((\partial_t + u\cdot\nabla)\tau - \tau(\nabla u)^T - (\nabla u)\tau\big) = 2\,\mathrm{D}(u). \tag{3.12}$$

By a formal expansion with respect to $\mathrm{Wi} \ll 1$, this model is found to agree to order Wi^k with a kth-order fluid model with some specific choice of parameters. In particular, to order Wi^2, we recover the second-order fluid model (3.3) with $\alpha_2 = -2\alpha_1$, which means in particular that the second normal stress coefficient vanishes, $\nu_{20} = 0$. We emphasize that this is not the case for the second-order fluid model that we shall derive from the Doi–Saintillan–Shelley theory. Hence, at small Wi, the viscoelastic effects of suspensions of rigid Brownian particles are not well described by an Oldroyd–B model. ◊

3.2. **Non-standard ordered fluid models at** $\mathrm{Pe} < \infty$**.** In the case of a particle suspension with finite Péclet number $\mathrm{Pe} < \infty$, as considered in this work, cf. (2.5), the Rivlin–Ericksen tensors (3.2) naturally need to be modified as follows,

$$A_1'(u) := A_1(u) = 2\,\mathrm{D}(u), \tag{3.13}$$

$$A_{n+1}'(u) := (\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u\cdot\nabla)A_n'(u) + (\nabla u)^T A_n'(u) + A_n'(u)(\nabla u), \qquad n \geq 1,$$

hence in particular $A_2'(u) := A_2(u) - \frac{1}{\mathrm{Pe}}\Delta A_1(u)$. The second-order fluid equations (3.1)–(3.3) are then replaced by

$$\begin{cases} \mathrm{Re}(\partial_t + u\cdot\nabla)u - \mathrm{div}(\sigma) + \nabla p = h, \\ \sigma = \eta_0 A_1(u) + \alpha_1 A_2'(u) + \alpha_2 A_1(u)^2, \\ \mathrm{div}(u) = 0. \end{cases} \tag{3.14}$$

For higher-order fluid models, in this diffusive setting $\mathrm{Pe} < \infty$, several additional tensors actually need to be included in the stress: for the third-order model, instead of (3.4), the stress rather needs to be chosen in general as

$$\begin{aligned} \sigma \;=\; &\eta_0 A_1(u) + \alpha_1 A_2'(u) + \alpha_2 A_1(u)^2 + \beta_1 A_3'(u) + \beta_1' B_3'(u) \\ &+ \beta_2\big(A_1(u)A_2'(u) + A_2'(u)A_1(u)\big) + \beta_3 A_1(u)\mathrm{tr}\big(A_1(u)^2\big), \end{aligned} \tag{3.15}$$

in terms of the following additional tensor,

$$B_3'(u) \;:=\; (\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u\cdot\nabla)A_1(u)^2 + \big((\nabla u)^T A_1(u)^2 + A_1(u)^2(\nabla u)\big).$$

At infinite Péclet number $\mathrm{Pe} = \infty$, this additional tensor is indeed redundant as it reduces to $B_3'(u) = A_1(u)A_2(u) + A_2(u)A_1(u) - A_1(u)^3$, so we recover (3.4).[5]

3.3. **Well-posedness of ordered fluid models.** We focus for shortness on the second-order fluid model. There has actually been a fair amount of confusion on the relevant range of parameters α_1, α_2: the sign condition $\alpha_1 < 0$ in (3.9) is motivated by experimental normal stress measurements, but it actually appears inconsistent with thermodynamics; see e.g. [DR95] for a detailed discussion from the physics perspective. From the mathematical point of view, this inconsistency leads to ill-posedness issues. The matter was investigated by Galdi [Gal95], who showed the following for the second-order fluid equations (3.1)–(3.3) at infinite Péclet number $\mathrm{Pe} = \infty$:

— the local-in-time well-posedness holds whenever $1/\alpha_1 > -\lambda_1$, where λ_1 stands for the Poincaré constant in $\mathbb{T}^d$, which is quite consistent with the choice (3.9) (although the case of a negative α_1 with a small absolute value is prohibited);

— the long-time well-posedness, as well as the stability of steady solutions, can only hold provided that $\alpha_1 > 0$.

For the corresponding system (3.14) with finite Péclet number $\mathrm{Pe} < \infty$, the situation is even worse: even local-in-time well-posedness actually fails whenever $\alpha_1 < 0$ because the equation then behaves like a backwards heat equation. Yet, even though the kinetic Doi model itself is known to be thermodynamically consistent (see [DE88, Chapter 8]), our analysis will precisely lead us to a second-order fluid with $\alpha_1 < 0$, and it is thus crucial to determine what meaning should be given to the model in that case. In fact, we shall derive (3.14) in the small-Wi limit, $\varepsilon := \mathrm{Wi} \ll 1$, with

$$(\alpha_1, \alpha_2) = (\varepsilon\gamma_1, \varepsilon\gamma_2), \qquad \text{for some } \gamma_1 < 0,\ \gamma_2 \in \mathbb{R}. \tag{3.16}$$

In this perturbative setting $\varepsilon \ll 1$, although the equation is ill-posed for fixed ε, there are several ways to rearrange the nonlinearity and define well-posed notions of approximate

[5]Recall that, as in (3.4), a further term $\beta_4 A_1(u)^3$ should always be included in the stress in arbitrary dimension d, but it reduces to $\frac{1}{2}\beta_4 A_1(u)\mathrm{tr}(A_1(u)^2)$ in dimensions $d \le 3$.

solutions that only satisfy the system (3.14) up to a higher-order $O(\varepsilon^2)$ remainder:

$$\begin{cases} \mathrm{Re}(\partial_t + \bar u_\varepsilon \cdot \nabla)\bar u_\varepsilon - \mathrm{div}(\bar\sigma_\varepsilon) + \nabla \bar p_\varepsilon = h + O(\varepsilon^2), \\ \bar\sigma_\varepsilon := \eta_0 A_1(\bar u_\varepsilon) + \varepsilon\gamma_1 A_2'(\bar u_\varepsilon) + \varepsilon\gamma_2 A_1(\bar u_\varepsilon)^2, \\ \mathrm{div}(\bar u_\varepsilon) = 0. \end{cases} \tag{3.17}$$

These notions of approximate solutions will be viewed as proper perturbative ways to interpret second-order fluid equations and settle ill-posedness and instability issues.

The simplest way is to define a notion of approximate *hierarchical solutions*. It is particularly convenient for us in this work due to its close relation to the Hilbert expansion method that we use for the hydrodynamic approximation. The proof is straightforward and is postponed to Appendix B.

Proposition 3.2 (Hierarchical solutions)**.** *Consider the system (3.17) in the regime $\varepsilon \ll 1$ with parameters $\eta_0, \mathrm{Pe} > 0$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. If v_0, v_1 are smooth solutions of the following two auxiliary systems,*

$$\begin{cases} \mathrm{Re}\,\big(\partial_t v_0 + (v_0\cdot\nabla)v_0\big) - \eta_0 \Delta v_0 + \nabla p_0 = h, \\ \mathrm{div}(v_0) = 0, \\ v_0|_{t=0} = u^\circ \quad \textit{if } \mathrm{Re} \neq 0, \quad \fint_{\mathbb{T}^d} v_0 = 0 \quad \textit{if } \mathrm{Re} = 0, \end{cases} \tag{3.18}$$

$$\begin{cases} \mathrm{Re}\,\big(\partial_t v_1 + (v_0\cdot\nabla)v_1 + (v_1\cdot\nabla)v_0\big) - \eta_0\Delta v_1 + \nabla p_1 \\ \qquad\qquad = \mathrm{div}\big(\gamma_1 A_2'(v_0) + \gamma_2 A_1(v_0)^2\big), \\ \mathrm{div}(v_1) = 0, \\ v_1|_{t=0} = 0 \quad \textit{if } \mathrm{Re}\neq 0, \quad \fint_{\mathbb{T}^d} v_1 = 0 \quad \textit{if } \mathrm{Re} = 0, \end{cases} \tag{3.19}$$

then the superposition $\bar u_\varepsilon := v_0 + \varepsilon v_1$ indeed satisfies the system (3.17) with some controlled error term $O(\varepsilon^2)$ (and with initial condition $\bar u_\varepsilon|_{t=0} = u^\circ$ if $\mathrm{Re}\neq 0$). For the well-posedness of (3.18) and (3.19), we separately consider the Stokes and Navier–Stokes cases:

(i) Stokes case $\mathrm{Re} = 0$, $d \le 3$:
Given $s > \frac{d}{2} - 1$ and $h \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^d)^d) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; H^{s-2}(\mathbb{T}^d)^d)$, there is a unique global solution $v_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+2}(\mathbb{T}^d)^d)$ of (3.18), and a unique global solution $v_1 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^d)^d)$ of (3.19), leading to $\bar u_\varepsilon = v_0 + \varepsilon v_1 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^d)^d)$.

(ii) Navier–Stokes case $\mathrm{Re} = 1$, $d = 2$:
Given $s > 0$, $h \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^2)^2)$, and $u^\circ \in H^{s+1}(\mathbb{T}^2)^2$ with $\mathrm{div}(u^\circ) = 0$, there is a unique global solution $v_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^2)^2) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+2}(\mathbb{T}^2)^2)$ of (3.18), and a unique global solution $v_1 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^2)^2) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^2)^2)$ of (3.19), leading to $\bar u_\varepsilon = v_0 + \varepsilon v_1 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^2)^2) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^2)^2)$. ◊

There are also non-hierarchical ways to perturbatively make sense of the ill-posed second-order fluid model (3.17), which may be more desirable in particular for stability issues. Comparing to corresponding ill-posedness issues in the Boussinesq theory for water waves, we recall that there is a standard way to rearrange the ill-posed Boussinesq equation perturbatively and make it well-posed, see [CMV96]: in a nutshell, the idea is to replace indefinite operators like $1 + \varepsilon\Delta$ by corresponding positive operators like $(1 - \varepsilon\Delta)^{-1}$ up to $O(\varepsilon^2)$ errors. We show that a similar so-called Boussinesq trick can be used in the present situation as well: for any value of γ_1, γ_2, both at finite and infinite Péclet number, it leads us to a perturbative rearrangement of the second-order fluid equation that is well-posed and is indeed equivalent to (3.14) up to $O(\varepsilon^2)$ terms. The so-defined solution is easily checked to differ from the corresponding hierarchical solution only by $O(\varepsilon^2)$. We focus here on the relevant ill-posed case $\gamma_1 \le 0$. Note that the procedure is easily extended to higher-order fluid equations (see also [ABV16; DGR23] in a different context). The proof is postponed to Appendix B.

Proposition 3.3 (Boussinesq-like solutions)**.** *Consider the system* (3.17) *in the regime* $\varepsilon \ll 1$ *with parameters* $\eta_0, \mathrm{Pe} > 0$, $\gamma_1 \le 0$, *and* $\gamma_2 \in \mathbb{R}$.

(i) Stokes case $\mathrm{Re} = 0$, $d \le 3$:
Given $s > \frac{d}{2}$, $T_0 > 0$, *and* $h \in \mathrm{L}^\infty(0, T_0; H^{s+1}(\mathbb{T}^d)^d) \cap W^{1,\infty}(0, T_0; H^{s-1}(\mathbb{T}^d)^d)$, *provided that* $\varepsilon \ll 1$ *is small enough (depending on* s, T_0, h *and on all parameters), the following nonlinear problem admits a unique solution* $\bar{u}_\varepsilon \in \mathrm{L}^\infty(0, T_0; H^{s+1}(\mathbb{T}^d)^d)$,

$$\begin{cases} -\eta_0 \Delta \bar{u}_\varepsilon + \nabla \bar{p}_\varepsilon = \big(1 - \varepsilon \frac{\gamma_1}{\eta_0}(\partial_t - \frac{1}{\mathrm{Pe}}\Delta)\big) h + \varepsilon \mathrm{div}(F_0(\bar{u}_\varepsilon)), \\ \mathrm{div}(\bar{u}_\varepsilon) = 0, \quad \fint_{\mathbb{T}^d} \bar{u}_\varepsilon = 0, \end{cases} \tag{3.20}$$

where we have set for abbreviation

$$F_0(u) := \gamma_1 (u \cdot \nabla) 2\mathrm{D}(u) + \gamma_1 \big((\nabla u)^T 2\mathrm{D}(u) + 2\mathrm{D}(u)(\nabla u)\big) + \gamma_2 (2\mathrm{D}(u))^2. \tag{3.21}$$

Moreover, the so-defined solution $\bar{u}_\varepsilon$ *satisfies* (3.17) *with* $\mathrm{Re} = 0$ *for some controlled error term* $O(\varepsilon^2)$.

(ii) Navier–Stokes case $\mathrm{Re} = 1$, $d = 2$:
Further assume $\eta_0 \ge \frac{1}{\mathrm{Pe}}$.[6] *Given* $s > 1$, $T_0 > 0$, $u^\circ \in H^s(\mathbb{T}^2)^2$ *with* $\mathrm{div}(u^\circ) = 0$, *and* $h \in \mathrm{L}^2(0, T_0; H^{s+1}(\mathbb{T}^2)^2)$, *provided that* $\varepsilon \ll 1$ *is small enough (depending on* s, T_0, u°, h *and on all parameters), the following nonlinear problem admits a unique solution* $\bar{u}_\varepsilon \in \mathrm{L}^\infty(0, T_0; H^s(\mathbb{T}^2)^2) \cap \mathrm{L}^2(0, T_0; H^{s+1}(\mathbb{T}^2)^2)$,

$$\begin{cases} (\partial_t + \bar{u}_\varepsilon \cdot \nabla)\bar{u}_\varepsilon - \eta_0 \Delta \bar{u}_\varepsilon - \varepsilon \gamma_1 (\eta_0 - \frac{1}{\mathrm{Pe}}) \Delta^2 \bar{u}_\varepsilon + \nabla \bar{p}_\varepsilon \\ \qquad\qquad = (1 + \varepsilon \gamma_1 \Delta) h + \varepsilon \mathrm{div}(F_1(\bar{u}_\varepsilon)), \\ \mathrm{div}(\bar{u}_\varepsilon) = 0, \\ \bar{u}_\varepsilon|_{t=0} = u^\circ, \end{cases} \tag{3.22}$$

where we have set for abbreviation

$$F_1(u) := 2\gamma_1 (\nabla u)^T (\nabla u) + \gamma_2 (2\mathrm{D}(u))^2.$$

Moreover, the so-defined solution $\bar{u}_\varepsilon$ *satisfies* (3.17) *with* $\mathrm{Re} = 1$ *with some controlled error term* $O(\varepsilon^2)$ *and with initial condition* $\bar{u}_\varepsilon|_{t=0} = u^\circ$. ◇

3.4. **Models for inhomogeneous suspensions.** The above formulation of ordered fluid models describes the behavior of spatially homogeneous suspensions, that is, the behavior of fluids with a constant density of suspended particles. This can be naturally generalized to an inhomogeneous setting to describe non-uniform particle suspensions. Surprisingly, we were not able to find any account of this generalization in the literature. We emphasize however that such an inhomogeneous setting should also arise naturally from the small-Wi expansion of the inhomogeneous Oldroyd–B model, which is the formal exact closure of the kinetic Hookean dumbbell model. Note that we are still considering an homogeneous solvent fluid. At infinite Péclet number $\mathrm{Pe} = \infty$, the adaptation is straightforward:

[6] For $\eta_0 < \frac{1}{\mathrm{Pe}}$, the equations would need to be rearranged differently; we skip it here for shortness.

the fluid equations (3.1) are simply coupled to a conservation equation for the particle density $\rho := \fint_{\mathbb{S}^{d-1}} f(\cdot, n)\,\mathrm{d}n$,[7]

$$(\partial_t + u\cdot\nabla)\rho = 0,$$

while the stress σ is now a function both of ρ and of the Rivlin–Ericksen tensors $\{A_k(u)\}_k$ where non-Newtonian corrections to the pure solvent viscosity η_0 are taken proportional to the suspended particle density ρ. More precisely, the inhomogeneous second-order fluid model takes the form

$$\begin{cases} \mathrm{Re}(\partial_t + u\cdot\nabla)u - \mathrm{div}(\sigma) + \nabla p = h, \\ (\partial_t + u\cdot\nabla)\rho = 0, \\ \sigma \,=\, (\eta_0 + \eta_1\rho)A_1(u) + \alpha_1\rho A_2(u) + \alpha_2\rho A_1(u)^2, \\ \mathrm{div}(u) = 0, \end{cases} \tag{3.23}$$

for some coefficients $\eta_1, \alpha_1, \alpha_2 \in \mathbb{R}$. Inhomogeneous versions of higher-order fluid models are formulated similarly.

At finite Péclet number $\mathrm{Pe} < \infty$, on the other hand, the particle density ρ is no longer simply transported by the fluid, but rather solves a transport-diffusion equation,

$$(\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u\cdot\nabla)\rho = 0.$$

In this diffusive setting, the structure of ordered fluid models becomes slightly more complicated: due to diffusion, the transport-diffusion operator that appears in the Rivlin–Ericksen tensors (3.13) at finite Péclet number does not commute with multiplication with the particle density ρ. The second-order fluid model then rather takes the form

$$\begin{cases} \mathrm{Re}(\partial_t + u\cdot\nabla)u - \mathrm{div}(\sigma) + \nabla p = h, \\ (\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u\cdot\nabla)\rho = 0, \\ \sigma \,=\, (\eta_0 + \eta_1\rho)A_1(u) + \alpha_1 A_2'(\rho,u) + \alpha_2\rho A_1(u)^2, \\ \mathrm{div}(u) = 0, \end{cases} \tag{3.24}$$

for some coefficients $\eta_1, \alpha_1, \alpha_2 \in \mathbb{R}$, in terms of the modified inhomogeneous second-order Rivlin–Ericksen tensor

$$A_2'(\rho,u) \,:=\, (\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u\cdot\nabla)(\rho A_1(u)) + \rho\big((\nabla u)^T A_1(u) + A_1(u)(\nabla u)\big). \tag{3.25}$$

Indeed, due to the diffusion, the latter quantity does not reduce to the Rivlin–Ericksen tensor defined in (3.13): we have $A_2'(\rho,u) \neq \rho A_2'(u)$ in general along solutions — in contrast with the case of infinite Péclet number.[8] Similarly, the inhomogeneous third-order fluid model amounts to (3.24) with stress

$$\sigma \,=\, (\eta_0 + \eta_1\rho)A_1(u) + \alpha_1 A_2'(\rho,u) + \alpha_2 A_1(u)^2 + \beta_1 A_3'(\rho,u) + \beta_1' B_3'(\rho,u) \\ + \beta_2\big(A_1(u)A_2'(\rho,u) + A_2'(\rho,u)A_1(u)\big) + \beta_3\rho A_1(u)\mathrm{tr}(A_1(u)^2),$$

in terms of the modified inhomogeneous third-order Rivlin–Ericksen tensor

$$A_3'(\rho,u) \,:=\, (\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u\cdot\nabla)A_2'(\rho,u) + (\nabla u)^T A_2'(\rho,u) + A_2'(\rho,u)(\nabla u),$$

[7] The physical particle density is rather given by $x \mapsto \int_{\mathbb{S}^{d-1}} f(x,n)\,\mathrm{d}n$, but for notational convenience we choose to normalize it by the area of $\mathbb{S}^{d-1}$. In particular, we have $\rho \in \frac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d)$.

[8] At infinite Péclet number $\mathrm{Pe} = \infty$, as the particle density ρ satisfies $(\partial_t + u\cdot\nabla)\rho = 0$, we indeed obtain $(\partial_t + u\cdot\nabla)(\rho A_1(u)) = \rho(\partial_t + u\cdot\nabla)A_1(u)$, so that the system (3.24) reduces to (3.23).

and in terms of the following additional quantity, which needs to be included similarly as in (3.15) at finite Péclet number,

$$B_3'(\rho,u) := (\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u\cdot\nabla)(\rho A_1(u)^2) + \rho\big((\nabla u)^T A_1(u)^2 + A_1(u)^2(\nabla u)\big).$$

We turn to the corresponding well-posedness question for the above inhomogeneous models. As in the homogeneous setting, we focus for shortness on the second-order model (3.24), which is again ill-posed whenever $\alpha_1 < 0$. We consider the perturbative case of a weak nonlinearity,

$$\alpha_1 = \varepsilon\gamma_1, \qquad \alpha_2 = \varepsilon\gamma_2, \qquad \varepsilon \ll 1,$$

and we shall define a well-posed notion of approximate solutions that only satisfy equation (3.24) up to higher-order $O(\varepsilon^2)$ remainder:

$$\begin{cases} \mathrm{Re}(\partial_t + \bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon - \mathrm{div}(\bar\sigma_\varepsilon) + \nabla\bar p_\varepsilon = h + O(\varepsilon^2),\\ (\partial_t + \bar u_\varepsilon\cdot\nabla)\bar\rho_\varepsilon - (\frac{1}{\mathrm{Pe}} + \varepsilon\mu_0)\Delta\bar\rho_\varepsilon = O(\varepsilon^2),\\ \bar\sigma_\varepsilon \,=\, (\eta_0+\eta_1\bar\rho_\varepsilon)A_1(\bar u_\varepsilon) + \varepsilon\gamma_1 A_2'(\bar\rho_\varepsilon,\bar u_\varepsilon) + \varepsilon\gamma_2 A_1(\bar u_\varepsilon)^2,\\ \mathrm{div}(\bar u_\varepsilon) = 0. \end{cases} \tag{3.26}$$

For later purposes, note that we henceforth increase the diffusion of the particle density by an additional constant $\mu_0 \ge 0$, which will appear in our setting as a possible effect of particle swimming velocities. Similarly as in Proposition 3.2 for the homogeneous case, the simplest notion of well-posed solutions takes the form of hierarchical solutions as described in the following statement. The proof is postponed to Appendix B. Note that the regularity theory for (3.27) below is quite delicate in the 3D Stokes case, as a particularly careful stepwise argument is needed to first cover low-regularity situations. The notion of Boussinesq-type solutions of Proposition 3.3 could also be easily extended to the present inhomogeneous setting, but we skip the detail for conciseness.

Proposition 3.4 (Hierarchical solutions)**.** *Consider equation (3.26) with parameters $\eta_0, \mathrm{Pe} > 0$, $\eta_1,\mu_0 \ge 0$, and $\gamma_1,\gamma_2 \in \mathbb{R}$. If $(u_0,\rho_0),(u_1,\rho_1)$ are smooth solutions of the following two auxiliary systems,*

$$\begin{cases} \mathrm{Re}(\partial_t + u_0\cdot\nabla)u_0 - \mathrm{div}\big(2(\eta_0+\eta_1\rho_0)\,\mathrm{D}(u_0)\big) + \nabla p_0 = h,\\ (\partial_t - \frac{1}{\mathrm{Pe}}\Delta + u_0\cdot\nabla)\rho_0 = 0,\\ \mathrm{div}(u_0) = 0,\\ u_0|_{t=0} = u^\circ \quad \text{if } \mathrm{Re}\ne 0, \quad \fint_{\mathbb{T}^d} u_0 = 0 \quad \text{if } \mathrm{Re} = 0,\\ \rho_0|_{t=0} = \rho^\circ, \end{cases} \tag{3.27}$$

$$\begin{cases} \mathrm{Re}\,\big((\partial_t + u_0\cdot\nabla)u_1 + (u_1\cdot\nabla)u_0\big) - \mathrm{div}\big(2(\eta_0+\eta_1\rho_0)\,\mathrm{D}(u_1)\big) + \nabla p_1\\ \qquad\qquad = \mathrm{div}\big(2\eta_1\rho_1\,\mathrm{D}(u_0) + \gamma_1 A_2'(\rho_0,u_0) + \gamma_2(2\,\mathrm{D}(u_0))^2\big),\\ (\partial_t - \frac{1}{\mathrm{Pe}}\Delta + u_0\cdot\nabla)\rho_1 = \mu_0\Delta\rho_0 - u_1\cdot\nabla\rho_0,\\ \mathrm{div}(u_1) = 0,\\ u_1|_{t=0} = 0 \quad \text{if } \mathrm{Re}\ne 0, \quad \fint_{\mathbb{T}^d} u_1 = 0 \quad \text{if } \mathrm{Re} = 0,\\ \rho_1|_{t=0} = 0, \end{cases} \tag{3.28}$$

then the superposition $(\bar u_\varepsilon,\bar\rho_\varepsilon) = (u_0+\varepsilon u_1, \rho_0+\varepsilon\rho_1)$ satisfies equation (3.26) with some controlled remainder $O(\varepsilon^2)$ and with initial condition $\bar\rho_\varepsilon|_{t=0} = \rho^\circ$ (and $\bar u_\varepsilon|_{t=0} = u^\circ$ if $\mathrm{Re}\ne 0$). For the well-posedness of (3.27) and (3.28), we separately consider the Stokes and Navier–Stokes cases:

(i) Stokes case $\mathrm{Re}=0$, $d\le 3$:
Given $\rho^\circ\in \mathrm{L}^2(\mathbb{T}^d)\cap\frac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d)$ *and* $h\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{-1}(\mathbb{T}^d)^d)$*, there exists a unique global solution* (u_0,ρ_0) *of* (3.27) *with*

$$\begin{array}{rcl} u_0 & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{T}^d)^d),\\ \rho_0 & \in & \mathrm{L}^\infty_{\mathrm{loc}}\left(\mathbb{R}^+;\mathrm{L}^2(\mathbb{T}^d)\cap\frac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d)\right)\cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{T}^d)).\end{array}$$

Moreover, for all integers $s\ge\frac d2+1$*, provided that* $\rho^\circ\in H^s(\mathbb{T}^d)$ *and that* h *belongs to* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{R}^+\times\mathbb{T}^d)^d)$*, this solution further satisfies*

$$\begin{array}{rcl} u_0 & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s+1}(\mathbb{T}^d)^d),\\ \rho_0 & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{T}^d))\cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s+1}(\mathbb{T}^d)).\end{array}$$

In that case, if furthermore $h\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+;H^{s-3}(\mathbb{T}^d)^d)$*, there exists a unique global solution* (u_1,ρ_1) *of* (3.28) *with*

$$\begin{array}{rcl} u_1 & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{T}^d)^d),\\ \rho_1 & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s-2}(\mathbb{T}^d))\cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{T}^d)).\end{array}$$

(ii) Navier–Stokes case $\mathrm{Re}=1$, $d=2$:
Given $\rho^\circ\in \mathrm{L}^2(\mathbb{T}^2)\cap\frac{1}{2\pi}\mathcal{P}(\mathbb{T}^2)$*,* $u^\circ\in \mathrm{L}^2(\mathbb{T}^2)^2$ *which satisfies* $\mathrm{div}(u^\circ)=0$*, and* $h\in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{-1}(\mathbb{T}^2)^2)$*, there exists a unique global solution* (u_0,ρ_0) *of* (3.27) *with*

$$\begin{array}{rcl} u_0 & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{T}^2)^2)\cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{T}^2)^2),\\ \rho_0 & \in & \mathrm{L}^\infty_{\mathrm{loc}}\left(\mathbb{R}^+;\mathrm{L}^2(\mathbb{T}^2)\cap\frac{1}{2\pi}\mathcal{P}(\mathbb{T}^2)\right)\cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{T}^2)).\end{array}$$

Moreover, for all integers $s\ge 2$*, provided that* $\rho^\circ\in H^s(\mathbb{T}^2)$*,* $u^\circ\in H^s(\mathbb{T}^2)^2$*, and* $h\in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{T}^2)^2)$*, this solution further satisfies*

$$\begin{array}{rcl} u_0 & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{T}^2)^2)\cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s+1}(\mathbb{T}^2)^2),\\ \rho_0 & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{T}^2))\cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s+1}(\mathbb{T}^2)).\end{array}$$

In that case, there exists a unique global solution (u_1,ρ_1) *of* (3.28) *with*

$$\begin{array}{rcl} u_1 & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s-2}(\mathbb{T}^2)^2)\cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{T}^2)^2),\\ \rho_1 & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s-2}(\mathbb{T}^2))\cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{T}^2)).\end{array}\qquad\Diamond$$

4. Statement of main results

We turn to the precise statement of our main result, that is, the rigorous hydrodynamic approximation of the Doi–Saintillan–Shelley theory in the small-Wi regime. For simplicity, we focus on the first-order approximation and the emergence of the second-order fluid model, but the same analysis can be pursued to arbitrary order (see Section 4.2). More precisely, we derive the second-order fluid model (3.26) with explicit coefficients

$$\begin{array}{lll} \eta_0=1, & \eta_1=\lambda\frac{(\theta+2)\omega_d}{2d(d+2)}, & \mu_0=\frac{1}{d(d-1)}U_0^2,\\ \gamma_1=-\lambda\theta\frac{\omega_d}{4d^2(d+2)}, & \gamma_2=\lambda\frac{\omega_d}{2d^2(d+4)}(\theta+\frac{2d}{d+2}), & \end{array}\tag{4.1}$$

where we recall that λ,θ,U_0 are parameters from the Doi–Saintillan–Shelley system (2.5). Before formulating a precise result, we introduce a suitable well-preparedness assumption for initial data. More precisely, in order to avoid initial boundary layers due to the $O(\frac{1}{\varepsilon})$

rotational diffusion, we first need to assume that to leading order the initial density is invariant under this rotational diffusion, which means that it is isotropic to leading order,

$$f_\varepsilon(x,n)|_{t=0} \;=\; f_\varepsilon^\circ(x,n) \;=\; \rho^\circ(x)+O(\varepsilon), \qquad \rho^\circ \in \tfrac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d).$$

Yet, this is not sufficient: in order to avoid initial boundary layers in the higher-order ε-expansion, we further need to assume that initial data are compatible with the formal limiting hierarchy, which is precisely the content of the assumption below. Similar issues are well known for higher-order hydrodynamic expansions in the Boltzmann theory, see e.g. [Caf80; SK83; Lac87].

Assumption 4.1 (Well-preparedness). *Let $h \in C^\infty(\mathbb{R}^+\times\mathbb{T}^d)^d$, let $\rho^\circ \in H^s(\mathbb{T}^d)\cap\frac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d)$ for some $s \gg 1$, and let also $u^\circ \in H^s(\mathbb{T}^d)^d$ with $\operatorname{div}(u^\circ)=0$ in the Navier–Stokes case. We assume that the initial condition $f_\varepsilon|_{t=0}=f_\varepsilon^\circ$ for the Doi–Saintillan–Shelley system (2.5) is well-prepared in the following sense: decomposing*

$$f_\varepsilon^\circ(x,n) \;=\; \rho_\varepsilon^\circ(x)+g_\varepsilon^\circ(x,n), \qquad \rho_\varepsilon^\circ \;:=\; \langle f_\varepsilon^\circ\rangle \;=\; \fint_{\mathbb{S}^{d-1}} f_\varepsilon^\circ(\cdot,n)\,dn \;\in\; \tfrac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d),$$

we have

$$\rho_\varepsilon^\circ=\rho^\circ \qquad \text{and} \qquad \varepsilon^{\frac12}\|g_\varepsilon^\circ-(\varepsilon g_1+\varepsilon^2 g_2)|_{t=0}\|_{\mathrm{L}^2_{x,n}} \;\le\; C_0\varepsilon^3,$$

for some constant $C_0<\infty$, where g_1 and g_2 are the solutions of the hierarchical equations (5.3) and (5.4) below with initial data ρ° in the Stokes case and (u°,ρ°) in the Navier–Stokes case. ♢

Note that this well-preparedness assumption is compatible with the positivity $f_\varepsilon|_{t=0}\ge 0$ for ε small enough, which is necessary to ensure well-posedness of the Doi–Saintillan–Shelley system (2.5), cf. Proposition 2.1. In these terms, we are now in position to state our main result, thus finally providing a more detailed statement of Theorem 1.1. The proof is given in Section 5.

Theorem 4.2 (Small-Wi expansion). *Let $h \in C^\infty(\mathbb{R}^+\times\mathbb{T}^d)^d$, let $\rho^\circ \in H^s(\mathbb{T}^d)\cap\frac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d)$ for some $s \gg 1$, and let also $u^\circ \in H^s(\mathbb{T}^d)^d$ with $\operatorname{div}(u^\circ)=0$ in the Navier–Stokes case. Denote by $(u_\varepsilon,f_\varepsilon)$ the global solution of the Doi–Saintillan–Shelley model (2.5) as given by Proposition 2.1 with initial condition $f_\varepsilon|_{t=0}=f_\varepsilon^\circ\in \mathrm{L}^2\cap\mathcal{P}(\mathbb{T}^d\times\mathbb{S}^{d-1})$ satisfying the well-preparedness of Assumption 4.1. Further assume that*

$$\lambda\theta(1+\mathrm{Pe})\|\rho^\circ\|_{\mathrm{L}^\infty_x} \;\ll\; 1$$

is smaller than some universal constant.

(i) Stokes case $\mathrm{Re}=0$, $d\le 3$:
Let $(\bar u_\varepsilon,\bar\rho_\varepsilon)$ be the unique global hierarchical solution of (3.26) as given by Proposition 3.4(i) with explicit coefficients (4.1). Then we have for all $t\ge 0$,

$$\begin{aligned}\|\nabla(u_\varepsilon-\bar u_\varepsilon)\|_{\mathrm{L}^2_t\mathrm{L}^2_x} &\;\lesssim\; \varepsilon^2,\\ \|\rho_\varepsilon-\bar\rho_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x}+\|\nabla(\rho_\varepsilon-\bar\rho_\varepsilon)\|_{\mathrm{L}^2_t\mathrm{L}^2_x} &\;\lesssim\; \varepsilon^2,\end{aligned}$$

where multiplicative constants depend on Pe *and on an upper bound on t,λ,U_0, and on controlled norms of h and ρ°.*

(ii) Navier–Stokes case $\mathrm{Re} = 0$, $d = 2$:
Let $(\bar u_\varepsilon, \bar\rho_\varepsilon)$ be the unique global hierarchical solution of (3.26) as given by Proposition 3.4(ii) with explicit coefficients (4.1). Then we have for all $t \ge 0$,

$$\|u_\varepsilon - \bar u_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^2_x} + \|\nabla(u_\varepsilon - \bar u_\varepsilon)\|_{\mathrm{L}^2_t \mathrm{L}^2_x} \lesssim \varepsilon^2,$$
$$\|\rho_\varepsilon - \bar\rho_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^2_x} + \|\nabla(\rho_\varepsilon - \bar\rho_\varepsilon)\|_{\mathrm{L}^2_t \mathrm{L}^2_x} \lesssim \varepsilon^2,$$

where multiplicative constants depend on Pe *and on an upper bound on t, λ, U_0, and on controlled norms of h, ρ°, and u°.* ♢

4.1. **Non-Newtonian properties of hydrodynamic approximation.** We comment on the rheological features of the obtained second-order fluid system (3.26) with the explicit parameters $\eta_0, \eta_1, \mu_0, \gamma_1, \gamma_2$ as defined in (4.1), briefly describing the resulting non-Newtonian properties and how they depend on microscopic features. We focus here on the physically relevant 3D setting, and we point out that in the passive case the parameter values agree with [HL72, Eqn (41) and Table 2] and [Bre74, Eqn (7.4)].

- *Effective spatial diffusion:*
The spatial diffusion $\frac{1}{\mathrm{Pe}}$ of the suspended particle density is enhanced by particles' activity even at infinite Péclet number: it is replaced by $\frac{1}{\mathrm{Pe}} + \varepsilon\mu_0$ with $\mu_0 = \frac{1}{6}U_0^2$. This naturally follows from the coupling of particles' swimming velocity with their rotational diffusion. This phenomenon of increased mixing has been observed in studies such as [SS07; Sai18].
- *Modified zero-shear viscosity:*
The presence of suspended particles leads to a non-trivial contribution to the zero-shear viscosity: in the homogeneous setting $\bar\rho_\varepsilon \equiv \frac{1}{\omega_d}$, we obtain a zero-shear viscosity
$$\tilde\eta_0 := \eta_0 + \tfrac{1}{\omega_d}\eta_1 = 1 + \tfrac{1}{30}\lambda(2+\theta).$$
In particular, in case of passive particles, this zero-shear viscosity is always larger than the plain fluid viscosity, $\tilde\eta_0 > 1$, while particles' activity can reverse this effect. For a precise description, first recall that $\theta = 6 + 2\alpha\frac{|V_0|\ell\mu_{\mathrm{fl}}}{k_B\Theta}$, cf. (2.4), where α characterizes the swimming mechanism:
— for passive particles $\alpha = 0$, the zero-shear viscosity is $\tilde\eta_0 = 1 + \frac{4}{15}\lambda > 1$;
— for so-called puller particles $\alpha > 0$, the zero-shear viscosity is even larger than for passive particles;
— for so-called pusher particles $\alpha < 0$, the zero-shear viscosity is smaller than for passive particles, and it can even be smaller than the plain fluid viscosity provided that the activity of the particles is strong enough: we find $\tilde\eta_0 < 1$ if $\alpha < -\frac{4k_B\Theta}{|V_0|\ell\mu_{\mathrm{fl}}}$.
This prediction is consistent with well-known experimental results, see e.g. [SA09; RJP10; Lóp+15], and it has been largely confirmed in the literature [HABK08; Hai+09; Sai10a; Sai10b; GLAB11; AMES16]. In particular, for E. coli bacteria (a typical pusher particle), we can assess the value of the parameter α using the experimental measurements performed in [Dre+11]: this yields $\alpha < -\frac{4k_B\Theta}{|V_0|\ell\mu_{\mathrm{fl}}}$ and the experimental findings of [Gac+13] then confirm our prediction that the effective zero-shear viscosity is smaller than the plain fluid viscosity.
- *Normal-stress differences:*
In the homogeneous setting $\bar\rho_\varepsilon \equiv \frac{1}{\omega_d}$, we obtain the following values for first and second-normal stress coefficients, as defined in (3.6),
$$\nu_{10} = \tfrac{\theta}{90}\varepsilon\lambda \qquad \text{and} \qquad \nu_{20} = \tfrac{3-\theta}{315}\varepsilon\lambda.$$

For passive particles ($\theta = 6$), we thus obtain $\nu_{10} > 0$, $\nu_{20} < 0$, and $\nu_{10}/|\nu_{20}| = 7$, which agrees with experiments as discussed in Section 3. The amplitude of these normal stress coefficients is even increased in case of puller particles. In contrast, for pusher particles, normal stress coefficients are reduced, and a very large activity could even result into opposite effects: we find $\nu_{10} < 0$ if $\theta < 0$, and $\nu_{20} > 0$ if $\theta < 3$. This behavior was also predicted in [Sai10b; PRB16], but has yet to be experimentally verified.

- *Elongational viscosity:*
  In the homogeneous setting $\bar\rho_\varepsilon \equiv \frac{1}{\omega_d}$, in a uniaxial elongational flow in the direction e_1, that is, $\bar u_\varepsilon = \kappa(x_1 e_1 - \frac12(x_2 e_2 + x_3 e_3))$, we obtain the following value for the elongational viscosity, as defined in (3.10),
  $$\eta_E = \big(3 + \tfrac{1}{10}\lambda(2+\theta)\big) + \kappa\tfrac{1}{140}\varepsilon\lambda(4+\theta).$$
  This shows that passive suspensions ($\theta = 6$) lead to a strain-thickening behavior, which is even increased in case of puller particles. In contrast, for pusher particles, the strain-thickening behavior is reduced, and a very large activity could even result in the opposite effect: the system becomes strain-thinning if $\theta < -4$ (that is, $\alpha < -\frac{5k_B\Theta}{|V_0|\ell\mu_{\mathrm{fl}}}$).

4.2. **Next-order description.** The above result is easily pursued to higher orders in the small-Wi expansion. For shortness, we stick here to a formal discussion. First, we show in Appendix C that the next-order description of the suspended particle density involves additional nontrivial transport and anisotropic diffusion terms depending on the surrounding fluid flow: we find
$$(\partial_t + \bar u_\varepsilon\cdot\nabla)\bar\rho_\varepsilon - \operatorname{div}\Big(\big(\tfrac{1}{\mathrm{Pe}} + \varepsilon\mu_0 + \varepsilon^2\mu_1\,\mathrm{D}(\bar u_\varepsilon)\big)\nabla\bar\rho_\varepsilon\Big) = \varepsilon^2\mu_2\operatorname{div}(\bar\rho_\varepsilon\Delta\bar u_\varepsilon) + O(\varepsilon^3), \qquad (4.2)$$
with explicit coefficients
$$\begin{aligned}\mu_0 &:= \tfrac{U_0^2}{d(d-1)},\\ \mu_1 &:= \tfrac{(3d+1)U_0^2}{d(d-1)^2(d+2)},\\ \mu_2 &:= \tfrac{U_0^2}{2d(d-1)(d+2)}.\end{aligned}$$
In particular, this shows that homogeneous spatial densities are still stable to order $O(\varepsilon^3)$, and we shall henceforth restrict for simplicity to the homogeneous setting,
$$\bar\rho_\varepsilon = \tfrac{1}{\omega_d} + O(\varepsilon^3).$$
In addition, we shall focus on the case of infinite Péclet number and of vanishing particle swimming velocity,
$$\mathrm{Pe} = \infty, \qquad U_0 = 0,$$
as this choice strongly simplifies the macroscopic equations and as it seems anyhow to be the most relevant setting physically, cf. Section 2 (recall however that our rigorous results do not hold for $\mathrm{Pe} = \infty$). In this setting, we formally derive in Appendix C the following third-order fluid equations,
$$\begin{cases}\mathrm{Re}(\partial_t + \bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon - \operatorname{div}(\bar\sigma_\varepsilon) + \nabla\bar p_\varepsilon = h + O(\varepsilon^3),\\ \operatorname{div}(\bar u_\varepsilon) = 0,\end{cases} \qquad (4.3)$$
where the stress is given by
$$\bar\sigma_\varepsilon = (1+\eta_1)A_1(\bar u_\varepsilon) + \varepsilon\gamma_1 A_2(\bar u_\varepsilon) + \varepsilon\gamma_2 A_1(\bar u_\varepsilon)^2$$

$$+\,\varepsilon^2\kappa_1 A_3(\bar u_\varepsilon)+\varepsilon^2\kappa_2\big(A_1(\bar u_\varepsilon)A_2(\bar u_\varepsilon)+A_2(\bar u_\varepsilon)A_1(\bar u_\varepsilon)\big)+\varepsilon^2\kappa_3 A_1(\bar u_\varepsilon)\mathrm{tr}(A_1(\bar u_\varepsilon)^2)$$

with explicit coefficients

$$\begin{aligned}
\eta_1 &:= \lambda\tfrac{1}{2d(d+2)}(\theta+2),\\
\gamma_1 &:= -\lambda\theta\tfrac{1}{4d^2(d+2)},\\
\gamma_2 &:= \lambda\tfrac{1}{2d^2(d+4)}(\theta+\tfrac{2d}{d+2}),\\
\kappa_1 &:= \lambda\theta\tfrac{1}{8d^3(d+2)},\\
\kappa_2 &:= -\lambda\tfrac{1}{8d^3(d+4)}(3\theta+\tfrac{2d}{d+2}),\\
\kappa_3 &:= \lambda\tfrac{1}{8d^3(d+2)^2(d+4)(d+6)}\Big(2d(3d^2+10d+6)+\theta(d+4)(3d^2+11d+12)\Big).
\end{aligned}$$

These third-order fluid coefficients coincide in the passive case ($\theta=6$) with those computed by Brenner [Bre74, Eq. (7.4)] (once the notation is properly identified). Regarding the non-Newtonian phenomena discussed in Section 3, the main observation is that this third-order fluid model describes the expected shear-thinning behavior of the suspension. Indeed, the shear-dependent viscosity is given in 3D as follows, cf. Section 3.1,

$$\begin{aligned}
\kappa \;&\mapsto\; 1+\eta_1+2\varepsilon^2(\kappa_2+\kappa_3)\kappa^2\\
&= 1+\lambda\tfrac{\theta+2}{30}-\varepsilon^2\lambda\tfrac{19\theta-12}{18900}\kappa^2,
\end{aligned}$$

which is decreasing in κ if and only if $\theta>\frac{12}{19}$. As expected, this shows that passive suspensions ($\theta=6$) lead to a shear-thinning behavior, which is even increased in case of puller particles. In contrast, for pusher particles, the shear-thinning behavior is reduced, and a very large activity could even result in the opposite effect: the system becomes shear-thickening if $\theta<\frac{12}{19}$ (that is, $\alpha<-\frac{51k_B\Theta}{19|V_0|\ell\mu_{\mathrm{fl}}}$). This possible shear-thickening effect was indeed measured experimentally in [Gac+13; Lóp+15] for suspensions of E. coli bacteria (pusher-type microswimmers) with strong enough activity. We also refer to [Hai+09; GLAB11; PRB16] for analytical and numerical results showing the same effect.

We note that for $U_0\neq0$ the corresponding fluid equation for $\bar u_\varepsilon$ would differ from the 3rd-order fluid model even at infinite Péclet number and at uniform particle density. In particular, an additional dispersive correction $-\varepsilon^2\kappa_4\Delta^2\bar u_\varepsilon$ needs to be included in the fluid equation. We skip the detail as the case $|U_0|\ll1$ seems to be the most relevant physically.

5. Small-Wi expansion of Doi–Saintillan–Shelley theory

This section is devoted to the small-ε expansion of the solution $(u_\varepsilon,f_\varepsilon)$ of the Doi–Saintillan–Shelley model (2.5), as well as to the identification of second-order fluid equations satisfied by the truncated expansion. We naturally split the particle density as

$$f_\varepsilon(x,n)\;=\;\rho_\varepsilon(x)+g_\varepsilon(x,n),\qquad \rho_\varepsilon\;:=\;\langle f_\varepsilon\rangle\;:=\;\fint_{\mathbb{S}^{d-1}}f_\varepsilon(\cdot,n)\,\mathrm{d}n\;\in\;\tfrac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d),$$

where ρ_ε stands for the spatial density and where $\langle g_\varepsilon\rangle=0$. Recall the well-preparedness condition of Assumption 4.1: we assume in particular

$$\rho_\varepsilon|_{t=0}=\rho^\circ,\qquad g_\varepsilon|_{t=0}=O(\varepsilon),\qquad \rho^\circ\in\tfrac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d). \tag{5.1}$$

meaning that we start from an initial density that is to leading order at equilibrium with respect to the strong rotational diffusion in (2.5). In this setting, we shall analyze the asymptotic behavior of the solution $(u_\varepsilon,f_\varepsilon)$ and derive a hydrodynamic approximation in the spirit of Hilbert's expansion method in the Boltzmann theory [Hil12; Caf80; Gol05; SR09]. We start from the ansatz

$$\begin{aligned} u_\varepsilon &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \\ \rho_\varepsilon &= \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \dots, \\ g_\varepsilon &= g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3 + \dots, \end{aligned} \tag{5.2}$$

with $\rho_0 \in \frac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d)$, with $\int_{\mathbb{T}^d} \rho_n = 0$ for $n \geq 1$, and with $\langle g_n \rangle = 0$ for all $n \geq 0$. Inserting it into the system (2.5), and identifying powers of ε, we are led formally to the following hierarchy of coupled equations:

- *Order $O(\varepsilon^{-1})$:* The equation for the particle density yields $\Delta_n g_0 = 0$, and therefore, as by definition $\langle g_0 \rangle = 0$, we must have

  $$g_0 = 0.$$

  On the other hand, the fluid equation yields $\operatorname{div}_x(\sigma_1[\rho_0 + g_0]) = 0$, which is then automatically satisfied as $g_0 = 0$ and as ρ_0 does not depend on n.
- *Order $O(\varepsilon^0)$:* The triplet (u_0, ρ_0, g_1) satisfies

$$\begin{cases} \mathrm{Re}(\partial_t + u_0 \cdot \nabla) u_0 - \Delta u_0 + \nabla p_0 = h + \operatorname{div}(\sigma_1[g_1]) + \operatorname{div}(\sigma_2[\rho_0, \nabla u_0]), \\ \Delta_n g_1 = U_0 n \cdot \nabla_x \rho_0 + \operatorname{div}_n\big(\pi_n^\perp(\nabla u_0) n \rho_0\big), \\ (\partial_t - \frac{1}{\mathrm{Pe}}\Delta + u_0 \cdot \nabla)\rho_0 = 0, \\ \operatorname{div}(u_0) = 0, \quad \langle g_1 \rangle = 0, \\ u_0|_{t=0} = u^\circ \ \text{ if } \mathrm{Re} \neq 0, \quad \int_{\mathbb{T}^d} u_0 = 0 \ \text{ if } \mathrm{Re} = 0, \\ \rho_0|_{t=0} = \rho^\circ. \end{cases} \tag{5.3}$$

- *Order $O(\varepsilon^1)$:* The triplet (u_1, ρ_1, g_2) satisfies

$$\begin{cases} \mathrm{Re}\,\big((\partial_t + u_0 \cdot \nabla) u_1 + (u_1 \cdot \nabla) u_0\big) - \Delta u_1 + \nabla p_1 \\ \qquad\qquad = \operatorname{div}(\sigma_1[g_2]) + \operatorname{div}\big(\sigma_2[\rho_0, \nabla u_1] + \sigma_2[\rho_1 + g_1, \nabla u_0]\big), \\ \Delta_n g_2 = (\partial_t - \frac{1}{\mathrm{Pe}}\Delta_x + u_0 \cdot \nabla_x) g_1 + P_1^\perp\big(U_0 n \cdot \nabla_x(\rho_1 + g_1)\big) \\ \qquad\qquad + \operatorname{div}_n\big(\pi_n^\perp(\nabla u_1) n \rho_0 + \pi_n^\perp(\nabla u_0) n (\rho_1 + g_1)\big), \\ (\partial_t - \frac{1}{\mathrm{Pe}}\Delta_x + u_0 \cdot \nabla_x)\rho_1 + u_1 \cdot \nabla_x \rho_0 + \langle U_0 n \cdot \nabla_x g_1 \rangle = 0, \\ \operatorname{div}(u_1) = 0, \quad \langle g_2 \rangle = 0, \\ u_1|_{t=0} = 0 \ \text{ if } \mathrm{Re} \neq 0, \quad \int_{\mathbb{T}^d} u_1 = 0 \ \text{ if } \mathrm{Re} = 0, \\ \rho_1|_{t=0} = 0. \end{cases} \tag{5.4}$$

- *Order $O(\varepsilon^2)$:* The triplet (u_2, ρ_2, f_3) satisfies

$$\begin{cases} \mathrm{Re}\,\big((\partial_t + u_0 \cdot \nabla) u_2 + (u_1 \cdot \nabla) u_1 + (u_2 \cdot \nabla) u_0\big) - \Delta u_2 + \nabla p_2 \\ \qquad = \operatorname{div}(\sigma_1[g_3]) + \operatorname{div}\big(\sigma_2[\rho_0, \nabla u_2] + \sigma_2[\rho_1 + g_1, \nabla u_1] + \sigma_2[\rho_2 + g_2, \nabla u_0]\big), \\ \Delta_n g_3 = (\partial_t - \frac{1}{\mathrm{Pe}}\Delta_x + u_0 \cdot \nabla_x) g_2 + u_1 \cdot \nabla_x g_1 + P_1^\perp\big(U_0 n \cdot \nabla_x(\rho_2 + g_2)\big) \\ \qquad + \operatorname{div}_n\big(\pi_n^\perp(\nabla u_2) n \rho_0 + \pi_n^\perp(\nabla u_1) n (\rho_1 + g_1) + \pi_n^\perp(\nabla u_0) n (\rho_2 + g_2)\big), \\ (\partial_t - \frac{1}{\mathrm{Pe}}\Delta_x + u_0 \cdot \nabla_x)\rho_2 + u_1 \cdot \nabla_x \rho_1 + u_2 \cdot \nabla_x \rho_0 + \langle U_0 n \cdot \nabla_x g_2 \rangle = 0, \\ \operatorname{div}(u_2) = 0, \quad \langle g_3 \rangle = 0, \\ u_2|_{t=0} = 0 \ \text{ if } \mathrm{Re} \neq 0, \quad \int_{\mathbb{T}^d} u_2 = 0 \ \text{ if } \mathrm{Re} = 0, \\ \rho_2|_{t=0} = 0. \end{cases} \tag{5.5}$$

In view of this hierarchy, we understand that the condition (5.1) for initial data needs to be further strengthened to avoid initial boundary layers: more precisely, we need to assume that the initial condition $f_\varepsilon|_{t=0} = f_\varepsilon^\circ$ is compatible with the above hierarchy, meaning that g_ε coincides initially with $\varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3$ up to higher order errors. To accuracy $O(\varepsilon^2)$, this well-preparedness is precisely the content of Assumption 4.1.

The well-posedness and the propagation of regularity for the above hierarchy are stated in the following result. We emphasize that we were not able to find references for the types of systems that naturally appear in this ε-expansion, and we believe that some of these new well-posedness results may be of independent interest (see for instance the system (5.18) below). Note that the hierarchy is triangular as we can eliminate the densities g_1, g_2, g_3 in terms of the velocity fields u_0, u_1, u_2 and of the spatial densities ρ_0, ρ_1, ρ_2. We focus on the Stokes case $\mathrm{Re} = 0$, while the 2D Navier–Stokes case follows up to straightforward adaptations and is omitted for shortness. The proof is displayed in Section 5.2.

Proposition 5.1 (Well-posedness of hierarchy)**.** *Consider the Stokes case* $\mathrm{Re} = 0$*,* $d \le 3$*.*

(i) Well-posedness for (u_0, ρ_0, g_1):
Given integer $s \ge \frac{d}{2} + 1$*,* $\rho^\circ \in H^s(\mathbb{T}^d) \cap \frac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d)$*, and* $h \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s-1}(\mathbb{T}^d)^d)$*, there exists a unique solution* (u_0, ρ_0, g_1) *of the Cauchy problem* (5.3) *with*

$$\begin{aligned} u_0 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^d)^d), \\ \rho_0 &\in \mathrm{L}^\infty_{\mathrm{loc}}\left(\mathbb{R}^+; H^s(\mathbb{T}^d) \cap \tfrac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d)\right) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^d)), \\ g_1 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s-1}(\mathbb{T}^d \times \mathbb{S}^{d-1})), \end{aligned}$$

and g_1 *is given by the explicit formula*

$$g_1(\cdot, n) = -\tfrac{1}{d-1} U_0 n \cdot \nabla\rho_0 + \tfrac{1}{2}(n \otimes n) : \rho_0 \, \mathrm{D}(u_0). \tag{5.6}$$

Moreover, for all $r \ge 0$*, provided that* $h \in W^{2,\infty}(\mathbb{R}^+; H^{s-1}(\mathbb{T}^d)^d)$ *and that* s *is chosen large enough, we also have*

$$\begin{aligned} \partial_t u_0,\, \partial_t^2 u_0 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^r(\mathbb{T}^d)^d), \\ \partial_t \rho_0,\, \partial_t^2 \rho_0 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^r(\mathbb{T}^d)). \end{aligned}$$

(ii) Well-posedness for (u_1, ρ_1, g_2):
Given $s \ge 0$*, provided that the solution* (u_0, ρ_0) *of item (i) is such that*

$$\begin{aligned} u_0 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+3}(\mathbb{T}^d)^d) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^d)^d), \\ \rho_0 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+2}(\mathbb{T}^d)) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^d)), \end{aligned}$$

there exists a unique solution (u_1, ρ_1, g_2) *of the Cauchy problem* (5.4) *with*

$$\begin{aligned} u_1 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^d)^d), \\ \rho_1 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^d)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^d)), \\ g_2 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^d \times \mathbb{S}^{d-1})), \end{aligned}$$

and g_2 *is given by the explicit formula*

$$\begin{aligned} g_2(\cdot, n) = &-\tfrac{1}{d-1} U_0 n \cdot \nabla\rho_1 - \tfrac{1}{2d}(n \otimes n - \tfrac{1}{d}\mathrm{Id}) : \Big(\tfrac{1}{4} A_2'(u_0, \rho_0) - \rho_0 \, \mathrm{D}(u_0)^2 \\ &\qquad - d\rho_0 \, \mathrm{D}(u_1) - d\rho_1 \, \mathrm{D}(u_0) - \tfrac{1}{d-1} U_0^2 \nabla^2 \rho_0\Big) \end{aligned}$$

$$-\tfrac{1}{3(d-1)}U_0(\nabla\rho_0)\cdot\Big(n\big((n\otimes n):\mathrm{D}(u_0)\big)+\tfrac{4}{d-1}\,\mathrm{D}(u_0)n\Big)$$
$$-\tfrac{1}{6(d+1)}U_0\rho_0\,\mathrm{div}_x\Big(n\big((n\otimes n):\mathrm{D}(u_0)\big)+\tfrac{4}{d-1}\,\mathrm{D}(u_0)n\Big)$$
$$+\tfrac{1}{8}\rho_0\Big(\big((n\otimes n):\mathrm{D}(u_0)\big)^2-\tfrac{2}{d(d+2)}\,\mathrm{tr}(\mathrm{D}(u_0)^2)\Big),\tag{5.7}$$

in terms of the (non-standard) Rivlin–Ericksen tensor A_2' defined in (3.25). Moreover, for all $r\ge 0$, provided that s is chosen large enough, we also have

$$\begin{aligned}\partial_t u_1 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^r(\mathbb{T}^d)^d),\\ \partial_t\rho_1 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^r(\mathbb{T}^d)).\end{aligned}$$

(iii) Well-posedness for (u_2,ρ_2,g_3):
Given $s\ge 0$, provided that the solutions (u_0,ρ_0) and (u_1,ρ_1) of items (i) and (ii) are such that

$$\begin{aligned}(u_0,\rho_0) &\in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^+;H^r(\mathbb{T}^d)^{d+1}),\\ (u_1,\rho_1) &\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+;H^r(\mathbb{T}^d)^{d+1}),\end{aligned}$$

for some r large enough, then there exists a unique solution (u_2,ρ_2,g_3) of the Cauchy problem (5.5) with

$$\begin{aligned}u_2 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s+1}(\mathbb{T}^d)^d),\\ \rho_2 &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{T}^d))\cap\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s+1}(\mathbb{T}^d)),\\ g_3 &\in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{T}^d\times\mathbb{S}^{d-1})).\end{aligned}$$

Associated to all the above well-posedness results are estimates of the corresponding norms of the solutions in terms of all the parameters and of the controlled norms of the data. ◊

With the above construction of the hierarchy $\{u_n,\rho_n,g_{n+1}\}_{n\ge0}$, we can now turn to the justification of the formal expansion (5.2). We stick to order $O(\varepsilon^3)$ for conciseness, but the proof could be pursued to arbitrary order. The proof is displayed in Section 5.3. Note that the well-preparedness assumption (5.8) below for initial data is one order stronger than in Assumption 4.1: indeed, while our main result in Section 4 focusses on $O(\varepsilon)$ effects, only deriving second-order fluid models, the present result further describes $O(\varepsilon^2)$ effects and therefore requires this strengthened well-preparedness condition. Although not needed for the purposes of our main result, the present next-order analysis is included to illustrate how the ε-expansion can be pursued to arbitrary order without additional mathematical difficulties, then leading to higher-order fluid models; we refer to Appendix C for the corresponding derivation of third-order fluid models.

Proposition 5.2 (Error estimates for ε-expansion). *Let $h\in C^\infty(\mathbb{R}^+\times\mathbb{T}^d)^d$, let $\rho^\circ\in H^s(\mathbb{T}^d)\cap\frac{1}{\omega_d}\mathcal{P}(\mathbb{T}^d)$ for some $s\gg 1$, and let also $u^\circ\in H^s(\mathbb{T}^d)^d$ with $\mathrm{div}(u^\circ)=0$ in the Navier–Stokes case. Denote by $(u_\varepsilon,f_\varepsilon)$ the solution of the Doi–Saintillan–Shelley model (2.5) as given by Proposition 2.1, and assume that the initial condition $f_\varepsilon|_{t=0}=f_\varepsilon^\circ$ is well-prepared in the following sense: decomposing $f_\varepsilon^\circ=\rho_\varepsilon^\circ+g_\varepsilon^\circ$ with $\rho_\varepsilon^\circ:=\langle f_\varepsilon^\circ\rangle$, we have in terms of the functions u_0,u_1,u_2,g_1,g_2,g_3 defined in Proposition 5.1 with data (h,u°,ρ°),*

$$\rho_\varepsilon^\circ=\rho^\circ\qquad\text{and}\qquad \varepsilon^{\frac12}\|g_\varepsilon^\circ-(\varepsilon g_1+\varepsilon^2g_2+\varepsilon^3g_3)|_{t=0}\|_{\mathrm{L}^2_{x,n}}\le C_0\varepsilon^4,\tag{5.8}$$

for some constant $C_0 < \infty$. Further assume that

$$\lambda\theta(1+\mathrm{Pe})\|\rho^\circ\|_{\mathrm{L}^\infty_x} \ll 1$$

is smaller than some universal constant.

(i) Stokes case $\mathrm{Re}=0$, $d \le 3$:
For all $t \ge 0$, we have

$$\begin{aligned}
\varepsilon^{\frac12}\|\nabla(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2)\|_{\mathrm{L}^\infty_t \mathrm{L}^2_x} + \|\nabla(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2)\|_{\mathrm{L}^2_t \mathrm{L}^2_x} &\le \mathcal{C}(t)\varepsilon^3,\\
\varepsilon^{\frac12}\|g_\varepsilon - \varepsilon g_1 - \varepsilon^2 g_2 - \varepsilon^3 g_3\|_{\mathrm{L}^\infty_t \mathrm{L}^2_{x,n}} + \|\nabla_n(g_\varepsilon - \varepsilon g_1 - \varepsilon^2 g_2 - \varepsilon^3 g_3)\|_{\mathrm{L}^2_t \mathrm{L}^2_{x,n}} &\le \mathcal{C}(t)\varepsilon^4,\\
\|\rho_\varepsilon - \rho_0 - \varepsilon\rho_1 - \varepsilon^2\rho_2\|_{\mathrm{L}^\infty_t \mathrm{L}^2_x} + \|\nabla(\rho_\varepsilon - \rho_0 - \varepsilon\rho_1 - \varepsilon^2\rho_2)\|_{\mathrm{L}^2_t \mathrm{L}^2_x} &\le \mathcal{C}(t)\varepsilon^3,
\end{aligned}$$

provided that $\varepsilon\mathcal{C}(t) \ll 1$ is small enough, where the multiplicative constant $\mathcal{C}(t)$ depends on Pe *and on an upper bound on*

$$\begin{gathered}
t,\ C_0,\ \lambda,\ U_0,\quad \|(\nabla u_0, \nabla u_1, \nabla u_2)\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty_x},\quad \|(\rho_0,\rho_1,\rho_2)\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty_x},\\
\|(g_1,g_2,g_3)\|_{\mathrm{L}^\infty_t W^{1,\infty}_{x,n}},\quad \|(\partial_t - \Delta_x)g_3\|_{\mathrm{L}^\infty_t \mathrm{L}^2_{x,n}}.
\end{gathered}$$

(ii) Navier–Stokes case $\mathrm{Re}=0$, $d=2$:
For all $t \ge 0$, we have

$$\begin{aligned}
\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2\|_{\mathrm{L}^\infty_t \mathrm{L}^2_x} + \|\nabla(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2)\|_{\mathrm{L}^2_t \mathrm{L}^2_x} &\le \mathcal{C}(t)\varepsilon^3,\\
\varepsilon^{\frac12}\|g_\varepsilon - \varepsilon g_1 - \varepsilon^2 g_2 - \varepsilon^3 g_3\|_{\mathrm{L}^\infty_t \mathrm{L}^2_{x,n}} + \|\nabla_n(g_\varepsilon - \varepsilon g_1 - \varepsilon^2 g_2 - \varepsilon^3 g_3)\|_{\mathrm{L}^2_t \mathrm{L}^2_{x,n}} &\le \mathcal{C}(t)\varepsilon^4,\\
\|\rho_\varepsilon - \rho_0 - \varepsilon\rho_1 - \varepsilon^2\rho_2\|_{\mathrm{L}^\infty_t \mathrm{L}^2_x} + \|\nabla(\rho_\varepsilon - \rho_0 - \varepsilon\rho_1 - \varepsilon^2\rho_2)\|_{\mathrm{L}^2_t \mathrm{L}^2_x} &\le \mathcal{C}(t)\varepsilon^3,
\end{aligned}$$

provided that $\varepsilon\mathcal{C}(t) \ll 1$ is small enough, where $\mathcal{C}(t)$ now depends Pe *and on an upper bound on*

$$\begin{gathered}
t,\ C_0,\ \lambda,\ U_0,\quad \|(u_0,u_1,u_2)\|_{\mathrm{L}^\infty_t W^{1,\infty}_x},\quad \|(\rho_0,\rho_1,\rho_2)\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty_x},\\
\|(g_1,g_2,g_3)\|_{\mathrm{L}^\infty_t W^{1,\infty}_{x,n}},\quad \|(\partial_t - \Delta_x)g_3\|_{\mathrm{L}^\infty_t \mathrm{L}^2_{x,n}}.
\end{gathered}$$ ◊

Finally, it remains to identify the equation satisfied by the truncated ε-expansion $(\bar u_\varepsilon, \bar\rho_\varepsilon) := (u_0 + \varepsilon u_1, \rho_0 + \varepsilon\rho_1)$: we show that this expansion precisely coincides with the hierarchical solution of the second-order fluid model (3.26) with some explicit choice of parameters. The proof is displayed in Section 5.4.

Proposition 5.3 (From hierarchy to second-order fluids)**.** *Given the solutions (u_0,ρ_0,g_1) and (u_1,ρ_1,g_2) of the hierarchy (5.3)–(5.4), as constructed in Proposition 5.1 in the Stokes case, the superposition $(\bar u_\varepsilon, \bar\rho_\varepsilon) := (u_0 + \varepsilon u_1, \rho_0 + \varepsilon\rho_1)$ coincides with the unique hierarchical solution of the second-order fluid model (3.26) in the sense of Proposition 3.2, with coefficients $\eta_0, \eta_1, \mu_0, \gamma_1, \gamma_2$ explicitly given by (4.1).* ◊

The combination of Propositions 5.2 and 5.3 completes the proof of Theorem 4.2. Note however that we only appeal to Assumption 4.1 in the statement of Theorem 4.2, which is one order weaker than the well-preparedness assumption (5.8) required in Proposition 5.2. Indeed, as explained, we focus in Theorem 4.2 on $O(\varepsilon)$ effects, only deriving the second-order fluid model, while in Proposition 5.2 we took care to further describe $O(\varepsilon^2)$ effects. For the purposes of Theorem 4.2, the well-preparedness assumption (5.8) can therefore simply be replaced by Assumption 4.1.

5.1. **Computational tools for spherical calculus.** In this section, we briefly recall several computational tools that will be used throughout this work to compute derivatives and integrals on the sphere. First, we recall that the Laplace–Beltrami operator Δ_n on the sphere $\mathbb{S}^{d-1}$ ($d \geq 2$) can be computed as follows: given a smooth function $g : \mathbb{S}^{d-1} \to \mathbb{R}$, we can extend it to $\mathbb{R}^d \setminus \{0\}$ by setting $G(x) := g\big(\frac{x}{|x|}\big)$, and we then have

$$\Delta_n g \;=\; (\Delta_x G)|_{\mathbb{S}^{d-1}}. \tag{5.9}$$

In particular, we can compute in this way

$$\begin{aligned}
\Delta_n(n_i) &= (1-d)n_i, \qquad (5.10)\\
\Delta_n(n_i n_j) &= 2\delta_{ij} - 2d n_i n_j,\\
\Delta_n(n_i n_j n_k) &= 2(\delta_{ij} n_k + \delta_{ik} n_j + \delta_{jk} n_i) - 3(d+1) n_i n_j n_k,\\
\Delta_n(n_i n_j n_k n_l) &= 2(\delta_{ij} n_k n_l + \delta_{kj} n_i n_l + \delta_{ki} n_j n_l + \delta_{il} n_j n_k + \delta_{lk} n_i n_j + \delta_{jl} n_i n_k)\\
&\quad -4(d+2) n_i n_j n_k n_l,
\end{aligned}$$

and so on for higher-order polynomials. These formulas can be used to explicitly invert Δ_n on mean-zero polynomial expressions: for any trace-free symmetric matrix $A \in \mathbb{R}^{d\times d}$, we find for instance,

$$\begin{aligned}
\Delta_n^{-1}(n) &= -\tfrac{1}{d-1} n, \qquad (5.11)\\
\Delta_n^{-1}\big(n\otimes n : A\big) &= -\tfrac{1}{2d}\, n\otimes n : A, \qquad (5.12)\\
\Delta_n^{-1}\Big(n(n\otimes n : A)\Big) &= -\tfrac{1}{3(d+1)}\Big(n(n\otimes n : A) + \tfrac{4}{d-1} A n\Big), \qquad (5.13)\\
\Delta_n^{-1}\left((n\otimes n : A)^2 - \tfrac{2}{d(d+2)}\operatorname{tr}(A^2)\right) &= -\tfrac{1}{4(d+2)}\Big((n\otimes n : A)^2 + \tfrac{4}{d} n\otimes n : A^2\\
&\qquad -\tfrac{2}{d}\big(\tfrac{1}{d+2} + \tfrac{2}{d}\big)\operatorname{tr}(A^2)\Big). \qquad (5.14)
\end{aligned}$$

Henceforth, the pseudo-inverse Δ_n^{-1} is chosen to be defined as an operator from mean-zero fields to mean-zero fields.

We also recall that the divergence of functions on $\mathbb{S}^{d-1}$ can be computed similarly as the Laplace–Beltrami operator (5.9) by an extension procedure: for any trace-free matrix A and any smooth function $g : \mathbb{S}^{d-1} \to \mathbb{R}$, we find for instance

$$\operatorname{div}_n(\pi_n^\perp A n g) \;=\; \nabla_n g \cdot A n - d\,(n\otimes n) : A g. \tag{5.15}$$

Finally, we further note that the above differential formulas (5.10) imply by direct integration the following elementary integral identities for polynomial expressions on the sphere,

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} n_i n_j \,\mathrm{d}n &= \tfrac{\omega_d}{d}\delta_{ij}, \qquad (5.16)\\
\int_{\mathbb{S}^{d-1}} n_i n_j n_k n_l \,\mathrm{d}n &= \tfrac{\omega_d}{d(d+2)}\big(\delta_{ij}\delta_{kl} + \delta_{kj}\delta_{il} + \delta_{ki}\delta_{jl}\big),\\
\int_{\mathbb{S}^{d-1}} n_i n_j n_k n_l n_m n_p \,\mathrm{d}n &= \tfrac{\omega_d}{d(d+2)(d+4)}\big(\delta_{ij}\delta_{kl}\delta_{mp} + \ldots\big),
\end{aligned}$$

and so on, where we recall the notation $\omega_d = |\mathbb{S}^{d-1}|$. These identities imply in particular, for any trace-free symmetric matrices $A, B \in \mathbb{R}^{d\times d}$,

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \big(n\otimes n - \tfrac1d \mathrm{Id}\big)\,\mathrm{d}n &= 0,\\ \int_{\mathbb{S}^{d-1}} \big(n\otimes n - \tfrac1d \mathrm{Id}\big)\,(n\otimes n : A)\,\mathrm{d}n &= \tfrac{2\omega_d}{d(d+2)}A,\\ \int_{\mathbb{S}^{d-1}} \big(n\otimes n - \tfrac1d \mathrm{Id}\big)\,(n\otimes n : A)^2\,\mathrm{d}n &= \tfrac{8\omega_d}{d(d+2)(d+4)}\big(A^2 - \tfrac1d \mathrm{tr}(A^2)\,\mathrm{Id}\big).\end{aligned}$$

5.2. **Proof of Proposition 5.1.** We focus on items (i) and (ii), while the argument for item (iii) easily follows by adapting that for item (ii). We split the proof into three main steps.

Step 1. Explicit formulas for g_1, g_2.
With the above identities (5.11)–(5.14), we can explicitly solve the successive equations for g_1 and g_2 in (5.3)–(5.4). Note that these equations can be solved for fixed (x,t), thus treating $\rho_0, \rho_1, \mathrm{D}(u_0), \mathrm{D}(u_1)$ and their derivatives as parameters. We start with the computation of g_1. Using (5.15), the defining equation for g_1 in (5.3) can be rewritten as

$$\Delta_n g_1 = U_0 n\cdot\nabla\rho_0 - d\,(n\otimes n) : \rho_0\,\mathrm{D}(u_0).$$

Hence, using (5.11) and (5.12), the explicit form (5.6) for g_1 follows.

We turn to the computation of g_2. Using (5.15), inserting the explicit form (5.6) for g_1, and noting in particular that $\nabla_n g_1 = -\frac{U_0}{d-1}\pi_n^\perp\nabla\rho_0 + \rho_0\,\pi_n^\perp\,\mathrm{D}(u_0)n$, the defining equation for g_2 in (5.4) can be rewritten as

$$\begin{aligned}\Delta_n g_2 &= (\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta_x + u_0\cdot\nabla_x)g_1 + P_1^\perp\big(U_0 n\cdot\nabla_x(\rho_1+g_1)\big)\\ &\quad -d\,(n\otimes n) : \rho_0\,\mathrm{D}(u_1) - d\,(n\otimes n) : (\rho_1+g_1)\,\mathrm{D}(u_0) + \nabla_n g_1\cdot(\nabla u_0)n\\ &= (\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta_x + u_0\cdot\nabla_x)\big(-\tfrac{1}{d-1}U_0 n\cdot\nabla\rho_0 + \tfrac12 (n\otimes n) : \rho_0\,\mathrm{D}(u_0)\big)\\ &\quad + U_0 n\cdot\big(\nabla\rho_1 - \tfrac{1}{d-1}(\nabla u_0)^T\nabla\rho_0\big) + \rho_0(n\otimes n) : (\mathrm{D}(u_0)(\nabla u_0))\\ &\quad -(n\otimes n - \tfrac1d\,\mathrm{Id}) : \big(d\rho_0\,\mathrm{D}(u_1) + d\rho_1\,\mathrm{D}(u_0) + \tfrac{1}{d-1}U_0^2\nabla^2\rho_0\big)\\ &\quad +\tfrac{d+1}{d-1}U_0(n\cdot\nabla\rho_0)\big((n\otimes n):\mathrm{D}(u_0)\big) + \tfrac12 U_0\rho_0 n\cdot\nabla_x\big((n\otimes n):\mathrm{D}(u_0)\big)\\ &\quad -\tfrac{d+2}{2}\rho_0\,\big((n\otimes n):\mathrm{D}(u_0)\big)^2.\end{aligned}$$

Recalling the following definition of Rivlin–Ericksen tensor,

$$A_2'(u_0,\rho_0) = \big(\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta_x + u_0\cdot\nabla_x\big)(2\rho_0\mathrm{D}(u_0)) + (\nabla u_0)^T(2\rho_0\mathrm{D}(u_0)) + (2\rho_0\mathrm{D}(u_0))(\nabla u_0),$$

and noting that $\mathrm{tr}(A_2'(u_0,\rho_0)) = 4\rho_0\,\mathrm{tr}(\mathrm{D}(u_0)^2)$, we can reformulate the above as

$$\begin{aligned}\Delta_n g_2 &= -\tfrac{1}{d-1}U_0 n\cdot\Big((\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u_0\cdot\nabla)\nabla\rho_0 + (\nabla u_0)^T\nabla\rho_0\Big) + U_0 n\cdot\nabla\rho_1\\ &\quad +(n\otimes n - \tfrac1d\,\mathrm{Id}) : \Big(\tfrac14 A_2'(u_0,\rho_0) - d\rho_0\,\mathrm{D}(u_1) - d\rho_1\,\mathrm{D}(u_0) - \tfrac{1}{d-1}U_0^2\nabla^2\rho_0\Big)\\ &\quad +\tfrac{d+1}{d-1}U_0(n\cdot\nabla\rho_0)\big((n\otimes n):\mathrm{D}(u_0)\big) + \tfrac12 U_0\rho_0 n\cdot\nabla_x\big((n\otimes n):\mathrm{D}(u_0)\big)\\ &\quad -\tfrac{d+2}{2}\rho_0\Big(\big((n\otimes n):\mathrm{D}(u_0)\big)^2 - \tfrac{2}{d(d+2)}\,\mathrm{tr}(\mathrm{D}(u_0)^2)\Big).\end{aligned}$$

The equation in (5.3) for ρ_0 entails that the first right-hand side term vanishes as

$$(\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u_0\cdot\nabla)\nabla\rho_0 + (\nabla u_0)^T\nabla\rho_0 = \nabla(\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + u_0\cdot\nabla)\rho_0 = 0. \tag{5.17}$$

Now appealing to (5.12), the second term reads

$$(n\otimes n-\tfrac{1}{d}\,\mathrm{Id}):\Big(\tfrac14 A_2'(u_0,\rho_0)-d\rho_0\,\mathrm{D}(u_1)-d\rho_1\,\mathrm{D}(u_0)-\tfrac{1}{d-1}U_0^2\nabla^2\rho_0\Big)$$
$$=-\tfrac{1}{2d}\Delta_n\bigg((n\otimes n):\Big(\tfrac14 A_2'(u_0,\rho_0)-d\rho_0\,\mathrm{D}(u_1)-d\rho_1\,\mathrm{D}(u_0)-\tfrac{1}{d-1}U_0^2\nabla^2\rho_0\Big)\,\mathrm{Id}\bigg).$$

Using (5.13), we further get for the third term,

$$\tfrac{d+1}{d-1}U_0(n\cdot\nabla\rho_0)\big((n\otimes n):\mathrm{D}(u_0)\big)$$
$$=-\tfrac{U_0}{3(d-1)}\Delta_n\bigg((\nabla\rho_0)\cdot\Big(n\big((n\otimes n):\mathrm{D}(u_0)\big)+\tfrac{4}{d-1}\,\mathrm{D}(u_0)n\Big)\bigg),$$

and for the fourth term,

$$\tfrac12 U_0\rho_0 n\cdot\nabla_x\big((n\otimes n):\mathrm{D}(u_0)\big)=-\tfrac{U_0}{6(d+1)}\Delta_n\bigg(\rho_0\,\mathrm{div}_x\Big(n\big((n\otimes n):\mathrm{D}(u_0)\big)\Big)\bigg).$$

Finally, using (5.14) for the last term, the explicit form (5.7) for g_2 follows.

Step 2. Proof of item (i).
In view of the explicit solution for the equation for g_1 obtained in Step 1, cf. (5.6), the elastic stress $\sigma_1[g_1]$ defined in (2.5) takes the form

$$\sigma_1[g_1]\;=\;\tfrac12\lambda\theta\int_{\mathbb{S}^{d-1}}(n\otimes n)\,\mathrm{D}(u_0)(n\otimes n)\rho_0\,\mathrm{d}n\;=\;\tfrac12\theta\,\sigma_2[\rho_0,\nabla u_0],$$

where we have used that integrals of monomials of odd degree on $\mathbb{S}^{d-1}$ vanish. By (5.16), this actually means

$$\sigma_1[g_1]\;=\;\tfrac{\omega_d}{d(d+2)}\lambda\theta\,\mathrm{D}(u_0)\rho_0\;=\;\tfrac12\theta\,\sigma_2[\rho_0,\nabla u_0].$$

The system (5.3) then becomes in the Stokes case $\mathrm{Re}=0$,

$$\begin{cases}-\mathrm{div}\big((1+c_0\rho_0)2\mathrm{D}(u_0)\big)+\nabla p_0=h,\\ (\partial_t-\tfrac{1}{\mathrm{Pe}}\Delta+u_0\cdot\nabla)\rho_0=0,\\ \mathrm{div}(u_0)=0,\quad \int_{\mathbb{T}^d}u_0=0,\\ \rho_0|_{t=0}=\rho^\circ,\end{cases}\tag{5.18}$$

where we have set for shortness $c_0:=\lambda\frac{\omega_d(\theta+2)}{2d(d+2)}$. Surprisingly, we could not find a reference for this natural system. We establish the well-posedness and the propagation of regularity for this system, which we believe to be of independent interest, and we split the proof into five further substeps. Note that the propagation of regularity requires some care: we need to treat low and high regularity separately in several steps.

Substep 2.1. Well-posedness of energy solutions for (5.18).
On the one hand, using the incompressibility constraint, the energy identity for the transport-diffusion equation for ρ_0 reads

$$\|\rho_0\|^2_{\mathrm{L}^\infty_t\mathrm{L}^2_x}+\tfrac{2}{\mathrm{Pe}}\|\nabla\rho_0\|^2_{\mathrm{L}^2_t\mathrm{L}^2_x}=\|\rho^\circ\|^2_{\mathrm{L}^2(\mathbb{T}^d)}.\tag{5.19}$$

On the other hand, for $\rho_0\ge0$, the energy identity for u_0 takes the form

$$\|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\;\le\;2\int_{\mathbb{T}^d}(1+c_0\rho_0)|\mathrm{D}(u_0)|^2\;=\;\int_{\mathbb{T}^d}hu_0,$$

and thus, using $\int_{\mathbb{T}^d} u_0 = 0$,

$$\|\nabla u_0\|_{\mathrm{L}^\infty_t \mathrm{L}^2_x} \le \|h\|_{H^{-1}(\mathbb{T}^d)}. \tag{5.20}$$

By a standard fixed-point approach, in view of these a priori estimates, given an initial condition $\rho^\circ \in \mathrm{L}^2(\mathbb{T}^d)$ with $\rho^\circ \ge 0$ and given $h \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{-1}(\mathbb{T}^d)^d)$, we easily check that the system (5.18) is globally well-posed with $u_0 \in \mathrm{L}^\infty(\mathbb{R}^+; H^1(\mathbb{T}^d)^d)$ and $\rho_0 \in \mathrm{L}^\infty(\mathbb{R}^+; \mathrm{L}^2(\mathbb{T}^d)) \cap \mathrm{L}^2(\mathbb{R}^+; H^1(\mathbb{T}^d)^d)$ with $\rho_0 \ge 0$; we skip the detail.

Substep 2.2. H^1 regularity for $(\nabla u_0, \rho_0)$: provided that the initial condition further satisfies $\rho^\circ \in H^1(\mathbb{T}^d)$, and provided that $h \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{T}^d)^d)$, we show that $\rho_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{T}^d)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^2(\mathbb{T}^d))$ and that $u_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; H^2(\mathbb{T}^2)^2)$, with

$$\|\nabla \rho_0\|_{\mathrm{L}^\infty_t \mathrm{L}^2_x} + \|\nabla^2 \rho_0\|_{\mathrm{L}^2_t \mathrm{L}^2_x} \le C(t, h, \rho^\circ), \tag{5.21}$$

$$\begin{cases} \|\Delta u_0\|_{\mathrm{L}^2_t \mathrm{L}^2_x} \lesssim C(t, h, \rho^\circ) & : \text{ if } d = 2, \\ \|\Delta u_0\|_{\mathrm{L}^1_t \mathrm{L}^2_x} \lesssim C(t, h, \rho^\circ) & : \text{ if } d = 3, \end{cases} \tag{5.22}$$

where henceforth $C(t, h, \rho^\circ)$ stands for a constant further depending on an upper bound on t, c_0, and on the controlled norms of the data h and ρ°.

We start with the proof of (5.21). Testing the equation for ρ_0 with $\Delta \rho_0$, and using the incompressibility of u_0, we find

$$\begin{aligned} \tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \|\nabla \rho_0\|^2_{\mathrm{L}^2_x} + \tfrac{1}{\mathrm{Pe}} \|\Delta \rho_0\|^2_{\mathrm{L}^2_x} &= \int_{\mathbb{T}^d} (u_0 \cdot \nabla \rho_0) \Delta \rho_0 \\ &= -\int_{\mathbb{T}^d} \nabla u_0 : (\nabla \rho_0 \otimes \nabla \rho_0) \\ &\le \|\nabla u_0\|_{\mathrm{L}^2_x} \|\nabla \rho_0\|^2_{\mathrm{L}^4_x}. \end{aligned}$$

In dimension $d < 4$, we can appeal to the Gagliardo–Nirenberg interpolation inequality

$$\|\nabla \rho_0\|_{\mathrm{L}^4_x} \lesssim \|\nabla \rho_0\|^{1-\frac{d}{4}}_{\mathrm{L}^2_x} \|\nabla^2 \rho_0\|^{\frac{d}{4}}_{\mathrm{L}^2_x}.$$

The above then becomes, further using the elliptic estimate $\|\nabla^2 \rho_0\|^2_{\mathrm{L}^2_x} \lesssim \|\Delta \rho_0\|^2_{\mathrm{L}^2_x}$,

$$\tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \|\nabla \rho_0\|^2_{\mathrm{L}^2_x} + \tfrac{1}{\mathrm{Pe}} \|\Delta \rho_0\|^2_{\mathrm{L}^2_x} \lesssim \|\nabla u_0\|_{\mathrm{L}^2_x} \|\nabla \rho_0\|^{2(1-\frac{d}{4})}_{\mathrm{L}^2_x} \|\Delta \rho_0\|^{\frac{d}{2}}_{\mathrm{L}^2_x},$$

where the last factor can now be absorbed in the left-hand side by Young's inequality, to the effect of

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \|\nabla \rho_0\|^2_{\mathrm{L}^2_x} + \tfrac{1}{\mathrm{Pe}} \|\Delta \rho_0\|^2_{\mathrm{L}^2_x} \lesssim \|\nabla u_0\|^{\frac{4}{4-d}}_{\mathrm{L}^2_x} \|\nabla \rho_0\|^2_{\mathrm{L}^2_x}.$$

By Grönwall's inequality and the a priori energy estimate (5.20), this precisely proves the claim (5.21).

Next, we turn to the proof of the corresponding estimate (5.22) for u_0. Testing the equation for u_0 with Δu_0, we find

$$\begin{aligned} \|\Delta u_0\|^2_{\mathrm{L}^2_x} &= -\int_{\mathbb{T}^d} h \cdot \Delta u_0 - 2c_0 \int_{\mathbb{T}^d} \operatorname{div}(\rho_0 \, \mathrm{D}(u_0)) \cdot \Delta u_0 \\ &= -\int_{\mathbb{T}^d} h \cdot \Delta u_0 - c_0 \int_{\mathbb{T}^d} \rho_0 |\Delta u_0|^2 - 2c_0 \int_{\mathbb{T}^d} (\mathrm{D}(u_0) \nabla \rho_0) \cdot \Delta u_0, \end{aligned}$$

and thus, by Young's inequality,

$$\|\Delta u_0\|_{\mathrm{L}^2_x} \lesssim \|h\|_{\mathrm{L}^2_x} + c_0 \|\mathrm{D}(u_0) \nabla \rho_0\|_{\mathrm{L}^2_x}.$$

In dimension $d<4$, appealing to the Gagliardo–Nirenberg interpolation inequality, we can estimate for all $2<p,q<\infty$ with $\frac1p+\frac1q=\frac12$ and $d\le p,q\le\frac{2d}{d-2}$,

$$\begin{aligned}\|\mathrm{D}(u_0)\nabla\rho_0\|_{\mathrm{L}^2_x} &\le \|\nabla u_0\|_{\mathrm{L}^p_x}\|\nabla\rho_0\|_{\mathrm{L}^q_x}\\ &\lesssim_{p,q} \|\nabla u_0\|_{\mathrm{L}^2_x}^{1-\frac dq}\|\nabla^2 u_0\|_{\mathrm{L}^2_x}^{\frac dq}\|\nabla\rho_0\|_{\mathrm{L}^2_x}^{1-\frac dp}\|\nabla^2\rho_0\|_{\mathrm{L}^2_x}^{\frac dp}.\end{aligned}$$

Inserting the a priori estimates (5.20) and (5.21), combining with the above, and using Young's inequality to absorb the H^2 norm of u_0, we are led to

$$\|\Delta u_0\|_{\mathrm{L}^2_x}\lesssim_{p,q}\|h\|_{\mathrm{L}^2_x}+C(t,h)\|\nabla\rho^\circ\|_{\mathrm{L}^2_x}^{\frac{q}{q-d}-\frac d2\frac{q-2}{q-d}}\|\nabla^2\rho_0\|_{\mathrm{L}^2_x}^{\frac d2\frac{q-2}{q-d}}.$$

Choosing any $q<\infty$ if $d=2$, and choosing $q=\frac{2d}{d-2}$ if $d=3$, we get

$$\|\Delta u_0\|_{\mathrm{L}^2_x}\lesssim_\gamma \|h\|_{\mathrm{L}^2_x}+C(t,h)\|\nabla\rho^\circ\|_{\mathrm{L}^2_x}^{\gamma}\|\nabla^2\rho_0\|_{\mathrm{L}^2_x}^{\frac{2}{4-d}},\tag{5.23}$$

where we can take any exponent $\gamma>0$ if $d=2$, and $\gamma=0$ if $d=3$. Combined with (5.21), this yields the claim (5.22).

Substep 2.3. H^2 regularity for $(\nabla u_0,\rho_0)$: provided that the initial condition further satisfies $\rho^\circ\in H^2(\mathbb{T}^d)$, and provided that $h\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{T}^d)^d)$, we show that $\rho_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^2(\mathbb{T}^d))\cap\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^3(\mathbb{T}^d))$ and that $u_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^2(\mathbb{T}^d)^d)\cap\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;H^3(\mathbb{T}^d)^d)$, with

$$\|\nabla^2\rho_0\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x}+\|\nabla^3\rho_0\|_{\mathrm{L}^2_t\mathrm{L}^2_x}+\|\nabla^2u_0\|_{\mathrm{L}^\infty_t\mathrm{L}^2(\mathbb{T}^d)}\le C(t,h,\rho^\circ),\tag{5.24}$$

$$\left\{\begin{array}{lll}\|\nabla^3u_0\|^2_{\mathrm{L}^\infty_t\mathrm{L}^2_x}\le C(t,h,\rho^\circ) &:& \text{if } d=2,\\ \|\nabla^3u_0\|^2_{\mathrm{L}^4_t\mathrm{L}^2_x}\le C(t,h,\rho^\circ) &:& \text{if } d=3.\end{array}\right.\tag{5.25}$$

We start with the proof of (5.24). Applying ∇^2 to both sides of the equation for ρ_0, testing with $\nabla^2\rho_0$, and integrating by parts, we find

$$\begin{aligned}\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\rho_0\|^2_{\mathrm{L}^2_x}+\|\nabla^3\rho_0\|^2_{\mathrm{L}^2_x} &= -\int_{\mathbb{T}^d}(\nabla^2_{ij}\rho_0)\nabla^2_{ij}\big((u_0)_k\nabla_k\rho_0\big)\\ &=\int_{\mathbb{T}^d}(\nabla^3\rho_0)_{ijk}(\nabla u_0)_{ki}(\nabla\rho_0)_j+\int_{\mathbb{T}^d}(\nabla^3\rho_0)_{ijj}(\nabla u_0)_{ki}(\nabla\rho_0)_k,\end{aligned}$$

and thus, using Young's inequality to absorb the factors $\nabla^3\rho_0$ in the right-hand side,

$$\partial_t\|\nabla^2\rho_0\|^2_{\mathrm{L}^2_x}+\|\nabla^3\rho_0\|^2_{\mathrm{L}^2_x}\lesssim\|\nabla u_0\nabla\rho_0\|^2_{\mathrm{L}^2_x}.$$

By the Gagliardo–Nirenberg interpolation inequality, we can estimate for all $2<p,q<\infty$ with $\frac1p+\frac1q=\frac12$ and $d\le p,q\le\frac{2d}{d-2}$,

$$\begin{aligned}\|\nabla u_0\nabla\rho_0\|_{\mathrm{L}^2_x} &\le \|\nabla u_0\|_{\mathrm{L}^p_x}\|\nabla\rho_0\|_{\mathrm{L}^q_x}\\ &\le \|\nabla u_0\|_{\mathrm{L}^2_x}^{1-\frac dq}\|\nabla^2u_0\|_{\mathrm{L}^2_x}^{\frac dq}\|\nabla\rho_0\|_{\mathrm{L}^2_x}^{1-\frac dp}\|\nabla^2\rho_0\|_{\mathrm{L}^2_x}^{\frac dp}.\end{aligned}$$

Inserting the a priori estimates (5.20), (5.21), and (5.23), and combining with the above, we are led to

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\rho_0\|^2_{\mathrm{L}^2_x}+\|\nabla^3\rho_0\|^2_{\mathrm{L}^2_x}\le C(t,h)\|\nabla\rho^\circ\|_{\mathrm{L}^2_x}^{2(1-\frac dp)}\Big(\|\nabla^2\rho_0\|_{\mathrm{L}^2_x}^{\frac{2d}{p}}+\|\nabla^2\rho_0\|_{\mathrm{L}^2_x}^{\frac{2d}{p}+\frac{4d}{q(4-d)}}\Big).$$

Choosing any $q<\infty$ if $d=2$, and choosing $q=\frac{2d}{d-2}$ if $d=3$, this proves

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\rho_0\|^2_{\mathrm{L}^2_x}+\|\nabla^3\rho_0\|^2_{\mathrm{L}^2_x}\le C(t,h,\rho^\circ)\times\left\{\begin{array}{lll}\|\nabla^2\rho_0\|^2_{\mathrm{L}^2_x}+1&:&\text{if } d=2,\\ \|\nabla^2\rho_0\|^2_{\mathrm{L}^2_x}+\|\nabla^2\rho_0\|^4_{\mathrm{L}^2_x}&:&\text{if } d=3.\end{array}\right.$$

By Grönwall's inequality, this yields

$$\begin{aligned}&\|\nabla^2\rho_0\|^2_{\mathrm{L}^\infty_t\mathrm{L}^2_x}+\|\nabla^3\rho_0\|^2_{\mathrm{L}^2_t\mathrm{L}^2_x}\\&\qquad\le C(t,h)\|\nabla^2\rho^\circ\|^2_{\mathrm{L}^2_x}\times\left\{\begin{array}{lll}1&:&\text{if } d=2,\\ \exp\big(C(t,h)\|\nabla^2\rho_0\|^2_{\mathrm{L}^2_t\mathrm{L}^2_x}\big)&:&\text{if } d=3.\end{array}\right.\end{aligned}$$

By the a priori estimate (5.21), this proves the claimed estimate on ρ_0. Combined with (5.23), this concludes the proof of (5.24).

We turn to the proof of (5.25). Applying ∇^2 to both sides of the equation for u_0, testing with $\nabla^2 u_0$, integrating by parts, and using Young's inequality, we find

$$\|\nabla^3 u_0\|_{\mathrm{L}^2_x}\lesssim\|\nabla h\|_{\mathrm{L}^2_x}+\|\nabla\rho_0\otimes\nabla^2u_0\|_{\mathrm{L}^2_x}+\|\nabla^2\rho_0\otimes\nabla u_0\|_{\mathrm{L}^2_x},$$

and thus, using as above the Gagliardo–Nirenberg interpolation inequality and the a priori estimates (5.20), (5.21), and (5.24), we obtain

$$\|\nabla^3u_0\|^2_{\mathrm{L}^2_x}\lesssim C(t,h,\rho^\circ)\Big(1+\|\nabla^3\rho_0\|^{d-2}_{\mathrm{L}^2_x}\Big).\tag{5.26}$$

Combined with (5.24), this proves (5.25).

Substep 2.4. H^s regularity for $(\nabla u_0,\rho_0)$: for all integers $s>2$, provided that the initial condition further satisfies $\rho^\circ\in H^s(\mathbb{T}^d)$, and provided that $h\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{T}^d)^d)$, we show that $\rho_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{T}^d))\cap\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^{s+1}(\mathbb{T}^d))$ and that $u_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s+1}(\mathbb{T}^d)^d)$. More precisely, we shall prove for all integers $s>2$,

$$\|\rho_0\|_{\mathrm{L}^\infty_tH^s_x}+\|\rho_0\|_{\mathrm{L}^2_tH^{s+1}_x}\lesssim_s\|\rho^\circ\|_{H^s_x}\exp\big(C_s\|u_0\|_{\mathrm{L}^1_tH^s_x}\big),\tag{5.27}$$

$$\|u_0\|_{\mathrm{L}^\infty_tH^{s+1}_x}\lesssim_s\|h\|_{\mathrm{L}^\infty_tH^{s-1}_x}+\|u_0\|_{\mathrm{L}^\infty_tH^s_x}\|\rho_0\|_{\mathrm{L}^\infty_tH^s_x},\tag{5.28}$$

and we note that these estimates indeed yield the conclusion by a direct iteration, starting from the results of Substep 2.3 for $s=3$.

Let $s>2$ be a fixed integer, which entails in particular $s>\frac d2+1$ as we consider dimension $d<4$. We start with the proof of (5.27). Applying $\langle\nabla\rangle^s:=(1-\Delta_x)^{s/2}$ to both sides of the equation for ρ_0, and testing it with $\langle\nabla\rangle^s\rho_0$, we find

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\langle\nabla\rangle^s\rho_0\|^2_{\mathrm{L}^2_x}+\tfrac{1}{\mathrm{Pe}}\|\nabla\langle\nabla\rangle^s\rho_0\|^2_{\mathrm{L}^2_x}=-\int_{\mathbb{T}^d}(\langle\nabla\rangle^s\rho_0)\langle\nabla\rangle^s(u_0\cdot\nabla\rho_0),$$

and thus, using the incompressibility constraint,

$$\begin{aligned}\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\langle\nabla\rangle^s\rho_0\|^2_{\mathrm{L}^2_x}+\tfrac{1}{\mathrm{Pe}}\|\nabla\langle\nabla\rangle^s\rho_0\|^2_{\mathrm{L}^2_x}&=-\int_{\mathbb{T}^d}(\langle\nabla\rangle^s\rho_0)[\langle\nabla\rangle^s,u_0\cdot\nabla]\rho_0\\&\le\|\langle\nabla\rangle^s\rho_0\|_{\mathrm{L}^2_x}\|[\langle\nabla\rangle^s,u_0\cdot\nabla]\rho_0\|_{\mathrm{L}^2_x}.\end{aligned}\tag{5.29}$$

To estimate the last factor, we appeal to the following form of the Kato–Ponce commutator estimate [KP88, Lemma X1] (see also [MB01, Lemma 3.4]): for all $u\in C^\infty(\mathbb{T}^d)^d$ and $\rho\in C^\infty(\mathbb{T}^d)$, we have

$$\|[\langle\nabla\rangle^s,u\cdot\nabla]\rho\|_{\mathrm{L}^2_x}\lesssim_s\|u\|_{H^s_x}\|\rho\|_{W^{1,\infty}_x}+\|u\|_{W^{1,\infty}_x}\|\rho\|_{H^s_x},$$

and thus, by the Sobolev embedding with $s > \frac{d}{2}+1$,

$$\|[\langle\nabla\rangle^s, u\cdot\nabla]\rho\|_{\mathrm{L}^2_x} \lesssim_s \|u\|_{H^s_x}\|\rho\|_{H^s_x}. \tag{5.30}$$

Using this to estimate the last factor in (5.29), we are led to

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_0\|^2_{H^s_x} + \|\nabla\rho_0\|^2_{H^s_x} \lesssim_s \|\rho_0\|^2_{H^s_x}\|u_0\|_{H^s_x}, \tag{5.31}$$

and the claim (5.27) follows by Grönwall's inequality.

We turn to the proof of (5.28). Applying $\langle\nabla\rangle^s$ to both sides of the equation for u_0, and testing it with $\langle\nabla\rangle^s u_0$, we find

$$\begin{aligned}\|\nabla\langle\nabla\rangle^s u_0\|^2_{\mathrm{L}^2_x} &= \int_{\mathbb{T}^d}\langle\nabla\rangle^s h\cdot\langle\nabla\rangle^s u_0 - 2c_0\int_{\mathbb{T}^d}\langle\nabla\rangle^s\,\mathrm{D}(u_0) : \langle\nabla\rangle^s(\rho_0\,\mathrm{D}(u_0))\\ &\le \int_{\mathbb{T}^d}\langle\nabla\rangle^s h\cdot\langle\nabla\rangle^s u_0 - 2c_0\int_{\mathbb{T}^d}\langle\nabla\rangle^s\,\mathrm{D}(u_0) : [\langle\nabla\rangle^s,\rho_0]\,\mathrm{D}(u_0),\end{aligned}$$

and thus, by Young's inequality,

$$\|u_0\|_{H^{s+1}_x} \lesssim \|h\|_{H^{s-1}_x} + \|[\langle\nabla\rangle^s,\rho_0]\,\mathrm{D}(u_0)\|_{\mathrm{L}^2_x}.$$

To estimate the last factor, we now appeal to the following form of the Kato–Ponce commutator estimate, instead of (5.30), with $s > \frac{d}{2}+1$,

$$\|[\langle\nabla\rangle^s,\rho]\,\mathrm{D}(u)\|_{\mathrm{L}^2_x} \lesssim_s \|u\|_{H^s_x}\|\rho\|_{H^s_x}.$$

Using this to estimate the last factor in the above, the claim (5.28) follows.

Substep 2.5. Time regularity.
For all $s\ge 0$, if $\rho_0\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s+2}(\mathbb{T}^d))$ and $u_0\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s+3}(\mathbb{T}^d)^d)$, then the equation for ρ_0 yields, by the Sobolev embedding in dimension $d<4$,

$$\partial_t\rho_0 = \tfrac{1}{\mathrm{Pe}}\Delta\rho_0 - u_0\cdot\nabla\rho_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{T}^d)).$$

Next, taking the time-derivative of both sides of the equation for u_0, we find

$$-\mathrm{div}\big((1+c_0\rho_0)2\mathrm{D}(\partial_t u_0)\big) + \nabla\partial_t p_0 = c_0\mathrm{div}\big((\partial_t\rho_0)2\mathrm{D}(u_0)\big) + \partial_t h.$$

For all $s\ge 0$, if $\rho_0\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s+2}(\mathbb{T}^d))$ and $u_0\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s+3}(\mathbb{T}^d)^d)$, we deduce by elliptic regularity that $\partial_t u_0\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s+1}(\mathbb{T}^d)^d)$ provided $h\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+;H^{s-1}(\mathbb{T}^d)^d)$. Higher time regularity is obtained similarly by direct induction.

Step 3. Proof of item (ii).
In the Stokes case $\mathrm{Re}=0$, the system (5.4) reads

$$\begin{cases}-\Delta u_1 + \nabla p_1 = \mathrm{div}(\sigma_1[g_2]) + \mathrm{div}\big(\sigma_2[\rho_0,\nabla u_1] + \sigma_2[\rho_1+g_1,\nabla u_0]\big),\\ (\partial_t - \frac{1}{\mathrm{Pe}}\Delta_x + u_0\cdot\nabla_x)\rho_1 + u_1\cdot\nabla_x\rho_0 + \langle U_0 n\cdot\nabla_x g_1\rangle = 0,\\ \mathrm{div}(u_1)=0,\quad \int_{\mathbb{T}^d} u_1 = 0,\\ \rho_1|_{t=0}=0,\end{cases} \tag{5.32}$$

where g_1,g_2 are given by (5.6) and (5.7), respectively, in view of the explicit computations in Step 1. Note that the form of g_1,g_2 ensures that this is a linear system for (u_1,ρ_1). We therefore focus on the proof of a priori energy estimates, while well-posedness and regularity properties easily follow. On the one hand, testing the equation for ρ_1 with ρ_1 itself, and using the incompressibility constraint, we find

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_1\|^2_{\mathrm{L}^2_x} + \tfrac{1}{\mathrm{Pe}}\|\nabla\rho_1\|^2_{\mathrm{L}^2_x} = -\int_{\mathbb{T}^d}\rho_1 u_1\cdot\nabla\rho_0 - \int_{\mathbb{T}^d}\Big(\int_{\mathbb{S}^{d-1}}\rho_1 U_0 n\cdot\nabla_x g_1(\cdot,n)\,\mathrm{d}n\Big)$$

$$\leq \|\rho_1\|_{\mathrm{L}^2_x}\Big(\|u_1\|_{\mathrm{L}^2_x}\|\nabla\rho_0\|_{\mathrm{L}^\infty_x}+|U_0|\|\nabla_x g_1\|_{\mathrm{L}^2_x}\Big),$$

and thus, by the explicit formula (5.6) for g_1 and by the Sobolev embedding,

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_1\|^2_{\mathrm{L}^2_x}+\tfrac{1}{\mathrm{Pe}}\|\nabla\rho_1\|^2_{\mathrm{L}^2_x}\lesssim\|\rho_1\|_{\mathrm{L}^2_x}\big(1+\|u_1\|_{\mathrm{L}^2_x}\big)\big(1+\|\rho_0\|_{H^3_x}+\|u_0\|_{H^2_x}\big)^2. \tag{5.33}$$

On the other hand, testing the equation for u_1 with u_1 itself, we find

$$\int_{\mathbb{T}^d}|\nabla u_1|^2=-\int_{\mathbb{T}^d}\nabla u_1:\sigma_1[g_2]-\int_{\mathbb{T}^d}\nabla u_1:\sigma_2[\rho_0,\nabla u_1]-\int_{\mathbb{T}^d}\nabla u_1:\sigma_2[\rho_1+g_1,\nabla u_0],$$

and thus, recalling the definition of the elastic and viscous stresses, inserting the explicit formulas (5.6) and (5.7) for g_1,g_2, collecting all quadratic terms in ∇u_1 in the left-hand side, and noting that integrals of monomials of odd degree on $\mathbb{S}^{d-1}$ vanish and using again the incompressibility constraint,

$$\begin{aligned}\int_{\mathbb{T}^d}|\nabla u_1|^2&+\lambda(1+\tfrac12\theta)\int_{\mathbb{T}^d}\Big(\int_{\mathbb{S}^{d-1}}(n\otimes n:\nabla u_1)^2\,\mathrm{d}n\Big)\rho_0\\&=\tfrac{1}{2d}\lambda\theta\int_{\mathbb{T}^d}\Big(\int_{\mathbb{S}^{d-1}}(n\otimes n:\nabla u_1)(n\otimes n-\tfrac1d\,\mathrm{Id})\,\mathrm{d}n\Big)\\&\qquad:\Big(\tfrac14A_2'(u_0,\rho_0)-\rho_0\,\mathrm{D}(u_0)^2-d\rho_1\,\mathrm{D}(u_0)-\tfrac{1}{d-1}U_0^2\nabla^2\rho_0\Big)\\&\quad-\tfrac18\lambda\theta\int_{\mathbb{T}^d}\Big(\int_{\mathbb{S}^{d-1}}(n\otimes n:\nabla u_1)(n\otimes n:\nabla u_0)^2\,\mathrm{d}n\Big)\rho_0-\int_{\mathbb{T}^d}\nabla u_1:\sigma_2[\rho_1+g_1,\nabla u_0].\end{aligned}$$

Noting that the second left-hand side term is nonnegative as $\rho_0\geq0$, we deduce by the Sobolev embedding,

$$\|\nabla u_1\|_{\mathrm{L}^2_x}\lesssim\|\rho_1\|_{\mathrm{L}^2_x}\|u_0\|_{H^3_x}+\|\partial_t\rho_0\|_{\mathrm{L}^2_x}\|u_0\|_{H^3_x}+\|\rho_0\|_{H^2_x}\|\partial_t u_0\|_{H^1_x}+\|\rho_0\|_{H^2_x}\big(1+\|u_0\|_{H^3_x}\big)^2.$$

Combining this with (5.33) and with the Poincaré and Grönwall inequalities, we deduce that u_1 is controlled in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{T}^d)^d)$ and that ρ_1 is controlled in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{T}^d))\cap\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{T}^d))$ provided that $u_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^3(\mathbb{T}^d)^d)\cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{T}^d)^d)$ and $\rho_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^2(\mathbb{T}^d))\cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{T}^d))$. □

5.3. **Proof of Proposition 5.2.** We aim to estimate the remainder terms $u_{k,\varepsilon},f_{k,\varepsilon},\rho_{k,\varepsilon},g_{k,\varepsilon}$ in the ε-expansion of the solution $(u_\varepsilon,f_\varepsilon)$, as defined through

$$\begin{aligned}u_\varepsilon&=\sum_{j=0}^{k-1}\varepsilon^ju_j+\varepsilon^ku_{k,\varepsilon},\qquad(5.34)\\\rho_\varepsilon&=\sum_{j=0}^{k-1}\varepsilon^j\rho_j+\varepsilon^k\rho_{k,\varepsilon},\\g_\varepsilon&=f_\varepsilon-\rho_\varepsilon=\sum_{j=1}^{k-1}\varepsilon^jg_j+\varepsilon^kg_{k,\varepsilon},\\f_{k,\varepsilon}&=\rho_{k,\varepsilon}+g_{k,\varepsilon},\end{aligned}$$

for any integer $k\geq1$. We split the proof into five steps. We shall constantly use the short-hand notation $\mathcal{C}=\mathcal{C}(t)$ for multiplicative constants as defined in the statement, the value of which may change from line to line.

Step 1. Energy estimate on the remainder $u_{3,\varepsilon}$.
Comparing equations for u_ε, u_0, u_1, cf. (2.5) and Proposition 5.1, using the linearity of σ_1 and the bilinearity of σ_2, the remainder term $u_{3,\varepsilon} = \varepsilon^{-3}(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2)$ as defined in (5.34) satisfies the linearized Navier–Stokes equation

$$\begin{aligned}
&\mathrm{Re}(\partial_t + u_\varepsilon\cdot\nabla)u_{3,\varepsilon} - \Delta u_{3,\varepsilon} - \operatorname{div}(\sigma_2[f_\varepsilon,\nabla u_{3,\varepsilon}]) + \nabla p_{3,\varepsilon}\\
&\qquad = -\operatorname{Re}\big[(u_{3,\varepsilon}\cdot\nabla)(u_0+\varepsilon u_1+\varepsilon^2 u_2) + (u_2\cdot\nabla)u_1 + ((u_1+\varepsilon u_2)\cdot\nabla)u_2\big]\\
&\qquad\qquad + \operatorname{div}(\sigma_1[g_{4,\varepsilon}]) + \operatorname{div}(\sigma_2[f_{3,\varepsilon},\nabla(u_0+\varepsilon u_1+\varepsilon^2 u_2)])\\
&\qquad\qquad\qquad + \operatorname{div}(\sigma_2[\rho_2+g_2,\nabla u_1]) + \operatorname{div}(\sigma_2[\rho_1+g_1+\varepsilon(\rho_2+g_2),\nabla u_2]).
\end{aligned}$$

Here, we have also used the fact that $\sigma_1[\tau]=0$ for any function τ depending only on x, which follows from the simple observation that $\int_{\mathbb{S}^{d-1}}(n\otimes n - \frac1d\,\mathrm{Id})\,\mathrm{d}n = 0$. Testing this equation with $u_{3,\varepsilon}$, using the incompressibility constraints, and inserting the definition of elastic and viscous stresses σ_1,σ_2, cf. (2.5), we get the following energy identity,

$$\begin{aligned}
&\mathrm{Re}\,\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|u_{3,\varepsilon}\|^2_{\mathrm{L}^2_x} + \|\nabla u_{3,\varepsilon}\|^2_{\mathrm{L}^2_x} + \lambda\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}}\big(n\otimes n:\nabla u_{3,\varepsilon}\big)^2 f_\varepsilon\\
&\quad= \;-\mathrm{Re}\int_{\mathbb{T}^d} u_{3,\varepsilon}\otimes u_{3,\varepsilon} : \nabla(u_0+\varepsilon u_1+\varepsilon^2 u_2)\\
&\qquad\;-\mathrm{Re}\int_{\mathbb{T}^d} u_{3,\varepsilon}\cdot\big((u_2\cdot\nabla)u_1 + ((u_1+\varepsilon u_2)\cdot\nabla)u_2\big)\\
&\qquad\;-\lambda\theta\int_{\mathbb{T}^d}\nabla u_{3,\varepsilon} : \Big(\int_{\mathbb{S}^{d-1}}(n\otimes n - \tfrac1d\,\mathrm{Id})\,g_{4,\varepsilon}\,\mathrm{d}n\Big)\\
&\qquad\;-\lambda\int_{\mathbb{T}^d}\nabla u_{3,\varepsilon} : \Big(\int_{\mathbb{S}^{d-1}}(n\otimes n)(\nabla(u_0+\varepsilon u_1+\varepsilon^2 u_2))(n\otimes n) f_{3,\varepsilon}\,\mathrm{d}n\Big)\\
&\qquad\;-\lambda\int_{\mathbb{T}^d}\nabla u_{3,\varepsilon} : \Big(\int_{\mathbb{S}^{d-1}}(n\otimes n)(\nabla u_1)(n\otimes n)(\rho_2+g_2)\,\mathrm{d}n\Big)\\
&\qquad\;-\lambda\int_{\mathbb{T}^d}\nabla u_{3,\varepsilon} : \Big(\int_{\mathbb{S}^{d-1}}(n\otimes n)(\nabla u_2)(n\otimes n)(\rho_1+g_1+\varepsilon(\rho_2+g_2))\,\mathrm{d}n\Big),
\end{aligned}$$

where we note that the viscous stress $\sigma_2[f_\varepsilon,\nabla u_{3,\varepsilon}]$ has led to an additional dissipation term in the left-hand side. Appealing to the Cauchy–Schwarz inequality and to Young's inequality, we may then deduce

$$\begin{aligned}
&\mathrm{Re}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\|u_{3,\varepsilon}\|^2_{\mathrm{L}^2_x} + \|\nabla u_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}\\
&\lesssim\;\; \mathrm{Re}\,\big(1+\|(\nabla u_0,\nabla u_1,\nabla u_2)\|_{\mathrm{L}^\infty_x}\big)\Big(\|u_{3,\varepsilon}\|^2_{\mathrm{L}^2_x} + \|(u_1,u_2)\|^2_{\mathrm{L}^2_x}\Big) + \lambda^2\theta^2\|g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}\\
&\qquad + \lambda^2\|(\nabla u_0,\nabla u_1,\nabla u_2)\|^2_{\mathrm{L}^\infty_x}\|(f_{3,\varepsilon},g_1,g_2,\rho_1,\rho_2)\|^2_{\mathrm{L}^2_{x,n}}.
\end{aligned}$$

Recalling the short-hand notation $\mathcal{C}$ for multiplicative constants, and further decomposing $f_{3,\varepsilon} = \rho_{3,\varepsilon} + g_3 + \varepsilon g_{4,\varepsilon}$, we get

$$\begin{aligned}
&\mathrm{Re}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\|u_{3,\varepsilon}\|^2_{\mathrm{L}^2_x} + \|\nabla u_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}\\
&\qquad\qquad \le\; \mathcal{C} + \mathcal{C}\Big(\mathrm{Re}\,\|u_{3,\varepsilon}\|^2_{\mathrm{L}^2_x} + \|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}\Big) + (\lambda^2\theta^2 + \mathcal{C}\varepsilon^2)\|g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}.
\end{aligned}\tag{5.35}$$

Step 2. Energy estimate for $\rho_{3,\varepsilon}$.
The equation satisfied by $\rho_{3,\varepsilon}=\varepsilon^{-3}(\rho_\varepsilon-\rho_0-\varepsilon\rho_1-\varepsilon^2\rho_2)$ takes the form

$$\begin{aligned}\partial_t\rho_{3,\varepsilon}-\tfrac{1}{\mathrm{Pe}}\Delta_x\rho_{3,\varepsilon} \;=\; &-u_\varepsilon\cdot\nabla\rho_{3,\varepsilon}-u_{3,\varepsilon}\cdot\nabla(\rho_0+\varepsilon\rho_1+\varepsilon^2\rho_2)\\ &-u_2\cdot\nabla\rho_1-(u_1+\varepsilon u_2)\cdot\nabla\rho_2+\langle U_0 n\cdot\nabla_x g_{3,\varepsilon}\rangle.\end{aligned}$$

Testing it with $\rho_{3,\varepsilon}$ itself, we can use the incompressibility constraint and several integrations by parts to get

$$\begin{aligned}&\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+\tfrac{1}{\mathrm{Pe}}\|\nabla\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}\\ &\quad\lesssim\Big(\|(\rho_0,\varepsilon\rho_1,\varepsilon^2\rho_2)\|_{\mathrm{L}^\infty_x}\|u_{3,\varepsilon}\|_{\mathrm{L}^2_x}+\|(\rho_1,\rho_2)\|_{\mathrm{L}^\infty_x}\|(u_1,u_2)\|_{\mathrm{L}^2_x}+\|g_{3,\varepsilon}\|_{\mathrm{L}^2_{x,n}}\Big)\|\nabla\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x},\end{aligned}$$

hence, after absorption in the spatial dissipation,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+\tfrac{1}{\mathrm{Pe}}\|\nabla\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}\lesssim\big(\mathrm{Pe}\,\|\rho_0\|^2_{\mathrm{L}^\infty_x}+\varepsilon^2\mathcal{C}\big)\|u_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+\mathcal{C}\big(1+\|g_{3,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}\big).$$

Using the maximum principle

$$\|\rho_0\|_{\mathrm{L}^\infty_{t,x}}\le\|\rho^\circ\|_{\mathrm{L}^\infty_x},\tag{5.36}$$

which holds for a (regular) solution ρ_0 of (5.3) with initial data ρ°, we deduce

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+\tfrac{1}{\mathrm{Pe}}\|\nabla\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}\lesssim\big(\mathrm{Pe}\,\|\rho^\circ\|^2_{\mathrm{L}^\infty_x}+\varepsilon^2\mathcal{C}\big)\|u_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+\mathcal{C}\big(1+\|g_{3,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}\big).\tag{5.37}$$

Step 3. Energy estimate for $g_{4,\varepsilon}$.
The equation satisfied by $g_{4,\varepsilon}=\varepsilon^{-4}(g_\varepsilon-\varepsilon g_1-\varepsilon^2g_2-\varepsilon^3g_3)$ takes the form

$$\varepsilon\partial_t g_{4,\varepsilon}-\Delta_n g_{4,\varepsilon}-\tfrac{\varepsilon}{\mathrm{Pe}}\Delta_x g_{4,\varepsilon}\;=\;\sum_{i=1}^7\mathrm{T}_i,\tag{5.38}$$

in terms of

$$\begin{aligned}T_1&:=-(\partial_t-\tfrac{1}{\mathrm{Pe}}\Delta_x)g_3-\mathrm{div}_x\big(u_1g_2+u_2(g_1+\varepsilon g_2)\big),\\ T_2&:=-\mathrm{div}_x(u_\varepsilon g_{3,\varepsilon}),\\ T_3&:=\varepsilon\,\mathrm{div}_x(u_{3,\varepsilon}(g_1+\varepsilon g_2)),\\ T_4&:=\mathrm{div}_xP_1^\perp\big(U_0nf_{3,\varepsilon}\big),\\ T_5&:=\mathrm{div}_n\big(\pi_n^\perp(\nabla u_\varepsilon)nf_{3,\varepsilon}\big),\\ T_6&:=\mathrm{div}_n\big(\pi_n^\perp(\nabla u_{3,\varepsilon})n(\rho_0+\varepsilon f_1+\varepsilon^2f_2)\big),\\ T_7&:=\mathrm{div}_n\big(\pi_n^\perp(\nabla u_1)nf_2+\pi_n^\perp(\nabla u_2)n(f_1+\varepsilon f_2)\big).\end{aligned}$$

Testing this equation with $g_{4,\varepsilon}$ itself, we separately analyze the effect of the seven different source terms. For T_1,T_6, and T_7, we integrate by parts, we use Poincaré's inequality on $\mathbb{S}^{d-1}$, recalling $\langle g_{4,\varepsilon}\rangle=0$, and we use the maximum principle (5.36), to the effect of

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}}T_1g_{4,\varepsilon}&\lesssim\mathcal{C}\|\nabla_ng_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}},\\ \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}}T_6g_{4,\varepsilon}&\lesssim\big(\|\rho^\circ\|_{\mathrm{L}^\infty_x}+\varepsilon\mathcal{C}\big)\|\nabla u_{3,\varepsilon}\|_{\mathrm{L}^2_x}\|\nabla_ng_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}},\\ \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}}T_7g_{4,\varepsilon}&\lesssim\mathcal{C}\|\nabla_ng_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}.\end{aligned}$$

For the term T_2, we use the incompressibility constraint, we further decompose $g_{3,\varepsilon} = g_3 + \varepsilon g_{4,\varepsilon}$, and we integrate by parts, to the effect of

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} T_2 g_{4,\varepsilon} &= -\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon}\,\mathrm{div}(u_\varepsilon g_{3,\varepsilon})\\ &= \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon} u_\varepsilon\cdot\nabla_x g_3 + \varepsilon\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon} u_\varepsilon\cdot\nabla_x g_{4,\varepsilon}\\ &= \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon} u_\varepsilon\cdot\nabla_x g_3,\end{aligned}$$

which is then estimated by

$$\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} T_2 g_{4,\varepsilon} \le \mathcal{C}\|u_\varepsilon\|_{\mathrm{L}^2_x}\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}},$$

again using Poincaré's inequality. For the term T_3, we get

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} T_3 g_{4,\varepsilon} &= \varepsilon\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon} u_{3,\varepsilon}\cdot\nabla_x(g_1+\varepsilon g_2)\\ &\le \varepsilon\mathcal{C}\|u_{3,\varepsilon}\|_{\mathrm{L}^2_x}\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}.\end{aligned}$$

For the term T_4, decomposing $f_{3,\varepsilon} = \rho_{3,\varepsilon} + g_3 + \varepsilon g_{4,\varepsilon}$, using that U_0 is a constant and that $P_1^\perp g_{4,\varepsilon} = g_{4,\varepsilon}$, and integrating by parts, we find

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} T_4 g_{4,\varepsilon} &= \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon}\mathrm{div}_x P_1^\perp\big(U_0 n f_{3,\varepsilon}\big)\\ &= \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon}\mathrm{div}_x\big(U_0 n(\rho_{3,\varepsilon}+g_3)\big) + \varepsilon U_0\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon}\mathrm{div}_x(n g_{4,\varepsilon})\\ &= \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon}\mathrm{div}_x\big(U_0 n(\rho_{3,\varepsilon}+g_3)\big),\end{aligned}$$

which is then estimated by

$$\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} T_4 g_{4,\varepsilon}\mathrm{d}x\mathrm{d}n \lesssim \big(\|\nabla\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x} + \mathcal{C}\big)\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}.$$

Finally, decomposing $f_{3,\varepsilon} = \rho_{3,\varepsilon} + g_3 + \varepsilon g_{4,\varepsilon}$ and $u_\varepsilon = u_0 + \varepsilon u_{1,\varepsilon}$, and integrating by parts, we split the remaining term T_5 as follows,

$$\begin{aligned}&\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} T_5 g_{4,\varepsilon}\\ &= \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon}\mathrm{div}_n\big(\pi_n^\perp(\nabla u_\varepsilon)n f_{3,\varepsilon}\big)\\ &= \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon}\mathrm{div}_n\big(\pi_n^\perp(\nabla u_\varepsilon)n(\rho_{3,\varepsilon}+g_3+\varepsilon g_{4,\varepsilon})\big)\\ &= \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} \rho_{3,\varepsilon} g_{4,\varepsilon}\mathrm{div}_n\big(\pi_n^\perp(\nabla u_\varepsilon)n\big) + \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon}\mathrm{div}_n\big(\pi_n^\perp(\nabla u_\varepsilon)n g_3\big)\\ &\quad+\frac{\varepsilon}{2}\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} |g_{4,\varepsilon}|^2\mathrm{div}_n(\pi_n^\perp(\nabla u_\varepsilon)n)\\ &= \iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} \rho_{3,\varepsilon} g_{4,\varepsilon}\mathrm{div}_n\big(\pi_n^\perp(\nabla u_0)n\big) + \varepsilon\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} \rho_{3,\varepsilon} g_{4,\varepsilon}\mathrm{div}_n\big(\pi_n^\perp(\nabla u_{1,\varepsilon})n\big)\end{aligned}$$

$$+\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{4,\varepsilon}\mathrm{div}_n\big(\pi_n^\perp(\nabla u_\varepsilon)ng_3\big)+\tfrac{\varepsilon}{2}\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}}|g_{4,\varepsilon}|^2\mathrm{div}_n(\pi_n^\perp(\nabla u_0)n)$$
$$+\tfrac{\varepsilon^2}{2}\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}}|g_{4,\varepsilon}|^2\mathrm{div}_n(\pi_n^\perp(\nabla u_{1,\varepsilon})n),$$

which is then estimated by

$$\iint_{\mathbb{T}^d\times\mathbb{S}^{d-1}} T_5 g_{4,\varepsilon}\lesssim \varepsilon\mathcal{C}\|g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}+\mathcal{C}\big(\|\nabla u_\varepsilon\|_{\mathrm{L}^2_x}+\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}\big)\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}$$
$$+\varepsilon\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}\big(\|g_{4,\varepsilon}\|^2_{\mathrm{L}^4_x\mathrm{L}^2_n}+\|\rho_{3,\varepsilon}\|_{\mathrm{L}^4_x}\|g_{4,\varepsilon}\|_{\mathrm{L}^4_x\mathrm{L}^2_n}\big).$$

Testing the equation (5.38) with $g_{4,\varepsilon}$ itself and using the above estimates on the different source terms, together with Young's inequality, we end up with

$$\varepsilon\tfrac{\mathrm{d}}{\mathrm{d}t}\|g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}+\|\nabla_n g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}+\tfrac{\varepsilon}{\mathrm{Pe}}\|\nabla_x g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}$$
$$\lesssim \mathcal{C}\big(1+\|u_\varepsilon\|^2_{H^1_x}+\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}\big)+\|\nabla\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+\big(\|\rho^\circ\|^2_{\mathrm{L}^\infty_x}+\varepsilon^2\mathcal{C}\big)\|u_{3,\varepsilon}\|^2_{H^1_x}$$
$$+\varepsilon\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}\big(\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^4_x}+\|g_{4,\varepsilon}\|^2_{\mathrm{L}^4_x\mathrm{L}^2_n}\big)+\varepsilon\mathcal{C}\|g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}. \quad (5.39)$$

In order to estimate the L^4_x norms in the right-hand side, we appeal to interpolation and to the Sobolev inequality on the torus in the following form: in dimension $d\le 3$, we have for any $\delta>0$,

$$\begin{aligned}\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}\|g_{4,\varepsilon}\|^2_{\mathrm{L}^4_x\mathrm{L}^2_n} &\le C\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}\|g_{4,\varepsilon}\|^{2(1-\frac{d}{4})}_{\mathrm{L}^2_{x,n}}\|g_{4,\varepsilon}\|^{\frac{d}{2}}_{H^1_x\mathrm{L}^2_n}\\ &\le \delta\|\nabla_x g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}+C\delta^{-\frac{d}{4-d}}\|\nabla u_{1,\varepsilon}\|^{\frac{4}{4-d}}_{\mathrm{L}^2_x}\|g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}\\ &\quad +C\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}\|g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}.\end{aligned}$$

Likewise, further noting that $\rho_{3,\varepsilon}$ is mean-free (given that the defining equations for ρ_0,ρ_1,ρ_2 ensure $\int_{\mathbb{T}^d}\rho_1=\int_{\mathbb{T}^d}\rho_2=0$ and $\int_{\mathbb{T}^d}\rho_\varepsilon=\int_{\mathbb{T}^d}\rho_0=1$), we have for all $\eta>0$,

$$\begin{aligned}\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^4_x} &\le C\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}\|\rho_{3,\varepsilon}\|^{2(1-\frac{d}{4})}_{\mathrm{L}^2_x}\|\nabla\rho_{3,\varepsilon}\|^{\frac{d}{2}}_{\mathrm{L}^2_x}\\ &\le \eta\|\nabla\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+C\eta^{-\frac{d}{4-d}}\|\nabla u_{1,\varepsilon}\|^{\frac{4}{4-d}}_{\mathrm{L}^2_x}\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}.\end{aligned}$$

Inserting these bounds into (5.39), and choosing $\delta\simeq\mathrm{Pe}^{-1}$ and $\eta\simeq 1$, we get for $\varepsilon\lesssim 1$

$$\varepsilon\tfrac{\mathrm{d}}{\mathrm{d}t}\|g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}+\|\nabla_n g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}$$
$$\lesssim \mathcal{C}\big(1+\|u_\varepsilon\|^2_{H^1_x}+\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}\big)+\|\nabla\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+\big(\|\rho^\circ\|^2_{\mathrm{L}^\infty_x}+\varepsilon^2\mathcal{C}\big)\|u_{3,\varepsilon}\|^2_{H^1_x}$$
$$+\varepsilon\|\nabla u_{1,\varepsilon}\|^{\frac{4}{4-d}}_{\mathrm{L}^2_x}\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+\varepsilon\mathcal{C}\Big(1+\|\nabla u_{1,\varepsilon}\|^{\frac{4}{4-d}}_{\mathrm{L}^2_x}\Big)\|g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}. \quad (5.40)$$

Step 4. Conclusion.
Combining estimates (5.37) and (5.40) in such a way that the term $\|\nabla\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}$ in the right-hand side of (5.40) can be absorbed into the left-hand side of (5.37), we obtain

$$\mathrm{Pe}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+\varepsilon\tfrac{\mathrm{d}}{\mathrm{d}t}\|g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}+\|\nabla\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}+\|\nabla_n g_{4,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}$$
$$\lesssim \mathcal{C}\big(1+\|u_\varepsilon\|^2_{H^1_x}+\|\rho_{3,\varepsilon}\|^2_{\mathrm{L}^2_x}\big)+\big((1+\mathrm{Pe})^2\|\rho^\circ\|^2_{\mathrm{L}^\infty_x}+\varepsilon^2\mathcal{C}\big)\|u_{3,\varepsilon}\|^2_{H^1_x}+\mathcal{C}\big(1+\|g_{3,\varepsilon}\|^2_{\mathrm{L}^2_{x,n}}\big)$$

$$+\,\varepsilon\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}^{\frac{4}{4-d}}\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\varepsilon\mathcal{C}\Big(1+\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}^{\frac{4}{4-d}}\Big)\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2.$$

Further using the fact that

$$\|u_\varepsilon\|_{H^1_x}^2\lesssim\mathcal{C}+\varepsilon^6\|u_{3,\varepsilon}\|_{H^1_x}^2,\qquad\|g_{3,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2\|\lesssim\mathcal{C}+\varepsilon^2\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2,$$

the above reduces to

$$\begin{aligned}\mathrm{Pe}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\varepsilon\tfrac{\mathrm{d}}{\mathrm{d}t}\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2+\|\nabla\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2\\ \lesssim\mathcal{C}\big(1+\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2\big)+\big((1+\mathrm{Pe})^2\|\rho_0\|_{\mathrm{L}^\infty_x}^2+\varepsilon^2\mathcal{C}\big)\|u_{3,\varepsilon}\|_{H^1_x}^2\\ +\,\varepsilon\mathcal{C}\Big(1+\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}^{\frac{4}{4-d}}\Big)\Big(\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2\Big).\end{aligned}\tag{5.41}$$

We now aim to combine this with (5.35) to conclude the proof, and we separately consider the Stokes and Navier–Stokes cases, splitting the proof into two further substeps.

Substep 4.1. Stokes case $\mathrm{Re}=0$, $d\le 3$.
In the Stokes case, the estimate (5.35) becomes, using Poincaré's inequality,

$$\begin{aligned}\|u_{3,\varepsilon}\|_{H^1_x}^2&\lesssim\mathcal{C}+\lambda^2\theta^2\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2+\mathcal{C}\big(\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\varepsilon^2\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2\big)\\&\lesssim\mathcal{C}+\lambda^2\theta^2\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2+\mathcal{C}\big(\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\varepsilon^2\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2\big).\end{aligned}\tag{5.42}$$

We now insert this estimate for $u_{3,\varepsilon}$ in the right-hand side of (5.41). Provided that

$$\lambda^2\theta^2(1+\mathrm{Pe})^2\big(\|\rho^\circ\|_{\mathrm{L}^\infty_x}^2+\varepsilon^2\mathcal{C}\big)\ll 1$$

is small enough, we can absorb the term involving $\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2$ in the estimate for $u_{3,\varepsilon}$, and we end up with

$$\begin{aligned}\mathrm{Pe}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\varepsilon\tfrac{\mathrm{d}}{\mathrm{d}t}\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2+\|\nabla\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2\\ \lesssim\mathcal{C}+\Big(\mathcal{C}+\varepsilon\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}^{\frac{4}{4-d}}\Big)\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\varepsilon\mathcal{C}\Big(1+\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}^{\frac{4}{4-d}}\Big)\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2.\end{aligned}$$

By Grönwall's inequality, we deduce

$$\begin{aligned}\|\rho_{3,\varepsilon}\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x}^2+\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_tH^1_x}^2+\varepsilon\|g_{4,\varepsilon}\|_{\mathrm{L}^\infty_t\mathrm{L}^2_{x,n}}^2+\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{t,x,n}}^2\\ \le\mathcal{C}(t)\Big(1+\|\rho_{3,\varepsilon}|_{t=0}\|_{\mathrm{L}^2_x}^2+\varepsilon\|g_{4,\varepsilon}|_{t=0}\|_{\mathrm{L}^2_{x,n}}^2\Big)\exp\Big(\mathcal{C}(t)\int_0^t\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}^{\frac{4}{4-d}}\Big).\end{aligned}$$

Now expanding $u_{1,\varepsilon}=u_1+\varepsilon u_2+\varepsilon^2u_{3,\varepsilon}$ and using once more (5.42), we get

$$\begin{aligned}&\|\rho_{3,\varepsilon}\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x}^2+\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_tH^1_x}^2+\varepsilon\|g_{4,\varepsilon}\|_{\mathrm{L}^\infty_t\mathrm{L}^2_{x,n}}^2+\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{t,x,n}}^2\\&\le\mathcal{C}(t)\Big(1+\|\rho_{3,\varepsilon}|_{t=0}\|_{\mathrm{L}^2_x}^2+\varepsilon\|g_{4,\varepsilon}|_{t=0}\|_{\mathrm{L}^2_{x,n}}^2\Big)\exp\Big(\varepsilon^4\mathcal{C}(t)\Big(\|\rho_{3,\varepsilon}\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x}+\|g_{4,\varepsilon}\|_{\mathrm{L}^\infty_t\mathrm{L}^2_{x,n}}\Big)^{\frac{4}{4-d}}\Big).\end{aligned}$$

The well-preparedness assumption (5.8) precisely ensures that the initial terms in the right-hand side are uniformly bounded. Using again Poincaré's inequality on $\mathbb{S}^{d-1}$ and appealing to a standard continuity argument, the stated estimates on $\rho_{3,\varepsilon}$ and $g_{4,\varepsilon}$ follow. The stated estimate on $u_{3,\varepsilon}$ is then deduced from (5.42).

Substep 4.2. Navier–Stokes case $\mathrm{Re}=0$, $d=2$.
In the Navier–Stokes case, provided that

$$\lambda^2\theta^2(1+\mathrm{Pe})^2\big(\|\rho^\circ\|_{\mathrm{L}^\infty_x}^2+\varepsilon^2\mathcal{C}\big)\ \ll\ 1$$

is small enough, combining (5.35) and (5.41) in such a way that the term $\|\nabla u_{3,\varepsilon}\|_{\mathrm{L}^2_x}$ in the right-hand side of (5.41) can be absorbed into the corresponding dissipation term in (5.35), while further absorbing the term $\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}\lesssim\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}$ in the right-hand side of (5.35) into the corresponding dissipation term in (5.41), we obtain

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\mathrm{Pe}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\varepsilon\tfrac{\mathrm{d}}{\mathrm{d}t}\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2+\|\nabla u_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\|\nabla\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\tfrac12\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2&\\ \lesssim\mathcal{C}\Big(1+\|u_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2\Big)+\varepsilon\mathcal{C}\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}^{\frac{4}{4-d}}\Big(\|\rho_{3,\varepsilon}\|_{\mathrm{L}^2_x}^2+\|g_{4,\varepsilon}\|_{\mathrm{L}^2_{x,n}}^2\Big)&,\end{aligned}$$

and thus, by Grönwall's inequality,

$$\begin{aligned}\|u_{3,\varepsilon}\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x}^2+\|\rho_{3,\varepsilon}\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x}^2+\varepsilon\|g_{4,\varepsilon}\|_{\mathrm{L}^\infty_t\mathrm{L}^2_{x,n}}^2+\|\nabla u_{3,\varepsilon}\|_{\mathrm{L}^2_{t,x}}^2+\|\nabla\rho_{3,\varepsilon}\|_{\mathrm{L}^2_{t,x}}^2+\|\nabla_n g_{4,\varepsilon}\|_{\mathrm{L}^2_{t,x,n}}^2&\\ \lesssim\mathcal{C}(t)\Big(1+\|u_{3,\varepsilon}|_{t=0}\|_{\mathrm{L}^2_x}^2+\|\rho_{3,\varepsilon}|_{t=0}\|_{\mathrm{L}^2_x}^2+\varepsilon\|g_{4,\varepsilon}|_{t=0}\|_{\mathrm{L}^2_{x,n}}^2\Big)\exp\Big(\mathcal{C}(t)\int_0^t\|\nabla u_{1,\varepsilon}\|_{\mathrm{L}^2_x}^{\frac{4}{4-d}}\Big)&.\end{aligned}$$

In the 2D case, the exponent in the exponential reduces to $\frac{4}{4-d}=2$. Then expanding again $u_{1,\varepsilon}=u_1+\varepsilon u_2+\varepsilon^2u_{3,\varepsilon}$, we can appeal to a continuity argument and the conclusion follows by noting that the well-preparedness assumption (5.8) precisely ensure the uniform boundedness of the initial terms. □

5.4. **Proof of Proposition 5.3.** The proof mainly consists in inserting the explicit expressions for g_1 and g_2 computed in Proposition 5.1 in terms of (u_0,ρ_0) and (u_1,ρ_1) inside the systems (5.3) and (5.4). Recalling the definition of stress tensors in (2.5), inserting the explicit expressions for g_1,g_2 in Proposition 5.1, and using the elementary integral identities in (5.16), we are led to

$$\sigma_1[g_1]\ =\ \lambda\theta\int_{\mathbb{S}^{d-1}}(n\otimes n-\tfrac1d\,\mathrm{Id})\,g_1\,\mathrm{d}n\ =\ \lambda\theta\tfrac{\omega_d}{d(d+2)}\rho_0\,\mathrm{D}(u_0),$$

and

$$\sigma_2[\rho_0,u_0]\ =\ \lambda\int_{\mathbb{S}^{d-1}}(n\otimes n)\nabla u_0(n\otimes n)\rho_0\,\mathrm{d}n\ =\ \lambda\tfrac{2\omega_d}{d(d+2)}\rho_0\,\mathrm{D}(u_0).$$

This proves that the couple (u_0,ρ_0) solves the system (3.27) with parameters $\eta_0=1$ and $\eta_1=\lambda\frac{\omega_d(\theta+2)}{2d(d+2)}$. Now turning to (u_1,ρ_1), we compute

$$\begin{aligned}\sigma_1[g_2]\ &=\ \lambda\theta\varepsilon\int_{\mathbb{S}^{d-1}}\big(n\otimes n-\tfrac1d\,\mathrm{Id}\big)g_2\,\mathrm{d}n\\ &=\ \lambda\theta\tfrac{\omega_d}{d(d+2)}\big(\rho_1\mathrm{D}(u_0)+\rho_0\mathrm{D}(u_1)\big)+\lambda\theta\tfrac{\omega_d}{d(d+2)}\Big(-\tfrac{1}{4d}A_2'(u_0,\rho_0)+2\tfrac{d+2}{d(d+4)}\rho_0\mathrm{D}(u_0)^2\\ &\qquad\qquad+\tfrac{1}{d(d-1)}U_0^2\nabla^2\rho_0-\tfrac{1}{d(d+4)}\mathrm{tr}(\mathrm{D}(u_0)^2)\,\mathrm{Id}\Big).\end{aligned}$$

We also compute

$$\sigma_2[\rho_0,\nabla u_1]\ =\ \lambda\int_{\mathbb{S}^{d-1}}(n\otimes n)(\nabla u_1)(n\otimes n)\,\rho_0\,\mathrm{d}n\ =\ \lambda\tfrac{2\omega_d}{d(d+2)}\rho_0\mathrm{D}(u_1),$$

as well as

$$\sigma_2[\rho_1+g_1,\nabla u_0] \;=\; \lambda\tfrac{2\omega_d}{d(d+2)}\rho_1\mathrm{D}(u_0)+\lambda\tfrac{2\omega_d}{d(d+2)(d+4)}\rho_0\Big(2\,\mathrm{D}(u_0)^2+\tfrac12\mathrm{tr}(\mathrm{D}(u_0)^2)\,\mathrm{Id}\Big).$$

Inserting these identities into (5.4), we verify that u_1 satisfies the fluid equation in (3.28) for some modified pressure field p_1 and with the expected coefficients

$$\gamma_1=-\lambda\theta\tfrac{\omega_d}{4d^2(d+2)}\qquad\text{and}\qquad\gamma_2=\lambda\tfrac{\omega_d}{2d^2(d+4)}\big(\theta+\tfrac{2d}{d+2}\big).$$

We turn to the derivation of the corresponding equation for ρ_1. By the defining equations for ρ_1 in (5.4), we can write

$$(\partial_t-\tfrac{1}{\mathrm{Pe}}\Delta+u_0\cdot\nabla)\rho_1=-u_1\cdot\nabla\rho_0-\langle U_0n\cdot\nabla_x g_1\rangle.$$

Inserting the explicit expressions for g_1 in Proposition 5.1, and using the above computations, we find

$$\langle U_0n\cdot\nabla_x g_1\rangle \;=\; -\tfrac{1}{d-1}U_0^2\langle n\otimes n:\nabla^2\rho_0\rangle \;=\; -\tfrac{1}{d(d-1)}U_0^2\Delta\rho_0,$$

and the conclusion follows for ρ_1. □

Appendix A. Well-posedness of the Doi–Saintillan–Shelley system

This appendix is devoted to the proof of Proposition 2.1. Let us assume that $h=0$ for simplicity, as well as $\mathrm{Pe}=\lambda=\theta=\varepsilon=1$, since these parameters play no role in the analysis. We then omit the subscript ε in the notation and we set $(u_\varepsilon,f_\varepsilon)\equiv(u,f)$. We focus on the Stokes case $\mathrm{Re}=0$ in dimension $d=3$, but standard adaptations of the proof below allow to treat the 2D Navier–Stokes case without important additional difficulty. For a given distribution function $f:\mathbb{R}^+\times\mathbb{T}^3\times\mathbb{S}^2\to\mathbb{R}^+$, we shall use the short-hand notation

$$\rho_f\;:=\;\int_{\mathbb{S}^2}f(\cdot,n)\,\mathrm{d}n.$$

We split the proof into four main steps.

Step 1. A priori energy estimates: we show that a smooth solution (u,f) of the system (2.5) satisfies for all $t\ge0$,

$$\|\nabla u\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x}\;\lesssim\;e^{Ct}\|\rho_{f^\circ}\|_{\mathrm{L}^2_x},\tag{A.1}$$

$$\|\rho_f\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x}+\|\nabla\rho_f\|_{\mathrm{L}^2_t\mathrm{L}^2_x}\;\lesssim\;e^{Ct}\|\rho_{f^\circ}\|_{\mathrm{L}^2_x},\tag{A.2}$$

$$\|f\|_{\mathrm{L}^\infty_t\mathrm{L}^2_{x,n}}+\|\nabla_x f\|_{\mathrm{L}^2_t\mathrm{L}^2_{x,n}}+\|\nabla_n f\|_{\mathrm{L}^2_t\mathrm{L}^2_{x,n}}\;\lesssim\;\exp\Big(te^{Ct}\big(1+\|\rho_{f^\circ}\|^4_{\mathrm{L}^2_x}\big)\Big)\|f^\circ\|_{\mathrm{L}^2_{x,n}}.\tag{A.3}$$

On the one hand, testing the equation for the fluid velocity u with u itself, using the incompressibility constraint, and inserting the form of σ_1,σ_2, we find

$$\|\nabla u\|^2_{\mathrm{L}^2_x}+\int_{\mathbb{T}^3\times\mathbb{S}^2}(\nabla u:n\otimes n)^2f\;=\;-\int_{\mathbb{T}^3\times\mathbb{S}^2}\nabla u:\big(n\otimes n-\tfrac13\,\mathrm{Id}\big)f,$$

and since the second left-hand side term is nonnegative,

$$\|\nabla u\|_{\mathrm{L}^2_x}\lesssim\|\rho_f\|_{\mathrm{L}^2_x}.\tag{A.4}$$

On the other hand, testing the kinetic equation for the particle density f with f itself, using again the incompressibility constraint, and integrating by parts, we find

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\mathrm{L}^2_{x,n}}+\|\nabla_x f\|^2_{\mathrm{L}^2_{x,n}}+\|\nabla_n f\|^2_{\mathrm{L}^2_{x,n}}\quad=\quad\int_{\mathbb{T}^3\times\mathbb{S}^2}\nabla_n f\cdot\pi_n^\perp(\nabla u)nf$$

$$\begin{aligned} &= -\tfrac{1}{2}\int_{\mathbb{T}^3\times\mathbb{S}^2} |f|^2 \mathrm{div}_n\big(\pi_n^\perp(\nabla u)n\big) \\ &\lesssim \|\nabla u\|_{\mathrm{L}^2_x}\|f\|^2_{\mathrm{L}^4_x\mathrm{L}^2_n}. \end{aligned} \tag{A.5}$$

By Ladyzhenskaya's inequality, we can estimate the last factor as

$$\|f\|^2_{\mathrm{L}^4_x\mathrm{L}^2_n} \lesssim \|f\|^{\frac12}_{\mathrm{L}^2_{x,n}}\|f\|^{\frac32}_{H^1_x\mathrm{L}^2_n} \lesssim \|f\|^2_{\mathrm{L}^2_{x,n}} + \|f\|^{\frac12}_{\mathrm{L}^2_{x,n}}\|\nabla_x f\|^{\frac32}_{\mathrm{L}^2_{x,n}}.$$

Inserting this into (A.5), and appealing to Young's inequality to absorb the norm of $\nabla_x f$, we deduce

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\mathrm{L}^2_{x,n}} + \|\nabla_x f\|^2_{\mathrm{L}^2_{x,n}} + \|\nabla_n f\|^2_{\mathrm{L}^2_{x,n}} \lesssim \big(\|\nabla u\|_{\mathrm{L}^2_x} + \|\nabla u\|^4_{\mathrm{L}^2_x}\big)\|f\|^2_{\mathrm{L}^2_{x,n}}. \tag{A.6}$$

Moreover, integrating the equation for f with respect to the angular variable, we get

$$\partial_t\rho_f + u\cdot\nabla\rho_f - \Delta\rho_f + U_0\,\mathrm{div}\Big(\int_{\mathbb{S}^2} nf\,\mathrm{d}n\Big) = 0,$$

and thus, testing this equation with ρ_f itself and using the incompressibility constraint,

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_f\|^2_{\mathrm{L}^2_x} + \|\nabla\rho_f\|^2_{\mathrm{L}^2_x} = U_0\int_{\mathbb{T}^3\times\mathbb{S}^2} nf\cdot\nabla\rho_f \lesssim \int_{\mathbb{T}^3}\rho_f|\nabla\rho_f|,$$

hence

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|\rho_f\|^2_{\mathrm{L}^2_x} + \|\nabla\rho_f\|^2_{\mathrm{L}^2_x} \lesssim \|\rho_f\|^2_{\mathrm{L}^2_x}.$$

By Grönwall's inequality, this proves (A.2), and the claim (A.1) then follows after combination with (A.4). Further combining with (A.6) and appealing again to Grönwall's inequality, the claim (A.3) also follows.

Step 2. Construction of approximate solutions.

In order to prove the existence of a weak solution for the system (2.5), we argue by means of a Galerkin approximation method. More precisely, given $k\in\mathbb{N}$, we introduce the following orthogonal projection on $\mathrm{L}^2(\mathbb{T}^3)$,

$$P_k : \mathrm{L}^2(\mathbb{T}^3) \twoheadrightarrow F_k := \big\{u\in\mathrm{L}^2(\mathbb{T}^3) : \hat u(l) = 0 \text{ for all } |l|>k\big\},$$

where $\{\hat u(l)\}_{l\in\mathbb{Z}^3}$ stands for the Fourier coefficients of a periodic function $u\in\mathrm{L}^2(\mathbb{T}^3)$. For all $u\in F_k$ and $s\geq 0$, we obviously have

$$\|u\|_{H^s_x} \leq \langle k\rangle^s\|u\|_{\mathrm{L}^2_x}. \tag{A.7}$$

Given an initial condition $f^\circ\in H^1\cap\mathcal{P}(\mathbb{T}^3\cap\mathbb{S}^2)$, we shall consider the following approximate system,

$$\begin{cases} -\Delta u_k + \nabla p_k = P_k\mathrm{div}(\sigma_1[f]) + P_k\mathrm{div}(\sigma_2[f_k,\nabla P_k u_k]),\\ \partial_t f_k + \mathrm{div}_x\big((u_k+U_0 n)f_k\big) + \mathrm{div}_n\big(\pi_n^\perp(\nabla P_k u_k)nf_k\big) = \Delta_x f_k + \Delta_n f_k,\\ \mathrm{div}(u_k) = 0, \quad \int_{\mathbb{T}^3} u_k = 0,\\ f_k|_{t=0} = f^\circ, \end{cases} \tag{A.8}$$

and we claim that this system admits a weak solution (u_k, f_k) with

$$\begin{aligned} u_k &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{T}^3)^3), &\text{(A.9)}\\ f_k &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2\cap\mathcal{P}(\mathbb{T}^3\times\mathbb{S}^2)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{T}^3\times\mathbb{S}^2)). &\text{(A.10)} \end{aligned}$$

To prove this, we argue by means of a Schauder fixed-point argument and we split the proof into four further substeps.

Substep 2.1. Fixed-point problem.
Given $T>0$, let

$$E\ :=\ \Big\{v\in\mathrm{L}^\infty(0,T;H^1(\mathbb{T}^3)^3):\mathrm{div}(v)=0,\ \textstyle\fint_{\mathbb{T}^3}v=0\Big\},$$

and for all $v\in E$ define $\Lambda_k(v):=u$ as the unique solution of

$$\begin{cases}-\Delta u+\nabla p=P_k\mathrm{div}(\sigma_1[g])+P_k\mathrm{div}(\sigma_2[g,\nabla P_k u]),\\ \partial_t g+\mathrm{div}_x\big((v+U_0n)g\big)+\mathrm{div}_n\big(\pi_n^\perp(\nabla P_k v)ng\big)=\Delta_x g+\Delta_n g,\\ \mathrm{div}(u)=0,\quad \fint_{\mathbb{T}^3}u=0,\\ g|_{t=0}=f^\circ.\end{cases}\tag{A.11}$$

Note that we use the velocity field u and not v in the viscous stress σ_2 in the equation for u so as to preserve its dissipative structure. The existence of a weak solution u_k for the approximate system (A.8) amounts to finding a fixed point $\Lambda_k(u_k)=u_k$.

Before going on with the fixed-point problem, we first check that the above system (A.11) is indeed well-posed and defines a map $\Lambda_k:E\to E$. First, given $v\in E$, by standard parabolic theory, the above kinetic equation for g admits a unique weak solution

$$g\ \in\ \mathrm{L}^\infty(0,T;\mathrm{L}^2\cap\mathcal{P}(\mathbb{T}^3\times\mathbb{S}^2))\cap\mathrm{L}^2(0,T;H^1(\mathbb{T}^3\times\mathbb{S}^2)).$$

Moreover, the a priori estimates of Step 1 can be repeated to the effect of

$$\begin{aligned}\|\rho_g\|_{\mathrm{L}^\infty_T\mathrm{L}^2_x}+\|\nabla\rho_g\|_{\mathrm{L}^2_T\mathrm{L}^2_x}\ &\lesssim\ e^{CT}\|\rho_{f^\circ}\|_{\mathrm{L}^2_x},\\ \|g\|_{\mathrm{L}^\infty_T\mathrm{L}^2_{x,n}}+\|\nabla_x g\|_{\mathrm{L}^2_T\mathrm{L}^2_{x,n}}+\|\nabla_n g\|_{\mathrm{L}^2_T\mathrm{L}^2_{x,n}}\ &\lesssim\ \exp\Big(CT\big(1+\|\nabla v\|^4_{\mathrm{L}^\infty_T\mathrm{L}^2_x}\big)\Big)\|f^\circ\|_{\mathrm{L}^2_x}.\end{aligned}\tag{A.12}$$

To ensure the well-posedness of the elliptic equation for u in (A.11), we first need to check that ρ_g has some additional regularity. Integrating the equation for g with respect to the angular variable, we find the following equation for ρ_g,

$$\partial_t\rho_g+v\cdot\nabla\rho_g-\Delta\rho_g+U_0\,\mathrm{div}\Big(\int_{\mathbb{S}^2}ng\,\mathrm{d}n\Big)=0,$$

and thus, testing the equation with $\Delta\rho_g$, we find

$$\begin{aligned}\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho_g\|^2_{\mathrm{L}^2_x}+\|\Delta\rho_g\|^2_{\mathrm{L}^2_x}\ &=\ -\int_{\mathbb{T}^3}(v\cdot\nabla\rho_g)\Delta\rho_g-U_0\int_{\mathbb{T}^3}(\Delta\rho_g)\,\mathrm{div}\Big(\int_{\mathbb{S}^2}ng\,\mathrm{d}n\Big)\\ &\lesssim\ \|v\|_{\mathrm{L}^4_x}\|\nabla\rho_g\|_{\mathrm{L}^4_x}\|\Delta\rho_g\|_{\mathrm{L}^2_x}+\|\Delta\rho_g\|_{\mathrm{L}^2_x}\|\nabla_x g\|_{\mathrm{L}^2_{x,n}}.\end{aligned}$$

By Ladyzhenskaya's and Poincaré's inequalities, the first right-hand side term can be bounded by

$$\begin{aligned}\|v\|_{\mathrm{L}^4_x}\|\nabla\rho_g\|_{\mathrm{L}^4_x}\|\Delta\rho_g\|_{\mathrm{L}^2_x}\ &\lesssim\ \|v\|^{\frac14}_{\mathrm{L}^2_x}\|\nabla v\|^{\frac34}_{\mathrm{L}^2_x}\|\nabla\rho_g\|^{\frac14}_{\mathrm{L}^2_x}\|\nabla^2\rho_g\|^{\frac74}_{\mathrm{L}^2_x}\\ &\lesssim\ \|\nabla v\|_{\mathrm{L}^2_x}\|\nabla\rho_g\|^{\frac14}_{\mathrm{L}^2_x}\|\nabla^2\rho_g\|^{\frac74}_{\mathrm{L}^2_x},\end{aligned}$$

and thus, inserting this into the above, recalling $\|\nabla^2\rho_g\|_{\mathrm{L}^2_x}\lesssim\|\Delta\rho_g\|_{\mathrm{L}^2_x}$, and using Young's inequality, we are led to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho_g\|^2_{\mathrm{L}^2_x}+\|\Delta\rho_g\|^2_{\mathrm{L}^2_x}\ \lesssim\ \|\nabla v\|^8_{\mathrm{L}^2_x}\|\nabla\rho_g\|^2_{\mathrm{L}^2_x}+\|\nabla_x g\|^2_{\mathrm{L}^2_{x,n}}.$$

By Grönwall's inequality, this implies

$$\|\nabla\rho_g\|_{\mathrm{L}^\infty_T\mathrm{L}^2_x}+\|\Delta\rho_g\|_{\mathrm{L}^2_T\mathrm{L}^2_x}\lesssim\Big(\|\nabla\rho_{f^\circ}\|_{\mathrm{L}^2_x}+\|\nabla_x g\|_{\mathrm{L}^2_T\mathrm{L}^2_{x,n}}\Big)\exp\Big(C\int_0^t\|\nabla v\|^8_{\mathrm{L}^2_x}\Big).$$

Combined with (A.12), this shows that $\rho_g \in \mathrm{L}^\infty(0,T;H^1(\mathbb{T}^3)) \cap \mathrm{L}^2(0,T;H^2(\mathbb{T}^3))$. In particular, we infer $\rho_g(t) \in \mathrm{L}^\infty(\mathbb{T}^3)$ for almost all $t \in [0,T]$. With this additional regularity result for ρ_g, we can now finally ensure the well-posedness of the elliptic equation for u in (A.11). Indeed, for all $t \in [0,T]$, the weak formulation of this equation for $u(t)$ can be written as follows: for all $w \in E$,

$$\begin{aligned} B_{t,k,g}(w,u(t)) := \int_{\mathbb{T}^3} \nabla w : \nabla u(t) + \int_{\mathbb{T}^3\times\mathbb{S}^2} \big(n\otimes n : \nabla P_k w\big)\big(n \otimes n : \nabla P_k u(t)\big)\, g(t) \\ = -\int_{\mathbb{T}^3} \nabla w \cdot P_k \sigma_1[g(t)]. \end{aligned}$$

For almost all t, as $\rho_g(t) \in \mathrm{L}^\infty(\mathbb{T}^3)$, we note that $B_{t,k,g}$ is a coercive continuous bilinear functional. As in addition we have $\|P_k\sigma_1[g(t)]\|_{\mathrm{L}^2_x} \lesssim \|\rho_g(t)\|_{\mathrm{L}^2_x}$, we can appeal to the Lax–Milgram theorem and deduce that there exists a unique solution $u \in \mathrm{L}^\infty(0,T;H^1(\mathbb{T}^3)^3)$ of (A.11). Further recalling (A.12), it satisfies

$$\|\nabla u\|_{\mathrm{L}^\infty_T\mathrm{L}^2_x} \lesssim \|\rho_g\|_{\mathrm{L}^\infty_T\mathrm{L}^2_x} \lesssim e^{CT}\|\rho_{f^\circ}\|_{\mathrm{L}^2_x}. \tag{A.13}$$

Letting $\Lambda_k(v) := u$ be the solution of (A.11), this shows that we are indeed led to a well-defined map $\Lambda_k : E \to E$.

Substep 2.2. Proof that the map $\Lambda_k : E \to E$ is compact.
Let $(v_r)_r$ be a bounded sequence in E, and for all r let $(u_r := \Lambda_k v_r, g_r)$ be the corresponding solution of the system (A.11) with v replaced by v_r. By (A.13), the sequence $(u_r)_r$ is bounded in $\mathrm{L}^\infty(0,T;H^1(\mathbb{T}^3)^3)$. By the Aubin–Lions lemma, in order to prove that it is actually precompact in $\mathrm{L}^\infty(0,T;H^1(\mathbb{T}^3)^3)$, it suffices to check that $(u_r)_r$ is also bounded in $\mathrm{L}^\infty(0,T;H^2(\mathbb{T}^3)^3)$ and that $(\partial_t u_r)_r$ is bounded for instance in $\mathrm{L}^{8/3}(0,T;H^1(\mathbb{T}^3)^3)$. On the one hand, noting that by definition we have $u_r = P_k u_r$, and appealing to (A.7), we directly find

$$\|u_r\|_{H^2_x} = \|P_k u_r\|_{H^2_x} \lesssim_k \|u_r\|_{H^1_x},$$

which shows that $(u_r)_r$ is indeed also bounded in $\mathrm{L}^\infty(0,T;H^2(\mathbb{T}^3)^3)$. We turn to the boundedness of time derivatives. Taking the time derivative of the elliptic equation for u_r, we find

$$\begin{aligned} -\Delta(\partial_t u_r) - P_k \mathrm{div}(\sigma_2[g_r, \nabla P_k(\partial_t u_r)]) + \nabla(\partial_t p_r) \\ = P_k\mathrm{div}(\sigma_1[\partial_t g_r]) + P_k \mathrm{div}(\sigma_2[\partial_t g_r, \nabla P_k u_r]). \end{aligned}$$

Arguing as for (A.13), we deduce

$$\|\partial_t u_r\|_{H^1_x} \lesssim \|P_k\sigma_1[\partial_t g_r]\|_{\mathrm{L}^2_x} + \|P_k\sigma_2[\partial_t g_r, \nabla P_k u_r]\|_{\mathrm{L}^2_x}. \tag{A.14}$$

To estimate the two right-hand side terms, we recall the definition of σ_1, σ_2, we insert the equation for g_r, we integrate by parts in the n-integrals, and we appeal to (A.7) again. For instance, for the term involving σ_1, we find

$$\begin{aligned} \|P_k\sigma_1[\partial_t g_r]\|_{\mathrm{L}^2_x} &= \lambda\theta \Big\| P_k \int_{\mathbb{S}^2} \big(n\otimes n - \tfrac{1}{d}\,\mathrm{Id}\big)\Big(\mathrm{div}_x\big((v_r + U_0 n) g_r\big) \\ &\qquad + \mathrm{div}_n\big(\pi_n^\perp(\nabla P_k v_r) n g_r\big) - \Delta_x g_r - \Delta_n g_r\Big)\,\mathrm{d}n \Big\|_{\mathrm{L}^2_x} \\ &= \lambda\theta \Big\| \sum_{i=1}^3 P_k \nabla_{x_i} \int_{\mathbb{S}^2} \big(n\otimes n - \tfrac{1}{d}\,\mathrm{Id}\big)\big((v_r+U_0 n)g_r\big)_i\,\mathrm{d}n \end{aligned}$$

$$
\begin{aligned}
&-P_k\int_{\mathbb{S}^2}\nabla_n(n\otimes n)\pi_n^{\perp}(\nabla P_k v_r)ng_r\,\mathrm{d}n\\
&-P_k\Delta_x\int_{\mathbb{S}^2}\big(n\otimes n-\tfrac{1}{d}\,\mathrm{Id}\,\big)\,g_r\,\mathrm{d}n-P_k\int_{\mathbb{S}^2}\Delta_n(n\otimes n)\,g_r\,\mathrm{d}n\bigg\|_{\mathrm{L}^2_x}\\
\lesssim_k\quad &\|v_r\|_{\mathrm{L}^4_x}\|g_r\|_{\mathrm{L}^4_x\mathrm{L}^2_n}+\|g_r\|_{\mathrm{L}^2_x\mathrm{L}^2_n}.
\end{aligned}
$$

Noting that we have $\|P_k(h_1P_kh_2)\|_{\mathrm{L}^2_x}\le\|(P_{2k}h_1)(P_kh_2)\|_{\mathrm{L}^2_x}$ for all $h_1,h_2\in\mathrm{L}^1(\mathbb{T}^3)$, we can argue similarly to estimate the second term in (A.14). Further using Jensen's inequality, we are led to

$$
\|\partial_t u_r\|_{H^1_x}\;\lesssim_k\;(1+\|u_r\|_{\mathrm{L}^2_x})(1+\|v_r\|_{\mathrm{L}^4_x})\|g_r\|_{\mathrm{L}^4_x\mathrm{L}^2_n}.
$$

By Ladyzhenskaya's inequality, this yields

$$
\|\partial_t u_r\|_{H^1_x}\;\lesssim_k\;(1+\|u_r\|_{\mathrm{L}^2_x})(1+\|v_r\|_{H^1_x})\|g_r\|_{\mathrm{L}^2_{x,n}}^{\frac14}\|g_r\|_{H^1_x\mathrm{L}^2_n}^{\frac34}.
$$

The a priori estimate (A.12) for $g=g_r$ then ensures that the sequence $(\partial_t u_r)_r$ is bounded in $\mathrm{L}^{8/3}(0,T;H^1(\mathbb{T}^3)^3)$, and the compactness of Λ_k follows.

Substep 2.3. Proof that the map $\Lambda_k:E\to E$ is continuous.
Given a sequence $(v_r)_r$ that converges (strongly) to some v in E, we need to check that the image $u_r:=\Lambda_k v_r$ converges to $\Lambda_k v$. By compactness of Λ_k, we already know that up to a subsequence $u_r=\Lambda_k v_r$ converges (strongly) to some w in E, and it remains to show that it satisfies $w=\Lambda_k v$. By definition of Λ_k, we recall that u_r satisfies the system (A.11) with v replaced by v_r, and we denote by g_r the corresponding density. Using the strong convergence of u_r and v_r along the extracted subsequence, recalling the a priori estimates for g_r, cf. (A.12), and using weak compactness for g_r, we can pass to the limit in the weak formulation of this system. This shows that the limit w satisfies the system defining $\Lambda_k v$, hence we have $w=\Lambda_k v$ by uniqueness.

Substep 2.4. Application of Schauder's fixed-point theorem.
Recall the a priori estimate (A.13), that is,

$$
\|\Lambda_k v\|_{\mathrm{L}^\infty_T H^1_x}\;\le\;Ce^{CT}\|\rho_{f^\circ}\|_{\mathrm{L}^2_x}.
$$

Considering the following non-empty closed convex subset of E,

$$
K\;:=\;\big\{u\in E:\|u\|_{\mathrm{L}^\infty_T H^1_x}\le Ce^{CT}\|\rho_{f^\circ}\|_{\mathrm{L}^2_x}\big\},
$$

the restriction of Λ_k defines a map $\Lambda_k|_K:K\to K$ that is compact and continuous by Steps 2.2 and 2.3. By Schauder's theorem, this map must then admit a fixed point $u_k\in K$, that is, $u_k=\Lambda_k(u_k)$. Combined with the a priori estimates (A.12), this concludes the proof of the existence of a weak solution of the approximate system (A.8) satisfying (A.9).

Step 3. Existence of a weak solution for the system (2.5).
We argue by means of a Galerkin method based on the approximations defined in Step 2. For that purpose, given an initial condition $f^\circ\in\mathrm{L}^2\cap\mathcal{P}(\mathbb{T}^3\times\mathbb{S}^2)$, we start by considering an approximating sequence of smooth initial conditions $(f^\circ_k)_k\subset C^\infty\cap\mathcal{P}(\mathbb{T}^3\times\mathbb{S}^2)$ that converges strongly to f° in $\mathrm{L}^2(\mathbb{T}^3\times\mathbb{S}^2)$. By Step 2, for all k, we may then consider the

solution (u_k, f_k) of the approximate system (A.8) with initial condition f_k°, that is,

$$\begin{cases} -\Delta u_k + \nabla p_k = P_k \mathrm{div}(\sigma_1[f]) + P_k \mathrm{div}(\sigma_2[f_k, \nabla P_k u_k]), \\ \partial_t f_k + \mathrm{div}_x\big((u_k + U_0 n) f_k\big) + \mathrm{div}_n\big(\pi_n^\perp(\nabla P_k u_k) n f_k\big) = \Delta_x f_k + \Delta_n f_k, \\ \mathrm{div}(u_k) = 0, \quad \fint_{\mathbb{T}^3} u_k = 0, \\ f_k|_{t=0} = f_k^\circ, \end{cases}$$

with

$$\begin{array}{rcl} u_k & \in & \mathrm{L}^\infty(0, T; H^1(\mathbb{T}^3)^3), \\ f_k & \in & \mathrm{L}^\infty(0, T; \mathrm{L}^2 \cap \mathcal{P}(\mathbb{T}^3 \times \mathbb{S}^2)) \cap \mathrm{L}^2(0, T; H^1(\mathbb{T}^3 \times \mathbb{S}^2)). \end{array}$$

From Step 2, we further learn that (u_k, f_k) is uniformly bounded in these spaces. By weak compactness, we deduce that up to a subsequence u_k converges weakly-* to some u in $\mathrm{L}^\infty(0, T; H^1(\mathbb{T}^3))$, and that f_k converges weakly-* to some f in $\mathrm{L}^\infty(0, T; \mathrm{L}^2(\mathbb{T}^3 \times \mathbb{S}^2))$ and weakly in $\mathrm{L}^2(0, T; H^1(\mathbb{T}^3 \times \mathbb{S}^2))$. In addition, examining the equation satisfied by f_k, a simple argument allows to check that $\partial_t f_k$ is bounded e.g. in $\mathrm{L}^\infty(0, T; W^{-2,1}(\mathbb{T}^3 \times \mathbb{S}^2))$, so that the Aubin–Lions lemma further entails that f_k converges strongly to f in $\mathrm{L}^2(0, T; \mathrm{L}^2(\mathbb{T}^3)^3)$. These convergences now allow to pass to the limit in the weak formulation of the system, and we easily conclude that the extracted limit (u, f) satisfies the limiting system (2.5).

Step 4. Weak-strong uniqueness for the system (2.5).
Let (u, f) and (u', f') be two weak solutions of (2.5) with common initial condition

$$f|_{t=0} = f'|_{t=0} = f^\circ,$$

and assume that (u', f') further satisfies

$$\begin{array}{rcl} u' & \in & \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; W^{1,\infty}(\mathbb{T}^3)^3), \\ f' & \in & \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{S}^2)). \end{array} \tag{A.15}$$

Consider the differences $U = u - u'$ and $F = f - f'$. On the one hand, the equations for u and u' yield

$$\begin{array}{rcl} -\Delta U + \nabla P & = & \mathrm{div}(\sigma_1[F]) + \mathrm{div}(\sigma_2[f, \nabla u] - \sigma_2[f', \nabla u']) \\ & = & \mathrm{div}(\sigma_1[F]) + \mathrm{div}(\sigma_2[f, \nabla U] + \sigma_2[F, \nabla u']). \end{array}$$

Testing this equation with U itself, using the incompressibility constraint, and taking advantage of the additional dissipation given by σ_2, we get

$$\begin{array}{rcl} \|\nabla U\|_{\mathrm{L}^2_x} & \le & \|\sigma_1[F]\|_{\mathrm{L}^2_x} + \|\sigma_2[F, \nabla u']\|_{\mathrm{L}^2_x} \\ & \le & \big(1 + \|\nabla u'\|_{\mathrm{L}^\infty_x}\big) \|F\|_{\mathrm{L}^2_{x,n}}. \end{array} \tag{A.16}$$

On the other hand, the equations for f and f' yield

$$\partial_t F + \mathrm{div}_x\big((u + U_0 n) F\big) + \mathrm{div}_n\big(\pi_n^\perp(\nabla u) n F\big) - \Delta_x F - \Delta_n F \\ = -\mathrm{div}_x(U f') - \mathrm{div}_n\big(\pi_n^\perp(\nabla U) n f'\big).$$

Testing this equation with F itself, and integrating by parts, we get

$$\tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \|F\|^2_{\mathrm{L}^2_{x,n}} + \|\nabla_x F\|^2_{\mathrm{L}^2_{x,n}} + \|\nabla_n F\|^2_{\mathrm{L}^2_{x,n}} \\ = -\tfrac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{S}^2} |F|^2 \mathrm{div}_n\big(\pi_n^\perp(\nabla u) n\big) + \int_{\mathbb{T}^3 \times \mathbb{S}^2} \nabla_x F \cdot U f' + \int_{\mathbb{T}^3 \times \mathbb{S}^2} \nabla_n F \cdot \pi_n^\perp(\nabla U) n f'$$

$$\lesssim \|\nabla u\|_{\mathrm{L}^2_x}\|F\|^2_{\mathrm{L}^4_x\mathrm{L}^2_n} + \|\nabla_x F\|_{\mathrm{L}^2_{x,n}}\|U\|_{\mathrm{L}^2_x}\|f'\|_{\mathrm{L}^\infty_{x,n}} + \|\nabla_n F\|_{\mathrm{L}^2_{x,n}}\|\nabla U\|_{\mathrm{L}^2_x}\|f'\|_{\mathrm{L}^\infty_{x,n}}.$$

By Ladyzhenskaya's inequality, the first right-hand side term can be bounded by

$$\begin{aligned}\|\nabla u\|_{\mathrm{L}^2_x}\|F\|^2_{\mathrm{L}^4_x\mathrm{L}^2_n} &\lesssim \|\nabla u\|_{\mathrm{L}^2_x}\|F\|^{\frac12}_{\mathrm{L}^2_{x,n}}\|F\|^{\frac32}_{H^1_x\mathrm{L}^2_n}\\ &\lesssim \|\nabla u\|_{\mathrm{L}^2_x}\|F\|^2_{\mathrm{L}^2_{x,n}} + \|\nabla u\|_{\mathrm{L}^2_x}\|F\|^{\frac12}_{\mathrm{L}^2_{x,n}}\|\nabla_x F\|^{\frac32}_{\mathrm{L}^2_{x,n}}.\end{aligned}$$

Inserting this into the above and appealing to Young's and Poincaré's inequalities, we are led to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|F\|^2_{\mathrm{L}^2_{x,n}} \lesssim \big(\|\nabla u\|_{\mathrm{L}^2_x} + \|\nabla u\|^4_{\mathrm{L}^2_x}\big)\|F\|^2_{\mathrm{L}^2_{x,n}} + \|\nabla U\|^2_{\mathrm{L}^2_x}\|f'\|^2_{\mathrm{L}^\infty_{x,n}},$$

and thus, recalling (A.16),

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|F\|^2_{\mathrm{L}^2_{x,n}} \lesssim \Big(\big(1+\|\nabla u\|^4_{\mathrm{L}^2_x}\big) + \big(1+\|\nabla u'\|^2_{\mathrm{L}^\infty_x}\big)\|f'\|^2_{\mathrm{L}^\infty_{x,n}}\Big)\|F\|^2_{\mathrm{L}^2_{x,n}}.$$

By the assumed regularity of u, u', f', this implies $F=0$ by Grönwall's inequality, hence also $U=0$ by (A.16). ☐

Appendix B. Perturbative well-posedness of ordered fluid equations

This section is devoted to the proof of Propositions 3.2, 3.3, and 3.4. In this section, multiplicative constants C are implicitly allowed to further depend on parameters $\mathrm{Pe}, \eta_0, \eta_1, \gamma_1, \gamma_2$.

B.1. **Proof of Proposition 3.2.** First note that, if v_0, v_1 satisfy equations (3.18) and (3.19), then their superposition $\bar u_\varepsilon = v_0 + \varepsilon v_1$ satisfies (3.17) in form of

$$\begin{cases}\mathrm{Re}(\partial_t + \bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon - \mathrm{div}(\bar\sigma_\varepsilon) + \nabla\bar p_\varepsilon = h + \varepsilon^2 R_\varepsilon,\\ \bar\sigma_\varepsilon := \eta_0 A_1(\bar u_\varepsilon) + \varepsilon\gamma_1 A_2'(\bar u_\varepsilon) + \varepsilon\gamma_2 A_1(\bar u_\varepsilon)^2,\\ \mathrm{div}(\bar u_\varepsilon) = 0,\end{cases}$$

in terms of the remainder

$$\begin{aligned}R_\varepsilon := (v_1\cdot\nabla)v_1 &- \gamma_1\mathrm{div}\Big((\partial_t - \tfrac{1}{\mathrm{Pe}}\Delta + v_0\cdot\nabla)2\,\mathrm{D}(v_1) + (\nabla v_0)^T 2\,\mathrm{D}(v_1) + 2\,\mathrm{D}(v_1)(\nabla v_0)\Big)\\ &- 2\gamma_1\mathrm{div}\Big((v_1\cdot\nabla)\,\mathrm{D}(\bar u_\varepsilon) + (\nabla v_1)^T\,\mathrm{D}(\bar u_\varepsilon) + \mathrm{D}(\bar u_\varepsilon)(\nabla v_1)\Big)\\ &- 4\gamma_2\mathrm{div}\Big(\big(\mathrm{D}(v_0)\,\mathrm{D}(v_1) + \mathrm{D}(v_1)\,\mathrm{D}(v_0)\big) + \varepsilon\,\mathrm{D}(v_1)^2\Big).\end{aligned}$$

We turn to the well-posedness and regularity theory for equations (3.18) and (3.19). In the Stokes case, given $s \ge 0$ and $h \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^d)^d) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; H^{s-2}(\mathbb{T}^d)^d)$, the Stokes theory ensures that equation (3.18) admits a unique global solution v_0 in the space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+2}(\mathbb{T}^d)^d) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^d)^d)$. Then using this v_0 in the source terms of equation (3.19), and using the Sobolev embedding with $s > \frac d2 - 1$, equation (3.18) admits a unique global solution $v_1 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^d)^d)$.

Next, in the 2D Navier–Stokes case, given $s \ge 0$, $h \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^2)^2)$, and $u^\circ \in H^{s+1}(\mathbb{T}^2)^2$, using Ladyzhenskaya's inequality, the Navier–Stokes theory ensures that equation (3.18) admits a unique global solution v_0 in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^2)^2) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+2}(\mathbb{T}^2)^2) \cap H^1_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^2)^2)$. Then using this v_0 in the source terms of equation (3.19), and using the 2D Sobolev embedding with $s > 0$, equation (3.19) admits a unique global solution $v_1 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{T}^2)^2) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{T}^2)^2)$. This ends the proof of Proposition 3.2. ☐

B.2. **Proof of Proposition 3.3 in Stokes case.** Let $\mathrm{Re}=0$ and $d\le 3$. We split the proof into three steps, first deriving the suitable reformulation of the equations, and then proving existence and uniqueness of smooth solutions.

Step 1. Reformulation.
Equation (3.20) implies in particular

$$h \;=\; -\eta_0\Delta\bar u_\varepsilon-\varepsilon\mathrm{div}(F_0(\bar u_\varepsilon))+\varepsilon\tfrac{\gamma_1}{\eta_0}(\partial_t-\tfrac{1}{\mathrm{Pe}})h+\nabla\bar p_\varepsilon,$$

and therefore, using this to replace $\varepsilon\frac{\gamma_1}{\eta_0}(\partial_t-\frac{1}{\mathrm{Pe}})h$ in the right-hand side of (3.20),

$$\begin{aligned}
&-\eta_0\Delta\bar u_\varepsilon+\nabla\bar p_\varepsilon=h+\varepsilon\mathrm{div}(F_0(\bar u_\varepsilon))\\
&\qquad\qquad-\varepsilon\tfrac{\gamma_1}{\eta_0}(\partial_t-\tfrac{1}{\mathrm{Pe}})\Big(-\eta_0\Delta\bar u_\varepsilon+\nabla\bar p_\varepsilon-\varepsilon\mathrm{div}(F_0(\bar u_\varepsilon))+\varepsilon\tfrac{\gamma_1}{\eta_0}(\partial_t-\tfrac{1}{\mathrm{Pe}})h\Big).
\end{aligned}$$

Recalling the definition of F_0, cf. (3.21), and reorganizing $O(\varepsilon)$ terms, we precisely deduce the second-order fluid equation

$$\begin{cases}-\mathrm{div}(\bar\sigma_\varepsilon)+\nabla\bar p_\varepsilon'=h+\varepsilon^2R_\varepsilon,\\ \bar\sigma_\varepsilon=\eta_0A_1(\bar u_\varepsilon)+\varepsilon\gamma_1A_2'(\bar u_\varepsilon)+\varepsilon\gamma_2A_1(\bar u_\varepsilon)^2,\end{cases}$$

for some modified pressure field $\bar p_\varepsilon'$ and some remainder term

$$R_\varepsilon \;:=\; \tfrac{\gamma_1}{\eta_0}(\partial_t-\tfrac{1}{\mathrm{Pe}})\Big(\mathrm{div}(F_0(\bar u_\varepsilon))-\tfrac{\gamma_1}{\eta_0}(\partial_t-\tfrac{1}{\mathrm{Pe}})h\Big).$$

Step 2. Existence.
Let $s>\frac d2$ be fixed. We proceed by an iterative scheme. We set $\bar u_0:=0$ and for all $n\ge0$, given $\bar u_n\in H^{s+1}(\mathbb T^d)^d$, we define $\bar u_{n+1}\in H^{s+1}(\mathbb T^d)^d$ as the unique solution of the linear problem

$$\begin{cases}-\eta_0\Delta\bar u_{n+1}-\varepsilon\gamma_1\mathrm{div}\big((\bar u_n\cdot\nabla)2\mathrm{D}(\bar u_{n+1})\big)+\nabla\bar p_{n+1}\\ \qquad\qquad\qquad=\big(1-\varepsilon\frac{\gamma_1}{\eta_0}(\partial_t-\Delta)\big)h+\varepsilon\mathrm{div}(G_0(\bar u_n)),\\ \mathrm{div}(\bar u_{n+1})=0,\end{cases}\tag{B.1}$$

where we have set

$$G_0(u) \;:=\; \gamma_1\big((\nabla u)^T2\mathrm{D}(u)+2\mathrm{D}(u)(\nabla u)\big)+\gamma_2(2\mathrm{D}(u))^2.$$

We split the proof into two further substeps.

Substep 2.1. Sobolev a priori estimates.
Applying $\langle\nabla\rangle^s=(1-\Delta)^{s/2}$ to both sides of the above equation, testing it with $\langle\nabla\rangle^s\bar u_{n+1}$, and using the incompressibility constraint, we find

$$\begin{aligned}
\eta_0\int_{\mathbb T^d}|\nabla\langle\nabla\rangle^s\bar u_{n+1}|^2 \;&=\; \int_{\mathbb T^d}\langle\nabla\rangle^s\bar u_{n+1}\cdot\big(1-\varepsilon\tfrac{\gamma_1}{\eta_0}(\partial_t-\Delta)\big)\langle\nabla\rangle^sh\\
&\quad-\varepsilon\int_{\mathbb T^d}\nabla\langle\nabla\rangle^s\bar u_{n+1}:\langle\nabla\rangle^sG_0(\bar u_n)-2\varepsilon\gamma_1\int_{\mathbb T^d}\mathrm{D}(\langle\nabla\rangle^s\bar u_{n+1}):[\langle\nabla\rangle^s,\bar u_n\cdot\nabla]\mathrm{D}(\bar u_{n+1}),
\end{aligned}$$

hence, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\|\nabla\bar u_{n+1}\|_{H^s_x}\;&\lesssim\;\|h\|_{H^{s-1}_x}+\varepsilon\|(\partial_t-\Delta)h\|_{H^{s-1}_x}+\varepsilon\|G_0(\bar u_n)\|_{H^s_x}\\
&\qquad+\varepsilon\|[\langle\nabla\rangle^s,\bar u_n\cdot\nabla]\mathrm{D}(\bar u_{n+1})\|_{\mathrm{L}^2_x}.
\end{aligned}\tag{B.2}$$

To estimate the last term, we use the following form of the Kato–Ponce commutator estimate [KP88, Lemma X1]: for all $u, v \in C^\infty(\mathbb{T}^d)^d$ and all $p, q \in [2, \infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, we have

$$\|[\langle\nabla\rangle^s, u \cdot \nabla]\mathrm{D}(v)\|_{\mathrm{L}^2_x} \lesssim_s \|u\|_{W^{s,q}_x}\|v\|_{W^{2,p}_x} + \|u\|_{W^{1,\infty}_x}\|v\|_{H^{s+1}_x}, \tag{B.3}$$

and thus, properly choosing p, q and appealing to the Sobolev embedding with $s > \frac{d}{2}$, we deduce

$$\|[\langle\nabla\rangle^s, u \cdot \nabla]\mathrm{D}(v)\|_{\mathrm{L}^2_x} \lesssim_s \|u\|_{H^{s+1}_x}\|v\|_{H^{s+1}_x}.$$

Using this to estimate the last right-hand side term in (B.2), inserting the definition of G_0, and further appealing to the Sobolev embedding with $s > \frac{d}{2}$, we are led to

$$\|\bar{u}_{n+1}\|_{H^{s+1}_x} \lesssim_s \|h\|_{H^{s-1}_x} + \varepsilon\|(\partial_t - \Delta)h\|_{H^{s-1}_x} + \varepsilon\|\bar{u}_n\|^2_{H^{s+1}_x} + \varepsilon\|\bar{u}_n\|_{H^{s+1}_x}\|\bar{u}_{n+1}\|_{H^{s+1}_x}.$$

Provided that $\|\bar{u}_n\|_{H^{s+1}_x} \le C_0$ and that $\varepsilon C_0 \ll_s 1$ is small enough, the last right-hand side term can be absorbed and we are led to

$$\|\bar{u}_{n+1}\|_{H^{s+1}_x} \lesssim_s \|h\|_{H^{s-1}_x} + \varepsilon\|(\partial_t - \Delta)h\|_{H^{s-1}_x} + \varepsilon C_0^2.$$

Choosing $C_0 := C_s(\|h\|_{H^{s-1}_x} + \varepsilon\|(\partial_t - \Delta)h\|_{H^{s-1}})$ for some $C_s \gg_s 1$ large enough, we then deduce, provided that $\varepsilon C_0 \ll_s 1$ is small enough,

$$\|\bar{u}_n\|_{H^{s+1}_x} \le C_0 \qquad \Longrightarrow \qquad \|\bar{u}_{n+1}\|_{H^{s+1}_x} \le C_0.$$

Recalling the choice $\bar{u}_0 = 0$, this proves by induction that we have for all $n \ge 0$, provided that $\varepsilon \ll_{s,h} 1$ is small enough,

$$\|\bar{u}_n\|_{H^{s+1}_x} \le C_0 \simeq_s \|h\|_{H^{s-1}_x} + \varepsilon\|(\partial_t - \Delta)h\|_{H^{s-1}_x}. \tag{B.4}$$

Substep 2.2. Contraction.
The difference $\bar{u}_{n+1} - \bar{u}_n$ satisfies

$$\begin{aligned} -\eta_0\Delta(\bar{u}_{n+1} - \bar{u}_n) + \nabla(\bar{p}_{n+1} - \bar{p}_n) &= \varepsilon\gamma_1\operatorname{div}\big((\bar{u}_n \cdot \nabla)2\mathrm{D}(\bar{u}_{n+1} - \bar{u}_n)\big) \\ &\quad + \varepsilon\gamma_1\operatorname{div}\big(((\bar{u}_n - \bar{u}_{n-1}) \cdot \nabla)2\mathrm{D}(\bar{u}_n)\big) + \varepsilon\operatorname{div}(G_0(\bar{u}_n) - G_0(\bar{u}_{n-1})). \end{aligned}$$

Testing this equation with $\bar{u}_{n+1} - \bar{u}_n$ and using the incompressibility constraint, we get similarly as in (B.2) above,

$$\|\nabla(\bar{u}_{n+1} - \bar{u}_n)\|_{\mathrm{L}^2_x} \lesssim \varepsilon\|(\bar{u}_n - \bar{u}_{n-1}) \cdot \nabla\,\mathrm{D}(\bar{u}_n)\|_{\mathrm{L}^2_x} + \varepsilon\|G_0(\bar{u}_n) - G_0(\bar{u}_{n-1})\|_{\mathrm{L}^2_x},$$

and thus, inserting the definition of G_0 and appealing to the Sobolev embedding with $s > \frac{d}{2}$,

$$\|\nabla(\bar{u}_{n+1} - \bar{u}_n)\|_{\mathrm{L}^2_x} \lesssim_s \varepsilon\|(\bar{u}_{n-1}, \bar{u}_n)\|_{H^{s+1}_x}\|\nabla(\bar{u}_n - \bar{u}_{n-1})\|_{\mathrm{L}^2_x}.$$

Provided that $\varepsilon \ll_{s,h} 1$ small enough, inserting the a priori estimate (B.4), we deduce

$$\|\nabla(\bar{u}_{n+1} - \bar{u}_n)\|_{\mathrm{L}^2_x} \le \tfrac{1}{2}\|\nabla(\bar{u}_n - \bar{u}_{n-1})\|_{\mathrm{L}^2_x}.$$

Together with the a priori bound (B.4), this contraction estimate entails that the sequence $(\bar{u}_n)_n$ converges weakly in $H^{s+1}(\mathbb{T}^d)^d$ to some limit $\bar{u}_\varepsilon \in H^{s+1}(\mathbb{T}^d)^d$. By the Rellich theorem, recalling $s > \frac{d}{2}$, this allows to pass to the limit in (B.1) and to deduce that the limit $\bar{u}_\varepsilon$ is a solution of the nonlinear equation (3.20). In addition, it automatically satisfies the a priori estimate

$$\|\bar{u}_\varepsilon\|_{H^{s+1}_x} \lesssim_s \|h\|_{H^{s-1}_x} + \varepsilon\|(\partial_t - \Delta)h\|_{H^{s-1}_x}. \tag{B.5}$$

Step 3. Uniqueness.
Let $s > \frac{d}{2}$ be fixed. Let $\bar u_\varepsilon, \bar v_\varepsilon \in H^{s+1}(\mathbb{R}^d)^d$ be two solutions of (3.20). Their difference then satisfies

$$\begin{aligned}-\eta_0\Delta(\bar u_\varepsilon-\bar v_\varepsilon)+\nabla\bar p_\varepsilon&=\varepsilon\gamma_1\mathrm{div}\big((\bar u_\varepsilon\cdot\nabla)2\mathrm{D}(\bar u_\varepsilon-\bar v_\varepsilon)\big)\\&\quad+\varepsilon\gamma_1\mathrm{div}\big(((\bar u_\varepsilon-\bar v_\varepsilon)\cdot\nabla)2\mathrm{D}(\bar v_\varepsilon)\big)+\varepsilon\mathrm{div}(G_0(\bar u_\varepsilon)-G_0(\bar v_\varepsilon)).\end{aligned}$$

Arguing as in Step 2.2 above, we easily deduce

$$\|\nabla(\bar u_\varepsilon-\bar v_\varepsilon)\|^2_{\mathrm{L}^2_x}\lesssim\varepsilon\|(\bar u_\varepsilon,\bar v_\varepsilon)\|_{H^{s+1}_x}\|\nabla(\bar u_\varepsilon-\bar v_\varepsilon)\|^2_{\mathrm{L}^2_x}.$$

By the a priori estimate (B.5) for $\bar u_\varepsilon, \bar v_\varepsilon$, we deduce $\bar u_\varepsilon=\bar v_\varepsilon$ provided that $\varepsilon \ll_{s,h} 1$ is small enough. □

B.3. **Proof of Proposition 3.3 in Navier–Stokes case.** Let $\mathrm{Re}=1$ and $d=2$. We skip the proof of well-posedness, which is a straightforward modification of the above proof in the corresponding Stokes case. It remains to derive the suitable reformulation of the equations. For that purpose, first note that equation (3.22) can be rewritten as

$$\begin{aligned}(\partial_t+\bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon-\eta_0\Delta\bar u_\varepsilon-\varepsilon\gamma_1(\eta_0-\tfrac{1}{\mathrm{Pe}})\Delta^2\bar u_\varepsilon+\nabla\bar p_\varepsilon&\\=h+\varepsilon\gamma_1\mathrm{div}(2\,\mathrm{D}(h))+\varepsilon\mathrm{div}(F_0(\bar u_\varepsilon))-\varepsilon\gamma_1\mathrm{div}\big(2\,\mathrm{D}((\bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon)\big),&\end{aligned}\tag{B.6}$$

where F_0 is defined in (3.21). This yields in particular the following relation,

$$\begin{aligned}h\;=\;&(\partial_t+\bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon-\eta_0\Delta\bar u_\varepsilon+\nabla\bar p_\varepsilon\\&-\varepsilon\gamma_1(\eta_0-\tfrac{1}{\mathrm{Pe}})\Delta^2\bar u_\varepsilon-\varepsilon\gamma_1\mathrm{div}(2\,\mathrm{D}(h))-\varepsilon\mathrm{div}(F_0(\bar u_\varepsilon))+\varepsilon\gamma_1\mathrm{div}\big(2\,\mathrm{D}((\bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon)\big).\end{aligned}$$

Using this to replace $\varepsilon\gamma_1\mathrm{div}(2\,\mathrm{D}(h))$ in the right-hand side of (B.6), we find after straightforward simplifications

$$\begin{aligned}(\partial_t+\bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon-\eta_0\Delta\bar u_\varepsilon+\nabla\bar p_\varepsilon&=h+\varepsilon\mathrm{div}(F_0(\bar u_\varepsilon))\\&\quad+\varepsilon\gamma_1\mathrm{div}\Big[2\,\mathrm{D}\Big((\partial_t-\tfrac{1}{\mathrm{Pe}}\Delta)\bar u_\varepsilon+\nabla\hat p_\varepsilon-\varepsilon\gamma_1(\eta_0-\tfrac{1}{\mathrm{Pe}})\Delta^2\bar u_\varepsilon\\&\qquad-\varepsilon\gamma_1\mathrm{div}(2\,\mathrm{D}(h))-\varepsilon\mathrm{div}(F_0(\bar u_\varepsilon))+\varepsilon\gamma_1\mathrm{div}\big(2\,\mathrm{D}((\bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon)\big)\Big)\Big].\end{aligned}$$

Recalling the definition of F_0, cf. (3.21), and reorganizing $O(\varepsilon)$ terms, we precisely deduce the second-order fluid equation

$$\begin{cases}(\partial_t+\bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon-\mathrm{div}(\bar\sigma_\varepsilon)+\nabla\bar p'_\varepsilon=h+\varepsilon^2R_\varepsilon,\\ \bar\sigma_\varepsilon=\eta_0A_1(\bar u_\varepsilon)+\varepsilon\gamma_1A'_2(\bar u_\varepsilon)+\varepsilon\gamma_2A_1(\bar u_\varepsilon)^2,\end{cases}$$

for some modified pressure field P'_ε and some remainder term

$$\begin{aligned}R_\varepsilon\;:=\;-\gamma_1\mathrm{div}\Big[2\,\mathrm{D}\Big(&\gamma_1(\eta_0-\tfrac{1}{\mathrm{Pe}})\Delta^2\bar u_\varepsilon+\gamma_1\mathrm{div}(2\,\mathrm{D}(h))+\mathrm{div}(F_0(\bar u_\varepsilon))\\&-\gamma_1\mathrm{div}\big(2\,\mathrm{D}((\bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon)\big)\Big)\Big].\end{aligned}$$

This concludes the proof. □

B.4. **Proof of Proposition 3.4.** Let (u_0,ρ_0) and (u_1,ρ_1) satisfy the systems (3.27) and (3.28), respectively. Their superposition $(\bar u_\varepsilon,\bar\rho_\varepsilon)=(u_0+\varepsilon u_1,\rho_0+\varepsilon\rho_1)$ is easily checked to satisfy the non-homogeneous second-order fluid equations (3.26) in the form

$$\left\{\begin{array}{l}\mathrm{Re}(\partial_t+\bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon-\operatorname{div}(\bar\sigma_\varepsilon)+\nabla\bar p_\varepsilon=h+\varepsilon^2\mathcal{R}_\varepsilon,\\ \partial_t\bar\rho_\varepsilon-(\frac{1}{\mathrm{Pe}}+\varepsilon\mu_0)\Delta\bar\rho_\varepsilon+\bar u_\varepsilon\cdot\nabla\bar\rho_\varepsilon=\varepsilon^2\mathcal{S}_\varepsilon,\\ \bar\sigma_\varepsilon\;=\;(\eta_0+\eta_1\bar\rho_\varepsilon)A_1(\bar u_\varepsilon)+\varepsilon\gamma_1A_2'(\bar\rho_\varepsilon,\bar u_\varepsilon)+\varepsilon\gamma_2A_1(\bar u_\varepsilon)^2,\\ \operatorname{div}(\bar u_\varepsilon)=0,\\ (\bar\rho_\varepsilon,\bar u_\varepsilon)|_{t=0}=(\rho^\circ,u^\circ).\end{array}\right.$$

with explicit remainders given by

$$\begin{array}{rcl}\mathcal{R}_\varepsilon&:=&-\lambda\theta\frac{\omega_d}{d(d+2)}\operatorname{div}(\rho_1\,\mathrm{D}(u_1))\\ &&-\gamma_1\frac{1}{\varepsilon}\operatorname{div}\big(A_2'(\bar u_\varepsilon,\bar\rho_\varepsilon)-A_2'(u_0,\rho_0)\big)-\lambda\theta\frac{2\omega_d}{d^2(d+4)}\frac{1}{\varepsilon}\operatorname{div}\big(\bar\rho_\varepsilon\,\mathrm{D}(\bar u_\varepsilon)^2-\rho_0\,\mathrm{D}(u_0)^2\big)\\ &&-\lambda\frac{2\omega_d}{d(d+2)(d+4)}\operatorname{div}\Big(\rho_0\big(\mathrm{D}(\bar u_\varepsilon)\,\mathrm{D}(u_1)+\mathrm{D}(u_1)\,\mathrm{D}(\bar u_\varepsilon)\big)+2\rho_1\,\mathrm{D}(\bar u_\varepsilon)^2\Big)\\ &&+\mathrm{Re}(u_1\cdot\nabla)u_1-\operatorname{div}(\sigma_2[\rho_1+g_1,\nabla u_1]),\\ \mathcal{S}_\varepsilon&:=&-\frac{1}{d(d-1)}U_0^2\Delta\rho_1+u_1\cdot\nabla\rho_1.\end{array}$$

In the proof of Proposition 5.2, we recall that we have actually shown that after elimination of g_1,g_2 the solutions (u_0,ρ_1) and (u_1,ρ_2) of the hierarchy (5.3)–(5.4) precisely satisfy equations (3.27)–(3.28) (for a specific set of parameters). Hence, the well-posedness of hierarchical solutions of (3.27)–(3.28) follows from the proof of Proposition 5.1, which indeed concerns the well-posedness of the hierarchy (5.3)–(5.4) in the Stokes case $\mathrm{Re}=0$. The proof is analogous in the 2D Navier–Stokes case and is skipped for conciseness. ☐

Appendix C. Derivation of third-order fluid equations

In this section, we extend Proposition 5.3 and derive the corresponding equations to next order as stated in Section 4.2 — we keep the derivation formal and skip detailed error estimates for shortness. By definition of $u_0,u_1,u_2,\rho_0,\rho_1,\rho_2$ in Proposition 5.1, we first note that $(\bar u_\varepsilon,\bar\rho_\varepsilon)$ now satisfies

$$\left\{\begin{array}{l}\mathrm{Re}(\partial_t+\bar u_\varepsilon\cdot\nabla)\bar u_\varepsilon-\Delta\bar u_\varepsilon+\nabla\bar p_\varepsilon=h+\operatorname{div}(\sigma_1[g_1+\varepsilon g_2+\varepsilon^2g_3])\\ \qquad\qquad\qquad\qquad+\operatorname{div}(\sigma_2[\bar\rho_\varepsilon+\varepsilon g_1+\varepsilon^2g_2,\nabla\bar u_\varepsilon])+O(\varepsilon^3),\\ (\partial_t-\frac{1}{\mathrm{Pe}}\Delta+\bar u_\varepsilon\cdot\nabla)\bar\rho_\varepsilon=-\langle U_0n\cdot\nabla_x(\varepsilon g_1+\varepsilon^2g_2)\rangle+O(\varepsilon^3),\end{array}\right.\tag{C.1}$$

and it remains to evaluate the different contributions of g_1,g_2,g_3 in the right-hand side.

C.1. **Equation for particle density.** Using $n=-\frac{1}{d-1}\Delta_nn$, we have

$$\begin{array}{rcl}\langle U_0n\cdot\nabla_x(\varepsilon g_1+\varepsilon^2g_2)\rangle&=&\varepsilon U_0\operatorname{div}\Big(\fint_{\mathbb{S}^{d-1}}n(g_1+\varepsilon g_2)\,dn\Big)\\ &=&-\varepsilon\frac{U_0}{d-1}\operatorname{div}\Big(\fint_{\mathbb{S}^{d-1}}n\Delta_n(g_1+\varepsilon g_2)\,dn\Big).\end{array}$$

Using the defining equations for g_1,g_2, we find

$$\begin{array}{rcl}\Delta_n(g_1+\varepsilon g_2)&=&U_0n\cdot\nabla_x\rho_0+\operatorname{div}_n\big(\pi_n^\perp(\nabla u_0)n\rho_0\big)\\ &&+\varepsilon(\partial_t-\frac{1}{\mathrm{Pe}}\Delta_x+u_0\cdot\nabla_x)g_1+\varepsilon P_1^\perp\big(U_0n\cdot\nabla_x(\rho_1+g_1)\big)\\ &&+\varepsilon\operatorname{div}_n\big(\pi_n^\perp(\nabla u_1)n\rho_0+\pi_n^\perp(\nabla u_0)n(\rho_1+g_1)\big).\end{array}$$

Inserting this identity into the above, as well as the explicit expression for g_1 in Proposition 5.1, then recalling (5.15) and (5.17), and using integral computations (5.16),

$$\begin{aligned}\langle U_0 n\cdot\nabla_x(\varepsilon g_1+\varepsilon^2 g_2)\rangle &= -\varepsilon\tfrac{U_0^2}{d(d-1)}\,\Delta\bar\rho_\varepsilon\\ &\quad-\varepsilon^2\tfrac{2U_0^2}{d(d-1)(d+2)}\,\mathrm{div}\Big(\tfrac{d+1}{d-1}\,\mathrm{D}(\bar u_\varepsilon)\nabla\bar\rho_\varepsilon+\tfrac12\mathrm{div}(\bar\rho_\varepsilon\,\mathrm{D}(\bar u_\varepsilon))\Big)+O(\varepsilon^3).\end{aligned}$$

Inserting this computation into (C.1), we precisely get the claimed equation (4.2).

C.2. **Equation for fluid velocity.** As the obtained equation (4.2) for $\bar\rho_\varepsilon$ shows that homogeneous spatial densities are stable to order $O(\varepsilon^3)$, we can focus for simplicity on the homogeneous setting,

$$\bar\rho_\varepsilon\;\equiv\;\tfrac{1}{\omega_d}+O(\varepsilon^3).$$

In addition, we shall focus on the case of infinite Peclet number and of vanishing particle swimming velocity,

$$\mathrm{Pe}=\infty,\qquad U_0=0,$$

which substantially simplifies the macroscopic fluid equations (recall however that our rigorous results do not hold for $\mathrm{Pe}=\infty$). We start by computing the contribution of the elastic stress σ_1 in (C.1). Using that

$$n\otimes n-\tfrac1d\,\mathrm{Id}=-\tfrac{1}{2d}\Delta_n(n\otimes n-\tfrac1d\,\mathrm{Id}),$$

we have by definition

$$\begin{aligned}\sigma_1[g_1+\varepsilon g_2+\varepsilon^2 g_3] &= \lambda\theta\int_{\mathbb S^{d-1}}(n\otimes n-\tfrac1d\,\mathrm{Id})\,(g_1+\varepsilon g_2+\varepsilon^2 g_3)(\cdot,n)\,dn\\ &= -\lambda\theta\tfrac{1}{2d}\int_{\mathbb S^{d-1}}(n\otimes n-\tfrac1d\,\mathrm{Id})\,\Delta_n(g_1+\varepsilon g_2+\varepsilon^2 g_3)(\cdot,n)\,dn.\end{aligned}$$

Using the defining equations for g_1,g_2,g_3 with $U_0=0$, in form of

$$\begin{aligned}\Delta_n(g_1+\varepsilon g_2+\varepsilon^2 g_3)&=\tfrac{1}{\omega_d}\mathrm{div}_n(\pi_n^\perp(\nabla\bar u_\varepsilon)n)+\varepsilon(\partial_t+\bar u_\varepsilon\cdot\nabla_x)(g_1+\varepsilon g_2)\\ &\qquad+\varepsilon\,\mathrm{div}_n(\pi_n^\perp(\nabla\bar u_\varepsilon)n(g_1+\varepsilon g_2))+O(\varepsilon^3),\end{aligned}$$

the above becomes, after straightforward simplifications and integrations by parts,

$$\begin{aligned}&\sigma_1[g_1+\varepsilon g_2+\varepsilon^2 g_3]_{ij}\\ &=\tfrac{1}{2\omega_d}\lambda\theta\,\mathrm{D}(\bar u_\varepsilon):\int_{\mathbb S^{d-1}}(n\otimes n)(n\otimes n-\tfrac1d\,\mathrm{Id})_{ij}\,dn\\ &\quad-\varepsilon\lambda\theta\tfrac{1}{2d}(\partial_t+\bar u_\varepsilon\cdot\nabla)\int_{\mathbb S^{d-1}}(n\otimes n-\tfrac1d\,\mathrm{Id})_{ij}(g_1+\varepsilon g_2)(\cdot,n)\,dn\\ &\quad+\varepsilon\lambda\theta\tfrac{1}{2d}(\nabla\bar u_\varepsilon)_{kl}\int_{\mathbb S^{d-1}}(\delta_{ik}n_jn_l+\delta_{jk}n_in_l-2n_in_jn_kn_l)\,(g_1+\varepsilon g_2)(\cdot,n)\,dn\\ &\quad+O(\varepsilon^3).\end{aligned}$$

Now inserting the explicit expressions for g_1,g_2 obtained in Proposition 5.1, in form of

$$\begin{aligned}(g_1+\varepsilon g_2)(\cdot,n)&=\tfrac{1}{2\omega_d}(n\otimes n-\tfrac1d\,\mathrm{Id}):\Big(\mathrm{D}(\bar u_\varepsilon)-\varepsilon\tfrac{1}{4d}A_2(\bar u_\varepsilon)+\varepsilon\tfrac1d\,\mathrm{D}(\bar u_\varepsilon)^2\Big)\\ &\quad+\varepsilon\tfrac{1}{8\omega_d}\Big(\big((n\otimes n):\mathrm{D}(\bar u_\varepsilon)\big)^2-\tfrac{2}{d(d+2)}\mathrm{tr}(\mathrm{D}(\bar u_\varepsilon)^2)\Big)+O(\varepsilon^2),\end{aligned}\tag{C.2}$$

using the explicit integrals (5.16), and using that $\mathrm{tr}(A_2(u))=4\mathrm{tr}(\mathrm{D}(u)^2)$, we are led to

$$\begin{aligned}
&\sigma_1[g_1+\varepsilon g_2+\varepsilon^2 g_3]\\
&\quad=\lambda\theta\tfrac{1}{2d(d+2)}A_1(\bar u_\varepsilon)-\varepsilon\lambda\theta\tfrac{1}{d^2(d+2)(d+4)}A_1(\bar u_\varepsilon)^2\\
&\qquad-\varepsilon\lambda\theta\tfrac{1}{4d^2(d+2)}\Big((\partial_t+\bar u_\varepsilon\cdot\nabla)A_1(\bar u_\varepsilon)-(\nabla\bar u_\varepsilon)A_1(\bar u_\varepsilon)-A_1(\bar u_\varepsilon)(\nabla\bar u_\varepsilon)^T\Big)\\
&\qquad+\varepsilon^2\lambda\theta\tfrac{1}{8d^3(d+2)}\Big((\partial_t+\bar u_\varepsilon\cdot\nabla)A_2(\bar u_\varepsilon)-(\nabla\bar u_\varepsilon)A_2(\bar u_\varepsilon)-A_2(\bar u_\varepsilon)(\nabla\bar u_\varepsilon)^T\Big)\\
&\qquad-\varepsilon^2\lambda\theta\tfrac{1}{4d^3(d+4)}\Big((\partial_t+\bar u_\varepsilon\cdot\nabla)A_1(\bar u_\varepsilon)^2-(\nabla\bar u_\varepsilon)A_1(\bar u_\varepsilon)^2-A_1(\bar u_\varepsilon)^2(\nabla\bar u_\varepsilon)^T\Big)\\
&\qquad+\varepsilon^2\lambda\theta\tfrac{1}{4d^3(d+2)(d+4)}\Big(A_1(\bar u_\varepsilon)A_2(\bar u_\varepsilon)+A_2(\bar u_\varepsilon)A_1(\bar u_\varepsilon)\Big)\\
&\qquad-\varepsilon^2\lambda\theta\tfrac{(5d+12)}{4d^3(d+2)(d+4)(d+6)}A_1(\bar u_\varepsilon)^3\\
&\qquad-\varepsilon^2\lambda\theta\tfrac{(d+3)}{4d^2(d+2)^2(d+4)(d+6)}A_1(\bar u_\varepsilon)\mathrm{tr}(A_1(\bar u_\varepsilon)^2)+R_\varepsilon\,\mathrm{Id}+O(\varepsilon^3),
\end{aligned}$$

for some scalar field R_ε that will be absorbed in the pressure field. Now recalling the definition of Rivlin–Ericksen tensors, cf. (3.2), and appealing to the following matrix identities, for any matrix B,

$$\begin{aligned}
-(\nabla u)B-B(\nabla u)^T&=(\nabla u)^TB+B(\nabla u)-\big(A_1(u)B+BA_1(u)\big),\\
(\partial_t+u\cdot\nabla)A_1(u)^2-(\nabla u)A_1(u)^2-A_1(u)^2(\nabla u)^T&=A_1(u)A_2(u)+A_2(u)A_1(u)-3A_1(u)^3,
\end{aligned}$$

we are led to

$$\begin{aligned}
&\sigma_1[g_1+\varepsilon g_2+\varepsilon^2 g_3]\\
&=\lambda\theta\tfrac{1}{2d(d+2)}A_1(\bar u_\varepsilon)-\varepsilon\lambda\theta\tfrac{1}{4d^2(d+2)}A_2(\bar u_\varepsilon)+\varepsilon\lambda\theta\tfrac{1}{2d^2(d+4)}A_1(\bar u_\varepsilon)^2\\
&\quad+\varepsilon^2\lambda\theta\tfrac{1}{8d^3(d+2)}A_3(\bar u_\varepsilon)-\varepsilon^2\lambda\theta\tfrac{3}{8d^3(d+4)}\Big(A_1(\bar u_\varepsilon)A_2(\bar u_\varepsilon)+A_2(\bar u_\varepsilon)A_1(\bar u_\varepsilon)\Big)\\
&\quad+\varepsilon^2\lambda\theta\tfrac{(3d^2+19d+24)}{4d^3(d+2)(d+4)(d+6)}A_1(\bar u_\varepsilon)^3-\varepsilon^2\lambda\theta\tfrac{(d+3)}{4d^2(d+2)^2(d+4)(d+6)}A_1(\bar u_\varepsilon)\mathrm{tr}(A_1(\bar u_\varepsilon)^2)\\
&\quad+R_\varepsilon\,\mathrm{Id}+O(\varepsilon^3).
\end{aligned}$$

Further recalling that $A_1(u)^3=\frac12A_1(u)\mathrm{tr}(A_1(u)^2)+\frac13\mathrm{tr}(A_1(u)^3)$ in dimension $d\le3$ by the Cayley–Hamilton theorem with $\mathrm{tr}(A_1(u))=0$, we obtain

$$\begin{aligned}
&\sigma_1[g_1+\varepsilon g_2+\varepsilon^2 g_3]\\
&=\lambda\theta\tfrac{1}{2d(d+2)}A_1(\bar u_\varepsilon)-\varepsilon\lambda\theta\tfrac{1}{4d^2(d+2)}A_2(\bar u_\varepsilon)+\varepsilon\lambda\theta\tfrac{1}{2d^2(d+4)}A_1(\bar u_\varepsilon)^2\\
&\quad+\varepsilon^2\lambda\theta\tfrac{1}{8d^3(d+2)}A_3(\bar u_\varepsilon)-\varepsilon^2\lambda\theta\tfrac{3}{8d^3(d+4)}\Big(A_1(\bar u_\varepsilon)A_2(\bar u_\varepsilon)+A_2(\bar u_\varepsilon)A_1(\bar u_\varepsilon)\Big)\\
&\quad+\varepsilon^2\lambda\theta\tfrac{3d^2+11d+12}{8d^3(d+2)^2(d+6)}A_1(\bar u_\varepsilon)\mathrm{tr}(A_1(\bar u_\varepsilon)^2)+R_\varepsilon\,\mathrm{Id}+O(\varepsilon^3),
\end{aligned}\tag{C.3}$$

up to modifying the scalar field R_ε.

Next, we turn to the computation of the contribution of the viscous stress σ_2 in (C.1). We have by definition

$$\sigma_2[\tfrac{1}{\omega_d}+\varepsilon g_1+\varepsilon^2 g_2,\nabla\bar u_\varepsilon]\;=\;\lambda\int_{\mathbb{S}^{d-1}}(n\otimes n)(\nabla\bar u_\varepsilon)(n\otimes n)\,(\tfrac{1}{\omega_d}+\varepsilon g_1+\varepsilon^2 g_2)\,\mathrm{d}n,$$

and thus, using the explicit integrals (5.16),

$$\sigma_2[\tfrac{1}{\omega_d} + \varepsilon g_1 + \varepsilon^2 g_2, \nabla \bar u_\varepsilon] \\ = \lambda \tfrac{2}{d(d+2)}\,\mathrm{D}(\bar u_\varepsilon) + \varepsilon\lambda \int_{\mathbb{S}^{d-1}} (n\otimes n)(\nabla \bar u_\varepsilon)(n\otimes n)\,(g_1+\varepsilon g_2)\,\mathrm{d}n + R_\varepsilon\,\mathrm{Id},$$

where again R_ε stands for some scalar field that can be absorbed in the pressure field. Inserting the explicit expressions for g_1, g_2 obtained in Proposition 5.1, in form of (C.2), and computing integrals as above, we easily find

$$\begin{aligned}\sigma_2[\tfrac{1}{\omega_d} + \varepsilon g_1 + \varepsilon^2 g_2, \nabla \bar u_\varepsilon] &= \lambda \tfrac{1}{d(d+2)} A_1(\bar u_\varepsilon) + \varepsilon\lambda \tfrac{1}{d(d+2)(d+4)} A_1(\bar u_\varepsilon)^2\\ &\quad - \varepsilon^2\lambda \tfrac{1}{4d^2(d+2)(d+4)} \big(A_1(\bar u_\varepsilon)A_2(\bar u_\varepsilon) + A_2(\bar u_\varepsilon)A_1(\bar u_\varepsilon)\big)\\ &\quad + \varepsilon^2\lambda \tfrac{(5d+12)}{4d^2(d+2)(d+4)(d+6)} A_1(\bar u_\varepsilon)^3\\ &\quad + \varepsilon^2\lambda \tfrac{(d^2-2d-12)}{8d^2(d+2)^2(d+4)(d+6)} A_1(\bar u_\varepsilon)\mathrm{tr}(A_1(\bar u_\varepsilon)^2) + O(\varepsilon^3) + R_\varepsilon\,\mathrm{Id}\,.\end{aligned}$$

Using again $A_1(u)^3 = \frac12 A_1(u)\mathrm{tr}(A_1(u)^2) + \frac13\mathrm{tr}(A_1(u)^3)$ in dimension $d \le 3$, this becomes

$$\begin{aligned}\sigma_2[\tfrac{1}{\omega_d} + \varepsilon g_1 + \varepsilon^2 g_2, \nabla \bar u_\varepsilon] &= \lambda \tfrac{1}{d(d+2)} A_1(\bar u_\varepsilon) + \varepsilon\lambda \tfrac{1}{d(d+2)(d+4)} A_1(\bar u_\varepsilon)^2\\ &\quad - \varepsilon^2\lambda \tfrac{1}{4d^2(d+2)(d+4)} \big(A_1(\bar u_\varepsilon)A_2(\bar u_\varepsilon) + A_2(\bar u_\varepsilon)A_1(\bar u_\varepsilon)\big)\\ &\quad + \varepsilon^2\lambda \tfrac{(3d^2+10d+6)}{4d^2(d+2)^2(d+4)(d+6)} A_1(\bar u_\varepsilon)\mathrm{tr}(A_1(\bar u_\varepsilon)^2) + R_\varepsilon\,\mathrm{Id} + O(\varepsilon^3).\end{aligned}$$

Inserting this together with (C.3) back into (C.1), we precisely get the claimed third-order fluid equation (4.3). ☐

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(Mitia Duerinckx) Université Libre de Bruxelles, Département de Mathématique, 1050 Brussels, Belgium
Email address: mitia.duerinckx@ulb.be

(Lucas Ertzbischoff - *corresponding author*) Department of Mathematics, Imperial College London, London, SW7 2AZ, United-Kingdom
Email address: l.ertzbischoff@imperial.ac.uk

(Alexandre Girodroux-Lavigne) University of Regensburg, Faculty of Mathematics, Regensburg, Germany
Email address: alexandre.girodroux-lavigne@mathematik.uni-regensburg.de

(Richard M. Höfer) University of Regensburg, Faculty of Mathematics, Regensburg, Germany
Email address: richard.hoefer@ur.de