ON THE LOWER SPECTRUM OF HETEROGENEOUS ACOUSTIC OPERATORS

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Abstract. This work relates quantitatively homogenization to Anderson localization for heterogeneous acoustic operators: we draw consequences on the spatial spreading of eigenstates in the lower spectrum (if any) from the long-time homogenization of the wave equation, through dispersive estimates. This gives an alternative proof (avoiding Floquet theory) that the lower spectrum of the acoustic operator is purely absolutely continuous in case of periodic coefficients, and it further provides nontrivial quantitative lower bounds on the spatial spreading of potential eigenstates in case of quasiperiodic and random coefficients.

1. Introduction

Given a coefficient field $A:\mathbb{R}^d\to\mathbb{R}^{d\times d}$ that is symmetric and uniformly elliptic in the sense of

$$e\cdot A(x)e\ge \tfrac{1}{C_0}|e|^2,\qquad |A(x)e|\le|e|,\qquad \text{for all } x,e\in\mathbb{R}^d, \tag{1.1}$$

for some $C_0\ge1$, we consider the associated acoustic operator

$$H_a:=-\nabla\cdot A\nabla,\qquad \text{on } \mathrm{L}^2(\mathbb{R}^d).$$

Depending on properties of A, the spectrum of H_a might be of different natures, and not much is actually known in general, except in two particular settings:

— If the coefficient field A is periodic, then the lower spectrum of H_a is absolutely continuous in any dimension $d\ge1$ (see e.g. [4, Theorem 1.5]). In fact, the whole spectrum of H_a is conjectured to be continuous in that case provided that A is smooth enough, see [20, Conjecture 6.13].

— If the coefficient field A is random (at least, in case of the so-called random displacement model [23, eqn (1.3)]), and if the space dimension is $d=1$, then the operator H_a has dense pure point spectrum with exponentially decaying eigenstates [23].

Compared to the case of Schrödinger operators $-\triangle+V$ with heterogeneous potential field V, the lower spectrum of H_a is actually special as the constant function 1 is always an extended state at energy $0=\inf\sigma(H_a)$. In this work, we focus on the spectrum of H_a close to this critical energy 0 and show that specific techniques such as homogenization theory can be used to bring more detailed information in that spectral region. More precisely, our main result below provides a nontrivial control on the spatial spreading of the mass density of eigenstates in the lower spectrum if they exist (as well as the spreading of their energy density, cf. Remark 1.2). This is expressed as a lower bound on the spatial 'width' of a normalized eigenstate ψ if it exists, which we define as

$$\ell_\eta(\psi)\;:=\;\inf\big\{r\ge0:\|\psi\|_{\mathrm{L}^2(B_r)}\ge1-\eta\big\},\qquad \text{given } \eta<\tfrac12. \tag{1.2}$$

Our results are related as follows with the previous results mentioned above in the periodic and random settings:

— In the periodic setting, a way to rule out the existence of eigenstates at low energy is for instance by proving that, if ψ was a normalized eigenstate at low enough energy, then its width would necessarily diverge, $\ell_\eta(\psi) = \infty$, which would indeed contradict the normalization of ψ. The absence of eigenstates then implies the absolute continuity of the lower spectrum using that singular continuous spectrum is always excluded by basic Floquet theory, e.g. [20, Section 6.3].

— In the 1D random setting, it is shown in [23] that almost surely there exists a dense subset $\Lambda \subset \sigma(H_a)$ and a family $(\psi_\lambda)_{\lambda\in\Lambda}$ of exponentially decaying eigenstates. Moreover, as 0 is a critical energy, the Lyapunov exponent at 0 vanishes and there holds $\lim_{\lambda\downarrow 0,\lambda\in\Lambda} \ell(\psi_\lambda) = \infty$. Our result below further provides a convergence rate for $\ell_\eta(\psi_\lambda)$, which reads in this 1D case as $\ell_\eta(\psi_\lambda) \ge \lambda^{\varepsilon-\frac23}$ for $\lambda \ll_{\varepsilon,\eta} 1$, for any $\varepsilon > 0$ and $\eta < \frac12$.

Our approach is quite robust and applies to essentially any kind of heterogeneity for the coefficient field A under suitable statistical spatial homogeneity assumptions — which we illustrate by considering periodic, quasi-periodic, and random coefficient fields in any dimension. In the random setting, we only treat the strongest mixing conditions for shortness.[1]

Theorem 1.1. *Let $d \ge 1$ and let the coefficient field $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be symmetric and uniformly elliptic in the sense of* (1.1)*.*

(i) Periodic setting:
Assume that A is periodic. Then there exists $\lambda_0 > 0$ (only depending on d, C_0) such that the operator H_a admits no eigenvalue in $[0, \lambda_0]$.

(ii) Quasiperiodic setting:
Assume that A takes the form $A(x) = A_0(Fx)$ for some frequency matrix $F \in \mathbb{R}^{M\times d}$, with $M > d$, and for some lifted map $A_0 : \mathbb{R}^M \to \mathbb{R}^{d\times d}$ that is periodic on $\mathbb{R}^M$ and is Gevrey-regular. Further assume that the frequency matrix F satisfies a Diophantine condition, that is, for some $C, \gamma > 0$,

$$|Fz| \ge \tfrac{1}{C}|z|^{-\gamma}, \qquad \text{for all } z \in \mathbb{Z}^M \setminus \{0\}.$$

Then, for all $\eta < \frac12$, there exists $K_\eta \ge 1$ with the following property: if H_a has an eigenvalue $\lambda \in [0, K_\eta^{-1}]$, any associated normalized eigenstate ψ_λ satisfies

$$\ell_\eta(\psi_\lambda) \ge \exp(\lambda^{-\theta}),$$

where we recall the notation (1.2) *for the spatial width of ψ_λ, and where the exponent $\theta > 0$ only depends on γ and on the Gevrey regularity of A_0.*

(iii) Random setting:
Assume that $A = \tilde A(\cdot, \omega_0)$ is a realization of a random field $\tilde A$: more precisely, consider a probability space $(\Omega, \mathbb{P})$ and a statistically spatially homogeneous, uniformly elliptic, random field $\tilde A : \Omega \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$, in the following sense,

— Random field: *$\tilde A$ is a jointly measurable map $\mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times d}$ such that for all $x \in \mathbb{R}^d$ the function $\tilde A(x, \cdot) : \Omega \to \mathbb{R}^{d\times d}$ is measurable.*

— Symmetric and uniformly elliptic: *for $\mathbb{P}$-almost all ω, the realization $\tilde A(\cdot, \omega)$ is a symmetric matrix field and satisfies the uniform ellipticity condition* (1.1)*.*

[1] Under weaker mixing conditions, we would be led to weaker lower bounds on $\ell_\eta(\psi_\lambda)$, and we refer to [16, 11] for the corresponding homogenization tools.

— Statistical spatial homogeneity: *the finite-dimensional law of A is shift-invariant, that is, for any finite set $E \subset \mathbb{R}^d$ the law of $\{\tilde{A}(x+y,\cdot)\}_{x\in E}$ does not depend on the shift $y \in \mathbb{R}^d$.*

Further assume that $\tilde{A}$ satisfies strong enough mixing assumptions, for instance:[2]

— Finite range of dependence: *for all finite subsets $E, F \subset \mathbb{R}^d$ with $\operatorname{dist}(E,F) \geq 1$, the families of random variables $(\tilde{A}(x,\cdot))_{x\in E}$ and $(\tilde{A}(x,\cdot))_{x\in F}$ are independent.*

Then, for all $\varepsilon \ll 1$ and $\eta < \frac{1}{2}$, there exists a random variable $\mathcal{K}_{\varepsilon,\eta} \geq 1$ with finite stretched exponential moments,

$$\mathbb{E}\big[\exp(C^{-1}\mathcal{K}_{\varepsilon,\eta}^{1/C})\big] < \infty, \qquad \text{for some } C < \infty \text{ (depending on } d, C_0, \varepsilon, \eta),$$

such that the following property holds: for $\mathbb{P}$-almost all ω_0, if for the coefficient field $A = \tilde{A}(\cdot,\omega_0)$ the operator H_a has an eigenvalue $\lambda \in [0, \mathcal{K}_{\varepsilon,\eta}(\omega_0)^{-1}]$, any associated normalized eigenstate ψ_λ satisfies

$$\ell_\eta(\psi_\lambda) \;\geq\; \left\{ \begin{array}{lll} \lambda^{\varepsilon-\frac{2}{3}} & : & d=1, \\ \lambda^{\varepsilon-\frac{1}{2}(\lfloor\frac{d}{2}\rfloor+1)} & : & d>1, \end{array} \right.$$

where $\lfloor\frac{d}{2}\rfloor$ stands for the integer part of $\frac{d}{2}$. $\Diamond$

Remark 1.2 (Energy distribution)**.** In the above result, in the definition (1.2) of the width of eigenstates, the local mass $\|\psi\|_{\mathrm{L}^2(B_r)}$ can be replaced by the local (rescaled) energy: more precisely, for a normalized eigenstate ψ_λ at energy λ, all the same estimates hold for $\ell_\eta(\psi_\lambda)$ replaced by

$$\ell'_\eta(\psi_\lambda) \;:=\; \inf\Big\{ r \geq 0 : \tfrac{1}{\sqrt{\lambda}}\|\sqrt{A}\nabla\psi_\lambda\|_{\mathrm{L}^2(B_r)} \geq 1-\eta \Big\}. \tag{1.3}$$

(Note that the rescaling of the local energy ensures $\frac{1}{\lambda}\int_{\mathbb{R}^d} \nabla\psi_\lambda \cdot A\nabla\psi_\lambda = 1$.) In other words, in our description, the spatial spreading of the mass density and of the energy density of eigenstates are comparable. A proof is included in Step 7 of Section 3. $\Diamond$

Before we describe the general strategy of the proof, let us further comment on the above result. In the periodic setting, item (i) is well-known to the expert and is part of the folklore of Floquet theory. Another proof of the absence of localized eigenstates in the lower spectrum was recently given in [4] by Armstrong, Kuusi, and Smart: it is based on periodic homogenization theory in form of a large-scale analyticity result and it significantly reduces the role of Floquet theory (which is only used in form of a general result by Kuchment that allows to pass from a statement valid for exponentially decaying eigenstates to a statement for any L^2 eigenstates, see [20, Theorem 6.15]). In the present work, our proof of item (i) also relies on homogenization theory (this time for the associated wave equation), but the main novelty is that it completely bypasses the use of Floquet theory and that it is quantitative. This latter point is crucial as it allows to extend the argument to quasiperiodic and random coefficient fields. Next, the results of items (ii) and (iii) are new and provide quantitative lower bounds on the spatial spreading of the

[2]The same result holds if, instead of the finite range of dependence, we rather assume that the random field $\tilde{A}$ has a Gaussian structure with integrable covariance — more precisely, this means to assume that $\tilde{A}$ can be written as $\tilde{A}(x,\omega) = A_0(G(x,\omega))$, where $G : \mathbb{R}^d \times \Omega \to \mathbb{R}^\kappa$ is a statistically spatially homogeneous Gaussian field, for some $\kappa \geq 1$, where $A_0 \in C_b^1(\mathbb{R}^\kappa; \mathbb{R}^{d\times d})$, and where we require the covariance function of G to be written as $\mathbb{E}\left[G(x,\cdot)\otimes G(y,\cdot)\right] = (C_0 * C_0)(x-y)$ for some even convolution kernel $C_0 : \mathbb{R}^d \to \mathbb{R}^{\kappa\times\kappa}$ satisfying the integrability condition $\int_{\mathbb{R}^d}(\sup_{B(x)}|C_0|)\,dx < \infty$.

mass (and energy) density of potential eigenstates in the lower spectrum. This is new even in the 1D case of [23], where only the qualitative result $\ell_\eta(\psi_\lambda) \to \infty$ was known as $\lambda \downarrow 0$. A particular advantage of our result is to hold for any eigenstate in $\mathrm{L}^2(\mathbb{R}^d)$ without the need to make further decay or regularity assumptions.

Let us now explain the main idea of the proof. We adopt a dynamical point of view and build on recent advances on the long-time homogenization of wave equations [8, 7, 2, 11], which show that the solution of the wave equation associated with the heterogeneous acoustic operator H_a can be described for quite long times by the solution of some constant-coefficient 'homogenized' wave equation. In order to derive spectral information from long-time homogenization results, our argument is the following: If H_a admits an eigenvalue, we consider the wave starting from the associated eigenstate, which of course consists of time oscillations around the eigenstate. Now, if the eigenvalue is close to 0, homogenization theory can be applied and ensures that the wave must behave for long times like the solution of an associated homogenized wave equation — which has constant coefficients and thus satisfies dispersive estimates. From these two pieces of information on the wave, we deduce that the eigenstate cannot be too localized: most of its mass (and energy) density cannot be supported in a ball of radius smaller than the maximal timescale up to which homogenization holds. For periodic coefficients, as homogenization holds up to arbitrarily long times, no eigenvalue can actually exist in the lower spectrum, thus proving item (i). For quasiperiodic and random coefficients, homogenization ultimately breaks down on long timescales, leaving the possibility of existence of eigenstates in the lower spectrum, and we are led to lower bounds on the spatial spreading of their mass (and energy) density as stated in items (ii) and (iii).

In other words, Theorem 1.1 relates quantitatively homogenization to Anderson localization for the acoustic operator in a neighborhood of the critical lowest energy 0. This result is essentially sharp in the sense that it is the best result one can obtain using homogenization techniques. The proof relies on the accurate description of the mass density of the wave by that of the solution of an associated homogenized wave equation, for which dispersive estimates hold.

The rest of the article is organized as follows. In Section 2, we recall the needed input from the recent results on the long-time homogenization of the wave equation, and we then proceed to the proof of Theorem 1.1 in Section 3.

Notation.

— We denote by $C \geq 1$ any constant that only depends on the dimension d and on the ellipticity constant C_0 in (1.1). We use the notation $\lesssim$ (resp. $\gtrsim$) for $\leq C\times$ (resp. $\geq \frac{1}{C}\times$) up to such a multiplicative constant C. We write $\ll$ (resp. $\gg$) for $\leq C\times$ (resp. $\geq C\times$) up to a sufficiently large multiplicative constant C. We add subscripts to indicate dependence of constants on other parameters.
— The ball centered at x of radius r in $\mathbb{R}^d$ is denoted by $B_r(x)$, and we set $B(x) = B_1(x)$, $B_r = B_r(0)$, and $B = B_1(0)$.
— Recall that $\lfloor r \rfloor$ denotes the integer part of r, that is, the largest integer smaller or equal to r, and we also denote by $\lceil r \rceil$ the smallest integer larger or equal to r.
— We set $\langle s \rangle := (1 + |s|^2)^{1/2}$.
— $\mathbb{N}$ stands for the set of positive integers.

2. Input from higher-order homogenization theory

Homogenization theory aims at describing the behavior of solutions of PDEs with heterogeneous coefficients in the regime when the typical spatial scale of the coefficients is much smaller than the spatial scale of external forces and initial data: in such a regime, one expects the coefficients to average out in some sense, so that solutions of the PDE would be described to leading order by solutions of a corresponding 'homogenized' constant-coefficient PDE (possibly of a different type). We refer the reader to [8] for a systematic treatment of homogenization of various PDEs with periodic coefficients.

For the purpose of the present work, we shall consider the wave equation associated with the heterogeneous acoustic operator $H_a = -\nabla \cdot A\nabla$,

$$\begin{cases} \partial_t^2 u_\lambda = \nabla \cdot A\nabla u_\lambda, \\ u_\lambda|_{t=0} = u_\lambda^\circ, \\ \partial_t u_\lambda|_{t=0} = 0, \end{cases} \tag{2.1}$$

with some (smooth) initial condition $u_\lambda^\circ \in \mathrm{L}^2(\mathbb{R}^d)$. If the initial condition is localized on very low frequencies $0 < \sqrt{\lambda} \ll 1$, then there is a scale separation with respect to variations of A on unit scale and homogenization is expected to hold under suitable statistical spatial homogeneity assumptions on the coefficient field A (for instance if A is periodic, quasi-periodic, or if A is a typical realization of a random field with statistical spatial homogeneity as in Theorem 1.1(iii)). More precisely, as shown e.g. in [9] in the periodic setting, it is a standard result of homogenization theory that u_λ is then close to the solution $\bar u_\lambda$ of the homogenized wave equation

$$\begin{cases} \partial_t^2 \bar u_\lambda = \nabla \cdot \bar A\nabla \bar u_\lambda, \\ \bar u_\lambda|_{t=0} = u_\lambda^\circ, \\ \partial_t \bar u_\lambda|_{t=0} = 0, \end{cases} \tag{2.2}$$

on any compact time set $[0,T]$, where the homogenized coefficient $\bar A$ is a constant matrix due to statistical spatial homogeneity. Recall that $\bar A$ only depends on A and is in particular independent of initial data. What is less standard and was only established recently, is the quantification of the maximal timescale up to which u_λ indeed remains close to $\bar u_\lambda$. In the periodic setting, as first understood in [22], this holds only up to times $t \ll \lambda^{-1}$, while on longer timescales the homogenized equation (2.2) is no longer accurate and dispersive corrections need to be added to the homogenized operator $-\nabla \cdot \bar A\nabla$. More precisely, as shown in [7], there exist higher-order constant tensors $\{\bar A^n\}_{n\ge1}$ with $\bar A^1 = \bar A$ such that for all $N \in \mathbb{N}$ the heterogeneous solution u_λ is well-approximated up to times $t \ll \sqrt{\lambda}^{-N-1}$ by the solution $\bar u_{\lambda,N}$ of the corrected homogenized wave equation

$$\begin{cases} \partial_t^2 \bar u_{\lambda,N} = \nabla \cdot \big(\sum_{n=1}^N \bar A^n_{j_1\ldots j_{n-1}} \nabla^{n-1}_{j_1\ldots j_{n-1}}\big)\nabla \bar u_{\lambda,N}, \\ \bar u_{\lambda,N}|_{t=0} = u_\lambda^\circ, \\ \partial_t \bar u_{\lambda,N}|_{t=0} = 0. \end{cases} \tag{2.3}$$

Of course, the maximal timescale up to which the result holds strongly depends on assumptions on A: whereas in the periodic setting one can take any $N \in \mathbb{N}$, the result can only hold up to some maximal exponent N_0 in the random setting due to Anderson localization (at least in 1D, cf. [23]).

In order to motivate the upcoming results, let us quickly describe informally why u_λ might indeed be described by the solution $\bar u_{\lambda,N}$ of a homogenized equation. To this aim, we follow the approach introduced in [7] and further developed in [11], based on so-called "approximate spectral theory". Starting point is to try diagonalizing the acoustic operator $H_a = -\nabla \cdot A\nabla$ at the bottom of its spectrum by means of extended states in form of Bloch waves $x \mapsto e^{ix\cdot\xi}\chi_\xi(x)$, where χ_ξ would have the same type of statistical spatial homogeneity as A. In the periodic case, this is precisely realized by Floquet theory, see e.g. [20], whereas in the random case statistically homogeneous Bloch waves are not expected to exist. With this failure of Floquet theory, we are led to rather look for an approximate diagonalization of H_a. More precisely, we start with the following observation: in the periodic setting, as was shown e.g. in [1], the gradient of the Bloch wave χ_ξ with respect to ξ at 0 coincides with the so-called corrector ϕ from elliptic homogenization theory — that is, the periodic solution (with vanishing average) of

$$-\nabla \cdot A(\nabla\phi + \mathrm{Id}) = 0.$$

In other words, one has a Taylor expansion $\chi_\xi = 1 + i\xi_j\phi_j + O(|\xi|^2)$ for the Bloch wave. As shown in [7], this expansion can be pursued to higher orders and we find

$$\chi_\xi = \sum_{n=0}^{N} (i\xi)^{\otimes n}_{j_1\ldots j_n} \phi^n_{j_1\ldots j_n} + O(|\xi|^{N+1})$$

for all $N \in \mathbb{N}$ in the periodic setting, where the coefficients $\{\phi^n\}_n$ are so-called higher-order (spectral) correctors and satisfy some hierarchy of PDEs. The main idea in [7] was then to replace the use of the Bloch wave χ_ξ by its higher-order expansions for all practical purposes: the advantage is that correctors $\{\phi^n\}_n$ can be defined in settings when Bloch waves do not exist (or are not known to exist) — for instance, all of them can easily be defined in the quasiperiodic setting (under some Diophantine condition), and at least a finite number of them in the random setting (under suitable mixing assumptions). The drawback is of course that these expansions do not exactly diagonalize $H_a = -\nabla \cdot A\nabla$: there is a remainder in the generalized eigenvalue equation (called eigendefect in [7]). Still, this yields an approximate diagonalization of H_a and we are led to an approximation of u_λ by means of a so-called two-scale expansion of the form

$$\sum_{n=0}^{N} \phi^n_{j_1\ldots j_n} \nabla^n_{j_1\ldots j_n} \bar{u}_{\lambda,N}, \tag{2.4}$$

for some (smooth) effective profile $\bar{u}_{\lambda,N}$. Inserting this ansatz into (2.1) easily shows that this profile $\bar{u}_{\lambda,N}$ must solve an homogenized wave equation of the form (2.3). For initial data u°_λ localized on low frequencies $0 < \sqrt{\lambda} \ll 1$, we find $\nabla^n \bar{u}_{\lambda,N} = O(\sqrt{\lambda}^n)$ in $\mathrm{L}^2(\mathbb{R}^d)$, so the above expansion (2.4) coincides with $\bar{u}_{\lambda,N}$ to leading order. This formally justifies why $\bar{u}_{\lambda,N}$ indeed provides a good description of u_λ in $\mathrm{L}^2(\mathbb{R}^d)$. Yet, the whole approach only works provided that we can control the error made by using an approximate diagonalization of H_a, and this depends on the size of the remainder in the eigenvalue equation: the smaller this error, the longer the time the approximate solution $\bar{u}_{\lambda,N}$ remains close to u_λ. If the corrector ϕ^N is a bounded function, the remainder in the eigenvalue equation is essentially $O(|k|^{N+1})$, and then using energy estimates on the wave equation leads us to an approximation error $u_\lambda - \bar{u}_{\lambda,N} = O(\sqrt{\lambda} + t\sqrt{\lambda}^{N+1})$ in $\mathrm{L}^2(\mathbb{R}^d)$. This is the origin for the timescale condition $t \ll \sqrt{\lambda}^{-N-1}$ mentioned above.

The quantitative analysis of corrector equations has been central in modern homogenization, and we have by now very accurate bounds on correctors $\{\phi^n\}_n$ in various settings, building up on [19] in the quasiperiodic setting, and on [17, 15, 21, 5] in the random setting. Before we state these bounds, let us give the precise form of the hierarchy of corrector equations. To cover all settings at once, we formulate these equations in the whole space $\mathbb{R}^d$, while implicitly understanding that we look for periodic solutions in the periodic setting, for quasi-periodic solutions in the quasi-periodic setting, and for statistically spatially homogeneous solutions (when possible) in the random setting. We also denote by $\mathbb{E}[\cdot]$ the 'mean' or infinite-volume average, which is understood as average over the periodicity cell in the periodic setting, average over the underlying high-dimensional periodicity cell in the quasiperiodic setting, and expectation in the random setting. With this notation, correctors $\phi^n = (\phi^n_{j_1\dots j_n})_{1\le j_1,\dots,j_n\le d} \in H^1_{\mathrm{loc}}(\mathbb{R}^d;(\mathbb{R}^d)^n)$ are defined inductively by $\phi^0=1$ and by the following equations, for all $n\ge 1$,

$$\begin{aligned} -\nabla\cdot A\nabla\phi^n_{j_1\dots j_n} \,=\, \nabla\cdot\big(A\phi^{n-1}_{j_1\dots j_{n-1}}e_{j_n}\big)+e_{j_n}\cdot A\big(\nabla\phi^{n-1}_{j_1\dots j_{n-1}}+\phi^{n-2}_{j_1\dots j_{n-2}}e_{j_{n-1}}\big)&\\ -\sum_{m=1}^{n-1}e_{j_n}\cdot\bar{A}^m_{j_1\dots j_{m-1}}\phi^{n-m-1}_{j_m\dots j_{n-2}}e_{j_{n-1}},& \quad (2.5)\end{aligned}$$

where we choose $\phi^n_{j_1\dots j_n}$ with zero mean $\mathbb{E}[\phi^n_{j_1\dots j_n}]=0$ (when possible), and where the constant tensors $\{\bar{A}^m\}_m$ are iteratively chosen to make sure that the right-hand side of (2.5) also has zero mean. More precisely, this amounts to defining for all $m\ge 1$,

$$\bar{A}^m_{j_1\dots j_{m-1}}e_{j_m} \,:=\, \mathbb{E}\big[A\big(\nabla\phi^m_{j_1\dots j_m}+\phi_{j_1\dots j_{m-1}}e_{j_m}\big)\big].$$

(We also implicitly set $\phi^{-1}=0$ for notational convenience.) As usual in homogenization theory, it is convenient to further introduce flux correctors $\sigma^n := (\sigma^n_{j_1\dots j_n})_{1\le j_1,\dots,j_n\le d}\in H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}\times(\mathbb{R}^d)^n)$: for all $n\ge 1$ we define $\sigma^n_{j_1\dots j_n}=\nabla\Phi^n_{j_1\dots j_n}$ where $\Phi^n_{j_1\dots j_n}\in H^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ satisfies

$$\triangle\Phi^n_{j_1\dots j_n}=A\big(\nabla\phi^n_{j_1\dots j_n}+\phi^{n-1}_{j_1\dots j_{n-1}}e_{j_n}\big)-\sum_{m=1}^{n}\bar{A}^m_{j_1\dots j_{m-1}}\phi^{n-m}_{j_m\dots j_{n-1}}e_{j_n}.$$

In particular, if well-defined, we note that this construction ensures for all $n\ge 1$,

$$\nabla\cdot\sigma^n_{j_1\dots j_n}=A\big(\nabla\phi^n_{j_1\dots j_n}+\phi^{n-1}_{j_1\dots j_{n-1}}e_{j_n}\big)-\sum_{m=1}^{n}\bar{A}^m_{j_1\dots j_{m-1}}\phi^{n-m}_{j_m\dots j_{n-1}}e_{j_n}. \qquad (2.6)$$

We also recall that $\bar{A}^n=0$ for n even, if well-defined, cf. [7, 11], which ensures the symmetry of the differential operator in the homogenized wave equation (2.3). Quantitative homogenization theory provides the following optimal corrector estimates; note how the result strongly varies between the periodic, quasiperiodic, and random settings.

Lemma 2.1 (Correctors in periodic setting; e.g. [8, 10])**.** *Assume that the coefficient field $A:\mathbb{R}^d\to\mathbb{R}^{d\times d}$ is symmetric and uniformly elliptic in the sense of* (1.1)*, and is periodic on $Q=[0,1)^d$. Then, for all $n\in\mathbb{N}$, the correctors (ϕ^n,σ^n) are uniquely defined as mean-zero Q-periodic functions and satisfy*

$$\|(\phi^n,\sigma^n)\|_{H^1(Q)}\,\le\,C^n,\qquad |\bar{A}^n|\,\le\,C^n. \qquad \Diamond$$

Lemma 2.2 (Correctors in quasiperiodic setting; e.g. [19, 14, 12])**.** *Assume that the coefficient field $A:\mathbb{R}^d\to\mathbb{R}^{d\times d}$ is symmetric and uniformly elliptic in the sense of* (1.1)*, and takes the form $A(x)=A_0(Fx)$ for some frequency matrix $F\in\mathbb{R}^{M\times d}$ with $M>d$, and for some lifted map $A_0:\mathbb{R}^M\to\mathbb{R}^{d\times d}$ that is periodic on $\mathbb{R}^M$ and Gevrey-regular. Further* *assume that the frequency matrix F satisfies a Diophantine condition, that is, for some $C,\gamma>0$,*

$$|Fz| \,\ge\, \tfrac{1}{C}|z|^{-\gamma}, \qquad \text{for all } z\in\mathbb{Z}^M\setminus\{0\}.$$

Then, for all $n\in\mathbb{N}$, the correctors (ϕ^n,σ^n) are uniquely defined as mean-zero quasiperiodic functions and satisfy

$$\|(\phi^n,\psi^n)\|_{W^{1,\infty}(\mathbb{R}^d)} \,\le\, (Cn^\theta)^n, \qquad |\bar A^n| \,\le\, (Cn^\theta)^n,$$

for an exponent $\theta>0$ depending both on γ and on the Gevrey regularity of A_0. $\Diamond$

Lemma 2.3 (Correctors in random setting; e.g. [18, 16, 5, 3, 13, 11])**.** *Assume that the coefficient field is given by a symmetric, uniformly elliptic, statistically spatially homogeneous random field $A:\mathbb{R}^d\times\Omega\to\mathbb{R}^{d\times d}$, constructed on some probability space $(\Omega,\mathbb{P})$, and further assume that it satisfies strong enough mixing assumptions, as for instance having finite range of dependence, in the sense of Theorem 1.1(iii). Then the corrector gradients $(\nabla\phi^n,\nabla\sigma^n)$ are uniquely defined as mean-zero statistically spatially homogeneous random fields for all $1\le n\le\lceil\frac d2\rceil$, such that the correctors (ϕ^n,σ^n) are themselves mean-zero statistically spatially homogeneous fields for $1\le n<\lceil\frac d2\rceil$, and such that corrector equations with coefficients $A(\cdot,\omega)$ are satisfied by $\{\phi^n(\cdot,\omega),\sigma^n(\cdot,\omega)\}_n$ for $\mathbb{P}$-almost all ω. The homogenized tensors $\bar A^m$ are then well-defined for all $1\le m\le\lceil\frac d2\rceil$. Moreover, for all $\varepsilon>0$ and $\delta>\frac12$, there exists a random variable $\mathcal{K}_{\varepsilon,\delta}$ with finite stretched-exponential moments,*

$$\mathbb{E}\big[\exp(C^{-1}\mathcal{K}_{\varepsilon,\delta}^{1/C})\big]<\infty, \qquad \text{for some } C<\infty \text{ (depending on } d,C_0,\varepsilon,\delta),$$

such that, $\mathbb{P}$-almost surely,

— for all $n\le\lfloor\frac d2\rfloor$,

$$|(\phi^n,\sigma^n)(x)|+|(\nabla\phi^n,\nabla\sigma^n)(x)| \,\le\, \mathcal{C}_{\varepsilon,\delta}(1+|x|)^\varepsilon;$$

— for d odd and $n=\lceil\frac d2\rceil$,

$$|(\phi^n,\sigma^n)(x)| \,\le\, \mathcal{K}_{\varepsilon,\delta}(1+|x|)^\delta, \qquad \text{and} \qquad |(\nabla\phi^n,\nabla\sigma^n)(x)| \,\le\, \mathcal{K}_{\varepsilon,\delta}(1+|x|)^\varepsilon. \quad \Diamond$$

3. Proof of Theorem 1.1

Assume that the heterogeneous acoustic operator $H_a=-\nabla\cdot A\nabla$ admits an eigenvalue $\lambda>0$ with normalized eigenstate $\psi_\lambda\in H^1(\mathbb{R}^d)$, that is,

$$-\nabla\cdot A\nabla\psi_\lambda=\lambda\psi_\lambda, \qquad \|\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d)}=1. \tag{3.1}$$

Given $0<\eta<\frac12$, recall our definition (1.2) of the width of ψ_λ, that is, the minimal radius

$$\ell_\eta(\psi_\lambda) \,:=\, \inf\big\{r\ge0:\|\psi_\lambda\|_{\mathrm{L}^2(B_r)}\ge1-\eta\big\}<\infty. \tag{3.2}$$

To prove Theorem 1.1, we take a dynamical approach and analyze the associated wave equation starting from ψ_λ. More precisely, we shall need to start from a suitably regularized version of ψ_λ: given a nonnegative cut-off function $\chi\in C^\infty(\mathbb{R}^d)$ supported in the unit ball B with $\chi=1$ on $\frac12B$, and given a nonnegative convolution kernel $\rho\in\mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d}\rho=1$, whose Fourier transform $\hat\rho$ is supported in B, we consider the rescalings

$$\chi_L:=\chi(\tfrac1L\cdot), \qquad \rho_\lambda:=\lambda^{\beta d}\rho(\lambda^\beta\cdot),$$

where $L \geq 1$ is a lengthscale and $0 < \beta < \frac{1}{2}$ is an exponent to be optimized later on in the proof, and we then define the regularized initial condition

$$\psi_{\lambda,L} \,:=\, \rho_\lambda * (\chi_L \psi_\lambda). \tag{3.3}$$

We proceed by investigating properties of the solution $u_{\lambda,L} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^d))$ of the associated wave equation

$$\begin{cases} \partial_t^2 u_{\lambda,L} = \nabla \cdot A \nabla u_{\lambda,L}, \\ u_{\lambda,L}|_{t=0} = \psi_{\lambda,L}, \\ \partial_t u_{\lambda,L}|_{t=0} = 0. \end{cases} \tag{3.4}$$

Using homogenization techniques, we shall show that $u_{\lambda,L}$ is close to the solution $\bar u_{\lambda,L,N}$ of some constant-coefficient homogenized equation; see (3.6) below. As the solution of the corresponding heterogeneous wave equation starting from ψ_λ is given by $t \mapsto \cos(t\sqrt{\lambda})\psi_\lambda$, the triangle inequality leads us to the following estimate, for all $t\sqrt{\lambda} \in 2\pi\mathbb{N}$ and $R \geq 1$,

$$\|\psi_\lambda\|_{\mathrm{L}^2(B_R)} \,\leq\, \|\psi_\lambda - \psi_{\lambda,L}\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|u^t_{\lambda,L} - \bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|\bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(B_R)}.$$

By our choice (3.3) for $\psi_{\lambda,L}$, we shall prove in Step 2 that

$$\|\psi_\lambda - \psi_{\lambda,L}\|_{\mathrm{L}^2(\mathbb{R}^d)} \,\leq\, \|\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d \setminus B_L)} + C\lambda^{\frac{1}{2}-\beta},$$

whereas by dispersive estimates for the homogenized equation we shall establish in Step 5 that for all $\alpha > 0$ (and under further milder conditions)

$$\|\bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(B_R)} \,\leq\, C_\alpha (LR)^{\frac{d}{2}} (1+t)^{\alpha-d}.$$

Choosing $L = R = \ell_\eta(\psi_\lambda)$ for some $\eta < \frac{1}{2}$, this entails for all $t\sqrt{\lambda} \in 2\pi\mathbb{N}$ large enough and $0 < \lambda \ll_{\beta,\eta} 1$ small enough,

$$t^\alpha \big(\tfrac{\ell_\eta(\psi_\lambda)}{t}\big)^d + \|u^t_{\lambda,L} - \bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)} \,\gtrsim_{\alpha,\eta}\, 1.$$

Provided that we can neglect the second right-hand side term up to some maximal timescale $T \gg 1$, the inequality turns into

$$T^\alpha \big(\tfrac{\ell_\eta(\psi_\lambda)}{T}\big)^d \,\gtrsim_{\alpha,\eta}\, 1,$$

which will precisely lead to the claimed lower bound on $\ell_\eta(\psi_\lambda)$. The proof of Theorem 1.1 is based on the above general strategy and is split into the first six steps below. The seventh and last step is devoted to the proof of the corresponding result for the energy density of eigenstates as stated in Remark 1.2.

Step 1. Higher-order correctors and homogenized equations.
We assume that we can iteratively construct $N \geq 1$ controlled correctors in the sense of Section 2 with the following pointwise bounds,

$$\begin{gathered} |\bar A^n| \leq K_n, \qquad \text{for all } 1 \leq n \leq N, \\ |(\phi^n, \sigma^n)(x)| + |(\nabla\phi^{n+1}, \nabla\sigma^{n+1})(x)| \leq K_n \langle x\rangle^\varepsilon, \quad \text{for all } 0 \leq n < N, \\ |(\phi^N, \sigma^N)(x)| \leq K_N \langle x\rangle^\delta. \end{gathered} \tag{3.5}$$

for some constants $K_0, K_1, \ldots, K_N \geq 1$ and some exponents $0 \leq \varepsilon \leq \delta < 1$. In the sequel of the proof, we shall argue by assuming that those properties are satisfied in this general form, before particularizing them to the specific case of periodic, quasiperiodic, and random coefficient fields, as described in Lemmas 2.1, 2.2, and 2.3. In a nutshell,

— in the periodic case, the above bounds hold with $N \uparrow \infty$, $\varepsilon = \delta = 0$, and $K_n = C^n$;

— in the quasiperiodic case, under a Diophantine condition and Gevrey regularity, the above bounds hold with $N \uparrow \infty$, $\varepsilon = \delta = 0$, and $K_n = (Cn^\theta)^n$ for some $\theta > 0$;

— in the random case, under suitable mixing assumptions, the above bounds hold with only $N = \lceil \frac{d}{2} \rceil$, with any $\varepsilon > 0$, and with any $\delta > 0$ if the spatial dimension d is even (resp. with any $\delta > \frac{1}{2}$ if d is odd).

Next, in terms of the homogenized tensors $\{\bar A^n\}_{1\le n\le N}$, we consider the associated (constant-coefficient) homogenized wave equation

$$\begin{cases} \partial_t^2 \bar u_{\lambda,L,N} = \nabla\cdot\big(\sum_{n=1}^N \bar A^n_{j_1\ldots j_{n-1}}\nabla^{n-1}_{j_1\ldots j_{n-1}}\big)\nabla \bar u_{\lambda,L,N}, \\ \bar u_{\lambda,L,N}|_{t=0} = \psi_{\lambda,L}, \\ \partial_t \bar u_{\lambda,L,N}|_{t=0} = 0. \end{cases} \tag{3.6}$$

This equation is however ill-posed for general initial conditions due to the lack of definiteness of the symbol

$$\mu_N(i\xi) := \xi\cdot\Big(\sum_{n=1}^N (i\xi)^{\otimes(n-1)}_{j_1\ldots j_{n-1}}\bar A^n_{j_1\ldots j_{n-1}}\Big)\xi.$$

To circumvent this, first recall that the uniform ellipticity condition (1.1) entails that $\bar A = \bar A^1$ also satisfies

$$\tfrac{1}{C_0}|e|^2 \,\le\, e\cdot \bar A e \,\le\, C_0|e|^2, \qquad \text{for all } e\in\mathbb R^d,$$

see e.g. [8], and therefore, using the above bounds (3.5), we find for the symbol μ_N, for all $\xi \in \lambda^\beta B$,

$$\mu_N(i\xi)\begin{cases} \ge \xi\cdot\bar A\xi\big(1 - C_0\sum_{n=2}^N K_n\lambda^{\beta(n-1)}\big) \,\ge\, \frac12\xi\cdot\bar A\xi \,\ge\, \frac{1}{2C_0}|\xi|^2, \\ \le \xi\cdot\bar A\xi\big(1 + C_0\sum_{n=2}^N K_n\lambda^{\beta(n-1)}\big) \,\le\, \frac32\xi\cdot\bar A\xi \,\le\, \frac{3C_0}{2}|\xi|^2, \end{cases} \tag{3.7}$$

provided that λ is small enough in the sense of

$$\sum_{n=2}^N K_n\lambda^{\beta(n-1)} \,\le\, \tfrac{1}{2C_0}. \tag{3.8}$$

(The symbol μ_N is obviously real-valued as $\bar A^n = 0$ for n even, cf. [7, 11].) Noting that by definition the initial Fourier transform $\hat\psi_{\lambda,L}$ is supported in $\lambda^\beta B$, just as $\hat\rho_\lambda$, and using that by (3.7) the symbol is definite and bounded in $\lambda^\beta B$, equation (3.6) admits a unique solution given by the Fourier formula

$$\bar u^t_{\lambda,L,N}(x) \,:=\, \int_{\mathbb R^d} e^{ix\cdot\xi}\cos\big(t\sqrt{\mu_N(i\xi)}\big)\,\hat\psi_{\lambda,L}(\xi)\,d^*\xi. \tag{3.9}$$

Step 2. Two-scale analysis for $u_{\lambda,L}$.
In terms of the above correctors, in the spirit of (2.4), we claim that the two-scale expansion

$$\sum_{n=0}^N \phi^n_{j_1\ldots j_n}\nabla^n_{j_1\ldots j_n}\bar u_{\lambda,L,N}$$

provides a good approximation for $u_{\lambda,L}$. To prove that, we proceed by examining the wave equation satisfied by the error. Applying the wave operator, we compute

$$\big(\partial_t^2 - \nabla\cdot A\nabla\big)\Big(u_{\lambda,L} - \sum_{n=0}^N \phi^n_{j_1\ldots j_n}\nabla^n_{j_1\ldots j_n}\bar u_{\lambda,L,N}\Big) \,=\, -\sum_{n=0}^N \phi^n_{j_1\ldots j_n}\nabla^n_{j_1\ldots j_n}\partial_t^2\bar u_{\lambda,L,N}$$

$$+\sum_{n=0}^{N}\nabla\cdot(A\nabla\phi^{n}_{j_1\ldots j_n})\nabla^{n}_{j_1\ldots j_n}\bar u_{\lambda,L,N}+\sum_{n=0}^{N}e_{j_{n+1}}\cdot A\nabla\phi^{n}_{j_1\ldots j_n}\nabla^{n+1}_{j_1\ldots j_{n+1}}\bar u_{\lambda,L,N}$$
$$+\sum_{n=0}^{N}\nabla\cdot(A\phi^{n}_{j_1\ldots j_n}e_{j_{n+1}})\nabla^{n+1}_{j_1\ldots j_{n+1}}\bar u_{\lambda,L,N}+\sum_{n=0}^{N}e_{j_{n+2}}\cdot A\phi^{n}_{j_1\ldots j_n}e_{j_{n+1}}\nabla^{n+2}_{j_1\ldots j_{n+2}}\bar u_{\lambda,L,N}.$$

Inserting the homogenized wave equation for $\bar u_{\lambda,L,N}$ in the first right-hand side term for $n<N$, relabeling the different sums, and recognizing the corrector equations (2.5), this becomes

$$\begin{aligned}
&\big(\partial_t^2-\nabla\cdot A\nabla\big)\Big(u_{\lambda,L}-\sum_{n=0}^{N}\phi^{n}_{j_1\ldots j_n}\nabla^{n}_{j_1\ldots j_n}\bar u_{\lambda,L,N}\Big)\\
&\quad=-\phi^{N}_{j_1\ldots j_N}\nabla^{N}_{j_1\ldots j_N}\partial_t^2\bar u_{\lambda,L,N}+e_{j_{N+1}}\cdot A\big(\nabla\phi^{N}_{j_1\ldots j_N}+\phi^{N-1}_{j_1\ldots j_{N-1}}e_{j_N}\big)\nabla^{N+1}_{j_1\ldots j_{N+1}}\bar u_{\lambda,L,N}\\
&\qquad+\nabla\cdot(A\phi^{N}_{j_1\ldots j_N}e_{j_{N+1}})\nabla^{N+1}_{j_1\ldots j_{N+1}}\bar u_{\lambda,L,N}+e_{j_{N+2}}\cdot A\phi^{N}_{j_1\ldots j_N}e_{j_{N+1}}\nabla^{N+2}_{j_1\ldots j_{N+2}}\bar u_{\lambda,L,N}\\
&\qquad\qquad-\sum_{n=N+1}^{2N}\sum_{m=n-N}^{N}e_{j_n}\cdot\phi^{n-m-1}_{j_1\ldots j_{n-m-1}}\bar A^{m}_{j_{n-m}\ldots j_{n-2}}e_{j_{n-1}}\nabla^{n}_{j_1\ldots j_n}\bar u_{\lambda,L,N},
\end{aligned}$$

or equivalently, after recombining the terms and using the definition (2.6) of the flux corrector σ^N,

$$\begin{aligned}
&\big(\partial_t^2-\nabla\cdot A\nabla\big)\Big(u_{\lambda,L}-\sum_{n=0}^{N}\phi^{n}_{j_1\ldots j_n}\nabla^{n}_{j_1\ldots j_n}\bar u_{\lambda,L,N}\Big)\\
&\qquad=\nabla\cdot\Big(\big(A\phi^{N}_{j_1\ldots j_N}+(\sigma^{N}_{j_1\ldots j_N})^T\big)\nabla\nabla^{N}_{j_1\ldots j_N}\bar u_{\lambda,L,N}\Big)\\
&\qquad-\partial_t\Big(\phi^{N}_{j_1\ldots j_N}\nabla^{N}_{j_1\ldots j_N}\partial_t\bar u_{\lambda,L,N}\Big)-\sigma^{N}_{j_1\ldots j_N}:\nabla^2\nabla^{N}_{j_1\ldots j_N}\bar u_{\lambda,L,N}\\
&\qquad-\sum_{n=N+2}^{2N}\sum_{m=n-N}^{N}e_{j_n}\cdot\phi^{n-m-1}_{j_1\ldots j_{n-m-1}}\bar A^{m}_{j_{n-m}\ldots j_{n-2}}e_{j_{n-1}}\nabla^{n}_{j_1\ldots j_n}\bar u_{\lambda,L,N}.
\end{aligned}\tag{3.10}$$

We now use the following standard estimate for the wave equation, see e.g. [11, Lemma B.1]: if v satisfies

$$\begin{cases}(\partial_t^2-\nabla\cdot A\nabla)v=\nabla\cdot f+\partial_t g+h,\\ v|_{t=0}=v^\circ,\\ \partial_t v|_{t=0}=0,\end{cases}$$

for some f,g,h,v° with $g|_{t=0}=0$, then we have

$$\|v^t\|_{\mathrm{L}^2(\mathbb{R}^d)}\lesssim\|v^\circ\|_{\mathrm{L}^2(\mathbb{R}^d)}+t\sup_{0\le s\le t}\|(f^s,g^s)\|_{\mathrm{L}^2(\mathbb{R}^d)}+t\sup_{0\le s\le t}\Big\|\int_0^s h\Big\|_{\mathrm{L}^2(\mathbb{R}^d)}.\tag{3.11}$$

Applying this to the above wave equation (3.10), we get

$$\Big\|u^t_{\lambda,L}-\sum_{n=0}^{N}\phi^{n}_{j_1\ldots j_n}\nabla^{n}_{j_1\ldots j_n}\bar u^t_{\lambda,L,N}\Big\|_{\mathrm{L}^2(\mathbb{R}^d)}\lesssim A^\circ_{\lambda,L,N}+t\big(B^t_{\lambda,L,N}+C^t_{\lambda,L,N}+D^t_{\lambda,L,N}\big),$$

and thus by the triangle inequality,

$$\|u^t_{\lambda,L}-\bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)}\lesssim A^\circ_{\lambda,L,N}+A^t_{\lambda,L,N}+t\big(B^t_{\lambda,L,N}+C^t_{\lambda,L,N}+D^t_{\lambda,L,N}\big),\tag{3.12}$$

where we have set for abbreviation

$$
\begin{aligned}
A^{\circ}_{\lambda,L,N} &:= \Big\| \sum_{n=1}^{N} \phi^n_{j_1\ldots j_n} \nabla^n_{j_1\ldots j_n} \psi_{\lambda,L} \Big\|_{\mathrm{L}^2(\mathbb{R}^d)}, \qquad (3.13)\\
A^{t}_{\lambda,L,N} &:= \Big\| \sum_{n=1}^{N} \phi^n_{j_1\ldots j_n} \nabla^n_{j_1\ldots j_n} \bar u^t_{\lambda,L,N} \Big\|_{\mathrm{L}^2(\mathbb{R}^d)},\\
B^{t}_{\lambda,L,N} &:= \sup_{0\le s\le t} \Big\| (\phi^N,\sigma^N)(\partial_s,\nabla)\nabla^N \bar u^s_{\lambda,L,N} \Big\|_{\mathrm{L}^2(\mathbb{R}^d)}\\
C^{t}_{\lambda,L,N} &:= \sup_{0\le s\le t} \Big\| \sigma^N \nabla^{N+1} \int_0^s \nabla \bar u_{\lambda,L,N} \Big\|_{\mathrm{L}^2(\mathbb{R}^d)},\\
D^{t}_{\lambda,L,N} &:= \sum_{n=N+2}^{2N} \sum_{m=n-N}^{N} \sup_{0\le s\le t} \Big\| \phi^{n-m-1} \bar A^m \nabla^{n-1} \int_0^s \nabla \bar u_{\lambda,L,N} \Big\|_{\mathrm{L}^2(\mathbb{R}^d)}.
\end{aligned}
$$

Step 3. Proof that, under assumption (3.8), we have for all $t \ge 0$,

$$
\begin{aligned}
A^{t}_{\lambda,L,N} &\lesssim \lambda^{\beta}(L+\lambda^{-\beta}+t)^{\varepsilon} + K_N \lambda^{\beta N}(L+\lambda^{-\beta}+t)^{\delta}, &(3.14)\\
B^{t}_{\lambda,L,N} &\lesssim K_N \lambda^{\beta(N+1)}(L+\lambda^{-\beta}+t)^{\delta}, &(3.15)\\
C^{t}_{\lambda,L,N} &\lesssim K_N \lambda^{\beta(N+1)}(L+\lambda^{-\beta}+t)^{\delta}, &(3.16)\\
D^{t}_{\lambda,L,N} &\lesssim M_{N,\lambda} \lambda^{\beta(N+1)}(L+\lambda^{-\beta}+t)^{\varepsilon}, &(3.17)
\end{aligned}
$$

where we have set for abbreviation

$$
M_{N,\lambda} := \sum_{n=1}^{N-1} \lambda^{\beta(n-1)} \sum_{m=1}^{N-n} K_{N-m} K_{n+m}, \tag{3.18}
$$

so that (3.12) becomes

$$
\begin{aligned}
\|u^t_{\lambda,L} - \bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)} \lesssim\ & \lambda^{\beta}(L+\lambda^{-\beta}+t)^{\varepsilon} + K_N \lambda^{\beta N}(L+\lambda^{-\beta}+t)^{\delta}\\
&+ (K_N + M_{N,\lambda})\lambda^{\beta(N+1)}(L+\lambda^{-\beta}+t)^{1+\delta}. \qquad (3.19)
\end{aligned}
$$

We start with the proof of (3.14). By definition (3.13) of $A_{\lambda,L,N}$, the corrector bounds (3.5) allow to estimate

$$
A^{t}_{\lambda,L,N} \lesssim \sum_{n=1}^{N-1} K_n \|\langle\cdot\rangle^{\varepsilon} \nabla^n \bar u_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)} + K_N \|\langle\cdot\rangle^{\delta} \nabla^N \bar u_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)}. \tag{3.20}
$$

Using the Fourier formula (3.9) for $\bar u_{\lambda,L,N}$, together with (3.7), we can compute

$$
\begin{aligned}
\|\nabla^n \bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)} &\le \|\nabla^n \psi_{\lambda,L}\|_{\mathrm{L}^2(\mathbb{R}^d)},\\
\||\cdot|\nabla^n \bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)} &\lesssim \|(|\cdot|+t)\nabla^n \psi_{\lambda,L}\|_{\mathrm{L}^2(\mathbb{R}^d)},
\end{aligned}
$$

and thus, by interpolation,

$$
\|\langle\cdot\rangle^{\varepsilon} \nabla^n \bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)} \lesssim \|(\langle\cdot\rangle+t)\nabla^n \psi_{\lambda,L}\|^{\varepsilon}_{\mathrm{L}^2(\mathbb{R}^d)} \|\nabla^n \psi_{\lambda,L}\|^{1-\varepsilon}_{\mathrm{L}^2(\mathbb{R}^d)}. \tag{3.21}
$$

Inserting this into (3.20), together with the definition (3.3) of $\psi_{\lambda,L}$, we get

$$A^t_{\lambda,L,N}\,\lesssim\,(L+\lambda^{-\beta}+t)^{\varepsilon}\sum_{n=1}^{N-1}K_n\lambda^{\beta n}+(L+\lambda^{-\beta}+t)^{\delta}K_N\lambda^{\beta N},$$

and the claim (3.14) follows under assumption (3.8). A similar argument leads to (3.15). We turn to the proof of (3.16). By definition (3.13) of $C_{\lambda,L,N}$, the corrector bounds (3.5) allow to estimate

$$C^t_{\lambda,L,N}\,\lesssim\,K_N\Big\|\langle\cdot\rangle^{\delta}\nabla^{N+1}\int_0^s\nabla\bar u_{\lambda,L,N}\Big\|_{\mathrm{L}^2(\mathbb{R}^d)}.$$

Noting that the Fourier formula (3.9) for $\bar u_{\lambda,L,N}$ yields

$$\int_0^s\nabla\bar u_{\lambda,L,N}\,=\,\int_{\mathbb{R}^d}e^{ix\cdot\xi}\tfrac{i\xi}{\sqrt{\mu_N(i\xi)}}\sin\big(s\sqrt{\mu_N(i\xi)}\big)\,\hat\psi_{\lambda,L}(\xi)\,d^*\xi,$$

a similar argument as above leads us to (3.16). Finally, for $D_{\lambda,L,N}$, a similar computation yields

$$D^t_{\lambda,L,N}\,\lesssim\,(L+\lambda^{-\beta}+t)^{\varepsilon}\sum_{n=N+2}^{2N}\sum_{m=n-N}^{N}K_{n-m-1}K_m\lambda^{\beta(n-1)},$$

and the claim (3.17) follows in terms of the short-hand notation (3.18).

Step 4. Dispersive estimate for the homogenized solution $\bar u_{\lambda,L,N}$: for all $k\ge d$, all $L,R\ge1$, and all $t\gg R+L+\lambda^{-\beta}$, we have

$$\|\bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(B_R)}\,\lesssim_k\,(LR)^{\frac d2}(\lambda^\beta t)^{\frac{d}{k+1}}\,t^{-d},\tag{3.22}$$

provided that λ is small enough in the sense that it satisfies the following strengthened form of (3.8),

$$\sum_{n=1}^{N}\tfrac{(n+1)!}{(n-k)!}K_n\lambda^{\beta(n-1)}\,\le\,\tfrac{1}{2C_0}.\tag{3.23}$$

As $\bar u_{\lambda,L,N}$ solves the homogenized wave equation (3.6) with initial data $\psi_{\lambda,L}=\rho_\lambda*(\chi_L\psi_\lambda)$, it can be represented as

$$\bar u_{\lambda,L,N}=G_{\lambda,N}*(\chi_L\psi_\lambda),$$

where $G_{\lambda,N}$ is the smoothened Green's function

$$\begin{cases}\partial_t^2G_{\lambda,N}=\nabla\cdot\big(\sum_{n=1}^N\bar A^n_{i_1\ldots i_{n-1}}\nabla^{n-1}_{i_1\ldots i_{n-1}}\big)\nabla G_{\lambda,N},\\ G_{\lambda,N}|_{t=0}=\rho_\lambda,\\ \partial_tG_{\lambda,N}|_{t=0}=0,\end{cases}$$

which is understood as in (3.9) in terms of the Fourier formula

$$G^t_{\lambda,N}(x)\,:=\,\int_{\mathbb{R}^d}e^{ix\cdot\xi}\cos\big(t\sqrt{\mu_N(i\xi)}\big)\,\hat\rho_\lambda(\xi)\,d^*\xi.\tag{3.24}$$

This representation of $\bar u_{\lambda,L,N}$ leads us to the estimate

$$\begin{aligned}\|\bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(B_R)}\;&\lesssim\;R^{\frac d2}\|\bar u^t_{\lambda,L,N}\|_{\mathrm{L}^\infty(B_R)}\\&\lesssim\;R^{\frac d2}\|\chi_L\psi_\lambda\|_{\mathrm{L}^1(\mathbb{R}^d)}\|G^t_{\lambda,N}\|_{\mathrm{L}^\infty(B_{R+L})}\\&\lesssim\;(LR)^{\frac d2}\|G^t_{\lambda,N}\|_{\mathrm{L}^\infty(B_{R+L})}.\end{aligned}\tag{3.25}$$

To exploit this estimate, we decompose $G_{\lambda,N}$ as

$$G_{\lambda,N} \;=\; \tfrac12\big(G_{\lambda,N,+}+G_{\lambda,N,-}\big),$$

where $G_{\lambda,N,+}$ and $G_{\lambda,N,-}$ are complex-valued and are defined by

$$G^t_{\lambda,N,\pm}(x) \;:=\; \int_{\mathbb{R}^d} e^{i\big(x\cdot\xi\pm t\sqrt{\mu_N(i\xi)}\big)}\,\hat\rho_\lambda(\xi)\,d^*\xi. \tag{3.26}$$

Given $0<\zeta<\lambda^\beta$ to be later optimized, recalling that $\hat\rho_\lambda=\hat\rho(\lambda^{-\beta}\cdot)$, we further decompose

$$G^t_{\lambda,N,\pm}(x) \;=\; G^t_{\lambda,N,\pm;\zeta,0}(x)+G^t_{\lambda,N,\pm;\zeta,1}(x), \tag{3.27}$$

in terms of

$$G^t_{\lambda,N,\pm;\zeta,\sigma}(x) \;:=\; \int_{\mathbb{R}^d} e^{i\big(x\cdot\xi\pm t\sqrt{\mu_N(i\xi)}\big)}\,\hat\rho_{\lambda;\zeta,\sigma}(\xi)\,d^*\xi, \tag{3.28}$$

$$\hat\rho_{\lambda;\zeta,0}(\xi) \;=\; \hat\rho(\lambda^{-\beta}\xi)\chi(\zeta^{-1}\xi),\quad \hat\rho_{\lambda;\zeta,1}(\xi) \;=\; \hat\rho(\lambda^{-\beta}\xi)\big(1-\chi(\zeta^{-1}\xi)\big).$$

In order to estimate these oscillating integrals, we argue as e.g. in [6, Section 8.1.3] and shall use that

$$e^{i\big(x\cdot\xi\pm t\sqrt{\mu_N(i\xi)}\big)} = \Big(1+t\big|\tfrac{x}{t}\pm\nu_N(\xi)\big|^2\Big)^{-1}\Big(1-i\big(\tfrac{x}{t}\pm\nu_N(\xi)\big)\cdot\nabla_\xi\Big)e^{i\big(x\cdot\xi\pm t\sqrt{\mu_N(i\xi)}\big)},$$

with the short-hand notation

$$\nu_N(\xi) \;:=\; \tfrac{\nabla_\xi\mu_N(i\xi)}{2\sqrt{\mu_N(i\xi)}}.$$

Inserting this into (3.28) and performing $k\ge0$ integrations by parts, we find

$$G^t_{\lambda,N,\pm;\zeta,\sigma}(x) \;=\; \int_{\mathbb{R}^d} e^{i\big(x\cdot\xi\pm t\sqrt{\mu_N(i\xi)}\big)}\,\big(D^k_{\xi;x,t}\hat\rho_{\lambda;\zeta,\sigma}\big)(\xi)\,d^*\xi, \tag{3.29}$$

provided that the integral makes sense, where $D_{\xi;x,t}$ stands for the first-order differential operator

$$\begin{aligned} D_{\xi;x,t} \;&:=\; \Big(1+i\nabla_\xi\cdot\big(\tfrac{x}{t}\pm\nu_N(\xi)\big)\Big)\Big(1+t\big|\tfrac{x}{t}\pm\nu_N(\xi)\big|^2\Big)^{-1}\\ &=\; \tfrac{1\pm i(\nabla\cdot\nu_N)(\xi)}{1+t|\frac{x}{t}\pm\nu_N(\xi)|^2}\mp 2t\tfrac{i(\frac{x}{t}\pm\nu_N(\xi))^{\otimes2}:\nabla\nu_N(\xi)}{(1+t|\frac{x}{t}\pm\nu_N(\xi)|^2)^2}+\tfrac{i(\frac{x}{t}\pm\nu_N(\xi))}{1+t|\frac{x}{t}\pm\nu_N(\xi)|^2}\cdot\nabla_\xi. \end{aligned} \tag{3.30}$$

For all $k\ge0$, provided that λ is small enough in the sense that it satisfies (3.23), a similar argument as in (3.7) yields for all $\xi\in\lambda^\beta B$ and $0\le j\le k$,

$$|\nu_N(\xi)|\;\simeq\;1,\qquad |\nabla^j_\xi\nu_N(\xi)|\;\lesssim_j\;|\xi|^{-j}.$$

Provided that $|x|\ll t$, we deduce in particular $|\frac{x}{t}\pm\nu_N(\xi)|\simeq1$ for $\xi\in\lambda^\beta B$. For all $k\ge0$, provided that (3.23) holds and that $|x|\ll t$, a direct computation starting from (3.30) then leads us to

$$\big|\big(D^k_{\xi;x,t}\hat\rho_{\lambda;\zeta,\sigma}\big)(\xi)\big|\;\lesssim_k\;(1+t)^{-k}(|\xi|^{-k}+\zeta^{-k})\mathbb{1}_{|\xi|\le\lambda^\beta},\qquad\text{for }\xi\ne0.$$

Inserting this into (3.27) and (3.29), choosing $k=d-1$ for $\sigma=0$ and any $k\ge d$ for $\sigma=1$, and using properties of ρ,χ, we deduce

$$\begin{aligned} \sup_{x:|x|\ll t}|G^t_{\lambda,N,\pm}(x)| \;&\lesssim\; \int_{\mathbb{R}^d}|D^{d-1}_{\xi;x,t}\hat\rho_{\lambda;\zeta,0}(\xi)|\,d\xi+\int_{\mathbb{R}^d}|D^k_{\xi;x,t}\hat\rho_{\lambda;\zeta,1}(\xi)|\,d\xi\\ &\lesssim_k\; (1+t)^{1-d}\int_{|\xi|\le\zeta}|\xi|^{1-d}d\xi+(1+t)^{-k}\zeta^{-k}\int_{\frac12\zeta\le|\xi|\le\lambda^\beta}d\xi \end{aligned}$$

$$\lesssim \quad (1+t)^{1-d}\zeta + (1+t)^{-k}\zeta^{-k}\lambda^{\beta d}.$$

Optimizing with respect to $0<\zeta<\lambda^\beta$ yields for all $t\ge\lambda^{-\beta}$ and $k\ge d$, provided that (3.23) holds,

$$\sup_{x:|x|\ll t}|G^t_{\lambda,N}(x)| \lesssim_k (\lambda^\beta t)^{\frac{d}{k+1}}\,t^{-d}. \tag{3.31}$$

Inserting this into (3.25) yields the claim (3.22).

Step 5. Estimate on $\ell_\eta(\psi_\lambda)$: given $\eta,\beta<\frac12$ and $k\ge d$, if (3.23) is satisfied and $\lambda\ll_{\eta,\beta}1$ is small enough, we have for all $t\in 2\pi\sqrt{\lambda}^{-1}\mathbb{N}$ (which implies both $\cos(\sqrt{\lambda}t)=1$ and $t\gg\lambda^{-\beta}$) with $t\gg\ell_\eta(\psi_\lambda)$,

$$\lambda^\beta t^\varepsilon + (K_N+M_{N,\lambda})\lambda^{\beta(N+1)}t^{1+\delta} + (\lambda^\beta t)^{\frac{d}{k+1}}\big(\tfrac{\ell_\eta(\psi_\lambda)}{t}\big)^d \gtrsim_{k,\eta} 1. \tag{3.32}$$

If the initial condition $\psi_{\lambda,L}$ were replaced by the eigenstate ψ_λ itself, the solution $u_{\lambda,L}$ of the wave equation (3.4) would simply be $t\mapsto\cos(t\sqrt{\lambda})\psi_\lambda$ since ψ_λ is an eigenstate. Estimating the difference, we then get for all $t\ge0$,

$$\|u^t_{\lambda,L}-\cos(t\sqrt{\lambda})\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\psi_\lambda-\psi_{\lambda,L}\|_{\mathrm{L}^2(\mathbb{R}^d)}. \tag{3.33}$$

The right-hand side is easily estimated: by definition (3.3) of $\psi_{\lambda,L}$, we find

$$\begin{aligned}\|\psi_\lambda-\psi_{\lambda,L}\|_{\mathrm{L}^2(\mathbb{R}^d)} &\le \|\psi_\lambda(1-\chi_L)\|_{\mathrm{L}^2(\mathbb{R}^d)}+\|\psi_\lambda-\rho_\lambda*\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d)}\\ &\le \|\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d\setminus B_L)}+\|\nabla\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d}|y|^2|\rho_\lambda(y)|\,dy\Big)^{\frac12}.\end{aligned}$$

As the uniform ellipticity condition (1.1) and the eigenvalue equation imply

$$\|\nabla\psi_\lambda\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le C_0\int_{\mathbb{R}^d}\nabla\psi_\lambda\cdot A\nabla\psi_\lambda = C_0\lambda,$$

we deduce

$$\|\psi_\lambda-\psi_{\lambda,L}\|_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d\setminus B_L)}+C\lambda^{\frac12-\beta}, \tag{3.34}$$

so that (3.33) becomes

$$\|u^t_{\lambda,L}-\cos(t\sqrt{\lambda})\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d\setminus B_L)}+C\lambda^{\frac12-\beta}.$$

Now comparing $u^t_{\lambda,L}$ to the corresponding homogenized solution $\bar u^t_{\lambda,L,N}$, we get by the triangle inequality,

$$\|\bar u^t_{\lambda,L,N}-\cos(t\sqrt{\lambda})\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d\setminus B_L)}+C\lambda^{\frac12-\beta}+\|u^t_{\lambda,L}-\bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)},$$

hence, for all $R\ge1$, further restricting norms to B_R, we obtain since $\cos(\sqrt{\lambda}t)=1$

$$\|\psi_\lambda\|_{\mathrm{L}^2(B_R)} \le \|\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d\setminus B_L)}+C\lambda^{\frac12-\beta}+\|u^t_{\lambda,L}-\bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(\mathbb{R}^d)}+\|\bar u^t_{\lambda,L,N}\|_{\mathrm{L}^2(B_R)}.$$

The last two right-hand side terms are precisely estimated in (3.12) and (3.22), which lead us to the following: given $\beta<\frac12$ and $k\ge d$, if $\lambda\ll_\beta1$ is small enough such that (3.23) is satisfied, we have for all $R,L\ge1$ and $t\gg R+L+\lambda^{-\beta}$,

$$\begin{aligned}\|\psi_\lambda\|_{\mathrm{L}^2(B_R)} \le{}& \|\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d\setminus B_L)}+C\lambda^{\frac12-\beta}+C\lambda^\beta t^\varepsilon\\ &+C(K_N+M_{N,\lambda})\lambda^{\beta(N+1)}t^{1+\delta}+C_k(LR)^{\frac d2}(\lambda^\beta t)^{\frac{d}{k+1}}\,t^{-d}.\end{aligned}$$

Choosing $R = L = \ell_\eta(\psi_\lambda)$, noting that for this choice we have

$$\|\psi_\lambda\|_{\mathrm{L}^2(B_R)} \geq 1-\eta, \qquad \|\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d\setminus B_L)} \leq \eta,$$

and choosing $\lambda \ll_{\eta,\beta} 1$ small enough such that $C\lambda^{\frac12-\beta} \leq \frac12(1-2\eta)$, the claim (3.32) follows.

Step 6. Conclusion.
It remains to optimize with respect to t in (3.32) to infer a lower bound on $\ell_\eta(\psi_\lambda)$. We split the proof into three further substeps, where we consider separately the periodic, quasiperiodic, and random settings.

Substep 6.1. Periodic setting.
In the periodic setting, by Lemma 2.1, we can choose N arbitrarily large and we have $\varepsilon = \delta = 0$ and $K_n = C^n$. The result (3.32) then reads as follows: given $\eta, \beta < \frac12$ and $k \geq d$, if $\lambda \ll_{\eta,\beta} 1$ is small enough, we have for all $N \geq 1$ and $t \in 2\pi\sqrt{\lambda}^{-1}\mathbb{N}$ with $t \gg \ell_\eta(\psi_\lambda)$,

$$t(C\lambda^\beta)^{N+1} + (\lambda^\beta t)^{\frac{d}{k+1}}\big(\tfrac{\ell_\eta(\psi_\lambda)}{t}\big)^d \gtrsim_{k,\eta} 1.$$

Taking in order the limits $N \uparrow \infty$ and $t \uparrow \infty$ yields a contradiction. This proves that the acoustic operator H_a has no eigenvalue $0 < \lambda \ll 1$.

Substep 6.2. Quasiperiodic setting.
In the quasiperiodic setting, by Lemma 2.2, under a Diophantine condition and Gevrey regularity, we can choose N arbitrarily large and we have $\varepsilon = \delta = 0$ and $K_n = (Cn^\theta)^n$, for some $\theta > 0$ determined by the Diophantine exponent and the Gevrey regularity. The result (3.32) then reads as follows: given $\eta, \beta < \frac12$ and $k \geq d$, if $\lambda \ll_{\eta,\beta} 1$ is small enough, we have for all $N \geq 1$ with $N^\theta\lambda^\beta \ll_k 1$, and for all $t \in 2\pi\sqrt{\lambda}^{-1}\mathbb{N}$ with $t \gg \ell_\eta(\psi_\lambda)$,

$$t(CN^{2\theta})^N\lambda^{\beta(N+1)} + (\lambda^\beta t)^{\frac{d}{k+1}}\big(\tfrac{\ell_\eta(\psi_\lambda)}{t}\big)^d \gtrsim_{k,\eta} 1.$$

Optimizing with respect to N with $N^{2\theta}\lambda^\beta \ll 1$, this inequality becomes

$$t\exp\big(-\tfrac{1}{C}\lambda^{-\frac{\beta}{2\theta}}\big) + (\lambda^\beta t)^{\frac{d}{k+1}}\big(\tfrac{\ell_\eta(\psi_\lambda)}{t}\big)^d \gtrsim_{k,\eta} 1.$$

Next, choosing $2\pi\sqrt{\lambda}^{-1}\mathbb{N} \ni t \simeq (\ell_\eta(\psi_\lambda) + \lambda^{-\frac12})^{1+\frac1k}$, this entails

$$\ell_\eta(\psi_\lambda) \gtrsim_{k,\eta} \exp\big(\tfrac{1}{C_k}\lambda^{-\frac{\beta}{2\theta}}\big),$$

which is the desired estimate up to renaming the exponent θ.

Substep 6.3. Random setting.
In the random setting, by Lemma 2.3, under suitable mixing assumptions, we can choose $N = \lceil\frac{d}{2}\rceil$, we can choose any $\varepsilon > 0$, and any $\delta > 0$ if d is even (resp. any $\delta > \frac12$ if d is odd), and we have $K_n \leq \mathcal{K}_{\varepsilon,\delta}$ for some random variable $\mathcal{K}_{\varepsilon,\delta}$ with finite stretched exponential moments. The result (3.32) then reads as follows: given $\eta, \beta < \frac12$ and $k \geq d$, if $\lambda \ll_{\eta,\beta} 1$ is small enough, we have for all $t \in 2\pi\sqrt{\lambda}^{-1}\mathbb{N}$ with $t \gg \ell_\eta(\psi_\lambda)$, provided $\mathcal{K}_{\varepsilon,\delta}\lambda^\beta \ll 1$,

$$\lambda^\beta t^\varepsilon + \mathcal{K}_{\varepsilon,\delta}^2\lambda^{\beta(\lceil\frac{d}{2}\rceil+1)}t^{1+\delta} + (\lambda^\beta t)^{\frac{d}{k+1}}\big(\tfrac{\ell_\eta(\psi_\lambda)}{t}\big)^d \gtrsim_{k,\eta} 1. \tag{3.35}$$

Choosing $2\pi\sqrt{\lambda}^{-1}\mathbb{N} \ni t \simeq (\ell_\eta(\psi_\lambda) + \lambda^{-\beta})^{1+\frac1k}$ and $k \geq d$ with $\frac1k = \varepsilon \ll 1$, we easily deduce the following, provided that $\lambda \ll \mathcal{K}_{\varepsilon,\delta}^{-C/\beta}$:

— if d is even, letting $\delta=\varepsilon$, we obtain

$$\ell_\eta(\psi_\lambda)\ \gtrsim_\eta\ (\mathcal{K}_{\varepsilon,\delta}^2\lambda^{\beta(\frac d2+1)})^{-\frac{1}{1+C\varepsilon}};$$

— if d is odd, letting $\delta=\frac12+\varepsilon$, we obtain

$$\ell_\eta(\psi_\lambda)\ \gtrsim_\eta\ (\mathcal{K}_{\varepsilon,\delta}^2\lambda^{\beta(\frac{d+1}{2}+1)})^{-\frac{2}{3+C\varepsilon}}.$$

This precisely yields the conclusion in the random setting both for $d=1$ and for d even. For d odd with $d>1$, in contrast, we rather appeal to (3.32) with $N=\lceil\frac d2\rceil$ replaced by $N'=\lceil\frac d2\rceil-1=\lfloor\frac d2\rfloor$: as the corrector $(\phi^{N'},\sigma^{N'})$ has arbitrarily small growth by Lemma 2.3, the corresponding version of (3.32) holds again with any $0<\delta=\varepsilon\ll1$ in that case, and the conclusion similarly follows.

Step 7. Control of the energy density of eigenstates.
The statement of Remark 1.2 is obtained as a post-processing of the result for the mass density using an exponentially-weighted energy estimate. Given $\alpha\ge1$, consider the exponential weight

$$\zeta_{\lambda,\alpha}(x)\ :=\ K_{\lambda,\alpha}^{-1}\exp(-\tfrac{\sqrt\lambda}{\alpha}|x|),\qquad K_{\lambda,\alpha}\ :=\ \int_{\mathbb{R}^d}\exp(-\tfrac{\sqrt\lambda}{\alpha}|x|).$$

For any $z\in\mathbb{R}^d$, testing the eigenvalue relation (3.1) with $\zeta_{\lambda,\alpha}(\cdot-z)\psi_\lambda$, we find

$$\begin{aligned}\lambda\int_{\mathbb{R}^d}\zeta_{\lambda,\alpha}(\cdot-z)|\psi_\lambda|^2\ =\ &\int_{\mathbb{R}^d}\zeta_{\lambda,\alpha}(\cdot-z)\nabla\psi_\lambda\cdot A\nabla\psi_\lambda\\ &-\tfrac{\sqrt\lambda}{\alpha}\int_{\mathbb{R}^d}\zeta_{\lambda,\alpha}(z-x)\psi_\lambda(x)\tfrac{z-x}{|z-x|}\cdot A\nabla\psi_\lambda(x)\,dx.\end{aligned}$$

By Young's inequality, with $\alpha\ge1$, we deduce

$$\begin{aligned}\lambda\int_{\mathbb{R}^d}\zeta_{\lambda,\alpha}(\cdot-z)|\psi_\lambda|^2\ &\le\ \tfrac{1+\frac{1}{2\alpha}}{1-\frac{1}{2\alpha}}\int_{\mathbb{R}^d}\zeta_{\lambda,\alpha}(\cdot-z)\nabla\psi_\lambda\cdot A\nabla\psi_\lambda\\ &\le\ (1+\tfrac2\alpha)\int_{\mathbb{R}^d}\zeta_{\lambda,\alpha}(\cdot-z)\nabla\psi_\lambda\cdot A\nabla\psi_\lambda.\end{aligned}\tag{3.36}$$

Next, we upgrade this estimate to get a comparison between the local norms $\lambda\int_{B_R}|\psi_\lambda|^2$ and $\int_{B_R}\nabla\psi_\lambda\cdot A\nabla\psi_\lambda$ (with non-smooth cut-off). For that purpose, using local averaging with the weight $\zeta_{\lambda,\alpha}$, we start by writing

$$\lambda\int_{|x|\ge R}|\psi_\lambda(x)|^2\,dx\ =\ \lambda\int_{\mathbb{R}^d}\Big(\int_{|x|\ge R}\zeta_{\lambda,\alpha}(x-z)|\psi_\lambda(x)|^2\,dx\Big)\,dz,\tag{3.37}$$

and we separately consider the near-field and far-field contributions in this integral. On the one hand, for the integral over $|z|\le R/2$, using that the conditions $|z|\le R/2$ and $|x|\ge R$ imply $|x-z|\ge R/2$, and using the normalization of ψ_λ, we find

$$\begin{aligned}\lambda\int_{|z|\le R/2}\Big(\int_{|x|\ge R}\zeta_{\lambda,\alpha}(x-z)|\psi_\lambda(x)|^2\,dx\Big)\,dz\ &\le\ C\lambda R^d\|\zeta_{\lambda,\alpha}\|_{\mathrm{L}^\infty(\mathbb{R}^d\setminus B_{R/2})}\\ &\le\ C\lambda\exp(-\tfrac{\sqrt\lambda}{4\alpha}R).\end{aligned}$$

On the other hand, for the integral over $|z|\ge R/2$, we appeal to (3.36) and obtain

$$\lambda\int_{|z|\ge R/2}\Big(\int_{|x|\ge R}\zeta_{\lambda,\alpha}(x-z)|\psi_\lambda(x)|^2\,dx\Big)\,dz$$

$$\begin{aligned} &\le \lambda \int_{|z|\ge R/2} \Big(\int_{\mathbb{R}^d} \zeta_{\lambda,\alpha}(\cdot - z)|\psi_\lambda|^2\Big)\,dz \\ &\le (1+\tfrac{2}{\alpha}) \int_{|z|\ge R/2} \Big(\int_{\mathbb{R}^d} \zeta_{\lambda,\alpha}(\cdot - z)\nabla\psi_\lambda \cdot A\nabla\psi_\lambda\Big)\,dz. \end{aligned}$$

We further split the last integral in bracket over $\mathbb{R}^d \setminus B_{R/4}$ and $B_{R/4}$, and we note respectively that

$$\int_{|z|\ge R/2} \Big(\int_{\mathbb{R}^d\setminus B_{R/4}} \zeta_{\lambda,\alpha}(\cdot - z)\nabla\psi_\lambda \cdot A\nabla\psi_\lambda\Big)\,dz \,\le\, \int_{\mathbb{R}^d\setminus B_{R/4}} \nabla\psi_\lambda \cdot A\nabla\psi_\lambda,$$

and that

$$\begin{aligned} &\int_{|z|\ge R/2} \Big(\int_{B_{R/4}} \zeta_{\lambda,\alpha}(\cdot - z)\nabla\psi_\lambda \cdot A\nabla\psi_\lambda\Big)\,dz \\ &\le \Big(\int_{|z|\ge R/2} \|\zeta_{\lambda,\alpha}\|_{\mathrm{L}^\infty(B_{R/4}(z))}\,dz\Big)\Big(\int_{\mathbb{R}^d} \nabla\psi_\lambda \cdot A\nabla\psi_\lambda\Big) \\ &\le C\lambda \exp(-\tfrac{\sqrt{\lambda}}{8\alpha}R). \end{aligned}$$

Combining these different estimates back into (3.37), we obtain

$$\lambda \int_{|x|\ge R} |\psi_\lambda(x)|^2\,dx \,\le\, (1+\tfrac{2}{\alpha}) \int_{\mathbb{R}^d\setminus B_{R/4}} \nabla\psi_\lambda \cdot A\nabla\psi_\lambda + C\lambda \exp(-\tfrac{\sqrt{\lambda}}{8\alpha}R). \tag{3.38}$$

We can now use this bound to compare the minimal radii $\ell_\eta(\psi_\lambda)$ and $\ell'_\eta(\psi_\lambda)$ defined in (1.2) and (1.3): for the choice $R := \ell_\eta(\psi_\lambda)$, the left-hand side in (3.38) is by definition equal to $\lambda\eta^2$, so we can deduce

$$\tfrac{1}{\sqrt{\lambda}}\|\sqrt{A}\nabla\psi_\lambda\|_{\mathrm{L}^2(\mathbb{R}^d\setminus B_{R/4})} \,\ge\, \big(1+\tfrac{2}{\alpha}\big)^{-\frac12}\big(\eta^2 - C\exp(-\tfrac{\sqrt{\lambda}}{8\alpha}R)\big)^{\frac12},$$

which means

$$\ell'_{\hat\eta}(\psi_\lambda) \,\ge\, \tfrac14 \ell_\eta(\psi_\lambda) \qquad \text{provided } \hat\eta \le \big(1+\tfrac{2}{\alpha}\big)^{-\frac12}\big(\eta^2 - C\exp(-\tfrac{\sqrt{\lambda}}{8\alpha}\ell_\eta(\psi_\lambda))\big)^{\frac12}.$$

In all the cases considered, as stated in Theorem 1.1, for any $\varepsilon > 0$ and $\eta < \frac12$, we have shown that we have at least $\ell_\eta(\psi_\lambda) \ge \lambda^{\varepsilon-2/3}$ for λ small enough, so the above becomes

$$\ell'_{\hat\eta}(\psi_\lambda) \,\ge\, \tfrac14 \ell_\eta(\psi_\lambda) \qquad \text{provided } \hat\eta \le \big(1+\tfrac{2}{\alpha}\big)^{-\frac12}\big(\eta^2 - C\exp(-\tfrac{\lambda^{-1/8}}{8\alpha})\big)^{\frac12}.$$

Choosing α arbitrarily large to reach any $\hat\eta < 1$ up to taking λ small enough, the statement of Remark 1.2 follows. ∎

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