

STOCHASTIC HOMOGENIZATION OF NONCONVEX UNBOUNDED INTEGRAL FUNCTIONALS WITH CONVEX GROWTH

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We consider the well-travelled problem of homogenization of random integral functionals. When the integrand has standard growth conditions, the qualitative theory is well-understood. When it comes to unbounded functionals, that is, when the domain of the integrand is not the whole space and may depend on the space-variable, there is no satisfactory theory. In this contribution we develop a complete qualitative stochastic homogenization theory for nonconvex unbounded functionals with convex growth. We first prove that if the integrand is convex and has p -growth from below (with $p > d$, the dimension), then it admits homogenization regardless of growth conditions from above. This result, that crucially relies on the existence and sublinearity at infinity of correctors, is also new in the periodic case. In the case of nonconvex integrands, we prove that a similar homogenization result holds provided the nonconvex integrand admits a two-sided estimate by a convex integrand (the domain of which may depend on the space variable) that itself admits homogenization. This result is of interest to the rigorous derivation of rubber elasticity from polymer physics, which involves the stochastic homogenization of such unbounded functionals.

CONTENTS

1. Introduction	2
2. Main results	4
2.1. Convex integrands	5
2.2. Nonconvex integrands with convex growth	8
2.3. Application to nonlinear elasticity	9
2.4. Some improved results	9
3. Proof of the results for convex integrands	13
3.1. Preliminaries	14
3.2. Γ -convergence of truncated energies	15
3.3. Γ -lim inf inequality	15
3.4. Commutation of truncation and homogenization	16
3.5. Proof of Theorem 2.2: Γ -convergence with Neumann boundary data	21
3.6. Proof of Corollary 2.4(i): lifting Dirichlet boundary data	25
3.7. Proof of Corollary 2.4(ii): soft buffer zone for Dirichlet boundary data	26
4. Proof of the results for nonconvex integrands	28
4.1. Definition of the homogenized energy density	28
4.2. Γ -lim inf inequality by blow-up	31
4.3. Γ -lim sup inequality with Neumann boundary data	35
5. Proof of the improved results	41
5.1. Subcritical case $1 < p \leq d$	41
5.2. Minimal soft buffer zone for Dirichlet boundary data	42
5.3. Approximation by periodization in law	45
Acknowledgements	51
A. Appendix	52
A.1. Normal random integrands	52
A.2. Stationary differential calculus in probability	52

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A.3. Ergodic Weyl decomposition	54
A.4. Measurability results	55
A.5. Approximation results	60
References	64

1. INTRODUCTION

Let O be a bounded Lipschitz domain of \mathbb{R}^d , $d, m \geq 1$. We consider the well-travelled problem of homogenization of random integral functionals $I_\varepsilon : W^{1,p}(O; \mathbb{R}^m) \rightarrow [0, \infty]$ given by

$$I_\varepsilon(u) = \int_O W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx,$$

where W is a random Borel function stationary in its first variable that satisfies for almost every $y \in \mathbb{R}^d$ and all $\Lambda \in \mathbb{R}^{m \times d}$ the two-sided estimate

$$\frac{1}{C}|\Lambda|^p - C \leq V(y, \Lambda) \leq W(y, \Lambda) \leq C(1 + V(y, \Lambda)), \quad (1.1)$$

for some $C > 0$, $p > 1$, and a random convex Borel function $V : \mathbb{R}^d \times \mathbb{R}^{m \times d} \rightarrow [0, \infty]$. The originality of the growth condition we consider here is that $V(y, \cdot)$ may take infinite values and that its domain may depend on y , so that the domain of the homogenized integrand \overline{W} (if it exists) is unknown a priori. The motivation for considering such a problem comes from the derivation of nonlinear elasticity from the statistical physics of polymer-chain networks, cf. [4, 26, 20]. Indeed, the free energy of the polymer-chain network is given by two contributions: a steric effect (for which proving homogenization is one of the most important open problems of the field), and the sum of free energies of the deformed chains. The free energy of a single chain is a convex increasing function of the square of the length of the deformed polymer-chain, which blows up at finite deformation. The corresponding problem in a continuum setting would be the homogenization of the nonconvex integrand

$$W(y, \Lambda) = V(y, \Lambda) + g(\det \Lambda) \leq C(1 + V(y, \Lambda)), \quad (1.2)$$

where V is an *infinite-valued convex stationary ergodic integrand* whose domain depends on the space variable, and g is a convex function (in this paper we assume that g is controlled by V , which unfortunately rules out the finite compressibility of matter).

Homogenization of multiple integrals has a long history, and we start with the state of the art when V and W are periodic in the first variable:

- (i) The first contribution (beyond the linear case) is due to Marcellini [34], who addressed the homogenization of *convex periodic* integrands satisfying a *polynomial standard growth condition*, that is (1.1) for $V(y, \Lambda) = |\Lambda|^p$.
- (ii) Marcellini's result was then generalized to *nonconvex periodic* integrands satisfying a *polynomial standard growth condition*, by Braides [11] (which covers in addition almost-periodic coefficients) and Müller [36, Theorem 1.3], independently.
- (iii) In [36, Theorem 1.5], Müller also addressed the case of a *convex periodic* integrand satisfying a *convex standard growth condition* (1.1) for $V(y, \Lambda) = \tilde{V}(\Lambda)$ with $\tilde{V} : \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^+$ a convex finite-valued map, and $p > d$.
- (iv) In [12, Chapter 21], Braides and Defranceschi treated the case of *nonconvex periodic* integrands (see also [18] in the convex case) satisfying (1.1) where V is *convex periodic* and satisfies the *polynomial non-standard growth condition*

$$\frac{1}{C}|\Lambda|^p - C \leq V(y, \Lambda) \leq C(1 + |\Lambda|^q)$$

for some $q < p^*$ (with p^* the Sobolev-conjugate of $p > 1$), and the *doubling property*

$$V(y, 2\Lambda) \leq C(1 + V(y, \Lambda)).$$

- (v) In [13], given a collection of well-separated periodic inclusions, Braides and Garroni treated the case of *nonconvex periodic* integrands satisfying a *polynomial standard growth condition* as well as the (strong) *doubling property*

$$W(y, 2\Lambda) \leq CW(y, \Lambda).$$

outside the inclusions, but only satisfying inside the inclusions a *convex standard growth condition* (1.1) for some possibly unbounded map $V(y, \Lambda)$ that is convex in the Λ -variable.

- (vi) More recently Anza Hafsa and Mandallena studied in [7] the homogenization of *quasiconvex periodic* integrands satisfying a *standard (unbounded) convex growth condition*, that is, (1.1) for $V(y, \Lambda) = \tilde{V}(\Lambda)$ with $\tilde{V} : \mathbb{R}^{m \times d} \rightarrow [0, \infty]$ a convex infinite-valued map such that $\tilde{V}(\Lambda) \geq |\Lambda|^p$, and with $p > d$. Note that in this case the domain is fixed.

When W is random, the results are more sparse:

- (vii) The first contribution beyond the linear case is due to Dal Maso and Modica, who addressed the homogenization of *convex random stationary* integrands satisfying a *polynomial standard growth condition* [19], generalizing Marcellini's result (ii) to the random setting.
- (viii) Messaoudi and Michaille later treated the homogenization of *quasiconvex stationary ergodic* integrands satisfying a *polynomial standard growth condition* [35], following Dal Maso and Modica's approach.
- (ix) In their monograph on homogenization, Jikov, Kozlov and Oleinik treated the case of *convex stationary ergodic* integrands satisfying a *polynomial non-standard growth condition*

$$\frac{1}{C}|\Lambda|^p - C \leq W(y, \Lambda) \leq C(1 + |\Lambda|^q)$$

for some $q < p^*$ (with p^* the Sobolev-conjugate of $p > 1$), see [31, Chapter 15].

For scalar functionals, that is, when $m = 1$, results are much more precise, and we refer the reader to the monograph [17] by Carbone and De Arcangelis (which is however only concerned with the periodic setting) and [15, 16]. When the domain of $V(y, \cdot)$ is not the whole of \mathbb{R}^d , the authors call I_ε an unbounded functional. The main technical tool for *scalar* unbounded functionals is truncation of test-functions (see also [31, Section 15.2]), which cannot be used for systems in general (see however the end of this introduction). In particular, such truncation arguments replace the Sobolev embedding we shall use for systems and allow one to relax the assumption $p > d$ for scalar problems.

In this contribution we give a far-reaching generalization of (i)–(iv) and (vi)–(ix) for systems in the random setting, by relaxing the assumption that the domain of $V(y, \cdot)$ in (1.1) is independent of y . Our contribution also generalizes (v) by relaxing all geometric assumptions. We argue in two steps. For convex integrands, our result shows that homogenization holds without any growth condition from above (cf. Theorem 2.2 for Neumann boundary conditions, and Corollary 2.4 for the more subtle case of Dirichlet boundary conditions), so that we may homogenize the bound V itself. We proceed by truncation of the energy density (following the approach by Müller in [36]), and first prove in Proposition 3.8 that homogenization and truncation commute at the level of the definition of the homogenized energy density. The proof relies on the existence of correctors with stationary gradients for convex problems and exploits quantitatively their sublinearity at infinity, see Lemma 3.4 (which is a substantial difference between the periodic and random cases, and makes the latter more subtle). The second main technical achievement is the construction of recovery sequences in Proposition 3.10 (a gluing argument based on affine boundary data trivially fails since the domain of $V(y, \cdot)$ depends on y — this difficulty is already present in the periodic setting, and prevents us from using the standard homogenization formula with Dirichlet boundary conditions). In the case of nonconvex integrands with a two-sided convex estimate (1.1), we show in Theorem 2.6 that homogenization reduces to the homogenization of the convex bound V . The first obstacle in this program is the definition of the homogenized energy density itself. Indeed, in the absence of correctors (which is a consequence of nonconvexity), one usually defines the homogenized energy density through an asymptotic limit with linear boundary data on increasing cubes. As pointed out above, such an approach fails in general

when the domain of the integrand is not fixed. Instead, in Lemma 4.1, we use the (well-defined) corrector of the associated *convex* problem as boundary data for the *nonconvex* problem on these increasing cubes. Next we argue by blow-up in Proposition 4.3 for the Γ -liminf inequality (following the approach by Fonseca and Müller in [25]), and make a systematic use of the corrector of the convex problem to control the nonconvex energy from above. Then, for the Γ -limsup we argue in Proposition 4.6 by relaxation (following the approach introduced by Fonseca in [24] for relaxation and first used in homogenization by Anza Hafsa and Mandallena in [7]), making a similar use of the corrector of the convex problem in the estimates.

To conclude this introduction, let us go back to our motivation, that is, the homogenization of (1.2). On the one hand, we have reduced the homogenization for such integrands to the homogenization for the convex integrand V . On the other hand, we have proved homogenization for convex integrands without growth condition from above, and therefore proved homogenization for (1.2). In the specific setting of (1.2), we can sharpen the general results described above, by simplifying the definition of the homogenized energy density \overline{W} , cf. Corollary 2.8. We believe a similar approach can be successfully implemented in the discrete setting considered in [4] for the derivation of nonlinear elasticity from polymer physics.

In the scalar case $m = 1$, combining our approach with truncation arguments, we may further refine our general results, in particular relaxing the condition $p > d$. Our approach then improves (and extends to the stochastic setting) some scalar results of [15, 17].

The main results are given in Section 2. The proofs of the results for convex integrands are displayed in Section 3, whereas Section 4 is dedicated to the proofs for nonconvex integrands. In Section 5 we turn to various possible improvements of our general results under additional assumptions. In the appendix we prove several standard and less standard results on approximation of functions, measurability of integral functionals, and Weyl decompositions in probability, that are abundantly used in this article.

2. MAIN RESULTS

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\tau := (\tau_y)_{y \in \mathbb{R}^d}$ be a measurable *ergodic* action of $(\mathbb{R}^d, +)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, that are fixed once and for all throughout the paper. We denote by \mathbb{E} the expectation on Ω with respect to \mathbb{P} .

Consider a map $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ that is τ -stationary in the sense that, for all $\Lambda \in \mathbb{R}^{m \times d}$, all $\omega \in \Omega$, and all $y, z \in \mathbb{R}^d$,

$$W(y, \Lambda, \tau_{-z}\omega) = W(y + z, \Lambda, \omega), \quad (2.1)$$

and assume that $W(y, \cdot, \omega)$ is lower semicontinuous on $\mathbb{R}^{m \times d}$ for almost all y, ω . We also assume in the rest of this paper that, for almost all ω , the map $y \mapsto W(y, \Lambda + u(y), \omega)$ is measurable for all $u \in \text{Mes}(\mathbb{R}^d; \mathbb{R}^{m \times d})$, and that, for almost all $y \in \mathbb{R}^d$, the map $\omega \mapsto W(y, \Lambda + v(\omega), \omega)$ is measurable for all $v \in \text{Mes}(\Omega; \mathbb{R}^{m \times d})$. Continuity in the second variable and joint measurability (in which case W is called a *Carathéodory* integrand) would ensure these properties; weaker sufficient conditions for these properties are given in Appendix A.1. Such integrands W will be called τ -stationary normal random integrands.

We further make the following additional measurability assumption on W :

Hypothesis 2.1. *For any jointly measurable function $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ and any bounded domain $O \subset \mathbb{R}^d$,*

$$\omega \mapsto \inf_{u \in W_0^{1,1}(O; \mathbb{R}^m)} \int_O W(y/\varepsilon, f(y, \omega) + \nabla u(y), \omega) dy$$

is \mathcal{F} -measurable on Ω . □

As discussed in Appendix A.4.2, this last hypothesis is always satisfied if W is convex in the second variable, and more generally if it is sup-quasiconvex (in the sense of Definition A.11), which includes e.g. the case of a sum of a convex integrand and of a “nice” nonconvex part.

For any bounded domain $O \subset \mathbb{R}^d$, we define the following family of random integral functionals, parametrized by $\varepsilon > 0$,

$$I_\varepsilon(\cdot, \cdot; O) : W^{1,1}(O; \mathbb{R}^m) \times \Omega \rightarrow [0, \infty] : (u, \omega) \mapsto I_\varepsilon(u, \omega; O) := \int_O W(y/\varepsilon, \nabla u(y), \omega) dy. \quad (2.2)$$

The aim of this paper is to prove homogenization for I_ε as $\varepsilon \downarrow 0$ under mild growth conditions on W , which we formulate in terms of Γ -convergence for the weak convergence of $W^{1,p}(O; \mathbb{R}^m)$ (for some $p > 1$). When $\Lambda \mapsto W(y, \Lambda, \omega)$ is convex for almost all y, ω , we say that W is a τ -stationary convex normal random integrand, and shall use the notation V and J_ε instead of W and I_ε , that is, for every bounded domain $O \subset \mathbb{R}^d$ and $\varepsilon > 0$,

$$J_\varepsilon(\cdot, \cdot; O) : W^{1,1}(O, \mathbb{R}^m) \times \Omega \rightarrow [0, \infty] : (u, \omega) \mapsto J_\varepsilon(u, \omega; O) := \int_O V(y/\varepsilon, \nabla u(y), \omega) dy. \quad (2.3)$$

The notation W and I_ε will be used for nonconvex integrands. We start our analysis with the case of convex integrands, then turn to nonconvex integrands, present an application to nonlinear elasticity, and conclude with a discussion of several possible improvements of these general results under additional assumptions.

2.1. Convex integrands. In this subsection we state homogenization results for J_ε with (essentially) no growth condition from above. We start with Neumann boundary conditions, and then address the more subtle case of Dirichlet boundary conditions.

2.1.1. Homogenization with Neumann boundary conditions. Our first result is as follows.

Theorem 2.2 (Convex integrands with Neumann boundary data). *Let $V : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ be a τ -stationary convex normal random integrand that satisfies the following uniform coercivity condition: there exist $C > 0$ and $p > d$ such that for almost all ω and y , we have for all Λ ,*

$$\frac{1}{C} |\Lambda|^p - C \leq V(y, \Lambda, \omega). \quad (2.4)$$

Assume that the convex function $M := \sup \text{ess}_{y, \omega} V(y, \cdot, \omega)$ has 0 in the interior of its domain. Then, for almost all $\omega \in \Omega$ and all bounded Lipschitz domains $O \subset \mathbb{R}^d$, the integral functionals $J_\varepsilon(\cdot, \omega; O)$ Γ -converge to the integral functional $J(\cdot; O) : W^{1,p}(O; \mathbb{R}^m) \rightarrow [0, \infty]$ defined by

$$J(u; O) = \int_O \bar{V}(\nabla u(y)) dy,$$

for some lower semicontinuous convex function $\bar{V} : \mathbb{R}^{m \times d} \rightarrow [0, \infty]$ characterized by the following three equivalent formulas:

(i) *Formula in probability: for all $\Lambda \in \mathbb{R}^{m \times d}$,*

$$\bar{V}(\Lambda) = \inf_{g \in F_{\text{pot}}^p(\Omega)^m} \mathbb{E}[V(0, \Lambda + g, \cdot)], \quad (2.5)$$

where the space of mean-zero potential random variables $F_{\text{pot}}^p(\Omega)$ is recalled in Section 3.1.2.

(ii) *Dirichlet formula with truncation: for any increasing sequence $V^k \uparrow V$ of τ -stationary convex random integrands that satisfy the standard p -growth condition*

$$\frac{1}{C} |\Lambda|^p - C \leq V^k(y, \Lambda, \omega) \leq C_k (1 + |\Lambda|^p) \quad (2.6)$$

for all y, ω, Λ , and some sequence $C_k < \infty$, we have for almost all ω , all Λ , and all bounded Lipschitz domains $O \subset \mathbb{R}^d$,

$$\bar{V}(\Lambda) = \lim_{k \uparrow \infty} \lim_{\varepsilon \downarrow 0} \inf_{\phi \in W_0^{1,p}(O/\varepsilon; \mathbb{R}^m)} \int_{O/\varepsilon} V^k(y, \Lambda + \nabla \phi(y), \omega) dy. \quad (2.7)$$

(iii) *Convexification formula: for all Λ and all bounded Lipschitz domains $O \subset \mathbb{R}^d$, we have, for almost all ω ,*

$$\bar{V}(\Lambda) = \lim_{t \uparrow 1} \lim_{\varepsilon \downarrow 0} \inf_{\substack{\phi \in W^{1,p}(O/\varepsilon; \mathbb{R}^m) \\ \int_{O/\varepsilon} \nabla \phi = 0}} \int_{O/\varepsilon} V(y, t\Lambda + \nabla \phi(y), \omega) dy. \quad (2.8)$$

As a consequence of convexity, the limit $t \uparrow 1$ can be omitted when $\Lambda \notin \partial \text{dom} \bar{V}$. \square

Comments are in order:

- The limit $t \uparrow 1$ cannot be omitted in (2.8) in general for $\Lambda \in \partial \text{dom} \bar{V}$. Indeed, let V coincide with a convex map $\tilde{V} : \mathbb{R}^{m \times d} \rightarrow [0, \infty]$ with a closed domain, and which is not lower semicontinuous at the boundary of its domain. In the interior of $\text{dom} \tilde{V}$, \bar{V} coincides with \tilde{V} . However, since \bar{V} is necessarily lower semicontinuous on its domain, it cannot coincide with \tilde{V} on $\partial \text{dom} \tilde{V}$.
- In the proof we take (2.7) as the defining formula for \bar{V} , following the approach by Müller in [36]. Formula (2.5) is interesting in two respects: first, it is intrinsic (no approximation is required), and second it is an exact formula (there is no asymptotic limit involved). The equivalence of both formulas, which can be interpreted as the commutation of truncation and homogenization, is the key to the proof of the Γ -convergence result.

We may extend Theorem 2.2 in two directions:

- The extension of Theorem 2.2 to the case of domains with holes (or more generally to soft inclusions, for which the coercivity assumption (2.4) does not hold everywhere) is straightforward provided we have a suitable extension result. When holes are well-separated, such extension results are standard (see e.g. [31, Sections 3.1]). For general situations however, this can become a subtle issue (see in particular [31, Sections 3.1 and 3.5]). In the particular case of the periodic setting, there is a very general extension result [1], which is used e.g. in [13].
- The assumption $p > d$ is crucial in the generality of Theorem 2.2, which is used in the form of the Sobolev embedding of $W^{1,p}(O; \mathbb{R}^m)$ in $L^\infty(O; \mathbb{R}^m)$. In the case $1 < p \leq d$, the conclusions of the theorem still hold true provided that $V(y, \Lambda, \omega) \leq M(\Lambda)$ for some convex function $M : \mathbb{R}^{m \times d} \rightarrow \mathbb{R}$ that satisfies the growth condition $\lim_{|\Lambda| \rightarrow \infty} M(\Lambda)/|\Lambda|^q < \infty$ for $q = dp/(d-p)$ if $p < d$ or for some $q < \infty$ if $p = d$. In the scalar case $m = 1$, the use of the Sobolev embedding can be avoided by a truncation argument, as explained in Corollary 2.9 below, see also [17, 15].

2.1.2. Dirichlet boundary conditions. We now discuss the homogenization result in the case of Dirichlet boundary conditions (the case of mixed boundary data can then be dealt with in a straightforward way, and we leave the details to the reader). A first remark is that Dirichlet data have to be well-prepared, as the following elementary example shows.

Example 2.3. Consider random unit spherical inclusions centered at the points of a Poisson point process, choose the integrand V to be equal to $|\Lambda|^p$ outside the inclusions and to have a bounded domain $\mathcal{D} \subset \mathbb{R}^{m \times d}$ inside the inclusions. Given a (nonempty) bounded open set $O \subset \mathbb{R}^d$, for almost all ω , the realization of the inclusions corresponding to ω intersects $\partial(O/\varepsilon)$ for infinitely many $\varepsilon > 0$, and hence $\limsup_{\varepsilon} J_{\varepsilon}(u + \Lambda \cdot x, \omega; O) = \infty$ for all $u \in W_0^{1,p}(O; \mathbb{R}^m)$, due to the Dirichlet boundary condition. In contrast, if the intensity of the underlying Poisson process is not too big, it is easily seen that the homogenized integral functional J defined in Theorem 2.2 is finite-valued. This proves that, for all $\Lambda \notin \mathcal{D}$ and almost all ω , $J_{\varepsilon}(\cdot + \Lambda \cdot x, \omega; O)$ cannot Γ -converge to $J(\cdot; O)$ on $W_0^{1,p}(O; \mathbb{R}^m)$, due to the intersection of some rigid inclusions with the boundary of the domain where the Dirichlet condition is imposed. \square

We propose two ways to prepare Dirichlet data:

- by relaxing the boundary data so that the energy remains finite for all $\varepsilon > 0$ while ensuring that the boundary data are recovered at the limit $\varepsilon \downarrow 0$ — we call “lifting” this procedure;
- by replacing the integrand V by a softer integrand on a neighborhood of the boundary where the boundary condition is imposed — we call this a “soft buffer zone”.

Corollary 2.4 (Convex integrands with Dirichlet boundary data). *Let V , M , J_ε , and J be as in Theorem 2.2 for some $p > d$. Then, for almost all $\omega \in \Omega$ and all bounded Lipschitz domains $O \subset \mathbb{R}^d$, we have*

(i) *For all boundary data $u \in W^{1,p}(O; \mathbb{R}^m)$ such that $J(\alpha u; O) < \infty$ for some $\alpha > 1$, there exists a lifted sequence $(u_\varepsilon)_\varepsilon$ with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O; \mathbb{R}^m)$, such that we have on $W_0^{1,p}(O; \mathbb{R}^m)$:*

$$\begin{aligned} J(\cdot + u; O) &= \Gamma\text{-}\lim_{t \uparrow 1} \Gamma\text{-}\liminf_{\varepsilon \downarrow 0} J_\varepsilon(\cdot + tu_\varepsilon, \omega; O) \\ &= \Gamma\text{-}\lim_{t \uparrow 1} \Gamma\text{-}\limsup_{\varepsilon \downarrow 0} J_\varepsilon(\cdot + tu_\varepsilon, \omega; O). \end{aligned}$$

In particular,

$$\begin{aligned} \inf_{v \in W_0^{1,p}(O)} J(v + u; O) &= \lim_{t \uparrow 1} \liminf_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O)} J_\varepsilon(v + tu_\varepsilon; O) \\ &= \lim_{t \uparrow 1} \limsup_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O)} J_\varepsilon(v + tu_\varepsilon; O). \end{aligned}$$

If u satisfies the additional condition $\int_O M(\nabla u(y)) dy < \infty$, then we may take $u_\varepsilon \equiv u$, and if this condition is strengthened to $\int_O M(\alpha \nabla u(y)) dy < \infty$ for some $\alpha > 1$ then the limit $t \uparrow 1$ can be omitted.

(ii) *For all boundary data $u \in W^{1,p}(O; \mathbb{R}^m)$ such that $J(u; O) < \infty$, we have on $W_0^{1,p}(O; \mathbb{R}^m)$:*

$$\begin{aligned} J(\cdot + u; O) &= \Gamma\text{-}\lim_{t \uparrow 1, \eta \downarrow 0} \Gamma\text{-}\liminf_{\varepsilon \downarrow 0} J_\varepsilon^\eta(\cdot + tu, \omega; O) \\ &= \Gamma\text{-}\lim_{t \uparrow 1, \eta \downarrow 0} \Gamma\text{-}\limsup_{\varepsilon \downarrow 0} J_\varepsilon^\eta(\cdot + tu, \omega; O), \end{aligned}$$

where J_ε^η is the following modification of J_ε on an η -neighborhood of ∂O :

$$\begin{aligned} J_\varepsilon^\eta(v, \omega; O) &:= \int_O V_\varepsilon^{O, \eta}(y, \nabla v(y), \omega) dy, \\ V_\varepsilon^{O, \eta}(y, \Lambda, \omega) &:= \begin{cases} V(y/\varepsilon, \Lambda, \omega), & \text{if } \text{dist}(y, \partial O) > \eta; \\ |\Lambda|^p, & \text{if } \text{dist}(y, \partial O) < \eta. \end{cases} \end{aligned} \quad (2.9)$$

In particular,

$$\begin{aligned} \inf_{v \in W_0^{1,p}(O)} J(v + u; O) &= \lim_{t \uparrow 1, \eta \downarrow 0} \liminf_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O)} J_\varepsilon^\eta(v + tu; O) \\ &= \lim_{t \uparrow 1, \eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O)} J_\varepsilon^\eta(v + tu; O). \end{aligned}$$

If u satisfies the additional condition $J(\alpha u; O) < \infty$ for some $\alpha > 1$, then the limit $t \uparrow 1$ can be omitted. \square

Comments are in order:

- The results of Corollary 2.4 are not completely satisfactory. Indeed, if one makes a diagonal extraction of t and η wrt ε to obtain a Γ -convergence in ε only, then the extraction for the Γ -limsup depends on the target function $v + u$ and not only on the boundary data u as we would hope for. This dependence is however restricted to a dilation parameter only in the case of the lifting. In the case of the buffer zone, the result can be (optimally) improved to $\eta_\varepsilon = \theta \varepsilon$ for any fixed $\theta > 0$, under some additional structural assumption in the form of the existence of stationary quasi-correctors (see Proposition 5.1). For specific examples for which this assumption holds, see Corollary 2.10 below.
- In the specific situation when $\text{dom} \bar{V} = \text{dom} M$ (this is trivially the case when the domain is fixed, i.e. $\text{dom} V(y, \cdot, \omega) = \text{dom} M$ for almost all y, ω), then the strong assumptions $\int_O M(\nabla u(y)) dy < \infty$ and $\int_O M(\alpha \nabla u(y)) dy < \infty$ (for some $\alpha > 1$) in part (i) of the statement reduce to the simpler assumptions $J(u; O) < \infty$ and $J(\alpha u; O) < \infty$ (for some $\alpha > 1$), respectively. In particular, in that situation, the lifting can always be chosen to be trivial, $u_\varepsilon := u$ for all ε .

- In [13] (see also [12, Chapter 20]), Braides and Garroni prepare the boundary data in a different way in the specific case of stiff inclusions. In particular they introduce an operator R^ε which acts on functions u as follows: on each stiff inclusion $R^\varepsilon(u)$ has value the average of u on the considered stiff inclusion, away from all inclusions $R^\varepsilon(u)$ coincides with u , and in between $R^\varepsilon(u)$ is an interpolation between u and the average of u on the inclusion. Such a construction can be used here as well, but seem to admit no natural generalization in other settings than stiff inclusions.

2.2. Nonconvex integrands with convex growth. In the case when W is nonconvex and admits a two-sided estimate by a convex function (which may depend on the space variable), we show that a Γ -convergence result similar to the convex case holds. Before we precisely state this result, let us recall the notion of radial uniform upper semicontinuity (which is trivially satisfied by convex maps).

Definition 2.5. A map $Z : \mathbb{R}^{m \times d} \rightarrow [0, \infty]$ is said to be *ru-usc* (i.e. *radially uniformly upper semicontinuous*) if there is some $\alpha \geq 0$ such that the function

$$\Delta_Z^\alpha(t) = \sup_{\Lambda \in \text{dom} Z} \frac{Z(t\Lambda) - Z(\Lambda)}{\alpha + Z(\Lambda)}$$

satisfies $\limsup_{t \uparrow 1} \Delta_Z^\alpha(t) \leq 0$. A τ -stationary normal random integrand W is said to be *ru-usc* if there exists a τ -stationary integrable random field $a : \mathbb{R}^d \times \Omega \rightarrow [0, \infty]$ such that the function

$$\Delta_W^a(t) := \sup_{y \in \mathbb{R}^d} \text{ess sup}_{\omega \in \Omega} \sup_{\Lambda \in \text{dom} W(y, \cdot, \omega)} \frac{W(y, t\Lambda, \omega) - W(y, \Lambda, \omega)}{a(y, \omega) + W(y, \Lambda, \omega)}$$

satisfies $\limsup_{t \uparrow 1} \Delta_W^a(t) \leq 0$. □

The following result is a far-reaching generalization of [36, Theorem 1.5] to a wide class of *random and nonconvex* integrands; it is also a substantial extension of [7, Corollary 2.2].

Theorem 2.6 (Nonconvex integrands with convex growth). *Let $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ be a (nonconvex) ru-usc τ -stationary normal random integrand satisfying Hypothesis 2.1. Assume that, for almost all ω , y , and for all Λ ,*

$$V(y, \Lambda, \omega) \leq W(y, \Lambda, \omega) \leq C(1 + V(y, \Lambda, \omega)), \quad (2.10)$$

for some $C > 0$ and some τ -stationary convex random integrand $V : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ that satisfies the assumptions of Theorem 2.2 for some $p > d$. Then, for almost all $\omega \in \Omega$ and all bounded Lipschitz domains $O \subset \mathbb{R}^d$, the integral functionals $I_\varepsilon(\cdot, \omega; O)$ Γ -converge to the integral functional $I(\cdot; O) : W^{1,p}(O; \mathbb{R}^m) \rightarrow [0, \infty]$ defined by

$$I(u; O) = \int_O \overline{W}(\nabla u(y)) dy,$$

for some ru-usc lower semicontinuous quasiconvex function $\overline{W} : \mathbb{R}^{m \times d} \rightarrow [0, \infty]$ that satisfies $\overline{V} \leq \overline{W} \leq C(1 + \overline{V})$, where \overline{V} is the homogenized integrand associated with V by Theorem 2.2. In addition, the results stated in Corollary 2.4 for J_ε also hold for I_ε .

For all $\Lambda \in \mathbb{R}^{m \times d}$, let g_Λ be the potential field in probability minimizing $\mathbb{E}[V(0, \Lambda + \cdot)]$ (cf. (2.5)), and note that $x \mapsto g_\Lambda(\tau_x \omega)$ is a gradient field on \mathbb{R}^d for almost all $\omega \in \Omega$, which we denote by $\nabla \varphi_\Lambda(x, \omega)$. The homogenized integrand \overline{W} is characterized for all $\Lambda \in \mathbb{R}^{m \times d}$ by

$$\overline{W}(\Lambda) = \liminf_{t \uparrow 1} \liminf_{\Lambda' \rightarrow \Lambda} \lim_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O/\varepsilon; \mathbb{R}^m)} \int_{O/\varepsilon} W(y, t\Lambda' + t\nabla \varphi_{\Lambda'}(y, \omega) + \nabla v(y), \omega) dy. \quad (2.11)$$

for any bounded Lipschitz domain $O \subset \mathbb{R}^d$ and almost every $\omega \in \Omega$. □

In general, we do not expect that the limits $t \uparrow 1$ and $\Lambda' \rightarrow \Lambda$ can be dropped in (2.11), see however Corollary 2.8 below under Hypothesis 2.7.

2.3. Application to nonlinear elasticity. In the example from the statistical physics of polymer-chain networks, the integrand has the specific decomposition (1.2). Moreover, the nonconvex part of the integrand satisfies the following assumption, which in particular implies that W satisfies Hypothesis 2.1 (see indeed Lemma A.12), as well as the ru-usc property.

Hypothesis 2.7. *There exists a τ -stationary convex map $V : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ and some $p > 1$ such that*

$$W(y, \Lambda, \omega) = V(y, \Lambda, \omega) + W^{nc}(y, \Lambda, \omega),$$

where $W^{nc} : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ is a (nonconvex) τ -stationary normal random integrand satisfying the p -th order upper bound and the local Lipschitz conditions, for some $C > 0$,

(i) for almost all y, ω , for all Λ ,

$$W^{nc}(y, \Lambda, \omega) \leq C(|\Lambda|^p + 1);$$

(ii) for almost all y, ω , for all Λ, Λ' ,

$$|W^{nc}(y, \Lambda, \omega) - W^{nc}(y, \Lambda', \omega)| \leq C(1 + |\Lambda|^{p-1} + |\Lambda'|^{p-1})|\Lambda - \Lambda'|.$$

□

Under this assumption, the following variant of Theorem 2.6 holds and yields in particular a simpler formula for the homogenized energy density.

Corollary 2.8. *Let $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ be a (nonconvex) τ -stationary normal random integrand satisfying Hypothesis 2.7. Assume that, for almost all ω, y , and for all Λ ,*

$$V(y, \Lambda, \omega) \leq W(y, \Lambda, \omega) \leq C(1 + V(y, \Lambda, \omega)), \quad (2.12)$$

for some $C > 0$, where $V : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ satisfies the assumptions of Theorem 2.2 for some $p > d$. Then the conclusions of Theorem 2.6 hold true. In addition we have for all $\Lambda \in \mathbb{R}^{m \times d}$ for almost all ω

$$\overline{W}(\Lambda) = \lim_{R \uparrow \infty} \inf_{v \in W_0^{1,p}(Q_R; \mathbb{R}^m)} \int_{Q_R} W(y, \Lambda + \nabla \varphi_\Lambda(y, \omega) + \nabla v(y), \omega) dy, \quad (2.13)$$

where φ_Λ is as in Theorem 2.6. □

The behavior of \overline{W} close to the boundary of its domain is of particular interest. From a mechanical point of view, in the case of (1.2) with V satisfying $V(y, \Lambda_n) \rightarrow \infty$ as $\text{dist}(\Lambda_n, \partial \text{dom} V(y, \cdot)) \rightarrow 0$, we expect that $\overline{W}(\Lambda_n) \rightarrow \infty$ as $\text{dist}(\Lambda_n, \partial \text{dom} \overline{W}) \rightarrow 0$. Except in a few explicit examples, we do not know under which condition and how to prove such a property. In the case of the geometry that saturates the Hashin-Shtrikman bound (obtained by a stationary and statistically isotropic Vitali covering of \mathbb{R}^d by balls, cf. [31, Section 6.2]), where each ball is the homothetic image of a unit ball with a given spherical inclusion, the corrector gradient field is explicit when Λ is a dilation, and $t \mapsto \overline{W}(t \text{Id})$ indeed blows up when t approaches the boundary of the domain.

2.4. Some improved results. The general results above naturally call for various questions concerning possible improvements:

- What about the subcritical case $1 < p \leq d$?
- What is the minimal size η_ε of the soft buffer zone needed in the presence of Dirichlet boundary conditions (see Corollary 2.4(ii))? Under which conditions can we take $\eta_\varepsilon = \theta \varepsilon$ for some constant $\theta > 0$?
- What about periodic boundary data? In particular, can one approximate \overline{V} by periodization (in law) in the spirit of [22]?

These three questions are partially addressed below under various additional assumptions.

2.4.1. *Subcritical case* $1 < p \leq d$. The first improvement concerns the growth condition from below in Theorem 2.2. It is relaxed here to any $p > 1$ in the convex scalar case $m = 1$ under the additional assumption that V has fixed domain. The idea is to avoid the use of Sobolev embedding by using suitable truncations (in the spirit of e.g. [17, proof of Lemma 13.1.5]), which are indeed only available in the scalar case with fixed domain. We recover in particular in this way the results of [17, Section 13.4] in Sobolev spaces.

Corollary 2.9 (Subcritical case). *Let V and M satisfy the assumptions of Theorem 2.2 in the scalar case $m = 1$, for some $p > 1$. Also assume that $\text{dom}V(y, \cdot, \omega) = \text{dom}M$ for almost all y, ω . Then the conclusions of Theorem 2.2 and Corollary 2.4 hold true for this $p > 1$. \square*

2.4.2. *Minimal soft buffer zone*. The second improvement concerns the size of the buffer zone for Dirichlet boundary data, at least for affine target functions. The minimal size $\eta_\varepsilon = \theta\varepsilon$, for any constant $\theta > 0$, is achieved under the technical structural assumption that stationary quasi-correctors exist, cf. Proposition 5.1. Understanding the validity of this assumption in general seems to be a difficult question of functional analytic nature. It is trivially satisfied in the periodic case. It is also valid provided that truncations are available, which holds in the scalar case with fixed domain.

Corollary 2.10 (Minimal soft buffer zone). *Let V, J_ε, J, M and W, I_ε, I be as in Theorems 2.2 and 2.6 for some $p > 1$. Also assume that one of the following holds:*

- (1) $p > d$, and, for all Λ, ω , $V(\cdot, \Lambda, \omega)$ and $W(\cdot, \Lambda, \omega)$ are Q -periodic;
- (2) $m = 1$, and $\text{dom}V(y, \cdot, \omega) = \text{dom}M$ is open for almost all y, ω .

Then, for all Λ , for almost all $\omega \in \Omega$,

$$\bar{V}(\Lambda) = \lim_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O)} \int_O V_\varepsilon^{O, \theta\varepsilon}(y, \Lambda + \nabla v(y), \omega) dy,$$

where $V_\varepsilon^{O, \theta\varepsilon}$ is defined as in (2.9) with $\eta = \theta\varepsilon$. The same result also holds for V, J_ε, J replaced by W, I_ε, I (for $p > d$). \square

2.4.3. *Approximation by periodization*. The last improvement concerns the approximation of \bar{V} by periodization. As for Dirichlet boundary conditions, periodic boundary conditions require “well-prepared data”. In this case, the well-preparedness is formalized in terms of periodization of the law of the energy density (as opposed to the more naive periodization in space, cf. Figure 1):

Definition 2.11. A collection $(V^R)_{R>0}$ of random maps $V^R : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ is said to be an *admissible periodization in law* for V if

- (i) Periodicity in law: for all $R > 0$, V^R is $Q_R = [-\frac{R}{2}, \frac{R}{2}]^d$ -periodic and is stationary with respect to translations on the torus $\mathbb{T}_R = (\mathbb{R}/R\mathbb{Z})^d$ (more precisely there exists a measurable action τ^R of \mathbb{T}_R on Ω such that, for all $y = y_1 + y_2$ with $y_1 \in (R\mathbb{Z})^d$ and $y_2 \in Q_R$, we have $V^R(y, \cdot, \omega) = V^R(y_2, \cdot, \omega) = V^R(0, \cdot, \tau_{-y_2}^R \omega)$);
- (ii) Stabilization property: for all $\theta \in (0, 1)$ and for almost all ω , there exists $R_\theta(\omega) > 0$ such that for all $R > R_\theta(\omega)$ we have $V^R(\cdot, \cdot, \omega)|_{Q_{\theta R} \times \mathbb{R}^{m \times d}} = V(\cdot, \cdot, \omega)|_{Q_{\theta R} \times \mathbb{R}^{m \times d}}$. \square

Remark 2.12. It may seem quite unnatural to make in this definition the action τ^R depend on R . Another definition would consist in keeping the same integrand V and action τ but letting the probability measure vary according to $\mathbb{P}_R := (T_R)_* \mathbb{P}$, which amounts to defining $V^R(y, \Lambda, \omega) = V(y, \Lambda, T_R(\omega))$, for some projection $T_R : \Omega \rightarrow \Omega$ with suitable properties such as the τ -invariance of the set $T_R(\Omega)$. As shown in the examples below, T_R will typically be the (natural) Q_R -periodization of an underlying Poisson process, and $T_R(\Omega)$ will be the set of Q_R -periodic realizations (that is of \mathbb{P} -measure 0). Note that this is a particular case of the formalism of Definition 2.11 above, for the conjugated action $\tau_y^R := \tau_y \circ T_R$.

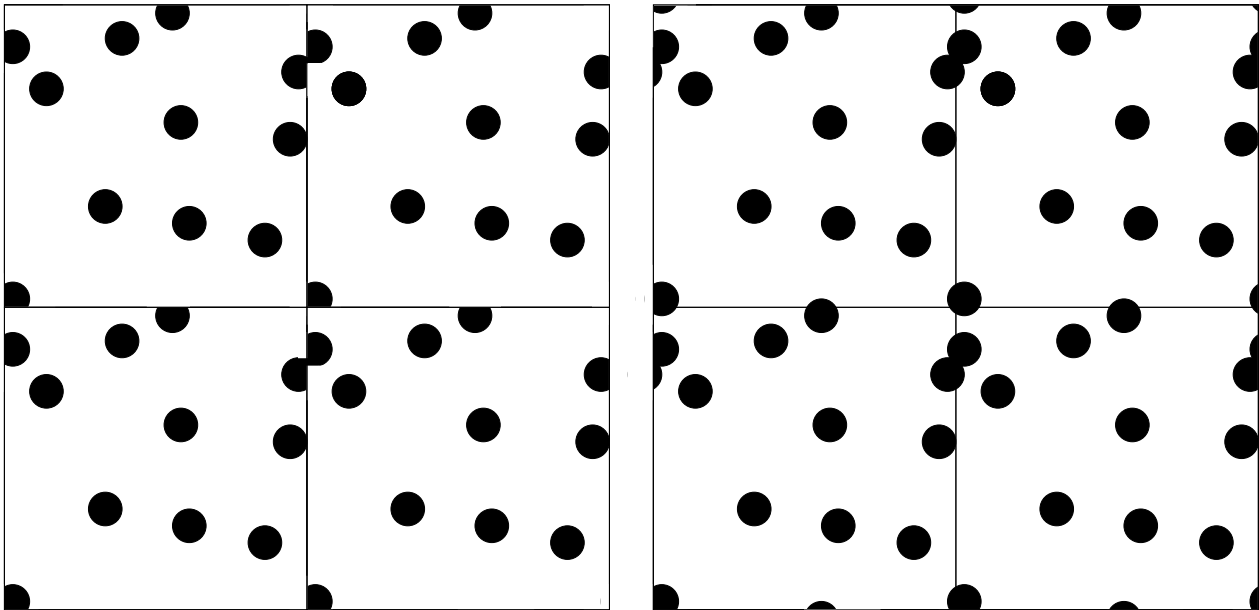


FIGURE 1. For Poisson random inclusions: periodization in space (left) versus periodization in law (right), four periods.

The main goal here is to prove an approximation formula for \bar{V} and \bar{W} by periodization in law, that is an asymptotic formula of the following form: for all $\Lambda \in \mathbb{R}^{m \times d}$, for almost all ω ,

$$\bar{V}(\Lambda) \sim \lim_{R \uparrow \infty} \inf_{u \in W_{\text{per}}^{1,p}(Q_R)} \int_{Q_R} V^R(y, \Lambda + \nabla u(y), \omega) dy, \quad (2.14)$$

and similarly for \bar{W} . Such a formula would be of particular interest for numerical approximations of \bar{V} and \bar{W} . The notion of periodization in law was introduced in [22]. It crucially differs from “periodization in space”, which would consist in replacing the argument of the limit in (2.14) by

$$\inf_{u \in W_{\text{per}}^{1,p}(Q_R)} \int_{Q_R} V(y, \Lambda + \nabla u(y), \omega) dy.$$

The difference between both types of periodization is illustrated on Figure 1 for Poisson random inclusions. Empirical results in [22] tend to show that periodization in law is more precise than periodization in space for the approximation of homogenized coefficients in the case of discrete linear elliptic equations, whereas [27, 29] provide with a complete numerical analysis (convergence rate, quantitative central limit theorem) of this method when the coefficients are independent and identically distributed. In the case of the unbounded integral functionals considered here, it is not clear whether the approximation by periodization *in space* converges in general. Note that for problems with discrete stationarity (like the random chessboard or periodic structures) such periodizations are not needed but the sequences ε, R have to be discrete: $\varepsilon_k = 1/k, R = k$ with $k \in \mathbb{N}$.

Before we state our results, let us shortly discuss periodization in law on the example of random inclusions. Let

$$V(y, \Lambda, \omega) := a(\Lambda) \mathbb{1}_{\mathbb{R}^d \setminus E(\omega)}(y) + b(\Lambda) \mathbb{1}_{E(\omega)}(y), \quad (2.15)$$

where $a, b : \mathbb{R}^{m \times d} \rightarrow [0, \infty]$ are convex functions and $E(\omega)$ is a stationary random set (to be thought of as a collection of inclusions). A typical choice for E is $E = \bigcup_{n=1}^{\infty} B(q_n)$, where $\rho = (q_n)_n$ is a stationary point process in \mathbb{R}^d . For $R > 0$, set $\rho^R = (q_{n,R})_n$, $E^R := \bigcup_{n=1}^{\infty} B(q_{n,R})$, and define

$V^R(y, \Lambda, \omega) := a(\Lambda)\mathbb{1}_{\mathbb{R}^d \setminus E^R(\omega)}(y) + b(\Lambda)\mathbb{1}_{E^R(\omega)}(y)$. For V^R to be an admissible periodization in law of V , it is sufficient that $(\rho^R)_R$ is an admissible periodization of ρ in the following sense:

- (i') Periodicity in law: for all $R > 0$, ρ^R is Q_R -periodic and τ^R -stationary with respect to translations on the torus $\mathbb{T}_R = (\mathbb{R}/R\mathbb{Z})^d$;
- (ii') Stabilization: for all $\theta \in (0, 1)$ and for almost all ω there is $R_\theta(\omega) > 0$ so that for all $R \geq R_\theta(\omega)$, $\rho^R(\omega) \cap Q_{\theta R} = \rho(\omega) \cap Q_{\theta R}$.

Let us now describe three typical examples, for which the periodization in law can be explicitly constructed:

- (a) *Poisson point process*. If ρ is a Poisson point process on \mathbb{R}^d , a suitable periodization in law is given by $\rho^R = \bigcup_{z \in \mathbb{Z}^d} (Rz + \rho \cap Q_R)$. Property (i') is satisfied by complete independence of the Poisson process, while property (ii') holds for $\theta = 1$.
- (b) *Random parking point process*. We now let ρ be the random parking measure on \mathbb{R}^d defined in [38]. Penrose's graphical construction is as follows: ρ is obtained as a transformation $\rho := T(\rho_0)$ of a Poisson point process ρ_0 on $\mathbb{R}^d \times \mathbb{R}^+$. Following [30, Remark 5], we may periodize the underlying Poisson process on Q_R as above by setting $\rho_0^R := \bigcup_{z \in \mathbb{Z}^d} ((Rz, 0) + \rho_0 \cap (Q_R \times \mathbb{R}^+))$, and define a periodization of ρ as $\rho^R := T(\rho_0^R)$. Property (i') holds by construction. Property (ii') is more subtle and relies on the stabilization properties of the random parking measure. By a union bound argument we may rephrase the exponential stabilization of [39, Lemma 3.5] as follows: There exist $K, \kappa > 0$ and for all $R > 0$ there exists a random variable t_R such that
 - $\rho \cap Q_R$ does not depend on $\rho_0 \cap ((\mathbb{R}^d \setminus Q_{t_R}) \times \mathbb{R}^+)$, and in particular, for any locally finite subset $F \subset (\mathbb{R}^d \setminus Q_{t_R}) \times \mathbb{R}^+$, we have

$$Q_R \cap T(F \cup (\rho_0 \cap (Q_{t_R} \times \mathbb{R}^+))) = Q_R \cap T(\rho_0 \cap (Q_{t_R} \times \mathbb{R}^+));$$

— for all $L \geq 0$, $\mathbb{P}[t_R \geq L] \leq R^d K e^{-\kappa(L-R)}$.

Hence, for all $\theta \in (0, 1)$,

$$\begin{aligned} & \sum_{R=1}^{\infty} \mathbb{P}[\rho \cap Q_{\theta R} \text{ does depend on } \rho_0 \cap ((\mathbb{R}^d \setminus Q_R) \times \mathbb{R}^+)] \\ & \leq \sum_{R=1}^{\infty} \mathbb{P}[t_{\theta R} \geq R] \leq \sum_{R=1}^{\infty} (\theta R)^d K e^{-\kappa(1-\theta)R} < \infty. \end{aligned}$$

By the Borel-Cantelli lemma, $\rho \cap Q_{\theta R}$ does eventually not depend on $\rho_0 \cap ((\mathbb{R}^d \setminus Q_R) \times \mathbb{R}^+)$ as $R \uparrow \infty$ almost surely. In particular, for all $\theta \in (0, 1)$ and for almost all ω , there exists some $R_\omega > 0$ with the following property: for all $R \geq R_\omega$, $\rho(\omega) \cap Q_{\theta R}$ does not depend on $\rho_0(\omega) \cap ((\mathbb{R}^d \setminus Q_R) \times \mathbb{R}^+)$, so that $\rho^R(\omega) \cap Q_{\theta R} = \rho(\omega) \cap Q_{\theta R}$, that is, (ii').

- (c) *Hardcore Poisson point process*. As noticed in [21, Step 1 of Section 4.2] (in a more general context), a hardcore Poisson point process ρ on \mathbb{R}^d can be obtained by applying the transformation T of (b) to a Poisson point process ρ_0 on $\mathbb{R}^d \times (0, 1)$, that is, $\rho = T(\rho_0)$. The same argument as in (b) (with however easier bounds) then yields the desired periodization in law of ρ .

In terms of applications, example (b) is of particular interest since it is used in [30, 26, 20] as a suitable random point set for the derivation of nonlinear elasticity from polymer physics started in [4].

We now turn to our Γ -convergence results with periodic boundary data under periodization in law (yielding in particular an approximation of the homogenized energy density of the form (2.14)). The main difficulty is the following: in order to carry out the analysis as before, we would somehow need the uniform sublinearity of the correctors associated to the family of periodized integrands $(V^R)_{R>0}$. It is not clear to us whether this uniformity holds in general. In what follows, we give an alternative argument in the scalar case with fix domain, as well as in the very particular example of well-separated random stiff inclusions in a soft matrix. Note that because of the separation assumption this does not include the example (a) of Poisson inclusions; however we believe that up to some technicalities

the proof could be adapted to that case. For simplicity, we consider spherical inclusions, but random shapes could of course be considered too.

Corollary 2.13 (Approximation by periodization). *Let V, J_ε, J, M and W, I_ε, I be as in Theorems 2.2 and 2.6 for some $p > 1$. Also assume that one of the following holds:*

- (1) $m = 1$, and $\text{dom}V(y, \cdot, \omega) = \text{dom}M$ for almost all y, ω ;
- (2) $p > d$, and $V(y, \Lambda, \omega) \leq C(1 + |\Lambda|^p)$ for all ω and all $y \notin E^\omega$, for some random stationary set $E^\omega = \bigcup_{n=1}^\infty B_{R_n^\omega}(q_n^\omega) \subset \mathbb{R}^d$ satisfying almost surely, for all n , and some constant $C > 0$,

$$\frac{1}{R_n^\omega} \inf_{m, m \neq n} \text{dist}(B_{R_m^\omega}(q_m^\omega), B_{R_n^\omega}(q_n^\omega)) \geq \frac{1}{C}, \quad R_n^\omega \leq C.$$

Let $(V^R)_{R>0}$ be an admissible periodization in law for V in the sense of Definition 2.11, and for all $\varepsilon > 0$ and all $\Lambda \in \mathbb{R}^{m \times d}$ denote by $J_\varepsilon^{\text{per}}(\cdot, \Lambda, \cdot)$ the following random integral functional on the unit cube $Q = [-\frac{1}{2}, \frac{1}{2}]^d$:

$$J_\varepsilon^{\text{per}}(u, \Lambda, \omega) := \int_Q V^{1/\varepsilon}(y/\varepsilon, \Lambda + \nabla u(y), \omega) dy, \quad u \in W_{\text{per}}^{1,p}(Q; \mathbb{R}^m),$$

where $W_{\text{per}}^{1,p}(Q; \mathbb{R}^m)$ denotes the closure in $W^{1,p}(Q; \mathbb{R}^m)$ of the set of smooth periodic functions on Q . Then, for almost all ω and for all Λ , the integral functionals $J_\varepsilon^{\text{per}}(\cdot, \Lambda, \omega)$ Γ -converge to $J(\cdot + \Lambda \cdot x, \omega)$ on the space $W_{\text{per}}^{1,p}(Q; \mathbb{R}^m)$. In particular, we have the following approximation of \bar{V} by periodization: for all $\Lambda \in \mathbb{R}^{m \times d}$, for almost all ω ,

$$\begin{aligned} \bar{V}(\Lambda) &= \lim_{t \uparrow 1} \lim_{R \uparrow \infty} \inf_{u \in W_{\text{per}}^{1,p}(Q_R)} \int_{Q_R} V^R(y, t\Lambda + \nabla u(y), \omega) dy \\ &= \lim_{t \uparrow 1} \lim_{R \uparrow \infty} \mathbb{E} \left[\inf_{u \in W_{\text{per}}^{1,p}(Q_R)} \int_{Q_R} V^R(y, t\Lambda + \nabla u(y), \cdot) dy \right]. \end{aligned}$$

By convexity, if $\Lambda \in \text{int dom } \bar{V}$, then the limits $t \uparrow 1$ can be dropped. Note that $\text{dom } \bar{V} = \mathbb{R}^{m \times d}$ holds in case (2). The same results also hold for V, J_ε, J replaced by W, I_ε, I , provided that $p > d$. \square

3. PROOF OF THE RESULTS FOR CONVEX INTEGRANDS

This section is dedicated to the proofs of Theorem 2.2, Corollary 2.4, and Corollaries 2.10 and 2.13. Let V be a convex τ -stationary normal random integrand. Up to the addition of a constant, we may restrict to the following stronger version of (2.4): for almost all ω, y , we have, for all Λ ,

$$\frac{1}{C} |\Lambda|^p \leq V(y, \Lambda, \omega), \tag{3.1}$$

for some $C > 0$ and $1 < p < \infty$. We assume that 0 belongs to the interior of the domain of the convex function $M := \text{sup ess}_{y, \omega} V(y, \cdot, \omega)$.

Following the strategy of [36, Theorem 1.5], we proceed by truncation of V . We let $(V^k)_k$ be an increasing sequence of τ -stationary convex normal random integrands $V^k : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow \mathbb{R}^+$ such that, for almost all ω, y , and for all Λ , we have

$$\lim_{k \uparrow \infty} V^k(y, \Lambda, \omega) = V(y, \Lambda, \omega), \quad \text{and} \quad \frac{1}{C} |\Lambda|^p \leq V^k(y, \Lambda, \omega) \leq C^k (|\Lambda|^p + 1), \tag{3.2}$$

for some $C > 0$ and some sequence $C^k \uparrow \infty$ (see [36, Lemma 3.4] for such a construction). Let $\Omega_0 \subset \Omega$ be a subset of maximal probability on which all these assumptions (about V and the V^k 's) are simultaneously pointwise satisfied.

We shall prove the existence of a subset $\Omega' \subset \Omega_0$ of maximal probability such that, for all $\omega \in \Omega'$ and all bounded Lipschitz domains $O \subset \mathbb{R}^d$, the functionals $J_\varepsilon(\cdot, \omega; O)$ Γ -converge to the functional $J(\cdot; O)$ on $W^{1,p}(O; \mathbb{R}^m)$, where we recall the definitions

$$J_\varepsilon(u, \omega; O) := \int_O V(y/\varepsilon, \nabla u(y), \omega) dy, \quad J(u; O) := \int_O \bar{V}(\nabla u(y)) dy.$$

As usual, the proof of Γ -convergence splits into two parts: the proof of a lower bound (Γ -lim inf inequality) and the explicit construction of a recovery sequence which achieves the lower bound (Γ -lim sup inequality).

3.1. Preliminaries. We first need to briefly recall the standard stationary differential calculus in probability (first introduced by [37, Section 2]), as well as some results on ergodic Weyl decompositions.

3.1.1. Stationary differential calculus in probability. Let $1 \leq p < \infty$. For all $1 \leq i \leq d$, consider the partial action $(T_h^i)_{h \in \mathbb{R}}$ of $(\mathbb{R}, +)$ on $L^p(\Omega)$, defined by $(T_h^i f)(\omega) = f(\tau_{-he_i}\omega)$, for $h \in \mathbb{R}$. The actions $(T_h^i)_{h \in \mathbb{R}}$ (for $1 \leq i \leq d$) commute with each other and are unitary and strongly continuous by Lemma A.5. For all i , we may then consider the infinitesimal generator D_i of $(T_h^i)_{h \in \mathbb{R}}$, defined by

$$D_i f = \lim_{h \rightarrow 0} \frac{T_h^i f - f}{h}, \quad f \in L^p(\Omega),$$

whenever the limit exists in the strong sense of $L^p(\Omega)$. By classical semigroup theory, the generators D_i are closed linear operators with dense domains $\mathcal{D}_i \subset L^p(\Omega)$, and the intersection $W^{1,p}(\Omega) := \bigcap_{i=1}^d \mathcal{D}_i$ is also dense in $L^p(\Omega)$. Moreover, $W^{1,p}(\Omega)$ is endowed with a natural Banach space structure.

For $f \in W^{1,p}(\Omega)$, its *stationary gradient* is then defined by $Df := (D_1 f, \dots, D_d f) \in L^p(\Omega; \mathbb{R}^d)$. Through the usual correspondence between random variables and τ -stationary random fields as recalled in Appendix A.2.1 (for all $g \in L^p(\Omega)$, write $g(x, \omega) := g(\tau_{-x}\omega)$, defining $g \in L_{\text{loc}}^p(\mathbb{R}^d; L^p(\Omega))$), we may define $Df(x, \omega) := Df(\tau_{-x}\omega)$ for all x . By unitarity of the action T , the operator D is skew-symmetric, so that the following ‘‘integration by parts formula’’ holds, for all $f \in W^{1,p}(\Omega)$ and $g \in W^{1,p'}(\Omega)$, $p' = p/(p-1)$,

$$\mathbb{E}[Df] = 0 \quad \text{and} \quad \mathbb{E}[f Dg] = -\mathbb{E}[g Df].$$

As explained in Section A.2.2 (see in particular Lemma A.7), for almost all ω , the function $Df(\cdot, \omega)$ is nothing but the distributional derivative of $f(\cdot, \omega) \in L_{\text{loc}}^p(\mathbb{R}^d)$, and the following identity holds:

$$W^{1,p}(\Omega) = \{f \in W_{\text{loc}}^{1,p}(\mathbb{R}^d; L^p(\Omega)) : f(x+y, \omega) = f(x, \tau_{-y}\omega), \forall x, y, \omega\}. \quad (3.3)$$

This justifies that in the sequel we simply use the notation $Df = \nabla f$.

3.1.2. Ergodic Weyl decomposition. Ergodicity of the measurable action τ of $(\mathbb{R}^d, +)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is crucial in the sequel. Let $1 < p < \infty$. In analogy with the classical Weyl subspaces of $L_{\text{loc}}^p(\mathbb{R}^d; \mathbb{R}^d)$, we define the subspaces of potential and solenoidal random fields with respect to the differential calculus associated with the group action in the following way: for $p' = p/(p-1)$,

$$\begin{aligned} L_{\text{pot}}^p(\Omega) &= \{f \in L^p(\Omega; \mathbb{R}^d) : \mathbb{E}[f \cdot (\nabla \times g)] = 0, \forall g \in W^{1,p'}(\Omega; \mathbb{R}^d)\}, \\ L_{\text{sol}}^p(\Omega) &= \{f \in L^p(\Omega; \mathbb{R}^d) : \mathbb{E}[f \cdot \nabla g] = 0, \forall g \in W^{1,p'}(\Omega)\}. \end{aligned} \quad (3.4)$$

Reinterpreting these definitions in physical space, we easily obtain the following reformulations in terms of stationary extensions:

$$\begin{aligned} L_{\text{pot}}^p(\Omega) &= \{f \in L^p(\Omega; \mathbb{R}^d) : \text{for almost all } \omega, x \mapsto f(\tau_x \omega) \in L_{\text{loc}}^p(\mathbb{R}^d; \mathbb{R}^d) \text{ is potential}\}, \\ L_{\text{sol}}^p(\Omega) &= \{f \in L^p(\Omega; \mathbb{R}^d) : \text{for almost all } \omega, x \mapsto f(\tau_x \omega) \in L_{\text{loc}}^p(\mathbb{R}^d; \mathbb{R}^d) \text{ is solenoidal}\}, \end{aligned} \quad (3.5)$$

where a function $h \in L_{\text{loc}}^p(\mathbb{R}^d)$ is said to be potential (resp. solenoidal) if $\nabla \times h = 0$ (resp. $\nabla \cdot f = 0$) in the distributional sense. Constant functions belong to both subspaces, and we further define

$$F_{\text{pot}}^p(\Omega) = \{f \in L_{\text{pot}}^p(\Omega) : \mathbb{E}[f] = 0\} \quad \text{and} \quad F_{\text{sol}}^p(\Omega) = \{f \in L_{\text{sol}}^p(\Omega) : \mathbb{E}[f] = 0\}. \quad (3.6)$$

The spaces $L_{\text{pot}}^p(\Omega)$, $L_{\text{sol}}^p(\Omega)$, $F_{\text{pot}}^p(\Omega)$ and $F_{\text{sol}}^p(\Omega)$ are all closed in $L^p(\Omega; \mathbb{R}^d)$, and the following Banach direct sum decomposition holds

$$L^p(\Omega; \mathbb{R}^d) = F_{\text{pot}}^p(\Omega) \oplus F_{\text{sol}}^p(\Omega) \oplus \mathbb{R}^d, \quad (3.7)$$

as well as the following density results:

$$\begin{aligned} F_{\text{pot}}^p(\Omega) &= \text{adh}_{L^p(\Omega; \mathbb{R}^d)} \{ \nabla g : g \in W^{1,p}(\Omega) \}, \\ F_{\text{sol}}^p(\Omega) &= \text{adh}_{L^p(\Omega; \mathbb{R}^d)} \{ \nabla \times g : g \in W^{1,p}(\Omega) \}. \end{aligned} \quad (3.8)$$

Since we did not find a suitable reference for these results (besides the Hilbert setting), we provide a proof of (3.7) and (3.8) in Appendix A.3.

3.2. Γ -convergence of truncated energies. Since the approximations V^k of V all satisfy standard polynomial growth conditions, we can appeal to the classical stochastic homogenization result of [19] (which could be reproved via a direct adaptation of the (periodic) arguments of [36, Theorem 1.3]). More precisely, there exists a subset $\Omega_1 \subset \Omega_0$ of maximal probability such that, for all $\omega \in \Omega_1$, all k , and all $\Lambda \in \mathbb{R}^{m \times d}$, the following limit exists (as a consequence of the Ackoglu-Krengel subadditive ergodic theorem) and defines the homogenized integrand \bar{V}^k :

$$\bar{V}^k(\Lambda) = \lim_{R \uparrow \infty} \inf_{\phi \in W_0^{1,p}(Q_R; \mathbb{R}^m)} \int_{Q_R} V^k(y, \Lambda + \nabla \phi(y), \omega) dy, \quad (3.9)$$

where $Q_R := [-\frac{R}{2}, \frac{R}{2}]^d$. By dominated convergence, this convergence also holds when taking the expectation of the infimum. In addition, for any bounded Lipschitz domain $O \subset \mathbb{R}^d$, and for all $\omega \in \Omega_1$ and all k , the functionals $J_\varepsilon^k(\cdot, \omega; O)$ Γ -converge, as $\varepsilon \downarrow 0$, to the functional $J^k(\cdot; O)$, defined by

$$J_\varepsilon^k(u, \omega; O) := \int_O V^k(y/\varepsilon, \nabla u(y), \omega) dy, \quad \text{and} \quad J^k(u; O) := \int_O \bar{V}^k(\nabla u(y)) dy.$$

Since $k \mapsto V^k$ is increasing, $k \mapsto \bar{V}^k$ is increasing as well, and for all $\Lambda \in \mathbb{R}^{m \times d}$ we may define $\bar{V}(\Lambda) := \lim_{k \rightarrow \infty} \bar{V}^k(\Lambda)$. In particular,

$$\bar{V}(\Lambda) = \sup_k \bar{V}^k(\Lambda) = \sup_k \lim_{R \uparrow \infty} \inf_{\phi \in W_0^{1,p}(Q_R; \mathbb{R}^m)} \int_{Q_R} V^k(y, \Lambda + \nabla \phi(y), \omega) dy. \quad (3.10)$$

It remains to pass to the limit $k \uparrow \infty$ in the Γ -convergence result. The key is to prove the commutation of homogenization and truncation, which we do in Subsection 3.4 below.

Alternative formulas for \bar{V}^k are obtained in Lemma 3.7. Since \bar{V}^k is convex and everywhere finite, it is continuous on $\mathbb{R}^{m \times d}$, and we may directly deduce from the definition $\bar{V}(\Lambda) := \sup_k \bar{V}^k(\Lambda)$:

Lemma 3.1. *The map $\bar{V} : \mathbb{R}^{m \times d} \rightarrow [0, \infty]$ is convex and lower semicontinuous.* \square

3.3. Γ -lim inf inequality. In view of the definition of \bar{V} , the Γ -lim inf inequality is an elementary consequence of the monotone convergence theorem:

Proposition 3.2 (Γ -lim inf inequality). *For all $\omega \in \Omega_1$, all bounded domains $O \subset \mathbb{R}^d$, and all sequences $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O; \mathbb{R}^m)$, we have*

$$\liminf_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O) \geq J(u; O).$$

\square

Proof. Let O , $(u_\varepsilon)_\varepsilon$, and u be as in the statement. Then, for all $\omega \in \Omega_1$ and all $k \in \mathbb{N}$, using the Γ -lim inf result for $J_\varepsilon^k(\cdot, \omega; O)$ towards $J^k(\cdot; O)$ (see Section 3.2), and that $V \geq V^k$,

$$\liminf_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O) \geq \liminf_{\varepsilon \downarrow 0} J_\varepsilon^k(u_\varepsilon, \omega; O) \geq J^k(u; O),$$

so that, by monotone convergence,

$$\liminf_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O) \geq \lim_{k \uparrow \infty} J^k(u; O) = J(u; O). \quad \square$$

From this Γ -lim inf result, we deduce the locality of recovery sequences, if they exist.

Corollary 3.3 (Locality of recovery sequences). *If for some $\omega \in \Omega_1$, some bounded domain $O \subset \mathbb{R}^d$, and some function $u \in W^{1,p}(O; \mathbb{R}^m)$, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(u_\varepsilon, \omega; O) \rightarrow J(u; O)$, then we also have $J_\varepsilon(u_\varepsilon, \omega; O') \rightarrow J(u; O')$ for any subdomain $O' \subset O$. Hence, by an extension result, the Γ -lim sup inequality on a bounded Lipschitz domain O implies the Γ -lim sup inequality on any Lipschitz subdomain $O' \subset O$. \square*

Proof. Choose a subdomain $O' \subset O$, and define $O'' := O \setminus O'$. We then have by assumption

$$\begin{aligned} J(u; O) &= \lim_{\varepsilon \downarrow 0} J(u_\varepsilon; O) = \lim_{\varepsilon \downarrow 0} (J_\varepsilon(u_\varepsilon, \omega; O') + J_\varepsilon(u_\varepsilon, \omega; O'')) \\ &\geq \liminf_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O') + \liminf_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O''). \end{aligned}$$

Now by Proposition 3.2 we have $\liminf_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O') \geq J(u; O')$ and $\liminf_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O'') \geq J(u; O'')$. The conclusion then follows from the identity $J(u; O') + J(u; O'') = J(u; O)$. \square

3.4. Commutation of truncation and homogenization. The crucial ingredient to prove the commutation of truncation and homogenization is the reformulation of the asymptotic homogenization formula in the probability space. For that purpose, we first introduce the following proxy for \bar{V} :

$$\bar{P}(\Lambda) := \inf_{f \in F_{\text{pot}}^p(\Omega)^m} \mathbb{E}[V(0, \Lambda + f, \cdot)]. \quad (3.11)$$

Likewise, for all $k \in \mathbb{N}$, we set

$$\bar{P}^k(\Lambda) := \inf_{f \in F_{\text{pot}}^p(\Omega)^m} \mathbb{E}[V^k(0, \Lambda + f, \cdot)]. \quad (3.12)$$

In this case, due to the growth condition (3.2), we may prove (see Lemma 3.7 below) that

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\inf_{\phi \in W_0^{1,p}(O/\varepsilon; \mathbb{R}^m)} \int_{O/\varepsilon} V^k(y, \Lambda + \nabla \phi(y), \cdot) dy \right] = \bar{P}^k(\Lambda).$$

However, for V itself, this equality has no chance to be true if $\Lambda \in \text{dom} \bar{P} \setminus \text{dom} M$ since the left-hand side could be infinite (because of the Dirichlet boundary condition) while the right-hand side is not — see Example 2.3. We thus rather use a “relaxed version” of the Dirichlet boundary conditions and set for all $\varepsilon > 0$

$$P^\varepsilon(\Lambda, \omega; O) := \inf_{\substack{\phi \in W^{1,p}(O/\varepsilon; \mathbb{R}^m) \\ \int_{O/\varepsilon} \nabla \phi = 0}} \int_{O/\varepsilon} V(y, \Lambda + \nabla \phi(y), \omega) dy. \quad (3.13)$$

As opposed to the case of Dirichlet boundary conditions, there is no natural subadditive property in this definition (two test-functions on disjoint domains cannot be glued together). This difficulty will be overcome by using a more sophisticated gluing argument that relies quantitatively on the following sublinearity property of the correctors.

Lemma 3.4 (Sublinearity of correctors). *For all $\Lambda \in \mathbb{R}^{m \times d}$, there exists a corrector field $\varphi_\Lambda \in \text{Mes}(\Omega; W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m))$ such that $\nabla \varphi_\Lambda(0, \cdot) \in F_{\text{pot}}^p(\Omega)^m$, and*

$$\bar{P}(\Lambda) = \mathbb{E}[V(0, \Lambda + \nabla \varphi_\Lambda(0, \cdot), \cdot)].$$

In addition, φ_Λ is sublinear at infinity in the sense that, for almost all $\omega \in \Omega$,

$$\varepsilon \varphi_\Lambda(\cdot/\varepsilon, \omega) \rightarrow 0 \quad (3.14)$$

weakly in $W^{1,p}(O; \mathbb{R}^m)$ for all bounded domains $O \subset \mathbb{R}^d$. \square

Remark 3.5. Although the space $\{\nabla g : g \in W^{1,p}(\Omega)\}$ is dense in $F_{\text{pot}}^p(\Omega)$ (see (3.8)), the infimum (3.11) defining $\bar{P}(\Lambda)$ cannot be replaced in general by an infimum over this smaller dense subspace because of a possible lack of strong continuity of the functional, see however Proposition 5.1. \square

Proof. Let $\Lambda \in \mathbb{R}^{m \times d}$ be fixed. By convexity and by the lower bound (3.1) on V , $\chi \mapsto \mathbb{E}[V(0, \Lambda + \chi, \cdot)]$ is lower semicontinuous and coercive on $F_{\text{pot}}^p(\Omega)^m$, and therefore attains its infimum. Let $g \in F_{\text{pot}}^p(\Omega)^m$ be a minimizer. The τ -stationary extension $(x, \omega) \mapsto g(\tau_x \omega)$ of g is a potential field on \mathbb{R}^d for almost every ω . Hence, there exists a map $\varphi_\Lambda \in \text{Mes}(\Omega; W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m))$ such that $g(\tau_x \omega) = \nabla \varphi_\Lambda(x, \omega)$ for almost all x, ω (see indeed Proposition A.10). The claim now follows from the combination of the following two applications of the Birkhoff-Khinchin ergodic theorem: for almost all ω ,

$$\nabla \varphi_\Lambda(\cdot/\varepsilon, \omega) \rightharpoonup 0, \quad (\text{weakly in } L^p(O; \mathbb{R}^m)), \quad (3.15)$$

$$\varepsilon \int_O \varphi_\Lambda(y/\varepsilon, \omega) dy \rightarrow 0. \quad (3.16)$$

Indeed, by Poincaré's inequality and (3.15), the sequence $y \mapsto \varepsilon \varphi_\Lambda(y/\varepsilon, \omega) - \varepsilon \int_O \varphi_\Lambda(z/\varepsilon, \omega) dz$ converges weakly to zero in $W^{1,p}(O; \mathbb{R}^m)$ for almost every ω . Combined with (3.16), this implies (3.14).

To conclude, we turn to the proofs of (3.15) and (3.16). The weak convergence (3.15) to zero is a direct consequence of the Birkhoff-Khinchin ergodic theorem in the form $\nabla \varphi_\Lambda(\cdot/\varepsilon, \omega) \rightharpoonup \mathbb{E}[\nabla \varphi_\Lambda(0, \cdot)] = 0$ in $L^p(O; \mathbb{R}^m)$. It remains to prove (3.16). We may assume wlog $\varphi_\Lambda(0, \cdot) = 0$ almost surely, so that

$$\begin{aligned} \left| \varepsilon \int_O \varphi_\Lambda(y/\varepsilon, \omega) dy \right| &= \left| \varepsilon \int_{O/\varepsilon} \varphi_\Lambda(\cdot, \omega) \right| = \left| \varepsilon \int_{O/\varepsilon} \int_0^1 x \cdot \nabla \varphi_\Lambda(tx, \omega) dt dx \right| \\ &\leq \int_0^{1/\varepsilon} \left| \int_O x \cdot \nabla \varphi_\Lambda(tx, \omega) dx \right| dt. \end{aligned} \quad (3.17)$$

For almost all ω , the function $\psi_\omega(t) := \int_O x \cdot \nabla \varphi_\Lambda(tx, \omega) dx$ is continuous on $(0, \infty)$. By (3.15), $\psi_\omega(t) \rightarrow 0$ as $t \uparrow \infty$ for almost all ω . By joint measurability and (local) integrability of $\nabla \varphi_\Lambda$, and by stationarity, 0 is a Lebesgue point of $\nabla \varphi_\Lambda(\cdot, \omega)$ for almost all ω , and hence $\limsup_{t \downarrow 0} |\psi_\omega(t)| < \infty$ for almost all ω . The result (3.16) then follows from (3.17). \square

For all $\Lambda \in \text{dom} \bar{P}$, let φ_Λ be defined as in Lemma 3.4, and let $\Omega_\Lambda \subset \Omega_1$ be a subset of maximal probability such that (3.14) holds on Ω_Λ for all bounded Lipschitz domains. Restricting Ω_Λ further, the Birkhoff-Khinchin ergodic ensures that, for all $\omega \in \Omega_\Lambda$, we have for all bounded subsets $O \subset \mathbb{R}^d$ and all $t \in \mathbb{Q}$,

$$\int_{O/\varepsilon} \nabla \varphi_{t\Lambda}(\cdot, \omega) \xrightarrow{\varepsilon \downarrow 0} 0 \quad (3.18)$$

and

$$\int_{O/\varepsilon} V(y, t\Lambda + \nabla \varphi_{t\Lambda}(y, \omega), \omega) dy \xrightarrow{\varepsilon \downarrow 0} \mathbb{E}[V(0, t\Lambda + \nabla \varphi_{t\Lambda}(0, \cdot))] = \bar{P}(t\Lambda). \quad (3.19)$$

We now turn to the proof that $\lim_\varepsilon P^\varepsilon(\Lambda, \omega; O) = \bar{P}(\Lambda)$ for all Λ for almost all $\omega \in \Omega$. The following inequality is the most subtle part.

Lemma 3.6. *For all $\Lambda \in \text{int dom} \bar{P}$ and all bounded domains $O \subset \mathbb{R}^d$, there exists a sequence $\psi_{\Lambda, O, \varepsilon} \in \text{Mes}(\Omega; W^{1,p}(O/\varepsilon; \mathbb{R}^m))$ such that, for all $\omega \in \Omega_\Lambda$, $\int_{O/\varepsilon} \nabla \psi_{\Lambda, O, \varepsilon}(\cdot, \omega) = 0$,*

$$\varepsilon \psi_{\Lambda, O, \varepsilon}(\cdot/\varepsilon, \omega) \rightharpoonup 0$$

weakly in $W^{1,p}(O; \mathbb{R}^m)$ as $\varepsilon \downarrow 0$, and

$$\bar{P}(\Lambda) \geq \limsup_{\varepsilon \downarrow 0} \int_{O/\varepsilon} V(y, \Lambda + \nabla \psi_{\Lambda, O, \varepsilon}(y, \omega), \omega) dy \geq \limsup_{\varepsilon \downarrow 0} P^\varepsilon(\Lambda, \omega; O). \quad (3.20)$$

\square

Proof. Let $\Lambda \in \text{int dom} \bar{P}$ be fixed, and let $\omega \in \Omega_\Lambda$. For all $t \in [0, 1) \cap \mathbb{Q}$ and $\varepsilon > 0$, set

$$\Lambda_{O, \varepsilon, t}^\omega := -t \int_{O/\varepsilon} \nabla \varphi_{\Lambda/t}(\cdot, \omega), \quad \text{and} \quad \psi_{\Lambda, O, \varepsilon, t}(x, \omega) := t \varphi_{\Lambda/t}(x, \omega) + \Lambda_{O, \varepsilon, t}^\omega \cdot x.$$

By definition, $\int_{O/\varepsilon} \nabla \psi_{\Lambda, O, \varepsilon, t} = 0$, and by Lemma 3.4, $\varepsilon \psi_{\Lambda, O, \varepsilon, t}(\cdot/\varepsilon, \omega) \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$ as $\varepsilon \downarrow 0$. Hence,

$$P^\varepsilon(\Lambda, \omega; O) \leq \int_{O/\varepsilon} V(y, \Lambda + \nabla \psi_{\Lambda, O, \varepsilon, t}(y, \omega), \omega) =: \widehat{P}_t^\varepsilon(\Lambda, \omega; O).$$

By convexity

$$\begin{aligned} \widehat{P}_t^\varepsilon(\Lambda, \omega; O) &= \int_{O/\varepsilon} V(y, \Lambda + t \nabla \varphi_{\Lambda/t}(y, \omega) + \Lambda_{O, \varepsilon, t}^\omega, \omega) dy \\ &\leq t \int_{O/\varepsilon} V(y, \Lambda/t + \nabla \varphi_{\Lambda/t}(y, \omega), \omega) dy + (1-t) \int_{O/\varepsilon} V\left(y, \frac{1}{1-t} \Lambda_{O, \varepsilon, t}^\omega\right) dy. \end{aligned}$$

Since $0 \in \text{int dom } M$, there exists $\delta > 0$ such that $\text{adh } B_\delta \subset \text{int dom } M$. As t is rational and $\omega \in \Omega_\Lambda$, we have $\Lambda_{O, \varepsilon, t}^\omega \rightarrow 0$ as $\varepsilon \downarrow 0$ by the Birkhoff-Khinchin ergodic theorem in the form of (3.18). Hence there exists $\varepsilon_{\Lambda, O, t}^\omega > 0$ such that, for all $0 < \varepsilon < \varepsilon_{\Lambda, O, t}^\omega$, we have

$$\left| \frac{1}{1-t} \Lambda_{O, \varepsilon, t}^\omega \right| < \delta,$$

and therefore,

$$\int_{O/\varepsilon} V\left(y, \frac{1}{1-t} \Lambda_{O, \varepsilon, t}^\omega\right) dy \leq \sup_{|\Lambda'| < \delta} M(\Lambda') < \infty.$$

This implies that

$$\limsup_{t \uparrow 1, t \in \mathbb{Q}} \limsup_{\varepsilon \downarrow 0} \widehat{P}_t^\varepsilon(\Lambda, \omega; O) \leq \limsup_{t \uparrow 1, t \in \mathbb{Q}} \limsup_{\varepsilon \downarrow 0} \int_{O/\varepsilon} V(y, \Lambda/t + \nabla \varphi_{\Lambda/t}(y, \omega), \omega) dy.$$

By the Birkhoff-Khinchin ergodic theorem in the form of (3.19) and the continuity of \bar{P} in the interior of its domain (as a consequence of convexity), this yields

$$\limsup_{t \uparrow 1, t \in \mathbb{Q}} \limsup_{\varepsilon \downarrow 0} \widehat{P}_t^\varepsilon(\Lambda, \omega; O) \leq \limsup_{t \uparrow 1, t \in \mathbb{Q}} \mathbb{E}[V(0, \Lambda/t + \nabla \varphi_{\Lambda/t}(0, \cdot), \cdot)] = \limsup_{t \uparrow 1, t \in \mathbb{Q}} \bar{P}(\Lambda/t) = \bar{P}(\Lambda).$$

We have thus proved:

$$\limsup_{t \uparrow 1, t \in \mathbb{Q}} \limsup_{\varepsilon \downarrow 0} \left(\widehat{P}_t^\varepsilon(\Lambda, \omega; O) - \bar{P}(\Lambda) \right)^+ + \|\varepsilon \psi_{\Lambda, O, \varepsilon, t}(\cdot/\varepsilon, \omega)\|_{L^p(O; \mathbb{R}^m)} = 0.$$

By Attouch's diagonalization lemma (see [8, Corollary 1.16]), this implies the existence of a sequence $(\psi_{\Lambda, O, \varepsilon})_\varepsilon$ with $\psi_{\Lambda, O, \varepsilon} \in W^{1,p}(O/\varepsilon; \mathbb{R}^m)$ such that $\int_{O/\varepsilon} \nabla \psi_{\Lambda, O, \varepsilon} = 0$, $\limsup_{\varepsilon} \widehat{P}_t^\varepsilon(\Lambda, \omega; O) \leq \bar{P}(\Lambda)$, and $\varepsilon \psi_{\Lambda, O, \varepsilon}(\cdot/\varepsilon, \omega) \rightarrow 0$ in $L^p(O; \mathbb{R}^m)$, for all $\omega \in \Omega_\Lambda$. By the choice of Λ , $\bar{P}(\Lambda) < \infty$, so that the lower bound (3.1) on V implies that the sequence $(\nabla \psi_{\Lambda, O, \varepsilon}(\cdot/\varepsilon, \omega))_\varepsilon$ is bounded in $L^p(O; \mathbb{R}^m)$. We thus conclude that $\varepsilon \psi_{\Lambda, O, \varepsilon}(\cdot/\varepsilon, \omega) \rightharpoonup 0$ weakly in $W^{1,p}(O; \mathbb{R}^m)$, as claimed. \square

In the case of standard growth conditions (thus e.g. for the V^k 's), the corresponding inequality (3.20) in Lemma 3.6 is indeed an equality. The following lemma gives equivalent definitions for the \bar{V}^k 's, which will be crucial in the sequel.

Lemma 3.7. *Let O be a bounded Lipschitz domain of \mathbb{R}^d . For all $\omega \in \Omega_1$, all k , and all $\Lambda \in \mathbb{R}^{m \times d}$, the following quantities are well-defined:*

$$\bar{V}_1^k(\Lambda) := \lim_{\varepsilon \downarrow 0} \inf_{\phi \in W_0^{1,p}(O/\varepsilon; \mathbb{R}^m)} \int_{O/\varepsilon} V^k(y, \Lambda + \nabla \phi(y), \omega) dy, \quad (3.21)$$

$$\bar{V}_2^k(\Lambda, \omega) := \lim_{\varepsilon \downarrow 0} \inf_{\substack{\phi \in W^{1,p}(O/\varepsilon; \mathbb{R}^m) \\ \int_{O/\varepsilon} \nabla \phi = 0}} \int_{O/\varepsilon} V^k(y, \Lambda + \nabla \phi(y), \omega) dy, \quad (3.22)$$

$$\bar{V}_3^k(\Lambda) := \inf_{\tilde{f} \in F_{\text{pot}}^p(\Omega)^m} \mathbb{E}[V^k(0, \Lambda + \tilde{f}, \cdot)], \quad (3.23)$$

and we have

$$\bar{V}^k(\Lambda) = \bar{V}_1^k(\Lambda) = \bar{V}_2^k(\Lambda) = \bar{V}_3^k(\Lambda).$$

□

This result is standard (see for instance [31, Chapter 15]) and we display its proof for completeness. Note that the formulas (3.22) and (3.23) for V will be shown to be equivalent to \bar{V} , whereas formula (3.21) is in general larger than \bar{V} .

Proof. Let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and $k \in \mathbb{N}$. By the definition of Γ -convergence for J^k on $W_0^{1,p}(O; \mathbb{R}^m)$ and the convergence of infima with Dirichlet boundary conditions, for all Λ and $\omega \in \Omega_1$ we have

$$\begin{aligned} \bar{V}^k(\Lambda) &= \frac{1}{|O|} \inf_{\phi \in W_0^{1,p}(O; \mathbb{R}^m)} \int_O \bar{V}^k(\Lambda + \nabla \phi) \\ &= \frac{1}{|O|} \lim_{\varepsilon \downarrow 0} \inf_{\phi \in W_0^{1,p}(O; \mathbb{R}^m)} \int_O V^k(y/\varepsilon, \Lambda + \nabla \phi(y), \cdot) dy = \bar{V}_1^k(\Lambda). \end{aligned} \quad (3.24)$$

Likewise, the Γ -convergence result holds on $\{u \in W^{1,p}(O) : \int_O \nabla u = 0\}$ so that the identity

$$\bar{V}^k(\Lambda) = \bar{V}_2^k(\Lambda)$$

also follows from the convergence of infima. Since Lemma 3.6 (applied to V^k instead of V) yields $\bar{V}_2^k(\Lambda) \leq \bar{V}_3^k(\Lambda)$, it remains to prove that $\bar{V}_3^k(\Lambda) \leq \bar{V}_1^k(\Lambda)$ for all Λ .

Let $O' \subset \mathbb{R}^d$ be a bounded domain. By the coercivity and the lower semicontinuity of the integral functional J^k (which follow from the growth condition (3.2) and the convexity of V^k), there exists a minimizer $\zeta \in L^\infty(\Omega; W_0^{1,p}(O'; \mathbb{R}^m))$ (where measurability follows from Proposition A.15) such that, for almost all ω ,

$$\int_{O'} V^k(y, \Lambda + \nabla \zeta(y, \omega), \omega) dy = \inf_{\phi \in W_0^{1,p}(O'; \mathbb{R}^m)} \int_{O'} V^k(y, \Lambda + \nabla \phi(y), \omega) dy.$$

Set

$$\xi(x, \omega) := \frac{1}{|O'|} \int_{\mathbb{R}^d} \zeta(x+z, \tau_z \omega) dz = \int_{-x+O'} \zeta(x+z, \tau_z \omega) dz.$$

Clearly, ξ is well-defined and stationary, belongs to $W^{1,p}(\Omega; \mathbb{R}^m)$, and

$$\nabla \xi(x, \omega) = \int_{-x+O'} \nabla \zeta(x+z, \tau_z \omega) dz.$$

Hence

$$\bar{V}_3^k(\Lambda) \leq \mathbb{E} \left[V^k(0, \Lambda + \nabla \xi(0, \cdot), \cdot) \right] = \mathbb{E} \left[V^k \left(0, \Lambda + \int_{O'} \nabla \zeta(z, \tau_z \cdot) dz, \cdot \right) \right],$$

and by convexity of V^k

$$\bar{V}_3^k(\Lambda) \leq \mathbb{E} \left[\int_{O'} V^k(0, \Lambda + \nabla \zeta(z, \tau_z \cdot), \cdot) dz \right].$$

By stationarity and the Fubini theorem, we may conclude

$$\bar{V}_3^k(\Lambda) \leq \int_{O'} \mathbb{E} [V^k(z, \Lambda + \nabla \zeta(z, \cdot), \cdot)] dz = \mathbb{E} \left[\inf_{\phi \in W_0^{1,p}(O'; \mathbb{R}^m)} \int_{O'} V^k(y, \Lambda + \nabla \phi(y), \cdot) dy \right].$$

With $O' := O/\varepsilon$, the claim $\bar{V}_3^k(\Lambda) \leq \bar{V}_1^k(\Lambda)$ follows by the dominated convergence theorem and the growth condition from above (3.2). □

The following result proves the equivalence between formulas (i), (ii) and (iii) in Theorem 2.2.

Proposition 3.8 (Commutation of limits). *For all bounded Lipschitz domains $O \subset \mathbb{R}^d$, and all $\Lambda \in \mathbb{R}^{m \times d}$, we have for almost all ω*

$$\overline{V}(\Lambda) = \overline{P}(\Lambda) = \lim_{t \uparrow 1} \lim_{\varepsilon \downarrow 0} P^\varepsilon(t\Lambda, \omega; O).$$

By convexity, for all $\Lambda \notin \partial \text{dom} \overline{V}$, this takes the form $\overline{V}(\Lambda) = \overline{P}(\Lambda) = \lim_{\varepsilon \downarrow 0} P^\varepsilon(\Lambda, \omega; O)$. \square

Remark 3.9. Although not stated explicitly, this result proves the commutation of truncation and homogenization. By monotone convergence (cf. the proof of $\overline{V} \equiv \overline{P}$ below) we have for all $\varepsilon > 0$ and almost every ω ,

$$\inf_{\substack{\phi \in W^{1,p}(O/\varepsilon; \mathbb{R}^m) \\ \int_{O/\varepsilon} \nabla \phi = 0}} \int_{O/\varepsilon} V(y, \Lambda + \nabla \phi(y), \omega) dy = \sup_k \inf_{\substack{\phi \in W_0^{1,p}(O/\varepsilon; \mathbb{R}^m) \\ \int_{O/\varepsilon} \nabla \phi = 0}} \int_{O/\varepsilon} V^k(y, \Lambda + \nabla \phi(y), \omega) dy$$

so that Proposition 3.8, combined with (3.22) in Lemma 3.7, yields the desired commutation result

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} \inf_k \inf_{\substack{\phi \in W^{1,p}(O/\varepsilon; \mathbb{R}^m) \\ \int_{O/\varepsilon} \nabla \phi = 0}} \int_{O/\varepsilon} V^k(y, \Lambda + \nabla \phi(y), \omega) dy \\ = \overline{V}(\Lambda) = \sup_k \overline{V}^k(\Lambda) = \sup_k \lim_{\varepsilon \downarrow 0} \inf_{\substack{\phi \in W^{1,p}(O/\varepsilon; \mathbb{R}^m) \\ \int_{O/\varepsilon} \nabla \phi = 0}} \int_{O/\varepsilon} V^k(y, \Lambda + \nabla \phi(y), \omega) dy. \end{aligned}$$

\square

Proof of Proposition 3.8. We split the proof into two steps.

Step 1. Proof of $\overline{V} \equiv \overline{P}$.

Let $\Lambda \in \text{dom} \overline{V}$. By (3.23) in Lemma 3.7, for all k ,

$$\overline{V}^k(\Lambda) = \inf_{f \in F_{\text{pot}}^p(\Omega)^m} \mathbb{E}[V^k(0, \Lambda + f)].$$

By convexity, $f \mapsto \mathbb{E}[V^k(0, \Lambda + f)]$ is lower semicontinuous on $F_{\text{pot}}^p(\Omega)^m$, and by coercivity, the infimum is attained. Hence there exists $g_k \in F_{\text{pot}}^p(\Omega)^m$ such that

$$\overline{V}^k(\Lambda) = \mathbb{E}[V^k(0, \Lambda + g_k)].$$

By the uniform growth condition from below (3.2), $(g_k)_k$ is bounded in $L^p(\Omega; \mathbb{R}^{m \times d})$:

$$\frac{1}{C} 2^{-p+1} \mathbb{E}[|g_k|^p] - \frac{1}{C} |\Lambda|^p \leq \frac{1}{C} \mathbb{E}[|\Lambda + g_k|^p] \leq \mathbb{E}[V^k(0, \Lambda + g_k)] = \overline{V}^k(\Lambda) \leq \overline{V}(\Lambda).$$

Let $g \in F_{\text{pot}}^p(\Omega)^m$ be a cluster point of (g_k) for the weak convergence of $L^p(\Omega; \mathbb{R}^{m \times d})$. We have along the subsequence

$$\overline{V}(\Lambda) = \sup_k \overline{V}^k(\Lambda) = \lim_{k \uparrow \infty} \mathbb{E}[V^k(0, \Lambda + g_k)].$$

Since $k \mapsto V^k$ is increasing and $f \mapsto \mathbb{E}[V^k(0, \Lambda + f)]$ is lower semicontinuous for the weak convergence of $L^p(\Omega; \mathbb{R}^{m \times d})$, this yields for all ℓ

$$\overline{V}(\Lambda) \geq \liminf_{k \uparrow \infty} \mathbb{E}[V^\ell(0, \Lambda + g_k)] \geq \mathbb{E}[V^\ell(0, \Lambda + g)].$$

We then conclude by monotone convergence that

$$\overline{V}(\Lambda) \geq \mathbb{E}[V(0, \Lambda + g)] \geq \inf_{f \in F_{\text{pot}}^p(\Omega)^m} \mathbb{E}[V(0, \Lambda + f)] = \overline{P}(\Lambda).$$

For $\Lambda \notin \text{dom} \overline{V}$, the above inequality is trivial so that $\overline{V}(\Lambda) \geq \overline{P}(\Lambda)$ for all $\Lambda \in \mathbb{R}^{m \times d}$. For the converse inequality, note that for all Λ ,

$$\overline{P}(\Lambda) \geq \sup_k \inf_{f \in F_{\text{pot}}^p(\Omega)^m} \mathbb{E}[V^k(0, \Lambda + f)] = \sup_k \overline{V}^k(\Lambda) = \overline{V}(\Lambda).$$

Hence, $\bar{V} \equiv \bar{P}$, as claimed.

Step 2. Proof of $\lim_{t \uparrow 1} \lim_{\varepsilon \downarrow 0} P^\varepsilon(t\Lambda, \omega; O) = \bar{V}(\Lambda)$.

Since for $\Lambda \in \text{dom} \bar{V}$ and $t \in [0, 1)$, $t\Lambda \in \text{int dom} \bar{V}$, Lemma 3.6 and Step 1 yield for almost all ω

$$\bar{V}(t\Lambda) = \bar{P}(t\Lambda) \geq \limsup_{\varepsilon \downarrow 0} P^\varepsilon(t\Lambda, \omega; O). \quad (3.25)$$

By (3.22) in Lemma 3.7, for all $\Lambda \in \mathbb{R}^{m \times d}$ and almost all ω ,

$$\liminf_{\varepsilon \downarrow 0} P^\varepsilon(\Lambda, \omega; O) \geq \sup_k \lim_{\varepsilon \downarrow 0} \inf_{\substack{\phi \in W^{1,p}(O/\varepsilon; \mathbb{R}^m) \\ \int_{O/\varepsilon} \nabla \phi = 0}} \int_{O/\varepsilon} V^k(y, \Lambda + \nabla \phi(y), \omega) dy = \sup_k \bar{V}^k(\Lambda) = \bar{V}(\Lambda). \quad (3.26)$$

Combined with (3.25), this yields $\lim_{\varepsilon \downarrow 0} P^\varepsilon(t\Lambda, \omega; O) = \bar{V}(t\Lambda)$ for almost all ω , for all $\Lambda \in \text{dom} \bar{V}$ and $t \in [0, 1)$. By convexity and lower semicontinuity of \bar{V} , this implies for all $\Lambda \in \text{dom} \bar{V}$

$$\lim_{t \uparrow 1} \lim_{\varepsilon \downarrow 0} P^\varepsilon(t\Lambda, \omega; O) = \lim_{t \uparrow 1} \bar{V}(t\Lambda) = \bar{V}(\Lambda), \quad (3.27)$$

and (3.26) ensures that this equality also holds for $\Lambda \notin \text{dom} \bar{V}$. By convexity and by (3.27), the function $\Lambda \mapsto \lim_{\varepsilon \downarrow 0} P^\varepsilon(\Lambda, \omega; O)$ is continuous outside $\partial \text{dom} \bar{V}$, so that the limit $t \uparrow 1$ can be omitted for $\Lambda \notin \partial \text{dom} \bar{V}$. \square

3.5. Proof of Theorem 2.2: Γ -convergence with Neumann boundary data. It only remains to prove the Γ -lim sup inequality.

Proposition 3.10 (Γ -lim sup inequality with Neumann boundary data). *Assume $p > d$. There exists a subset $\Omega' \subset \Omega_1$ of maximal probability with the following property: for all $\omega \in \Omega'$, all bounded Lipschitz domains $O \subset \mathbb{R}^d$, and all $u \in W^{1,p}(O; \mathbb{R}^m)$, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(u_\varepsilon, \omega; O) \rightarrow J(u; O)$.* \square

Proof. We split the proof into three steps. We first treat the case of affine functions, then the case of continuous piecewise affine functions, and finally the general case. The novelty of our approach is the careful gluing argument needed to pass from affine to piecewise affine functions.

Step 1. Recovery sequence for affine functions.

In this step, we consider the case when $u = \Lambda \cdot x$ is an affine function. More precisely, we prove the existence of a subset $\Omega' \subset \Omega_1$ of maximal probability with the following property: given a bounded Lipschitz domain $O \subset \mathbb{R}^d$, for all $\omega \in \Omega'$ and all $\Lambda \in \text{int dom} \bar{V}$, there exists a sequence $(u_{\Lambda, \varepsilon}^\omega)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ with $u_{\Lambda, \varepsilon}^\omega \rightharpoonup \Lambda \cdot x$ weakly in $W^{1,p}(O; \mathbb{R}^m)$ such that, for all Lipschitz subdomains $O' \subset O$, we have $J_\varepsilon(u_{\Lambda, \varepsilon}^\omega, \omega; O') \rightarrow J(\Lambda \cdot x; O')$. By Corollary 3.3, it suffices to prove this for $O' = O$.

By Lemma 3.4 and Proposition 3.8, there exists a sequence $\varphi_\Lambda \in \text{Mes}(\Omega; W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m))$ such that, for all $\omega \in \Omega_\Lambda$, we have $\varepsilon \varphi_\Lambda(\cdot/\varepsilon, \omega) \rightharpoonup 0$ weakly in $W^{1,p}(O; \mathbb{R}^m)$ and, by the Birkhoff-Khinchin ergodic theorem in the form of (3.19),

$$\bar{V}(\Lambda) = \bar{P}(\Lambda) = \lim_{\varepsilon \downarrow 0} \int_{O/\varepsilon} V(y, \Lambda + \nabla \varphi_\Lambda(y, \omega), \omega) dy.$$

In particular, by a change of variables, this yields

$$J(\Lambda \cdot x; O) = |O| \bar{V}(\Lambda) = \lim_{\varepsilon \downarrow 0} J_\varepsilon(\Lambda \cdot x + \varepsilon \varphi_\Lambda(\cdot/\varepsilon, \omega), \omega; O).$$

The function $u_\varepsilon^{\Lambda, \omega}(x) := \Lambda \cdot x + \varepsilon \varphi_\Lambda(x/\varepsilon, \omega)$ thus satisfies $u_\varepsilon^{\Lambda, \omega} \rightharpoonup \Lambda \cdot x$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(u_\varepsilon^{\Lambda, \omega}, \omega; O) \rightarrow J(\Lambda \cdot x; O)$ as $\varepsilon \downarrow 0$, for all $\omega \in \Omega_\Lambda$.

We then define $\Omega' \subset \Omega_1$ as the (countable) intersection of all Ω_Λ 's with $\Lambda \in \mathbb{Q}^{m \times d} \cap \text{int dom} \bar{V}$, which is still of maximal probability. Let $\Lambda \in \text{int dom} \bar{V}$ and $\omega \in \Omega'$ be fixed. Choose a sequence $(\Lambda_n)_n \subset \mathbb{Q}^{m \times d} \cap \text{int dom} \bar{V}$ such that $\Lambda_n \rightarrow \Lambda$. For all n , we have already constructed a sequence

$(u_{\varepsilon,n}^\omega)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ such that $u_{\varepsilon,n}^\omega \rightharpoonup \Lambda_n \cdot x$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(u_{\varepsilon,n}^\omega, \omega; O) \rightarrow J(\Lambda_n \cdot x; O)$. Since by convexity, \bar{V} is continuous on $\text{int dom } \bar{V}$, we have

$$\begin{aligned} & \limsup_{n \uparrow \infty} \limsup_{\varepsilon \downarrow 0} (|J_\varepsilon(u_{\varepsilon,n}^\omega, \omega; O) - J(\Lambda \cdot x; O)| + \|u_{\varepsilon,n}^\omega - \Lambda \cdot x\|_{L^p(O; \mathbb{R}^m)}) \\ &= \limsup_{n \uparrow \infty} (|J(\Lambda_n \cdot x; O) - J(\Lambda \cdot x; O)| + \|\Lambda_n \cdot x - \Lambda \cdot x\|_{L^p(O; \mathbb{R}^m)}) \\ &\leq \limsup_{n \uparrow \infty} (|O| |\bar{V}(\Lambda_n) - \bar{V}(\Lambda)| + C_O |\Lambda_n - \Lambda|) = 0. \end{aligned}$$

By the Attouch diagonalization lemma (see [8, Corollary 1.16]), this implies the existence of a sequence $(v_\varepsilon^\omega)_\varepsilon$ such that $J_\varepsilon(v_\varepsilon^\omega, \omega; O) \rightarrow J(\Lambda \cdot x; O)$ and $v_\varepsilon^\omega \rightarrow \Lambda \cdot x$ in $L^p(O; \mathbb{R}^m)$ for all $\omega \in \Omega'$. By the p -th order lower bound for V , we conclude that v_ε^ω converges weakly to $\Lambda \cdot x$ in $W^{1,p}(O; \mathbb{R}^m)$.

Step 2. Recovery sequence for continuous piecewise affine functions.

Let $\omega \in \Omega'$, $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and u be a continuous piecewise affine function on O such that $\nabla u \in \text{int dom } \bar{V}$ pointwise. We shall prove that there exists a sequence $(u_\varepsilon^\omega)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ with $u_\varepsilon^\omega \rightharpoonup u$ weakly in $W^{1,p}(O; \mathbb{R}^m)$, such that $J_\varepsilon(u_\varepsilon^\omega, \omega; O) \rightarrow J(u; O)$. For that purpose, the major issue consists in gluing the recovery sequences for the different affine parts together, which requires a particularly careful treatment.

Consider the open partition $O = \bigsqcup_{l=1}^k O_l$ associated with u (note that the O_l 's have piecewise flat boundary outside ∂O), and define $c_l + \Lambda_l \cdot x := u|_{O_l}$, with $\Lambda_l \in \text{int dom } \bar{V}$, for all $1 \leq l \leq k$. Let $\mathcal{M} := (\bigcup_{l=1}^k \partial O_l) \setminus \partial O$ be the interior boundary of the partition of O , and for all $r > 0$ set $\mathcal{M}_r := (\mathcal{M} + B_r) \cap O = \{x \in O : \text{dist}(x, \mathcal{M}) < r\}$, the r -neighborhood of this interior boundary. By Proposition A.17, for all $0 < \kappa \leq 1$ and $r > 0$, there exists a continuous piecewise affine function $u_{\kappa,r}$ on O with the following properties:

(i) $\nabla u_{\kappa,r} = \nabla u$ pointwise on $O \setminus \mathcal{M}_r$, and

$$\limsup_{r \downarrow 0} \sup_{0 < \kappa \leq 1} \|u_{\kappa,r} - u\|_{L^\infty(O)} = 0; \quad (3.28)$$

(ii) $\nabla u_{\kappa,r} \in \text{conv}(\{\Lambda_l : 1 \leq l \leq k\}) \Subset \text{int dom } \bar{V}$ pointwise (where $\text{conv}(\cdot)$ denotes the convex hull);

(iii) denoting by $O := \bigsqcup_{l=1}^{n_{\kappa,r}} O_{\kappa,r}^l$ the open partition associated with $u_{\kappa,r}$, and denoting $c_{\kappa,r}^l + \Lambda_{\kappa,r}^l \cdot x := u_{\kappa,r}|_{O_{\kappa,r}^l}$ for all l , then, for any i, j with $\partial O_{\kappa,r}^i \cap \partial O_{\kappa,r}^j \neq \emptyset$, we have $|\Lambda_{\kappa,r}^i - \Lambda_{\kappa,r}^j| \leq \kappa$.

We shall approximate u with these refined continuous piecewise affine functions $u_{\kappa,r}$ having smoother variations; in the sequel, we shall successively take the limits $\kappa \downarrow 0$ and $r \downarrow 0$.

Since $\omega \in \Omega'$ and $O \subset \mathbb{R}^d$ are fixed in the argument, we drop them from our notation. Fix $\kappa, r > 0$. By Step 1, for all $1 \leq i \leq n_{\kappa,r}$ there exists a sequence $(u_{\varepsilon,\kappa,r}^i)_\varepsilon \subset W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$ with $u_{\varepsilon,\kappa,r}^i \rightharpoonup c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot x$ in $W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$ and such that, for all Lipschitz subdomains $O' \subset O$, we have $J_\varepsilon(u_{\varepsilon,\kappa,r}^i, \omega; O') \rightarrow J(\Lambda_{\kappa,r}^i \cdot x; O')$. For all $\eta > 0$ and all $1 \leq i \leq n_{\kappa,r}$, define the sets

$$\begin{aligned} O_{\kappa,r,\eta}^{i+} &:= \{x \in O : \text{dist}(x, O_{\kappa,r}^i) < \eta\} = O \cap (O_{\kappa,r}^i + B_\eta), \\ O_{\kappa,r,\eta}^{i-} &:= \{x \in O_{\kappa,r}^i : \text{dist}(x, \partial O_{\kappa,r}^i) > \eta\} \end{aligned}$$

Let then $\sum_{i=1}^{n_{\kappa,r}} \chi_{\kappa,r,\eta}^i = 1$ be a partition of unity on O , where, for all $1 \leq i \leq n_{\kappa,r}$, the smooth cut-off function $\chi_{\kappa,r,\eta}^i$ has values in $[0, 1]$, equals 1 on $O_{\kappa,r,\eta}^{i-}$ and vanishes outside $O_{\kappa,r,\eta}^{i+}$, and satisfies the bound $|\nabla \chi_{\kappa,r,\eta}^i| \leq C'/\eta$ pointwise for some constant $C' > 0$. We now set

$$u_{\varepsilon,\kappa,r,\eta} := u_{\kappa,r} + \sum_{i=1}^{n_{\kappa,r}} (u_{\varepsilon,\kappa,r}^i - (c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot x)) \chi_{\kappa,r,\eta}^i.$$

By the Sobolev compact embedding for $p > d$, we have $u_{\varepsilon,\kappa,r}^i \rightarrow c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot x$ in $L^\infty(O; \mathbb{R}^m)$ as $\varepsilon \downarrow 0$, and hence $\limsup_{\eta} \limsup_{\varepsilon} \|u_{\varepsilon,\kappa,r,\eta} - u_{\kappa,r}\|_{L^\infty(O)} = 0$, so that (3.28) yields

$$\lim_{t \uparrow 1} \limsup_{r \downarrow 0} \limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} \|tu_{\varepsilon,\kappa,r,\eta} - u\|_{L^\infty(O)} = 0. \quad (3.29)$$

Let us now evaluate the integral functional $J_\varepsilon(\cdot, \omega; O)$ at $tu_{\varepsilon,\kappa,r,\eta}$ for $t \in [0, 1)$. Since

$$\begin{aligned} t \nabla u_{\varepsilon,\kappa,r,\eta} &= \sum_{i=1}^{n_{\kappa,r}} t \chi_{\kappa,r,\eta}^i \nabla u_{\varepsilon,\kappa,r}^i \\ &\quad + (1-t) \frac{t}{1-t} \sum_{i=1}^{n_{\kappa,r}} \left((u_{\varepsilon,\kappa,r}^i - (c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot x)) \nabla \chi_{\kappa,r,\eta}^i + (\nabla u_{\varepsilon,\kappa,r} - \Lambda_{\kappa,r}^i) \chi_{\kappa,r,\eta}^i \right), \end{aligned}$$

and $(1-t) + \sum_{i=1}^{n_{\kappa,r}} t \chi_{\kappa,r,\eta}^i = 1$, we have by convexity and non-negativity of V

$$\begin{aligned} J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta}, \omega; O) &\leq (1-t) E_{\varepsilon,\kappa,r,\eta,t} + t \sum_{i=1}^{n_{\kappa,r}} \int_{O_{\kappa,r,\eta}^{i+}} \chi_{\kappa,r,\eta}^i(y) V(y/\varepsilon, \nabla u_{\varepsilon,\kappa,r}^i(y), \omega) dy \\ &\leq (1-t) E_{\varepsilon,\kappa,r,\eta,t} + \sum_{i=1}^{n_{\kappa,r}} J_\varepsilon(u_{\varepsilon,\kappa,r}^i, \omega; O_{\kappa,r,\eta}^{i+}), \end{aligned} \quad (3.30)$$

where the error term reads

$$\begin{aligned} E_{\varepsilon,\kappa,r,\eta,t} &= \int_O V\left(y/\varepsilon, \frac{t}{1-t} \sum_{i=1}^{n_{\kappa,r}} \left((u_{\varepsilon,\kappa,r}^i(y) - (c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot y)) \nabla \chi_{\kappa,r,\eta}^i(y) \right. \right. \\ &\quad \left. \left. + (\nabla u_{\varepsilon,\kappa,r}(y) - \Lambda_{\kappa,r}^i) \chi_{\kappa,r,\eta}^i(y) \right), \omega\right) dy. \end{aligned}$$

For all i , set $N_{\kappa,r,\eta}^i := \{j : j \neq i, O_{\kappa,r,\eta}^{j+} \cap O_{\kappa,r,\eta}^{i+} \neq \emptyset\}$. We then rewrite the argument of the energy density in the error term as

$$\begin{aligned} S_{\varepsilon,\kappa,r,\eta}(y) &:= \left| \sum_{i=1}^{n_{\kappa,r}} \left((u_{\varepsilon,\kappa,r}^i(y) - (c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot y)) \nabla \chi_{\kappa,r,\eta}^i(y) + (\nabla u_{\varepsilon,\kappa,r}(y) - \Lambda_{\kappa,r}^i) \chi_{\kappa,r,\eta}^i(y) \right) \right| \\ &\leq \frac{C'}{\eta} \sum_{i=1}^{n_{\kappa,r}} \|u_{\varepsilon,\kappa,r}^i - (c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot x)\|_{L^\infty(O)} + \sup_{1 \leq i \leq n_{\kappa,r}} \sup_{j \in N_{\kappa,r,\eta}^i} |\Lambda_{\kappa,r}^j - \Lambda_{\kappa,r}^i|. \end{aligned}$$

Since by definition $\limsup_{\eta \downarrow 0} \sup_{j \in N_{\kappa,r,\eta}^i} |\Lambda_{\kappa,r}^j - \Lambda_{\kappa,r}^i| \leq \kappa$ for all i , we have

$$\limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} S_{\varepsilon,\kappa,r,\eta}(y) = 0$$

for all $r, \eta > 0$. By assumption, there exists $\delta > 0$ such that $\text{adh } B_\delta \subset \text{int dom } M$. Hence, for all $r, t > 0$ there exists $\kappa_{r,t} > 0$ such that for all $0 < \kappa < \kappa_{r,t}$ there exists $\eta_{\kappa,r} > 0$ such that for all $0 < \eta < \eta_{\kappa,r}$ there exists $\varepsilon_{\kappa,r,\eta,t} > 0$ with the following property: for all $0 < \varepsilon < \varepsilon_{\kappa,r,\eta,t}$, we have

$$\left\| \frac{t}{1-t} S_{\varepsilon,\kappa,r,\eta} \right\|_{L^\infty(O)} < \delta.$$

This yields the bound

$$E_{\varepsilon,\kappa,r,\eta,t} \leq |O| \sup_{|\Lambda'| < \delta} M(\Lambda') < \infty,$$

and proves

$$\lim_{t \uparrow 1} \limsup_{r \downarrow 0} \limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} (1-t) E_{\varepsilon,\kappa,r,\eta,t} = 0,$$

so that (3.30) turns into

$$\begin{aligned} & \limsup_{t \uparrow 1} \limsup_{r \downarrow 0} \limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} J_\varepsilon(tu_{\varepsilon, \kappa, r, \eta}, \omega; O) \\ & \leq \limsup_{r \downarrow 0} \limsup_{\kappa \downarrow 0} \sum_{i=1}^{n_{\kappa, r}} \limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} J_\varepsilon(u_{\varepsilon, \kappa, r}^i, \omega; O_{\kappa, r, \eta}^{i+}). \end{aligned} \quad (3.31)$$

For all i , we have by construction

$$\lim_{\varepsilon \downarrow 0} J_\varepsilon(u_{\varepsilon, \kappa, r}^i, \omega; O_{\kappa, r, \eta}^{i+}) = |O_{\kappa, r, \eta}^{i+}| \bar{V}(\Lambda_{\kappa, r}^i),$$

so that, by definition of $O_{\kappa, r, \eta}^{i+}$,

$$\lim_{\eta \downarrow 0} \lim_{\varepsilon \downarrow 0} J_\varepsilon(u_{\varepsilon, \kappa, r}^i, \omega; O_{\kappa, r, \eta}^{i+}) = |O_{\kappa, r}^i| \bar{V}(\Lambda_{\kappa, r}^i).$$

Hence, summing over i , $1 \leq i \leq n_{\kappa, r}$, yields

$$\sum_{i=1}^{n_{\kappa, r}} \lim_{\eta \downarrow 0} \lim_{\varepsilon \downarrow 0} J_\varepsilon(u_{\varepsilon, \kappa, r}^i, \omega; O_{\kappa, r, \eta}^{i+}) = \sum_{i=1}^{n_{\kappa, r}} |O_{\kappa, r}^i| \bar{V}(\Lambda_{\kappa, r}^i).$$

On the one hand, $\nabla u_{\kappa, r} = \nabla u$ holds on $O \setminus \mathcal{M}_r$. On the other hand, for all i, κ, r , $\Lambda_{\kappa, r}^i \in K := \text{conv}(\{\Lambda_l : 1 \leq l \leq k\})$, which is a compact subset of $\text{int dom } \bar{V}$. Using in addition the non-negativity of the energy density, one may then turn the above equality into

$$\begin{aligned} \sum_{i=1}^{n_{\kappa, r}} \lim_{\eta \downarrow 0} \lim_{\varepsilon \downarrow 0} J_\varepsilon(u_{\varepsilon, \kappa, r}^i, \omega; O_{\kappa, r, \eta}^{i+}) &= J(u_{\kappa, r}; O) = J(u; O \setminus \mathcal{M}_r) + J(u_{\kappa, r}; \mathcal{M}_r) \\ &\leq J(u; O) + |\mathcal{M}_r| \sup_K \bar{V}. \end{aligned}$$

Combined with (3.31), this yields

$$\limsup_{t \uparrow 1} \limsup_{r \downarrow 0} \limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} J_\varepsilon(tu_{\varepsilon, \kappa, r, \eta}, \omega; O) \leq J(u; O). \quad (3.32)$$

We are now in position to conclude. By coercivity of V , the sequence $\nabla(tu_{\varepsilon, \kappa, r, \eta})$ is bounded in $L^p(O; \mathbb{R}^{m \times d})$. Combined with (3.29) (convergence in $L^\infty(O; \mathbb{R}^m)$), this shows that any weakly converging subsequence of $(tu_{\varepsilon, \kappa, r, \eta})_{\varepsilon, \eta, \kappa, r, t}$ in $W^{1,p}(O; \mathbb{R}^m)$ converges to u . Hence the Γ -lim inf inequality of Proposition 3.2 yields

$$\liminf_{t \uparrow 1} \liminf_{r \downarrow 0} \liminf_{\kappa \downarrow 0} \liminf_{\eta \downarrow 0} \liminf_{\varepsilon \downarrow 0} J_\varepsilon(tu_{\varepsilon, \kappa, r, \eta}, \omega; O) \geq J(u; O).$$

These last two inequalities combine to

$$\limsup_{t \uparrow 1} \limsup_{r \downarrow 0} \limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} (|J_\varepsilon(tu_{\varepsilon, \kappa, r, \eta}, \omega; O) - J(u; O)| + \|tu_{\varepsilon, \kappa, r, \eta} - u\|_{L^p(O; \mathbb{R}^m)}) = 0,$$

and we conclude as before by the Attouch diagonalization lemma.

Step 3. Recovery sequence for general functions.

We claim that, for all $\omega \in \Omega'$, all bounded Lipschitz domains $O \subset \mathbb{R}^d$ and all $u \in W^{1,p}(O; \mathbb{R}^m)$, there is a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(u_\varepsilon, \omega; O) \rightarrow J(u; O)$. By the locality of recovery sequences (cf. Corollary 3.3), we may consider that O is a ball of \mathbb{R}^d to which we may apply the approximation result of Proposition A.16. By the Γ -lim inf inequality of Proposition 3.2, we can further assume that $u \in W^{1,p}(O; \mathbb{R}^m)$ satisfies

$$J(u; O) = \int_O \bar{V}(\nabla u(y)) dy < \infty,$$

so that $\nabla u \in \text{dom } \bar{V}$ almost everywhere. Let u be such a function and let $\omega \in \Omega'$ be fixed.

Since O is bounded, Lipschitz and strongly star-shaped, \bar{V} is convex, and $0 \in \text{int dom } \bar{V}$, Proposition A.16(ii) shows that there exists a sequence $(u_n)_n$ of continuous piecewise affine functions with

$\nabla u_n \in \text{int dom } \bar{V}$ pointwise such that $u_n \rightarrow u$ (strongly) in $W^{1,p}(O; \mathbb{R}^m)$ and $J(u_n; O) \rightarrow J(u; O)$ as $n \uparrow \infty$. By Step 2, for all n , there exists a sequence $(u_{\varepsilon,n})_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ such that $u_{\varepsilon,n} \rightarrow u_n$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(u_{\varepsilon,n}, \omega; O) \rightarrow J(u_n; O)$, as $\varepsilon \downarrow 0$. In particular,

$$\begin{aligned} \lim_{n \uparrow \infty} \lim_{\varepsilon \downarrow 0} (|J_\varepsilon(u_{\varepsilon,n}, \omega; O) - J(u; O)| + \|u_{\varepsilon,n} - u\|_{L^p(O; \mathbb{R}^m)}) \\ = \lim_{n \uparrow \infty} (|J(u_n; O) - J(u; O)| + \|u_n - u\|_{L^p(O; \mathbb{R}^m)}) = 0. \end{aligned}$$

We then conclude as before by the Attouch diagonalization argument. \square

3.6. Proof of Corollary 2.4(i): lifting Dirichlet boundary data. We split the proof into two steps. We first consider the case when $J(\alpha u; O) < \infty$ for some $\alpha > 1$, and then turn to the case when in addition $\int_O M(\nabla u) < \infty$ or $\int_O M(\alpha \nabla u) < \infty$ for some $\alpha > 1$.

Step 1. Case when $J(\alpha u; O) < \infty$, for some $\alpha > 1$.

As $v \in u + W_0^{1,p}(O; \mathbb{R}^m)$ and $J(\alpha u; O) < \infty$, Proposition A.16(ii)(a) yields the existence of a sequence $(v_k)_k \subset u + C_c^\infty(O; \mathbb{R}^m)$ with $v_k \rightarrow v$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J(v_k; O) \rightarrow J(v; O)$. For all $r > 0$, set $O_r^1 := \{x \in O : \text{dist}(x, \partial O) > 2r\}$, $O_r^2 := \{x \in O : \text{dist}(x, \partial O) > r\}$, and choose smooth cut-off functions χ_r^1, χ_r^2 with the following properties: the functions take values in $[0, 1]$, χ_r^1 equals 1 on O_r^1 and 0 on $\mathbb{R}^d \setminus O_r^2$, χ_r^2 equals 1 on O_r^2 and 0 on $\mathbb{R}^d \setminus O$, and $|\nabla \chi_r^1|, |\nabla \chi_r^2| \leq C'/r$ for some constant C' . For all $\omega \in \Omega'$, Proposition 3.10 provides sequences $(u_\varepsilon^\omega)_\varepsilon$ and $(v_{\varepsilon,r,k}^\omega)_\varepsilon$ in $W^{1,p}(O; \mathbb{R}^m)$ such that $u_\varepsilon^\omega \rightarrow u$ and $v_{\varepsilon,r,k}^\omega \rightarrow \chi_r^1 v_k + (1 - \chi_r^1)u$ in $W^{1,p}(O; \mathbb{R}^m)$, and $J_\varepsilon(u_\varepsilon^\omega, \omega; O') \rightarrow J(u; O')$ and $J_\varepsilon(v_{\varepsilon,r,k}^\omega, \omega; O') \rightarrow J(\chi_r^1 v_k + (1 - \chi_r^1)u; O')$, for any subdomain $O' \subset O$. We then set $w_{\varepsilon,r,k}^\omega := \chi_r^2 v_{\varepsilon,r,k}^\omega + (1 - \chi_r^2)u_\varepsilon^\omega$. Given $t \in [0, 1)$, using the decomposition

$$t \nabla w_{\varepsilon,r,k}^\omega = t \chi_r^2 \nabla v_{\varepsilon,r,k}^\omega + t(1 - \chi_r^2) \nabla u_\varepsilon^\omega + (1 - t) \frac{t}{1 - t} \nabla \chi_r^2 (v_{\varepsilon,r,k}^\omega - u_\varepsilon^\omega),$$

and convexity, we obtain

$$J_\varepsilon(t w_{\varepsilon,r,k}^\omega, \omega; O) \leq (1 - t) E_{\varepsilon,r,k,t}^\omega + J_\varepsilon(v_{\varepsilon,r,k}^\omega, \omega; O) + J_\varepsilon(u_\varepsilon^\omega, \omega; O \setminus O_r^2), \quad (3.33)$$

where the error term reads

$$E_{\varepsilon,r,k,t}^\omega := \int_O V \left(y/\varepsilon, \frac{t}{1 - t} \nabla \chi_r^2(y) (v_{\varepsilon,r,k}^\omega(y) - u_\varepsilon^\omega(y)), \omega \right) dy.$$

For all $y \in O \setminus O_r^2$, since $\chi_r^1(y) = 0$, we have

$$|v_{\varepsilon,r,k}^\omega(y) - u_\varepsilon^\omega(y)| \leq \|v_{\varepsilon,r,k}^\omega - (\chi_r^1 v_k + (1 - \chi_r^1)u)\|_{L^\infty(O)} + \|u_\varepsilon^\omega - u\|_{L^\infty(O)}.$$

By assumption, there is some $\delta > 0$ with $\text{adh } B_\delta \subset \text{int dom } M$. Hence, for all fixed r, k, t , there exists $\varepsilon_{r,k,t} > 0$ such that for all $0 < \varepsilon < \varepsilon_{r,k,t}$ we have

$$\left\| \frac{t}{1 - t} \nabla \chi_r^2 (v_{\varepsilon,r,k}^\omega - u_\varepsilon^\omega) \right\|_{L^\infty(O)} < \delta,$$

and therefore

$$\limsup_{\varepsilon \downarrow 0} E_{\varepsilon,r,k,t}^\omega \leq |O| \sup_{|\Lambda'| < \delta} M(\Lambda') < \infty.$$

Inequality (3.33) then turns into

$$\begin{aligned} \limsup_{t \uparrow 1} \limsup_{k \uparrow \infty} \limsup_{r \downarrow 0} \limsup_{\varepsilon \downarrow 0} J_\varepsilon(t w_{\varepsilon,r,k}^\omega, \omega; O) \\ \leq \limsup_{k \uparrow \infty} \limsup_{r \downarrow 0} J(\chi_r^1 v_k + (1 - \chi_r^1)u; O) + \limsup_{r \downarrow 0} J(u; O \setminus O_r^2). \end{aligned}$$

The second term in the right-hand side vanishes since $J(u; O) < \infty$, and it only remains to study the first term. By definition, for fixed k , we have $v_k \in u + C_c^\infty(O; \mathbb{R}^m)$, so that for all $r > 0$ small enough, $\chi_r^1 v_k + (1 - \chi_r^1)u = v_k$ pointwise on O . This implies

$$\limsup_{k \uparrow \infty} \limsup_{r \downarrow 0} J(\chi_r^1 v_k + (1 - \chi_r^1)u; O) = \limsup_{k \uparrow \infty} J(v_k; O) = J(v; O),$$

and thus

$$\limsup_{t \uparrow 1} \limsup_{k \uparrow \infty} \limsup_{r \downarrow 0} \limsup_{\varepsilon \downarrow 0} J_\varepsilon(tw_{\varepsilon,r,k}^\omega; \omega; O) \leq J(v; O).$$

Combined with the Γ -lim inf inequality of Proposition 3.2 and a diagonalization argument, this proves the first part of the statement.

Step 2. Cases when $\int_O M(\nabla u) < \infty$ or $\int_O M(\alpha \nabla u) < \infty$ for some $\alpha > 1$.

If u is chosen in such a way that $\int_O M(\nabla u(y))dy < \infty$, then we can repeat the argument of Step 1 with $w_\varepsilon^\omega := u$, and bound the last term in the right-hand side of (3.33) by

$$\limsup_{r \downarrow 0} \limsup_{\varepsilon \downarrow 0} J_\varepsilon(u, \omega; O \setminus O_r^2) \leq \limsup_{r \downarrow 0} \int_{O \setminus O_r^2} M(\nabla u(y))dy = 0.$$

We conclude with the case when u satisfies $\int_O M(\alpha \nabla u(y))dy < \infty$ for some $\alpha > 1$. Let v_k, χ_r^1, χ_r^2 be chosen as in Step 1, and let $\omega \in \Omega'$ be fixed. For any $t \in [0, 1)$, Proposition 3.10 shows the existence of a sequence $(v_{\varepsilon,r,k,t}^\omega)_\varepsilon$ in $W^{1,p}(O; \mathbb{R}^m)$ such that $v_{\varepsilon,r,k,t}^\omega \rightharpoonup \chi_r^1 v_k + (1 - \chi_r^1)u/t$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(v_{\varepsilon,r,k,t}^\omega; \omega; O') \rightarrow J(\chi_r^1 v_k + (1 - \chi_r^1)u/t; O')$, for any subdomain $O' \subset O$. Set $w_{\varepsilon,r,k,t}^\omega := \chi_r^2 v_{\varepsilon,r,k,t}^\omega + (1 - \chi_r^2)u/t$. As before, we obtain by convexity

$$\begin{aligned} J_\varepsilon(tw_{\varepsilon,r,k,t}^\omega; \omega) &\leq J_\varepsilon(v_{\varepsilon,r,k,t}^\omega; \omega; O) + J_\varepsilon(u/t; \omega; O \setminus O_r^2) \\ &\quad + (1-t) \int_O M\left(\frac{t}{1-t} \nabla \chi_r^2(y)(v_{\varepsilon,r,k,t}^\omega(y) - u(y)/t)\right) dy, \end{aligned}$$

and the conclusion then follows, using the convexity once more in the following form: for $t > 1/\alpha$,

$$\begin{aligned} \limsup_{r \downarrow 0} \limsup_{\varepsilon \downarrow 0} J_\varepsilon(u/t; \omega; O \setminus O_r^2) &\leq \limsup_{r \downarrow 0} \int_{O \setminus O_r^2} M(\nabla u(y)/t) dy \\ &\leq \frac{1}{\alpha t} \limsup_{r \downarrow 0} \int_{O \setminus O_r^2} M(\alpha \nabla u(y)) dy + \limsup_{r \downarrow 0} |O \setminus O_r^2| M(0) = 0. \end{aligned}$$

This completes the proof. \square

Remark 3.11. As can be seen in the proof, the assumption that $J(\alpha u; O) < \infty$ can be relaxed to $J(\alpha u; O') < \infty$ for some open neighborhood $O' \subset O$ of ∂O in O . \square

3.7. Proof of Corollary 2.4(ii): soft buffer zone for Dirichlet boundary data. We split the proof into two steps. For all $s > 0$ and $O \subset \mathbb{R}^d$, we use the notation $O_s := \{x \in O : \text{dist}(x, \partial O) > s\}$.

Step 1. Γ -lim inf inequality.

Let $\omega \in \Omega'$, let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, let $u \in W^{1,p}(O; \mathbb{R}^m)$ with $J(u; O) < \infty$, and let $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ be a sequence with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O; \mathbb{R}^m)$. By the Γ -lim inf inequality for J_ε in Proposition 3.2,

$$\liminf_{\varepsilon \downarrow 0} J_\varepsilon^\eta(u_\varepsilon; \omega; O) \geq \liminf_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon; \omega; O_\eta) \geq J(u; O_\eta) = \int_{O_\eta} \bar{V}(\nabla u(y)) dy,$$

that is, using that $\int_{O \setminus O_\eta} \bar{V}(\nabla u(y)) dy \rightarrow 0$ as $\eta \downarrow 0$,

$$\liminf_{\eta \downarrow 0} \liminf_{\varepsilon \downarrow 0} J_\varepsilon^\eta(u_\varepsilon; \omega; O) \geq J(u; O).$$

Step 2. Γ -lim sup inequality.

Let $\omega \in \Omega'$, let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $u \in W^{1,p}(O; \mathbb{R}^m)$ with $J(u; O) < \infty$. By Proposition 3.10, there exists a sequence $(w_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ such that $w_\varepsilon \rightarrow 0$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(u + w_\varepsilon, \omega; O) \rightarrow J(u; O)$. Given $\eta > 0$, choose a cut-off function χ_η with values in $[0, 1]$, such that χ_η equals 1 on O_η and 0 outside O , and that satisfies $|\nabla \chi_\eta| \leq C'/\eta$ for some constant $C' > 0$. Set $v_{\varepsilon,\eta} := \chi_\eta w_\varepsilon \in W_0^{1,p}(O; \mathbb{R}^m)$. For all $t \in [0, 1]$, we have

$$t\nabla u + t\nabla v_{\varepsilon,\eta} = t\chi_\eta \nabla(u + w_\varepsilon) + t(1 - \chi_\eta)\nabla u + (1 - t)\frac{t}{1-t}w_\varepsilon \nabla \chi_\eta,$$

so that by convexity and the definition of $V_\varepsilon^{O,\eta}$,

$$\begin{aligned} J_\varepsilon^\eta(tu + tv_{\varepsilon,\eta}, \omega; O) &\leq (1-t)E_{\varepsilon,\eta,t} + J_\varepsilon^\eta(u + w_\varepsilon, \omega; O) + \int_O (1 - \chi_\eta(y))V_\varepsilon^{O,\eta}(y, \nabla u(y), \omega) dy \\ &\leq (1-t)E_{\varepsilon,\eta,t} + J_\varepsilon(u + w_\varepsilon, \omega; O) + \int_{O \setminus O_\eta} |\nabla u(y)|^p dy, \end{aligned} \quad (3.34)$$

where the error is defined by

$$\begin{aligned} E_{\varepsilon,\eta,t} &:= \int_O V_\varepsilon^{O,\eta} \left(y, \frac{t}{1-t}w_\varepsilon(y)\nabla \chi_\eta(y), \omega \right) dy \\ &\leq |O_\eta|M(0) + \int_{O \setminus O_\eta} \left| \frac{t}{1-t}w_\varepsilon(y)\nabla \chi_\eta(y) \right|^p dy. \end{aligned} \quad (3.35)$$

By the Rellich-Kondrachov theorem, $w_\varepsilon \rightarrow 0$ (strongly) in $L^p(O)$, so that $\limsup_\varepsilon E_{\varepsilon,\eta,t} \leq |O|M(0)$ for all t, η . Passing to the limit in inequality (3.34) thus yields

$$\limsup_{t \uparrow 1} \limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} J_\varepsilon^\eta(tu + tv_\varepsilon, \omega; O, \eta) \leq \limsup_{\varepsilon \downarrow 0} J_\varepsilon(u + w_\varepsilon, \omega; O) = J(u; O).$$

We then conclude by the same diagonalization argument as before and Step 1. This proves the first part of the statement.

Now consider the case when u satisfies $J(\alpha u; O) < \infty$ for some $\alpha > 1$. Then, for all $t \in [0, 1]$, Proposition 3.10 provides a sequence $(w_{\varepsilon,t})_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ such that $w_{\varepsilon,t} \rightarrow 0$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(u/t + w_{\varepsilon,t}, \omega; O) \rightarrow J(u/t; O)$ as $\varepsilon \downarrow 0$. Define $v_{\varepsilon,t,\eta} := \chi_\eta w_{\varepsilon,t}$, where χ_η is the same cut-off function as above. We then have now

$$\nabla u + t\nabla v_{\varepsilon,t,\eta} = t\chi_\eta \nabla(u/t + w_{\varepsilon,t}) + t(1 - \chi_\eta)\nabla u/t + (1 - t)\frac{t}{1-t}w_{\varepsilon,t} \nabla \chi_\eta,$$

so that by convexity and definition of $V_\varepsilon^{O,\eta}$,

$$J_\varepsilon^\eta(u + tv_{\varepsilon,t,\eta}, \omega; O) \leq (1-t)E'_{\varepsilon,\eta,t} + J_\varepsilon(u/t + w_{\varepsilon,t}, \omega; O) + t^{1-p} \int_{O \setminus O_\eta} |\nabla u(y)|^p dy, \quad (3.36)$$

where the error is defined by

$$E'_{\varepsilon,\eta,t} := |O_\eta|M(0) + \int_{O \setminus O_\eta} \left| \frac{t}{1-t}w_{\varepsilon,t}(y)\nabla \chi_\eta(y) \right|^p dy.$$

By the Rellich-Kondrachov theorem, $w_{\varepsilon,t} \rightarrow 0$ (strongly) in $L^p(O)$ for all t , so that $\limsup_\varepsilon E'_{\varepsilon,\eta,t} = |O_\eta|M(0)$ for all t, η . Passing to the limit in inequality (3.36) then yields

$$\limsup_{t \uparrow 1} \limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} J_\varepsilon^\eta(u + tv_{\varepsilon,t,\eta}, \omega; O) \leq \limsup_{t \uparrow 1} \limsup_{\varepsilon \downarrow 0} J_\varepsilon(u/t + w_{\varepsilon,t}, \omega; O) = \limsup_{t \uparrow 1} J(u/t; O).$$

Since u satisfies $J(\alpha u; O) < \infty$, we deduce by convexity that the map $t \mapsto J(u/t; O)$ is continuous on $(1/\alpha, 1]$. This implies $\limsup_{t \uparrow 1} J(u/t; O) = J(u; O)$, and the conclusion follows. \square

Remark 3.12. As can be seen in the proof, the assumption that $J(\alpha u; O) < \infty$ can be relaxed to $J(\alpha u; O') < \infty$ for some open neighborhood $O' \subset O$ of ∂O in O . \square

4. PROOF OF THE RESULTS FOR NONCONVEX INTEGRANDS

In this section, we study the case when W is nonconvex but admits a two-sided estimate by a convex function (which may depend on the space variable) and prove Theorem 2.6. Let W be a (nonconvex) τ -stationary normal random integrand, which is further assumed to be ru-usc (in the sense of Definition 2.5, with respect to some τ -stationary integrable random field a). Up to a translation, for simplicity of notation, we can restrict to the following stronger version of (2.10) and (2.4): for almost all ω , y , we have, for all Λ ,

$$\frac{1}{C}|\Lambda|^p \leq V(y, \Lambda, \omega) \leq W(y, \Lambda, \omega) \leq C(V(y, \Lambda, \omega) + 1), \quad (4.1)$$

for some $C > 0$ and $d < p < \infty$, and for some convex τ -stationary normal random integrand V . Also assume that 0 belongs to the interior of the domain of the convex function $M := \sup \operatorname{ess}_{y, \omega} V(y, \cdot, \omega)$. We can then apply Theorem 2.2 to V , yielding an homogenized energy density \bar{V} with the following property: defining

$$J_\varepsilon(u, \omega; O) = \int_O V(y/\varepsilon, \nabla u(y), \omega) dy, \quad J(u; O) = \int_O \bar{V}(\nabla u(y)) dy,$$

for almost all ω , the integral functionals $J_\varepsilon(\cdot, \omega; O)$ Γ -converge to $J(\cdot; O)$ on $W^{1,p}(O; \mathbb{R}^m)$, for any bounded Lipschitz domain $O \subset \mathbb{R}^d$. Let $\Omega_0 \subset \Omega$ be a subset of maximal probability on which all these assumptions and properties (of V, W) are simultaneously pointwise satisfied.

4.1. Definition of the homogenized energy density. We need to define in this section a candidate for the homogenized energy density \bar{W} . As before, the standard homogenization formula with Dirichlet boundary conditions does not hold because of the generality of the growth conditions considered here. Instead, we use the corrector for the convex problem as a boundary condition for the nonconvex problem, which is indeed admissible because of the two-sided growth condition (4.1).

More precisely, for all $\Lambda \in \mathbb{R}^{m \times d}$, Lemma 3.4 yields a function $\varphi_\Lambda \in \operatorname{Mes}(\Omega; W_{\operatorname{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m))$ such that $\nabla \varphi_\Lambda(0, \cdot) \in F_{\operatorname{pot}}^p(\Omega)^m$ and

$$\bar{V}(\Lambda) = \mathbb{E}[V(0, \Lambda + \nabla \varphi_\Lambda(0, \cdot), \cdot)].$$

Now, for any $t \in [0, 1)$, consider the function μ_Λ^t defined by

$$\mu_\Lambda^t(O, \omega) := \inf_{v \in W_0^{1,p}(O; \mathbb{R}^m)} \int_O W(y, t\Lambda + t\nabla \varphi_\Lambda(y, \omega) + \nabla v(y), \omega) dy.$$

As this quantity is stationary and subadditive, the Ackoglu-Krengel ergodic theorem implies:

Lemma 4.1 (Definition of the homogenized energy density). *Let $t \in [0, 1)$ be fixed. Then, there exists a function $\bar{W}_t : \operatorname{dom} \bar{V} \rightarrow [0, \infty)$ such that, for all $\Lambda \in \operatorname{dom} \bar{V}$, the following holds for almost all $\omega \in \Omega_0$: for all bounded Lipschitz domains $O \subset \mathbb{R}^d$,*

$$\bar{W}_t(\Lambda) = \lim_{\varepsilon \downarrow 0} \frac{\mu_{\Lambda'}^t(O/\varepsilon, \omega)}{|O/\varepsilon|}, \quad (4.2)$$

where convergence also holds for expectations. Now define $\bar{W}(\Lambda) := \liminf_{t \uparrow 1} \liminf_{\Lambda' \rightarrow \Lambda} \bar{W}_t(\Lambda')$ for any $\Lambda \in \operatorname{dom} \bar{V}$, and further set $\bar{W}(\Lambda) = \infty$ for all $\Lambda \notin \operatorname{dom} \bar{V}$. Then, \bar{W} satisfies $\bar{V} \leq \bar{W} \leq C(1 + \bar{V})$ on the whole of $\mathbb{R}^{m \times d}$, and for all $\Lambda \in \mathbb{R}^{m \times d}$ the following holds for almost all ω and all bounded Lipschitz domain $O \subset \mathbb{R}^d$:

$$\bar{W}(\Lambda) = \liminf_{t \uparrow 1} \liminf_{\Lambda' \rightarrow \Lambda} \lim_{\varepsilon \downarrow 0} \frac{\mu_{\Lambda'}^t(O/\varepsilon, \omega)}{|O/\varepsilon|}, \quad (4.3)$$

where the \liminf as $t \uparrow 1$ can further be restricted to $t \in \mathbb{Q}$. Finally, in the particular case when W is convex, then \bar{W} coincides with the various definitions for the homogenized integrand as given by Theorem 2.2. \square

Proof. We split the proof into three steps.

Step 1. Definition of $\overline{W}_t(\Lambda)$ and proof of (4.2).

First consider the case when $\Lambda \in \text{dom} \overline{V}$. Let $t \in [0, 1)$ be fixed. The upper bound in (4.1) then implies $\mathbb{E}[\mu_\Lambda^t(O, \cdot)] \leq C|O|(1 + t\overline{V}(\Lambda) + (1-t)M(0)) < \infty$. As the function μ_Λ^t is obviously stationary and subadditive, and as $\mu_\Lambda^t(O, \cdot)$ is measurable by Hypothesis 2.1, the Ackoglu-Krengel subadditive ergodic theorem (see e.g. [33, Section 6.2]) can be applied and asserts the existence of some $\overline{W}_t(\Lambda) \in [0, \infty)$ such that, for almost all ω , we have

$$\overline{W}_t(\Lambda) = \lim_{n \uparrow \infty} \frac{\mu_\Lambda^t(I_n, \omega)}{|I_n|},$$

for any regular sequence $(I_n)_n \subset \mathcal{I} := \{[a, b) : a, b \in \mathbb{Z}^d\}$ such that $\lim_{n \uparrow \infty} I_n = \mathbb{R}^d$ (in the usual sense of [33, Section 6.2]), and moreover this convergence also holds for expectations. In particular, we easily see that the same result must hold for the choice $I_n = nQ_0$, where Q_0 is any cube aligned with the axes. Further note that, for all bounded Lipschitz subsets $O' \subset O \subset \mathbb{R}^d$, we can estimate, as $W_0^{1,p}(O'/\varepsilon; \mathbb{R}^m) \subset W_0^{1,p}(O/\varepsilon; \mathbb{R}^m)$,

$$\begin{aligned} \varepsilon^d \mu_\Lambda^t(O/\varepsilon, \omega) &\leq \varepsilon^d \mu_\Lambda^t(O'/\varepsilon, \omega) + \varepsilon^d \int_{(O \setminus O')/\varepsilon} W(y, t\Lambda + t\nabla\varphi_\Lambda(y, \omega), \omega) dy \\ &\leq \varepsilon^d \mu_\Lambda^t(O'/\varepsilon, \omega) + C|O \setminus O'| \left(1 + \int_{(O \setminus O')/\varepsilon} V(y, \Lambda + \nabla\varphi_\Lambda(y, \omega), \omega) dy \right), \end{aligned}$$

where the last expression in brackets converges to $1 + \overline{V}(\Lambda) < \infty$ as $\varepsilon \downarrow 0$. Now based on this estimate, an easy approximation argument (see e.g. [30, Step 4 of the proof of Theorem 3.1]) allows us to conclude as follows: for almost all ω (for all $\omega \in \Omega_\Lambda$, for some subset $\Omega_\Lambda \subset \Omega$ of maximal probability, say), we have for all bounded Lipschitz domains $O \subset \mathbb{R}^d$ and all $t \in (0, 1)$

$$\lim_{\varepsilon \downarrow 0} \frac{\mu_\Lambda^t(O/\varepsilon, \omega)}{|O/\varepsilon|} = \overline{W}_t(\Lambda). \quad (4.4)$$

Step 2. Definition of \overline{W} and proof of the bounds $\overline{V}(\Lambda) \leq \overline{W}(\Lambda) \leq C(1 + \overline{V}(\Lambda))$ for $\Lambda \in \text{dom} \overline{V}$.

Let $\Lambda \in \text{dom} \overline{V}$ be fixed, and let $O \subset \mathbb{R}^d$ be some bounded Lipschitz domain. We define $\overline{W}(\Lambda) = \liminf_t \liminf_{\Lambda' \rightarrow \Lambda} \overline{W}_t(\Lambda')$. The bounds $\overline{V}(\Lambda) \leq \overline{W}(\Lambda) \leq C(1 + \overline{V}(\Lambda))$ directly follow from the two-sided estimate (4.1) together with the following equality, for almost all ω ,

$$\overline{V}(\Lambda) = \overline{V}_0(\Lambda, \omega), \quad (4.5)$$

where we have defined

$$\overline{V}_0(\Lambda, \omega) := \liminf_{t \uparrow 1} \liminf_{\Lambda' \rightarrow \Lambda} \lim_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O; \mathbb{R}^m)} \int_O V(y/\varepsilon, t\Lambda' + t\nabla\varphi_{\Lambda'}(y/\varepsilon, \omega) + \nabla v(y), \omega) dy.$$

Let us give the argument for (4.5). On the one hand, we can estimate

$$\overline{V}_0(\Lambda, \omega) \geq \liminf_{t \uparrow 1} \liminf_{\Lambda' \rightarrow \Lambda} \lim_{\varepsilon \downarrow 0} \inf_{\substack{v \in W_0^{1,p}(O; \mathbb{R}^m) \\ \int_O \nabla v = 0}} \int_O V(y/\varepsilon, t\Lambda' + t\Lambda'_\varepsilon(\omega) + \nabla v(y), \omega) dy,$$

where we have set $\Lambda'_\varepsilon(\omega) := \int_O \nabla\varphi_{\Lambda'}(\cdot/\varepsilon, \omega)$. For almost all ω , since $\Lambda'_\varepsilon(\omega) \rightarrow 0$, we can write, for any $\kappa > 0$,

$$\overline{V}_0(\Lambda, \omega) \geq \inf_{\Lambda': |\Lambda' - \Lambda| \leq \kappa} \liminf_{t \uparrow 1} \lim_{\varepsilon \downarrow 0} \inf_{\substack{v \in W_0^{1,p}(O; \mathbb{R}^m) \\ \int_O \nabla v = 0}} \int_O V(y/\varepsilon, t\Lambda' + \nabla v(y), \omega) dy,$$

so that formula (2.8) yields $\overline{V}_0(\Lambda, \omega) \geq \inf_{\Lambda': |\Lambda' - \Lambda| \leq \kappa} \overline{V}(\Lambda')$. Passing to the limit $\kappa \downarrow 0$, the lower semicontinuity of \overline{V} directly gives $\overline{V}_0(\Lambda, \omega) \geq \overline{V}(\Lambda)$. On the other hand, the convexity of V , the

Birkhoff-Khinchin ergodic theorem and the definition of $\varphi_{\Lambda'}$ give for all $t \in [0, 1]$ and all $\Lambda' \in \mathbb{R}^{m \times d}$

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O; \mathbb{R}^m)} \int_O V(y/\varepsilon, t\Lambda' + t\nabla\varphi_{\Lambda'}(y/\varepsilon, \omega) + \nabla v(y), \omega) dy \\ & \leq \lim_{\varepsilon \downarrow 0} \int_O V(y/\varepsilon, \Lambda' + \nabla\varphi_{\Lambda'}(y/\varepsilon, \omega), \omega) dy + (1-t)M(0) = \bar{V}(\Lambda') + (1-t)M(0). \end{aligned}$$

Passing to the limit $\Lambda' \rightarrow \Lambda$ and $t \uparrow 1$, and using the lower semicontinuity of \bar{V} in the form of $\bar{V}(\Lambda) = \liminf_{\Lambda' \rightarrow \Lambda} \bar{V}(\Lambda')$, this gives $\bar{V}_0(\Lambda, \omega) \leq \bar{V}(\Lambda)$. The desired identity (4.5) is proven.

Step 3. Case when $\Lambda \notin \text{dom } \bar{V}$.

For $\Lambda \notin \text{dom } \bar{V}$, arguing as in Step 2 above, we can estimate, using the pointwise bound $V \leq W$,

$$\liminf_{t \uparrow 1} \liminf_{\Lambda' \rightarrow \Lambda} \liminf_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O; \mathbb{R}^m)} \int_O W(y/\varepsilon, t\Lambda + t\nabla\varphi_{\Lambda}(y/\varepsilon, \omega) + \nabla v(y), \omega) dy \geq \bar{V}(\Lambda) = \infty,$$

so that (4.3) trivially holds with $\bar{W}(\Lambda) := \infty$. Moreover, the bounds $\bar{V} \leq \bar{W} \leq C(1 + \bar{V})$ holds as well. \square

Although the definition of the homogenized energy density $\bar{W}(\Lambda)$ may a priori depend on the choice of a corrector φ_{Λ} , it would follow a posteriori from the Γ -convergence result that the value of $\bar{W}(\Lambda)$ is independent of that choice. As this independence will actually be useful in the proof of the Γ -lim sup inequality (see the proof of Lemma 4.4(c)), we display a direct proof.

Lemma 4.2 (Independence upon the choice of a corrector). *Assume $p > d$, and let $t \in [0, 1]$ and $\Lambda \in \text{dom } \bar{V}$ be fixed. For almost all ω , given a bounded domain $O \subset \mathbb{R}^d$, if $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ satisfies $\|u_\varepsilon\|_{L^\infty(O)} \rightarrow 0$ and $\limsup_\varepsilon \int_D V(y/\varepsilon, \Lambda + \nabla u_\varepsilon(y), \omega) dy \leq C_\Lambda |D|$ for all subdomains $D \subset O$ and some constant $C_\Lambda > 0$, then, for all Lipschitz subdomains $D \subset O$,*

$$\bar{W}_t(\Lambda) = \lim_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(D; \mathbb{R}^m)} \int_D W(y/\varepsilon, t\Lambda + t\nabla u_\varepsilon(y) + \nabla v(y), \omega) dy, \quad (4.6)$$

where the limit is well-defined. \square

Proof. Let $t \in [0, 1]$ and $\Lambda \in \text{dom } \bar{V}$ be fixed. Let $\omega \in \Omega$ be fixed such that (4.2) holds on all bounded Lipschitz domains and such that moreover, for all bounded domains $D \subset \mathbb{R}^d$,

$$\|\varepsilon\varphi_\Lambda(\cdot/\varepsilon, \omega)\|_{L^\infty(D)} \rightarrow 0, \quad \int_D V(y/\varepsilon, \Lambda + \nabla\varphi_\Lambda(y/\varepsilon, \omega), \omega) dy \rightarrow \bar{V}(\Lambda),$$

which follows almost surely from Lemma 3.4, the Sobolev embedding and the Birkhoff-Khinchin ergodic theorem. Let $(u_\varepsilon)_\varepsilon$ be as in the statement of the lemma. Also denote $v_\varepsilon^\omega := \varepsilon\varphi_\Lambda(\cdot/\varepsilon, \omega) \in W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$. By the choice of ω , the sequence $(v_\varepsilon^\omega)_\varepsilon$ satisfies the same properties as u_ε on any bounded domain, with C_Λ replaced by $C'_\Lambda = \bar{V}(\Lambda)$, and moreover, for all bounded Lipschitz domains $D \subset \mathbb{R}^d$,

$$\bar{W}_t(\Lambda) = \lim_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(D; \mathbb{R}^m)} \int_D W(y/\varepsilon, t\Lambda + t\nabla v_\varepsilon^\omega(y) + \nabla v(y), \omega) dy. \quad (4.7)$$

Let $D \subset O$ be some fixed Lipschitz subdomain. On the one hand, define

$$\bar{W}'_t(\Lambda, \omega; D) = \limsup_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(D; \mathbb{R}^m)} \int_D W(y/\varepsilon, t\Lambda + t\nabla u_\varepsilon(y) + \nabla v(y), \omega) dy. \quad (4.8)$$

Given $\eta > 0$, set $D_\eta := \{x \in D : \text{dist}(x, \partial D) > \eta\}$ and consider the difference

$$\begin{aligned} \Delta_{\varepsilon, t, \eta}^\omega & := \inf_{v \in W_0^{1,p}(D; \mathbb{R}^m)} \int_D W(y/\varepsilon, t\Lambda + t\nabla u_\varepsilon(y) + \nabla v(y), \omega) dy \\ & \quad - \inf_{w \in W_0^{1,p}(D_\eta; \mathbb{R}^m)} \int_{D_\eta} W(y/\varepsilon, t\Lambda + t\nabla v_\varepsilon^\omega(y) + \nabla w(y), \omega) dy. \end{aligned} \quad (4.9)$$

Choose a smooth cut-off function χ_η such that χ_η equals 1 on D_η and vanishes outside D , with $|\nabla\chi_\eta| \leq C'/\eta$ for some constant $C' > 0$, and define $w_{\varepsilon,\eta}^\omega := \chi_\eta v_\varepsilon^\omega + (1 - \chi_\eta)u_\varepsilon$. Restricting the first infimum in (4.9) to those v 's that are equal to $t(v_\varepsilon - u_\varepsilon)$ on ∂D_η , we obtain

$$\Delta_{\varepsilon,t,\eta}^\omega \leq \inf_{v \in W_0^{1,p}(D \setminus D_\eta; \mathbb{R}^m)} \int_{D \setminus D_\eta} W(y/\varepsilon, t\Lambda + t\nabla w_{\varepsilon,\eta}^\omega(y) + \nabla v(y), \omega) dy.$$

Hence, choosing $v = 0$, using the upper bound $W \leq C(1 + V)$ and decomposing

$$t\nabla w_{\varepsilon,\eta}^\omega = t\chi_\eta \nabla v_\varepsilon^\omega + t(1 - \chi_\eta)\nabla u_\varepsilon + (1 - t)\frac{t}{1-t}\nabla\chi_\eta(v_\varepsilon^\omega - u_\varepsilon),$$

we obtain by convexity

$$\begin{aligned} \Delta_{\varepsilon,t,\eta}^\omega \leq C|D \setminus D_\eta| & \left(1 + \int_{D \setminus D_\eta} V(y/\varepsilon, \Lambda + \nabla u_\varepsilon(y), \omega) dy \right. \\ & \left. + \int_{D \setminus D_\eta} V(y/\varepsilon, \Lambda + \nabla v_\varepsilon^\omega(y), \omega) dy + E_{\varepsilon,t,\eta}^\omega \right), \end{aligned}$$

where the error reads

$$E_{\varepsilon,t,\eta}^\omega := \int_{D \setminus D_\eta} V\left(y/\varepsilon, \frac{t}{1-t}\nabla\chi_\eta(y)(v_\varepsilon^\omega(y) - u_\varepsilon(y)), \omega\right) dy.$$

Since v_ε^ω and u_ε go to 0 in $L^\infty(D; \mathbb{R}^m)$, we can prove that, for any $t \in (0, 1)$,

$$\limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} \Delta_{\varepsilon,t,\eta}^\omega \leq \limsup_{\eta \downarrow 0} C|D \setminus D_\eta|(1 + C_\Lambda + C'_\Lambda) = 0.$$

In view of equalities (4.7) and (4.10), this implies $\overline{W}'_t(\Lambda, \omega; D) \leq \overline{W}_t(\Lambda)$.

On the other hand, define

$$\overline{W}''_t(\Lambda, \omega; D) = \liminf_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(D; \mathbb{R}^m)} \int_D W(y/\varepsilon, t\Lambda + t\nabla u_\varepsilon(y) + \nabla v(y), \omega) dy, \quad (4.10)$$

and repeat the same argument as above with $D^\eta := \{x : \text{dist}(x, D) < \eta\}$ and

$$\begin{aligned} \tilde{\Delta}_{\varepsilon,t,\eta}^\omega & := \inf_{v \in W_0^{1,p}(D^\eta; \mathbb{R}^m)} \int_{D^\eta} W(y/\varepsilon, t\Lambda + t\nabla v_\varepsilon^\omega(y) + \nabla v(y), \omega) dy \\ & \quad - \inf_{w \in W_0^{1,p}(D; \mathbb{R}^m)} \int_D W(y/\varepsilon, t\Lambda + t\nabla u_\varepsilon(y) + \nabla w(y), \omega) dy, \end{aligned}$$

which then yields $\overline{W}''_t(\Lambda, \omega; D) \geq \overline{W}_t(\Lambda)$. This shows that $\overline{W}''_t(\Lambda, \omega; D) = \overline{W}'_t(\Lambda, \omega; D) = \overline{W}_t(\Lambda)$, and the result is proven. \square

Let $\Omega_1 \subset \Omega_0$ be a subset of maximal probability such that (4.2) holds for all $\omega \in \Omega_1$, $t \in \mathbb{Q} \cap [0, 1)$ and $\Lambda \in \mathbb{Q}^{m \times d} \cap \text{dom} \overline{V}$, such that (4.3) holds for all $\omega \in \Omega_1$ and $\Lambda \in \mathbb{Q}^{m \times d}$, and such that we further have, for all $\omega \in \Omega_1$, $\Lambda \in \mathbb{Q}^{m \times d}$, and all bounded domains $O \subset \mathbb{R}^d$,

$$\overline{V}(\Lambda) = \lim_{\varepsilon \downarrow 0} \int_{O/\varepsilon} V(y, \Lambda + \nabla \varphi_\Lambda(y, \omega), \omega) dy.$$

4.2. Γ -lim inf inequality by blow-up. In this section, we prove the Γ -lim inf inequality for Theorem 2.6 by adapting the blow-up method introduced by [25] (see also [7, Section 4.1] and [14]). In the present context, a subtle use of the corrector for the convex problem is needed.

Proposition 4.3 (Γ -lim inf inequality). *For any $\omega \in \Omega_1$, any bounded Lipschitz domain $O \subset \mathbb{R}^d$, and any sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O; \mathbb{R}^m)$, we have*

$$\liminf_{\varepsilon \downarrow 0} I_\varepsilon(u_\varepsilon, \omega; O) \geq I(u; O).$$

\square

Proof. For all $r > 0$ and $x \in \mathbb{R}^d$, define $Q_r(x) = x + rQ$ and $S_{r,\kappa}(x) = Q_r(x) \setminus Q_{r\kappa}(x)$ for all $\kappa > 0$. For all $\varepsilon > 0$ and all Λ, ω , define $\chi_{\varepsilon,\Lambda}^\omega = \varepsilon\varphi_\Lambda(\cdot/\varepsilon, \omega)$. For all $\omega \in \Omega_1$ and $\Lambda \in \mathbb{Q}^{m \times d}$, the sequence $(\chi_{\varepsilon,\Lambda}^\omega)_\varepsilon$ satisfies $\chi_{\varepsilon,\Lambda}^\omega \rightarrow 0$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(\chi_{\varepsilon,\Lambda}^\omega + \Lambda \cdot x, \omega; O') \rightarrow J(\Lambda \cdot x; O') = |O'| \bar{V}(\Lambda)$ as $\varepsilon \downarrow 0$, for any subdomain $O' \subset O$.

From now on, let $\omega \in \Omega_1$ be *fixed*, let $O \subset \mathbb{R}^d$ be some bounded Lipschitz subset, and let $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ be some fixed sequence with $u_\varepsilon \rightarrow u$ in $W^{1,p}(O; \mathbb{R}^m)$. We need to prove

$$\liminf_{\varepsilon \downarrow 0} I_\varepsilon(u_\varepsilon, \omega; O) \geq I(u; O). \quad (4.11)$$

It does not restrict generality to assume $\liminf_\varepsilon I_\varepsilon(u_\varepsilon, \omega; O) = \lim_\varepsilon I_\varepsilon(u_\varepsilon, \omega; O) < \infty$ and also $\sup_\varepsilon I_\varepsilon(u_\varepsilon, \omega; O) < \infty$. Hence, $\nabla u_\varepsilon(x) \in \text{dom}W(x/\varepsilon, \cdot, \omega) = \text{dom}V(x/\varepsilon, \cdot, \omega)$ for almost all x . Furthermore, the Γ -convergence result for V yields $J(u; O) \leq \liminf_\varepsilon J_\varepsilon(u_\varepsilon, \omega; O) \leq \lim_\varepsilon I_\varepsilon(u_\varepsilon, \omega; O) < \infty$, so that $\nabla u(x) \in \text{dom}\bar{V}$ for almost all x .

Step 1. Localization by blow up: we prove that it suffices to show for almost all x_0 that

$$\liminf_{t \uparrow 1} \liminf_{r \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, t\nabla u_\varepsilon(y), \omega) dy \geq \bar{W}(\nabla u(x_0)). \quad (4.12)$$

For all $\varepsilon > 0$, consider the positive Radon measure on O defined by $d\rho_\varepsilon(x) = W(x/\varepsilon, \nabla u_\varepsilon(x), \omega) dx$. As $\sup_\varepsilon \rho_\varepsilon(\text{adh } O) < \infty$ by hypothesis, the Prokhorov theorem asserts the convergence $\rho_\varepsilon \xrightarrow{*} \rho$ up to extraction of a subsequence, for some positive Radon measure ρ on $\text{adh } O$. (The extraction will remain implicit in our notation in the sequel.) By Lebesgue's decomposition theorem, we can consider the absolutely continuous part of the positive measure ρ , and the Radon-Nikodym theorem allows to define the density $f \in L^1(U)$ of the latter. As O is open, we then have by the portmanteau theorem (see e.g. [10, Theorem 2.1])

$$\liminf_\varepsilon I_\varepsilon(u_\varepsilon, \omega; O) = \liminf_\varepsilon \rho_\varepsilon(O) \geq \rho(O) \geq \int_O f(x) dx.$$

Hence, in order to prove (4.11), it suffices to show that $f(x) \geq \bar{W}(\nabla u(x))$ for almost all x . Since $\rho(\text{adh } O) < \infty$, we have $\rho(\partial Q_r(x)) = 0$ for all $r \in (0, 1) \setminus D_x$, where D_x is at most countable, so that, for almost all x , Lebesgue's differentiation theorem and the portmanteau theorem successively give

$$f(x) = \lim_{\substack{r \downarrow 0 \\ r \notin D_x}} \frac{\rho(Q_r(x))}{r^d} = \lim_{\substack{r \downarrow 0 \\ r \notin D_x}} \lim_{\varepsilon \downarrow 0} \frac{\rho_\varepsilon(Q_r(x))}{r^d}.$$

Hence, it suffices to show that, for almost all x_0 ,

$$\liminf_{r \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y), \omega) dy \geq \bar{W}(\nabla u(x_0)).$$

Using the ru-use assumption on W , we easily deduce the following inequality:

$$\limsup_{t \uparrow 1} \liminf_{r \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, t\nabla u_\varepsilon(y), \omega) dy \leq \liminf_{r \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y), \omega) dy. \quad (4.13)$$

Indeed, as $\nabla u_\varepsilon(y) \in \text{dom}W(y/\varepsilon, \cdot, \omega)$ for almost all y , we can write

$$\begin{aligned} \int_{Q_r(x_0)} W(y/\varepsilon, t\nabla u_\varepsilon(y), \omega) dy &\leq (1 + \Delta_W^a(t)) \int_{Q_r(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y), \omega) dy \\ &\quad + \Delta_W^a(t) \int_{Q_r(x_0)} a(y/\varepsilon, \omega) dy, \end{aligned}$$

and thus, by τ -stationarity of a , the Birkhoff-Khinchin ergodic theorem yields

$$\begin{aligned} &\liminf_{r \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, t\nabla u_\varepsilon(y), \omega) dy \\ &\leq (1 + \Delta_W^a(t)) \liminf_{r \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y), \omega) dy + \Delta_W^a(t) \mathbb{E}[a(0, \cdot)], \end{aligned}$$

so that inequality (4.13) now directly follows from the ru-usc assumption with respect to a (meaning indeed that $\limsup_{t \uparrow 1} \Delta_W^a(t) \leq 0$). Using (4.13), we finally conclude that it is sufficient to show (4.12) for almost all x_0 .

Step 2. Proof of (4.12) by truncation.

The idea is to truncate u_ε at the boundary, in order to make appear in the left-hand side of (4.12) precisely $\bar{W}(t\nabla u(x_0))$, which will then allow us to conclude.

Let $t, \kappa \in (0, 1)$ be fixed. Since $p > d$, the Sobolev embedding yields $u_\varepsilon \rightarrow u$ in $L^\infty(O; \mathbb{R}^m)$. Moreover, combining the Lebesgue differentiation theorem for ∇u and the Sobolev embedding for $p > d$, we deduce that, for all $x_0 \notin \mathcal{N}$ (for some null set $\mathcal{N} \subset \mathbb{R}^d$, $|\mathcal{N}| = 0$),

$$\lim_{r \downarrow 0} \frac{1}{r} \|u - u(x_0) - \nabla u(x_0) \cdot (x - x_0)\|_{L^\infty(Q_r(x_0))} = 0. \quad (4.14)$$

Enlarging the null set \mathcal{N} , we can also assume that $\nabla u(x_0) \in \text{dom } \bar{V}$ for any $x_0 \notin \mathcal{N}$. From now on, let $x_0 \in O \setminus \mathcal{N}$ be fixed and write for simplicity $\Lambda := \nabla u(x_0)$. Since \bar{V} is convex and lower semicontinuous, we have $\bar{V}(\Lambda) = \liminf_{\Lambda' \rightarrow \Lambda, \Lambda' \in \mathbb{Q}^{m \times d}} \bar{V}(\Lambda')$, and a diagonalization argument then allows us to choose a sequence $(\Lambda_r)_r \subset \mathbb{Q}^{m \times d}$ such that $\Lambda_r \rightarrow \Lambda$ and $\bar{V}(\Lambda_r) \rightarrow \bar{V}(\Lambda)$ as $r \downarrow 0$, and simultaneously

$$\lim_{r \downarrow 0} \frac{1}{r} \|u - u(x_0) - \Lambda_r \cdot (x - x_0)\|_{L^\infty(Q_r(x_0))} = 0. \quad (4.15)$$

Let $\phi_{r,\kappa}$ be a smooth cut-off function with values in $[0, 1]$, such that $\phi_{r,\kappa}$ equals 1 on $Q_{r\kappa}(x_0)$, vanishes outside $Q_r(x_0)$, and satisfies $\|\nabla \phi_{r,\kappa}\|_{L^\infty} \leq \frac{2}{r(1-\kappa)}$. We then set

$$v_{\varepsilon,r,\kappa} := \phi_{r,\kappa} u_\varepsilon + (1 - \phi_{r,\kappa})(u(x_0) + \Lambda_r \cdot (x - x_0) + \chi_{\varepsilon,\Lambda_r}^\omega(x)).$$

Since $v_{\varepsilon,r,\kappa}$ coincides with u_ε on $Q_{r\kappa}(x_0)$ and $0 \leq W \leq C(1 + V)$, we have

$$\begin{aligned} & \int_{Q_r(x_0)} W(y/\varepsilon, t\nabla v_{\varepsilon,r,\kappa}(y), \omega) dy \\ & \leq \int_{Q_r(x_0)} W(y/\varepsilon, t\nabla u_\varepsilon(y), \omega) dy + \frac{C}{r^d} \int_{S_{r,\kappa}(x_0)} V(y/\varepsilon, t\nabla v_{\varepsilon,r,\kappa}(y), \omega) dy + C(1 - \kappa^d). \end{aligned} \quad (4.16)$$

Defining $\Psi_{\varepsilon,r,\kappa}(x) := \nabla \phi_{r,\kappa}(x) \otimes (u_\varepsilon(x) - u(x_0) - \Lambda_r \cdot (x - x_0) - \chi_{\varepsilon,\Lambda_r}^\omega)$ and decomposing

$$t\nabla v_{\varepsilon,r,\kappa} = t\phi_{r,\kappa} \nabla u_\varepsilon + t(1 - \phi_{r,\kappa})(\Lambda_r + \nabla \chi_{\varepsilon,\Lambda_r}^\omega) + (1 - t) \frac{t}{1 - t} \Psi_{\varepsilon,r,\kappa},$$

we obtain by convexity of V

$$\begin{aligned} V(y/\varepsilon, t\nabla v_{\varepsilon,r,\kappa}(y), \omega) & \leq t\phi_{r,\kappa} V(y/\varepsilon, \nabla u_\varepsilon(y), \omega) + t(1 - \phi_{r,\kappa}) V(y/\varepsilon, \Lambda_r + \nabla \chi_{\varepsilon,\Lambda_r}^\omega(y), \omega) \\ & \quad + (1 - t) V\left(y/\varepsilon, \frac{t}{1 - t} \Psi_{\varepsilon,r,\kappa}(y), \omega\right) \\ & \leq W(y/\varepsilon, \nabla u_\varepsilon(y), \omega) + V(y/\varepsilon, \Lambda_r + \nabla \chi_{\varepsilon,\Lambda_r}^\omega(y), \omega) \\ & \quad + (1 - t) V\left(y/\varepsilon, \frac{t}{1 - t} \Psi_{\varepsilon,r,\kappa}(y), \omega\right). \end{aligned} \quad (4.17)$$

Combined with

$$\begin{aligned} \|\Psi_{\varepsilon,r,\kappa}\|_{L^\infty(S_{r,\kappa}(x_0))} & \leq \frac{2}{r(1-\kappa)} \left(\|u_\varepsilon - u\|_{L^\infty(O)} + \|u - u(x_0) - \Lambda_r \cdot (x - x_0)\|_{L^\infty(Q_r(x_0))} \right. \\ & \quad \left. + \|\chi_{\varepsilon,\Lambda_r}^\omega\|_{L^\infty(O)} \right), \end{aligned}$$

the convergences $u_\varepsilon \rightarrow u$ and $\chi_{\varepsilon,\Lambda_r}^\omega \rightarrow 0$ in $L^\infty(O; \mathbb{R}^m)$ as $\varepsilon \downarrow 0$, and (4.15) yield

$$\limsup_{r \downarrow 0} \limsup_{\varepsilon \downarrow 0} \|\Psi_{\varepsilon,r,\kappa}\|_{L^\infty(S_{r,\kappa}(x_0))} = 0.$$

By assumption, we can find $\delta > 0$ with $\text{adh } B_\delta \subset \text{int dom } M$. Hence, for all $t, \kappa \in (0, 1)$, there exists $r_{\kappa, t} > 0$ such that, for all $0 < r < r_{\kappa, t}$, there exists some $\varepsilon_{r, \kappa, t} > 0$ such that for all $0 < \varepsilon < \varepsilon_{r, \kappa, t}$

$$\left\| \frac{t}{1-t} \Psi_{\varepsilon, r, \kappa} \right\|_{L^\infty(S_{r, \kappa}(x_0))} < \delta.$$

This implies

$$\int_{S_{r, \kappa}(x_0)} V\left(y/\varepsilon, \frac{t}{1-t} \Psi_{\varepsilon, r, \kappa}(y), \omega\right) dy \leq |S_{r, \kappa}(x_0)| \sup_{|\Lambda'| < \delta} M(\Lambda') = r^d (1 - \kappa^d) \sup_{|\Lambda'| < \delta} M(\Lambda'),$$

where the supremum is finite, by virtue of the convexity of M and our choice of $\delta > 0$. Combined with inequality (4.17) and the definition of the correctors $\chi_{\varepsilon, \Lambda_r}^\omega$ (with $\lim_r \bar{V}(\Lambda_r) = \bar{V}(\Lambda) < \infty$), this yields

$$\begin{aligned} & \liminf_{\kappa \uparrow 1} \liminf_{t \uparrow 1} \liminf_{r \downarrow 0} \liminf_{\varepsilon \downarrow 0} \int_{S_{r, \kappa}(x_0)} V(y/\varepsilon, t \nabla v_{\varepsilon, r, \kappa}(y), \omega) dy \\ & \leq \liminf_{\kappa \uparrow 1} \liminf_{r \downarrow 0} \liminf_{\varepsilon \downarrow 0} \int_{S_{r, \kappa}(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y), \omega) dy. \end{aligned}$$

This turns inequality (4.16) into

$$\begin{aligned} & \liminf_{\kappa \uparrow 1} \liminf_{t \uparrow 1} \liminf_{r \downarrow 0} \liminf_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, t \nabla v_{\varepsilon, r, \kappa}(y), \omega) dy \\ & \leq \liminf_{t \uparrow 1} \liminf_{r \downarrow 0} \liminf_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, t \nabla u_\varepsilon(y), \omega) dy \\ & \quad + \liminf_{\kappa \uparrow 1} \liminf_{r \downarrow 0} \liminf_{\varepsilon \downarrow 0} \frac{C}{r^d} \int_{S_{r, \kappa}(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y), \omega) dy. \end{aligned} \tag{4.18}$$

Since we have chosen $\omega \in \Omega_1$, $t \in \mathbb{Q} \cap (0, 1)$ and $\Lambda_r \in \mathbb{Q}^{m \times d}$, (4.2) holds and reads

$$\bar{W}_t(\Lambda_r) = \liminf_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O; \mathbb{R}^m)} \int_{Q_r(x_0)} W(y/\varepsilon, t \Lambda_r + t \nabla \varphi_{\Lambda_r}(\cdot/\varepsilon), \omega) + \nabla v(y), \omega) dy.$$

Hence, since $v_{\varepsilon, r, \kappa} - u(x_0) - \Lambda_r \cdot (x - x_0) \in \chi_{\varepsilon, \Lambda_r}^\omega + W_0^{1,p}(Q_r(x_0); \mathbb{R}^m)$ with $\chi_{\varepsilon, \Lambda_r}^\omega = \varepsilon \varphi_{\Lambda_r}(\cdot/\varepsilon, \omega)$, (4.18) yields

$$\begin{aligned} \bar{W}(\nabla u(x_0)) &= \bar{W}(\Lambda) \leq \liminf_{t \uparrow 1} \liminf_{r \downarrow 0} \bar{W}_t(\Lambda_r) \\ & \leq \liminf_{t \uparrow 1} \liminf_{r \downarrow 0} \liminf_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, t \nabla v_{\varepsilon, r, \kappa}(y), \omega) dy \\ & \leq \liminf_{t \uparrow 1} \liminf_{r \downarrow 0} \liminf_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, t \nabla u_\varepsilon(y), \omega) dy \\ & \quad + \limsup_{\kappa \uparrow 1} \limsup_{r \downarrow 0} \limsup_{\varepsilon \downarrow 0} \frac{C}{r^d} \int_{S_{r, \kappa}(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y), \omega) dy. \end{aligned} \tag{4.19}$$

It remains to get rid of the second term of the right-hand side of (4.19). By the portmanteau theorem,

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} \frac{1}{r^d} \int_{S_{r, \kappa}(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y), \omega) dy &= \limsup_{\varepsilon \downarrow 0} \frac{\rho_\varepsilon(S_{r, \kappa}(x_0))}{r^d} \\ &\leq \limsup_{\varepsilon \downarrow 0} \frac{\rho_\varepsilon(\text{adh } S_{r, \kappa}(x_0))}{r^d} \leq \frac{\rho(\text{adh } S_{r, \kappa}(x_0))}{r^d}. \end{aligned}$$

Since the singular part of ρ must be supported in a closed subset of $\text{adh } O$ of measure 0, we deduce, for almost all $x_0 \in O \setminus \mathcal{N}$, the existence of some $r_0 > 0$ sufficiently small such that $\text{adh } Q_r(x_0)$ has no

intersection with that support for all $0 < r < r_0$. Hence, for all $0 < r < r_0$,

$$\begin{aligned} & \limsup_{\varepsilon \downarrow 0} \frac{1}{r^d} \int_{S_{r,\kappa}(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y), \omega) dy \\ & \leq \frac{\rho(\text{adh } S_{r,\kappa}(x_0))}{r^d} = \frac{1}{r^d} \int_{S_{r,\kappa}(x_0)} f(y) dy = \int_{Q_r(x_0)} f(y) dy - \kappa^d \int_{Q_{r\kappa}(x_0)} f(y) dy, \end{aligned}$$

where, for almost all x_0 , the right-hand side converges to $(1 - \kappa^d)f(x_0)$ as $r \downarrow 0$ by Lebesgue's differentiation theorem. Hence, for almost all x_0 , (4.19) turns into

$$\overline{W}(\nabla u(x_0)) \leq \limsup_{t \uparrow 1} \liminf_{r \downarrow 0} \liminf_{\varepsilon \downarrow 0} \int_{Q_r(x_0)} W(y/\varepsilon, t \nabla u_\varepsilon(y), \omega) dy,$$

and the desired result (4.12) is proven. \square

4.3. Γ -lim sup inequality with Neumann boundary data. In this section, we prove the Γ -lim sup inequality, first considering the affine case, and then deducing the general case by approximation. For this approximation argument to hold, we would however need to know a priori that the homogenized energy \overline{W} satisfies good regularity properties (i.e. lower semicontinuity on $\mathbb{R}^{m \times d}$ and continuity on $\text{int dom } \overline{W}$). Since this is not clear at all a priori, our strategy (inspired by [7]) consists in introducing some relaxations of \overline{W} that enjoy the required properties, and then in deducing a posteriori from Γ -convergence (or a weaker form of it) the equality of \overline{W} with its relaxations (so that \overline{W} itself has all the desired properties). Motivated by the work of Fonseca [24] (see also [7]), we thus consider the following relaxation of \overline{W} :

$$\mathcal{Z}W(\Lambda) := \inf \left\{ \int_O \overline{W}(\Lambda + \nabla \phi(y)) dy : \phi \text{ continuous piecewise affine on } O \text{ and } \phi|_{\partial O} = 0 \right\},$$

where the definition does clearly not depend on the chosen underlying (nonempty) bounded Lipschitz domain $O \subset \mathbb{R}^d$. Also write $\widehat{\mathcal{Z}}W$ for the lower semicontinuous envelope of $\mathcal{Z}W$ (defined by $\widehat{\mathcal{Z}}W(\Lambda) := \liminf_{\Lambda' \rightarrow \Lambda} \mathcal{Z}W(\Lambda')$ for all Λ). Now define the integral functionals corresponding to all these relaxed integrands: for any bounded domain $O \subset \mathbb{R}^d$ and $u \in W^{1,p}(O; \mathbb{R}^m)$,

$$\mathcal{Z}I(u; O) := \int_O \mathcal{Z}W(\nabla u(y)) dy, \quad \widehat{\mathcal{Z}}I(u; O) := \int_O \widehat{\mathcal{Z}}W(\nabla u(y)) dy.$$

The following result gives some properties of these relaxations, which will be crucial in the sequel:

Lemma 4.4 (Properties of relaxations). *Assume $p > d$. Then the following holds:*

- (a) $\mathcal{Z}W$ (and thus also $\widehat{\mathcal{Z}}W$) is continuous on $\text{int dom } \mathcal{Z}W$.
- (b) $\overline{V} \leq \widehat{\mathcal{Z}}W \leq \overline{W} \leq C(1 + \overline{V})$.
- (c) \overline{W} and $\mathcal{Z}W$ are ru-usc.
- (d) For any $t \in (0, 1)$, we have $t \text{ adh dom } \mathcal{Z}W \subset \text{int dom } \mathcal{Z}W$, and the following representation result holds:

$$\widehat{\mathcal{Z}}W(\Lambda) = \liminf_{t \rightarrow 1} \mathcal{Z}W(t\Lambda) = \begin{cases} \mathcal{Z}W(\Lambda), & \text{if } \Lambda \in \text{int dom } \mathcal{Z}W; \\ \lim_{t \uparrow 1} \mathcal{Z}W(t\Lambda), & \text{if } \Lambda \in \partial \text{ dom } \mathcal{Z}W; \\ \infty, & \text{otherwise;} \end{cases}$$

where, in particular, the limit in the second line does exist.

- (e) Let $\Lambda \in \mathbb{R}^{m \times d}$ and let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, there exists a sequence $(\phi_k)_k \subset W_0^{1,p}(O; \mathbb{R}^m)$ of piecewise affine functions such that $\phi_k \rightarrow 0$ in $L^\infty(O; \mathbb{R}^m)$ and

$$\lim_{k \uparrow \infty} \int_O \overline{W}(\Lambda + \nabla \phi_k(y)) dy = \mathcal{Z}W(\Lambda).$$

\square

Proof. Continuity of $\mathcal{Z}W$ on $\text{int dom } \mathcal{Z}W$ is a result due to [24], which yields part (a) (even without any ru-usc assumption on W). The inequalities stated in part (b) directly follow from the definitions of \bar{V} , \bar{W} and $\hat{\mathcal{Z}}W$. Part (e) is standard (see [7, Proposition 3.17] for details). It remains to prove properties (c) and (d).

Step 1. Proof of (c).

We first prove that \bar{W}_t is ru-usc, for any fixed $t \in [0, 1)$. Let $s > 0$, $\Lambda \in \text{dom } \bar{W} = \text{dom } \bar{V}$, and let $O \subset \mathbb{R}^d$ be some bounded Lipschitz domain. For almost all ω , note that by convexity, for all subdomains $D \subset O$,

$$\begin{aligned} & \limsup_{\varepsilon \downarrow 0} \int_D V(y/\varepsilon, s\Lambda + s\nabla\varphi_\Lambda(y/\varepsilon, \omega), \omega) dy \\ & \leq \lim_{\varepsilon \downarrow 0} \int_D V(y/\varepsilon, \Lambda + \nabla\varphi_\Lambda(y/\varepsilon, \omega), \omega) dy + (1-s)M(0) = \bar{V}(\Lambda) + (1-s)M(0) < \infty. \end{aligned}$$

Hence for almost all ω we can apply equality (4.6) at $s\Lambda$ with $u_\varepsilon = s\varepsilon\varphi_\Lambda(\cdot/\varepsilon, \omega)$, which yields

$$\bar{W}_t(s\Lambda) = \lim_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O; \mathbb{R}^m)} \int_O W(y/\varepsilon, st\Lambda + st\nabla\varphi_\Lambda(y/\varepsilon, \omega) + \nabla v(y), \omega) dy.$$

Given $\omega \in \Omega$ such that this convergence and (4.2) both hold, and choosing a sequence $(v_\varepsilon^\omega)_\varepsilon \subset W_0^{1,p}(O; \mathbb{R}^d)$ such that

$$\bar{W}_t(\Lambda) = \lim_{\varepsilon \downarrow 0} \int_O W(y/\varepsilon, t\Lambda + t\nabla\varphi_\Lambda(y/\varepsilon, \omega) + \nabla v_\varepsilon^\omega(y), \omega) dy,$$

we deduce

$$\begin{aligned} \bar{W}_t(s\Lambda) - \bar{W}_t(\Lambda) & \leq \lim_{\varepsilon \downarrow 0} \int_O \left(W(y/\varepsilon, s(t\Lambda + t\nabla\varphi_\Lambda(y/\varepsilon, \omega) + \nabla v_\varepsilon^\omega(y)), \omega) \right. \\ & \quad \left. - W(y/\varepsilon, t\Lambda + t\nabla\varphi_\Lambda(y/\varepsilon, \omega) + \nabla v_\varepsilon^\omega(y), \omega) \right) dy \\ & \leq \Delta_W^a(s) \lim_{\varepsilon \downarrow 0} \int_O (a(y/\varepsilon, \omega) + W(y/\varepsilon, t\Lambda + t\nabla\varphi_\Lambda(y/\varepsilon, \omega) + \nabla v_\varepsilon^\omega(y), \omega)) dy \\ & = \Delta_W^a(s) (\mathbb{E}[a(0, \cdot)] + \bar{W}_t(\Lambda)), \end{aligned} \tag{4.20}$$

using the Birkhoff-Khinchin ergodic theorem for the stationary field a . As \bar{W}_t and the field a are nonnegative, as $\alpha := \mathbb{E}[a(0, \cdot)]$ is finite and as $\limsup_{s \uparrow 1} \Delta_W^a(s) \leq 0$ by assumption, we deduce that \bar{W}_t is also ru-usc. Now rewriting inequality (4.20) in the form

$$\bar{W}_t(s\Lambda) \leq \alpha \Delta_W^a(s) + (1 + (-1) \vee \Delta_W^a(s)) \bar{W}_t(\Lambda),$$

and taking the suitable \liminf , we directly deduce $\bar{W}(s\Lambda) - \bar{W}(\Lambda) \leq (-1) \vee \Delta_W^a(s) (\alpha + \bar{W}(\Lambda))$ for all $\Lambda \in \text{dom } \bar{W}$ and $s \in [0, 1)$, proving that \bar{W} is itself ru-usc with $\Delta_{\bar{W}}^\alpha = (-1) \vee \Delta_W^a(s)$.

We now show that $\mathcal{Z}W$ is also ru-usc. Take $s > 0$ and $\Lambda \in \text{dom } \bar{W}$. By definition, there exists a sequence of piecewise affine functions $(\phi_k)_k \subset W_0^{1,p}(O)$ such that

$$\mathcal{Z}W(\Lambda) = \lim_{k \uparrow \infty} \int_O \bar{W}(\Lambda + \nabla\phi_k(y)) dy.$$

As $\Lambda \in \text{dom } \bar{W}$, the left-hand side is finite, and we can thus assume $\Lambda + \nabla\phi_k \in \text{dom } \bar{W}$ almost everywhere. Hence the ru-usc property satisfied by \bar{W} gives

$$\begin{aligned} \mathcal{Z}W(s\Lambda) - \mathcal{Z}W(\Lambda) & \leq \lim_{k \uparrow \infty} \int_O (\bar{W}(s(\Lambda + \nabla\phi_k(y))) - \bar{W}(\Lambda + \nabla\phi_k(y))) dy \\ & \leq \Delta_{\bar{W}}^\alpha(s) \lim_{k \uparrow \infty} \int_O (\alpha + \bar{W}(\Lambda + \nabla\phi_k(y))) dy = \Delta_{\bar{W}}^\alpha(s) (\alpha + \mathcal{Z}W(\Lambda)). \end{aligned}$$

Step 2. Proof of (d).

Since $\text{dom } \mathcal{Z}W = \text{dom } \overline{V}$ is a convex set containing 0, it is clear that, for all $t \in [0, 1)$, $t \text{ adh dom } \mathcal{Z}W$ is contained in $\text{int dom } \overline{V}$. We first show that the limit $\lim_{t \uparrow 1} \mathcal{Z}W(t\Lambda)$ exists for all $\Lambda \in \text{adh dom } \overline{V}$. Given some fixed $\Lambda \in \text{adh dom } \overline{V}$, choose two sequences $s_n \uparrow 1$ and $t_n \uparrow 1$ with $t_n/s_n \uparrow 1$ such that

$$\lim_{n \uparrow \infty} \mathcal{Z}W(s_n \Lambda) = \liminf_{t \uparrow 1} \mathcal{Z}W(t\Lambda) \quad \text{and} \quad \lim_{n \uparrow \infty} \mathcal{Z}W(t_n \Lambda) = \limsup_{t \uparrow 1} \mathcal{Z}W(t\Lambda).$$

As $s_n \Lambda, t_n \Lambda \in \text{dom } \overline{V}$ for all n , and as $\mathcal{Z}W$ is ru-usc, we have

$$\begin{aligned} \lim_{n \uparrow \infty} \mathcal{Z}W(t_n \Lambda) &\leq \limsup_{n \uparrow \infty} (\alpha + \mathcal{Z}W(s_n \Lambda)) \Delta_W^a(t_n/s_n) + \lim_{n \uparrow \infty} \mathcal{Z}W(s_n \Lambda) \\ &\leq \lim_{n \uparrow \infty} \mathcal{Z}W(s_n \Lambda) \leq \lim_{n \uparrow \infty} \mathcal{Z}W(t_n \Lambda), \end{aligned}$$

which thus proves the existence of the limit $\lim_{t \uparrow 1} \mathcal{Z}W(t\Lambda)$ for all $\Lambda \in \text{adh dom } \overline{V}$.

We now prove the claimed representation result. First, if $\Lambda \in \text{int dom } \overline{V}$, then $\liminf_{t \rightarrow 1} \mathcal{Z}W(t\Lambda) = \mathcal{Z}W(\Lambda) = \widehat{\mathcal{Z}}W(\Lambda)$ follows from part (a). Second, if $\Lambda \notin \text{adh dom } \overline{V}$, then $\mathcal{Z}W(t\Lambda) = \infty$ for any t sufficiently close to 1, and thus $\liminf_{t \rightarrow 1} \mathcal{Z}W(t\Lambda) = \infty = \widehat{\mathcal{Z}}W(\Lambda)$. Now it only remains to consider $\Lambda \in \partial \text{dom } \mathcal{Z}W$. Then $\mathcal{Z}W(t\Lambda) = \infty$ whenever $t > 1$, so that we simply have $\liminf_{t \rightarrow 1} \mathcal{Z}W(t\Lambda) = \liminf_{t \uparrow 1} \mathcal{Z}W(t\Lambda) = \lim_{t \uparrow 1} \mathcal{Z}W(t\Lambda)$, since we have already proven the existence of this limit. Hence, it suffices to prove that $\widehat{\mathcal{Z}}W(\Lambda) = \liminf_{t \uparrow 1} \mathcal{Z}W(t\Lambda)$. By definition of the lower semicontinuous envelope $\widehat{\mathcal{Z}}W$ of $\mathcal{Z}W$, this equality would follow if we could show that, for any sequence $\Lambda_n \rightarrow \Lambda$, we have

$$\liminf_{n \uparrow \infty} \mathcal{Z}W(\Lambda_n) \geq \liminf_{t \uparrow 1} \mathcal{Z}W(t\Lambda). \quad (4.21)$$

It is of course sufficient to assume $\liminf_n \mathcal{Z}W(\Lambda_n) = \lim_n \mathcal{Z}W(\Lambda_n) < \infty$ and $\sup_n \mathcal{Z}W(\Lambda_n) < \infty$. Hence, $\Lambda_n \in \text{dom } \overline{V}$ for all n , and thus, for all $t \in [0, 1)$, $t\Lambda \in \text{int dom } \overline{V}$, so that, using part (a) as well as the ru-usc property satisfied by $\mathcal{Z}W$, we have

$$\mathcal{Z}W(t\Lambda) = \lim_{n \uparrow \infty} \mathcal{Z}W(t\Lambda_n) \leq \lim_{n \uparrow \infty} \mathcal{Z}W(\Lambda_n) + \Delta_W^a(t) \lim_{n \uparrow \infty} (\alpha + \mathcal{Z}W(\Lambda_n)).$$

This yields

$$\limsup_{t \uparrow 1} \mathcal{Z}W(t\Lambda) \leq \liminf_{n \uparrow \infty} \mathcal{Z}W(\Lambda_n),$$

and proves (4.21). \square

Combining the Γ -lim inf inequality for I_ε towards I with a Γ -lim sup argument, we prove the following a priori surprising equality of \overline{W} with its relaxations.

Lemma 4.5 (Regularity of the homogenized energy density). *Assume $p > d$. Then $\overline{W}(\Lambda) = \mathcal{Z}W(\Lambda) = \widehat{\mathcal{Z}}W(\Lambda)$ for all $\Lambda \in \mathbb{R}^{m \times d}$. In particular, \overline{W} is lower semicontinuous on $\mathbb{R}^{m \times d}$ and is continuous on $\text{int dom } \overline{V}$. \square*

Proof. We split the proof into four steps.

Step 1. Recovery sequence for $I(\Lambda \cdot x; O)$.

Let $\Lambda \in \text{int dom } \overline{V}$ and let $t \in [0, 1)$. In this step, for almost all ω , for all bounded Lipschitz domain $O \subset \mathbb{R}^d$, we prove the existence of sequences $t_\varepsilon \uparrow 1$, $\Lambda_\varepsilon \rightarrow \Lambda$ and $(w_\varepsilon)_\varepsilon \subset W_0^{1,p}(O; \mathbb{R}^m)$ such that $\varepsilon \varphi_{\Lambda_\varepsilon}(\cdot/\varepsilon, \omega) \rightarrow 0$, $w_\varepsilon \rightarrow 0$ in $W^{1,p}(O; \mathbb{R}^m)$ and $I_\varepsilon(t_\varepsilon \Lambda_\varepsilon \cdot x + \varepsilon t_\varepsilon \varphi_{\Lambda_\varepsilon}(\cdot/\varepsilon, \omega) + w_\varepsilon, \omega; O) \rightarrow |O| \overline{W}(\Lambda) = I(\Lambda \cdot x; O)$.

By definition of \overline{W} and a diagonalization argument, for almost all ω and all bounded Lipschitz domains $O \subset \mathbb{R}^d$, it suffices to prove the existence of a sequence $(v_\varepsilon)_\varepsilon \subset W_0^{1,p}(O; \mathbb{R}^m)$ such that $v_\varepsilon \rightarrow 0$ in $W^{1,p}(O; \mathbb{R}^m)$ and $I_\varepsilon(t\Lambda \cdot x + \varepsilon t \varphi_\Lambda(\cdot/\varepsilon, \omega) + v_\varepsilon, \omega; O) \rightarrow |O| \overline{W}_t(\Lambda)$ as $\varepsilon \downarrow 0$.

Let O be some fixed bounded Lipschitz domain. Given $\varepsilon > 0$, consider the cubes of the form $k(z + Q)$, $z \in \mathbb{Z}^d$, that are contained in O/ε , and denote by $z_j \in \mathbb{Z}^d$, $j = 1, \dots, N_{\varepsilon,k}$, the centers of these cubes (the enumeration of which can be chosen independent of ε, k). Since O is Lipschitz,

we have $N_{\varepsilon,k}(\varepsilon k)^d \rightarrow |O|$ as $\varepsilon \downarrow 0$, for all k . For all j, ω , we can choose a sequence $(v_k^{j,\omega})_k$ with $v_k^{j,\omega} \in W_0^{1,p}(k(z_j + Q); \mathbb{R}^m)$ such that

$$\begin{aligned} & \int_{k(z_j+Q)} W(y, t\Lambda + t\nabla\varphi_\Lambda(y, \omega) + \nabla v_k^{j,\omega}(y), \omega) dy \\ & \leq \frac{1}{k} + \inf_{v \in W_0^{1,p}(k(z_j+Q); \mathbb{R}^m)} \int_{k(z_j+Q)} W(y, t\Lambda + t\nabla\varphi_\Lambda(y, \omega) + \nabla v(y), \omega) dy. \end{aligned}$$

For all ε, k, ω , we then consider the function $v_{\varepsilon,k}^\omega := \sum_{j=1}^{N_{\varepsilon,k}} v_k^{j,\omega} \mathbb{1}_{k(z_j+Q)} \in W_0^{1,p}(O/\varepsilon; \mathbb{R}^m)$, and we define $w_{\varepsilon,k}^\omega := \varepsilon v_{\varepsilon,k}^\omega(\cdot/\varepsilon) \in W_0^{1,p}(O; \mathbb{R}^m)$. Up to a diagonalization argument, it suffices to show that, for almost all ω (independent of the choice of O , as it is clear in the proof below),

$$\limsup_{k \uparrow \infty} \limsup_{\varepsilon \downarrow 0} \left(|I_\varepsilon(t\Lambda \cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon, \omega) + w_{\varepsilon,k}^\omega, \omega; O) - |O|\overline{W}_t(\Lambda)| + \|w_{\varepsilon,k}^\omega\|_{L^p(O)} \right) = 0. \quad (4.22)$$

First we argue that, for almost all ω ,

$$\limsup_{k \uparrow \infty} \limsup_{\varepsilon \downarrow 0} I_\varepsilon(t\Lambda \cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon, \omega) + w_{\varepsilon,k}^\omega, \omega; O) \leq |O|\overline{W}_t(\Lambda). \quad (4.23)$$

Indeed, by definition of $w_{\varepsilon,k}^\omega$,

$$\begin{aligned} & I_\varepsilon(t\Lambda \cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon, \omega) + w_{\varepsilon,k}^\omega, \omega; O) \\ & \leq \frac{1}{k}(\varepsilon k)^d N_{\varepsilon,k} + (\varepsilon k)^d \sum_{j=1}^{N_{\varepsilon,k}} \inf_{v \in W_0^{1,p}(k(z_j+Q); \mathbb{R}^m)} \int_{k(z_j+Q)} W(y, t\Lambda + t\nabla\varphi_\Lambda(y, \omega) + \nabla v(y), \omega) dy \\ & \quad + \varepsilon^d \int_{(O/\varepsilon) \setminus \bigcup_{j=1}^{N_{\varepsilon,k}} k(z_j+Q)} W(y, t\Lambda + t\nabla\varphi_\Lambda(y, \omega), \omega) dy. \end{aligned}$$

Since $W \leq C(1 + V)$, the last term of the right-hand side goes to 0 as $\varepsilon \downarrow 0$ for almost all ω by construction of the cubes $k(z_j + Q)$ and definition of φ_Λ . The Birkhoff-Khinchin ergodic theorem (which we apply to a measurable map by Hypothesis 2.1) then gives, for almost all ω ,

$$\begin{aligned} & \limsup_{\varepsilon \downarrow 0} I_\varepsilon(t\Lambda \cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon, \omega) + w_{\varepsilon,k}^\omega, \omega; O) \\ & \leq \frac{|O|}{k} + |O| \mathbb{E} \left[\inf_{v \in W_0^{1,p}(kQ; \mathbb{R}^m)} \int_{kQ} W(y, t\Lambda + t\nabla\varphi_\Lambda(y, \cdot) + \nabla v(y), \cdot) dy \right]. \end{aligned} \quad (4.24)$$

Lemma 4.1 then yields the desired result (4.23) as $k \uparrow \infty$. On the other hand, by definition (4.2) of \overline{W}_t , for all k and almost all ω , we have

$$\begin{aligned} & \liminf_{\varepsilon \downarrow 0} I_\varepsilon(t\Lambda \cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon, \omega) + w_{\varepsilon,k}^\omega(\cdot, \omega), \omega; O) \\ & \geq |O| \liminf_{\varepsilon \downarrow 0} \inf_{v \in W_0^{1,p}(O; \mathbb{R}^m)} \int_{O/\varepsilon} W(y, t\Lambda + t\nabla\varphi_\Lambda(y, \omega) + \nabla v(y), \omega; O) = |O|\overline{W}_t(\Lambda). \end{aligned} \quad (4.25)$$

We now show that $w_{\varepsilon,k}^\omega \rightarrow 0$ in $L^p(O; \mathbb{R}^m)$ as $\varepsilon \downarrow 0$, for almost all ω . Combining inequality (4.24) with the bound $W \leq C(1 + V)$, the p -th order lower bound for W and the convexity of V , we indeed have

$$\limsup_{\varepsilon \downarrow 0} \|t\Lambda + t\nabla\varphi_\Lambda(\cdot/\varepsilon, \omega) + \nabla w_{\varepsilon,k}^\omega\|_{L^p(O)}^p \leq \frac{|O|}{k} + C|O|(1 + \overline{V}(\Lambda) + (1-t)M(0)) < \infty.$$

For almost all ω , the weak L^p convergence of the sequence $(\nabla\varphi_\Lambda(\cdot/\varepsilon, \omega))_\varepsilon$ to 0 implies the boundedness of this sequence in $L^p(O; \mathbb{R}^{m \times d})$, so that $(\nabla w_{\varepsilon,k}^\omega)_\varepsilon$ is also bounded in $L^p(O; \mathbb{R}^{m \times d})$, for any fixed k . By Poincaré's inequality on cubes of side length $k\varepsilon$, this implies

$$\|w_{\varepsilon,k}^\omega\|_{L^p(O)} \leq C_k(\omega)\varepsilon,$$

for some (random) constant $C_k(\omega)$. Combined with (4.23) and (4.25), this proves (4.22).

Step 2. Recovery sequence for $\mathcal{Z}I(\Lambda \cdot x; O)$.

Let $\Lambda \in \text{int dom } \bar{V}$ and let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain. In this step, for almost all ω , we prove the existence of a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$ and $I_\varepsilon(\Lambda \cdot x + u_\varepsilon, \omega; O) \rightarrow \mathcal{Z}I(\Lambda \cdot x; O)$.

Lemma 4.4(e) gives a sequence $(\phi_k)_k \subset W_0^{1,p}(O; \mathbb{R}^m)$ of piecewise affine functions such that $\phi_k \rightarrow 0$ in $L^p(O; \mathbb{R}^m)$ (and even in $L^\infty(O; \mathbb{R}^m)$), and

$$I(\phi_k + \Lambda \cdot x; O) = \int_O \bar{W}(\Lambda + \nabla \phi_k(y)) dy \xrightarrow{k \uparrow \infty} |O| \mathcal{Z}W(\Lambda).$$

Denote by $(P_k^i)_{i=1}^{n_k}$ the partition of O associated with the piecewise affine function ϕ_k . For all k and $1 \leq i \leq n_k$, considering $\Lambda_k^i := \Lambda + \nabla \phi_k|_{P_k^i}$ on P_k^i , Step 1 above gives, for almost all ω , a sequence $t_\varepsilon \uparrow 1$, a sequence $\Lambda_{k,\varepsilon}^i \rightarrow \Lambda_k^i$ and a sequence $(v_{\varepsilon,k}^i)_\varepsilon \subset W_0^{1,p}(P_k^i; \mathbb{R}^m)$ such that $\varepsilon \varphi_{\Lambda_{k,\varepsilon}^i}(\cdot/\varepsilon, \omega), v_{\varepsilon,k}^i \rightarrow 0$ in $W^{1,p}(P_k^i; \mathbb{R}^m)$ and $I_\varepsilon(t_\varepsilon \Lambda_{k,\varepsilon}^i \cdot x + \varepsilon t_\varepsilon \varphi_{\Lambda_{k,\varepsilon}^i}(\cdot/\varepsilon, \omega) + v_{\varepsilon,k}^i, \omega; P_k^i) \rightarrow I(\Lambda_k^i \cdot x; P_k^i) = I(\phi_k + \Lambda \cdot x; P_k^i)$. As the $v_{\varepsilon,k}^i$'s satisfy Dirichlet boundary conditions, they can be directly glued together, while for the $\varphi_{\Lambda_{k,\varepsilon}^i}$'s we need to repeat the more complicated gluing argument of Step 2 of the proof of Proposition 3.10, with $p > d$. Although the functional I_ε is not convex here, as in the proof of Proposition 4.3, the idea is to use the bound $W \leq C(1 + V)$ at all points where the cut-off functions are different from 1 or 0, then use the convexity of V and estimate the corresponding error terms as before. We leave the details to the reader.

Step 3. Recovery sequence for $\widehat{\mathcal{Z}}I(\Lambda \cdot x; O)$.

Let $\Lambda \in \text{dom } \bar{V}$ and let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain. In this step, for almost all ω , we prove the existence of a sequence $(u_\varepsilon^\omega)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ such that $u_\varepsilon^\omega \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$ and $I_\varepsilon(\Lambda \cdot x + u_\varepsilon^\omega, \omega; O) \rightarrow \widehat{\mathcal{Z}}I(\Lambda \cdot x; O)$.

By Lemma 4.4(d), $\mathcal{Z}W$ and $\widehat{\mathcal{Z}}W$ coincide on $\text{int dom } \bar{V}$, and hence the result on $\text{int dom } \bar{V}$ already follows from Step 2. Let now $\Lambda \in \partial \text{dom } \mathcal{Z}W$. Lemma 4.4(d) then asserts $\widehat{\mathcal{Z}}W(\Lambda) = \lim_{t \uparrow 1} \mathcal{Z}W(t\Lambda)$. By convexity of $\text{dom } \bar{V}$, for all $t \in [0, 1)$, we have $t\Lambda \in \text{int dom } \bar{V}$, and hence, for almost all ω , Step 2 above gives a sequence $(u_{\varepsilon,t})_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ such that $u_{\varepsilon,t} \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$ and $I_\varepsilon(t\Lambda \cdot x + u_{\varepsilon,t}, \omega; O) \rightarrow \mathcal{Z}I(t\Lambda \cdot x; O) = |O| \mathcal{Z}W(t\Lambda)$. The conclusion then follows from a diagonalization argument.

Step 4. Conclusion.

Let $\Lambda \in \text{dom } \bar{V}$, let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $(u_\varepsilon^\omega)_\varepsilon$ be the sequence given by Step 3 above. As $u_\varepsilon^\omega \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$, the Γ -lim inf inequality (see Proposition 4.3) gives, for almost all ω ,

$$|O| \widehat{\mathcal{Z}}W(\Lambda) = \lim_{\varepsilon \downarrow 0} I_\varepsilon(\Lambda \cdot x + u_\varepsilon^\omega, \omega; O) \geq I(\Lambda \cdot x; O) = |O| \bar{W}(\Lambda).$$

This being true for any $\Lambda \in \text{dom } \bar{V}$, we conclude that $\widehat{\mathcal{Z}}W = \bar{W}$ everywhere. \square

With Lemma 4.5 at hands, we may prove the Γ -lim sup inequality.

Proposition 4.6 (Γ -lim sup inequality). *Assume $p > d$. There exists a subset $\Omega' \subset \Omega_1$ of maximal probability with the following property: for all $\omega \in \Omega'$, all strongly star-shaped (in the sense of Proposition A.16) bounded Lipschitz domains $O \subset \mathbb{R}^d$ and all $u \in W^{1,p}(O; \mathbb{R}^m)$, there exist a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ and a sequence $(v_\varepsilon)_\varepsilon \subset W_0^{1,p}(O; \mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup u$ and $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$, and such that $I_\varepsilon(u_\varepsilon + v_\varepsilon, \omega; O) \rightarrow I(u; O)$ and $J_\varepsilon(u_\varepsilon, \omega; O) \rightarrow J(u; O)$ as $\varepsilon \downarrow 0$. \square*

Recall that the Γ -lim inf inequality implies the locality of recovery sequences (see the proof of Corollary 3.3). Hence, the Γ -convergence result on a Lipschitz domain D for Neumann boundary conditions follows from the Γ -lim sup on a ball $B \supset D$ and the Γ -lim inf inequality on $B \setminus D$. For the adaptation of Corollary 2.4, the approach is similar and we leave the details to the reader.

Proof. *Step 1.* Recovery sequence for affine functions.

In this step, we consider the case when $u = \Lambda \cdot x$ is an affine function. More precisely, we prove the existence of a subset $\Omega' \subset \Omega_1$ of maximal probability with the following property: given a bounded Lipschitz domain $O \subset \mathbb{R}^d$, for all $\omega \in \Omega'$ and all $\Lambda \in \text{int dom } \overline{W}$, there exist sequences $(u_\varepsilon)_\varepsilon \subset W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$ and $(v_\varepsilon)_\varepsilon \subset W_0^{1,p}(O; \mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup \Lambda \cdot x$ in $W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$ and $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$, and such that $I_\varepsilon(u_\varepsilon + v_\varepsilon, \omega; O) \rightarrow I(\Lambda \cdot x; O)$ and $J_\varepsilon(u_\varepsilon, \omega; O') \rightarrow J(\Lambda \cdot x; O')$ for all bounded domains $O' \subset \mathbb{R}^d$.

Let $\Lambda \in \text{int dom } \overline{V}$. For almost all $\omega \in \Omega_1$, and all bounded domains $O' \subset \mathbb{R}^d$, we have by convexity, the Birkhoff-Khinchin ergodic theorem, definition of $\varphi_{\Lambda'}$, and continuity of \overline{V} at Λ :

$$\begin{aligned} \limsup_{t \uparrow 1, \Lambda' \rightarrow \Lambda} \lim_{\varepsilon \downarrow 0} J_\varepsilon(t\Lambda' \cdot x + \varepsilon t \varphi_{\Lambda'}(\cdot/\varepsilon, \omega), \omega; O') &\leq \limsup_{\Lambda' \rightarrow \Lambda} \lim_{\varepsilon \downarrow 0} J_\varepsilon(\Lambda' \cdot x + \varepsilon \varphi_{\Lambda'}(\cdot/\varepsilon, \omega), \omega; O') \\ &= |O'| \lim_{\Lambda' \rightarrow \Lambda} \overline{V}(\Lambda') = |O'| \overline{V}(\Lambda) = J(\Lambda \cdot x; O'). \end{aligned}$$

Combined with the Γ -lim inf inequality for $J_\varepsilon(\cdot, \omega; O')$ towards $J(\cdot; O')$ (for $\omega \in \Omega_1$), this yields

$$\lim_{t \uparrow 1, \Lambda' \rightarrow \Lambda} \lim_{\varepsilon \downarrow 0} J_\varepsilon(t\Lambda' \cdot x + \varepsilon t \varphi_{\Lambda'}(\cdot/\varepsilon, \omega), \omega; O') = J(\Lambda \cdot x; O'). \quad (4.26)$$

By definition of \overline{W} , we may choose sequences $\Lambda_n \rightarrow \Lambda$ and $t_n \uparrow 1$ such that $\overline{W}_{t_n}(\Lambda_n) \rightarrow \overline{W}(\Lambda)$. For this choice, (4.26) yields for almost all ω and all bounded domain $O' \subset \mathbb{R}^d$

$$\lim_{n \uparrow \infty} \lim_{\varepsilon \downarrow 0} J_\varepsilon(t_n \Lambda_n \cdot x + \varepsilon t_n \varphi_{\Lambda_n}(\cdot/\varepsilon, \omega), \omega; O') = J(\Lambda \cdot x; O').$$

For all n and almost all ω , set $u_{\varepsilon,n}^\omega := t_n \Lambda_n \cdot x + \varepsilon t_n \varphi_{\Lambda_n}(\cdot/\varepsilon, \omega)$. By Step 1 of the proof of Lemma 4.5, for any bounded Lipschitz domains $O \subset \mathbb{R}^d$, there exists a sequence $(v_{\varepsilon,n}^\omega)_\varepsilon \subset W_0^{1,p}(O; \mathbb{R}^m)$ such that $v_{\varepsilon,n}^\omega \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$ and $I_\varepsilon(u_{\varepsilon,n}^\omega + v_{\varepsilon,n}^\omega, \omega; O) \rightarrow |O| \overline{W}_{t_n}(\Lambda_n)$.

By a diagonalization argument, we then conclude that for almost all ω and all bounded Lipschitz domains $O \subset \mathbb{R}^d$ there exist sequences $(u_\varepsilon)_\varepsilon \subset W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$ and $(v_\varepsilon)_\varepsilon \subset W_0^{1,p}(O; \mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup \Lambda \cdot x$ and $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}$, and such that $I_\varepsilon(u_\varepsilon + v_\varepsilon, \omega; O) \rightarrow I(\Lambda \cdot x; O)$ and $J_\varepsilon(u_\varepsilon, \omega; O') \rightarrow J(\Lambda \cdot x; O')$ for all bounded domains $O' \subset \mathbb{R}^d$.

Now define $\Omega' \subset \Omega_1$ as a subset of maximal probability such that this result holds for all $\Lambda \in \mathbb{Q}^{m \times d} \cap \text{int dom } \overline{V}$ and all $\omega \in \Omega'$. Arguing as in the end of Step 1 of the proof of Proposition 3.10, and using the continuity of both \overline{W} and \overline{V} in the interior of the domain (see Lemma 4.5), the conclusion follows.

Step 2. Recovery sequence for continuous piecewise affine functions.

We now show that, for any $\omega \in \Omega'$, any bounded Lipschitz domain $O \subset \mathbb{R}^d$, and any continuous piecewise affine function u on O with $\nabla u \in \text{int dom } \overline{V}$ pointwise, there exist a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ and a sequence $(v_\varepsilon)_\varepsilon \subset W_0^{1,p}(O; \mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup u$ and $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$, and such that $I_\varepsilon(u_\varepsilon + v_\varepsilon, \omega; O) \rightarrow I(u; O)$ and $J_\varepsilon(u_\varepsilon, \omega; O) \rightarrow J(\Lambda \cdot x; O)$. This follows from an immediate adaptation of Step 2 of the proof of Proposition 3.10. Again, the functional I_ε is not convex, but we may use the bound $W \leq C(1 + V)$ at all points where the cut-off functions are different from 1 or 0, and use the convexity of V to estimate the corresponding error terms. We leave the details to the reader.

Step 3. Recovery sequence for general functions.

We show that, for all $\omega \in \Omega'$, all strongly star-shaped bounded Lipschitz domains $O \subset \mathbb{R}^d$, and all $u \in W^{1,p}(O; \mathbb{R}^m)$, there exist a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ and a sequence $(v_\varepsilon)_\varepsilon \subset W_0^{1,p}(O; \mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup u$ and $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$, and such that $I_\varepsilon(u_\varepsilon + v_\varepsilon, \omega; O) \rightarrow I(u; O)$ and $J_\varepsilon(u_\varepsilon, \omega; O) \rightarrow J(u; O)$. Let $O \subset \mathbb{R}^d$ be some fixed strongly star-shaped bounded Lipschitz domain. By the Γ -lim inf inequality of Proposition 4.3, we can restrict attention to those $u \in W^{1,p}(O; \mathbb{R}^m)$ that satisfy

$$I(u; O) = \int_O \overline{W}(\nabla u(y)) dy < \infty,$$

so that $\nabla u \in \text{dom } \overline{V}$ almost everywhere. Let u be such a function and let $\omega \in \Omega'$ be fixed.

Let $t \in (0, 1)$. Since O is Lipschitz and strongly star-shaped, and since \overline{V} is lower semicontinuous on $\mathbb{R}^{m \times d}$, continuous on $\text{int dom } \overline{V}$, ru-usc, and satisfies $\overline{V} \leq \overline{W} \leq C(1 + \overline{V})$ (see indeed Lemmas 4.4 and 4.5), the nonconvex approximation result of Proposition A.16(ii)(c) yields a sequence $(u_n)_n$ of continuous piecewise affine functions such that $u_n \rightarrow u$ (strongly) in $W^{1,p}(O; \mathbb{R}^m)$, $I(u_n; O) \rightarrow I(u; O)$ and $J(u_n; O) \rightarrow J(u; O)$ as $n \uparrow \infty$, and such that $\nabla u_n \in \text{int dom } \overline{V}$ pointwise. Now Step 2 above gives, for any n , sequences $(u_{\varepsilon,n})_{\varepsilon} \subset W^{1,p}(O; \mathbb{R}^m)$ and $(v_{\varepsilon,n})_{\varepsilon} \subset W_0^{1,p}(O; \mathbb{R}^m)$ such that $u_{\varepsilon,n} \rightarrow u_n$ and $v_{\varepsilon,n} \rightarrow 0$ in $W^{1,p}(O; \mathbb{R}^m)$, and such that $I_{\varepsilon}(u_{\varepsilon,n} + v_{\varepsilon,n}, \omega; O) \rightarrow I(u_n; O)$ and $J_{\varepsilon}(u_{\varepsilon,n}, \omega; O) \rightarrow J(u_n; O)$ as $\varepsilon \downarrow 0$. The result then follows from a diagonalization argument. \square

5. PROOF OF THE IMPROVED RESULTS

5.1. Subcritical case $1 < p \leq d$. In this section, we prove Corollary 2.9. We shall use truncations in place of the Sobolev compact embedding. For such truncation arguments to work, we need to restrict to the scalar case and to assume that the domain is fixed, i.e. $\text{dom } V(y, \cdot, \omega) = \text{dom } M$ for almost all y, ω .

Proof of Corollary 2.9. In the proof of Theorem 2.2 and Corollary 2.4, the Sobolev compact embedding into bounded functions is used both in Step 2 of the proof of Proposition 3.10 and in Step 1 of the proof of Corollary 2.4(i) (see Section 3.6). We only display the argument for Proposition 3.10 (the argument for Corollary 2.4(i) is similar).

We use the notation of Step 2 of the proof of Proposition 3.10. For all $s > 0$, define the truncation map $T_s : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$T_s(x) = \text{sign}(x)|x| \wedge s = \begin{cases} s, & \text{if } x \geq s; \\ x, & \text{if } -s \leq x \leq s; \\ -s, & \text{if } x \leq -s; \end{cases} \quad (5.1)$$

and for all $s > 0$ consider the following s -truncation of $u_{\varepsilon, \kappa, r, \eta}$:

$$u_{\varepsilon, \kappa, r, \eta, s} := T_s(u_{\varepsilon, \kappa, r, \eta} - u_{\kappa, r}) + u_{\kappa, r} \in W^{1,p}(O; \mathbb{R}). \quad (5.2)$$

Since $|tu_{\varepsilon, \kappa, r, \eta, s} - u| \leq s + |tu_{\kappa, r} - u|$, we may replace (3.29) by

$$\lim_{t \uparrow 1} \limsup_{r \downarrow 0} \limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{s \downarrow 0} \limsup_{\varepsilon \downarrow 0} \|tu_{\varepsilon, \kappa, r, \eta, s} - u\|_{L^\infty(O)} = 0.$$

Since $\nabla u_{\varepsilon, \kappa, r, \eta, s} = T'_s(u_{\varepsilon, \kappa, r, \eta} - u_{\kappa, r}) \nabla u_{\varepsilon, \kappa, r, \eta} + (1 - T'_s(u_{\varepsilon, \kappa, r, \eta} - u_{\kappa, r})) \nabla u_{\kappa, r}$, we deduce by convexity, noting that T'_s takes values in $[0, 1]$,

$$\begin{aligned} J_{\varepsilon}(tu_{\varepsilon, \kappa, r, \eta, s}, \omega; O) &\leq \int_O T'_s(u_{\varepsilon, \kappa, r, \eta}(y) - u_{\kappa, r}(y)) V(y/\varepsilon, t \nabla u_{\varepsilon, \kappa, r, \eta}, \omega) dy \\ &\quad + \int_O (1 - T'_s(u_{\varepsilon, \kappa, r, \eta}(y) - u_{\kappa, r}(y))) V(y/\varepsilon, t \nabla u_{\kappa, r}, \omega) dy \\ &\leq J_{\varepsilon}(tu_{\varepsilon, \kappa, r, \eta}, \omega; O) + |\{y \in O : |u_{\varepsilon, \kappa, r, \eta}(y) - u_{\kappa, r}(y)| > s\}| \max_{1 \leq l \leq k} M(t\Lambda_l) \\ &\leq J_{\varepsilon}(tu_{\varepsilon, \kappa, r, \eta}, \omega; O) + s^{-p} \|u_{\varepsilon, \kappa, r, \eta} - u_{\kappa, r}\|_{L^p(O)}^p \left((1-t)M(0) + \max_{1 \leq l \leq k} M(\Lambda_l) \right). \end{aligned} \quad (5.3)$$

Since by definition (and by the Rellich-Kondrachov theorem) we have $u_{\varepsilon, \kappa, r, \eta} \rightarrow u_{\kappa, r}$ (strongly) in $L^p(O)$ as $\varepsilon \downarrow 0$, since the Λ_l 's all belong to $\text{dom } \overline{V}$, and since by assumption $\text{dom } M = \text{dom } \overline{V}$, we deduce, combining this with (3.32), that

$$\lim_{t \uparrow 1} \limsup_{r \downarrow 0} \limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{s \downarrow 0} \limsup_{\varepsilon \downarrow 0} J_{\varepsilon}(tu_{\varepsilon, \kappa, r, \eta, s}, \omega; O) \leq J(u; O).$$

The rest of the proof is unchanged. \square

5.2. Minimal soft buffer zone for Dirichlet boundary data. In this section, we prove Corollary 2.10. In view of the error term (3.35) in the proof of Corollary 2.4, it seems that the speed of convergence of η to 0 with respect to ε must depend quantitatively on the speed of convergence of w_ε to 0 in $L^\infty(O; \mathbb{R}^m)$. In the case when the target function is affine ($\Lambda \cdot x$, say), then $w_\varepsilon := \varepsilon \varphi_\Lambda(\cdot/\varepsilon, \cdot)$ is the rescaling of the corrector and its convergence to zero is strictly related to the sublinearity of φ_Λ at infinity, cf. Lemma 3.4. Even in the linear scalar case when $V(y, \Lambda) = \Lambda \cdot A(y)\Lambda$ for some matrix random field A , this sublinear growth can be arbitrary, and we expect that for all $\gamma < 1$, there exists a field A such that $\mathbb{E}[|\varphi_\Lambda(x)|^2]^{\frac{1}{2}} \sim |x|^\gamma$ as $|x| \gg 1$ (see recent results in [28] and the example of Gaussian fields with non-integrable covariance). Yet, if instead of using the corrector φ_Λ itself, which is in general not stationary and well-behaved, we use a proxy that is stationary, then the size of the buffer zone can be (optimally) reduced, at least for affine target functions, as the following proposition shows.

Proposition 5.1. *If for all $\Lambda \in \mathbb{R}^{m \times d}$ we have*

$$\bar{V}(\Lambda) = \inf_{\phi \in W^{1,p}(\Omega; \mathbb{R}^m)} \mathbb{E}[V(0, \Lambda + D\phi, \cdot)], \quad (5.4)$$

then the conclusion of Corollary 2.10 holds (and we can further replace $\theta\varepsilon$ by any sequence $\eta_\varepsilon \downarrow 0$ satisfying $\liminf_\varepsilon \eta_\varepsilon/\varepsilon > 0$). \square

Identity (5.4) is essentially a regularity statement on quasi-minimizers of $f \mapsto \mathbb{E}[V(0, \Lambda + f, \cdot)]$ on $L^p_{\text{pot}}(\Omega)^m$. By Poincaré's inequality, periodic gradients with mean-value zero are gradients of periodic functions, and hence in that case the space $F^p_{\text{pot}}(\Omega)$ coincide with $\{D\phi : \phi \in W^{1,p}_\tau(\Omega)^m\}$, so that (5.4) is trivially satisfied. This already proves Corollary 2.10 under the additional assumption (1).

On the other hand, the following result shows that (5.4) is also satisfied in the scalar case $m = 1$ if the domain of V is fixed, in which case truncations are available. This proves Corollary 2.10 under the additional assumption (2).

Lemma 5.2. *If $m = 1$ and if $\text{dom}V(y, \cdot, \omega) = \text{dom}M$ is open for almost all y, ω , then (5.4) holds true for all Λ .* \square

Proof of Proposition 5.1. For all $\Lambda \in \mathbb{R}^{m \times d}$, by assumption (5.4), for all $\delta > 0$, there exists a stationary random field $\varphi_{\Lambda, \delta} \in W^{1,p}(\Omega; \mathbb{R}^m)$ such that $\mathbb{E}[V(0, \Lambda + \nabla\varphi_{\Lambda, \delta}(0, \cdot), \cdot)] \leq \bar{V}(\Lambda) + \delta$. Set $u_\varepsilon^{\Lambda, \delta, \omega} := \varepsilon \varphi_{\Lambda, \delta}(\cdot/\varepsilon, \omega)$. By stationarity, for almost all ω , the Birkhoff-Khinchin ergodic theorem asserts that, for any bounded domain $O \subset \mathbb{R}^d$,

$$\lim_{\varepsilon \downarrow 0} \int_O |u_\varepsilon^{\Lambda, \delta, \omega}/\varepsilon|^p = |O| \mathbb{E}[|\varphi_{\Lambda, \delta}|^p], \quad \lim_{\varepsilon \downarrow 0} \int_O |\nabla u_\varepsilon^{\Lambda, \delta, \omega}|^p = |O| \mathbb{E}[|\nabla\varphi_{\Lambda, \delta}|^p]. \quad (5.5)$$

Let $O \subset \mathbb{R}^d$ be some fixed bounded domain. For $\eta > 0$, set $O_\eta := \{x \in O : \text{dist}(x, \partial O) > \eta\}$. For any sequence $\eta_\varepsilon \downarrow 0$, (5.5) yields

$$\lim_{\varepsilon \downarrow 0} \int_{O \setminus O_{\eta_\varepsilon}} |u_\varepsilon^{\Lambda, \delta, \omega}/\varepsilon|^p = 0 = \lim_{\varepsilon \downarrow 0} \int_{O \setminus O_{\eta_\varepsilon}} |\nabla u_\varepsilon^{\Lambda, \delta, \omega}|^p. \quad (5.6)$$

Fix such a sequence $\eta_\varepsilon \downarrow 0$. As in Step 1 of the proof of Proposition 3.10, for all $\Lambda \in \mathbb{R}^{m \times d}$, we obtain the following, for some subset $\Omega_\Lambda \subset \Omega$ of maximal probability: for all $\omega \in \Omega_\Lambda$ and all $\delta > 0$, there is a sequence $(u_\varepsilon^{\Lambda, \delta, \omega})_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ such that $u_\varepsilon^{\Lambda, \delta, \omega} \rightarrow 0$ in $W^{1,p}(O; \mathbb{R}^m)$ as $\varepsilon \downarrow 0$, $\limsup_\varepsilon J_\varepsilon(\Lambda \cdot x + u_\varepsilon^{\Lambda, \delta, \omega}, \omega; O) \leq J(\Lambda \cdot x; O) + \delta$, and such that (5.6) is satisfied.

Let Λ and $\omega \in \Omega_\Lambda$ be fixed, and let $(u_\varepsilon^{\Lambda, \delta, \omega})_\varepsilon$ be as above. For all $\varepsilon > 0$, choose a smooth cut-off function χ_ε with values in $[0, 1]$, equal to 1 on $O_{\eta_\varepsilon} = \{x \in O : d(x, \partial O) > \eta_\varepsilon\}$, vanishing outside O ,

and with $|\nabla\chi_\varepsilon| \leq C'/\eta_\varepsilon$ for some constant C' . Defining $v_\varepsilon^{\Lambda,\delta,\omega} := \chi_\varepsilon u_\varepsilon^{\Lambda,\delta,\omega} \in W_0^{1,p}(O; \mathbb{R}^m)$, we obtain

$$\begin{aligned} J_\varepsilon^{\eta_\varepsilon}(\Lambda \cdot x + v_\varepsilon^{\Lambda,\delta,\omega}, \omega; O) &= J_\varepsilon(\Lambda \cdot x + v_\varepsilon^{\Lambda,\delta,\omega}, \omega; O_{\eta_\varepsilon}) + \int_{O \setminus O_{\eta_\varepsilon}} |\Lambda + \nabla v_\varepsilon^{\Lambda,\delta,\omega}|^p \\ &\leq J_\varepsilon(\Lambda \cdot x + v_\varepsilon^{\Lambda,\delta,\omega}, \omega; O) + 3^{p-1} |\Lambda|^p |O \setminus O_{\eta_\varepsilon}| \\ &\quad + 3^{p-1} \int_{O \setminus O_{\eta_\varepsilon}} (|\nabla u_\varepsilon^{\Lambda,\delta,\omega}|^p + |C' u_\varepsilon^{\Lambda,\delta,\omega} / \eta_\varepsilon|^p), \end{aligned}$$

and hence, if the sequence $\eta_\varepsilon \downarrow 0$ is further chosen such that $\liminf_\varepsilon \eta_\varepsilon / \varepsilon > 0$,

$$\limsup_{\varepsilon \downarrow 0} J_\varepsilon^{\eta_\varepsilon}(\Lambda \cdot x + v_\varepsilon^{\Lambda,\delta,\omega}, \omega; O) \leq \limsup_{\varepsilon \downarrow 0} J_\varepsilon(\Lambda \cdot x + v_\varepsilon^{\Lambda,\delta,\omega}, \omega; O) \leq J(\Lambda \cdot x; O) + \delta.$$

Therefore, $\limsup_\delta \limsup_\varepsilon J_\varepsilon^{\eta_\varepsilon}(\Lambda \cdot x + v_\varepsilon^{\Lambda,\delta,\omega}, \omega; O) \leq J(\Lambda \cdot x; O)$. Combined with the Γ -lim inf inequality of Proposition 3.2 and a diagonalization argument, this proves the result. \square

Proof of Lemma 5.2. We split the proof into two steps.

Step 1. Preliminary: we claim that it suffices to prove that, for all $\Lambda \in \text{dom}\bar{V}$,

$$\limsup_{t \uparrow 1} \inf_{\phi \in W^{1,p}(\Omega)} \mathbb{E}[V(0, t\Lambda + D\phi, \cdot)] \leq \bar{V}(\Lambda). \quad (5.7)$$

Define $\bar{V}'(\Lambda) := \inf_{\phi \in W^{1,p}(\Omega)} \mathbb{E}[V(0, \Lambda + D\phi, \cdot)]$. By definition $\bar{V}'(\Lambda) \geq \bar{V}(\Lambda)$ for all Λ , and hence property (5.7) together with the lower semicontinuity of \bar{V} directly yields $\bar{V}(\Lambda) = \lim_{t \uparrow 1} \bar{V}'(t\Lambda)$ for all Λ (and in particular the limit exists). Since \bar{V}' is obviously convex, it is continuous on the interior of its domain. Since the domain is assumed to be open, this yields $\bar{V}(\Lambda) = \bar{V}'(\Lambda)$ for all Λ .

Step 2. Proof of (5.7).

Let $\Lambda \in \text{dom}\bar{V}$ be fixed. Lemma 3.4 gives a measurable corrector $u := \varphi_\Lambda \in \text{Mes}(\Omega; W_{\text{loc}}^{1,p}(\mathbb{R}^d))$ such that $\nabla u \in F_{\text{pot}}^p(\Omega)$ and $\bar{V}(\Lambda) = \mathbb{E}[V(0, \Lambda + \nabla u(0, \cdot), \cdot)]$. For all $R > r > 0$, choose a smooth cut-off function $\chi_{R,r}$ taking values in $[0, 1]$, equal to 1 on Q_{R-r} , vanishing outside Q_R and satisfying $|\nabla\chi_{R,r}| \leq 2/r$. Also recall the definition (5.1) of the truncation T_s . We then set

$$u_R(x, \omega) = u(x, \omega) - \int_{Q_R} u(\cdot, \omega), \quad v_{R,r}^s(x, \omega) = \chi_{R,r}(x) T_s u_R(x, \omega),$$

and

$$w_{R,r}^s(x, \omega) = \frac{1}{|Q_R|} \int_{\mathbb{R}^d} v_{R,r}^s(x + y, \tau_y \omega) dy = \int_{-x+Q_R} v_{R,r}^s(x + y, \tau_y \omega) dy.$$

Clearly, $w_{R,r}^s$ is well-defined, stationary, and belongs to $W^{1,p}(\Omega)$, with

$$\nabla w_{R,r}^s(x, \omega) = \int_{-x+Q_R} \nabla v_{R,r}^s(x + y, \tau_y \omega) dy.$$

Let $t \in [0, 1)$. By Jensen's inequality,

$$\begin{aligned} K_{R,r}^s(t) &:= \mathbb{E}[V(0, t\Lambda + t\nabla w_{R,r}^s(0, \cdot), \cdot)] = \mathbb{E} \left[V \left(0, t\Lambda + t \int_{Q_R} \nabla v_{R,r}^s(y, \tau_y \cdot) dy, \cdot \right) \right] \\ &\leq \mathbb{E} \left[\int_{Q_R} V(0, t\Lambda + t\nabla v_{R,r}^s(y, \tau_y \cdot), \cdot) dy \right], \end{aligned}$$

and hence, by stationarity and the Fubini theorem,

$$K_{R,r}^s(t) \leq \mathbb{E} \left[\int_{Q_R} V(y, t\Lambda + t\nabla v_{R,r}^s(y, \cdot), \cdot) dy \right].$$

Decomposing

$$\begin{aligned} t\Lambda + t\nabla v_{R,r}^s(y, \omega) &= t\Lambda + t\chi_{R,r}(y)\nabla T_s u_R(y, \omega) + (1-t)\frac{t}{1-t}\nabla\chi_{R,r}(y)T_s u_R(y, \omega) \\ &= t\chi_{R,r}(y)T'_s(u_R(y, \omega))(\Lambda + \nabla u(y, \omega)) + t(1-\chi_{R,r}(y))T'_s(u_R(y, \omega))\Lambda \\ &\quad + t(1-T'_s(u_R(y, \omega)))\Lambda + (1-t)\frac{t}{1-t}\nabla\chi_{R,r}(y)T_s u_R(y, \omega), \end{aligned}$$

with T'_s taking values in $[0, 1]$, we may then bound by convexity

$$\begin{aligned} K_{R,r}^s(t) &\leq \mathbb{E} \left[\int_{Q_R} V(y, \Lambda + \nabla u(y, \cdot), \cdot) dy \right] + (1-t)E_{R,r}^s(t) \\ &\quad + M(\Lambda) \int_{Q_R} (1-\chi_{R,r}) + M(\Lambda) \mathbb{E} \left[\int_{Q_R} (1-T'_s(u_R(y, \cdot))) \right], \end{aligned} \quad (5.8)$$

where the error term reads

$$E_{R,r}^s(t) = \mathbb{E} \left[\int_{Q_R} M \left(\frac{t}{1-t} \nabla\chi_{R,r}(y)T_s u_R(y, \omega) \right) dy \right].$$

By stationarity of ∇u , note that

$$\mathbb{E} \left[\int_{Q_R} V(y, \Lambda + \nabla u(y, \cdot), \cdot) dy \right] = \mathbb{E}[V(0, \Lambda + \nabla u(0, \cdot), \cdot)] = \bar{V}(\Lambda). \quad (5.9)$$

For the error term, note that

$$\left\| \frac{t}{1-t} \nabla\chi_{R,r}(\cdot)T_s u_R(\cdot, \omega) \right\|_{L^\infty(O)} \leq \frac{2t}{1-t} \frac{s}{r}. \quad (5.10)$$

It remains to treat the last two terms of (5.8). Noting that $\int_{Q_R} (1-\chi_{R,r}) = R^{-d}(R^d - (R-r)^d) \leq dr/R$, we obtain

$$\begin{aligned} \int_{Q_R} (1-\chi_{R,r}) + \mathbb{E} \left[\int_{Q_R} (1-T'_s(u_R(y, \cdot))) \right] &\leq \frac{dr}{R} + \mathbb{E} \left[\int_{Q_R} \mathbb{1}_{|u_R(y)| \geq s} dy \right] \\ &\leq \frac{dr}{R} + \int_Q \mathbb{P} \left[\frac{1}{R} \left| u(Ry, \cdot) - \int_Q u(Rz, \cdot) dz \right| \geq \frac{s}{R} \right] dy. \end{aligned} \quad (5.11)$$

Lemma 3.4 (together with the Rellich-Kondrachov theorem) gives $\frac{1}{R}|u(R\cdot, \omega) - \int_Q u(Rz, \omega) dz| \rightarrow 0$ (strongly) in $L^p(Q)$ as $R \uparrow \infty$, for almost all ω . Hence up to an extraction in R (implicit in the sequel) we deduce that, for almost all $y \in Q$, $\frac{1}{R}|u(Ry, \cdot) - \int_Q u(Rz, \cdot) dz| \rightarrow 0$ almost surely. Since almost sure convergence implies convergence in probability, we deduce by dominated convergence, for all $\varepsilon > 0$,

$$\lim_{R \uparrow \infty} \int_Q \mathbb{P} \left[\frac{1}{R} \left| u(Ry, \cdot) - \int_Q u(Rz, \cdot) dz \right| \geq \varepsilon \right] dy = 0.$$

A diagonalization argument then gives a sequence $\varepsilon_R \downarrow 0$ such that

$$\lim_{R \uparrow \infty} \int_Q \mathbb{P} \left[\frac{1}{R} \left| u(Ry, \cdot) - \int_Q u(Rz, \cdot) dz \right| \geq \varepsilon_R \right] dy = 0. \quad (5.12)$$

Choose $s = s_R := R\varepsilon_R$ and $r = r_R := R\sqrt{\varepsilon_R}$. By assumption, there exists some $\delta > 0$ with $\text{adh } B_\delta \subset \text{int dom } M$. By (5.10), for all $t \in [0, 1]$, there is some $R_t > 0$ such that for all $R > R_t$

$$\left\| \frac{t}{1-t} \nabla\chi_{R,r_R}(\cdot)T_{s_R} u_R(\cdot, \omega) \right\|_{L^\infty(O)} \leq \frac{2t}{1-t} \sqrt{\varepsilon_R} < \delta. \quad (5.13)$$

Combining this with (5.8), (5.9), (5.11), (5.12), and noting that $M(\Lambda) < \infty$ follows from the choice $\Lambda \in \text{dom } \bar{V}$, we obtain

$$\limsup_{t \uparrow 1} \limsup_{R \uparrow \infty} K_{R,r_R}^{s_R}(t) \leq \bar{V}(\Lambda),$$

and the result (5.7) follows. \square

5.3. Approximation by periodization in law. This last section is devoted to the proof of Corollary 2.13. For that purpose, we first prove the following version of the Γ -lim sup inequality of Proposition 3.10 in expectation. As Lemma 3.4 only gives almost sure (and not $L^1(\Omega)$) control of the sublinearity of the corrector, we need to use truncations to prove this result (and therefore restrict to the scalar case with fixed domain).

Proposition 5.3 (Γ -lim sup inequality in expectation with Neumann boundary data). *Let V, J_ε, J, M be as in Theorem 2.2 for some $p > 1$. Also assume that we have $m = 1$ and $\text{dom}V(y, \cdot, \omega) = \text{dom}M$ for almost all y, ω . Then, for all bounded Lipschitz domains $O \subset \mathbb{R}^d$ and all $u \in W^{1,p}(O)$, there exists a sequence $(u_\varepsilon)_\varepsilon \subset \text{Mes}(\Omega; W^{1,p}(O))$ such that $\mathbb{E}[J_\varepsilon(u_\varepsilon, \cdot; O)] \rightarrow J(u; O)$ and $u_\varepsilon \rightarrow u$ in $L^\infty(\Omega; L^\infty(O))$. \square*

Proof. We split the proof into three steps.

Step 1. Preliminary.

We claim that, for all bounded domains $O \subset \mathbb{R}^d$ and all $\Lambda \in \text{int dom } \bar{V}$, there exists a sequence $(u_{\Lambda, \varepsilon})_\varepsilon \subset \text{Mes}(\Omega; W^{1,p}(O))$ with $u_{\Lambda, \varepsilon}(\cdot, \omega) \rightharpoonup \Lambda \cdot x$ weakly in $W^{1,p}(O)$ for almost all ω and $\mathbb{E}[J_\varepsilon(u_{\Lambda, \varepsilon}, \cdot; O)] \rightarrow J(\Lambda \cdot x; O')$ for all subdomains $O' \subset O$. This is indeed a simple reformulation of Lemma 3.4 with the notation $u_{\Lambda, \varepsilon}(x, \omega) := \varepsilon \varphi_\Lambda(x/\varepsilon, \omega)$.

Step 2. Recovery sequence for continuous piecewise affine functions.

Let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let u be a continuous piecewise affine function on O such that $\nabla u \in \text{int dom } \bar{V}$ pointwise. We prove the existence of a sequence $(u_\varepsilon)_\varepsilon \subset \text{Mes}(\Omega; W^{1,p}(O))$ with $u_\varepsilon \rightarrow u$ in $L^\infty(\Omega; L^\infty(O))$ and $\mathbb{E}[J_\varepsilon(u_\varepsilon, \cdot; O)] \rightarrow J(u; O)$.

Let us modify u as in Step 2 of the proof of Proposition 3.10 to make its variations smoother (using Proposition A.17), and use the same notation. By Step 1, for all $1 \leq i \leq n_{\kappa, r}$, there exists a sequence $(u_{\varepsilon, \kappa, r}^i)_\varepsilon \subset \text{Mes}(\Omega; W_{\text{loc}}^{1,p}(\mathbb{R}^d))$ with $u_{\varepsilon, \kappa, r}^i(\cdot, \omega) \rightharpoonup c_{\kappa, r}^i + \Lambda_{\kappa, r}^i \cdot x$ in $W_{\text{loc}}^{1,p}(\mathbb{R}^d)$ for almost all ω and such that, for all Lipschitz subdomains $O' \subset O$, we have $\mathbb{E}[J_\varepsilon(u_{\varepsilon, \kappa, r}^i, \cdot; O')] \rightarrow J(\Lambda_{\kappa, r}^i \cdot x; O')$. Consider the same partition of unity $\sum_{i=1}^{n_{\kappa, r}} \chi_{\kappa, r, \eta}^i$ as in Step 2 of the proof of Proposition 3.10, and also recall the definition (5.1) of the truncation T_s . We now set, for $s > 0$,

$$u_{\varepsilon, \kappa, r, \eta, s} := u_{\kappa, r} + \sum_{i=1}^{n_{\kappa, r}} \chi_{\kappa, r, \eta}^i T_s(u_{\varepsilon, \kappa, r}^i - (c_{\kappa, r}^i + \Lambda_{\kappa, r}^i \cdot x)) \in \text{Mes}(\Omega; W^{1,p}(O)).$$

On the one hand, since $|tu_{\varepsilon, \kappa, r, \eta} - u| \leq s + |tu_{\kappa, r} - u|$ pointwise, we deduce

$$\lim_{t \uparrow 1} \limsup_{r \downarrow 0} \limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{s \downarrow 0} \limsup_{\varepsilon \downarrow 0} \|tu_{\varepsilon, \kappa, r, \eta} - u\|_{L^\infty(\Omega; L^\infty(O))} = 0. \quad (5.14)$$

On the other hand, since

$$\begin{aligned} t \nabla u_{\varepsilon, \kappa, r, \eta, s} &= \sum_{i=1}^{n_{\kappa, r}} t \chi_{\kappa, r, \eta}^i T_s'(u_{\varepsilon, \kappa, r}^i - (c_{\kappa, r}^i + \Lambda_{\kappa, r}^i \cdot x)) \nabla u_{\varepsilon, \kappa, r}^i \\ &\quad + \sum_{i=1}^{n_{\kappa, r}} t \chi_{\kappa, r, \eta}^i (1 - T_s'(u_{\varepsilon, \kappa, r}^i - (c_{\kappa, r}^i + \Lambda_{\kappa, r}^i \cdot x))) \Lambda_{\kappa, r}^i + (1 - t) S_{\varepsilon, \kappa, r, \eta, s, t}, \end{aligned}$$

where we have set

$$S_{\varepsilon, \kappa, r, \eta, s, t} := \frac{t}{1-t} \sum_{i=1}^{n_{\kappa, r}} T_s(u_{\varepsilon, \kappa, r}^i - (c_{\kappa, r}^i + \Lambda_{\kappa, r}^i \cdot x)) \nabla \chi_{\kappa, r, \eta}^i + \frac{t}{1-t} \sum_{i=1}^{n_{\kappa, r}} \chi_{\kappa, r, \eta}^i (\nabla u_{\kappa, r} - \Lambda_{\kappa, r}^i),$$

we deduce by convexity

$$\begin{aligned} \mathbb{E}[J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta,s}, \cdot; O)] &\leq (1-t)E_{\varepsilon,\kappa,r,\eta,s,t} + \sum_{i=1}^{n_{\kappa,r}} \mathbb{E}[J_\varepsilon(u_{\varepsilon,\kappa,r}^i, \cdot; O_{\kappa,r,\eta}^{i+})] \\ &\quad + \sum_{i=1}^{n_{\kappa,r}} M(\Lambda_{\kappa,r}^i) \mathbb{E}[|\{y \in O_{\kappa,r,\eta}^{i+} : |u_{\varepsilon,\kappa,r}^i(y, \omega) - (c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot y)| \geq s\}|] \end{aligned} \quad (5.15)$$

where the error reads

$$E_{\varepsilon,\kappa,r,\eta,s,t} = \mathbb{E} \left[\int_O V(y/\varepsilon, S_{\varepsilon,\kappa,r,\eta,s,t}(y, \cdot), \cdot) dy \right].$$

For all i , set $N_{\kappa,r,\eta}^i := \{j : j \neq i, O_{\kappa,r,\eta}^{j+} \cap O_{\kappa,r,\eta}^{i+} \neq \emptyset\}$. We estimate $S_{\varepsilon,\kappa,r,\eta,s,t}$ for all y, ω as follows:

$$|S_{\varepsilon,\kappa,r,\eta,s,t}(y, \omega)| \leq \frac{t}{1-t} \frac{C' n_{\kappa,r}}{\eta} s + \frac{t}{1-t} \sup_{1 \leq i \leq n_{\kappa,r}} \sup_{j \in N_{\kappa,r,\eta}^i} |\Lambda_{\kappa,r}^j - \Lambda_{\kappa,r}^i|,$$

and hence, since $\limsup_{\eta \downarrow 0} \sup_{j \in N_{\kappa,r,\eta}^i} |\Lambda_{\kappa,r}^j - \Lambda_{\kappa,r}^i| \leq \kappa$ for all i ,

$$\limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{s \downarrow 0} \limsup_{\varepsilon \downarrow 0} \|S_{\varepsilon,\kappa,r,\eta,s,t}\|_{L^\infty(\Omega; L^\infty(O))} = 0. \quad (5.16)$$

The last term of the right-hand side of (5.15) is estimated by

$$\mathbb{E}[|\{y \in O_{\kappa,r,\eta}^{i+} : |u_{\varepsilon,\kappa,r}^i(y, \omega) - (c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot y)| \geq s\}|] \leq \int_O \mathbb{P}[|u_{\varepsilon,\kappa,r}^i(y, \omega) - (c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot y)| \geq s] dy.$$

For almost all ω , the Rellich-Kondrachov theorem shows that $u_{\varepsilon,\kappa,r}^i(\cdot, \omega) \rightarrow c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot x$ (strongly) in $L^p(O)$, and hence, up to an extraction in ε (implicit in the sequel), this convergence holds almost surely, almost everywhere on O . The dominated convergence theorem then yields

$$\limsup_{\varepsilon \downarrow 0} \mathbb{E}[|\{y \in O_{\kappa,r,\eta}^{i+} : |u_{\varepsilon,\kappa,r}^i(y, \omega) - (c_{\kappa,r}^i + \Lambda_{\kappa,r}^i \cdot y)| \geq s\}|] = 0. \quad (5.17)$$

Finally we note that by construction,

$$\lim_{\varepsilon \downarrow 0} \sum_{i=1}^{n_{\kappa,r}} \mathbb{E}[J_\varepsilon(u_{\varepsilon,\kappa,r}^i, \cdot; O_{\kappa,r,\eta}^{i+})] = \sum_{i=1}^{n_{\kappa,r}} J(\Lambda_{\kappa,r}^i \cdot x; O_{\kappa,r,\eta}^{i+}) = |O_{\kappa,r,\eta}^{i+}| \bar{V}(\Lambda_{\kappa,r}^i).$$

Hence, arguing as in Step 2 of the proof of Proposition 3.10, combining this with (5.15), (5.16) and (5.17), and recalling that $\text{dom} \bar{V} = \text{dom} M$, we obtain

$$\lim_{t \uparrow 1} \limsup_{r \downarrow 0} \limsup_{\kappa \downarrow 0} \limsup_{\eta \downarrow 0} \limsup_{s \downarrow 0} \limsup_{\varepsilon \downarrow 0} \mathbb{E}[J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta,s}, \cdot; O)] \leq J(u; O), \quad (5.18)$$

so that the conclusion follows as in Step 2 of the proof of Proposition 3.10 (note that by Fatou's lemma the Γ -lim inf inequality also holds in expectation).

Step 3. Conclusion. The existence of recovery sequences for general functions can now be deduced as in Step 3 of the proof of Proposition 3.10, using the result of Step 2 above. \square

We now turn to the proof of Corollary 2.13 itself. In case (1), we use Proposition 5.3 to control the energy close to the boundary of the cube, where the periodization in law may have modified the integrand (which however is the same in law locally and can therefore be handled by taking the expectation). In case (2), we use the particular geometric structure to explicitly estimate the energy density near the boundary and prove its uniform integrability.

Proof of Corollary 2.13. Let $(V^R)_{R>0}$ be an admissible periodization in law for V in the sense of Definition 2.11, and for all $\varepsilon > 0$ and all $\Lambda \in \mathbb{R}^{m \times d}$ denote by $J_\varepsilon^{\text{per}}(\cdot, \Lambda, \cdot)$ the following random integral functional on the unit cube $Q = [-\frac{1}{2}, \frac{1}{2}]^d$:

$$J_\varepsilon^{\text{per}}(u, \Lambda, \omega) := \int_Q V^{1/\varepsilon}(y/\varepsilon, \Lambda + \nabla u(y), \omega) dy, \quad u \in W_{\text{per}}^{1,p}(Q; \mathbb{R}^m).$$

We split the proof into three main steps.

Step 1. Γ -lim inf inequality. The results of this step hold without the additional assumptions (1) and (2). Let $\Omega'' \subset \Omega'$ be a subset of maximal probability such that the almost sure property of Definition 2.11(ii) holds pointwise on Ω'' . We claim that for all $u \in W_{\text{per}}^{1,p}(Q; \mathbb{R}^m)$, all $u_\varepsilon \rightharpoonup u$ in $W_{\text{per}}^{1,p}(Q)$ and all $\Lambda_\varepsilon \rightarrow \Lambda$, we have $\liminf_{\varepsilon \downarrow 0} J_\varepsilon^{\text{per}}(u_\varepsilon, \Lambda_\varepsilon, \omega) \geq J(u + \Lambda \cdot x; Q)$ for all $\omega \in \Omega''$.

For all $\theta \in (0, 1)$ and $\omega \in \Omega''$, we thus have for all $\varepsilon > 0$ small enough

$$J_\varepsilon^{\text{per}}(u_\varepsilon, \Lambda_\varepsilon, \omega) = J_\varepsilon(u_\varepsilon + \Lambda_\varepsilon \cdot x, \omega; Q_\theta) + \int_{Q \setminus Q_\theta} V^{1/\varepsilon}(y/\varepsilon, \Lambda_\varepsilon + \nabla u_\varepsilon(y), \omega) dy.$$

By the Γ -lim inf inequality for $J_\varepsilon(\cdot, \omega; Q_\theta)$ (see Proposition 3.2) and by the non-negativity of $V^{1/\varepsilon}$, this turns into

$$\liminf_{\varepsilon \downarrow 0} J_\varepsilon^{\text{per}}(u_\varepsilon, \Lambda_\varepsilon, \omega) \geq J(u + \Lambda \cdot x; Q_\theta),$$

and the result follows by the arbitrariness of $\theta < 1$ and the monotone convergence theorem.

Step 2. Γ -lim sup inequality under assumption (1). We assume $m = 1$ and $\text{dom}V(y, \cdot, \omega) = \text{dom}M$ for almost all y, ω . We prove the existence of a subset $\Omega''' \subset \Omega''$ of maximal probability such that, for all $u \in W_{\text{per}}^{1,p}(Q)$, all $\Lambda \in \mathbb{R}^{1 \times d}$, and all $\omega \in \Omega'''$, there exists a sequence $(u_\varepsilon)_\varepsilon \subset \text{Mes}(\Omega; W_{\text{per}}^{1,p}(Q))$ such that $u_\varepsilon \rightarrow 0$ in $L^\infty(\Omega; L^\infty(Q))$ and

$$\lim_{t \uparrow 1} \liminf_{\varepsilon \downarrow 0} J_\varepsilon^{\text{per}}(tu_\varepsilon(\cdot, \omega) + tu, t\Lambda, \omega) = \lim_{t \uparrow 1} \limsup_{\varepsilon \downarrow 0} J_\varepsilon^{\text{per}}(tu_\varepsilon(\cdot, \omega) + tu, t\Lambda, \omega) = J(u + \Lambda \cdot x; Q).$$

Moreover, the limits $t \uparrow 1$ can be dropped if we have $J(\alpha u + \alpha \Lambda \cdot x; Q) < \infty$ for some $\alpha > 1$. We split the proof of this step into three parts.

Step 2.1. Local recovery sequence for affine functions.

For all $R > 0$, define the event

$$\Omega_R := \{\omega \in \Omega : V^r(\cdot, \cdot, \omega)|_{Q_{\theta_0 r} \times \mathbb{R}^{m \times d}} = V(\cdot, \cdot, \omega)|_{Q_{\theta_0 r} \times \mathbb{R}^{m \times d}}, \text{ for all } r > R\},$$

for some fixed $\theta_0 > 0$. Definition 2.11(ii) ensures that $\mathbb{P}[\Omega_R] \uparrow 1$ as $R \uparrow \infty$. Let $\Lambda \in \mathbb{R}^{1 \times d}$, let $O \subset Q$ be a domain of diameter smaller than $\theta_0/2$, and let $R > 0$. By the diameter condition, there exists $x_0 \in Q$ such that $O \subset x_0 + \theta_0 Q$. We claim that there is a sequence $(u_\varepsilon)_\varepsilon \subset \text{Mes}(\Omega; W^{1,p}(O))$ such that $u_\varepsilon \rightarrow 0$ in $L^\infty(\Omega; L^\infty(O))$ and $\mathbb{E}[\mathbf{1}_{\tau_{x_0/\varepsilon} \Omega_R} \int_O V^{1/\varepsilon}(\cdot/\varepsilon, \Lambda + \nabla u_\varepsilon, \cdot)] \rightarrow \mathbb{P}[\Omega_R] J(\Lambda \cdot x; O)$.

By definition, for all $\omega \in \Omega_R$ we have $V^{1/\varepsilon}(\cdot, \cdot, \omega)|_{Q_{\theta_0/\varepsilon} \times \mathbb{R}^{m \times d}} = V(\cdot, \cdot, \omega)|_{Q_{\theta_0/\varepsilon} \times \mathbb{R}^{m \times d}}$ for all $\varepsilon < 1/R$. Therefore, applying Proposition 5.3 on $-x_0 + O$ restricted to Ω_R yields the following: there exists a sequence $(u_\varepsilon)_\varepsilon \subset \text{Mes}(\Omega; W^{1,p}(-x_0 + O))$ with $u_\varepsilon \rightarrow 0$ in $L^\infty(\Omega; L^\infty(-x_0 + O))$ and

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\mathbf{1}_{\Omega_R} \int_{-x_0+O} V^{1/\varepsilon}(\cdot/\varepsilon, \Lambda + \nabla u_\varepsilon, \cdot) \right] &= \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\mathbf{1}_{\Omega_R} \int_{-x_0+O} V(\cdot/\varepsilon, \Lambda + \nabla u_\varepsilon, \cdot) \right] \\ &= \mathbb{P}[\Omega_R] J(\Lambda \cdot x; -x_0 + O) = \mathbb{P}[\Omega_R] J(\Lambda \cdot x; O). \end{aligned}$$

Now define $u'_\varepsilon := u_\varepsilon(-x_0 + \cdot, \tau_{-x_0/\varepsilon}^{1/\varepsilon} \cdot) \in \text{Mes}(\Omega; W^{1,p}(-x_0 + O))$ for all ε . By construction $u'_\varepsilon \rightarrow 0$ in $L^\infty(\Omega; L^\infty(O))$, and the stationarity property of Definition 2.11(i) further gives

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\mathbf{1}_{\tau_{x_0/\varepsilon}^{1/\varepsilon} \Omega_R} \int_O V^{1/\varepsilon}(y/\varepsilon, \Lambda + \nabla u'_\varepsilon(y, \cdot), \cdot) dy \right] \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\mathbf{1}_{\Omega_R} \int_O V^{1/\varepsilon}(y/\varepsilon, \Lambda + \nabla u_\varepsilon(-x_0 + y, \cdot), \tau_{x_0/\varepsilon}^{1/\varepsilon} \cdot) dy \right] \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\mathbf{1}_{\Omega_R} \int_O V^{1/\varepsilon}((-x_0 + y)/\varepsilon, \Lambda + \nabla u_\varepsilon(-x_0 + y, \cdot), \cdot) dy \right] \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\mathbf{1}_{\Omega_R} \int_{-x_0+O} V^{1/\varepsilon}(y/\varepsilon, \Lambda + \nabla u_\varepsilon(y, \cdot), \cdot) dy \right] = \mathbb{P}[\Omega_R] J(\Lambda \cdot x; O). \end{aligned}$$

Step 2.2. Global recovery sequence for (piecewise) affine functions.

Let $\Lambda \in \mathbb{R}^{1 \times d}$ and let u be a continuous piecewise affine Q -periodic function on \mathbb{R}^d with $\Lambda + \nabla u \in \text{int dom } \bar{V}$. We claim that there exists a sequence $(u_\varepsilon)_\varepsilon \subset \text{Mes}(\Omega; W_{\text{per}}^{1,p}(Q))$ such that $u_\varepsilon \rightarrow u$ in $L^\infty(\Omega; L^\infty(Q))$, and, for almost all ω ,

$$\begin{aligned} & \lim_{t \uparrow 1} \liminf_{\varepsilon \downarrow 0} J_\varepsilon^{\text{per}}(tu_\varepsilon(\cdot, \omega) + tu, t\Lambda, \omega) \\ &= \lim_{t \uparrow 1} \limsup_{\varepsilon \downarrow 0} J_\varepsilon^{\text{per}}(tu_\varepsilon(\cdot, \omega) + tu, t\Lambda, \omega) = J(u + \Lambda \cdot x; Q). \end{aligned} \quad (5.19)$$

Consider the partition of Q associated with u , and let $Q = \biguplus_{l=1}^k O_l$ be a refined partition such that all the O_l 's have diameter at most $\theta_0/2$. For all l , define $c_l + \Lambda_l \cdot x := \nabla u|_{O_l}$ with $\Lambda + \Lambda_l \in \text{int dom } \bar{V}$, and choose $x_l \in Q$ such that $O_l \subset x_l + \theta_0 Q$. Given $R > 0$, define

$$\chi_{\varepsilon, R} := \prod_{l=1}^k \mathbb{1}_{\tau_{x_l/\varepsilon} \Omega_R}.$$

Step 2.1 then gives a sequence $(u_\varepsilon^l)_\varepsilon \subset \text{Mes}(\Omega; W^{1,p}(O_l))$ such that $u_\varepsilon^l \rightarrow 0$ in $L^\infty(\Omega; L^\infty(O_l))$ and

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} \mathbb{E} \left[\chi_{\varepsilon, R} \int_{O_l} V^{1/\varepsilon}(\cdot/\varepsilon, \Lambda + \Lambda_l + \nabla u_\varepsilon^l, \cdot) \right] &\leq \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\mathbb{1}_{\tau_{x_l/\varepsilon} \Omega_R} \int_{O_l} V^{1/\varepsilon}(\cdot/\varepsilon, \Lambda + \Lambda_l + \nabla u_\varepsilon^l, \cdot) \right] \\ &= \mathbb{P}[\Omega_R] J(u + \Lambda \cdot x; O_l). \end{aligned}$$

We can then repeat the argument of Step 2 of the proof of Proposition 3.10, and glue the recovery sequences for the (small) affine parts, using this time Proposition A.18 instead of Proposition A.17 (applied to the function u). This allows to deduce the following: there exists a sequence $(u_\varepsilon)_\varepsilon \subset \text{Mes}(\Omega; W_{\text{per}}^{1,p}(O))$ such that $u_\varepsilon \rightarrow 0$ in $L^\infty(\Omega; L^\infty(Q))$ and

$$\limsup_{t \uparrow 1} \limsup_{\varepsilon \downarrow 0} \mathbb{E}[\chi_{\varepsilon, R} J_\varepsilon^{\text{per}}(tu_\varepsilon + tu, t\Lambda, \cdot)] \leq \mathbb{P}[\Omega_R] J(u + \Lambda \cdot x; Q). \quad (5.20)$$

Now the Γ -lim inf inequality of Step 1 yields $\liminf_t \liminf_\varepsilon J_\varepsilon^{\text{per}}(u_\varepsilon(\cdot, \omega) + u, t\Lambda, \omega) \geq J(u + \Lambda \cdot x; Q)$ for almost all ω . Assume by contradiction that this inequality is strict for all ω in some set of positive probability (even for some subsequence in ε, t). In this case we would find $\varepsilon_0, \delta > 0$ and $t_0 \in (0, 1)$ such that $\mathbb{P}[\Omega_{\varepsilon_0, t_0}^\delta] > 0$, where we have defined the event

$$\Omega_{\varepsilon_0, t_0}^\delta := \{\omega \in \Omega : J_\varepsilon^{\text{per}}(tu_\varepsilon(\cdot, \omega) + tu, t\Lambda, \omega) > \delta + J(u + \Lambda \cdot x; Q), \text{ for all } \varepsilon < \varepsilon_0, t > t_0\},$$

By definition we have

$$\begin{aligned} & \liminf_{t \uparrow 1} \liminf_{\varepsilon \downarrow 0} \mathbb{E}[\chi_{\varepsilon, R} J_\varepsilon^{\text{per}}(tu_\varepsilon + tu, t\Lambda, \cdot)] \\ &\geq J(u + \Lambda \cdot x; Q) \liminf_{\varepsilon \downarrow 0} \mathbb{E}[\chi_{\varepsilon, R}] + \liminf_{t \uparrow 1} \liminf_{\varepsilon \downarrow 0} \mathbb{E}[\chi_{\varepsilon, R} (J_\varepsilon^{\text{per}}(tu_\varepsilon + tu, t\Lambda, \cdot) - J(u + \Lambda \cdot x; Q))] \\ &\geq J(u + \Lambda \cdot x; Q) \liminf_{\varepsilon \downarrow 0} \mathbb{E}[\chi_{\varepsilon, R}] + \delta \liminf_{\varepsilon \downarrow 0} \mathbb{E}[\mathbb{1}_{\Omega_{\varepsilon_0, t_0}^\delta} \chi_{\varepsilon, R}] \\ &\quad + \liminf_{t \uparrow 1} \liminf_{\varepsilon \downarrow 0} \mathbb{E}[\mathbb{1}_{\Omega \setminus \Omega_{\varepsilon_0, t_0}^\delta} \chi_{\varepsilon, R} (J_\varepsilon^{\text{per}}(tu_\varepsilon + tu, t\Lambda, \cdot) - J(u + \Lambda \cdot x; Q))]. \end{aligned}$$

Now a union bound argument yields

$$\mathbb{E}[\chi_{\varepsilon, R}] \geq 1 - k\mathbb{P}[\Omega \setminus \Omega_R], \quad \mathbb{E}[\mathbb{1}_{\Omega_{\varepsilon_0, t_0}^\delta} \chi_{\varepsilon, R}] \geq \mathbb{P}[\Omega_{\varepsilon_0, t_0}^\delta] - k\mathbb{P}[\Omega \setminus \Omega_R],$$

and hence, by Fatou's lemma and by the Γ -lim inf inequality of Step 1, we obtain

$$\begin{aligned} & \liminf_{t \uparrow 1} \liminf_{\varepsilon \downarrow 0} \mathbb{E}[\chi_{\varepsilon, R} J_\varepsilon^{\text{per}}(tu_\varepsilon + tu, t\Lambda, \cdot)] \\ &\geq J(u + \Lambda \cdot x; Q)(1 - k\mathbb{P}[\Omega \setminus \Omega_R]) + \delta(\mathbb{P}[\Omega_{\varepsilon_0, t_0}^\delta] - k\mathbb{P}[\Omega \setminus \Omega_R]). \end{aligned}$$

However, taking the limit $R \uparrow \infty$, and recalling $\mathbb{P}[\Omega_R] \uparrow 1$, this contradicts (5.20). The result (5.19) is then proven.

Step 2.3. Recovery sequence for general target functions.

For all $\Lambda \in \text{int dom } \bar{V}$, let $\Omega_\Lambda \subset \Omega''$ be a subset of maximal probability such that equalities (5.19) hold for all $\omega \in \Omega_\Lambda$ for the choice $u = 0$. We then define $\Omega''' \subset \Omega''$ as the (countable) intersection of all Ω_Λ 's with $\Lambda \in \mathbb{Q}^{1 \times d} \cap \text{int dom } \bar{V}$. Repeating on this basis the gluing argument of Step 2 of the proof of Proposition 3.10, for all $\Lambda \in \mathbb{R}^{1 \times d}$ and for all continuous piecewise affine Q -periodic functions u on \mathbb{R}^d with $\Lambda + \nabla u \in \mathbb{Q}^{1 \times d} \cap \text{int dom } \bar{V}$, we may prove that (5.19) holds for all $\omega \in \Omega'''$.

Let now $u \in W_{\text{per}}^{1,p}(Q)$ and let $\Lambda \in \mathbb{R}^{1 \times d}$. By Step 1, we may assume $J(u + \Lambda \cdot x; Q) < \infty$, so that by convexity $\Lambda \in \text{dom } \bar{V}$. By convexity and lower semicontinuity of \bar{V} , $\lim_{t \uparrow 1} J(tu + t\Lambda \cdot x; Q) = J(u + \Lambda \cdot x; Q)$. For all $t \in (0, 1)$, since $\Lambda \in \text{dom } \bar{V}$, we have $0 \in \text{int dom } \bar{V}(\cdot + t\Lambda)$ and Proposition A.16(ii)(b) then gives a sequence $(u_{n,t})_n$ of continuous piecewise affine Q -periodic functions such that $\Lambda + \nabla u_{n,t} \in \mathbb{Q}^{1 \times d} \cap \text{int dom } \bar{V}$, $u_{n,t} \rightarrow tu$ in $W^{1,p}(Q)$, and $J(u_{n,t} + t\Lambda \cdot x; Q) \rightarrow J(tu + t\Lambda \cdot x; Q)$. We can then apply the result for the (rational) continuous piecewise affine approximations on Ω''' , and the conclusion then follows from a diagonalization argument.

In the case when $J(\alpha u + \alpha \Lambda \cdot x; Q) < \infty$ for some $\alpha > 1$, the limits $t \uparrow 1$ can be dropped. Similarly as in the proof of Corollary 2.4 (see Sections 3.6 and 3.7), it is enough to replace u, Λ by $u/t, \Lambda/t$ for $t \in [1/\alpha, 1)$ and to adapt Steps 2.2 and 2.3 above accordingly. We omit the details.

Step 3. Γ -lim sup inequality under assumption (2). We assume $p > d$, and $V(y, \Lambda, \omega) \leq C(1 + |\Lambda|^p)$ for all ω and all $y \notin E(\omega)$, for some random stationary set $E^\omega = \bigcup_{n=1}^\infty B_{R_n^\omega}(q_n^\omega)$ satisfying almost surely, for all n , for some constant $C > 0$,

$$2\delta_n^\omega := \frac{1}{R_n^\omega} \inf_{m, m \neq n} \text{dist}(B_{R_m^\omega}(q_m^\omega), B_{R_n^\omega}(q_n^\omega)) \geq \frac{1}{C}, \quad R_n^\omega \leq C. \quad (5.21)$$

We prove the existence of a subset $\Omega''' \subset \Omega''$ of maximal probability such that, for all $u \in W_{\text{per}}^{1,p}(Q)$, all $\Lambda \in \mathbb{R}^{m \times d}$, and all $\omega \in \Omega'''$, there exists a sequence $u_\varepsilon^\omega \rightarrow 0$ in $W_{\text{per}}^{1,p}(Q; \mathbb{R}^m)$ such that

$$\lim_{t \uparrow 1} \liminf_{\varepsilon \downarrow 0} J_\varepsilon^{\text{per}}(tu_\varepsilon^\omega + tu, t\Lambda, \omega) = \lim_{t \uparrow 1} \limsup_{\varepsilon \downarrow 0} J_\varepsilon^{\text{per}}(tu_\varepsilon^\omega + tu, t\Lambda, \omega) = J(u + \Lambda \cdot x; Q).$$

Moreover, the limits $t \uparrow 1$ can be dropped if we have $J(\alpha u + \alpha \Lambda \cdot x; Q) < \infty$ for some $\alpha > 1$. Finally, we have for all $\Lambda \in \mathbb{R}^{m \times d}$ and almost all ω

$$\bar{V}(\Lambda) = \lim_{\varepsilon \downarrow 0} \inf_{u \in W_{\text{per}}^{1,p}(Q; \mathbb{R}^m)} \int_Q V^{1/\varepsilon}(y/\varepsilon, \Lambda + \nabla u(y), \omega) dy, \quad (5.22)$$

where the convergence also holds in expectation. We split the proof of this step into three parts.

Step 3.1. Recovery sequence for affine functions. We prove that for all $\Lambda \in \mathbb{R}^{m \times d}$ for almost all ω there is a sequence $u_\varepsilon^\omega \rightarrow 0$ in $W_{\text{per}}^{1,p}(Q; \mathbb{R}^m)$ such that $J_\varepsilon^{\text{per}}(\Lambda, u_\varepsilon^\omega, \omega) \rightarrow J(\Lambda \cdot x; Q)$.

Let $\Lambda \in \mathbb{R}^{m \times d}$ be fixed. Lemma 3.4 and the Birkhoff-Khinchin ergodic theorem give a random field $\varphi_\Lambda \in \text{Mes}(\Omega; W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m))$ such that, for almost all ω , $\varepsilon \varphi_\Lambda(\cdot/\varepsilon, \omega) \rightarrow 0$ weakly in $W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$ and also $J_\varepsilon(\Lambda \cdot x + \varepsilon \varphi_\Lambda(\cdot/\varepsilon, \omega), \omega; O) \rightarrow J(\Lambda \cdot x; O)$, for all bounded domains $O \subset \mathbb{R}^d$. Let $\omega \in \Omega''$ be fixed such that both properties hold, and set $u_\varepsilon^\omega := \Lambda \cdot x + \varepsilon \varphi_\Lambda(\cdot/\varepsilon, \omega)$.

For all $\varepsilon > 0$, consider the Q/ε -periodic random set $E_\varepsilon^\omega = \bigcup_{n=1}^\infty B_{R_{\varepsilon,n}^\omega}(q_{\varepsilon,n}^\omega)$ associated with the periodization in law $V^{1/\varepsilon}$ at scale $1/\varepsilon$. By stationarity, these periodized inclusions also satisfy (5.21), that is $2\delta_{\varepsilon,n}^\omega \geq 1/C$ and $R_{\varepsilon,n}^\omega \leq C$ almost surely. For all ε, n , define $B_{\varepsilon,n}^\omega := B_{R_{\varepsilon,n}^\omega}(q_{\varepsilon,n}^\omega)$ and $\tilde{B}_{\varepsilon,n}^\omega := B_{R_{\varepsilon,n}^\omega(1+1 \wedge \delta_{\varepsilon,n}^\omega)}(q_{\varepsilon,n}^\omega)$. By definition, the $\tilde{B}_{\varepsilon,n}^\omega$'s are all disjoint. For all ε, n , choose a smooth cut-off function $\chi_{\varepsilon,n}^\omega$ that equals 1 on $\varepsilon B_{\varepsilon,n}^\omega$, vanishes outside $\varepsilon \tilde{B}_{\varepsilon,n}^\omega$ and satisfies $|\nabla \chi_{\varepsilon,n}^\omega| \leq 2/(\varepsilon R_{\varepsilon,n}^\omega(1 \wedge \delta_{\varepsilon,n}^\omega)) \leq 4C/(\varepsilon R_{\varepsilon,n}^\omega)$. We choose these cut-off functions in such a way that $\sum_n \chi_{\varepsilon,n}^\omega$ is Q -periodic.

Let $\theta \in (0, 1)$. Denote by $N_{\varepsilon,\theta}^\omega$ (resp. M_ε^ω) the set of all $n \geq 1$ such that $(Q \setminus \theta Q) \cap \varepsilon \tilde{B}_{\varepsilon,n}^\omega \neq \emptyset$ (resp. $\partial Q \cap \varepsilon \tilde{B}_{\varepsilon,n}^\omega \neq \emptyset$). Choose a smooth cut-off function χ_θ that equals 1 on θQ , vanishes outside Q and satisfies $|\nabla \chi_\theta| \leq 2/(1 - \theta)$. Set

$$\chi_{\varepsilon,\theta}^\omega(y) := \chi_\theta(y) - \chi_\theta(y) \sum_{n \in M_\varepsilon^\omega} \chi_{\varepsilon,n}^\omega(y) + \sum_{n \in N_{\varepsilon,\theta}^\omega \setminus M_\varepsilon^\omega} \chi_{\varepsilon,n}^\omega(y) \int_{\varepsilon \tilde{B}_{\varepsilon,n}^\omega} (\chi_\theta(z) - \chi_\theta(y)) dz.$$

This defines a smooth cut-off function $\chi_{\varepsilon,\theta}^\omega$ that equals 1 on $(\theta - 2C\varepsilon)Q$, vanishes outside Q , is constant on each inclusion $B_{\varepsilon,n}$, and satisfies $|\nabla\chi_{\varepsilon,\theta}^\omega| \leq C/(1 - \theta)$ for some constant $C > 0$. We then define

$$\begin{aligned} u_{\varepsilon,\theta}^\omega(y) &= (1 - \chi_{\varepsilon,\theta}^\omega(y))\Lambda \cdot y + (1 - \chi_{\varepsilon,\theta}^\omega(y)) \sum_{n \in N_{\varepsilon,\theta}^\omega} \chi_{\varepsilon,n}^\omega(y) \int_{\varepsilon\tilde{B}_{\varepsilon,n}^\omega} \Lambda \cdot (z - y) dz \\ &\quad + \chi_{\varepsilon,\theta}^\omega(y) \left(1 - \sum_{n \in N_{\varepsilon,\theta}^\omega} \chi_{\varepsilon,n}^\omega(y) \right) u_\varepsilon^\omega(y) + \chi_{\varepsilon,\theta}^\omega(y) \sum_{n \in N_{\varepsilon,\theta}^\omega} \chi_{\varepsilon,n}^\omega(y) \int_{\varepsilon\tilde{B}_{\varepsilon,n}^\omega} u_\varepsilon^\omega. \end{aligned}$$

By construction $u_{\varepsilon,\theta}^\omega \in \Lambda \cdot x + W_{\text{per}}^{1,p}(Q; \mathbb{R}^m)$. Since $u_{\varepsilon,\theta}^\omega = u_\varepsilon^\omega$ at least on $(\theta - 2\varepsilon C)Q$, and since $\nabla u_{\varepsilon,\theta}^\omega = 0$ on all $\varepsilon\tilde{B}_{\varepsilon,n}^\omega$ with $n \in N_{\varepsilon,\theta}^\omega$, we have, for all $\varepsilon > 0$ small enough,

$$\begin{aligned} \int_Q V^{1/\varepsilon}(\cdot/\varepsilon, \nabla u_{\varepsilon,\theta}^\omega, \omega) &\leq \int_{(\theta - 2\varepsilon C)Q} V(\cdot/\varepsilon, \nabla u_\varepsilon^\omega, \omega) \\ &\quad + M(0)|Q \setminus (\theta - 2\varepsilon C)Q| + C \int_{Q \setminus (\theta - 2\varepsilon C)Q} (1 + |\nabla u_{\varepsilon,\theta}^\omega|^p) \\ &\leq J_\varepsilon(u_\varepsilon^\omega, \omega; Q) + C(1 - \theta + 2\varepsilon C) + C \int_{Q \setminus (\theta - 2\varepsilon C)Q} |\nabla u_{\varepsilon,\theta}^\omega|^p. \end{aligned} \quad (5.23)$$

We need to estimate the last term. A straightforward calculation yields

$$\begin{aligned} \int_{Q \setminus (\theta - 2\varepsilon C)Q} |\nabla u_{\varepsilon,\theta}^\omega|^p &\leq C(1 - \theta + 2\varepsilon C)|\Lambda|^p + \frac{C}{(1 - \theta)^p} \int_Q |u_\varepsilon^\omega(y) - \Lambda \cdot y|^p dy + C \int_{Q \setminus (\theta - 2\varepsilon C)Q} |\nabla u_\varepsilon^\omega|^p \\ &\quad + C \sum_{n \in M_\varepsilon^\omega} ((1 - \theta)^{-p} + (\varepsilon R_{\varepsilon,n}^\omega)^{-p}) \int_{\varepsilon\tilde{B}_{\varepsilon,n}^\omega} \left| \int_{\varepsilon\tilde{B}_{\varepsilon,n}^\omega} \Lambda \cdot (z - y) dz \right|^p dy \\ &\quad + C \sum_{n \in N_{\varepsilon,\theta}^\omega} ((1 - \theta)^{-p} + (\varepsilon R_{\varepsilon,n}^\omega)^{-p}) \int_{\varepsilon\tilde{B}_{\varepsilon,n}^\omega} \left| u_\varepsilon^\omega - \int_{\varepsilon\tilde{B}_{\varepsilon,n}^\omega} u_\varepsilon^\omega \right|^p, \end{aligned}$$

and hence, by the Poincaré inequality and the estimate $\sum_{n \in M_\varepsilon^\omega} |\varepsilon\tilde{B}_{\varepsilon,n}^\omega| \leq |Q \setminus (\theta - 2\varepsilon C)Q| \leq 2d\varepsilon C$,

$$\begin{aligned} \int_{Q \setminus (\theta - 2\varepsilon C)Q} |\nabla u_{\varepsilon,\theta}^\omega|^p &\leq C(1 - \theta + 2\varepsilon C)|\Lambda|^p + \frac{C}{(1 - \theta)^p} \int_Q |\varepsilon\varphi_\Lambda(\cdot/\varepsilon, \omega)|^p \\ &\quad + C \left(1 + \frac{\varepsilon^p}{(1 - \theta)^p} \right) \int_{Q \setminus (\theta - 2\varepsilon C)Q} (|\Lambda|^p + |\nabla\varphi_\Lambda(\cdot/\varepsilon, \omega)|^p). \end{aligned} \quad (5.24)$$

For all $\varepsilon_0 > 0$, the Birkhoff-Khinchin ergodic theorem yields, for almost all ω ,

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} \int_{Q \setminus (\theta - 2\varepsilon C)Q} (|\Lambda|^p + |\nabla\varphi_\Lambda(\cdot/\varepsilon, \omega)|^p) &\leq \lim_{\varepsilon \downarrow 0} \int_{Q \setminus (\theta - 2\varepsilon_0 C)Q} (|\Lambda|^p + |\nabla\varphi_\Lambda(\cdot/\varepsilon, \omega)|^p) \\ &\leq |Q \setminus (\theta - 2\varepsilon_0 C)Q| (|\Lambda|^p + \mathbb{E}[|\nabla\varphi_\Lambda|^p]) \\ &\leq C(1 - \theta + 2\varepsilon_0 C)(|\Lambda|^p + \mathbb{E}[|\nabla\varphi_\Lambda|^p]). \end{aligned}$$

Combined with (5.24) and the sublinearity of the corrector in the form of $\varepsilon\varphi_\Lambda(\cdot/\varepsilon, \omega) \rightarrow 0$ in $L^p(Q; \mathbb{R}^m)$ (see Lemma 3.4 and the Rellich-Kondrachov theorem), this yields, for almost all ω ,

$$\limsup_{\varepsilon \downarrow 0} \int_{Q \setminus (\theta - 2\varepsilon C)Q} |\nabla u_{\varepsilon,\theta}^\omega|^p \leq C(1 - \theta)|\Lambda|^p + C(1 - \theta + 2\varepsilon_0 C)\mathbb{E}[|\Lambda + \nabla\varphi_\Lambda|^p].$$

We then insert this estimate in (5.23) and to pass to the limits $\theta \uparrow 1$ and $\varepsilon_0 \downarrow 0$ to obtain for almost all ω

$$\limsup_{\theta \uparrow 1} \limsup_{\varepsilon \downarrow 0} \int_Q V^{1/\varepsilon}(\cdot/\varepsilon, \nabla u_{\varepsilon,\theta}^\omega, \omega) \leq \lim_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon^\omega, \omega; Q) = \bar{V}(\Lambda) = J(\Lambda \cdot x; Q).$$

The conclusion then follows from Step 1 and a diagonalization argument.

Step 3.2. Recovery sequence for general target functions.

As we assume here $p > d$, we may use the gluing argument of Step 2 of the proof of Proposition 3.10 to pass from affine to piecewise affine functions. The conclusion follows by approximation similarly as in Step 2.3 above. We omit the details.

Step 3.3. Proof that the almost sure convergence (5.22) also holds in expectation.

Using the same notation and cut-off functions as in Step 3.1, we define $u_\varepsilon^\omega \in W_{\text{per}}^{1,p}(Q; \mathbb{R}^m)$ as

$$u_\varepsilon^\omega(y) := \sum_n \chi_{\varepsilon,n}^\omega(y) \int_{\varepsilon \tilde{B}_{\varepsilon,n}^\omega} \Lambda \cdot (z - y) dz.$$

Testing the infimum in (5.22) with $u = u_\varepsilon^\omega$, we obtain by similar calculations as in (5.23) and (5.24) that for all $\varepsilon > 0$ and all ω

$$\begin{aligned} & \inf_{u \in W_{\text{per}}^{1,p}(Q; \mathbb{R}^m)} \int_Q V^{1/\varepsilon}(\cdot/\varepsilon, \Lambda + \nabla u, \omega) \\ & \leq C(1 + |\Lambda|^p) + C \sum_n (\varepsilon R_{\varepsilon,n}^\omega)^{-p} \int_{Q \cap \varepsilon \tilde{B}_{\varepsilon,n}^\omega} \left| \int_{\varepsilon \tilde{B}_{\varepsilon,n}^\omega} \Lambda \cdot (z - y) dz \right|^p dy \leq C(1 + |\Lambda|^p). \end{aligned}$$

The conclusion then follows by dominated convergence. \square

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A. APPENDIX

A.1. Normal random integrands. In this appendix, we briefly recall the precise definition of normal random integrands (as defined e.g. in [23, Section VIII.1.3]) and we prove their main properties, mentioned in the beginning of Section 2 and used throughout this paper. Let (Ω, \mathcal{F}, P) be a complete probability space. We denote by $\mathcal{B}(\mathbb{R}^k)$ the (not completed) Borel σ -algebra on \mathbb{R}^k .

Definition A.1. A *normal random integrand* is a map $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ such that

- (a) W is jointly measurable (i.e. with respect to the completion of $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{m \times d}) \times \mathcal{F}$);
- (b) for almost all ω , there exists a map $V_\omega : \mathbb{R}^d \times \mathbb{R}^{m \times d} \rightarrow [0, \infty]$ that is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{m \times d})$ -measurable and such that $W(y, \cdot, \omega) = V_\omega(y, \cdot)$ for almost all y ;
- (c) for almost all y , there exists a map $V_y : \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ that is $\mathcal{B}(\mathbb{R}^{m \times d}) \times \mathcal{F}$ -measurable and such that $W(y, \cdot, \omega) = V_y(\cdot, \omega)$ for almost all ω ;
- (d) for almost all y, ω , the map $W(y, \cdot, \omega)$ is lower semicontinuous on $\mathbb{R}^{m \times d}$.

It is said to be τ -stationary if it satisfies (2.1) for all Λ, y, z, ω . □

As shown e.g. in [23, Section VIII.1.3], a simple example of normal random integrands is given by the so-called *Carathéodory random integrands*, that are maps $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ such that $W(y, \cdot, \omega)$ is continuous on $\mathbb{R}^{m \times d}$ for almost all y, ω , and such that $W(\cdot, \Lambda, \cdot)$ is jointly measurable on $\mathbb{R}^d \times \Omega$ for all Λ .

As already advertised in the beginning of Section 2, the reason for these technical assumptions is that they are particularly weak but still guarantee the following properties:

Lemma A.2. Let $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ be a normal random integrand. Then,

- (i) for almost all ω , the map $y \mapsto W(y, u(y), \omega)$ is measurable for all $u \in \text{Mes}(\mathbb{R}^d, \mathbb{R}^{m \times d})$;
- (ii) for almost all y , the map $\omega \mapsto W(y, u(\omega), \omega)$ is measurable for all $u \in \text{Mes}(\Omega, \mathbb{R}^{m \times d})$. □

Proof. For almost all ω , part (b) of Definition A.1 gives a $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{m \times d})$ -measurable map V_ω on $\mathbb{R}^d \times \mathbb{R}^{m \times d}$ such that $W(y, \cdot, \omega) = V_\omega(y, \cdot)$ for almost all y . Hence, for $u \in \text{Mes}(\mathbb{R}^d; \mathbb{R}^{m \times d})$, the map $y \mapsto W(y, u(y), \omega)$ is equal almost everywhere to the map $y \mapsto V_\omega(y, u(y))$, which is necessarily measurable since $\text{Id} \times u : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^{m \times d}$ is measurable and since V_ω is Borel-measurable. This proves (i), and (ii) is similar. □

If W is τ -stationary, we may write $W(y, \Lambda, \omega) = W(0, \Lambda, \tau_{-y}\omega)$, which thus receives a pointwise meaning in the first variable, and Lemma A.2(ii) may obviously be strengthened as follows.

Lemma A.3. Let $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ be a τ -stationary normal random integrand. For all y and all $u \in \text{Mes}(\Omega; \mathbb{R}^{m \times d})$, the map $\omega \mapsto W(y, u(\omega), \omega)$ is measurable. □

A.2. Stationary differential calculus in probability. In this appendix, we precisely define the measurable action τ that is used throughout this paper to induce the stationarity, and we discuss the properties of the stationary derivatives defined in Section 3.1.1, and prove in particular the useful identity (3.3).

A.2.1. Stationary random fields. As usual, the standard notion of stationarity of random fields (defined as the translation invariance of all the finite-dimensional distributions) is strictly equivalent to a formulation of stationarity as the invariance under some (measure-preserving) action of the group of translations $(\mathbb{R}^d, +)$ on the probability space (see e.g. [32, Section 16.1]). This point of view is of great interest, since it puts us into the realm of ergodic theory.

Because we focus on jointly measurable random fields, which is standard in stochastic homogenization theory (see also Remark A.6 below), a further measurability requirement is added in our definition of an action, as e.g. in [31, Section 7.1],

Definition A.4. A *measurable action* of the group $(\mathbb{R}^d, +)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection $\tau := (\tau_x)_{x \in \mathbb{R}^d}$ of measurable transformations of Ω such that

- (i) $\tau_x \circ \tau_y = \tau_{x+y}$ for all $x, y \in \mathbb{R}^d$;
- (ii) $\mathbb{P}[\tau_x A] = \mathbb{P}[A]$ for all $x \in \mathbb{R}^d$ and all $A \in \mathcal{F}$;
- (iii) the map $\mathbb{R}^d \times \Omega \rightarrow \Omega : (x, \omega) \mapsto \tau_x \omega$ is measurable. \square

For any random variable $f \in \text{Mes}(\Omega; \mathbb{R})$, we may define its τ -stationary extension $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ by $f(x, \omega) := f(\tau_{-x}\omega)$, which is a τ -stationary random field on \mathbb{R}^d (in the sense that $f(x+y, \omega) = f(x, \tau_{-y}\omega)$ for all x, y, ω) and which is by definition jointly measurable on $\mathbb{R}^d \times \Omega$. For $f \in L^p(\Omega)$, $1 \leq p < \infty$, the τ -stationary extension f belongs to $L^p(\Omega; L^p_{\text{loc}}(\mathbb{R}^d))$. In this way, we get a bijection between the random variables (resp. in $L^p(\Omega)$) and the τ -stationary random fields (resp. in $L^p(\Omega; L^p_{\text{loc}}(\mathbb{R}^d))$).

We may also naturally consider the associated action $T := (T_x)_{x \in \mathbb{R}^d}$ of $(\mathbb{R}^d, +)$ on $\text{Mes}(\Omega; \mathbb{R})$, defined by $(T_x f)(\omega) = f(\tau_{-x}\omega)$ for all $\omega \in \Omega$ and $f \in \text{Mes}(\Omega; \mathbb{R})$. Let $1 \leq p < \infty$. The following gives elementary properties of this action (see e.g. [31, Section 7.1]):

Lemma A.5. *The action T defined above is unitary and strongly continuous on $L^p(\Omega)$.* \square

In the context of stochastic homogenization theory, the measurability hypotheses made above (as in e.g. [31, Section 7.1]) are sometimes replaced by stochastic continuity hypotheses (see e.g. [37, Section 2]). As the following remark shows, both are actually equivalent.

Remark A.6 (Measurability or continuity). It should be noted that the additional measurability assumption (iii) in Definition A.4 above is not inoffensive at all. Indeed, a stochastic version of the Lusin theorem can easily be proven: a random field h on \mathbb{R}^d is jointly measurable if and only if, for almost all $x \in \mathbb{R}^d$, for all $\delta > 0$,

$$\lim_{y \rightarrow 0} \mathbb{P}[|h(x+y, \omega) - h(x, \omega)| > \delta] = 0.$$

Hence, for a *stationary* random field h on \mathbb{R}^d , joint measurability is actually equivalent to stochastic continuity (and even to continuity in the p -th mean, in the case when $h(0, \cdot) \in L^p(\Omega)$). In the same vein, the measurability property (iii) in Definition A.4 is equivalent to the strong continuity of the action T of $(\mathbb{R}^d, +)$ on $L^p(\Omega)$, and also to the property that all τ -stationary extensions are stochastically continuous. \square

A.2.2. *Stationary Sobolev spaces.* Let $1 \leq p < \infty$, and let the stationary gradient D and the space $W^{1,p}(\Omega)$ be defined as in Section 3.1.1. Now we present another useful vision for derivatives of stationary random fields.

Given a random variable $f \in L^p(\Omega)$, the τ -stationary extension is an element $f \in L^p_{\text{loc}}(\mathbb{R}^d; L^p(\Omega)) = L^p(\Omega; L^p_{\text{loc}}(\mathbb{R}^d))$ and can thus be seen as an $L^p(\Omega)$ -valued distribution on \mathbb{R}^d . We may then define its distributional gradient ∇f in the usual way. Note that by definition, for almost all ω , $\nabla f(\cdot, \omega)$ is nothing but the usual distributional gradient of $f(\cdot, \omega) \in L^p_{\text{loc}}(\mathbb{R}^d)$. As usual, the Sobolev space $W^{1,p}_{\text{loc}}(\mathbb{R}^d; L^p(\Omega))$ is defined as the space of functions $f \in L^p_{\text{loc}}(\mathbb{R}^d; L^p(\Omega))$ such that $\nabla f \in L^p_{\text{loc}}(\mathbb{R}^d; L^p(\Omega; \mathbb{R}^d))$, and in that case ∇f is called the *weak gradient*. The following result shows the link with stationary gradients and with the space $W^{1,p}(\Omega)$, in particular proving identity (3.3).

Lemma A.7. *Modulo the correspondence between random variables and τ -stationary random fields, we have*

$$W^{1,p}(\Omega) = \{f \in W^{1,p}_{\text{loc}}(\mathbb{R}^d; L^p(\Omega)) : f(x+y, \omega) = f(x, \tau_{-y}\omega), \forall x, y, \omega\},$$

and moreover $\nabla f = Df$ for all $f \in W^{1,p}(\Omega)$. \square

Proof. Denote for simplicity $E_p := \{f \in W^{1,p}_{\text{loc}}(\mathbb{R}^d; L^p(\Omega)) : f(x+y, \omega) = f(x, \tau_{-y}\omega), \forall x, y, \omega\}$. For all i , the stationary derivative $D_i f$ is defined as the strong derivative of the map $\mathbb{R} \rightarrow L^p(\Omega) : h \mapsto T_{-he_i} f$, so that

$$W^{1,p}(\Omega) = \{f \in C^1(\mathbb{R}^d; L^p(\Omega)) : f(x+y, \omega) = f(x, \tau_{-y}\omega), \forall x, y, \omega\}.$$

Hence, for all $f \in W^{1,p}(\Omega)$, we have $f \in E_p$, and $\nabla f = Df$, since weak derivatives are generalizations of strong derivatives.

We now turn to the converse statement. Let $f \in W^{1,p}(\Omega)$. As in [31, Section 7.2], choose a nonnegative even function $\rho \in C_c^\infty(\mathbb{R}^d)$ with $\int \rho = 1$ and $\text{supp } \rho \subset B_1$, write $\rho_\delta(x) = \delta^{-d}\rho(x/\delta)$ for all $\delta > 0$, and define a regularization $R_\delta[f] \in L^p(\Omega)$ by

$$R_\delta[f](\omega) = \int_{\mathbb{R}^d} \rho_\delta(y) f(y, \omega) dy, \quad (\text{A.1})$$

or equivalently, as ρ_δ is even,

$$R_\delta[f](x, \omega) = \int_{\mathbb{R}^d} \rho_\delta(y) f(x+y, \omega) dy = \int_{\mathbb{R}^d} \rho_\delta(y-x) f(y, \omega) dy = (\rho_\delta * f(\cdot, \omega))(x).$$

Clearly, $R_\delta[f] \rightarrow f$ in $L^p(\Omega)$, and hence by stationarity $R_\delta[f](x, \cdot) \rightarrow f(x, \cdot)$ in $L^p(\Omega)$ uniformly in x . As by definition $R_\delta[f] \in C^\infty(\mathbb{R}^d; L^p(\Omega))$, we have $R_\delta[f] \in W^{1,p}(\Omega)$ and the stationary gradient is simply $DR_\delta[f] = R_\delta[\nabla f]$. This proves $DR_\delta[f] \rightarrow \nabla f$ in $L^p(\Omega; \mathbb{R}^d)$, and thus $DR_\delta[f](x, \cdot) \rightarrow \nabla f(x, \cdot)$ in $L^p(\Omega; \mathbb{R}^d)$ uniformly in x . Hence $R_\delta[f] \rightarrow f$ in $C^1(\mathbb{R}^d; L^p(\Omega))$, so $f \in C^1(\mathbb{R}^d; L^p(\Omega))$, from which we conclude $f \in W^{1,p}(\Omega)$. \square

A.3. Ergodic Weyl decomposition. In this appendix, we discuss the various properties of the ergodic Weyl spaces recalled in Section 3.1.2. In particular, we prove the ergodic Weyl decomposition (3.7) as well as the density result (3.8).

Let τ be a measurable action of $(\mathbb{R}^d, +)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $1 < p < \infty$, and let $L_{\text{pot}}^p(\Omega)$, $L_{\text{sol}}^p(\Omega)$, $F_{\text{pot}}^p(\Omega)$ and $F_{\text{sol}}^p(\Omega)$ be defined by (3.4) (or equivalently (3.5)) and (3.6). First we prove the ergodic Weyl decomposition (3.7):

Proposition A.8 (Ergodic Weyl decomposition). *Let τ be ergodic and let $1 < p < \infty$. Then the following Banach direct sum decompositions hold:*

$$L^p(\Omega; \mathbb{R}^d) = L_{\text{pot}}^p(\Omega) \oplus F_{\text{sol}}^p(\Omega) = F_{\text{pot}}^p(\Omega) \oplus L_{\text{sol}}^p(\Omega) = F_{\text{pot}}^p(\Omega) \oplus F_{\text{sol}}^p(\Omega) \oplus \mathbb{R}^d.$$

\square

Proof. Let $1 < p < \infty$ and let $q := p' = p/(p-1)$. The following weaker form of the result is proven e.g. in [31, Lemma 15.1]:

$$\begin{aligned} (F_{\text{pot}}^p(\Omega))^\perp &= L_{\text{sol}}^q(\Omega) = F_{\text{sol}}^q(\Omega) \oplus \mathbb{R}^d, \\ (F_{\text{sol}}^p(\Omega))^\perp &= L_{\text{pot}}^q(\Omega) = F_{\text{pot}}^q(\Omega) \oplus \mathbb{R}^d, \end{aligned} \quad (\text{A.2})$$

in the sense of duality between $L^p(\Omega; \mathbb{R}^d)$ and $L^q(\Omega; \mathbb{R}^d)$. In the Hilbert case $p = 2$, this is already enough to conclude by the orthogonal decomposition theorem. For the general case, we will need to use more subtle (nonlinear) decomposition results in Banach spaces.

As the Banach space $L^p(\Omega; \mathbb{R}^d)$ is uniformly smooth and uniformly convex, the following (nonlinear) direct sum decomposition (see e.g. [3, Theorem 2.13]) holds for any closed subspace $M \subset L^p(\Omega; \mathbb{R}^d)$:¹

$$L^p(\Omega; \mathbb{R}^d) = M \oplus J(M^\perp),$$

where $M^\perp \subset L^q(\Omega; \mathbb{R}^d)$ denotes the orthogonal in the sense of duality, and where the (nonlinear) map $J : L^q(\Omega; \mathbb{R}^d) \rightarrow L^p(\Omega; \mathbb{R}^d)$ is defined by $J(u) := \|u\|_{L^q}^{2-q} |u|^{q-2} u$. Let us apply this result to the choice $M = F_{\text{pot}}^p(\Omega)$. By (A.2) we have $M^\perp = L_{\text{sol}}^q(\Omega)$, and we may compute $DJ(u) = (q-1)\|u\|_{L^q}^{2-q} |u|^{q-2} Du$, so that $JM^\perp \subset L_{\text{sol}}^p(\Omega)$. On the other hand, given $u \in L_{\text{sol}}^p(\Omega)$, we may write $u = J(v)$ with $v := \|u\|_{L^p}^{2-p} |u|^{p-2} u \in L_{\text{sol}}^q(\Omega) = M^\perp$. This proves equality $J(M^\perp) = L_{\text{sol}}^p(\Omega)$, and the result follows. \square

The density result (3.8) is now easily obtained, using the same (nonlinear) direct sum decomposition in Banach spaces:

1. The direct sum means here that any element of $L^p(\Omega; \mathbb{R}^d)$ can be written in a unique way as the sum of an element of M and an element of $J(M^\perp)$, and that moreover $M \cap J(M^\perp) = \{0\}$. As shown in [3], this is actually even a direct sum in Birkhoff-James' sense, that is $\|u\|_{L^p(\Omega)} \leq \|u + tv\|_{L^p(\Omega)}$ for all $u \in M$, $v \in J(M^\perp)$ and $t \in \mathbb{R}$.

Proposition A.9. *Let τ be ergodic, and let $1 < p < \infty$. Then,*

$$\begin{aligned} F_{\text{pot}}^p(\Omega) &= \text{adh}_{L^p(\Omega; \mathbb{R}^d)} \{D\chi : \chi \in W^{1,p}(\Omega)\}, \\ F_{\text{sol}}^p(\Omega) &= \text{adh}_{L^p(\Omega; \mathbb{R}^d)} \{D \times \chi : \chi \in W^{1,p}(\Omega)\}. \end{aligned}$$

□

Proof. Using the same notation as in the proof of Proposition A.8, the (nonlinear) direct sum decomposition of [3, Theorem 2.13] gives

$$L^p(\Omega; \mathbb{R}^d) = \text{adh } DW^{1,p}(\Omega) \oplus J((\text{adh } DW^{1,p}(\Omega))^\perp).$$

By definition $(\text{adh } DW^{1,p}(\Omega))^\perp = L_{\text{sol}}^q(\Omega)$, while as in the proof of Proposition A.8 we find $J(L_{\text{sol}}^q(\Omega)) = L_{\text{sol}}^p(\Omega)$. This yields

$$L^p(\Omega; \mathbb{R}^d) = \text{adh } DH^\infty(\Omega) \oplus L_{\text{sol}}^p(\Omega).$$

The obvious inclusion $\text{adh } DW^{1,p}(\Omega) \subset F_{\text{pot}}^p(\Omega)$ and the direct sum decomposition of Proposition A.8 then imply the equality $\text{adh } DW^{1,p}(\Omega) = F_{\text{pot}}^p(\Omega)$. The result for F_{sol}^p is obtained similarly. □

A.4. Measurability results. This appendix is concerned with various measurability properties.

A.4.1. Measurable potentials for random fields. The following result complements the equivalent definition (3.5) of $L_{\text{pot}}^p(\Omega)$, and shows that potentials associated with potential random fields may be chosen in a measurable way with respect to the alea.

Proposition A.10. *Let τ be an ergodic measurable action on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $1 < p < \infty$. For all $f \in L_{\text{pot}}^p(\Omega)$ there exists a random field $\phi \in \text{Mes}(\Omega; W_{\text{loc}}^{1,p}(\mathbb{R}^d))$ such that $f(\cdot, \omega) = \nabla \phi(\cdot, \omega)$ for almost all ω .* □

Proof. Let $f \in L_{\text{pot}}^p(\Omega)$ be fixed. For all $n, k \geq 1$, define the space

$$X_{n,k} := \left\{ \phi \in W^{1,p}(B_n) : \|\nabla \phi\|_{L^p(B_n)} \leq k, \int_{B_n} \phi = 0 \right\},$$

endowed with the weak topology. By Poincaré's inequality and by the Banach-Alaoglu theorem, this space is metrizable and compact, hence Polish. Consider the multifunction $\Gamma_{n,k} : \Omega \rightrightarrows X_{n,k}$ defined by

$$\Gamma_{n,k}(\omega) := \{ \phi \in X_{n,k} : \nabla \phi|_{B_n} = f(\cdot, \omega)|_{B_n} \}.$$

Clearly $\Gamma_{n,k}(\omega)$ is closed for all ω . We first prove further properties of this multifunction, and the conclusion will then follow by applying the Rokhlin–Kuratowski–Ryll Nardzewski theorem.

Step 1. For all $n \geq 1$, we claim the existence of an increasing sequence of events $\Omega_{n,k} \subset \Omega$ such that $\mathbb{P}[\Omega_{n,k}] \uparrow 1$ as $k \uparrow \infty$ for fixed n , and such that $\Gamma_{n,k}(\omega) \neq \emptyset$ for all $\omega \in \Omega_{n,k}$.

By Definition 3.5, there is a subset $\Omega' \subset \Omega$ of maximal probability, such that for all $\omega \in \Omega'$ the function $f(\cdot, \omega)$ is a potential field in $L_{\text{loc}}^p(\mathbb{R}^d; \mathbb{R}^d)$, and hence there exists $\phi^\omega \in W_{\text{loc}}^{1,p}(\mathbb{R}^d)$ such that $f(\cdot, \omega) = \nabla \phi^\omega$. For all $n \geq 1$, define $\phi_n^\omega := \phi^\omega - \int_{B_n} \phi^\omega(z) dz \in W^{1,p}(B_n)$. By definition, $\int_{B_n} \phi_n^\omega = 0$ and $\nabla \phi_n^\omega = f(\cdot, \omega)$ on B_n . Moreover, $\|\nabla \phi_n^\omega\|_{L^p(B_n)} \leq M$ holds for all $\omega \in \Omega_{n,k}$, where we define the event

$$\Omega_{n,k} := \{ \omega \in \Omega' : \|f(\cdot, \omega)\|_{L^p(B_n)} \leq k \}.$$

Integrability and stationarity of f easily imply that $\mathbb{P}[\Omega_{n,k}] \uparrow 1$ as $k \uparrow \infty$.

Step 2. Proof that $\Gamma_{n,k}$ is measurable, in the sense that $\Gamma_{n,k}^{-1}(O) \in \mathcal{F}$ for all open subset $O \subset X_{n,k}$, where we have set

$$\Gamma_{n,k}^{-1}(O) := \{ \omega \in \Omega : \Gamma_{n,k}(\omega) \cap O \neq \emptyset \}.$$

As $X_{n,k}$ is metrizable, it suffices to check $\Gamma_{n,k}^{-1}(F) \in \mathcal{F}$ for all closed subset $F \subset X_{n,k}$ (see e.g. [5, Lemma 18.2]). Given a closed subset $F \subset X_{n,k}$, we may write, using Poincaré's inequality and the weak lower semicontinuity of the norm,

$$\begin{aligned} \Gamma_{n,k}^{-1}(F) &= \{\omega \in \Omega : \exists \phi \in F, \nabla \phi = f(\cdot, \omega)|_{B_n}\} \\ &= \bigcap_{j=1}^{\infty} \{\omega \in \Omega : \exists \phi \in F, \|\nabla \phi - f(\cdot, \omega)\|_{L^p(B_n)} \leq 1/j\}. \end{aligned}$$

Separability of the Polish space $X_{n,k}$ implies that F is itself separable, and there exists a countable dense subset $F_0 \subset F$. Hence

$$\Gamma_{n,k}^{-1}(F) = \bigcap_{j=1}^{\infty} \bigcup_{\phi \in F_0} \{\omega \in \Omega : \|\nabla \phi - f(\cdot, \omega)\|_{L^p(B_n)} \leq 1/j\},$$

and measurability of $\Gamma_{n,k}^{-1}(F)$ then follows from measurability of f .

Step 3. Conclusion.

By steps 1 and 2, for all $n, k \geq 1$, the restricted multifunction $\Gamma_{n,k}|_{\Omega_{n,k}} : \Omega_{n,k} \rightrightarrows X_{n,k}$ is measurable and has nonempty closed values. As $X_{n,k}$ is a Polish space, we may apply the Rokhlin–Kuratowski–Ryll–Nardzewski theorem (see e.g. [5, Theorem 18.13]), which gives a measurable function $\phi_{n,k} : \Omega_{n,k} \rightarrow X_{n,k}$ such that $\phi_{n,k}(\omega) \in \Gamma_{n,k}(\omega)$, that is $\nabla \phi_{n,k}(\cdot, \omega) = f(\cdot, \omega)|_{B_n}$ for all $\omega \in \Omega_{n,k}$. For all $n > 0$, define a measurable function $\phi_n : \Omega \rightarrow W^{1,p}(B_n)$ by

$$\phi_n(\omega) = \mathbf{1}_{\Omega_1}(\omega)\phi_{1,n}(\omega) + \sum_{k=2}^{\infty} \mathbf{1}_{\Omega_k \setminus \Omega_{k-1}}(\omega)\phi_{n,k}(\omega).$$

By definition, we have $\nabla \phi_n(\cdot, \omega) = f(\cdot, \omega)|_{B_n}$ for all $\omega \in \Omega_n$, where $\Omega_n := \bigcup_{k=1}^{\infty} \Omega_{n,k}$ is a subset of maximal probability. Denote $\Omega'' := \bigcap_{n=1}^{\infty} \Omega_n$.

Let $n \geq 1$. By definition, $\nabla \phi_n - \nabla \phi_1$ vanishes on B_1 , hence the difference $\delta_n(\omega) := \phi_n(\cdot, \omega) - \phi_1(\cdot, \omega)$ is constant on B_1 for all $\omega \in \Omega''$ and defines a measurable function $\delta_n : \Omega'' \rightarrow \mathbb{R}$. Then consider the measurable function $\psi_n : \Omega \rightarrow W^{1,p}(B_n)$ defined by $\psi_n(x, \omega) := \phi_n(x, \omega) - \delta_n(\omega)$. By construction, for all $m > n \geq 1$, we have $\psi_n = \psi_m$ on B_n , so the ψ_n 's can be glued together and yield a measurable function $\psi : \Omega'' \rightarrow W_{\text{loc}}^{1,p}(\mathbb{R}^d)$ such that $\nabla \psi(\cdot, \omega) = f(\cdot, \omega)$ for all $\omega \in \Omega''$. \square

A.4.2. Sufficient conditions for Hypothesis 2.1. As will be shown below, the measurability Hypothesis 2.1 is automatically satisfied if the integrand is quasiconvex and has the following nice approximation property, introduced by [6]:

Definition A.11. A normal random integrand $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ is said to be *quasiconvex* if $W(y, \cdot, \omega)$ is quasiconvex for almost all y, ω . Given $p \geq 1$, it is further said to be *p-sup-quasiconvex* if there exists a sequence $(W_k)_k$ of quasiconvex normal random integrands such that $W_k(y, \Lambda, \omega) \uparrow W(y, \Lambda, \omega)$ pointwise as $k \uparrow \infty$, and such that, for all k , for almost all y, ω and for all Λ, Λ' ,

$$\frac{1}{C}|\Lambda|^p - C \leq W_k(y, \Lambda, \omega) \leq C_k(1 + |\Lambda|^p), \quad (\text{A.3})$$

for some constants $C, C_k > 0$. \square

Note that Tartar [40] has proven the existence of quasiconvex functions that are not p -sup-quasiconvex properties for any $p \geq 1$. Before stating our measurability result, let us examine some important particular cases:

Lemma A.12. *Let $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ be a normal random integrand. Assume that there exist $C > 0$ and $p > 1$ such that, for almost all ω, y and for all Λ ,*

$$\frac{1}{C}|\Lambda|^p - C \leq W(y, \Lambda, \omega). \quad (\text{A.4})$$

Also assume that one of the following holds:

- (1) $W = V$ is a convex normal random integrand, and the convex function $M := \sup \operatorname{ess}_{y,\omega} V(y, \cdot, \omega)$ has 0 in the interior of its domain;
- (2) W satisfies Hypothesis 2.7, with a convex part V such that $M := \sup \operatorname{ess}_{y,\omega} V(y, \cdot, \omega)$ has 0 in the interior of its domain.

Then, W is a p -sup-quasiconvex normal random integrand. \square

Proof. Case (2) directly follows from the approximation result of case (1) applied to the convex part V . We may thus focus on case (1). Let $W = V$ be a convex normal random integrand. For all $k \geq 0$, consider the following Yosida transform:

$$V_k(y, \Lambda, \omega) = \inf_{\Lambda' \in \mathbb{R}^{m \times d}} (V(y, \Lambda', \omega) + k|\Lambda - \Lambda'|^p). \quad (\text{A.5})$$

For almost all y, ω , convexity of $V_k(y, \cdot, \omega)$ easily follows from convexity of $V(y, \cdot, \omega)$. For almost all y, ω , the lower semicontinuity of $V(y, \cdot, \omega)$ ensures that $V_k(y, \cdot, \omega) \uparrow V(y, \cdot, \omega)$ pointwise as $k \uparrow \infty$. Moreover, by definition, $V_k(y, \Lambda, \omega) \leq M(0) + k|\Lambda|^p$, while the lower bound (A.4) implies $V_k(y, \Lambda, \omega) \geq \frac{1}{C}|\Lambda|^p - C$.

It remains to check that the V_k 's are normal random integrands. For almost all y, ω , the function $V(y, \cdot, \omega)$ is convex and lower semicontinuous, hence it is continuous on its domain $D_{y,\omega}$ (not only on the interior). As by assumption $0 \in \operatorname{int} D_{y,\omega}$, the set $D_{y,\omega}$ is a convex subset of maximal dimension, and hence points with rational coordinates are dense in $D_{y,\omega}$. The infimum (A.5) defining V_k may thus be restricted to $\mathbb{Q}^{m \times d}$. As a countable infimum, the required measurability properties follow. \square

We now turn to the validity of the measurability Hypothesis 2.1 for p -sup-quasiconvex integrands.

Proposition A.13. *Let $O \subset \mathbb{R}^d$ be a bounded domain, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and let $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ be a p -sup-quasiconvex normal random integrand for some $p > 1$ (in the sense of Definition A.11). Given some fixed function $f \in L^p(\Omega; L^p(O; \mathbb{R}^{m \times d}))$, consider the random integral functional $I : W^{1,p}(O; \mathbb{R}^m) \times \Omega \rightarrow [0, \infty]$ defined by*

$$I(u, \omega) = \int_O W(y, f(y, \omega) + \nabla u(y), \omega) dy. \quad (\text{A.6})$$

Then, I is weakly lower semicontinuous on $W^{1,p}(O; \mathbb{R}^m)$. Moreover, for all weakly closed subsets $F \subset W_0^{1,p}(O; \mathbb{R}^m)$ or $F \subset \{u \in W^{1,p}(O; \mathbb{R}^m) : \int_O u = 0\}$, the function $\omega \mapsto \inf_{v \in F} I(v, \omega)$ is \mathcal{F} -measurable. In particular, Hypothesis 2.1 is satisfied. \square

Proof. For all k , define the approximated random functional $I_k : W^{1,p}(O; \mathbb{R}^m) \times \Omega \rightarrow [0, \infty]$ by

$$I_k(u, \omega) = \int_O W_k(y, f(y, \omega) + \nabla u(y), \omega) dy.$$

As the W_k 's are nonnegative, monotone convergence yields that $I_k \uparrow I$ pointwise. Moreover, for all k , and almost all ω , the quasiconvexity and the upper bound (A.3) satisfied by $W_k(\cdot, \cdot, \omega)$ imply the weak lower semicontinuity of $I_k(\cdot, \omega)$ on $W^{1,p}(O; \mathbb{R}^m)$ (see [2]). As a pointwise supremum of weakly lower semicontinuous functions, we deduce that $I(\cdot, \omega)$ is itself weakly lower semicontinuous on $W^{1,p}(O; \mathbb{R}^m)$.

Combining the weak lower semicontinuity of I_k with the uniform coercivity assumption (cf. lower bound in (A.3)) and with Poincaré's inequality, we easily conclude, for any weakly closed subset $F \subset W_0^{1,p}(O; \mathbb{R}^m)$ or $F \subset \{u \in W^{1,p}(O; \mathbb{R}^m) : \int_O u = 0\}$,

$$\liminf_{k \uparrow \infty} \inf_{v \in F} I_k(v, \omega) = \inf_{v \in F} I(v, \omega). \quad (\text{A.7})$$

For all k , as W_k is quasiconvex hence rank-1 convex in its second variable (see [9]), the p -growth condition (A.3) implies the following local Lipschitz condition: for almost all y, ω , for all Λ, Λ' ,

$$|W_k(y, \Lambda, \omega) - W_k(y, \Lambda', \omega)| \leq C_k |\Lambda - \Lambda'| (1 + |\Lambda|^{p-1} + |\Lambda'|^{p-1}),$$

for some constant $C_k > 0$. The Hölder inequality then gives, for all u, v , for almost all ω ,

$$|I_k(u, \omega) - I_k(v, \omega)| \leq C_k C \|\nabla(u - v)\|_{L^p(O)} (1 + \|\nabla u\|_{L^p(O)}^{p-1} + \|\nabla v\|_{L^p(O)}^{p-1} + \|f(\cdot, \omega)\|_{L^p(O)}^{p-1}).$$

This proves that the map $I_k(\cdot, \omega)$ is strongly continuous on $W^{1,p}(O; \mathbb{R}^m)$, for almost all ω . Given a weakly (hence strongly) closed subset $F \subset W_0^{1,p}(O; \mathbb{R}^m)$ or $F \subset \{u \in W^{1,p}(O; \mathbb{R}^m) : \int_O u = 0\}$, strong separability of $W^{1,p}(O; \mathbb{R}^m)$ implies strong separability of F , so there exists a countable strongly dense subset $F_0 \subset F$. Therefore, the map

$$\omega \mapsto \inf_{v \in F} I_k(v, \omega) = \inf_{v \in F_0} I_k(v, \omega)$$

is \mathcal{F} -measurable, and the conclusion follows from (A.7). \square

A.4.3. Measurable minimizers. We show that in the convex case (or more generally in the p -sup-quasiconvex case) the random functional (A.6) admits a measurable minimizer. For that purpose, we begin with the following useful reformulation of the Rokhlin–Kuratowski–Ryll Nardzewski theorem, which essentially asserts that the measurability of the infimum implies the measurability of minimizers.

Lemma A.14. *Let X be a Polish space, let $(\Omega, \mathcal{F}, \mu)$ be a complete measure space, and let $I : X \times \Omega \rightarrow [0, \infty]$. Assume that*

- (i) for all ω , $I(\cdot, \omega)$ is lower semicontinuous on X ;
- (ii) for all ω , $I(\cdot, \omega)$ is coercive on X (i.e. the sublevel sets $\{u \in X : I(u, \omega) \leq c\}$ are compact for all $c > 0$);
- (iii) for all closed subset $F \subset X$, the map $\phi_F : \Omega \rightarrow [0, \infty]$ defined by $\phi_F(\omega) := \min_{v \in F} I(v, \omega)$ is \mathcal{F} -measurable.

Then, there exists an \mathcal{F} -measurable map $u : \Omega \rightarrow X$ such that, for all $\omega \in \Omega$,

$$I(u(\omega), \omega) = \min_{v \in X} I(v, \omega) = \phi_X(\omega).$$

\square

Proof. By coercivity and lower semicontinuity, the minima of $I(\cdot, \omega)$ are always attained on all closed subsets F , so that the function ϕ_F is always well-defined.

Consider the multifunction $\Gamma : \Omega \rightrightarrows X$ defined by $\Gamma(\omega) = \{u \in X : I(u, \omega) = \phi_X(\omega)\}$. By lower semicontinuity, $\Gamma(\omega) \subset X$ is nonempty and closed for all ω . Moreover, we claim that Γ is measurable, in the sense that

$$\Gamma^{-1}(O) := \{\omega \in \Omega : \Gamma(\omega) \cap O \neq \emptyset\}$$

belongs to \mathcal{F} for all open subsets $O \subset X$. As X is metrizable, it actually suffices to check $\Gamma^{-1}(F) \in \mathcal{F}$ for all closed subsets $F \subset X$ (see e.g. [5, Lemma 18.2]). By the coercivity and the lower semicontinuity of I , we may write

$$\begin{aligned} \Gamma^{-1}(F) &= \{\omega \in \Omega : \exists u \in F, I(u, \omega) = \phi_X(\omega)\} \\ &= \bigcap_{n=1}^{\infty} \left\{ \omega \in \Omega : \exists u \in F, I(u, \omega) \leq \phi_X(\omega) + \frac{1}{n} \right\} \\ &= \bigcap_{n=1}^{\infty} \left\{ \omega \in \Omega : \phi_F(\omega) \leq \phi_X(\omega) + \frac{1}{n} \right\}, \end{aligned}$$

where the right-hand side belongs to \mathcal{F} , by measurability of ϕ_X and ϕ_F . Hence, Γ is measurable, and we may thus apply the Rokhlin–Kuratowski–Ryll Nardzewski measurable selection theorem (see e.g. [5, Theorem 18.13]), which states the existence of a \mathcal{F} -measurable map $u : \Omega \rightarrow X$ such that $u(\omega) \in \Gamma(\omega)$ for all ω . \square

Combining this measurable selection lemma with the measurability result of Proposition A.13, we obtain the following:

Proposition A.15. *Let O be some bounded domain, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and let $W : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \rightarrow [0, \infty]$ be a normal random integrand. Assume that there exists $C > 0$ and $p > 1$ such that, for almost all y, ω and for all Λ ,*

$$\frac{1}{C}|\Lambda|^p - C \leq W(y, \Lambda, \omega). \quad (\text{A.8})$$

Also assume that W satisfies Hypothesis 2.1 and that $I(\cdot, \omega)$ is weakly lower semicontinuous on $W^{1,p}(O; \mathbb{R}^m)$ for almost all ω (in particular this is the case if W is convex or p -sup-quasiconvex in the sense of Definition A.11). Given some fixed function $f \in L^p(\Omega; L^p(O; \mathbb{R}^{m \times d}))$, consider the random integral functional $I : W^{1,p}(O; \mathbb{R}^m) \times \Omega \rightarrow [0, \infty]$ defined by

$$I(u, \omega) = \int_O W(y, f(y, \omega) + \nabla u(y), \omega) dy.$$

Then, for all nonempty weakly closed subsets $F \subset W_0^{1,p}(O; \mathbb{R}^m)$ or $F \subset \{u \in W^{1,p}(O; \mathbb{R}^m) : \int_O u = 0\}$, there exists a \mathcal{F} -measurable map $u : \Omega \rightarrow F$ such that, for almost all ω ,

$$I(u(\omega), \omega) = \inf_{v \in F} I(v, \omega).$$

□

Proof. Let X denote the Banach space $W_0^{1,p}(O; \mathbb{R}^m)$ or $\{u \in W^{1,p}(O; \mathbb{R}^m) : \int_O u = 0\}$, endowed with the weak topology, and, for all $k \geq 1$, consider the subset $X_k := \{u \in X : \|\nabla u\|_{L^p(O)} \leq k\}$, endowed with the induced weak topology. By Poincaré's inequality and by the Banach-Alaoglu theorem, X_k is easily seen to be metrizable and compact, hence Polish. Let $F \subset X$ be a nonempty (weakly) closed subset.

Let $\Omega' \subset \Omega$ denote a subset of maximal probability such that $I(\cdot, \omega)$ is weakly lower semicontinuous on $W^{1,p}(O; \mathbb{R}^m)$ for all $\omega \in \Omega'$. Since $X_k \subset X$ is (weakly) closed, the intersection $X_k \cap F$ is also (weakly) closed, and hence Hypothesis 2.1 asserts that the map $\omega \rightarrow \inf_{v \in F \cap X_k} I(v, \omega)$ is \mathcal{F} -measurable. Applying Lemma A.14 on the compact Polish space X_k and on Ω' then yields a \mathcal{F} -measurable map $u_k : \Omega' \rightarrow X_k$ such that, for all $\omega \in \Omega'$,

$$I(u_k(\omega), \omega) = \min_{v \in F \cap X_k} I(v, \omega).$$

The lower bound (A.8) and the triangle inequality give

$$\int_O |\nabla u|^p \leq C^2 2^{p-1} + C 2^{p-1} I(u, \omega) + 2^{p-1} \int_O |f(\cdot, \omega)|^p.$$

Define for all $k \geq 1$,

$$\Omega_k := \left\{ \omega \in \Omega' : C^2 2^{p-1} + C 2^{p-1} \inf_{v \in F} I(v, \omega) + 2^{p-1} \int_O |f(\cdot, \omega)|^p \leq k \right\},$$

and note that $\Omega_k \in \mathcal{F}$ by Hypothesis 2.1. By definition, for all $\omega \in \Omega_k$ we have

$$I(u_k(\omega), \omega) = \min_{v \in F \cap X_k} I(v, \omega) = \min_{v \in F} I(v, \omega).$$

The sequence $(\Omega_k)_k$ is increasing, $\Omega_k \uparrow \Omega'' := \bigcup_{k=1}^{\infty} \Omega_k$. By integrability of f , $\Omega'' \subset \Omega'''$ and $\mathbb{P}[\Omega''' \setminus \Omega''] = 0$, where we have defined the event $\Omega''' := \{\omega \in \Omega : \inf_{v \in F} I(v, \omega) < \infty\}$. Given some fixed $w \in F$, the measurable map $u : \Omega \rightarrow X$ defined by

$$u(\omega) := \mathbf{1}_{\Omega \setminus \Omega''} w + \mathbf{1}_{\Omega_1} u_1(\omega) + \sum_{k=2}^{\infty} \mathbf{1}_{\Omega_k \setminus \Omega_{k-1}} u_k(\omega)$$

satisfies by definition $I(u(\omega), \omega) = \inf_{v \in F} I(v, \omega)$ for all $\omega \in \Omega'' \cup (\Omega \setminus \Omega''')$. □

A.5. Approximation results. In the present appendix, we prove two general approximation results that are crucially needed in this paper. The first one is an extension of [36, Lemma 3.6] and [23, Proposition 2.6 of Chapter X].

Proposition A.16. *Let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, which is also strongly star-shaped, in the sense that there exists $x_0 \in O$ such that*

$$\overline{-x_0 + O} \subset \alpha(-x_0 + O), \quad \text{for all } \alpha > 1.$$

Let $\Theta : \mathbb{R}^{m \times d} \rightarrow [0, \infty]$ be a convex lower semicontinuous function with $0 \in \text{int dom } \Theta$, and let $u \in W^{1,1}(O; \mathbb{R}^m)$ such that $\int_O \Theta(\nabla u) < \infty$. Then,

(i) *there is a sequence $(v_n)_n \subset C^\infty(\text{adh } O; \mathbb{R}^m)$ such that $\nabla v_n \in \text{int dom } \Theta$ pointwise,*

$$v_n \rightarrow u \quad \text{in } W^{1,1}(O; \mathbb{R}^m), \quad \text{and} \quad \int_O \Theta(\nabla v_n(y)) dy \rightarrow \int_O \Theta(\nabla u(y)) dy;$$

(ii) *there is a sequence $(w_n)_n$ of (continuous) piecewise affine functions such that $\nabla w_n \in \mathbb{Q}^{m \times d} \cap \text{int dom } \Theta$ pointwise,*

$$w_n \rightarrow u \quad \text{in } W^{1,1}(O; \mathbb{R}^m), \quad \text{and} \quad \int_U \Theta(\nabla w_n(y)) dy \rightarrow \int_O \Theta(\nabla u(y)) dy.$$

If in addition u belongs to $W^{1,p}(O; \mathbb{R}^m)$ for some $1 \leq p < \infty$, then the sequences $(v_n)_n$ and $(w_n)_n$ can be chosen such that $v_n \rightarrow u$ and $w_n \rightarrow u$ in $W^{1,p}(O; \mathbb{R}^m)$. Moreover,

(a) *if u belongs to $W_0^{1,1}(O; \mathbb{R}^m)$, then we can choose $v_n \in C_c^\infty(O; \mathbb{R}^m)$ and $w_n|_{\partial O} = 0$ (and in that case the assumption that O be strongly star-shaped can be relaxed);*

(b) *if $O = Q$ and $u \in W_{\text{per}}^{1,1}(Q; \mathbb{R}^m)$, then v_n and w_n can be both chosen to be Q -periodic;*

(c) *if $\Xi : \mathbb{R}^{m \times d} \rightarrow [0, \infty]$ is a (nonconvex) ru-usc lower semicontinuous function which is continuous on $\text{int dom } \Theta$ and satisfies $0 \leq \Xi \leq \Theta$ pointwise, then the sequences $(v_n)_n$ and $(w_n)_n$ can be chosen in such a way that $\int_O \Xi(\nabla v_n) \rightarrow \int_O \Xi(\nabla u)$ and $\int_O \Xi(\nabla w_n) \rightarrow \int_O \Xi(\nabla u)$. \square*

Proof. We divide the proof into four steps.

Step 1. Proof of (i).

First, we show that we can assume $\nabla u \in \text{int dom } \Theta$ almost everywhere. Indeed, assume that (i) is proven for such u 's, and let us deduce the general case. Given $u \in W^{1,p}(O; \mathbb{R}^m)$ with $\int_O \Theta(\nabla u) < \infty$, we have $\nabla u \in \text{dom } \Theta$ almost everywhere, and hence by convexity $t\nabla u \in \text{dom } \Theta$ almost everywhere for all $t \in [0, 1)$. As convexity also implies $\int_O \Theta(t\nabla u) \leq t \int_O \Theta(\nabla u) + (1-t)\Theta(0) < \infty$, we can apply the result (i) to tu , for any $t \in [0, 1)$: this gives a sequence $(v_{n,t})_n \subset C^\infty(\text{adh } O; \mathbb{R}^m)$ such that $v_{n,t} \rightarrow tu$ in $W^{1,1}(O; \mathbb{R}^m)$ and $\int_O \Theta(\nabla v_{n,t}(y)) dy \rightarrow \int_O \Theta(t\nabla u(y)) dy$. Weak lower semicontinuity of the integral functional $u \mapsto \int_O \Theta(\nabla u)$ on $W^{1,1}(O; \mathbb{R}^m)$ (which follows from convexity and lower semicontinuity of Θ) implies $\liminf_{t \uparrow 1} \int_O \Theta(t\nabla u) \geq \int_O \Theta(\nabla u)$. As the converse inequality follows from convexity, we obtain

$$\begin{aligned} & \limsup_{t \uparrow 1} \limsup_{n \uparrow \infty} \left(\|v_{n,t} - u\|_{W^{1,1}(O)} + \left| \int_O \Theta(\nabla v_{n,t}) - \int_O \Theta(\nabla u) \right| \right) \\ &= \limsup_{t \uparrow 1} \left(\|tu - u\|_{W^{1,1}(O)} + \left| \int_O \Theta(t\nabla u) - \int_O \Theta(\nabla u) \right| \right) = 0, \end{aligned}$$

and hence a standard diagonalization argument gives a sequence $(v_n)_n \subset C^\infty(\text{adh } O; \mathbb{R}^m)$ such that $\nabla v_n \in \text{int dom } \Theta$ almost everywhere, $v_n \rightarrow u$ in $W^{1,1}(O; \mathbb{R}^m)$, and $\int_O \Theta(\nabla v_n) \rightarrow \int_O \Theta(\nabla u)$, and proves the general version of (i).

Hence, from now on, we assume $\nabla u \in \text{int dom } \Theta$ almost everywhere. Moreover, without loss of generality, we may also assume that O is strongly star-shaped with respect to $x_0 = 0$. Choose $(\alpha_k)_k \subset (1, \infty)$ a decreasing sequence of positive numbers converging to 1, sufficiently slowly so that $\frac{1}{k} < \frac{1}{2} \text{dist}(\text{adh } O, \partial(\alpha_k O))$ for all k , and define

$$u_k : \alpha_k O \rightarrow \mathbb{R}^m : x \mapsto u_k(x) = u(x/\alpha_k).$$

Take $\rho \in C_c^\infty(\mathbb{R}^d)$ such that $\int \rho = 1$, $\rho \geq 0$ and $\text{supp } \rho \subset B(0, 1)$, and write $\rho_k(x) = k^{-d}\rho(kx)$, for all $k \geq 1$. Consider the sequence $(v_k)_k$ defined by $v_k = \rho_k * (\alpha_k u_k)$ (which is well-defined by virtue of the condition on the α_k 's). Note that $v_k \in C^\infty(\text{adh } O; \mathbb{R}^m)$ and $\nabla v_k = \rho_k * (\nabla u)_k$, where we use the notation $(\nabla u)_k(x) = \nabla u(x/\alpha_k)$. Moreover, we then observe $v_k \rightarrow u$ in $W^{1,1}(O; \mathbb{R}^m)$ since $u_k \rightarrow u$ et $(\nabla u)_k \rightarrow \nabla u$ in $L^1(O; \mathbb{R}^m)$. Now, Jensen's inequality yields

$$0 \leq \Theta(\nabla v_k) = \Theta(\rho_k * (\nabla u)_k) \leq \rho_k * (\Theta((\nabla u)_k)) = \rho_k * (\Theta(\nabla u))_k.$$

As the sequence $(\rho_k * (\Theta(\nabla u))_k)_k$ converges to $\Theta \circ \nabla u$ in $L^1(O; \mathbb{R}^m)$, it is uniformly integrable, and the same thus holds for the sequence $(\Theta(\nabla v_k))_k$. As $\nabla v_k \rightarrow \nabla u$ in $L^1(O; \mathbb{R}^m)$, the convergence holds almost everywhere up to an extraction. Since by convexity Θ is continuous on $\text{int dom } \Theta$, since $\nabla u \in \text{int dom } \Theta$ almost everywhere, and since $\nabla v_k \rightarrow \nabla u$ almost everywhere up to an extraction, we deduce $\Theta(\nabla v_k) \rightarrow \Theta(\nabla u)$ almost everywhere up to an extraction. By uniform integrability the latter convergence also holds in $L^1(O; \mathbb{R}^m)$, which allows us to get rid of the extraction. In particular,

$$\int_O \Theta(\nabla v_k) \rightarrow \int_O \Theta(\nabla u).$$

Moreover, $(\nabla u)_k \in \text{int dom } \Theta$ almost everywhere, and thus $\nabla v_k = \rho_k * (\nabla u)_k \in \text{int dom } \Theta$ everywhere for all k , since $\text{int dom } \Theta$ is a convex set containing 0. This proves part (i).

Let us now assume that $u \in W^{1,p}(O; \mathbb{R}^m)$ for some $1 \leq p < \infty$, and consider the sequences $(u_k)_k$, $((\nabla u)_k)_k$ and $(v_k)_k$ defined above. First, Lebesgue's dominated convergence theorem implies $u_k \rightarrow u$ and $(\nabla u)_k \rightarrow \nabla u$ in $L^p(O; \mathbb{R}^m)$. Further, since the Jensen inequality gives

$$\int_O |v_k - u|^p = \int_O |\alpha_k \rho_k * u_k - u|^p \leq \int_O \int_{B_{1/k}} \rho_k(t) |\alpha_k u_k(x-t) - u(x)|^p dt dx,$$

and likewise for gradients, we conclude that the sequence $(v_k)_k$ converges to u in $W^{1,p}(O; \mathbb{R}^m)$. This proves part (i) in the case when $u \in W^{1,p}(O; \mathbb{R}^m)$.

Step 2. Proof of (ii).

For all k , since $v_k \in C^\infty(\text{adh } O; \mathbb{R}^m)$, there exists a sequence $(w_{k,j})_j$ of piecewise affine functions such that $w_{k,j} \rightarrow v_k$ in $W^{1,\infty}(O; \mathbb{R}^m)$ as $j \uparrow \infty$ and $\|\nabla w_{k,j}\|_{L^\infty} \leq \|\nabla v_k\|_{L^\infty}$ for all j, k (see e.g. [23, Proposition 2.1 of Chapter X]). Further, these functions $w_{k,j}$ can (simply remember their construction by triangulation) be chosen taking their values in $\text{int dom } \Theta$, since we have constructed $\nabla v_k \in \text{int dom } \Theta$ everywhere for all k . Another approximation argument further allows us to choose $w_{k,j}$ such that $\nabla w_{k,j}$ only takes rational values. The desired result then follows from Step 1 and a diagonalization argument. Finally, the particular case when u belongs to $W^{1,p}(O; \mathbb{R}^m)$ is obtained similarly as in Step 1.

Step 3. Proof of the additional statements.

It remains to address the particular cases (a) and (b). First assume that u belongs to $W_0^{1,1}(O; \mathbb{R}^m)$. For the corresponding result, we refer to [23, Proposition 2.6 of Chapter X]. The only difference is that the argument in [23] requires continuity of Θ . Instead, we replace u by tu for $t < 1$ as in Step 1, so that by convexity $t\nabla u \in \text{int dom } \Theta$ almost everywhere, hence all the constructed quantities have almost all their values in $\text{int dom } \Theta$, on which Θ is continuous by convexity. No further continuity assumption is then needed.

Finally, if we assume $O = Q$ with $u \in W_{\text{per}}^{1,1}(Q; \mathbb{R}^m)$, then we can consider the periodic extension of u on \mathbb{R}^d and repeat the arguments in such a way that periodicity is conserved.

Step 4. Proof in the nonconvex case.

Let $u \in W^{1,p}(O; \mathbb{R}^m)$ be such that $\int_U \Theta(\nabla u) < \infty$, which implies $\nabla u \in \text{dom } \Theta$ almost everywhere. Let $t \in (0, 1)$. The approximation result given by point (i) gives a sequence $(u_{n,t})_n$ of smooth functions such that $u_{n,t} \rightarrow tu$ (strongly) in $W^{1,p}(O; \mathbb{R}^m)$ and $\int_O \Theta(\nabla u_{n,t}) \rightarrow \int_O \Theta(t\nabla u)$ as $n \uparrow \infty$, and such that $\nabla u_{n,t} \in \text{int dom } \Theta$ pointwise. Up to an extraction, we have $\nabla u_{n,t} \rightarrow t\nabla u$ almost everywhere,

and thus $\Xi(\nabla u_{n,t}) \rightarrow \Xi(t\nabla u)$ almost everywhere, which follows from continuity of Ξ on the interior of its domain, with indeed $t\nabla u \in \text{int dom } \Xi$ almost everywhere. Then noting that

$$0 \leq \Xi(\nabla u_{n,t}) \leq C(1 + \Theta(\nabla u_{n,t})),$$

and invoking both Lebesgue's dominated convergence theorem (for $\Xi(\nabla u_{n,t})$) and its converse (for $\Theta(\nabla u_{n,t})$), we deduce convergence $\int_O \Xi(\nabla u_{n,t}) \rightarrow \int_O \Xi(t\nabla u)$ as $n \uparrow \infty$. As Ξ is lower semi-continuous and also ru-usc, we compute, by Fatou's lemma,

$$\begin{aligned} \int_O \Xi(\nabla u(y)) dy &\leq \int_O \liminf_{t \uparrow 1} \Xi(t\nabla u(y)) dy \leq \liminf_{t \uparrow 1} \int_O \Xi(t\nabla u(y)) dy \\ &\leq \limsup_{t \uparrow 1} \int_O \Xi(t\nabla u(y)) dy \leq \int_O \Xi(\nabla u(y)) dy. \end{aligned}$$

Hence,

$$\lim_{t \uparrow 1} \lim_{n \uparrow \infty} \int_O \Xi(\nabla u_{n,t}) = \lim_{t \uparrow 1} \int_O \Xi(t\nabla u) = \int_O \Xi(\nabla u),$$

and similarly

$$\lim_{t \uparrow 1} \lim_{n \uparrow \infty} \int_O \Theta(\nabla u_{n,t}) = \int_O \Theta(\nabla u),$$

so that the conclusion follows from a standard diagonalization argument. The other properties are deduced in a similar way. \square

For technical reasons, we need in the proof of Proposition 3.10 to further approximate piecewise affine functions by refined ones with smoother variations. The precise approximation result we need is the following.

Proposition A.17. *Let u be an \mathbb{R}^m -valued continuous piecewise affine function on a bounded Lipschitz domain $O \subset \mathbb{R}^d$. Consider the open partition $O = \biguplus_{l=1}^k O^l$ associated with u (i.e. u is affine on each piece O^l). Define $M := (\bigcup_{l=1}^k \partial O^l) \setminus \partial O$ the interior boundary of this partition of O , and, for fixed $r > 0$, also define $M_r := (M + B_r) \cap O$ the r -neighborhood of this interior boundary. Then, for all $\kappa > 0$, there exists a continuous piecewise affine function $u_{\kappa,r}$ on O with the following properties:*

- (i) $\nabla u_{\kappa,r} = \nabla u$ pointwise on $O \setminus M_r$, and $\limsup_{r \downarrow 0} \sup_{0 < \kappa \leq 1} \|u_{\kappa,r} - u\|_{L^\infty(O)} = 0$;
- (ii) $\nabla u_{\kappa,r} \in \text{conv}(\{\nabla u(x) : x \in O\})$ pointwise (where $\text{conv}(\cdot)$ denotes the convex hull);
- (iii) denoting by $O := \biguplus_{l=1}^{n_{\kappa,r}} O_{\kappa,r}^l$ the open partition associated with $u_{\kappa,r}$, and $\Lambda_{\kappa,r}^l := \nabla u_{\kappa,r}|_{O_{\kappa,r}^l}$ for all l , we have $|\Lambda_{\kappa,r}^i - \Lambda_{\kappa,r}^j| \leq \kappa$ for all i, j with $\partial O_{\kappa,r}^i \cap \partial O_{\kappa,r}^j \neq \emptyset$. \square

Proof. Let u , O and r be fixed. Without loss of generality, we can assume $0 \in O$. Denote $r_0 := \text{dist}(0, \partial O)$ and $R_0 := \max_{x \in \partial O} \text{dist}(0, x)$, and define $\alpha_r > 1$ by $(\alpha_r - 1)R_0 = r/2$. Choose a nonnegative smooth function ρ_r supported in $B_{(\alpha_r - 1)r_0}$ with $\int \rho_r = 1$, and consider the smooth function u_r on O defined by $u_r = \rho_r * [\alpha_r u(\cdot/\alpha_r)]$. Since by definition inequality $|\alpha_r x - x| \leq (\alpha_r - 1)R_0 = r/2$ holds for any $x \in O$, we easily check $\nabla u_r = \nabla u$ on the set $O \setminus M_r$. As u_r is smooth, we can consider $L_r := \max_{x \in \text{adh } O} |\nabla \nabla u_r(x)| < \infty$. Choose a triangulation $(O_{\kappa,r}^l)_{l=1}^{n_{\kappa,r}}$ of O which is a refinement of the partition $\{O^l \setminus M_r : l = 1, \dots, k\} \cup \{M_r\}$, such that the diameter of each of the $O_{\kappa,r}^l$'s is at most $\kappa/(2L_r)$. Now construct as usual the piecewise affine approximation $u_{\kappa,r}$ of u_r with respect to the triangulation $(O_{\kappa,r}^l)_l$ (see e.g. [23, Proposition 2.1 of Chapter X]). By construction, $\nabla u_{\kappa,r} = \nabla u_r = \nabla u$ on $O \setminus M_r$, since u_r is affine on each connected component of $O \setminus M_r$. Also note that the construction ensures that $\nabla u_{\kappa,r}$ belongs pointwise to the set $\{\nabla u_r(x) : x \in O\}$ (as a consequence of the mean value theorem), which is by definition included in the convex hull $\text{conv}(\{\nabla u(x) : x \in O\})$. Moreover, if $\partial O_{\kappa,r}^i \cap \partial O_{\kappa,r}^j \neq \emptyset$, it implies that $O_{\kappa,r}^i \cup O_{\kappa,r}^j$ has diameter bounded by κ/L_r . The construction of $u_{\kappa,r}$ then gives

$$\left| \nabla u_{\kappa,r}|_{O_{\kappa,r}^i} - \nabla u_{\kappa,r}|_{O_{\kappa,r}^j} \right| \leq \sup_{x \in O_{\kappa,r}^i} \sup_{y \in O_{\kappa,r}^j} |\nabla u_r(x) - \nabla u_r(y)| \leq \frac{\kappa}{L_r} \sup_{x \in O} |\nabla \nabla u_r(x)| \leq \kappa,$$

which proves property (iii). Finally, the last property of (i) directly follows from the construction. \square

To treat the case of periodic boundary conditions, we further need the following periodic version of the previous result. The proof is omitted, since it is a direct adaptation on the torus of the proof above.

Proposition A.18. *Let u be an \mathbb{R}^m -valued continuous piecewise affine Q -periodic function on \mathbb{R}^d . Consider the periodic partition $\mathbb{R}^d = \bigsqcup_{l=1}^{\infty} O^l$ associated with u (i.e. u is affine on each piece O^l). Define $M := \bigcup_{l=1}^{\infty} \partial O^l$ the boundary of this partition, and, for fixed $r > 0$, also define $M_r := (M + B_r) \cap Q$ the r -neighborhood of this boundary. Then, for all $\kappa > 0$, there exists a continuous piecewise affine Q -periodic function $u_{\kappa,r}$ on \mathbb{R}^d with the following properties:*

- (i) $\nabla u_{\kappa,r} = \nabla u$ pointwise on $\mathbb{R}^d \setminus M_r$, and $\limsup_{r \downarrow 0} \sup_{0 < \kappa \leq 1} \|u_{\kappa,r} - u\|_{L^\infty(Q)} = 0$;
- (ii) $\nabla u_{\kappa,r} \in \text{conv}(\{\nabla u(x) : x \in Q\})$ pointwise (where $\text{conv}(\cdot)$ denotes the convex hull);
- (iii) denoting by $\mathbb{R}^d := \bigsqcup_{l=1}^{\infty} T_{\kappa,r}^l$ the Q -periodic partition associated with $u_{\kappa,r}$, and $\Lambda_{\kappa,r}^l := \nabla u_{\kappa,r}|_{T_{\kappa,r}^l}$ for all l , we have $|\Lambda_{\kappa,r}^i - \Lambda_{\kappa,r}^j| \leq \kappa$ for all i, j with $\partial T_{\kappa,r}^i \cap \partial T_{\kappa,r}^j \neq \emptyset$. \square

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