CORRECTOR EQUATIONS IN FLUID MECHANICS: EFFECTIVE VISCOSITY OF COLLOIDAL SUSPENSIONS

MITIA DUERINCKX AND ANTOINE GLORIA

ABSTRACT. Consider a colloidal suspension of rigid particles in a steady Stokes flow. In a celebrated work, Einstein argued that in the regime of dilute particles the system behaves at leading order like a Stokes fluid with some explicit effective viscosity. In the present contribution, we rigorously define a notion of effective viscosity, regardless of the dilute regime assumption. More precisely, we establish a homogenization result when particles are distributed according to a given stationary and ergodic random point process. The main novelty is the introduction and analysis of suitable corrector equations.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

1.1. **General overview.** This article is devoted to the large-scale behavior of the steady Stokes equation for a fluid with a dense colloidal suspension of small rigid particles that are randomly distributed. The fluid and the particles interact via the action-reaction principle, and satisfy a no-slip condition at the particle boundaries. Suspended particles then act as obstacles, hindering the fluid flow and therefore increasing the flow resistance, that is, the viscosity. The system is naturally expected to behave on large scales approximately like a Stokes fluid with some effective viscosity. Our main result in this contribution makes this statement precise and rigorously defines the effective viscosity in terms of a stochastic homogenization result.

Let us first describe previous contributions on the topic, and emphasize our main motivation. In his PhD thesis, Einstein [14] was the first to analyze this effective viscosity problem: focussing on a dilute regime (that is, assuming that particles are scarce), he argued that the fluid indeed behaves at leading order like a Stokes fluid with some effective viscosity and that the latter can be explicitly computed at first order in the particle concentration in form of the so-called Einstein's formula, which played a key role in the physics community at that time as it served as a basis for Perrin's celebrated experiment to estimate the Avogadro number. Various contributions followed, in particular going beyond the first order, e.g. [7, 25, 3, 4]. From a rigorous perspective, several recent contributions stand out.

In [20] (see also the refined version [24]), Haines and Mazzucato provide bounds on the difference between a heuristic notion of effective viscosity (defined as some integral ratio with the correct dimensionality) and Einstein's formula. In [16] (see also [17]), Gérard-Varet and Hillairet took another approach, considering the solution of the Stokes problem and proving its closeness to the Stokes flow associated with some effective viscosity (described at higher order than Einstein's formula) — a quantified consistency result. In both works, for the effective behavior of a sequence of solutions, the authors establish error estimates that only get sharp in the dilute regime. On the one hand, the analysis in [20, 24, 16, 17] requires sophisticated arguments (reflection method, renormalized energy method, etc.) in order to get quantitative statements. On the other hand, their applicability is limited by the dilute regime assumption that allows to construct "explicit" approximate solutions. In particular, the very notion of effective viscosity is not defined independently of the dilute regime. Our main motivation is to remedy this issue by taking yet another approach and distinguishing two independent questions:

- the definition of an effective viscosity in full generality in the setting of homogenization theory in terms of a suitable corrector problem;
- the asymptotic analysis of the effective viscosity in the dilute regime — in the spirit of the so-called Clausius-Mossotti formula for homogenization of electrostatics and linear elasticity, cf. [10].

The present contribution answers the first question, while the second one is the object of a forthcoming work [13].

In a nutshell, our approach is in the pure tradition of homogenization theory. We reformulate the problem as the study of a family of solutions of fluid mechanics equations in a perforated domain associated with the spatial rescaling of some stationary and ergodic random array of inclusions, and we prove that this family converges to the solution of some effective (deterministic) fluid mechanics equation. Periodic homogenization in fluid mechanics is not new, dating back to Sánchez-Palencia [27], Tartar [28], and Allaire [1, 2], to cite but a few. We also refer to the early work of Cioranescu and Saint Jean Paulin [9], where a related scalar problem is considered in form of the so-called torsion problem. In the random setting, we refer to the contributions by Beliaev and Kozlov [8], by Basson and Gérard-Varet [5], and more recently by Giunti and Höfer [18]. We further refer to the works of Jikov [21, 22] on the closely related homogenization problem for stiff inclusions in linear elasticity, see also [23, Chapter 8.6]. In the present work, the homogenization result that is established in the general stationary and ergodic random framework (independently of the dilute regime) is new even in the periodic setting due to the specificity of the considered boundary conditions.

In terms of insight, the main novelty of this contribution is the introduction and analysis of suitable corrector equations in a context where this had not been done before. From a mathematical perspective, the divergence-free constraint for the fluid velocity yields technical difficulties and makes the analysis quite subtle — although still solely based on soft, qualitative arguments. As usual, the proof of the homogenization result splits into two parts: the construction of correctors, and the convergence result using Tartar's method of oscillating test functions [29]. The development of a corresponding *quantitative* homogenization theory, which is postponed to a forthcoming work [11], requires a suitable strong mixing condition on the particle distribution.

Before turning to the actual statement of the main results, let us mention that an additional motivation stems from the sedimentation problem for rigid particles in a Stokes flow, e.g. [6]. This concerns the case of particles that are heavier than the fluid, and therefore settle in the fluid. In the corrector equation, this yields an additional force on the particles, which pumps energy into the system and entails a crucial lack of compactness. We refer to our very recent work [12] (see also [19]) for a thorough discussion of the behavior of such sedimenting suspensions; although inspired by the present contribution, the analysis is much more involved and happens to require a strong mixing condition on the particle distribution even for qualitative results. Among other things, we show in [12] that the corresponding effective viscosity coincides with that for a non-sedimenting suspension, hence only depends on the geometry of the suspension.

1.2. **Main results.** Throughout, we place ourselves in dimension $d \geq 2$. We start with a suitable description of the random suspension of particles. Let $\{x_n^\omega\}_n$ denote a stationary and ergodic random point process on the ambient space $\mathbb{R}^d$, constructed on a given probability space $(\Omega, \mathbb{P})$; see Remark 1.2 below for a proper definition of stationarity. Define the corresponding spherical inclusion process

$$\mathcal{I}^\omega := \bigcup_n I_n^\omega, \qquad I_n^\omega := B(x_n^\omega),$$

where $B(x_n^\omega)$ denotes the unit ball centered at x_n^ω, and assume that it satisfies the hardcore condition

$$\inf_{m \neq n} \operatorname{dist}(I_n^\omega, I_m^\omega) > \delta \quad \text{almost surely},$$

for some fixed $\delta \in (0,1)$. Note that spherical inclusions could be replaced by random shapes under a uniform C^2 regularity assumption. In addition, the deterministic lower bound on the minimal interparticle distance can be relaxed into a lower bound of the type

$$\mathbb{E}\left[\mathbb{1}_{|x_n|<1} \sup_{m:m\neq n} \operatorname{dist}(I_n, I_m)^{-p} \right] < \infty,$$

for some large enough power $p \geq 1$, at the price of tracking down random constants in the proof and using Meyers-type estimates on solutions. We do however not pursue in this direction here; we believe that such conditions could be further improved, possibly in the spirit of [22], see also [23, Section 8.6].

Given a reference bounded Lipschitz domain U, we consider the set $\mathcal{N}_\varepsilon^\omega(U)$ of all indices n such that $\varepsilon(I_n^\omega + \delta B) \subset U$, and we define the corresponding rescaled inclusion process $\mathcal{I}_\varepsilon^\omega(U)$ in U,

$$\mathcal{I}_\varepsilon^\omega(U) := \bigcup_{n \in \mathcal{N}_\varepsilon^\omega(U)} \varepsilon I_n^\omega.$$

Note that balls of this collection are at distance at least $\varepsilon\delta$ from one another and from the boundary ∂U. This inclusion process represents a random suspension of particles in the reference domain U. We then consider these particles as suspended in a solvent described by the steady Stokes equation: the fluid velocity u_ε^ω satisfies

$$-\triangle u_\varepsilon^\omega + \nabla P_\varepsilon^\omega = 0, \qquad \operatorname{div} u_\varepsilon^\omega = 0, \qquad \text{in } U \setminus \mathcal{I}_\varepsilon^\omega(U),$$

and $u_\varepsilon^\omega = 0$ on ∂U. As the pressure is defined up to a constant, we choose for instance

$$\int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} P_\varepsilon^\omega = 0.$$

Next, no-slip boundary conditions are imposed at particle boundaries; since particles are constrained to have rigid motions, this amounts to letting the velocity field u_ε^ω be extended inside particles, with the rigidity constraint

$$\mathrm{D}(u_\varepsilon^\omega) = 0, \qquad \text{in } \mathcal{I}_\varepsilon^\omega(U),$$

where $\mathrm{D}(u_\varepsilon^\omega)$ denotes the symmetrized gradient of u_ε^ω. In other words, this condition means that u_ε^ω coincides with a rigid motion $V_{\varepsilon,n}^\omega + \Theta_{\varepsilon,n}^\omega(x - \varepsilon x_n^\omega)$ inside each inclusion εI_n^ω, for some $V_{\varepsilon,n}^\omega \in \mathbb{R}^d$ and skew-symmetric matrix $\Theta_{\varepsilon,n}^\omega \in \mathbb{R}^{d \times d}$. Finally, assuming that the particles have the same mass density as the fluid, buoyancy forces vanish, hence the force and torque balances on each particle take the form

$$\int_{\varepsilon \partial I_n^\omega} \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega)\nu = 0, \tag{1.1}$$

$$\int_{\varepsilon \partial I_n^\omega} \Theta(x - \varepsilon x_n^\omega) \cdot \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega)\nu = 0, \quad \text{for all } \Theta \in \mathbb{M}^{\mathrm{skew}},$$

where $\mathbb{M}^{\mathrm{skew}} \subset \mathbb{R}^{d \times d}$ denotes the subspace of skew-symmetric matrices, $\sigma(u_\varepsilon^\omega, P_\varepsilon^\omega)$ is the usual Cauchy stress tensor,

$$\sigma(u_\varepsilon^\omega, P_\varepsilon^\omega) = 2\,\mathrm{D}(u_\varepsilon^\omega) - P_\varepsilon^\omega \,\mathrm{Id},$$

and ν stands for the outward unit normal vector at the particle boundaries. In the physically relevant three-dimensional case $d = 3$, skew-symmetric matrices $\Theta \in \mathbb{M}^{\mathrm{skew}}$ are equivalent to cross products $\theta \times$ with $\theta \in \mathbb{R}^3$, and equations recover their more standard form.

In this context, modeling a dense suspension of small rigid particles in a viscous fluid with the same mass density, our homogenization result takes on the following guise.

Theorem 1. *Given a bounded Lipschitz domain $U \subset \mathbb{R}^d$ and given a forcing $f \in \mathrm{L}^2(U)$, consider for all $\varepsilon > 0$ and $\omega \in \Omega$ the unique weak solution $(u_\varepsilon^\omega, P_\varepsilon^\omega) \in H_0^1(U) \times \mathrm{L}^2(U \setminus \mathcal{I}_\varepsilon^\omega(U))$ of the Stokes problem introduced above, that is,*

$$\begin{cases} -\triangle u_\varepsilon^\omega + \nabla P_\varepsilon^\omega = f, & \text{in } U \setminus \mathcal{I}_\varepsilon^\omega(U), \\ \mathrm{div}\, u_\varepsilon^\omega = 0, & \text{in } U \setminus \mathcal{I}_\varepsilon^\omega(U), \\ u_\varepsilon^\omega = 0, & \text{on } \partial U, \\ \mathrm{D}(u_\varepsilon^\omega) = 0, & \text{in } \mathcal{I}_\varepsilon^\omega(U), \\ \int_{\varepsilon \partial I_n^\omega} \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega)\nu = 0, & \forall n \in \mathcal{N}_\varepsilon^\omega(U), \\ \int_{\varepsilon \partial I_n^\omega} \Theta(x - \varepsilon x_n^\omega) \cdot \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega)\nu = 0, & \forall n \in \mathcal{N}_\varepsilon^\omega(U),\, \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \\ \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} P_\varepsilon^\omega = 0, \end{cases} \tag{1.2}$$

and denote by $\lambda := \mathbb{E}\left[\mathbb{1}_\mathcal{I}\right]$ the volume fraction of the suspension. Then for almost all ω there holds

$$u_\varepsilon^\omega - \bar{u} \;\rightharpoonup\; 0, \qquad \text{weakly in } H_0^1(U),$$

$$(P_\varepsilon^\omega - \bar{P} - \bar{\boldsymbol{b}} : \mathrm{D}(\bar{u}))\mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \;\rightharpoonup\; 0, \qquad \text{weakly in } \mathrm{L}^2(U),$$

where $(\bar{u}, \bar{P}) \in H_0^1(U) \times \mathrm{L}^2(U)$ is the unique weak solution of the homogenized Stokes flow

$$\begin{cases} -\mathrm{div}\, 2\bar{\boldsymbol{B}}\,\mathrm{D}(\bar{u}) + \nabla \bar{P} = (1 - \lambda)f, & \text{in } U, \\ \mathrm{div}\, \bar{u} = 0, & \text{in } U, \\ \bar{u} = 0, & \text{on } \partial U, \\ \int_U \bar{P} = 0, \end{cases} \tag{1.3}$$

and the effective constants are as follows:

- *the effective diffusion tensor $\bar{\boldsymbol{B}}$ is a positive definite symmetric linear map on symmetric trace-free matrices $\mathbb{M}_0^{\mathrm{sym}} \subset \mathbb{R}^{d \times d}$, and is defined for all $E \in \mathbb{M}_0^{\mathrm{sym}}$ by*

$$E : \bar{\boldsymbol{B}}E := \mathbb{E}\left[|\operatorname{D}(\psi_E) + E|^2\right];\qquad(1.4)$$

- *$\bar{\boldsymbol{b}}$ is a symmetric trace-free matrix and is given for all $E \in \mathbb{M}_0^{\mathrm{sym}}$ by*

$$\bar{\boldsymbol{b}} : E := \frac{1}{d}\mathbb{E}\left[\sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \int_{\partial I_n} (x - x_n) \cdot \sigma(\psi_E + Ex, \Sigma_E)\nu\right];\qquad(1.5)$$

where $\nabla\psi_E \in \operatorname{L}^2(\Omega; \operatorname{L}^2_{\mathrm{loc}}(\mathbb{R}^d)^{d \times d})$ is the unique stationary gradient solution with vanishing expectation and $\Sigma_E \in \operatorname{L}^2(\Omega; \operatorname{L}^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \mathcal{I}))$ is the unique associated stationary pressure with vanishing expectation for the following infinite-volume corrector problem, cf. Proposition 2.1: for almost all ω,

$$\begin{cases} -\triangle\psi_E^\omega + \nabla\Sigma_E^\omega = 0, & in\ \mathbb{R}^d \setminus \mathcal{I}^\omega, \\ \operatorname{div}\psi_E^\omega = 0, & in\ \mathbb{R}^d \setminus \mathcal{I}^\omega, \\ \operatorname{D}(\psi_E^\omega + Ex) = 0, & in\ \mathcal{I}^\omega, \\ \int_{\partial I_n^\omega} \sigma(\psi_E^\omega + Ex, \Sigma_E^\omega)\nu = 0, & \forall n, \\ \int_{\partial I_n^\omega} \Theta(x - \varepsilon x_n^\omega) \cdot \sigma(\psi_E^\omega + Ex, \Sigma_E^\omega)\nu = 0, & \forall n,\ \forall\Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}\qquad(1.6)$$

Moreover, provided $f \in \operatorname{L}^p(U)$ for some $p > d$, for almost all ω, we have a corrector result for the velocity field,

$$\left\|u_\varepsilon^\omega - \bar{u} - \varepsilon\sum_{E \in \mathcal{E}} \psi_E^\omega(\tfrac{\cdot}{\varepsilon})\nabla_E\bar{u}\right\|_{H^1(U)} \to 0,$$

and for the pressure field,

$$\inf_{\kappa \in \mathbb{R}} \left\|P_\varepsilon^\omega - \bar{P} - \bar{\boldsymbol{b}} : \operatorname{D}(\bar{u}) - \sum_{E \in \mathcal{E}}(\Sigma_E^\omega\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon})\nabla_E\bar{u} - \kappa\right\|_{\operatorname{L}^2(U \setminus \mathcal{I}_\varepsilon^\omega(U))} \to 0,$$

where the sums run over an orthonormal basis $\mathcal{E}$ of $\mathbb{M}_0^{\mathrm{sym}}$. $\diamond$

Remark 1.1 (Buoyancy and sedimentation problem). If particles do not have the same mass density as the solvent fluid, a nontrivial buoyancy must be taken into account in the force balance (1.1): denoting by $g \in C_b(U)^d$ the buoyancy, this equation is replaced by

$$\frac{1}{\varepsilon} \int_{\varepsilon I_n^\omega} g + \int_{\varepsilon\partial I_n^\omega} \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega)\nu = 0.\qquad(1.7)$$

The scaling in ε is such that surface and volumetric forces have the same order uniformly in ε (it is equivalent, in sedimentation experiments, to increasing the size of the tank, rather than decreasing the size of the particles). Since an a priori diverging amount $O(\frac{1}{\varepsilon})$ of energy is then pumped into the system, it needs to be compensated by modifying the definition of correctors (1.6). This is fully analyzed in our companion article [11] under strong mixing conditions, where we show in particular that the effective viscosity is not affected by the settling process. A weak sedimentation regime can however be considered as a direct adaptation of our present analysis, replacing (1.7) by

$$\int_{\varepsilon I_n^\omega} g + \int_{\varepsilon\partial I_n^\omega} \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega)\nu = 0,$$

in which case the buoyancy vanishes in the limit (as the quotient of a volumetric over a surfacic term in the limit of small particles), and the effective equation is then obtained by adding a forcing term to (1.3) in form of

$$\begin{cases} -\operatorname{div} 2\bar{\boldsymbol{B}}\,\mathrm{D}(\bar{u}) + \nabla\bar{P} = (1-\lambda)f + \lambda g, & \text{in } U, \\ \operatorname{div}\bar{u} = 0, & \text{in } U, \\ \bar{u} = 0, & \text{on } \partial U, \\ \int_U \bar{P} = 0. \end{cases}$$

This simpler problem is however strictly distinct from the proper sedimentation regime.   ◇

Remark 1.2 (Stationary setting). We briefly recall the standard formulation of the stationary setting, make precise probabilistic assumptions, and recall some useful notation and constructions for stationary random fields.

  (i) *Stationarity and probabilistic assumptions.* As is customary in stochastic homogenization theory, e.g. [23, Section 7], stationarity is most conveniently defined via a measurable action $\{\tau_x\}_{x\in\mathbb{R}^d}$ of the translation group $(\mathbb{R}^d, +)$ on the underlying probability space $(\Omega, \mathbb{P})$. More precisely, the space is endowed with measurable maps $\tau_x : \Omega \to \Omega$ that satisfy
      - $\tau_x \circ \tau_y = \tau_{x+y}$ for all $x, y \in \mathbb{R}^d$;
      - $\mathbb{P}[\tau_x A] = \mathbb{P}[A]$ for all $x \in \mathbb{R}^d$ and measurable $A \subset \Omega$;
      - the map $\mathbb{R}^d \times \Omega \to \Omega : (x, \omega) \mapsto \tau_x \omega$ is jointly measurable;
      and this action is assumed to be ergodic in the sense that any random variable $\tilde{\phi} \in \mathrm{L}^1(\Omega)$ that is τ-invariant (i.e., $\tilde{\phi}(\tau_x \cdot) = \tilde{\phi}$ almost surely for all x) is almost surely constant. The point process $\{x_n^\omega\}_n$ is then said to be stationary (with respect to τ) if $\{x_n^{\tau_x\omega}\}_n = \{x + x_n^\omega\}_n$ for all x, ω.

  (ii) *Stationary extensions.* A function $\phi : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is said to be stationary if there exists a measurable map $\tilde{\phi} : \Omega \to \mathbb{R}$ such that $\phi(x, \omega) = \tilde{\phi}(\tau_{-x}\omega)$ for all x, ω. The joint measurability assumption for the action then ensures that ϕ is jointly measurable, which in view of a result by von Neumann is equivalent to stochastic continuity, that is, $\mathbb{P}[|\phi(x+y, \cdot) - \phi(x, \cdot)| > \delta] \to 0$ as $y \to 0$ for all x and $\delta > 0$, cf. [23, Section 7]. Stationarity then yields a bijection between random variables $\tilde{\phi} : \Omega \to \mathbb{R}$ and stationary measurable functions $\phi : \mathbb{R}^d \times \Omega \to \mathbb{R}$. The function ϕ is referred to as the stationary extension of the random variable $\tilde{\phi}$. The subspace of stationary functions $\phi \in \mathrm{L}^2(\Omega; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d))$ is then identified with the Hilbert space $\mathrm{L}^2(\Omega)$, and the (spatial) weak gradient ∇ on locally square integrable functions turns into a linear operator on $\mathrm{L}^2(\Omega)$. We also define $H^s(\Omega)$ as the subspace of random variables $\tilde{\phi} \in \mathrm{L}^2(\Omega)$ with stationary extension $\phi \in \mathrm{L}^2(\Omega; H^s_{\mathrm{loc}}(\mathbb{R}^d))$. We often use the shorthand notation $\phi^\omega(x) := \phi(x, \omega)$.   ◇

Notation.
  - For vector fields u, u' and matrix fields T, T', we set $(\nabla u)_{ij} = \nabla_j u_i$, $(\operatorname{div} T)_i = \nabla_j T_{ij}$, $T : T' = T_{ij}T'_{ij}$, $(u \otimes u')_{ij} = u_i u'_j$, $(T^s)_{ij} = \frac{1}{2}(T_{ij} + T_{ji})$, $\mathrm{D}(u) = (\nabla u)^s$. For a vector field u and a matrix E, we also write $\nabla_E u = E : \nabla u$. We systematically use Einstein's summation convention on repeated indices.
  - We denote by $\mathbb{M} = \mathbb{R}^{d \times d}$ the space of $d \times d$ matrices, by $\mathbb{M}_0^{\mathrm{sym}}$ the subset of symmetric trace-free matrices, and by $\mathbb{M}^{\mathrm{skew}}$ the subset of skew-symmetric matrices.

- We denote by $C \geq 1$ any constant that only depends on the dimension d, on the reference domain U, and on the hardcore constant $\delta \in (0,1)$. We use the notation $\lesssim$ (resp. $\gtrsim$) for $\leq C\times$ (resp. $\geq \frac{1}{C}\times$) up to such a multiplicative constant C. We add subscripts to $C, \lesssim, \gtrsim$ in order to indicate dependence on other parameters.

- The ball centered at x of radius r in $\mathbb{R}^d$ is denoted by $B_r(x)$, and we simply write $B(x) = B_1(x)$, $B_r := B_r(0)$, and $B = B_1(0)$.

2. Construction of correctors

This section is devoted to the construction of a suitable solution to the Stokes corrector equation (1.6).

Proposition 2.1. *Under the assumptions and notation of Theorem 1, for all $E \in \mathbb{M}_0^{\mathrm{sym}}$, there exist a unique random field $\psi_E \in \mathrm{L}^2(\Omega; H_{\mathrm{loc}}^1(\mathbb{R}^d)^d)$ and a unique pressure field $\Sigma_E \in \mathrm{L}^2(\Omega; \mathrm{L}_{\mathrm{loc}}^2(\mathbb{R}^d \setminus \mathcal{I}))$ such that*

(i) For almost all ω the realizations $\psi_E^\omega \in H_{\mathrm{loc}}^1(\mathbb{R}^d)$ and $\Sigma_E^\omega \in \mathrm{L}_{\mathrm{loc}}^2(\mathbb{R}^d \setminus \mathcal{I}^\omega)$ satisfy

$$\begin{cases} -\triangle \psi_E^\omega + \nabla \Sigma_E^\omega = 0, & in \ \mathbb{R}^d \setminus \mathcal{I}^\omega, \\ \mathrm{div}\, \psi_E^\omega = 0, & in \ \mathbb{R}^d \setminus \mathcal{I}^\omega, \\ \mathrm{D}(\psi_E^\omega + Ex) = 0, & in \ \mathcal{I}^\omega, \\ \fint_{\partial I_n^\omega} \sigma(\psi_E^\omega + Ex, \Sigma_E^\omega)\nu = 0, & \forall n, \\ \fint_{\partial I_n^\omega} \Theta(x - \varepsilon x_n^\omega) \cdot \sigma(\psi_E^\omega + Ex, \Sigma_E^\omega)\nu = 0, & \forall n, \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \qquad (2.1)$$

(ii) The corrector gradient $\nabla \psi_E$ and the pressure $\Sigma_E \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}$ are stationary[1], with

$$\mathbb{E}\big[\nabla \psi_E\big] = 0, \quad \mathbb{E}\big[\Sigma_E \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big] = 0,$$
$$\mathbb{E}\big[|\nabla \psi_E|^2\big] + \mathbb{E}\big[\Sigma_E^2 \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big] \lesssim |E|^2,$$

and we choose the anchoring $\int_B \psi_E^\omega = 0$ for the corrector.
In addition, the following properties hold:

(iii) Ergodic theorem for averages of corrector gradient and pressure: for almost all ω,

$$\begin{aligned} (\nabla \psi_E^\omega)(\tfrac{\cdot}{\varepsilon}) &\rightharpoonup \mathbb{E}\big[\nabla \psi_E\big] = 0 & \text{weakly in } \mathrm{L}_{\mathrm{loc}}^2(\mathbb{R}^d) \text{ as } \varepsilon \downarrow 0, \\ (\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon}) &\rightharpoonup \mathbb{E}\big[\Sigma_E \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big] = 0 & \text{weakly in } \mathrm{L}_{\mathrm{loc}}^2(\mathbb{R}^d) \text{ as } \varepsilon \downarrow 0. \end{aligned}$$

(iv) Sublinearity of the corrector: for almost all ω, for all $q < \frac{2d}{d-2}$,

$$\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \to 0 \quad \text{strongly in } \mathrm{L}_{\mathrm{loc}}^q(\mathbb{R}^d) \text{ as } \varepsilon \downarrow 0. \qquad \diamondsuit$$

Proof. We start by defining suitable functional subspaces of $\mathrm{L}^2(\Omega)^{d \times d}$ that are tailored for the study of the corrector equation (2.1). We first consider the subspace of potential fields with vanishing trace,

$$\mathcal{L}^2(\Omega) := \big\{ \tilde{\Psi} \in \mathrm{L}^2(\Omega)^{d \times d} : \mathbb{E}\big[\tilde{\Psi}\big] = 0, \ \mathrm{tr}\, \tilde{\Psi} = 0, \ \mathbb{E}\big[\tilde{\Psi} : (\nabla \times \tilde{\chi})\big] = 0 \ \forall \tilde{\chi} \in H^1(\Omega)^d \big\}.$$

---

[1]That is, $\nabla \psi_E^\omega(x+y) = \nabla \psi_E^{\tau_{-y}\omega}(x)$ and $\Sigma_E^\omega(x+y)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega}(x+y) = \Sigma_E^{\tau_{-y}\omega}(x)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^{\tau_{-y}\omega}}(x)$ for all x, y, ω, cf. Remark 1.2.

Using stationary extensions, cf. Remark 1.2, it is well-known (e.g. [23, Section 7]) that this space is equivalently given by

$$\mathcal{L}^2(\Omega) = \big\{ \tilde{\Psi} \in \mathrm{L}^2(\Omega)^{d \times d} : \mathbb{E}\big[\tilde{\Psi}\big] = 0, \text{ and } \exists \psi \in \mathrm{L}^2(\Omega; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)^d)$$
$$\text{with } \Psi = \nabla\psi \text{ and } \operatorname{div}\psi = 0\big\},$$

where the differential constraints are more clearly interpreted. We further incorporate the specific boundary conditions of the corrector equation (2.1) into the functional space, defining for $E \in \mathbb{M}_0$ the convex set

$$\mathcal{L}^2_E(\Omega) := \big\{ \tilde{\Psi} \in \mathcal{L}^2(\Omega) : \exists \psi \in \mathrm{L}^2(\Omega; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)^d) \text{ with } \Psi = \nabla\psi,$$
$$\text{and with } \mathrm{D}(\psi^\omega + Ex) = 0 \text{ in } \mathcal{I}^\omega \ \forall\omega\big\}.$$

As we shall check in Substep 3.1 below, $\mathcal{L}^2_E(\Omega)$ is not empty. Differences of elements of $\mathcal{L}^2_E(\Omega)$ belong to the vector space

$$\mathcal{L}^2_0(\Omega) := \big\{ \tilde{\Psi} \in \mathcal{L}^2(\Omega) : \exists \psi \in \mathrm{L}^2(\Omega; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)^d) \text{ with } \Psi = \nabla\psi,$$
$$\text{and with } \mathrm{D}(\psi^\omega) = 0 \text{ in } \mathcal{I}^\omega \ \forall\omega\big\}.$$

A well-known density result (e.g. [23, Section 7]) ensures that

$$\mathcal{L}^2(\Omega) = \mathrm{adh}_{\mathrm{L}^2(\Omega)^{d \times d}}\big\{ \nabla\tilde{\psi} : \tilde{\psi} \in H^1(\Omega)^d, \operatorname{div}\tilde{\psi} = 0 \big\}.$$

Likewise,

$$\mathcal{L}^2_0(\Omega) = \mathrm{adh}_{\mathrm{L}^2(\Omega)^{d \times d}} \mathcal{K}^2_0(\Omega), \tag{2.2}$$

with

$$\mathcal{K}^2_0(\Omega) := \big\{ \nabla\tilde{\psi} : \tilde{\psi} \in H^1(\Omega)^d, \operatorname{div}\tilde{\psi} = 0, \text{ and } \mathrm{D}(\psi^\omega) = 0 \text{ in } \mathcal{I}^\omega \ \forall\omega\big\}.$$

Once these spaces are introduced, the structure of the proof is as follows. We first show that for a solution (ψ_E, Σ_E) of (i)–(ii) the gradient $\nabla\psi_E$ is the unique Lax-Milgram solution in $\mathcal{L}^2_E(\Omega)$ of an abstract coercive problem on the probability space. We then argue that conversely this unique solution indeed provides a solution of (i)–(ii) in a weak sense in the physical space. Finally, from such a weak formulation, we reconstruct the pressure and establish the desired estimates (iii)–(iv). The proof is split into five main steps.

Step 1. From (i)–(ii) to an abstract problem in $\mathcal{L}^2_E(\Omega)$.
Let ψ_E be a solution of (i)–(ii). In particular, $\Psi_E := \nabla\psi_E$ is stationary and defines an element $\tilde{\Psi}_E \in \mathcal{L}^2_E(\Omega)$. We claim that it satisfies

$$\mathbb{E}\big[\tilde{\Phi} : \tilde{\Psi}_E\big] = 0, \qquad \text{for all } \tilde{\Phi} \in \mathcal{L}^2_0(\Omega). \tag{2.3}$$

By density (2.2), it is enough to prove (2.3) for all $\tilde{\Phi} \in \mathcal{K}^2_0(\Omega)$. Let $\tilde{\Phi} \in \mathcal{K}^2_0(\Omega)$ be given by $\tilde{\Phi} = \nabla\tilde{\phi}$ for some $\tilde{\phi} \in H^1(\Omega)^d$ with $\operatorname{div}\tilde{\phi} = 0$ and with $\mathrm{D}(\phi^\omega) = 0$ in $\mathcal{I}^\omega$ for all ω. In view of the hardcore condition, for all $R > 0$, we can construct a cut-off function η^ω_R supported in B_{R+3} with $\eta^\omega_R = 1$ on B_R and with $|\nabla\eta^\omega_R| \lesssim_\delta 1$, such that η^ω_R is constant in $I^\omega_n + \frac{\delta}{4}B$ for all n. Since ψ^ω_E is divergence-free, an integration by parts yields

$$\int_{\mathbb{R}^d} \nabla(\eta^\omega_R \phi^\omega) : \nabla\psi^\omega_E = 2 \int_{\mathbb{R}^d} \nabla(\eta^\omega_R \phi^\omega) : \mathrm{D}(\psi^\omega_E),$$

and thus, since $\int_{\mathbb{R}^d} \nabla(\eta^\omega_R \phi^\omega) = 0$ and since $\mathrm{D}(\psi^\omega_E) + E = 0$ in $\mathcal{I}^\omega$,

$$\int_{\mathbb{R}^d} \nabla(\eta^\omega_R \phi^\omega) : \nabla\psi^\omega_E = 2 \int_{\mathbb{R}^d \setminus \mathcal{I}^\omega} \nabla(\eta^\omega_R \phi^\omega) : (\mathrm{D}(\psi^\omega_E) + E).$$

Recalling the definition $\sigma(\psi_E^\omega + Ex, \Sigma_E^\omega) = 2(\mathrm{D}(\psi_E^\omega) + E) - \Sigma_E^\omega \mathrm{Id}$, integrating by parts, and using the corrector equation (2.1), we obtain

$$\int_{\mathbb{R}^d} \nabla(\eta_R^\omega \phi^\omega) : \nabla\psi_E^\omega - \int_{\mathbb{R}^d\setminus\mathcal{I}^\omega} \mathrm{div}(\eta_R^\omega \phi^\omega)\, \Sigma_E^\omega \;=\; \int_{\mathbb{R}^d\setminus\mathcal{I}^\omega} \nabla(\eta_R^\omega \phi^\omega) : \sigma(\psi_E^\omega + Ex, \Sigma_E^\omega)$$
$$= \; -\sum_n \int_{\partial I_n^\omega} \eta_R^\omega \phi^\omega \cdot \sigma(\psi_E^\omega + Ex, \Sigma_E^\omega)\nu. \quad (2.4)$$

For all n, since η_R^ω is constant in I_n^ω and since ϕ^ω takes the special form $\kappa_n^\omega + \Theta_n^\omega(x - x_n^\omega)$ in I_n^ω for some $\kappa_n^\omega \in \mathbb{R}^d$ and $\Theta_n^\omega \in \mathbb{M}^{\mathrm{skew}}$, the boundary condition in (2.1) for ψ_E^ω on ∂I_n^ω precisely yields

$$\int_{\partial I_n^\omega} \eta_R^\omega \, \phi^\omega \cdot \sigma(\psi_E^\omega + Ex, \Sigma_E^\omega)\nu \;=\; 0.$$

The weak form (2.4) of the equation thus becomes, after expanding the gradients and recalling that $\mathrm{div}\,\phi^\omega = 0$,

$$\int_{\mathbb{R}^d} \eta_R^\omega \nabla\phi^\omega : \nabla\psi_E^\omega \;=\; -\int_{\mathbb{R}^d} \phi^\omega \otimes \nabla\eta_R^\omega : \nabla\psi_E^\omega + \int_{\mathbb{R}^d\setminus\mathcal{I}^\omega} \Sigma_E^\omega \nabla\eta_R^\omega \cdot \phi^\omega,$$

which by the properties of η_R^ω we rewrite as

$$\int_{B_R} \nabla\phi^\omega : \nabla\psi_E^\omega \;=\; -\int_{B_{R+3}\setminus B_R} \eta_R^\omega \nabla\phi^\omega : \nabla\psi_E^\omega$$
$$-\int_{B_{R+3}\setminus B_R} \phi^\omega \otimes \nabla\eta_R^\omega : \nabla\psi_E^\omega + \int_{B_{R+3}\setminus B_R} \Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}^\omega} \nabla\eta_R^\omega \cdot \phi^\omega.$$

Taking the expectation, using the stationarity of ϕ, $\nabla\psi_E$, and Σ_E, as well as the a priori bounds (ii) and the boundedness of η_R, we obtain from Cauchy-Schwarz' inequality for all $R \geq 1$,

$$\left| \mathbb{E}\big[\nabla\tilde{\phi} : \tilde{\Psi}_E\big] \right| \;\lesssim\; \frac{1}{R}|E|\, \mathbb{E}\left[|\phi|^2 + |\nabla\phi|^2 \right]^{\frac{1}{2}},$$

and the claim follows from the arbitrariness of R.

Step 2. Well-posedness of the abstract problem (2.3).
In this step, we argue that there exists a unique solution $\tilde{\Psi}_E \in \mathcal{L}_E^2(\Omega)$ to the problem (2.3). As we shall check in Substep 3.1 below, the convex set $\mathcal{L}_E^2(\Omega)$ is not empty, so that we may choose a reference field $\tilde{\Psi}_E^0 \in \mathcal{L}_E^2(\Omega)$. Writing $\tilde{\Psi}_E = \tilde{\Psi}_E^0 + \tilde{\Psi}_E^1$ for some $\tilde{\Psi}_E^1 \in \mathcal{L}_0^2(\Omega)$, the equation (2.3) for $\tilde{\Psi}_E$ is equivalent to the following equation for $\tilde{\Psi}_E^1$,

$$\mathbb{E}\big[\tilde{\Phi} : \tilde{\Psi}_E^1\big] = -\mathbb{E}\big[\tilde{\Phi} : \tilde{\Psi}_E^0\big] \qquad \text{for all } \tilde{\Phi} \in \mathcal{L}_0^2(\Omega). \quad (2.5)$$

The existence and uniqueness of the solution $\tilde{\Psi}_E^1$ to this equation then follow from the Lax-Milgram theorem in the Hilbert space $\mathcal{L}_0^2(\Omega)$.

Step 3. From the abstract problem (2.3) to a weak formulation of (i).
Let $\tilde{\Psi}_E \in \mathcal{L}_E^2$ denote the unique solution of (2.3) as constructed in Step 2, which can be written as $\Psi_E = \nabla\psi_E$ in terms of the almost surely unique random field $\psi_E \in \mathrm{L}^2(\Omega; H^1_{\mathrm{loc}}(\mathbb{R}^d)^d)$ that satisfies the anchoring condition $\int_B \psi_E = 0$ at the origin. By construction, $\mathrm{div}\,\psi_E^\omega = 0$, and $\mathrm{D}(\psi_E^\omega + Ex) = 0$ in $\mathcal{I}^\omega$ for all ω. Next, we prove that ψ_E

satisfies the following weak formulation of (2.1): for almost all ω,

$$\int_{\mathbb{R}^d} \nabla\phi : \nabla\psi_E^\omega = 0, \qquad\qquad (2.6)$$

for all test functions ϕ in the class

$$\mathcal{C}^\omega := \left\{ \phi \in H^1(\mathbb{R}^d)^d \,:\, \phi \text{ has compact support, } \operatorname{div}\phi = 0, \text{ and } \mathrm{D}(\phi) = 0 \text{ in } \mathcal{I}^\omega \right\}.$$

We split the proof of (2.6) into two further substeps.

Substep 3.1. Definition of a suitable map $\mathfrak{M}^\omega : H^1_{c,\mathrm{div}}(\mathbb{R}^d)^d \to \mathcal{C}^\omega$, where $H^1_{c,\mathrm{div}}(\mathbb{R}^d)^d$ stands for the subspace $\{\zeta \in H^1(\mathbb{R}^d)^d : \zeta \text{ has compact support and } \operatorname{div}\zeta = 0\}$ of $H^1(\mathbb{R}^d)^d$.

Choose a map $\mathfrak{M}_\circ : H^1_{\mathrm{div}}(B_{1+\delta/2})^d \to H^1_{\mathrm{div}}(B_{1+\delta/2})^d$ that satisfies for all $\zeta \in H^1_{\mathrm{div}}(B_{1+\delta/2})^d$:

(1) $\mathfrak{M}_\circ\zeta - \zeta \in H^1_0(B_{1+\delta/2})$;
(2) $\mathrm{D}(\mathfrak{M}_\circ\zeta) = 0$ in B;
(3) if $\mathrm{D}(\zeta) = 0$ in B, then $\mathfrak{M}_\circ\zeta = \zeta$;
(4) $\|\nabla\mathfrak{M}_\circ\zeta\|_{\mathrm{L}^2(B_{1+\delta/2})} \lesssim \|\nabla\zeta\|_{\mathrm{L}^2(B_{1+\delta/2})}$.

Such a map $\mathfrak{M}_\circ$ can for instance be constructed as follows,

$$\mathfrak{M}_\circ\zeta := \operatorname{arginf}\left\{ \|\nabla\xi - \nabla\zeta\|^2_{\mathrm{L}^2(B_{1+\delta/2})} \,:\, \xi \in \zeta + H^1_0(B_{1+\delta/2}),\ \operatorname{div}\xi = 0, \right.$$
$$\left. \text{and } \mathrm{D}(\xi) = 0 \text{ in } B\right\}. \quad (2.7)$$

Since this is the minimization of a strictly convex lower-semicontinuous functional on a convex set, the infimum is attained and unique provided the convex set is nonempty. Choosing $\kappa = \fint_{B_{1+\delta/2}\setminus B} \zeta$ and $\Theta = 0$, it suffices to check that there exists $\xi \in \zeta + H^1_0(B_{1+\delta/2})$ with

$$\operatorname{div}\xi = 0 \qquad \text{and} \qquad \xi|_B = \kappa. \qquad\qquad (2.8)$$

For that purpose, choose $u_\zeta \in H^1_0(B_{1+\delta/2})$ that coincides with $-\zeta + \kappa$ on B. In view of the compatibility condition

$$\int_{B_{1+\delta/2}\setminus B} \operatorname{div} u_\zeta = \int_{\partial B_{1+\delta/2}} u_\zeta \cdot \nu - \int_{\partial B} u_\zeta \cdot \nu = \int_{\partial B} \zeta \cdot \nu = \int_B \operatorname{div}\zeta = 0,$$

a standard use of the Bogovskii operator in form of [15, Theorem III.3.1] ensures that this u_ζ can be modified in $B_{1+\delta/2}\setminus B$ (without changing its boundary values) to be divergence-free in $B_{1+\delta/2}\setminus B$ (hence in the whole of $B_{1+\delta/2}$), with the estimate

$$\|\nabla u_\zeta\|_{\mathrm{L}^2(B_{1+\delta/2}\setminus B)} \lesssim \|\zeta - \kappa\|_{H^{\frac{1}{2}}(\partial B)}.$$

In particular, by a trace estimate and Poincaré's inequality, this yields

$$\|\nabla u_\zeta\|_{\mathrm{L}^2(B_{1+\delta/2}\setminus B)} \lesssim \|\zeta - \kappa\|_{H^1(B_{1+\delta/2}\setminus B)} \lesssim_\delta \|\nabla\zeta\|_{\mathrm{L}^2(B_{1+\delta/2}\setminus B)}.$$

The function $\xi_\zeta := \zeta + u_\zeta \in \zeta + H^1_0(B_{1+\delta/2})$ then satisfies (2.8) and

$$\|\nabla\xi_\zeta\|_{\mathrm{L}^2(B_{1+\delta/2})} \lesssim_\delta \|\nabla\zeta\|_{\mathrm{L}^2(B_{1+\delta/2})}.$$

This implies that $\mathfrak{M}_\circ$ in (2.7) is well-defined and indeed satisfies the properties (1)–(4).

Next, for $\zeta \in H^1_{c,\mathrm{div}}(\mathbb{R}^d)^d$, we extend $\mathfrak{M}_\circ\zeta$ by ζ outside $B_{1+\delta/2}$, and for all $x \in \mathbb{R}^d$ we denote by $\mathfrak{M}_x$ the corresponding operator when the origin 0 is replaced by x. For all ω, we then define the operator $\mathfrak{M}^\omega := \prod_n \mathfrak{M}_{x_n^\omega}$, which indeed maps $H^1_{c,\mathrm{div}}(\mathbb{R}^d)^d$ to $\mathcal{C}^\omega$ as desired.

We conclude this construction of $\mathfrak{M}^\omega$ with a weak continuity result: for all bounded domains D and all sequences $(\zeta_n)_n$ of divergence-free functions compactly supported in D, if $\zeta_n \rightharpoonup \zeta$ weakly in $H^1(D)$, then for all ω we have $\mathfrak{M}^\omega \zeta_n \rightharpoonup \mathfrak{M}^\omega \zeta$ in $H^1(D)$. In view of the above construction of $\mathfrak{M}^\omega$, it is enough to prove this continuity result at the level of the elementary map $\mathfrak{M}_\circ$. Since the sequence $(\mathfrak{M}_\circ \zeta_n)_n$ is bounded in $H^1(B_{1+\delta/2})$, it converges to some ξ along a subsequence (not relabelled), which is necessarily an admissible test function for the minimization problem (2.7) for $\mathfrak{M}_\circ \zeta$. It remains to argue that it coincides with the desired minimizer $\mathfrak{M}_\circ \zeta$. To this aim, we use that the unique minimizers $\mathfrak{M}_\circ \zeta_n$ of (2.7) are characterized by the following Euler-Lagrange equations: for all $\xi' \in H_0^1(B_{1+\delta/2})$ with $\operatorname{div} \xi' = 0$ and with $\mathrm{D}(\xi') = 0$ in B,

$$\int_{B_{1+\delta/2}} (\nabla \mathfrak{M}_\circ \zeta_n - \nabla \zeta_n) : \nabla \xi' = 0,$$

in which we may pass to the limit in n in form of

$$\int_{B_{1+\delta/2}} (\nabla \xi - \nabla \zeta) : \nabla \xi' = 0,$$

thus recovering the Euler-Lagrange equation for $\mathfrak{M}_\circ \zeta$. This entails $\xi = \mathfrak{M}_\circ \zeta$ and ensures the convergence of the whole sequence.

We now quickly argue that a similar argument ensures that the convex set $\mathcal{L}_E^2(\Omega)$ is not empty. Choose $u_E \in H_0^1(B_{1+\delta/2})$ that coincides with $x \mapsto -Ex$ in B. In view of the compatibility condition

$$\int_{B_{1+\delta/2} \setminus B} \operatorname{div} u_E = \int_{\partial B_{1+\delta/2}} u_E \cdot \nu - \int_{\partial B} u_E \cdot \nu = \int_{\partial B} Ex \cdot \nu = |B| \operatorname{tr} E = 0,$$

a standard use of the Bogovskii operator in form of [15, Theorem III.3.1] ensures that u_E can be chosen divergence-free in $B_{1+\delta/2} \setminus B$ (hence in the whole of $B_{1+\delta/2}$), with the estimate

$$\|\nabla u_E\|_{\mathrm{L}^2(B_{1+\delta/2} \setminus B)} \lesssim |E|.$$

We may then define the stationary function $\phi = \sum_n u_E(\cdot - x_n)$, which is such that $\tilde{\Phi} = \nabla \tilde{\phi}$ belongs to $\mathcal{L}_E^2(\Omega)$ by construction.

Substep 3.2. Proof of (2.6).
Given a vector field $\phi \in H_{c,\mathrm{div}}^1(\mathbb{R}^d)^d$ and given a random variable $\tilde{\chi} \in \mathrm{L}^2(\Omega)$, we define Φ as the stationarization of the product $\tilde{\chi} \nabla \mathfrak{M} \phi$, that is,

$$\Phi(x, \omega) := \int_{\mathbb{R}^d} \tilde{\chi}(\tau_y \omega) \nabla (\mathfrak{M}^{\tau_y \omega} \phi)(x + y) \, dy,$$

which is well-defined in $\mathrm{L}^2(\Omega, \mathrm{L}_{\mathrm{loc}}^2(\mathbb{R}^d)^{d \times d})$ since ϕ (hence $\sup_\omega |\mathfrak{M}^\omega \phi|$) is compactly supported. On the one hand, Φ is obviously a stationary random field: for all x, z, ω,

$$\begin{aligned}
\Phi(x + z, \omega) &= \int_{\mathbb{R}^d} \tilde{\chi}(\tau_y \omega) \nabla \mathfrak{M}^{\tau_y \omega} \phi(x + z + y) \, dy \\
&= \int_{\mathbb{R}^d} \tilde{\chi}(\tau_{y-z} \omega) \nabla \mathfrak{M}^{\tau_{y-z} \omega} \phi(x + y) \, dy \\
&= \Phi(x, \tau_{-z} \omega).
\end{aligned}$$

On the other hand, the definition of $\mathfrak{M}$ ensures that $\tilde{\Phi}$ belongs to $\mathcal{L}_0^2$, which makes it an admissible test function for (2.3). By stationarity of $\Psi_E = \nabla \psi_E$ and of $\mathcal{I}$ in the form $(\Psi_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega})(0) = (\Psi_E^{\tau_y \omega} \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^{\tau_y \omega}})(y)$, and since the group action preserves the probability measure, we find

$$
\begin{aligned}
0 &= \mathbb{E}\big[\tilde{\Phi} : \tilde{\Psi}_E\big] \\
&= \int_\Omega \left(\int_{\mathbb{R}^d} \tilde{\chi}(\tau_y \omega) \nabla(\mathfrak{M}^{\tau_y \omega} \phi)(y)\, dy : \tilde{\Psi}_E^\omega \right) d\mathbb{P}(\omega) \\
&= \int_\Omega \left(\int_{\mathbb{R}^d} \tilde{\chi}(\tau_y \omega) \nabla(\mathfrak{M}^{\tau_y \omega} \phi)(y) : \nabla \psi_E^{\tau_y \omega}(y)\, dy \right) d\mathbb{P}(\omega) \\
&= \mathbb{E}\left[\tilde{\chi} \int_{\mathbb{R}^d} \nabla(\mathfrak{M}\phi) : \nabla \psi_E \right].
\end{aligned}
$$

By the arbitrariness of $\tilde{\chi}$, this implies that for any compactly supported vector field $\phi \in H^1_{\mathrm{div}}(\mathbb{R}^d)^d$ there holds for almost all ω,

$$
\int_{\mathbb{R}^d} \nabla(\mathfrak{M}^\omega \phi) : \nabla \psi_E^\omega = 0.
$$

By a density argument together with the weak continuity of $\mathfrak{M}^\omega$ as established in Substep 3.1, we deduce that for almost all ω this actually holds for all compactly supported vector fields $\phi \in H^1_{\mathrm{div}}(\mathbb{R}^d)^d$. Given ω, for ϕ^ω in the (realization-dependent) class $\mathcal{C}^\omega$, there holds $\mathfrak{M}^\omega \phi^\omega = \phi^\omega$ and the conclusion (2.6) follows.

Step 4. Reconstruction of the pressure.

In Step 3, we proved that the unique solution $\Psi_E = \nabla \psi_E$ of the abstract problem (2.3) also satisfies the weak formulation (2.6) of the corrector equation (2.1). In addition, note that the construction of Step 3 yields the bound $\mathbb{E}\big[|\nabla \psi_E|^2\big] \lesssim |E|^2$. In the present step, we show that one can construct a stationary pressure field Σ_E such that for almost all ω the vector field ψ_E^ω is a classical solution of the corrector equation (2.1), and that Σ_E and $\nabla \psi_E$ satisfy (ii). We split the proof into five further substeps.

Substep 4.1. Reconstruction of a pressure field $\bar{\Sigma}_E$.

For $R \geq 2$, consider the bounded Lipschitz domain

$$
D_R^\omega := B_R \cup \bigcup_{n : I_n^\omega \cap B_R \neq \varnothing} (I_n^\omega + \tfrac{\delta}{2}B).
$$

In view of (2.6), for almost all ω, ψ_E^ω satisfies for all vector fields $\phi \in H^1_{c,\mathrm{div}}(\mathbb{R}^d)^d$ that vanish on $\mathcal{I}^\omega$ and outside D_R^ω,

$$
\int_{D_R^\omega} \nabla \phi : \nabla \psi_E^\omega = 0.
$$

We deduce that ψ_E^ω is a weak solution of

$$
\begin{cases}
-\triangle \psi_E^\omega + \nabla \bar{\Sigma}_E^\omega = 0, & \text{in } D_R^\omega \setminus \mathcal{I}^\omega, \\
\mathrm{div}\, \psi_E^\omega = 0, & \text{in } D_R^\omega, \\
\mathrm{D}(\psi_E^\omega + Ex) = 0, & \text{in } \mathcal{I}^\omega \cap D_R^\omega,
\end{cases}
\tag{2.9}
$$

in the sense of [15, Definition IV.1.1]. Hence, by [15, Lemma IV.1.1], there exists a unique pressure field $\bar{\Sigma}_E^\omega \in \mathrm{L}^2(D_R^\omega \setminus \mathcal{I}^\omega)$ with the anchoring condition $\int_{2B} \bar{\Sigma}_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega} = 0$, such that (2.9) holds in the usual weak sense (that is, for all test functions $\phi \in H_0^1(D_R^\omega \setminus \mathcal{I}^\omega)^d$ without divergence-free constraint). In addition, by [15, Theorems IV.4.3 and IV.5.2],

both ψ_E^ω and $\bar{\Sigma}_E^\omega$ are smooth in $B_{R/2} \setminus \mathcal{I}^\omega$. By the arbitrariness of R, this implies that the pressure field $\bar{\Sigma}_E^\omega$ is well-defined in $\mathrm{L}_{\mathrm{loc}}^2(\mathbb{R}^d \setminus \mathcal{I}^\omega)$ and that ψ_E^ω and $\bar{\Sigma}_E^\omega$ are smooth on $\mathbb{R}^d \setminus \mathcal{I}^\omega$. In particular, the solutions are classical and the boundary conditions in (2.1) are satisfied in a pointwise sense. Note that the joint measurability of $\bar{\Sigma}_E$ on $\mathbb{R}^d \times \Omega$ easily follows from the reconstruction procedure for the pressure in [15]; details are omitted.

Substep 4.2. Proof that for all $R \geq 5$,

$$\fint_{B_R \setminus \mathcal{I}^\omega} \left(\bar{\Sigma}_E^\omega - \fint_{B_R \setminus \mathcal{I}^\omega} \bar{\Sigma}_E^\omega \right)^2 \lesssim \fint_{B_R \setminus \mathcal{I}^\omega} |\nabla \psi_E^\omega|^2. \tag{2.10}$$

As usual for pressure estimates for the Stokes equation, we first need to construct a map $\zeta_R^\omega \in H_0^1(B_R)$ such that

$$\mathrm{div}\, \zeta_R^\omega = \left(\bar{\Sigma}_E^\omega - \fint_{B_R \setminus \mathcal{I}^\omega} \bar{\Sigma}_E^\omega \right) \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega}, \tag{2.11}$$

$$\|\nabla \zeta_R^\omega\|_{\mathrm{L}^2(B_R)} \lesssim \left\| \bar{\Sigma}_E - \fint_{B_R \setminus \mathcal{I}^\omega} \bar{\Sigma}_E^\omega \right\|_{\mathrm{L}^2(B_R \setminus \mathcal{I}^\omega)}, \tag{2.12}$$

with the slight twist that $\zeta_R^\omega|_{I_n^\omega}$ further needs to be constant for all n. Testing (2.1) with such a ζ_R^ω then yields

$$\int_{\mathbb{R}^d \setminus \mathcal{I}^\omega} \nabla \zeta_R^\omega : \nabla \psi_E^\omega - \int_{\mathbb{R}^d \setminus \mathcal{I}^\omega} \bar{\Sigma}_E^\omega \,\mathrm{div}\, \zeta_R^\omega = 0,$$

which entails in view of the choice (2.11) of ζ_R^ω,

$$\int_{B_R \setminus \mathcal{I}^\omega} \left| \bar{\Sigma}_E^\omega - \fint_{B_R \setminus \mathcal{I}^\omega} \bar{\Sigma}_E^\omega \right|^2 \leq \int_{B_R \setminus \mathcal{I}^\omega} |\nabla \zeta_R^\omega| |\nabla \psi_E^\omega|,$$

and (2.10) follows from (2.12).

It remains to construct such a map ζ_R^ω. First define $\xi_R^\omega \in H_0^1(B_R)^d$ (extended to zero outside B_R) as a solution of the divergence problem

$$\mathrm{div}\, \xi_R^\omega = \left(\bar{\Sigma}_E^\omega - \fint_{B_R \setminus \mathcal{I}^\omega} \bar{\Sigma}_E^\omega \right) \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega},$$

$$\|\nabla \xi_R^\omega\|_{\mathrm{L}^2(B_R)} \lesssim \left\| \bar{\Sigma}_E^\omega - \fint_{B_R \setminus \mathcal{I}^\omega} \bar{\Sigma}_E^\omega \right\|_{\mathrm{L}^2(B_R \setminus \mathcal{I}^\omega)},$$

as provided by [15, Theorem III.3.1], where we emphasize that the multiplicative constant in the estimate is uniformly bounded in R. Next, we need to modify ξ_R^ω in the inclusions I_n^ω's that intersect B_R without changing $\mathrm{div}\, \xi_R^\omega$ and without increasing the norm of $\nabla \xi_R^\omega$ too much. This is performed by constructing suitable compactly supported corrections around the inclusions. For inclusions I_n^ω's contained in B_R with $\mathrm{dist}(I_n^\omega, \partial B_R) \geq \delta$, arguing as in Substep 3.1, we can construct a divergence-free vector field $\xi_{R,n}^\omega \in H_0^1(I_n^\omega + \frac{\delta}{2} B)^d$ that coincides with $-\xi_R^\omega + \fint_{(I_n^\omega + \frac{\delta}{2} B) \setminus I_n^\omega} \xi_R^\omega$ on I_n^ω such that

$$\|\nabla \xi_{R,n}^\omega\|_{\mathrm{L}^2((I_n^\omega + \frac{\delta}{2} B) \setminus I_n^\omega)} \lesssim \|\nabla \xi_R^\omega\|_{\mathrm{L}^2((I_n^\omega + \frac{\delta}{2} B) \setminus I_n^\omega)}.$$

We turn to inclusions I_n^ω's that intersect B_R such that $\mathrm{dist}(I_n^\omega, \partial B_R) < \delta$, for which we construct a divergence-free vector field $\xi_{R,n}^\omega \in H_0^1(B_R \cap (I_n^\omega + \frac{\delta}{2} B))^d$ that coincides with $-\xi_R^\omega$ on $B_R \cap I_n^\omega$ (that is indeed divergence-free there). Such a vector field can be constructed as

an application of the Bogovskii operator on $B_R \cap (I_n^\omega + \frac{\delta}{2}B) \setminus I_n^\omega$, in view of the compatibility condition

$$
\begin{aligned}
\int_{B_R \cap (I_n^\omega + \frac{\delta}{2}B) \setminus I_n^\omega} \operatorname{div} \xi_{R,n}^\omega &= \int_{\partial(B_R \cap (I_n^\omega + \frac{\delta}{2}B))} \xi_{R,n}^\omega \cdot \nu - \int_{\partial(B_R \cap I_n^\omega)} \xi_{R,n}^\omega \cdot \nu \\
&= \int_{\partial(B_R \cap I_n^\omega)} \xi_R^\omega \cdot \nu = \int_{B_R \cap I_n^\omega} \operatorname{div} \xi_R^\omega = 0, \qquad (2.13)
\end{aligned}
$$

and it satisfies

$$
\|\nabla \xi_{R,n}^\omega\|_{\mathrm{L}^2(B_R \cap (I_n^\omega + \frac{\delta}{2}B) \setminus I_n^\omega)} \lesssim \|\xi_R^\omega\|_{H^{\frac{1}{2}}(\partial(B_R \cap I_n^\omega))}.
$$

Hence, by a trace estimate (with ∂I_n^ω at distance at most δ from ∂B_R, on which ξ_R^ω vanishes) and Poincaré's inequality,

$$
\begin{aligned}
\|\nabla \xi_{R,n}^\omega\|_{\mathrm{L}^2(B_R \cap (I_n^\omega + \frac{\delta}{2}B) \setminus I_n^\omega)} &\lesssim \|\xi_R^\omega\|_{H^1(B_R \cap (I_n^\omega + 2\delta B))} \\
&\lesssim \|\nabla \xi_R^\omega\|_{\mathrm{L}^2(B_R \cap (I_n^\omega + 2\delta B))}.
\end{aligned}
$$

We finally define

$$
\zeta_R^\omega := \xi_R^\omega + \sum_{n : I_n^\omega \cap B_R \neq \varnothing} \xi_{R,n}^\omega,
$$

which by construction is constant in each of the inclusions I_n^ω's and satisfies the required properties (2.11) and (2.12).

Substep 4.3. Extension of $\bar\Sigma_E$ to $\mathbb{R}^d$ and estimate of $\nabla\bar\Sigma_E$.
In this substep, we extend $\bar\Sigma_E$ to $\mathbb{R}^d$ in such a way that $\bar\Sigma_E \in \mathrm{L}^2(\Omega; H^1_{\mathrm{loc}}(\mathbb{R}^d))$, that $\nabla\bar\Sigma_E$ is stationary, and that we have for all $R \geq 5$,

$$
\mathbb{E}\left[\fint_{B_R} \left(\bar\Sigma_E - \fint_{B_R} \bar\Sigma_E \right)^2 \right] + \mathbb{E}\left[|\nabla\bar\Sigma_E|^2 \right] \lesssim |E|^2. \qquad (2.14)
$$

We start by proving that $(\nabla\bar\Sigma_E^\omega)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}$ is a stationary field and satisfies

$$
\mathbb{E}\left[|(\nabla\bar\Sigma_E)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}|^2 \right] \lesssim |E|^2. \qquad (2.15)
$$

By the Stokes equation in form of $(\triangle\psi_E^\omega)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}} = (\nabla\bar\Sigma_E^\omega)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega}$, it suffices to prove that $(\triangle\psi_E)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}} \in \mathrm{L}^2(\Omega)$ satisfies $\mathbb{E}\left[|(\triangle\psi_E)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}|^2 \right] \lesssim |E|^2$. Since $(\nabla\psi_E)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}$ is stationary and since ψ_E is of class C^2 up to the boundaries ∂I_n, it is enough to prove that for almost all ω,

$$
\limsup_{R \uparrow \infty} \fint_{B_R} |(\nabla^2 \psi_E^\omega)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}|^2 \lesssim |E|^2. \qquad (2.16)
$$

To this aim, it suffices to show that for all $x \in \mathbb{R}^d$,

$$
\int_{B_{\delta/8}(x)} (|\nabla^2 \psi_E^\omega|^2 + |\nabla\Sigma_E^\omega|^2)\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}} \lesssim_\delta \int_{B_5(x)} |\nabla\psi_E^\omega|^2, \qquad (2.17)
$$

since the desired estimate (2.16) then follows in combination with the ergodic theorem and the bound $\mathbb{E}\left[|\nabla\psi_E|^2 \right] \lesssim |E|^2$. First consider the case when $x \in \mathbb{R}^d$ satisfies $\operatorname{dist}(x, \mathcal{I}^\omega) > \delta/4$, for which $B_{\delta/4}(x) \subset \mathbb{R}^d \setminus \mathcal{I}^\omega$. By interior regularity for the Stokes equation in form

of [15, Theorems IV.4.1], by (2.10), and by Poincaré's inequality, we then have with the choice $c_1^\omega = \fint_{B_{\delta/2}(x)} \psi_E^\omega$ and $c_2^\omega = \fint_{B_5(x)\setminus\mathcal{I}^\omega} \bar{\Sigma}_E^\omega$,

$$\int_{B_{\delta/8}(x)} (|\nabla^2\psi_E^\omega|^2 + |\nabla\bar{\Sigma}_E^\omega|^2)\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}^\omega} \lesssim_\delta \int_{B_{\delta/4}(x)} (|\nabla\psi_E^\omega|^2 + |\psi_E^\omega - c_1^\omega|^2 + |\bar{\Sigma}_E^\omega - c_2^\omega|^2)$$

$$\lesssim \int_{B_5(x)} |\nabla\psi_E^\omega|^2,$$

that is, (2.17). Next consider the case when $x \in \mathbb{R}^d$ satisfies $\mathrm{dist}(x, \mathcal{I}^\omega) \le \delta/4$, and let I_n^ω be the unique ball such that $\mathrm{dist}(x, I_n^\omega) < \delta/4$. By the boundary regularity theory for the Stokes equation in form of [15, Theorems IV.5.1–5.3], we then have with the choice $c_1^\omega = \fint_{I_n^\omega + \frac{\delta}{2}B} \psi_E^\omega$ and $c_2^\omega = \fint_{B_5(x)\setminus\mathcal{I}^\omega} \bar{\Sigma}_E^\omega$,

$$\int_{B_{\delta/8}(x)} (|\nabla^2\psi_E^\omega|^2 + |\nabla\bar{\Sigma}_E^\omega|^2)\mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}^\omega} \lesssim_\delta \|\psi_E^\omega|_{I_n^\omega} - c_1^\omega\|_{H^{\frac{3}{2}}(\partial I_n^\omega)}^2$$

$$+ \|\bar{\Sigma}_E^\omega - c_2^\omega\|_{\mathrm{L}^2((I_n^\omega + \frac{\delta}{2}B)\setminus I_n^\omega)}^2 + \|\psi_E^\omega - c_1^\omega\|_{H^1((I_n^\omega + \frac{\delta}{2}B)\setminus I_n^\omega)}.$$

Since ψ_E^ω is affine on I_n^ω, we have

$$\|\psi_E^\omega|_{I_n^\omega} - c_1^\omega\|_{H^{\frac{3}{2}}(\partial I_n^\omega)} \lesssim \|\psi_E^\omega|_{I_n^\omega} - c_1^\omega\|_{H^2(I_n^\omega)} = \|\psi_E^\omega - c_1^\omega\|_{H^1(I_n^\omega)}$$

$$\le \|\psi_E^\omega - c_1^\omega\|_{H^1(I_n^\omega + \frac{\delta}{2}B)},$$

while Poincaré's inequality with mean-value zero yields

$$\|\psi_E^\omega - c_1^\omega\|_{H^1(I_n^\omega + \frac{\delta}{2}B)} \lesssim_\delta \|\nabla\psi_E^\omega\|_{\mathrm{L}^2(I_n^\omega + \frac{\delta}{2}B)},$$

so that in combination with (2.10) the above turns into (2.17).

It remains to extend $\bar{\Sigma}_E$ on the inclusions. We simply choose $\bar{\Sigma}_E^\omega|_{B_{1/2}(x_n^\omega)} = \fint_{(I_n^\omega + \frac{\delta}{2}B)\setminus I_n^\omega} \bar{\Sigma}_E^\omega$, and we extend $\bar{\Sigma}_E^\omega$ radially linearly between ∂I_n^ω and $\partial B_{1/2}(x_n^\omega)$ (recall that $\bar{\Sigma}_E^\omega \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}^\omega}$ is continuous up to the boundary). So defined, $\bar{\Sigma}_E^\omega$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\nabla\bar{\Sigma}_E$ is stationary on $\mathbb{R}^d$. We conclude by establishing (2.14). Noting that the choice of the extension ensures

$$\int_{I_n^\omega} |\nabla\Sigma_E^\omega|^2 \lesssim_\delta \int_{(I_n^\omega + \frac{\delta}{4}B)\setminus I_n^\omega} |\nabla\Sigma_E^\omega|^2,$$

the gradient estimate in (2.14) simply follows from (2.15), and it remains to check the other part. By the definition of the extension, with $c^\omega = \fint_{B_{R+2}\setminus\mathcal{I}} \bar{\Sigma}_E^\omega$, we find using (2.10)

and (2.15),

$$
\begin{aligned}
\fint_{B_R} \Big(\bar\Sigma_E^\omega - \fint_{B_R} \bar\Sigma_E^\omega \Big)^2 &\lesssim \fint_{B_R} (\bar\Sigma_E^\omega - c^\omega)^2 \\
&\lesssim \fint_{B_R} (\bar\Sigma_E^\omega - c^\omega)^2 \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega} + R^{-d} \int_{B_R \cap \mathcal{I}} (\bar\Sigma_E^\omega - c^\omega)^2 \\
&\lesssim_\delta \fint_{B_R} (\bar\Sigma_E^\omega - c^\omega)^2 \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega} + R^{-d} \sum_{n : I_n^\omega \cap B_R \neq \varnothing} \int_{(I_n^\omega + \frac{\delta}{2} B) \setminus I_n^\omega} (|\bar\Sigma_E^\omega - c^\omega|^2 + |\nabla \bar\Sigma_E^\omega|^2) \\
&\lesssim \fint_{B_{R+2}} (\bar\Sigma_E^\omega - c^\omega)^2 \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega} + \fint_{B_{R+2}} |\nabla \bar\Sigma_E^\omega|^2 \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega} \\
&\lesssim \fint_{B_{R+2}} |\nabla \psi_E^\omega|^2,
\end{aligned}
$$

and the estimate (2.14) follows.

Substep 4.4. Construction of a stationary pressure field Σ_E.
Let $\chi \in C_c^\infty(B)$ satisfy $\int_B \chi = 1$, consider the rescaled kernel $\chi_r = \frac{1}{r^d} \chi(\frac{\cdot}{r})$ for $r \geq 1$, and define $P_r := \bar\Sigma_E - \chi_r * \bar\Sigma_E$. By construction, P_r is stationary, and we claim that

$$
\mathbb{E}\big[P_r^2 + |\nabla P_r|^2\big] \;\lesssim\; |E|^2, \tag{2.18}
$$

$$
\lim_{r \uparrow \infty} \mathbb{E}\big[|\nabla P_r - \nabla \bar\Sigma_E|^2 \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big] \;=\; 0. \tag{2.19}
$$

From (2.18), we deduce by weak compactness that there exists some $\tilde P \in H^1(\Omega)$ such that $(\tilde P_r, \nabla \tilde P_r) \rightharpoonup (\tilde P, \nabla \tilde P)$ weakly in $L^2(\Omega)$ along some subsequence (not relabelled), with

$$
\mathbb{E}\big[P^2 + |\nabla P|^2\big] \;\lesssim\; |E|^2. \tag{2.20}
$$

From (2.19) and the weak lower-semicontinuity of the $L^2(\Omega)$-norm, we then deduce

$$
\mathbb{E}\big[|\nabla P - \nabla \bar\Sigma_E|^2 \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big] \;\leq\; \liminf_{r \uparrow \infty} \mathbb{E}\big[|\nabla P_r - \nabla \bar\Sigma_E|^2 \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big] \;=\; 0.
$$

Hence, for almost all ω, the limit P^ω coincides with $\bar\Sigma_E^\omega$ up to an additive constant on the connected set $\mathbb{R}^d \setminus \mathcal{I}^\omega$. We then define the stationary pressure as $\Sigma_E \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}} := \big(P - \mathbb{E}\big[P \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big]\big) \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}$, which satisfies $\mathbb{E}\big[\Sigma_E \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big] = 0$ and the a priori estimate (ii).

It remains to give the arguments in favor of (2.18) and (2.19). We start with the former. For all $R \geq r \geq 1$, for $c^\omega = \fint_{B_{R+r}} \bar\Sigma_E^\omega$, we have

$$
\begin{aligned}
\fint_{B_R} (P_r^\omega)^2 + |\nabla P_r^\omega|^2 &= \fint_{B_R} (\bar\Sigma_E^\omega - \chi_r * \bar\Sigma_E^\omega)^2 + |\nabla \bar\Sigma_E^\omega - \chi_r * \nabla \bar P_E^\omega|^2 \\
&\lesssim \fint_{B_R} (\bar\Sigma_E^\omega - c^\omega)^2 + (\chi_r * (\bar\Sigma_E^\omega - c^\omega))^2 + |\nabla \bar\Sigma_E^\omega|^2 + |\chi_r * \nabla \bar\Sigma_E^\omega|^2 \\
&\lesssim R^{-d} \int_{B_{R+r}} |\nabla \bar\Sigma_E^\omega|^2 + (\bar\Sigma_E^\omega - c^\omega)^2.
\end{aligned}
$$

Taking the expectation and using (2.14) then yields by stationarity of P_r,

$$
\mathbb{E}\big[P_r^2 + |\nabla P_r|^2\big] \;\lesssim\; \frac{(R+r)^d}{R^d} |E|^2,
$$

from which (2.18) follows by taking the limit $R \uparrow \infty$. We turn to (2.19). By definition of P_r and since $|\nabla \chi_r| \lesssim \frac{1}{r^{d+1}} \mathbb{1}_{B_R}$ for all $R \geq r \geq 1$, we have for $c^\omega = \fint_{B_{R+r}} \bar{\Sigma}_E^\omega$,

$$\fint_{B_R} |\nabla P_r^\omega - \nabla \bar{\Sigma}_E^\omega|^2 \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega} \leq \fint_{B_R} |\nabla \chi_r * \bar{\Sigma}_E^\omega|^2 = \fint_{B_R} |\nabla \chi_r * (\bar{\Sigma}_E^\omega - c^\omega)|^2$$

$$\lesssim \frac{1}{r} \frac{(R+r)^d}{R^d} \fint_{B_{R+r}} (\bar{\Sigma}_E^\omega - c^\omega)^2.$$

As before, taking the expectation, recalling that $(\nabla P_r - \nabla \bar{\Sigma}_E) \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}$ is stationary, using (2.14), and letting $R \uparrow \infty$, we deduce

$$\mathbb{E}\big[|\nabla P_r - \nabla \bar{\Sigma}_E|^2 \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\big] \lesssim \frac{1}{r} |E|^2,$$

from which the claim (2.19) follows.

Substep 4.5. Proof of existence and uniqueness for (i)–(ii).
In Step 1, we have shown that if ψ_E is a solution of (i)–(ii), then $\Psi_E = \nabla \psi_E$ satisfies the abstract problem (2.3), for which existence and uniqueness is proved in Step 2. In Step 3, we considered the unique solution Ψ_E of (2.3) and proved that $\Psi_E = \nabla \psi_E$ is automatically a weak solution of (2.1) in form of (2.6). In Substeps 4.1–4.4, we reconstructed a unique stationary pressure field Σ_E (with vanishing expectation) such that ψ_E is a classical solution of (2.1). Uniqueness for (i)–(ii) then follows from uniqueness for (2.3). For the existence part for (i)–(ii), it remains to note that Σ_E and ψ_E satisfy (ii) as shown in Substep 4.4.

Step 5. Proof of (iii)–(iv).
The convergences in (iii) are a standard application of the ergodic theorem. The sublinearity (iv) of the corrector ψ_E^ω at infinity is also a standard result for random fields the gradients of which are stationary and have vanishing expectation, cf. [26, 23]. □

3. PROOF OF THE HOMOGENIZATION RESULT

This section is devoted to the proof of Theorem 1, making use of the correctors $(\psi_E)_E$ defined in Proposition 2.1 and adapting the classical oscillating test function method by Tartar [29]. We split the proof into eight different steps.

Step 1. Reformulation of the equations.
We show that the solution u_ε^ω of (1.2) satisfies in the weak sense in the whole domain U,

$$-\triangle u_\varepsilon^\omega + \nabla(P_\varepsilon^\omega \mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)}) = f \mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} - \sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \delta_{\varepsilon \partial I_n^\omega} \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega) \nu, \qquad (3.1)$$

while the corrector ψ_E^ω satisfies in the whole space $\mathbb{R}^d$,

$$-\triangle \psi_E^\omega + \nabla(\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega}) = -\sum_n \delta_{\partial I_n^\omega} \sigma(\psi_E^\omega + Ex, \Sigma_E^\omega) \nu. \qquad (3.2)$$

We focus on (3.1), and leave the proof of (3.2) (which is similar) to the reader. Since u_ε^ω is divergence-free, an integration by parts yields

$$\int_U \nabla \zeta : \nabla u_\varepsilon^\omega = 2 \int_U \nabla \zeta : \mathrm{D}(u_\varepsilon^\omega),$$

and thus, since $\mathrm{D}(u_\varepsilon^\omega) = 0$ in $\mathcal{I}_\varepsilon^\omega(U)$,

$$\int_U \nabla\zeta : \nabla u_\varepsilon^\omega \,=\, 2 \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \nabla\zeta : \mathrm{D}(u_\varepsilon^\omega).$$

Recalling the definition $\sigma(u_\varepsilon^\omega, P_\varepsilon^\omega) = 2\,\mathrm{D}(u_\varepsilon^\omega) - P_\varepsilon^\omega\,\mathrm{Id}$, integrating by parts, and using equation (1.2), we obtain

$$
\begin{aligned}
\int_U \nabla\zeta : \nabla u_\varepsilon^\omega - \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} (\mathrm{div}\,\zeta)\,P_\varepsilon^\omega \;&=\; \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \nabla\zeta : \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega) \\
&=\; \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \zeta \cdot f - \sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} \zeta \cdot \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega)\nu,
\end{aligned}
$$

that is, (3.1).

Step 2. Energy estimates.
We now show that for almost all ω the solution u_ε^ω of (1.2) satisfies

$$\int_U |\nabla u_\varepsilon^\omega|^2 + \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} |P_\varepsilon^\omega|^2 \;\lesssim_\delta\; \int_U |f|^2. \tag{3.3}$$

For almost all ω, by weak compactness, this allows us to consider $\bar{u}^\omega \in H_0^1(U)^d$ and $\bar{Q}^\omega \in \mathrm{L}^2(U)$ such that, along a subsequence (not relabelled) as $\varepsilon \downarrow 0$,

$$u_\varepsilon^\omega \rightharpoonup \bar{u}^\omega \quad \text{in } H_0^1(U), \qquad \text{and} \qquad P_\varepsilon^\omega \mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \rightharpoonup \bar{Q}^\omega \quad \text{in } \mathrm{L}^2(U). \tag{3.4}$$

In particular, by Rellich's theorem, $u_\varepsilon^\omega \to \bar{u}^\omega$ in $\mathrm{L}^2(U)$ strongly.

Here comes the argument for (3.3). For all $v \in H_0^1(U)$ with $\mathrm{div}\,v = 0$ in U and with $\mathrm{D}(v) = 0$ in $\mathcal{I}_\varepsilon(U)$, testing the formulation (3.1) of the Stokes equation with v yields

$$\int_U \nabla v : \nabla u_\varepsilon^\omega = \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} v \cdot f,$$

which for the choice $v = u_\varepsilon^\omega$ yields

$$\int_U |\nabla u_\varepsilon^\omega|^2 = \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} f \cdot u_\varepsilon^\omega \;\lesssim\; \Big(\int_U |f|^2 \Big)^{\frac{1}{2}} \Big(\int_U |\nabla u_\varepsilon^\omega|^2 \Big)^{\frac{1}{2}} \tag{3.5}$$

by Poincaré's inequality in $H_0^1(U)$, that is, (3.3) for $\nabla u_\varepsilon^\omega$. The corresponding estimate for the pressure is obtained by a similar argument as in Substep 4.2 of the proof of Proposition 2.1.

Step 3. A priori estimates at inclusion boundaries.
We claim that the solution u_ε^ω of (1.2) and the corrector ψ_E^ω satisfy for almost all ω,

$$\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} |u_\varepsilon^\omega|^2 \;\lesssim\; \frac{1}{\varepsilon} \int_U |f|^2, \tag{3.6}$$

$$\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} |\nabla u_\varepsilon^\omega|^2 + |P_\varepsilon^\omega|^2 \;\lesssim_\delta\; \frac{1}{\varepsilon} \int_U |f|^2, \tag{3.7}$$

$$\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} |\psi_E^\omega(\tfrac{\cdot}{\varepsilon})|^2 \;\lesssim_\delta\; \frac{1}{\varepsilon} \int_U |\psi_E^\omega(\tfrac{\cdot}{\varepsilon})|^2, \tag{3.8}$$

$$\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} |\nabla \psi_E^\omega(\tfrac{\cdot}{\varepsilon})|^2 + |\Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})|^2 \quad \lesssim_\delta \quad \frac{1}{\varepsilon} \int_U |\nabla \psi_E^\omega(\tfrac{\cdot}{\varepsilon})|^2 + |(\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon})|^2. \quad (3.9)$$

We start with the proof of (3.6). For all $n \in \mathcal{N}_\varepsilon^\omega(U)$, since u_ε^ω is affine in εI_n^ω, there holds

$$\int_{\varepsilon \partial I_n^\omega} |u_\varepsilon^\omega|^2 \lesssim \frac{1}{\varepsilon} \int_{\varepsilon I_n^\omega} |u_\varepsilon^\omega|^2,$$

so that

$$\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} |u_\varepsilon^\omega|^2 \lesssim \frac{1}{\varepsilon} \int_U |u_\varepsilon^\omega|^2,$$

and the claim (3.6) follows from Poincaré's inequality and (3.3). Likewise, for all n, since ψ_E^ω is affine in I_n^ω, we find

$$\int_{\partial I_n^\omega} |\psi_E^\omega|^2 \lesssim \int_{I_n^\omega} |\psi_E^\omega|^2,$$

and the claim (3.8) follows after summing and rescaling. We turn to the proof of (3.7). By scaling, it suffices to check that $\hat{u}_\varepsilon^\omega := \varepsilon^{-2} u_\varepsilon^\omega(\varepsilon \cdot)$ and $\hat{P}_\varepsilon^\omega := \varepsilon^{-1} P_\varepsilon^\omega(\varepsilon \cdot)$ satisfy

$$\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\partial I_n^\omega} |\nabla \hat{u}_\varepsilon^\omega|^2 + |\hat{P}_\varepsilon^\omega|^2 \lesssim_\delta \frac{1}{\varepsilon^2} \int_{\frac{1}{\varepsilon} U} |f(\varepsilon \cdot)|^2. \quad (3.10)$$

Given $n \in \mathcal{N}_\varepsilon^\omega(U)$, a trace estimate yields

$$\int_{\partial I_n^\omega} |\nabla \hat{u}_\varepsilon^\omega|^2 + |\hat{P}_\varepsilon^\omega|^2 \lesssim_\delta \|(\nabla \hat{u}_\varepsilon^\omega, \hat{P}_\varepsilon^\omega)\|^2_{H^1((I_n^\omega + \frac{\delta}{4} B) \setminus I_n^\omega)}.$$

Recalling that the inclusion I_n^ω is at distance at least $\delta > 0$ from other inclusions and from $\frac{1}{\varepsilon} \partial U$ so that $-\triangle \hat{u}_\varepsilon^\omega + \nabla \hat{P}_\varepsilon^\omega = f(\varepsilon \cdot)$ is satisfied in the annulus $(I_n^\omega + \delta B) \setminus I_n^\omega$, the regularity theory for the Stokes equation near a boundary in form of [15, Theorems IV.5.1–5.3] leads to the following, with $c_{n,\varepsilon}^\omega := \fint_{I_n^\omega + \frac{\delta}{2} B} \hat{u}_\varepsilon^\omega$,

$$\int_{\partial I_n^\omega} |\nabla \hat{u}_\varepsilon^\omega|^2 + |\hat{P}_\varepsilon^\omega|^2 \lesssim_\delta \|\hat{u}_\varepsilon^\omega|_{I_n^\omega} - c_{n,\varepsilon}^\omega\|^2_{H^{\frac{3}{2}}(\partial I_n^\omega)} + \|f(\varepsilon \cdot)\|^2_{\mathrm{L}^2(I_n^\omega + \frac{\delta}{2} B)}$$
$$+ \|\hat{P}_\varepsilon^\omega\|^2_{\mathrm{L}^2((I_n^\omega + \frac{\delta}{2} B) \setminus I_n^\omega)} + \|\hat{u}_\varepsilon^\omega - c_{n,\varepsilon}^\omega\|_{H^1((I_n^\omega + \frac{\delta}{2} B) \setminus I_n^\omega)}.$$

Since $\hat{u}_\varepsilon^\omega$ is affine on I_n^ω, we have

$$\|\hat{u}_\varepsilon^\omega|_{I_n^\omega} - c_{n,\varepsilon}^\omega\|^2_{H^{\frac{3}{2}}(\partial I_n^\omega)} \lesssim \|\hat{u}_\varepsilon^\omega - c_{n,\varepsilon}^\omega\|^2_{H^2(I_n^\omega)} = \|\hat{u}_\varepsilon^\omega - c_{n,\varepsilon}^\omega\|^2_{H^1(I_n^\omega)},$$

while Poincaré's inequality with mean-value zero yields

$$\|\hat{u}_\varepsilon^\omega - c_{n,\varepsilon}^\omega\|_{H^1(I_n^\omega + \frac{\delta}{2} B)} \lesssim \|\nabla \hat{u}_\varepsilon^\omega\|^2_{\mathrm{L}^2(I_n^\omega + \frac{\delta}{2} B)},$$

so that the above turns into

$$\int_{\partial I_n^\omega} |\nabla \hat{u}_\varepsilon^\omega|^2 + |\hat{P}_\varepsilon^\omega|^2 \lesssim_\delta \|f(\varepsilon \cdot)\|^2_{\mathrm{L}^2(I_n^\omega + \frac{\delta}{2} B)} + \|(\nabla \hat{u}_\varepsilon^\omega, \hat{P}_\varepsilon^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega})\|^2_{\mathrm{L}^2(I_n^\omega + \frac{\delta}{2} B)}.$$

Since the balls of the collection $\{I_n^\omega + \frac{\delta}{2} B\}_n$ are all disjoint, the rescaled version of the energy estimate (3.3) leads to

$$\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\partial I_n^\omega} |\nabla \hat{u}_\varepsilon^\omega|^2 + |\hat{P}_\varepsilon^\omega|^2 \lesssim_\delta \int_{\frac{1}{\varepsilon} U} |f(\varepsilon \cdot)|^2 + |\nabla \hat{u}_\varepsilon^\omega|^2 + |\hat{P}_\varepsilon^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega}|^2 \lesssim \frac{1}{\varepsilon^2} \int_{\frac{1}{\varepsilon} U} |f(\varepsilon \cdot)|^2,$$

that is, (3.10). It remains to establish (3.9). Applying as above a trace estimate together with the regularity theory for the Stokes equation near a boundary (cf. Substep 4.3 in the proof of Proposition 2.1), we obtain

$$\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\partial I_n^\omega} |\nabla \psi_E^\omega|^2 + |\Sigma_E^\omega|^2 \lesssim_\delta \int_{\frac{1}{\varepsilon}U} |\nabla \psi_E^\omega|^2 + |\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega}|^2,$$

and the claim (3.9) follows after rescaling.

Step 4. Oscillating test function method.
We show that for all test functions $\bar{v} \in C_c^\infty(U)^d$ with $\operatorname{div} \bar{v} = 0$ we have for almost all ω, along a subsequence (not relabelled),

$$2 \int_U \mathrm{D}(\bar{v}) : \mathrm{D}(\bar{u}^\omega) + \lim_{\varepsilon \downarrow 0} \sum_{E \in \mathcal{E}} 2 \int_U (\nabla_E \bar{v}) \, \mathrm{D}(\psi_E^\omega)(\tfrac{\cdot}{\varepsilon}) : \mathrm{D}(u_\varepsilon^\omega) = (1 - \lambda) \int_U \bar{v} \cdot f, \qquad (3.11)$$

where the sum runs over an orthonormal basis $\mathcal{E}$ of symmetric trace-free matrices $\mathbb{M}_0^{\mathrm{sym}}$, and where the limit in the left-hand side indeed exists (and is computed in the next step).

Let a typical $\omega \in \Omega$ be fixed such that the bounds of Steps 1–2 hold as well as the convergence (3.4) along a subsequence (not relabelled), and such that for all $E \in \mathbb{M}_0$ the corrector ψ_E^ω and corresponding pressure Σ_E^ω satisfy the corrector equation (1.6) in the classical sense as well as the properties (iii)–(iv) of Proposition 2.1. Given a test function $\bar{v} \in C_c^\infty(U)^d$ with $\operatorname{div} \bar{v} = 0$, we follow Tartar's ideas and define its oscillatory version $v_\varepsilon^\omega \in H_0^1(U)^d$ via

$$v_\varepsilon^\omega := \bar{v} + \sum_{E \in \mathcal{E}} \varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \nabla_E \bar{v},$$

where we recall the notation $\nabla_E \bar{v} = E : \nabla \bar{v}$. Testing equation (3.1) with v_ε^ω leads to

$$\int_U \nabla v_\varepsilon^\omega : \nabla u_\varepsilon^\omega - \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} (\operatorname{div} v_\varepsilon^\omega) P_\varepsilon^\omega$$

$$= \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} v_\varepsilon^\omega \cdot f - \sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} v_\varepsilon^\omega \cdot \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega) \nu, \quad (3.12)$$

and it remains to examine each of the four terms appearing in this identity.

• First, an integration by parts with $\operatorname{div} u_\varepsilon^\omega = 0$ yields

$$\int_U \nabla v_\varepsilon^\omega : \nabla u_\varepsilon^\omega = 2 \int_U \mathrm{D}(v_\varepsilon^\omega) : \mathrm{D}(u_\varepsilon^\omega),$$

and then inserting the definition of v_ε^ω,

$$\int_U \nabla v_\varepsilon^\omega : \nabla u_\varepsilon^\omega = 2 \int_U \mathrm{D}(\bar{v}) : \mathrm{D}(u_\varepsilon^\omega) + \sum_{E \in \mathcal{E}} 2 \int_U \varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \otimes \nabla \nabla_E \bar{v} : \mathrm{D}(u_\varepsilon^\omega)$$

$$+ \sum_{E \in \mathcal{E}} 2 \int_U (\nabla_E \bar{v}) \, \mathrm{D}(\psi_E^\omega)(\tfrac{\cdot}{\varepsilon}) : \mathrm{D}(u_\varepsilon^\omega).$$

By Step 2, the first right-hand side term converges to $2 \int_U \mathrm{D}(\bar{v}) : \mathrm{D}(\bar{u}^\omega)$. By sublinearity of ψ_E^ω (cf. Proposition 2.1(iv)), together with the boundedness of $\nabla u_\varepsilon^\omega$ in $\mathrm{L}^2(U)$ (cf. Step 2), the second right-hand side term converges to 0.

- Second, the definition of v_ε^ω with $\operatorname{div} \bar{v} = 0$ and $\operatorname{div} \psi_E^\omega = 0$ leads to

$$\int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} (\operatorname{div} v_\varepsilon^\omega) P_\varepsilon^\omega = \sum_{E \in \mathcal{E}} \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \cdot P_\varepsilon^\omega \nabla \nabla_E \bar{v},$$

which converges to 0 in view of the sublinearity of ψ_E^ω (cf. Proposition 2.1(iv)) together with the boundedness of P_ε^ω in $\mathrm{L}^2(U)$ (cf. Step 2).

- Third, the sublinearity of ψ_E^ω (cf. Proposition 2.1(iv)) implies $v_\varepsilon^\omega \to \bar{v}$ in $\mathrm{L}^2(U)$ and the ergodic theorem for the inclusion process yields $\mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \rightharpoonup \mathbb{E}[\mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}] \mathbb{1}_U = (1 - \lambda) \mathbb{1}_U$ weakly-* in $\mathrm{L}^\infty(U)$ for typical ω, so that $\int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} v_\varepsilon^\omega \cdot f \to (1 - \lambda) \int_U \bar{v} \cdot f$.

- Fourth, for $n \in \mathcal{N}_\varepsilon^\omega(U)$, the oscillating test function v_ε^ω can be expanded as follows, for all $x \in \varepsilon \partial I_n^\omega$,

$$\Big| v_\varepsilon^\omega(x) - \bar{v}(\varepsilon x_n^\omega) - \nabla \bar{v}(\varepsilon x_n^\omega)(x - \varepsilon x_n^\omega) - \sum_{E \in \mathcal{E}} \varepsilon \psi_E(\tfrac{x}{\varepsilon}) \nabla_E \bar{v}(\varepsilon x_n^\omega) \Big|$$
$$\lesssim \varepsilon^2 \|\nabla^2 \bar{v}\|_{\mathrm{L}^\infty} \max_{E \in \mathcal{E}} (1 + |\psi_E(\tfrac{x}{\varepsilon})|).$$

Setting for abbreviation $\Theta_{\varepsilon,n}^\omega := (\nabla \bar{v} - \mathrm{D}(\bar{v}))(\varepsilon x_n^\omega) \in \mathbb{M}^{\mathrm{skew}}$, and recalling the choice $\operatorname{tr} \mathrm{D}(\bar{v}) = \operatorname{div} \bar{v} = 0$, this can be reorganized as

$$\Big| v_\varepsilon^\omega(x) - \bar{v}(\varepsilon x_n^\omega) - \Theta_{\varepsilon,n}^\omega(x - \varepsilon x_n^\omega) - \sum_{E \in \mathcal{E}} \varepsilon \big(\psi_E^\omega(\tfrac{x}{\varepsilon}) + E(\tfrac{x}{\varepsilon} - x_n^\omega) \big) \nabla_E \bar{v}(\varepsilon x_n^\omega) \Big|$$
$$\lesssim \varepsilon^2 \|\nabla^2 \bar{v}\|_{\mathrm{L}^\infty} \max_{E \in \mathcal{E}} (1 + |\psi_E(\tfrac{x}{\varepsilon})|).$$

Inserting this approximation of v_ε^ω on $\varepsilon \partial I_n^\omega$, and recalling that $\psi_E^\omega + E(\cdot - x_n^\omega)$ is a rigid motion on I_n^ω, the boundary conditions for u_ε^ω on $\varepsilon \partial I_n^\omega$ lead to

$$\Big| \int_{\varepsilon \partial I_n^\omega} v_\varepsilon^\omega \cdot \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega) \nu \Big| \lesssim \varepsilon^2 \|\nabla^2 \bar{v}\|_{\mathrm{L}^\infty} \max_{E \in \mathcal{E}} \int_{\varepsilon \partial I_n^\omega} (1 + |\psi_E^\omega(\tfrac{\cdot}{\varepsilon})|) (|\nabla u_\varepsilon^\omega| + |P_\varepsilon^\omega|).$$

Summing over n and using Cauchy-Schwarz' inequality and the estimates (3.7) and (3.8) of Step 3, we obtain

$$\Big| \sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} v_\varepsilon^\omega \cdot \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega) \nu \Big| \lesssim \|\nabla^2 \bar{v}\|_{\mathrm{L}^\infty} \Big(\max_{E \in \mathcal{E}} \int_U (\varepsilon + |\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon})|)^2 \Big)^{\frac{1}{2}} \Big(\int_U |f|^2 \Big)^{\frac{1}{2}},$$

where the right-hand side tends to 0 by the sublinearity of ψ_E^ω at infinity (cf. Proposition 2.1(iv)).

Inserting the above estimates into (3.12), the claim (3.11) follows.

Step 5. Computation of the limit in (3.11) by compensated compactness.
For all $\bar{v} \in C_c^\infty(U)^d$ with $\operatorname{div} \bar{v} = 0$, we claim that for almost all ω,

$$\lim_{\varepsilon \downarrow 0} \sum_{E \in \mathcal{E}} 2 \int_U (\nabla_E \bar{v}) \, \mathrm{D}(\psi_E^\omega)(\tfrac{\cdot}{\varepsilon}) : \mathrm{D}(u_\varepsilon^\omega) = \sum_{E \in \mathcal{E}} \mathbb{E}[Z_E] : \int_U \bar{u}^\omega \otimes \nabla \nabla_E \bar{v}, \qquad (3.13)$$

in terms of the (matrix-valued) stationary random field Z_E defined componentwise by

$$Z_E^\omega := -\sum_n \frac{\mathbb{1}_{I_n^\omega}}{|I_n^\omega|} \int_{\partial I_n^\omega} \sigma(\psi_E^\omega + Ex, \Sigma_E^\omega) \nu \otimes (x - x_n^\omega). \qquad (3.14)$$

Integrating by parts, using equation (3.2) for the corrector, and the constraint div $u_\varepsilon^\omega = 0$, we may rewrite the product $\mathrm{D}(\psi_E^\omega)(\frac{\cdot}{\varepsilon}) : \mathrm{D}(u_\varepsilon^\omega)$ of two weakly convergent sequences as

$$2 \int_U (\nabla_E \bar{v}) \, \mathrm{D}(\psi_E^\omega)(\tfrac{\cdot}{\varepsilon}) : \mathrm{D}(u_\varepsilon^\omega) \;=\; -\int_{U \setminus \varepsilon \mathcal{I}^\omega} (u_\varepsilon^\omega \otimes \nabla \nabla_E \bar{v}) : \big(2\,\mathrm{D}(\psi_E^\omega) - \Sigma_E^\omega\big)(\tfrac{\cdot}{\varepsilon})$$
$$-\sum_n \int_{U \cap \varepsilon \partial I_n^\omega} (\nabla_E \bar{v}) \, u_\varepsilon^\omega \cdot \sigma\big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\nu.$$

By the ergodic theorem in form of $\nabla \psi_E^\omega(\frac{\cdot}{\varepsilon}) \rightharpoonup 0$ and $\Sigma_E^\omega(\frac{\cdot}{\varepsilon}) \rightharpoonup 0$ in $\mathrm{L}^2(U)$ (cf. Proposition 2.1(iii)) and by the strong convergence $u_\varepsilon^\omega \to \bar{u}^\omega$ in $\mathrm{L}^2(U)$, the first right-hand side term converges to 0. Hence, the limit of interest (which exists by (3.11)) takes the form

$$L^\omega := \lim_{\varepsilon \downarrow 0} \sum_{E \in \mathcal{E}} 2 \int_U (\nabla_E \bar{v}) \, \mathrm{D}(\psi_E^\omega)(\tfrac{\cdot}{\varepsilon}) : \mathrm{D}(u_\varepsilon^\omega) \;=\; \lim_{\varepsilon \downarrow 0} I_\varepsilon^\omega, \tag{3.15}$$

where I_ε^ω denotes the third and main right-hand side term in the above,

$$I_\varepsilon^\omega := -\sum_{E \in \mathcal{E}} \sum_n \int_{U \cap \varepsilon \partial I_n^\omega} (\nabla_E \bar{v}) \, u_\varepsilon^\omega \cdot \sigma\big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\nu.$$

Since $\bar{v}$ is compactly supported in U, we may restrict to ε small enough such that $\bar{v}$ is supported in $\{x \in U : d(x, \partial U) > \varepsilon\}$, so that $\bar{v}$ vanishes on $U \cap \varepsilon \partial I_n^\omega$ for $n \notin \mathcal{N}_\varepsilon^\omega(U)$. The above thus becomes

$$I_\varepsilon^\omega = -\sum_{E \in \mathcal{E}} \sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} (\nabla_E \bar{v}) \, u_\varepsilon^\omega \cdot \sigma\big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\nu.$$

For $n \in \mathcal{N}_\varepsilon^\omega(U)$, since u_ε^ω is a rigid motion in εI_n^ω, the boundary conditions for the corrector ψ_E^ω on ∂I_n^ω ensure that

$$\int_{\varepsilon \partial I_n^\omega} u_\varepsilon^\omega \cdot \sigma\big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\nu = 0,$$

which allows to reformulate I_ε^ω as

$$I_\varepsilon^\omega = -\sum_{E \in \mathcal{E}} \sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} \big(\nabla_E \bar{v} - \nabla_E \bar{v}(\varepsilon x_n^\omega)\big) u_\varepsilon^\omega \cdot \sigma\big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\nu. \tag{3.16}$$

For $n \in \mathcal{N}_\varepsilon^\omega(U)$, since u_ε^ω is affine in εI_n^ω, we can write on $\varepsilon \partial I_n^\omega$,

$$u_\varepsilon^\omega = \Big(\fint_{\varepsilon I_n^\omega} u_\varepsilon^\omega \Big) + (x - \varepsilon x_n^\omega)_i \nabla_i u_\varepsilon^\omega,$$

so that

$$\Big| \big(\nabla_E \bar{v} - \nabla_E \bar{v}(\varepsilon x_n^\omega)\big) u_\varepsilon^\omega - (x - \varepsilon x_n^\omega)_i \Big(\fint_{\varepsilon I_n^\omega} u_\varepsilon^\omega \, \nabla_i \nabla_E \bar{v} \Big) \Big|$$
$$\lesssim \varepsilon^2 (|u_\varepsilon^\omega| + |\nabla u_\varepsilon^\omega|)\big(\|\nabla^3 \bar{v}\|_{\mathrm{L}^\infty(U)} + \|\nabla^2 \bar{v}\|_{\mathrm{L}^\infty(U)} \big). \tag{3.17}$$

Next, appealing to the estimates (3.6), (3.7), and (3.9) of Step 3, we obtain

$$\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \varepsilon^2 \int_{\varepsilon \partial I_n^\omega} \left| \sigma \big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon}) \big) \right| \big(|u_\varepsilon^\omega| + |\nabla u_\varepsilon^\omega| \big)$$

$$\lesssim \quad \varepsilon \bigg(\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \varepsilon \int_{\varepsilon \partial I_n^\omega} |\nabla \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + E|^2 + |\Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})|^2 \bigg)^{\frac{1}{2}} \bigg(\sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \varepsilon \int_{\varepsilon \partial I_n^\omega} |u_\varepsilon^\omega|^2 + |\nabla u_\varepsilon^\omega|^2 \bigg)^{\frac{1}{2}}$$

$$\lesssim_\delta \quad \varepsilon \, \|f\|_{\mathrm{L}^2(U)} \Big(\int_U |E|^2 + |\nabla \psi_E^\omega(\tfrac{\cdot}{\varepsilon})|^2 + |(\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon})|^2 \Big)^{\frac{1}{2}}. \tag{3.18}$$

Inserting (3.17) into (3.15) and (3.16), and using the above to estimate the errors together with the boundedness statement of Proposition 2.1(iii), we are led to

$$L^\omega = \lim_{\varepsilon \downarrow 0} I_\varepsilon^\omega = -\lim_{\varepsilon \downarrow 0} \sum_{E \in \mathcal{E}} \sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \bigg(\fint_{\varepsilon I_n^\omega} u_\varepsilon^\omega \, \nabla_i \nabla_E \bar{v} \bigg)$$

$$\cdot \bigg(\int_{\varepsilon \partial I_n^\omega} (x - \varepsilon x_n^\omega)_i \, \sigma \big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon}) \big) \nu \bigg).$$

Recalling that for ε small enough the test function $\bar{v}$ vanishes on εI_n^ω for $n \notin \mathcal{N}_\varepsilon^\omega(U)$, we can rewrite

$$L^\omega = \lim_{\varepsilon \downarrow 0} \sum_{E \in \mathcal{E}} \int_U (u_\varepsilon^\omega \otimes \nabla \nabla_E \bar{v}) : Z_E^\omega(\tfrac{\cdot}{\varepsilon}),$$

in terms of the (matrix-valued) stationary field Z_E defined in (3.14). Since Z_E is stationary and bounded in $\mathrm{L}^2(\Omega)$, the ergodic theorem ensures $Z_E^\omega(\tfrac{\cdot}{\varepsilon}) \rightharpoonup \mathbb{E}[Z_E]$ in $\mathrm{L}^2(U)$ for typical ω. Combining this with the strong convergence $u_\varepsilon^\omega \to \bar{u}^\omega$ in $\mathrm{L}^2(U)$, the claim (3.13) follows.

Step 6. Identification of $\mathbb{E}[Z_E]$: for all $E \in \mathbb{M}_0^{\mathrm{sym}}$,

$$\mathbb{E}[Z_E] = 2(\mathrm{Id} - \bar{\boldsymbol{B}})E - (\bar{\boldsymbol{b}} : E)\,\mathrm{Id}, \tag{3.19}$$

where $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{b}}$ are defined in (1.4) and (1.5).

Let $E \in \mathbb{M}_0^{\mathrm{sym}}$ be fixed. First note that for any skew-symmetric matrix $E' \in \mathbb{M}^{\mathrm{skew}}$ the definition of Z_E and the boundary conditions for ψ_E entail $E' : Z_E = 0$ almost surely. Also note that the definition (1.5) of $\bar{\boldsymbol{b}}$ takes the form $\bar{\boldsymbol{b}} : E = -\frac{1}{d}\mathbb{E}[\mathrm{tr}\, Z_E]$. It then suffices to prove (3.19) when testing with symmetric trace-free matrices, that is, for all $E' \in \mathbb{M}_0^{\mathrm{sym}}$,

$$E' : \mathbb{E}[Z_E] = E' : 2(\mathrm{Id} - \bar{\boldsymbol{B}})E. \tag{3.20}$$

Let $E' \in \mathbb{M}_0^{\mathrm{sym}}$ be fixed. For $\eta > 0$, choose a cut-off function $\chi_\eta \in C_c^\infty(B)$ with $0 \le \chi_\eta \le 1$ pointwise, with $\chi_\eta = 1$ on $B_{1-\eta}$, and with $|\nabla \chi_\eta| \lesssim \frac{1}{\eta}$. For $0 < \varepsilon < \frac{1}{4}\eta$, in view of the hardcore condition, we can construct a modification $\chi_{\varepsilon,\eta}^\omega \in C_c^\infty(B)$ of χ_η that satisfies the same properties as χ_η, such that in addition $\chi_{\varepsilon,\eta}^\omega$ is constant in each inclusion of the collection $\{\varepsilon I_n^\omega\}_n$, vanishes in inclusions εI_n^ω with $n \notin \mathcal{N}_\varepsilon(B)$, and such that $\chi_{\varepsilon,\eta}^\omega \to \chi_\eta$ in $\mathrm{L}^\infty(B)$ as $\varepsilon \downarrow 0$. The ergodic theorem yields for almost all ω,

$$E' : \mathbb{E}[Z_E] = \lim_{\eta \downarrow 0} \lim_{\varepsilon \downarrow 0} \fint_B \chi_{\varepsilon,\eta}^\omega (E' : Z_E^\omega)(\tfrac{\cdot}{\varepsilon}).$$

Injecting the definition (3.14) of Z_E yields

$$E' : \mathbb{E}\left[Z_E\right]$$
$$= -\frac{1}{|B|} \lim_{\eta \downarrow 0} \lim_{\varepsilon \downarrow 0} \sum_{n \in \mathcal{N}_\varepsilon(B)} \left(\fint_{\varepsilon I_n^\omega} \chi_{\varepsilon,\eta}^\omega \right) \left(\int_{\varepsilon \partial I_n^\omega} E'(x - \varepsilon x_n^\omega) \cdot \sigma\left(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\right)\nu \right).$$

Since for all n the corrector $\psi_{E'}^\omega$ has the form $\kappa_n^\omega + \Theta_n^\omega(x - x_n^\omega) - E'(x - x_n^\omega)$ on I_n^ω for some $\kappa_n^\omega \in \mathbb{R}^d$ and $\Theta_n^\omega \in \mathbb{M}^{\mathrm{skew}}$, the boundary conditions for ψ_E^ω on ∂I_n^ω allow to rewrite

$$E' : \mathbb{E}\left[Z_E\right] = \frac{1}{|B|} \lim_{\eta \downarrow 0} \lim_{\varepsilon \downarrow 0} \sum_{n \in \mathcal{N}_\varepsilon(B)} \left(\fint_{\varepsilon I_n^\omega} \chi_{\varepsilon,\eta}^\omega \right) \left(\varepsilon \int_{\varepsilon \partial I_n^\omega} \psi_{E'}^\omega(\tfrac{\cdot}{\varepsilon}) \cdot \sigma\left(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\right)\nu \right).$$

Since $\chi_{\varepsilon,\eta}^\omega$ is constant in each inclusion and vanishes in inclusions εI_n^ω with $n \notin \mathcal{N}_\varepsilon(B)$, this is equivalently written as

$$E' : \mathbb{E}\left[Z_E\right] = \frac{1}{|B|} \lim_{\eta \downarrow 0} \lim_{\varepsilon \downarrow 0} \sum_n \varepsilon \int_{\varepsilon \partial I_n^\omega} \chi_{\varepsilon,\eta}^\omega \, \psi_{E'}^\omega(\tfrac{\cdot}{\varepsilon}) \cdot \sigma\left(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\right)\nu.$$

Using equation (3.2) for the corrector ψ_E^ω together with $\mathrm{div}\,\psi_{E'}^\omega = 0$, in form of

$$\sum_n \varepsilon \int_{\varepsilon \partial I_n^\omega} \chi_{\varepsilon,\eta}^\omega \, \psi_{E'}^\omega(\tfrac{\cdot}{\varepsilon}) \cdot \sigma\left(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\right)\nu$$
$$= -\int_B \chi_{\varepsilon,\eta}^\omega \, \nabla \psi_{E'}^\omega(\tfrac{\cdot}{\varepsilon}) : \nabla \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) - \int_B \varepsilon \psi_{E'}^\omega(\tfrac{\cdot}{\varepsilon}) \otimes \nabla \chi_{\varepsilon,\eta}^\omega : (\nabla \psi_E^\omega - \Sigma_E^\omega \,\mathrm{Id}\, \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon}),$$

and noting that the second right-hand side term converges to 0 as $\varepsilon \downarrow 0$ in view of the sublinearity of $\psi_{E'}$ (cf. Proposition 2.1(iv)) and in view of the boundedness statement of Proposition 2.1(iii), we deduce

$$E' : \mathbb{E}\left[Z_E\right] = -\lim_{\eta \downarrow 0} \lim_{\varepsilon \downarrow 0} \fint_B \chi_{\varepsilon,\eta}^\omega \, \nabla \psi_{E'}^\omega(\tfrac{\cdot}{\varepsilon}) : \nabla \psi_E^\omega(\tfrac{\cdot}{\varepsilon}).$$

Equivalently, again integrating by parts and using that $\mathrm{div}\,\psi_{E'}^\omega = 0$, we have

$$E' : \mathbb{E}\left[Z_E\right] = -\lim_{\eta \downarrow 0} \lim_{\varepsilon \downarrow 0} 2 \fint_B \chi_{\varepsilon,\eta}^\omega \, \mathrm{D}(\psi_{E'}^\omega)(\tfrac{\cdot}{\varepsilon}) : \mathrm{D}(\psi_E^\omega)(\tfrac{\cdot}{\varepsilon}).$$

In view of the ergodic theorem and of the definition (1.4) of $\bar{\boldsymbol{B}}$, this yields the claim (3.20) in form of

$$E' : \mathbb{E}\left[Z_E\right] = -2\,\mathbb{E}\left[\mathrm{D}(\psi_{E'}) : \mathrm{D}(\psi_E)\right] = E' : 2(\mathrm{Id} - \bar{\boldsymbol{B}})E. \tag{3.21}$$

Step 7. Conclusion: convergence result.
Combining the results of Steps 4–6, we conclude that for almost all ω there holds $u_\varepsilon^\omega \rightharpoonup \bar{u}^\omega$ weakly in $H_0^1(U)$ as $\varepsilon \downarrow 0$ along a subsequence, where the limit $\bar{u}^\omega$ satisfies $\mathrm{div}\,\bar{u}^\omega = 0$ and, for all $\bar{v} \in C_c^\infty(U)^d$ with $\mathrm{div}\,\bar{v} = 0$,

$$\int_U \mathrm{D}(\bar{v}) : 2\bar{\boldsymbol{B}}\,\mathrm{D}(\bar{u}^\omega) = (1 - \lambda) \int_U \bar{v} \cdot f,$$

where we recall that $\bar{\boldsymbol{B}}$ is defined in (1.4). Note that $\bar{\boldsymbol{B}}$ is positive definite on $\mathbb{M}_0^{\mathrm{sym}}$: by linearity of the corrector $E \mapsto \psi_E$ with $\mathbb{E}\left[\nabla \psi_E\right] = 0$, we compute for all $E \in \mathbb{M}_0^{\mathrm{sym}}$,

$$E : \bar{\boldsymbol{B}}E = |E|^2 + \mathbb{E}\left[|\nabla \psi_E|^2\right] \geq |E|^2.$$

Hence, $\bar{u}^\omega \in H_0^1(U)$ is a weak solution of the following well-posed steady Stokes equation in U,

$$-\operatorname{div} 2\bar{\boldsymbol{B}} \operatorname{D}(\bar{u}^\omega) + \nabla \bar{P}^\omega = (1-\lambda)f, \qquad \operatorname{div} \bar{u}^\omega = 0,$$

in the sense of [15, Definition IV.1.1]. In addition, by [15, Lemma IV.1.1], there exists a unique pressure field $\bar{P}^\omega \in \operatorname{L}^2(U)$ with $\int_U \bar{P}^\omega = 0$ such that this equation holds in the usual weak sense. By uniqueness for the above problem (e.g. [15, Theorem IV.1.1]), the solution $(\bar{u}^\omega, \bar{P}^\omega) = (\bar{u}, \bar{P})$ is independent of ω and the whole sequence converges. The convergence for the pressure field follows from the corrector result below combined with an approximation argument, cf. Substep 8.4.

Step 8. Corrector results.
We finally turn to the additional corrector results, which we obtain by a suitable recycling of the above computations. We consider the following two-scale expansion errors,

$$
\begin{aligned}
w_\varepsilon^\omega &:= u_\varepsilon^\omega - \bar{u} - \varepsilon \sum_{E \in \mathcal{E}} \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \nabla_E \bar{u}, \\
Q_\varepsilon^\omega &:= P_\varepsilon^\omega \mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} - \bar{P} - \bar{\boldsymbol{b}} : \operatorname{D}(\bar{u}) - (\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \varepsilon \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon}) \nabla_E \bar{u},
\end{aligned}
$$

and we split the proof into four further substeps: we start with the short proof of the corrector result for the velocity field, that is, $w_\varepsilon^\omega \to 0$ in $H^1(U)$, based on the convergence of the energy, and then we establish a suitable equation for w_ε^ω, from which we deduce a bound on the pressure Q_ε^ω and the corresponding corrector result. In the first three substeps, we assume for simplicity that the homogenized solution $\bar{u}$ belongs to $W^{3,\infty}(U)^d$, an assumption that we relax in the last substep.

Substep 8.1. Corrector result for the velocity field.
First, combining (3.5) with the strong convergence $u_\varepsilon^\omega \to \bar{u}$ in $\operatorname{L}^2(U)$ yields for almost all ω the convergence of energies in the form

$$2 \int_U |\operatorname{D}(u_\varepsilon^\omega)|^2 = \int_U |\nabla u_\varepsilon^\omega|^2 = \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} f \cdot u_\varepsilon^\omega \to (1-\lambda) \int_U f \cdot \bar{u} = 2 \int_U \operatorname{D}(\bar{u}) : \bar{\boldsymbol{B}} \operatorname{D}(\bar{u}).$$

Second, using the constraint $\operatorname{tr} \nabla \bar{u} = \operatorname{div} \bar{u} = 0$ in the form $\operatorname{D}(\bar{u}) = \sum_{E \in \mathcal{E}} (\nabla_E \bar{u})E$, and appealing to the stationarity of $\nabla \psi_E$, the ergodic theorem, and the sublinearity of ψ_E (cf. Proposition 2.1(iv)), together with the additional regularity of $\bar{u}$, we find for almost all ω,

$$\int_U \left| \operatorname{D}\left(\bar{u} + \varepsilon \sum_{E \in \mathcal{E}} \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \nabla_E \bar{u}\right) \right|^2 = \int_U \left| \sum_{E \in \mathcal{E}} (\operatorname{D}(\psi_E^\omega) + E)(\tfrac{\cdot}{\varepsilon}) \nabla_E \bar{u} + \sum_{E \in \mathcal{E}} \left(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \otimes \nabla \nabla_E \bar{u}\right)^s \right|^2$$

$$\to \int_U \operatorname{D}(\bar{u}) : \bar{\boldsymbol{B}} \operatorname{D}(\bar{u}).$$

Third, choosing $\bar{v} = \bar{u}$ as a test function, the computations of Steps 5–6 together with the regularity of $\bar{u}$ precisely yield for almost all ω,

$$\int_U \mathrm{D}\left(\bar{u} + \varepsilon \sum_{E \in \mathcal{E}} \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \nabla_E \bar{u}\right) : \mathrm{D}(u_\varepsilon^\omega)$$

$$= \sum_{E \in \mathcal{E}} \int_U (\nabla_E \bar{u})\,(\mathrm{D}(\psi_E^\omega) + E)(\tfrac{\cdot}{\varepsilon}) : \mathrm{D}(u_\varepsilon^\omega) + \sum_{E \in \mathcal{E}} \int_U \varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \otimes \nabla \nabla_E \bar{u} : \mathrm{D}(u_\varepsilon^\omega)$$

$$\to \int_U \mathrm{D}(\bar{u}) : \bar{\boldsymbol{B}}\,\mathrm{D}(\bar{u}).$$

Combining the above and reconstructing the square lead to the stated corrector result for the velocity field.

Substep 8.2. Equation for the two-scale expansion error.
We claim that $(w_\varepsilon^\omega, Q_\varepsilon^\omega)$ satisfies in the weak sense in U,

$$-\triangle w_\varepsilon^\omega + \nabla Q_\varepsilon^\omega = - \sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \delta_{\varepsilon \partial I_n^\omega} \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega) \nu$$

$$- \operatorname{div} 2(\bar{\boldsymbol{B}} - \mathrm{Id})\,\mathrm{D}(\bar{u}) - \nabla(\bar{\boldsymbol{b}} : \mathrm{D}(\bar{u})) + \sum_{E \in \mathcal{E}} \nabla_E \bar{u} \sum_n \delta_{\varepsilon \partial I_n^\omega} \sigma\big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big) \nu$$

$$+ (\lambda - \mathbb{1}_{\mathcal{I}_\varepsilon^\omega(U)}) f - \sum_{E \in \mathcal{E}} (\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \varepsilon \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon}) \nabla \nabla_E \bar{u} + \operatorname{div}\left(\sum_{E \in \mathcal{E}} \varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \otimes \nabla \nabla_E \bar{u}\right). \quad (3.22)$$

Combining equations (3.1) and (3.2) indeed yields

$$-\triangle w_\varepsilon^\omega + \nabla Q_\varepsilon^\omega = \triangle \bar{u} - \nabla \bar{P} - \nabla(\bar{\boldsymbol{b}} : \mathrm{D}(\bar{u})) + f \mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} - \sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \delta_{\varepsilon \partial I_n^\omega} \sigma(u_\varepsilon^\omega, P_\varepsilon^\omega) \nu$$

$$+ \sum_{E \in \mathcal{E}} \nabla_E \bar{u} \sum_n \delta_{\varepsilon \partial I_n^\omega} \sigma\big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big) \nu$$

$$- \sum_{E \in \mathcal{E}} (\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon}) \nabla \nabla_E \bar{u} + \sum_{E \in \mathcal{E}} \nabla \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \nabla \nabla_E \bar{u} + \varepsilon \sum_{E \in \mathcal{E}} \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \triangle \nabla_E \bar{u},$$

and the claim follows after inserting the equation for $\bar{u}$ and recombining the last two right-hand side terms.

Substep 8.3. Corrector result for the pressure field.
As in Substep 4.2 of the proof of Proposition 2.1, for almost all ω, we can construct a map $\zeta_\varepsilon^\omega \in H_0^1(U)$ such that $\zeta_\varepsilon^\omega|_{\varepsilon I_n^\omega}$ is constant for all $n \in \mathcal{N}_\varepsilon^\omega(U)$ and such that

$$\operatorname{div} \zeta_\varepsilon^\omega = \left(Q_\varepsilon^\omega - \fint_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} Q_\varepsilon^\omega\right) \mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)}, \quad (3.23)$$

$$\|\nabla \zeta_\varepsilon^\omega\|_{\mathrm{L}^2(U)} \lesssim \left\|Q_\varepsilon^\omega - \fint_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} Q_\varepsilon^\omega\right\|_{\mathrm{L}^2(U \setminus \mathcal{I}_\varepsilon^\omega(U))}. \quad (3.24)$$

Testing equation (3.22) with ζ_ε^ω, using the boundary conditions for u_ε^ω at inclusion boundaries, and recalling that ζ_ε^ω is constant on each inclusion εI_n^ω with $n \in \mathcal{N}_\varepsilon^\omega(U)$, we find

$$\int_U \nabla \zeta_\varepsilon^\omega : \nabla w_\varepsilon^\omega - \int_U (\operatorname{div} \zeta_\varepsilon^\omega) Q_\varepsilon^\omega = \int_U D(\zeta_\varepsilon^\omega) : 2(\bar{\boldsymbol{B}} - \operatorname{Id}) D(\bar{u}) + \int_U (\operatorname{div} \zeta_\varepsilon^\omega) \bar{\boldsymbol{b}} : D(\bar{u})$$

$$+ \sum_{E \in \mathcal{E}} \sum_n \int_{\varepsilon \partial I_n^\omega} (\nabla_E \bar{u}) \zeta_\varepsilon^\omega \cdot \sigma\big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\nu$$

$$+ \int_U (\lambda - \mathbb{1}_{\mathcal{I}_\varepsilon^\omega(U)}) \zeta_\varepsilon^\omega \cdot f - \sum_{E \in \mathcal{E}} \int_U (\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon}) \zeta_\varepsilon^\omega \cdot \nabla \nabla_E \bar{u} - \int_U \nabla \zeta_\varepsilon^\omega : \sum_{E \in \mathcal{E}} \varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) \otimes \nabla \nabla_E \bar{u}.$$

In view of properties (3.23) and (3.24) of the test function ζ_ε^ω, we deduce after reorganizing the terms,

$$\int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \Big(Q_\varepsilon^\omega - \fint_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} Q_\varepsilon^\omega \Big)^2 \lesssim \sum_{j=1}^5 T_{\varepsilon,j}^\omega, \qquad (3.25)$$

where

$$T_{\varepsilon,1}^\omega := \int_U |\nabla w_\varepsilon^\omega|^2,$$

$$T_{\varepsilon,2}^\omega := \Big| \int_U (\lambda - \mathbb{1}_{\mathcal{I}_\varepsilon^\omega(U)}) \zeta_\varepsilon^\omega \cdot f \Big| + \sum_{E \in \mathcal{E}} \Big| \int_U (\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \varepsilon \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon}) \zeta_\varepsilon^\omega \cdot \nabla \nabla_E \bar{u} \Big|,$$

$$T_{\varepsilon,3}^\omega := \sum_{E \in \mathcal{E}} \int_U \varepsilon |\psi_E^\omega(\tfrac{\cdot}{\varepsilon})| |\nabla \zeta_\varepsilon^\omega| |\nabla \nabla_E \bar{u}|,$$

$$T_{\varepsilon,4}^\omega := \sum_{E \in \mathcal{E}} \sum_{n \notin \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} |\nabla_E \bar{u}| |\zeta_\varepsilon^\omega| \big| \sigma\big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big) \big|,$$

$$T_{\varepsilon,5}^\omega := \Big| \int_U D(\zeta_\varepsilon^\omega) : 2(\bar{\boldsymbol{B}} - \operatorname{Id}) D(\bar{u}) + \int_U (\operatorname{div} \zeta_\varepsilon^\omega) \bar{\boldsymbol{b}} : D(\bar{u})$$

$$+ \sum_{E \in \mathcal{E}} \sum_{n \in \mathcal{N}_\varepsilon^\omega(U)} \int_{\varepsilon \partial I_n^\omega} (\nabla_E \bar{u}) \zeta_\varepsilon^\omega \cdot \sigma\big(\varepsilon \psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\nu \Big|.$$

We successively estimate these different terms. First, the corrector result for the velocity field in Step 8.1 yields $T_{\varepsilon,1}^\omega \to 0$ for almost all ω. We turn to the second term $T_{\varepsilon,2}^\omega$. In view of (3.3) and of the boundedness statement of Proposition 2.1(iii), with the regularity of $\bar{u}$, we deduce that for almost all ω the pressure Q_ε^ω is bounded in $L^2(U)$ uniformly in ε, hence in view of (3.24) the test function ζ_ε^ω is bounded in $H_0^1(U)$. For almost all ω, by weak compactness, there exists $\bar{\zeta}^\omega \in H_0^1(U)^d$ such that $\zeta_\varepsilon^\omega \rightharpoonup \bar{\zeta}^\omega$ in $H_0^1(U)$ along some subsequence (not relabelled), hence also $\zeta_\varepsilon^\omega \to \bar{\zeta}^\omega$ in $L^2(U)$ by Rellich's theorem. Combining this strong convergence with the ergodic theorem in form of $(\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \varepsilon \mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon}) \rightharpoonup 0$ in $L^2(U)$ (cf. Proposition 2.1(iii)) and in form of $\mathbb{1}_{\mathcal{I}_\varepsilon^\omega(U)} \rightharpoonup \lambda \mathbb{1}_U$ weakly-* in $L^\infty(U)$, together with the regularity of $\bar{u}$, we deduce $T_{\varepsilon,2}^\omega \to 0$ for almost all ω. Similarly, in view of the sublinearity of ψ_E (cf. Proposition 2.1(ii)), we find $T_{\varepsilon,3}^\omega \to 0$.

We turn to the boundary term $T_{\varepsilon,4}^\omega$. For $n \notin \mathcal{N}_\varepsilon^\omega(U)$ with $\varepsilon I_n^\omega \cap U \neq \varnothing$, since $\varepsilon \partial I_n^\omega$ is at distance at most ε from ∂U, on which ζ_ε^ω vanishes, we deduce from a trace estimate,

$$\|\zeta_\varepsilon^\omega\|_{L^2(\varepsilon \partial I_n^\omega \cap U)} \lesssim \varepsilon^{\frac{1}{2}} \|\nabla \zeta_\varepsilon^\omega\|_{L^2(\varepsilon(I_n^\omega + B) \cap U)},$$

hence by Cauchy-Schwarz' inequality,

$$T_{\varepsilon,4}^\omega \lesssim \|\nabla\bar{u}\|_{\mathrm{L}^\infty(U)}\|\nabla\zeta_\varepsilon^\omega\|_{\mathrm{L}^2(U)} \sum_{E\in\mathcal{E}} \Big(\sum_{n\notin\mathcal{N}_\varepsilon^\omega(U)} \varepsilon \int_{\varepsilon\partial I_n^\omega} \big|\sigma\big(\varepsilon\psi_E(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\big|^2 \Big)^{\frac{1}{2}}.$$

As in the proof of (3.9), we appeal to a trace estimate and to the regularity theory for the Stokes equation near a boundary in the form

$$\int_{\partial I_n^\omega} |\nabla\psi_E^\omega|^2 + |\Sigma_E^\omega|^2 \lesssim \|(\nabla\psi_E^\omega, \Sigma_E^\omega)\|_{\mathrm{L}^2(I_n^\omega + \frac{\delta}{2}B)}^2,$$

so that the above yields

$$T_{\varepsilon,4}^\omega \lesssim \|\nabla\bar{u}\|_{\mathrm{L}^\infty(U)}\|\nabla\zeta_\varepsilon^\omega\|_{\mathrm{L}^2(U)} \sum_{E\in\mathcal{E}} \Big(\int_{(\partial U)+3\varepsilon B} |E|^2 + |\nabla\psi_E^\omega(\tfrac{\cdot}{\varepsilon})|^2 + |(\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}^\omega})(\tfrac{\cdot}{\varepsilon})|^2 \Big)^{\frac{1}{2}},$$

where the right-hand side converges to 0 for almost all ω as a consequence of the ergodic theorem, cf. Proposition 2.1(iii).

It remains to estimate $T_{\varepsilon,5}^\omega$, and we use the short-hand notation

$$J_\varepsilon^\omega := -\sum_{E\in\mathcal{E}} \sum_{n\in\mathcal{N}_\varepsilon(U)} \int_{\varepsilon\partial I_n^\omega} (\nabla_E\bar{u})\,\zeta_\varepsilon^\omega \cdot \sigma\big(\varepsilon\psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\nu.$$

As shown in Step 5, in view of the boundary conditions for ψ_E at the inclusion boundaries, together with the regularity of $\bar{u}$, an approximation argument for $\nabla_E\bar{u}$ leads to

$$\lim_{\varepsilon\downarrow 0} \Big| J_\varepsilon^\omega + \sum_{E\in\mathcal{E}} \sum_{n\in\mathcal{N}_\varepsilon^\omega(U)} \Big(\fint_{\varepsilon I_n^\omega} \zeta_\varepsilon^\omega \nabla_i\nabla_E\bar{u} \Big)$$
$$\cdot \Big(\int_{\varepsilon\partial I_n^\omega} (x - \varepsilon x_n^\omega)_i\, \sigma\big(\varepsilon\psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\nu \Big) \Big| = 0. \quad (3.26)$$

Applying Cauchy-Schwarz' inequality in the form

$$\sum_{n\notin\mathcal{N}_\varepsilon^\omega(U)} \Big| \Big(\fint_{\varepsilon I_n^\omega} \zeta_\varepsilon^\omega \nabla_i\nabla_E\bar{u} \Big) \cdot \Big(\int_{\varepsilon\partial I_n^\omega} (x - \varepsilon x_n^\omega)_i\, \sigma\big(\varepsilon\psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\nu \Big) \Big|$$
$$\lesssim \|\nabla^2\bar{u}\|_{\mathrm{L}^\infty(U)}\|\zeta_\varepsilon^\omega\|_{\mathrm{L}^2(U)} \Big(\sum_{n\notin\mathcal{N}_\varepsilon^\omega(U)} \varepsilon \int_{\varepsilon\partial I_n^\omega} \big|\sigma\big(\varepsilon\psi_E^\omega(\tfrac{\cdot}{\varepsilon}) + Ex, \Sigma_E^\omega(\tfrac{\cdot}{\varepsilon})\big)\big|^2 \Big)^{\frac{1}{2}},$$

and noting that the estimate on $T_{\varepsilon,4}^\omega$ above ensures that the right-hand side converges to 0, we deduce that the restriction to $n\in\mathcal{N}_\varepsilon^\omega(U)$ can be removed from the sum in (3.26). In terms of the random field Z_E defined in (3.14), we are thus led to

$$\lim_{\varepsilon\downarrow 0} \Big| J_\varepsilon^\omega - \sum_{E\in\mathcal{E}} \int_U (\zeta_\varepsilon^\omega \otimes \nabla\nabla_E\bar{u}) : Z_E^\omega(\tfrac{\cdot}{\varepsilon}) \Big| = 0.$$

Appealing to the ergodic theorem for Z_E, to the identification of $\mathbb{E}[Z_E]$ in Step 6, and to the strong convergence $\zeta_\varepsilon^\omega \to \bar{\zeta}^\omega$ in $\mathrm{L}^2(U)$, together with the regularity of $\bar{u}$, we deduce

$$\lim_{\varepsilon\downarrow 0} \Big| J_\varepsilon^\omega - \int_U \mathrm{D}(\bar{\zeta}^\omega) : 2(\bar{\boldsymbol{B}} - \mathrm{Id})\,\mathrm{D}(\bar{u}) - \int_U (\mathrm{div}\,\bar{\zeta}^\omega)\,\bar{\boldsymbol{b}} : \mathrm{D}(\bar{u}) \Big| = 0,$$

that is, $T_{\varepsilon,5}^\omega \to 0$. We conclude that the whole right-hand side in (3.25) converges to 0 for almost all ω, and the corrector result follows.

Substep 8.4. Relaxing the regularity assumption.

Assume that $f \in \mathrm{L}^p(U)$ for some $p > d$ and note that in view of the regularity theory for the homogenized Stokes equation (1.3) in form of [15, Lemma IV.6.1] this implies $\bar{u} \in W_0^{2,p}(U)^d$ and $\bar{P} \in W^{1,p}(U)$. Choosing an approximating sequence $(f^r)_r \subset C_b^\infty(U)$ with $f^r \to f$ in $\mathrm{L}^p(U)$ as $r \downarrow 0$, we deduce by linearity that the corresponding solution $(\bar{u}^r, \bar{P}^r)$ of the homogenized equation satisfies $\bar{u}^r \to \bar{u}$ in $W^{2,p}(U)$, hence $\bar{u}^r \to \bar{u}$ in $W^{1,\infty} \cap W^{2,d}(U)$ by the Sobolev embedding. In addition, in view of the energy estimate (3.3), the corresponding solution $(u_\varepsilon^{r,\omega}, P_\varepsilon^{r,\omega})$ of (1.2) satisfies

$$\sup_{\varepsilon>0} \int_U |\nabla(u_\varepsilon^{r,\omega} - u_\varepsilon^\omega)|^2 + \sup_{\varepsilon>0} \int_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} |P_\varepsilon^{r,\omega} - P_\varepsilon^\omega|^2 \lesssim \int_U |f^r - f|^2 \to 0, \qquad (3.27)$$

as $r \downarrow 0$. Since for fixed $r > 0$ the approximation f^r is smooth, the regularity theory for the homogenized Stokes equation ensures that $\bar{u}^r$ belongs at least to $W^{3,\infty}(U)^d$, hence the above Steps 8.1–8.3 show that the corrector results indeed hold for the r-approximations. Since $\nabla \psi_E^\omega(\frac{\cdot}{\varepsilon})$ and $(\Sigma_E^\omega \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}^\omega})(\frac{\cdot}{\varepsilon})$ are bounded in $\mathrm{L}^2(U)$ for almost all ω (cf. Proposition 2.1(iii)), and since the Sobolev embedding also ensures the boundedness of $\varepsilon \psi_E^\omega(\frac{\cdot}{\varepsilon})$ in $\mathrm{L}^{2d/(d-2)}(U)$, the above convergences precisely allow to get rid of approximations.

We conclude with the argument for the weak convergence of the pressure for $f \in \mathrm{L}^2(U)$. Choose an approximating sequence $(f^r)_r \subset C_b^\infty(U)$ with $f^r \to f$ in $\mathrm{L}^2(U)$, and denote by $(u_\varepsilon^{r,\omega}, P_\varepsilon^{r,\omega})$ and by $(\bar{u}^r, \bar{P}^r)$ the corresponding solutions of (1.2) and of (1.3). Starting from the corrector result for the pressure for the regularized data f^r, appealing to the ergodic theorem of Proposition 2.1(iii), and noting that the weak-* convergence $\mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \rightharpoonup 1 - \lambda$ in $\mathrm{L}^\infty(U)$ entails $\int_U (\bar{P}^r + \bar{\boldsymbol{b}} : \mathrm{D}(\bar{u}^r)) \mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \to (1 - \lambda) \int_U (\bar{P}^r + \bar{\boldsymbol{b}} : \mathrm{D}(\bar{u}^r)) = 0$, we obtain for all r, for almost all ω,

$$(P_\varepsilon^{r,\omega} - \bar{P}^r - \bar{\boldsymbol{b}} : \mathrm{D}(\bar{u}^r)) \mathbb{1}_{U \setminus \mathcal{I}_\varepsilon^\omega(U)} \rightharpoonup 0, \quad \text{weakly in } \mathrm{L}^2(U), \quad \text{as } \varepsilon \downarrow 0.$$

Next, the same argument as above yields $(\bar{u}^r, \bar{P}^r) \to (\bar{u}, \bar{P})$ in $H_0^1(U) \times \mathrm{L}^2(U)$, as well as (3.27), which allows to get rid of approximations in this weak convergence result. $\square$

REFERENCES

[1] G. Allaire. Homogenization of the Stokes flow in a connected porous medium. *Asymptotic Anal.*, 2(3):203–222, 1989.

[2] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. *Arch. Rational Mech. Anal.*, 113(3):209–259, 1990.

[3] Y. Almog and H. Brenner. Global homogenization of a dilute suspension of sphere. Unpublished manuscript, 1998.

[4] H. Ammari, P. Garapon, H. Kang, and H. Lee. Effective viscosity properties of dilute suspensions of arbitrarily shaped particles. *Asymptot. Anal.*, 80(3-4):189–211, 2012.

[5] A. Basson and D. Gérard-Varet. Wall laws for fluid flows at a boundary with random roughness. *Comm. Pure Appl. Math.*, 61(7):941–987, 2008.

[6] G. K. Batchelor. Sedimentation in a dilute dispersion of spheres. *J. Fluids Mech.*, 52(2):245–268, 1972.

[7] G. L. Batchelor and J. T. Green. The determination of the bulk stress in suspension of spherical particles to order c^2. *J. Fluid Mech.*, 56:401–427, 1972.

[8] A. Yu. Beliaev and S. M. Kozlov. Darcy equation for random porous media. *Comm. Pure Appl. Math.*, 49(1):1–34, 1996.

[9] D. Cioranescu and J. Saint Jean Paulin. Homogenization in open sets with holes. *J. Math. Anal. Appl.*, 71(2):590–607, 1979.

[10] M. Duerinckx and A. Gloria. Analyticity of homogenized coefficients under Bernoulli perturbations and the Clausius-Mossotti formulas. *Arch. Ration. Mech. Anal.*, 220(1):297–361, 2016.

[11] M. Duerinckx and A. Gloria. Quantitative homogenization theory for suspensions in steady Stokes flow. In preparation.

[12] M. Duerinckx and A. Gloria. Sedimentation of random suspensions and the effect of hyperuniformity. Preprint, arXiv:2004.03240.

[13] M. Duerinckx and A. Gloria. On Einstein's effective viscosity formula. Preprint, 2020.

[14] A. Einstein. Eine neue Bestimmung der Moleküldimensionen. *Ann. Phys.*, 19(2):289–306, 1906.

[15] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems.* Springer Monographs in Mathematics. Springer, New York, second edition, 2011.

[16] D. Gérard-Varet and M. Hillairet. Analysis of the viscosity of dilute suspensions beyond Einstein's formula. Preprint, arXiv:1905.08208.

[17] D. Gérard-Varet and A. Mecherbet. On the correction to Einstein's formula for the effective viscosity. Preprint, arXiv:2004.05601.

[18] A. Giunti and R. M. Höfer. Homogenization for the Stokes equations in randomly perforated domains under almost minimal assumptions on the size of the holes. Preprint, arXiv:1809.04491.

[19] A. Gloria. A scalar version of the Caflisch-Luke paradox. Preprint, arXiv:1907.08182.

[20] B. M. Haines and A. L. Mazzucato. A proof of Einstein's effective viscosity for a dilute suspension of spheres. *SIAM J. Math. Anal.*, 44(3):2120–2145, 2012.

[21] V. V. Jikov. Averaging of functionals in the calculus of variations and elasticity, *Math. USSR, Izvestiya*, 29:33–66, 1987.

[22] V. V. Jikov. Some problems of extension of functions arising in connection with the homogenization theory, *Diff. Uravnenia*, 26(1):39–51, 1990.

[23] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals.* Springer-Verlag, Berlin, 1994.

[24] B. Niethammer and R. Schubert. A local version of Einstein's formula for the effective viscosity of suspensions. Preprint, arXiv:1903.08554.

[25] K. Nunan and J. Keller. Effective viscosity of a periodic suspension. *J. Fluid Mech.*, 142:269–287, 1984.

[26] G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam, 1981.

[27] E. Sánchez-Palencia. *Nonhomogeneous media and vibration theory*, volume 127 of *Lecture Notes in Physics*. Springer-Verlag, Berlin-New York, 1980.

[28] L. Tartar. Incompressible fluid flow in a porous medium: convergence of the homogenization process. In *Nonhomogeneous media and vibration theory (E. Sánchez-Palencia)*, volume 127 of *Lecture Notes in Physics*, pages 368–377. Springer-Verlag, Berlin-New York, 1980.

[29] L. Tartar. *The general theory of homogenization*, volume 7 of *Lecture Notes of the Unione Matematica Italiana*. Springer-Verlag, Berlin; UMI, Bologna, 2009.

(Mitia Duerinckx) Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, 91405 Orsay, France & Université Libre de Bruxelles, Département de Mathématique, 1050 Brussels, Belgium

Email address: mduerinc@ulb.ac.be

(Antoine Gloria) Sorbonne Université, CNRS, Université de Paris, Laboratoire Jacques-Louis Lions, 75005 Paris, France & Université Libre de Bruxelles, Département de Mathématique, 1050 Brussels, Belgium

Email address: gloria@ljll.math.upmc.fr