THE STRUCTURE OF FLUCTUATIONS
IN STOCHASTIC HOMOGENIZATION

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ABSTRACT. Four quantities are fundamental in homogenization of elliptic systems in divergence form and in its applications: the field and the flux of the solution operator (applied to a general deterministic right-hand side), and the field and the flux of the corrector. Homogenization is the study of the large-scale properties of these objects. In case of random coefficients, these quantities fluctuate and their fluctuations are a priori unrelated. Depending on the law of the coefficient field, and in particular on the decay of its correlations on large scales, these fluctuations may display different scalings and different limiting laws (if any). In this contribution, we identify another crucial intrinsic quantity, motivated by H-convergence, which we refer to as the *homogenization commutator* and is related to variational quantities first considered by Armstrong and Smart. In the simplified setting of the random conductance model, we show what we believe to be a general principle, namely that the homogenization commutator drives at leading order the fluctuations of each of the four other quantities in a strong norm in probability, which is expressed in form of a suitable two-scale expansion and reveals the *pathwise structure* of fluctuations in stochastic homogenization. In addition, we show that the (rescaled) homogenization commutator converges in law to a Gaussian white noise, and we analyze to which precision the covariance tensor that characterizes the latter can be extracted from the representative volume element method. This collection of results constitutes a new theory of fluctuations in stochastic homogenization that holds in any dimension and yields optimal rates. Extensions to the (non-symmetric) continuum setting are also discussed, the details of which are postponed to forthcoming works.

CONTENTS

1. INTRODUCTION

This article constitutes the first part of a series of works that develop a new theory of fluctuations in stochastic homogenization of elliptic (non-necessarily symmetric) systems. In this first contribution, we provide a full picture of our theory with optimal convergence rates in the simplified setting of the random conductance model, that is, for discrete elliptic

equations with independent and identically distributed (iid) conductances. The extension of our results to more general frameworks is shortly described in Section 1.3 below and is postponed to forthcoming works, and a thorough discussion of the literature is provided in Section 1.4.

1.1. **General overview.** Although in the sequel we shall focus on the case of discrete elliptic equations, we use non-symmetric continuum notation in this introduction. Let $\boldsymbol{a}$ be a stationary and ergodic measurable random coefficient field on $\mathbb{R}^d$ that satisfies the boundedness and ellipticity properties, for some $\lambda > 0$,

$$|\boldsymbol{a}(x)\xi| \leq |\xi|, \qquad \xi \cdot \boldsymbol{a}(x)\xi \geq \lambda|\xi|^2, \qquad \text{for all } x, \xi \in \mathbb{R}^d.$$

We denote by $(\Omega, \mathbb{P})$ the underlying probability space and by $\mathbb{E}[\cdot]$ the associated expectation. Alternatively, we may view Ω as the space of coefficient fields that satisfy the above properties, and $\mathbb{P}$ as a suitable cylindrical probability measure on that space.

For $\varepsilon > 0$, we set $\boldsymbol{a}_\varepsilon := \boldsymbol{a}(\frac{\cdot}{\varepsilon})$, and for all deterministic vector fields $f \in C_c^\infty(\mathbb{R}^d)^d$, we consider the random family $(u_\varepsilon)_{\varepsilon > 0}$ of unique Lax-Milgram solutions in $\mathbb{R}^d$ (which in the rest of this article means the unique weak solutions in $\dot{H}^1(\mathbb{R}^d)$) of the rescaled problems

$$-D \cdot \boldsymbol{a}_\varepsilon D u_\varepsilon = D \cdot f, \tag{1.1}$$

where D denotes the continuum gradient (while the notation ∇ is reserved in the sequel for the discrete gradient). It is known since the pioneering work of Papanicolaou and Varadhan [56] and of Kozlov [41] that, almost surely, u_ε converges weakly in $\dot{H}^1(\mathbb{R}^d)$ as $\varepsilon \downarrow 0$ to the unique Lax-Milgram solution $\bar{u}$ in $\mathbb{R}^d$ of

$$-D \cdot \bar{\boldsymbol{a}} D \bar{u} = D \cdot f, \tag{1.2}$$

where $\bar{\boldsymbol{a}}$ is a deterministic and constant matrix that only depends on $\boldsymbol{a}$. More precisely, for any direction $e \in \mathbb{R}^d$, the projection $\bar{\boldsymbol{a}}e$ is the expectation of the flux of the corrector in the direction e,

$$\bar{\boldsymbol{a}}e = \mathbb{E}[\boldsymbol{a}(D\phi_e + e)],$$

where the corrector ϕ_e is the unique (up to a random additive constant) almost sure solution of the corrector equation in $\mathbb{R}^d$,

$$-D \cdot \boldsymbol{a}(D\phi_e + e) = 0,$$

in the class of functions the gradient of which is stationary, has vanishing expectation, and has finite second moment. We denote by $\phi = (\phi_i)_{i=1}^d$ the vector field the entries of which are the correctors ϕ_i in the canonical directions e_i of $\mathbb{R}^d$. Note that the convergence of u_ε to $\bar{u}$ in $\dot{H}^1(\mathbb{R}^d)$ is only weak since Du_ε typically displays spatial oscillations at scale ε, which are not captured by the limit $D\bar{u}$. These oscillations are however well-described by those of the corrector field $D\phi(\frac{\cdot}{\varepsilon})$ through the following two-scale expansion,[1]

$$Du_\varepsilon \approx (D\phi_i(\tfrac{\cdot}{\varepsilon}) + e_i) D_i \bar{u}, \tag{1.3}$$

in the sense that $Du_\varepsilon - (D\phi_i(\frac{\cdot}{\varepsilon}) + e_i) D_i \bar{u}$ converges strongly to zero in $L^2(\mathbb{R}^d)^d$. In the random setting, this theory of oscillations was recently optimally quantified in [26, 29, 33, 3] (see also [4, Chapter 6]).

As opposed to periodic homogenization, which boils down to the sole understanding of the (spatial) oscillations of Du_ε, the stochastic setting involves the (random) fluctuations

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[1]We systematically use Einstein's summation rule on repeated indices.

of Du_ε next to its oscillations. More precisely, whereas oscillations reflect the (almost sure) lack of strong compactness for Du_ε in $L^2(\mathbb{R}^d)^d$, fluctuations are concerned with the leading-order probabilistic behavior of weak-type expressions of the form $\int_{\mathbb{R}^d} g \cdot Du_\varepsilon$ for $g \in C_c^\infty(\mathbb{R}^d)^d$. Let us emphasize that in the case of a weakly correlated coefficient field $\boldsymbol{a}$ the error in the two-scale expansion (1.3) is of order ε in $L^2(\mathbb{R}^d)^d$ (or $\varepsilon|\log \varepsilon|^{\frac{1}{2}}$ for $d = 2$) while fluctuations of Du_ε display the central limit theorem (CLT) scaling $\varepsilon^{\frac{d}{2}}$, so that (1.3) is not expected to be accurate in that scaling. This was indeed first realized in dimension $d \geq 3$ by Gu and Mourrat [36, Section 3.2] (see also the last item in Remarks 2.12 below for $d = 2$), who further argue that accuracy in (1.3) in the fluctuation scaling cannot even be reached by the use of higher-order correctors. The corrector field $D\phi$ is therefore the driving quantity for oscillations but a priori not for fluctuations.

In the present article, we develop a new theory of fluctuations in stochastic homogenization that builds on the known theory of oscillations, and our main achievement is the rigorous identification of the corresponding driving quantity for fluctuations. The key consists in focusing on the *homogenization commutator of the solution*,

$$\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}} Du_\varepsilon, \tag{1.4}$$

and in studying its relation to the *(standard) homogenization commutator* $\Xi := (\Xi_i)_{i=1}^d$ defined by

$$\Xi_i := \boldsymbol{a}(D\phi_i + e_i) - \bar{\boldsymbol{a}}(D\phi_i + e_i), \qquad \Xi_{ij} := (\Xi_i)_j. \tag{1.5}$$

We first briefly comment on the special form of these quantities. As well-known in applications, homogenization is the rigorous version of averaging fields and fluxes in a consistent way, as made precise in the very definition of H-convergence by Murat and Tartar [50], which indeed requires both weak convergence of the fields $Du_\varepsilon \rightharpoonup D\bar{u}$ and of the fluxes $\boldsymbol{a}_\varepsilon Du_\varepsilon \rightharpoonup \bar{\boldsymbol{a}} D\bar{u}$ in $L^2(\mathbb{R}^d)^d$, to the effect of

$$\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}} Du_\varepsilon \rightharpoonup 0.$$

This weak convergence of the homogenization commutator (1.4) is the mathematical formulation of the so-called Hill-Mandel relation in mechanics [39, 40]. Applied to the corrector, this justifies the definition of the (standard) homogenization commutator Ξ in (1.5), which is thus seen as a natural and intrinsic measure of the accuracy of homogenization for large-scale averages. In addition, as pointed out in Section 1.4, this commutator turns out to be related to variational quantities first considered by Armstrong and Smart [6]. In these terms, our theory of fluctuations consists of the following three main principles.

(I) First and most importantly, the two-scale expansion of the homogenization commutator of the solution

$$\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}} Du_\varepsilon - \mathbb{E}\left[\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}} Du_\varepsilon\right] \approx \Xi_i(\tfrac{\cdot}{\varepsilon}) D_i\bar{u} \tag{1.6}$$

is (generically) accurate in the fluctuation scaling, in the sense of the following relative estimate,

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} g \cdot \left(\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}} Du_\varepsilon - \mathbb{E}\left[\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}} Du_\varepsilon\right]\right) - \int_{\mathbb{R}^d} g \cdot \Xi_i(\tfrac{\cdot}{\varepsilon}) D_i\bar{u}\right|^2\right]^{\frac{1}{2}}$$

$$\leq o(1)\,\mathbb{E}\left[\left|\int_{\mathbb{R}^d} g \cdot \Xi_i(\tfrac{\cdot}{\varepsilon}) D_i\bar{u}\right|^2\right]^{\frac{1}{2}}, \quad (1.7)$$

where $o(1) \downarrow 0$ as $\varepsilon \downarrow 0$, for all $g \in C_c^\infty(\mathbb{R}^d)^d$. Let us emphasize again that this property is nontrivial and is due to the special form of the commutator — in particular, it is not true for (1.3).

(II) Second, both the fluctuations of the field Du_ε and of the flux $\boldsymbol{a}_\varepsilon Du_\varepsilon$ can be recovered through *deterministic* projections of the fluctuations of the homogenization commutator (1.4), which shows that no information is lost by passing to the commutator. More precisely, the following elementary identities are easily checked,

$$\int_{\mathbb{R}^d} g \cdot (Du_\varepsilon - D\bar{u}) \;=\; -\int_{\mathbb{R}^d} (\bar{\mathcal{P}}_H^* g) \cdot (\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}} Du_\varepsilon),$$

$$\int_{\mathbb{R}^d} g \cdot (\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}} D\bar{u}) \;=\; \int_{\mathbb{R}^d} (\bar{\mathcal{P}}_L^* g) \cdot (\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}} Du_\varepsilon), \qquad (1.8)$$

in terms of the Helmholtz and Leray projections in $\mathrm{L}^2(\mathbb{R}^d)^d$,

$$\bar{\mathcal{P}}_H := D(D \cdot \bar{\boldsymbol{a}} D)^{-1} D\cdot, \qquad \bar{\mathcal{P}}_L := \mathrm{Id} - \bar{\mathcal{P}}_H \bar{\boldsymbol{a}},$$

$$\bar{\mathcal{P}}_H^* := D(D \cdot \bar{\boldsymbol{a}}^* D)^{-1} D\cdot, \qquad \bar{\mathcal{P}}_L^* := \mathrm{Id} - \bar{\mathcal{P}}_H \bar{\boldsymbol{a}}^*, \qquad (1.9)$$

where $\bar{\boldsymbol{a}}^*$ denotes the transpose of $\bar{\boldsymbol{a}}$. Similarly, the fluctuations of the field $D\phi$ and of the flux $\boldsymbol{a} D\phi$ of the corrector are determined by those of the standard commutator Ξ itself: indeed, the definition of Ξ yields $-D \cdot \bar{\boldsymbol{a}} D\phi_i = D \cdot \Xi_i$ and $\boldsymbol{a}(D\phi_i + e_i) - \bar{\boldsymbol{a}} e_i = \Xi_i + \bar{\boldsymbol{a}} D\phi_i$, to the effect of $D\phi_i = -\bar{\mathcal{P}}_H \Xi_i$ and $\boldsymbol{a}(D\phi_i + e_i) - \bar{\boldsymbol{a}} e_i = (\mathrm{Id} - \bar{\boldsymbol{a}} \bar{\mathcal{P}}_H) \Xi_i$ in the stationary sense, hence formally,

$$\int_{\mathbb{R}^d} F : D\phi(\tfrac{\cdot}{\varepsilon}) \;=\; -\int_{\mathbb{R}^d} (\bar{\mathcal{P}}_H^* F) : \Xi(\tfrac{\cdot}{\varepsilon}),$$

$$\int_{\mathbb{R}^d} F : \big(\boldsymbol{a}_\varepsilon(D\phi(\tfrac{\cdot}{\varepsilon}) + \mathrm{Id}) - \bar{\boldsymbol{a}}\big) \;=\; \int_{\mathbb{R}^d} (\bar{\mathcal{P}}_L^* F) : \Xi(\tfrac{\cdot}{\varepsilon}), \qquad (1.10)$$

where $\bar{\mathcal{P}}_H^*$ and $\bar{\mathcal{P}}_L^*$ act on the second index of the tensor field F; a suitable sense to these identities is given as part of Corollary 2.4 below.

(III) Third, the standard homogenization commutator Ξ is an approximately local function of the coefficients $\boldsymbol{a}$, which allows to infer the large-scale behavior of Ξ from the large-scale behavior of $\boldsymbol{a}$ itself. While this locality property does not hold for the corrector $D\phi$ itself, it makes Ξ a particularly relevant quantity for fluctuations. This property is best seen when formally computing the so-called "vertical" derivatives of Ξ with respect to $\boldsymbol{a}$: Letting ϕ^* denote the corrector associated with the pointwise transpose coefficient field $\boldsymbol{a}^*$, and letting σ^* denote the corresponding flux corrector (cf. (3.1) below), we obtain (cf. (3.18))

$$\frac{\partial}{\partial \boldsymbol{a}(x)} \Xi_{ij} = (D\phi_j^* + e_j) \cdot \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{a}(x)} (D\phi_i + e_i)$$

$$- D \cdot \left(\phi_j^* \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{a}(x)} (D\phi_i + e_i) \right) - D \cdot \left((\phi_j^* \boldsymbol{a} + \sigma_j^*) \frac{\partial D\phi_i}{\partial \boldsymbol{a}(x)} \right). \quad (1.11)$$

In view of $\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{a}(x)} = \delta(\cdot - x)$, the first right-hand side term reveals an exactly local dependence upon $\boldsymbol{a}$. The second term is exactly local as well, but since it is written in divergence form its contribution is negligible when integrating on large scales. The only non-local effect comes from the last term due to $\frac{\partial D\phi}{\partial \boldsymbol{a}}$, which is given by the mixed derivative of the Green's function for $-D \cdot \boldsymbol{a} D$ and thus is expected to have

only borderline integrable decay. However, it also appears inside a divergence, hence it is negligible when integrated on large scales. In fact, in this work, the accuracy in (1.6) is established relying on a similar representation of the vertical derivative $\frac{\partial}{\partial \boldsymbol{a}}(\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}}Du_\varepsilon - \Xi_i(\frac{\cdot}{\varepsilon})D_i\bar{u})$, cf. Lemma 3.2.

In view of these main principles, our fluctuation results split into two independent parts. We focus on the case of weakly correlated coefficient fields.

(R$_1$) We start from (II) in form of identity (1.8),

$$\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot (Du_\varepsilon - D\bar{u}) = -\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} (\bar{\mathcal{P}}_H^* g) \cdot (\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}}Du_\varepsilon),$$

where $\bar{\mathcal{P}}_H^* g$ is a deterministic transformation of g. While Du_ε cannot be accurately replaced by its two-scale expansion in the left-hand side (cf. [36]), property (I) states that this strategy is possible in the right-hand side due to the special structure of the homogenization commutator $\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}}Du_\varepsilon$. More precisely, in case of a weakly correlated coefficient field, an $O(\varepsilon)$ error estimate is obtained for $d > 2$,

$$\text{Var}\left[\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} (\bar{\mathcal{P}}_H^* g) \cdot (\boldsymbol{a}_\varepsilon Du_\varepsilon - \bar{\boldsymbol{a}}Du_\varepsilon) - \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} (\bar{\mathcal{P}}_H^* g) \cdot \Xi_i(\tfrac{\cdot}{\varepsilon})D_i\bar{u}\right]^{\frac{1}{2}} \lesssim \varepsilon,$$

in terms of the standard homogenization commutator Ξ. Leading-order fluctuations of Du_ε are thus determined by the scaling limit of this canonical object Ξ in form of a suitable two-scale expansion,

$$\text{Var}\left[\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot Du_\varepsilon + \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} (\bar{\mathcal{P}}_H^* g) \cdot \Xi_i(\tfrac{\cdot}{\varepsilon})D_i\bar{u}\right]^{\frac{1}{2}} \lesssim \varepsilon.$$

Corresponding results hold for $\boldsymbol{a}_\varepsilon Du_\varepsilon$, $D\phi(\frac{\cdot}{\varepsilon})$, and $\boldsymbol{a}_\varepsilon D\phi(\frac{\cdot}{\varepsilon})$. In order to emphasize that it is not only a comparison of limiting laws, but that random fluctuations are considered and compared as measurable functions on the space Ω of realizations of the coefficient field, with an error estimated in the strong L^2 norm on that space, we henceforth refer to this two-scale relation as the *pathwise structure of fluctuations*, in analogy with the language of SPDEs.

(R$_2$) If $\boldsymbol{a}$ is weakly correlated, the locality property (III) formally ensures that the homogenization commutator Ξ is also weakly correlated, so that the CLT would ensure that the rescaling $\varepsilon^{-\frac{d}{2}}\Xi(\frac{\cdot}{\varepsilon})$ converges in law to a Gaussian white noise Γ. We can further deduce optimal rates for this scaling limit.

The combination of the pathwise result (R$_1$) with the scaling limit result (R$_2$) leads to the following joint convergence in law,

$$\left(\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F : \Xi(\tfrac{\cdot}{\varepsilon}) , \ \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot (Du_\varepsilon - \mathbb{E}[Du_\varepsilon]) , \ \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot (\boldsymbol{a}_\varepsilon Du_\varepsilon - \mathbb{E}[\boldsymbol{a}_\varepsilon Du_\varepsilon]) , \right.$$

$$\left. \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F : D\phi(\tfrac{\cdot}{\varepsilon}) , \ \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F : \left(\boldsymbol{a}_\varepsilon(D\phi(\tfrac{\cdot}{\varepsilon}) + \text{Id}) - \bar{\boldsymbol{a}}\right)\right)$$

$$\to \left(\Gamma(F) , \ \Gamma(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g) , \ -\Gamma(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_L^* g) , \ -\Gamma(\bar{\mathcal{P}}_H^* F) , \ \Gamma(\bar{\mathcal{P}}_L^* F)\right), \qquad (1.12)$$

where the pathwise structure is manifested by the fact that the same random field Γ describes at the same time the different marginals of the limiting joint law. Whereas this scaling limit (1.12) is essentially already contained in previous contributions on the

topic [49, 36, 48] (cf. Section 1.4), the pathwise result (R$_1$) characterizes not only the (joint) law of fluctuations, but actually provides a corrector result for fluctuations in form of an accurate description in L$^2(\Omega)$ of leading-order fluctuations of a general solution Du_ε by means of an off-line procedure using the suitable intrinsic object Ξ. In parallel with the classical two-scale expansion (1.3), which similarly describes oscillations in terms of the corrector, this reduction of complexity for fluctuations in stochastic homogenization is bound to affect multi-scale computing and uncertainty quantification in an essential way.

One could be tempted to reduce our results (R$_1$)–(R$_2$) to an (optimal) quantification of the joint convergence in law (1.12). However, as emphasized in Section 1.3, this is in general not correct: for instance, in the case of coefficient fields with thick tails, the pathwise result (R$_1$) still holds true whereas the scaling limit result may fail. This advocates that (R$_1$) and (R$_2$) are of different natures and should be treated separately. From a broader perspective, the main novelty of the present contribution does not rely on scaling limit results per se, but rather in uncovering the mechanism that leads to them. As discussed in Section 1.4, similar results had been conjectured by Armstrong, Gu, and Mourrat in [36, 47] based on the variational and renormalization perspective of [6, 5, 2, 3, 4]; this heuristic has not been made rigorous yet.

1.2. **Main results.** In order to present our theory of fluctuations, we place ourselves in the simplest setting possible and focus on the random conductance model, that is, the discrete setting with iid conductances, which we think of as the prototype for weakly correlated coefficient fields. Although conceptually simpler on the stochastic side, the discrete setting adds some technical inconveniences on the deterministic side, including a discretization error. Our main result takes on the following guise; precise notation and assumptions on the model are postponed to Section 2, as well as many remarks and corollaries. Items (i) and (ii) below (together with the non-degeneracy in (iv)) imply property (I) in form of the relative estimate (1.7) with the optimal rate $o(1) \simeq_{f,g} \varepsilon\mu_d(\frac{1}{\varepsilon})^{\frac{1}{2}}$. Items (iii) and (iv) are manifestations of the locality property (III). Throughout, the notation $\lesssim_\gamma$ (with possibly a subscript γ) stands for $\leq$ up to a multiplicative constant C_γ that only depends on d, λ, and γ (through a suitable norm of γ, should γ be a function).

Theorem 1. *Consider the (iid) random conductance model, and assume that the law of conductances is non-degenerate. Then the following hold for all $\varepsilon > 0$,*

(i) CLT scaling: *For all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$,*

$$\mathbb{E}\left[\left|\varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d} F : \Xi(\tfrac{\cdot}{\varepsilon})\right|^2\right]^{\frac{1}{2}} \lesssim_F 1.$$

(ii) Pathwise structure (with optimal error estimate): *For all $f, g \in C_c^\infty(\mathbb{R}^d)^d$, letting u_ε and $\bar{u}$ denote the solutions of (the discrete version of) (1.1) and of (1.2),*

$$\mathbb{E}\left[\left|\varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d} g \cdot \left(\boldsymbol{a}_\varepsilon\nabla_\varepsilon u_\varepsilon - \bar{\boldsymbol{a}}\nabla_\varepsilon u_\varepsilon - \mathbb{E}\left[\boldsymbol{a}_\varepsilon\nabla_\varepsilon u_\varepsilon - \bar{\boldsymbol{a}}\nabla_\varepsilon u_\varepsilon\right]\right) - \varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d} g \cdot \Xi_i(\tfrac{\cdot}{\varepsilon})D_i\bar{u}\right|^2\right]^{\frac{1}{2}}$$
$$\lesssim_{f,g} \varepsilon\mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}}, \quad (1.13)$$

where we set for all $r > 0$,

$$\mu_d(r) := \begin{cases} r & : \quad d = 1, \\ \log(2 + r) & : \quad d = 2, \\ 1 & : \quad d > 2. \end{cases} \tag{1.14}$$

(iii) Asymptotic normality (with optimal rate): *For all $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$,*

$$\delta_{\mathcal{N}}\left(\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F : \Xi(\tfrac{\cdot}{\varepsilon}) \right) \lesssim_F \varepsilon^{\frac{d}{2}} \log(2 + \tfrac{1}{\varepsilon}),$$

where for a random variable $X \in \mathrm{L}^2(\Omega)$ its distance to normality is defined by

$$\delta_{\mathcal{N}}(X) := \mathrm{d}_{\mathrm{W}}\left(\frac{X}{\mathrm{Var}\,[X]^{\frac{1}{2}}}, \mathcal{N} \right) + \mathrm{d}_{\mathrm{K}}\left(\frac{X}{\mathrm{Var}\,[X]^{\frac{1}{2}}}, \mathcal{N} \right), \tag{1.15}$$

with $\mathcal{N}$ a standard Gaussian random variable and with $\mathrm{d}_{\mathrm{W}}\left(\cdot, \cdot\right)$ and $\mathrm{d}_{\mathrm{K}}\left(\cdot, \cdot\right)$ the Wasserstein and Kolmogorov metrics, that is, for random variables $X, Y \in \mathrm{L}^1(\Omega)$,

$$\begin{aligned} \mathrm{d}_{\mathrm{W}}\left(X, Y\right) &:= \sup \left\{ \mathbb{E}\left[f(X) - f(Y)\right] : f \in C^1(\mathbb{R}), \|\nabla f\|_{\mathrm{L}^\infty} \le 1 \right\}, \\ \mathrm{d}_{\mathrm{K}}\left(X, Y\right) &:= \sup \left\{ \left| \mathbb{P}\left[X \le x\right] - \mathbb{P}\left[Y \le x\right] \right| : x \in \mathbb{R}^d \right\}. \end{aligned}$$

(iv) Convergence of the covariance structure (with optimal rate): *There exists a non-degenerate symmetric 4-tensor $\mathcal{Q}$ such that for all $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$,*

$$\left| \mathrm{Var}\left[\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F : \Xi(\tfrac{\cdot}{\varepsilon}) \right] - \int_{\mathbb{R}^d} F : \mathcal{Q}\, F \right| \lesssim_F \varepsilon \mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}}.$$

In particular, combined with item (iii), this yields the convergence in law of $\varepsilon^{-\frac{d}{2}} \Xi(\tfrac{\cdot}{\varepsilon})$ to a (matrix-valued) Gaussian white noise Γ with covariance structure $\mathcal{Q}$, and the (discrete version of the) joint convergence result (1.12) follows. ◇

This fluctuation theory is complemented by the following characterization of the fluctuation tensor $\mathcal{Q}$ by periodization in law. This characterization comes in form of a representative volume element (RVE) method, for which we give the optimal error estimate. In particular, comparing with the results for the RVE approximation $\bar{\boldsymbol{a}}_{L,N}$ of the homogenized coefficients $\bar{\boldsymbol{a}}$ (cf. [31, 32, 28]), and choosing $N \simeq L^d$ below, we may conclude that an RVE approximation for $\mathcal{Q}$ with accuracy $O(L^{-\frac{d}{2}})$ (up to logarithmic corrections) is extracted at the same cost as an RVE approximation for $\bar{\boldsymbol{a}}$ with accuracy $O(L^{-d})$. Precise assumptions and notation are again postponed to Section 2.

Theorem 2. *Consider the (iid) random conductance model. Define*

$$\bar{\boldsymbol{a}}_L e_i := \fint_{Q_L} \boldsymbol{a}_L (\nabla \phi_{L,i} + e_i), \tag{1.16}$$

in terms of the L-periodized coefficient field $\boldsymbol{a}_L$ and corrector ϕ_L. Then the fluctuation tensor $\mathcal{Q}$ defined in Theorem 1(iv) satisfies

$$\mathcal{Q} = \lim_{L \uparrow \infty} \mathrm{Var}\left[L^{\frac{d}{2}} \bar{\boldsymbol{a}}_L^* \right]. \tag{1.17}$$

In addition, considering iid realizations $(\boldsymbol{a}_L^{(n)})_{n=1}^N$ of $\boldsymbol{a}_L$ and setting $\bar{\boldsymbol{a}}_L^{(n)} := \bar{\boldsymbol{a}}_L(\boldsymbol{a}_L^{(n)})$, we define the RVE approximation as the square of the sample standard deviation

$$\mathcal{Q}_{L,N} := \frac{L^d}{N-1} \sum_{n=1}^N \left(\bar{\boldsymbol{a}}_L^{(n)} - \bar{\boldsymbol{a}}_{L,N}\right)^* \otimes \left(\bar{\boldsymbol{a}}_L^{(n)} - \bar{\boldsymbol{a}}_{L,N}\right)^*, \qquad \bar{\boldsymbol{a}}_{L,N} := \frac{1}{N}\sum_{n=1}^N \bar{\boldsymbol{a}}_L^{(n)}, \quad (1.18)$$

and for all $L, N \geq 2$ there holds

$$|\mathrm{Var}\,[\mathcal{Q}_{L,N}]\,|^{\frac{1}{2}} \lesssim N^{-\frac{1}{2}}, \qquad |\mathbb{E}\,[\mathcal{Q}_{L,N}] - \mathcal{Q}| \lesssim L^{-\frac{d}{2}} \log^{\frac{d}{2}} L. \qquad\qquad \diamond$$

The ingredients to the proof of the above results are mainly twofold. The first one consists of specific concentration inequalities for iid conductances, which reduce various questions to sensitivity estimates, that is, to estimates of "vertical derivatives" with respect to the coefficient field $\boldsymbol{a}$ (which quantify how random solutions are affected by changing the conductance at one edge). This line of argument in stochastic homogenization originates in an unpublished work by Naddaf and Spencer [51] (see also [52]), and was considerably exploited in recent years starting with the early contributions of the last two authors [31, 32]. More precisely, items (i) and (ii) in Theorem 1 are established using a Poincaré inequality in the probability space (cf. Lemma 3.1), item (iii) using a second-order Poincaré inequality due to Chatterjee [10, 11] (cf. Lemma 4.1), and item (iv) using (an iid version of) the so-called Helffer-Sjöstrand representation formula for variances [38, 58, 52] (cf. Lemma 5.1). Apart from these convenient tools specific to the random conductance model (which can be either extended [17, 18, 19] or avoided in some settings, cf. Section 1.3), the proofs only rely on arguments that extend to the continuum setting and to the case of systems. The second main ingredient consists of large-scale regularity theory in form of a large-scale weighted Calderón-Zygmund theory for the random elliptic operator $-\nabla\cdot\boldsymbol{a}\nabla$ (cf. [1, 27, 4]), which is exploited here to properly estimate vertical derivatives of quantities of interest. Large-scale regularity originates in the work of Armstrong and Smart [6] (see also annealed regularity in [46]) and has been recently fully developed in [5, 2, 3, 4] and in [27, 29, 33].

1.3. **Extensions and robustness of results.** The present contribution presents our new theory of fluctuations in the simplest possible setting of the iid conductance model. As briefly described below, this theory will be extended to more general settings and in different directions in the companion articles [20, 14, 15, 22, 16]. These developments highlight the robustness of the pathwise structure of fluctuations unravelled here.

A first extension concerns the (non-symmetric) continuous setting. In [20, 14], in the spirit of [17, 18, 19], we consider the case when the coefficient field satisfies some proxy for the specific concentration inequalities of the iid setting; this actually covers all the models considered in the reference textbook [60] on random heterogeneous materials. In addition, we analyze the case of strongly correlated coefficient fields. More precisely, we mainly focus on the model framework of a coefficient field given by (a local function of) a Gaussian field that has an algebraically decaying (not necessarily integrable) covariance function c, say at some fixed rate $c(x) \simeq (1+|x|)^{-\beta}$ parametrized by $\beta > 0$. For such coefficient fields, we extend in [20] the validity of the two-scale expansion (1.13) of the homogenization commutator in the suitable fluctuation scaling (that is, $\varepsilon^{-\frac{d}{2}}$ is replaced by $\varepsilon^{-\frac{1}{2}(\beta\wedge d)}$, with a logarithmic correction when $\beta = d$). The proof relies on a weighted version of a Poincaré inequality in the probability space (cf. [17, 18]), which can alternatively be reformulated in terms of Malliavin calculus, together with the available quantitative

homogenization theory in that specific setting [27, 29]. This result illustrates the robustness of the pathwise structure with respect to the large-scale behavior of the homogenization commutator. Indeed, in dimension $d = 1$ (in which case the quantities under investigation are simpler and explicit), two typical behaviors have been identified in terms of the scaling limit of Ξ, depending on the parameter β (cf. [7]):

- For $\beta > d = 1$, the commutator Ξ displays the CLT scaling and $\varepsilon^{-\frac{d}{2}}\Xi(\frac{\cdot}{\varepsilon})$ converges to a Gaussian white noise (Gaussian fluctuations, local limiting covariance structure), but the convergence rate is arbitrarily slow as β gets closer to $d = 1$.
- For $0 < \beta < d = 1$, the suitable rescaling $\varepsilon^{-\frac{\beta}{2}}\Xi(\frac{\cdot}{\varepsilon})$ generically converges along a subsequence to a fractional Gaussian field (Gaussian fluctuations, nonlocal limiting covariance structure, potentially no uniqueness of the limit). (Note that a different, non-Gaussian behavior may also occur in this case, cf. [35, 45].)

In particular, the pathwise result (1.13) is shown to hold in both cases with rate $\varepsilon^{\frac{\beta}{2} \wedge 1}$ (up to a logarithmic correction) whereas the rescaled homogenization commutator does not necessarily converge to white noise, may converge arbitrarily slowly, or may even have no limit at all, which shows that the pathwise structure of fluctuations is much more robust than scaling limit results. The identification of the scaling limit in higher dimensions with optimal rates is addressed in [14] for the whole range of values of $\beta > 0$ (see also [22] for $\beta > d$), where we further exploit the locality of Ξ with respect to the coefficient field together with tools from Malliavin calculus; this work extends [7] to dimensions $d \geq 2$.

A second extension concerns the continuous setting without product space assumption (that is, without using any proxy for the specific concentration inequalities of the iid setting), and more specifically we focus on random coefficient fields with finite range of dependence. The convergence in law of the rescaled homogenization commutator $\varepsilon^{-\frac{d}{2}}\Xi(\frac{\cdot}{\varepsilon})$ to a Gaussian white noise (albeit without optimal convergence rate) was obtained in that case independently in [3] and [33]. The proof of the validity of the two-scale expansion (1.13) of the homogenization commutator in the CLT scaling is more involved and will be presented in [15] based on the semi-group approach of [33], as well as an optimal convergence rate for the scaling limit of the commutator based on [24].

A third extension concerns higher-order corrections to fluctuation theory: in [22], a suitable higher-order version of the homogenization commutator is identified and its higher-order two-scale expansion is shown to indeed have higher-order accuracy in the CLT scaling, which in particular leads to a higher-order description of fluctuations of the solution operator, while the scaling limit of the higher-order standard homogenization commutator is again easily computed in view of a corresponding higher-order locality property. In other words, the key properties (I)–(III) are fully extended to higher order, and this theory is in line with the usual higher-order theory of oscillations based on higher-order correctors.

A last extension concerns fluctuations for solutions of the wave equation with random coefficients. The homogenization commutator (1.4) is also relevant in that setting and similarly leads to a corresponding pathwise theory of fluctuations [16]. This theory can further be extended to higher orders in terms of suitable higher-order hyperbolic correctors introduced in [21].

1.4. **Relation to previous works.** The description of fluctuations in stochastic homogenization has been the most central open question in the field since its very beginning in the

late 1970s, and it has been a particularly active topic in the last few years. We organize the discussion of previous works on fluctuations in separate parts addressing different aspects of the theory for the random conductance model.

- *CLT scaling.*
  The CLT scaling for the solution operator in form of uniform moment bounds on $\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot (\nabla_\varepsilon u_\varepsilon - \mathbb{E}\left[\nabla_\varepsilon u_\varepsilon\right])$ was first investigated by Conlon and Naddaf [12] and by the second author [25], and it was established in its optimal form in all dimensions by Marahrens and the third author [46] for the random conductance model.

- *Asymptotic normality.*
  The first asymptotic normality result in stochastic homogenization is due to Nolen [54, 55] (which contains the first use of second-order Poincaré inequalities in the field), see also Rossignol [57], Biskup, Salvi, and Wolff [9].

- *Scaling limits.*
  The covariance structure of the corrector was identified by Mourrat and the third author [49] (for $d > 2$), and the limiting variance of the energy of the periodized corrector (which a posteriori is nothing but the average of the periodized homogenization commutator) by the second author and Nolen [30]. These works were then extended to the solution operator by Gu and Mourrat [36] (for $d > 2$), which essentially yields as a corollary the joint convergence in law (1.12). These works were based on Chatterjee's second-order Poincaré inequalities [10, 11] (cf. Lemma 4.1) and on the Helffer-Sjöstrand representation formula [38, 58, 52] (cf. Lemma 5.1) together with tools from the early quantitative homogenization theory as developed by the last two authors [31, 32] (inspired by the unpublished work by Naddaf and Spencer [51]), the last two authors and Neukamm [28, 26], and Marahrens and the last author [46]. Note that the present contribution originated in the attempt to upgrade the results in [30] into a functional CLT for a suitable energy density.

- *Genesis of the homogenization commutator.*
  Although strongly motivated by the very notion of H-convergence as defined by Murat and Tartar [50], the explicit apparition of homogenization commutators and the discovery of their remarkable properties are much more recent. A variational quantity related to the standard commutator can first be traced back in the work of Armstrong and Smart [6] and has been the driving quantity in the subsequent developments by Armstrong, Kuusi, and Mourrat [2, 3, 4]; the explicit link with the standard commutator in the form (1.5) is provided e.g. in [4, Lemma 4.25]. The idea that such a quantity might be important for fluctuations was originally formulated by Armstrong, Gu, and Mourrat in form of a heuristic [36, 47]. The early version of the present work provided the first rigorous result in that direction, as well as a proof of the locality of the standard commutator and its convergence to white noise. This scaling limit result for the standard commutator was also independently obtained in [3] and in [33] for coefficient fields with finite range of dependence (although without optimal rates); in view of the obvious relation (1.10), note that this characterizes fluctuations of the corrector. Importantly, instead of a variational interpretation, the homogenization commutator is given here a more intrinsic and practical definition motivated by H-convergence.

- *Pathwise structure of fluctuations.*
  The pathwise structure in the restricted form of the joint convergence in law (1.12) was essentially observed by Gu and Mourrat in [36], and the first attempt at a general

explanation of this structure was proposed by Armstrong, Gu, and Mourrat in form of a heuristic [36, 47]. No explanation was however suggested why fluctuations of the solution operator could be expressed in terms of the same intrinsic object as fluctuations of the corrector (that is, in terms of the standard commutator), which constitutes the main gap in that work. As we show here, this question can be reformulated as the validity of a suitable two-scale expansion principle for fluctuations of the solution operator. Since the two-scale expansion of the solution operator itself is known not to be accurate in the fluctuation scaling [36], it was unclear whether a two-scale expansion principle could actually hold for fluctuations. Our main contribution in this work precisely fills this gap in form of the accuracy of the two-scale expansion of commutators. This key missing part in the heuristic proposed in [36, 47] has still not been filled within a variational and renormalization perspective.

To sum up, our main achievement is the clarification of the connection between fluctuations and two-scale expansions by means of the notion of homogenization commutator. The pathwise structure in form of the two-scale expansion of commutators is to fluctuations what the usual two-scale expansion is to oscillations. The whole mechanism that drives fluctuations, as summarized in properties (I)–(III), is made precise and rigorous here for the first time in any setting. This new theory of fluctuations turns out to be remarkably robust, as shown by the various extensions that it has triggered. Even for the more specific question of scaling limits, this theory provides new optimal convergence rates.

2. Main results

In this section, we introduce notation and assumptions on the random conductance model, we state precise versions of the main results (in particular including explicit norms of the test functions in the estimates), and we discuss various corollaries.

2.1. **Notation and assumptions.** We start by introducing the random conductance model on the integer lattice $\mathbb{Z}^d$, which is the framework of our main results. We denote by $\{e_i\}_{i=1}^d$ the canonical basis of $\mathbb{R}^d$, and we regard $\mathbb{Z}^d$ as a graph with (unoriented) edge set $\mathcal{B} = \{(x, z) \in \mathbb{Z}^d \times \mathbb{Z}^d : |x - z| = 1\}$. For edges $(x, z) \in \mathcal{B}$, we write $x \sim z$. We define the set of conductances $\{a(b)\}_{b \in \mathcal{B}}$ by $\Omega = [\lambda, 1]^{\mathcal{B}}$ for some fixed $0 < \lambda \leq 1$. We endow Ω with the σ-algebra generated by cylinder sets and with a probability measure $\mathbb{P}$. We denote by $\mathbb{E}[\cdot]$, $\mathrm{Var}[\cdot]$, and $\mathrm{Cov}[\cdot; \cdot]$ the associated expectation, variance, and covariance. A realization $a \in \Omega$ is by definition a collection $\{a(b)\}_{b \in \mathcal{B}}$ of conductances. A random field $u : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is said to be stationary if it is shift-covariant, in the sense of $u(x, a(\cdot - z)) = u(x - z, a)$ for all $x, z \in \mathbb{R}^d$ and $a \in \Omega$. In this contribution, we focus on the case when the probability measure $\mathbb{P}$ is a product measure, that is, when the conductances $\{a(b)\}_{b \in \mathcal{B}}$ are iid random variables, and we shall make use of available functional inequalities in this product probability space.

Let ∇ denote the forward discrete gradient $(u : \mathbb{Z}^d \to \mathbb{R}) \mapsto (\nabla u : \mathbb{Z}^d \to \mathbb{R}^d)$ defined componentwise by $\nabla_i u(x) = u(x + e_i) - u(x)$ for $1 \leq i \leq d$, and let ∇^* denote the backward discrete gradient $(u : \mathbb{Z}^d \to \mathbb{R}) \mapsto (\nabla^* u : \mathbb{Z}^d \to \mathbb{R}^d)$ defined componentwise by $\nabla_i^* u(x) = u(x) - u(x - e_i)$ for $1 \leq i \leq d$. The operator $-\nabla^* \cdot$ is thus the adjoint of ∇ on $\ell^2(\mathbb{Z}^d)$, and we consider the elliptic operator $-\nabla^* \cdot \boldsymbol{a}\nabla$ with coefficients

$$\boldsymbol{a} : x \mapsto \boldsymbol{a}(x) := \mathrm{diag}\left[a(x, x + e_1), \ldots, a(x, x + e_d)\right],$$

acting on functions $u : \mathbb{Z}^d \to \mathbb{R}$ as

$$-\nabla^* \cdot \boldsymbol{a} \nabla u(x) := \sum_{z : z \sim x} a(x, z)(u(x) - u(z)).$$

In order to state the standard qualitative homogenization result [43, 42] for the corresponding discrete elliptic equation, we consider for all $\varepsilon > 0$ the rescaled operator $-\nabla_\varepsilon^* \cdot \boldsymbol{a}_\varepsilon \nabla_\varepsilon$, where $\boldsymbol{a}_\varepsilon(\cdot) := \boldsymbol{a}(\frac{\cdot}{\varepsilon})$, and where ∇_ε and ∇_ε^* act on functions $u_\varepsilon : \mathbb{Z}_\varepsilon^d := \varepsilon \mathbb{Z}^d \to \mathbb{R}$, and are defined componentwise by $\nabla_{\varepsilon,i} u_\varepsilon(x) = \varepsilon^{-1}(u_\varepsilon(x + \varepsilon e_i) - u_\varepsilon(x))$ and $\nabla_{\varepsilon,i}^* u_\varepsilon(x) = \varepsilon^{-1}(u_\varepsilon(x) - u_\varepsilon(x - \varepsilon e_i))$ for all i. We shall also let ∇_ε and ∇_ε^* act on continuous functions $u : \mathbb{R}^d \to \mathbb{R}$, so that $\nabla_\varepsilon u$ and $\nabla_\varepsilon^* u$ are continuous functions as well. If $u \in C^1(\mathbb{R}^d)$, then $\nabla_\varepsilon u(x)$ and $\nabla_\varepsilon^* u(x)$ converge to the continuum gradient $Du(x)$ for all $x \in \mathbb{R}^d$ as $\varepsilon \downarrow 0$. In what follows, for all $m \geq 1$, we systematically extend maps $v : \mathbb{Z}^d \to \mathbb{R}^m$ to piecewise constant maps $\mathbb{R}^d \to \mathbb{R}^m$ (still denoted by v) by setting $v|_{Q(x)} := v(x)$ for all $x \in \mathbb{Z}^d$ (where $Q(x) := x + [-\frac{1}{2}, \frac{1}{2})^d$ is the unit cube centered at x), and we use this notation e.g. for $\boldsymbol{a} : \mathbb{Z}^d \to \mathbb{R}^{d \times d}$ (but also for $\phi : \mathbb{Z}^d \to \mathbb{R}^d$ and $\Xi : \mathbb{Z}^d \to \mathbb{R}^{d \times d}$ defined below). This systematic extension of functions from the lattice $\mathbb{Z}^d$ to $\mathbb{R}^d$ allows to state all discrete results in a form that would hold mutatis mutandis in the continuum setting. In addition, although in the discrete setting it is more natural to consider a symmetric coefficient field $\boldsymbol{a}$, we use non-symmetric notation in the statement of the results in view of the extension to the non-symmetric continuum setting, and we denote by $\boldsymbol{a}^*$ the pointwise transpose field associated with $\boldsymbol{a}$.

Qualitative stochastic homogenization [43, 42] ensures that, for all $f, g \in C_c^\infty(\mathbb{R}^d)^d$, almost surely, the unique Lax-Milgram solutions u_ε and v_ε in $\mathbb{R}^d$ of[2]

$$- \nabla_\varepsilon^* \cdot \boldsymbol{a}_\varepsilon \nabla_\varepsilon u_\varepsilon = \nabla_\varepsilon^* \cdot f, \qquad -\nabla_\varepsilon^* \cdot \boldsymbol{a}_\varepsilon^* \nabla_\varepsilon v_\varepsilon = \nabla_\varepsilon^* \cdot g, \tag{2.1}$$

converge weakly as $\varepsilon \downarrow 0$ to the unique Lax-Milgram solutions $\bar{u}$ and $\bar{v}$ in $\mathbb{R}^d$ of the (continuum) elliptic equations

$$-D \cdot \bar{\boldsymbol{a}} D \bar{u} = D \cdot f, \qquad -D \cdot \bar{\boldsymbol{a}}^* D \bar{v} = D \cdot g, \tag{2.2}$$

respectively, where $\bar{\boldsymbol{a}}$ is the homogenized matrix characterized by

$$\bar{\boldsymbol{a}} e_i = \mathbb{E}\left[\boldsymbol{a}(\nabla \phi_i + e_i)\right], \tag{2.3}$$

for all $1 \leq i \leq d$, and where ϕ_i is the so-called corrector in direction e_i. It is defined, for almost every realization $\boldsymbol{a}$, as the unique solution in $\mathbb{Z}^d$ of

$$- \nabla^* \cdot \boldsymbol{a}(\nabla \phi_i + e_i) = 0, \tag{2.4}$$

with $\nabla \phi_i$ stationary and having vanishing expectation and finite second moment, and with the anchoring $\phi_i(0) = 0$ at the origin. We then set $\phi := (\phi_i)_{i=1}^d$. Note that $\overline{(\boldsymbol{a}^*)} = (\bar{\boldsymbol{a}})^*$. For symmetric coefficient fields, $\boldsymbol{a}^* = \boldsymbol{a}$ and $\bar{\boldsymbol{a}}^* = \bar{\boldsymbol{a}}$.

We consider the fluctuations of the field ∇u_ε and of the flux $\boldsymbol{a}_\varepsilon \nabla u_\varepsilon$, as encoded in the random linear functionals $I_1^\varepsilon : (f, g) \mapsto I_1^\varepsilon(f, g)$ and $I_2^\varepsilon : (f, g) \mapsto I_2^\varepsilon(f, g)$ defined for all

---

[2] These equations are understood as follows: for all $x \in Q$ the function $u_\varepsilon(\varepsilon x + \cdot)$ on $\mathbb{Z}_\varepsilon^d$ is the solution of the discrete elliptic equation with coefficient $\boldsymbol{a}_\varepsilon$ and with right-hand side $\nabla_\varepsilon^* \cdot f(\varepsilon x + \cdot)$. This definition allows to state results in a form that holds in the continuum setting, and in terms of norms of the right-hand side that do not necessarily have to embed into the space of continuous functions.

$f, g \in C_c^\infty(\mathbb{R}^d)^d$ by

$$
\begin{aligned}
I_1^\varepsilon(f, g) &:= \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla_\varepsilon(u_\varepsilon - \mathbb{E}[u_\varepsilon]), \\
I_2^\varepsilon(f, g) &:= \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \big(\boldsymbol{a}_\varepsilon \nabla_\varepsilon u_\varepsilon - \mathbb{E}[\boldsymbol{a}_\varepsilon \nabla_\varepsilon u_\varepsilon]\big).
\end{aligned}
$$

We further encode the fluctuations of the corrector field $\nabla\phi$ and flux $\boldsymbol{a}(\nabla\phi + \mathrm{Id})$ in the random linear functionals $J_1^\varepsilon : F \mapsto J_1^\varepsilon(F)$ and $J_2^\varepsilon : F \mapsto J_2^\varepsilon(F)$ defined for all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ by

$$
\begin{aligned}
J_1^\varepsilon(F) &:= \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F(x) : \nabla\phi(\tfrac{x}{\varepsilon})\, dx, \\
J_2^\varepsilon(F) &:= \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F(x) : \big(\boldsymbol{a}_\varepsilon(x)(\nabla\phi(\tfrac{x}{\varepsilon}) + \mathrm{Id}) - \bar{\boldsymbol{a}}\big)\, dx.
\end{aligned}
$$

As explained above, a crucial role is played by the (standard) homogenization commutator, which in the present discrete setting takes the form $\Xi := (\Xi_i)_{i=1}^d$ with

$$
\Xi_i := \boldsymbol{a}(\nabla\phi_i + e_i) - \bar{\boldsymbol{a}}(\nabla\phi_i + e_i), \qquad \Xi_{ij} := (\Xi_i)_j, \tag{2.5}
$$

and by the error in the two-scale expansion of the homogenization commutator of the solution. These quantities are encoded in the random linear functionals $J_0^\varepsilon : F \mapsto J_0^\varepsilon(F)$ and $E^\varepsilon : (f, g) \mapsto E^\varepsilon(f, g)$ defined for all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ and all $f, g \in C_c^\infty(\mathbb{R}^d)^d$ by

$$
J_0^\varepsilon(F) := \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F(x) : \Xi(\tfrac{x}{\varepsilon})\, dx,
$$

$$
E^\varepsilon(f, g) := \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \big(\boldsymbol{a}_\varepsilon \nabla_\varepsilon u_\varepsilon - \bar{\boldsymbol{a}} \nabla_\varepsilon u_\varepsilon - \mathbb{E}[\boldsymbol{a}_\varepsilon \nabla_\varepsilon u_\varepsilon - \bar{\boldsymbol{a}} \nabla_\varepsilon u_\varepsilon]\big) - \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \Xi_i(\tfrac{\cdot}{\varepsilon}) D_i \bar{u}.
$$

Since the case $d = 1$ is much simpler and well-understood [34], we shall only focus in the sequel on dimensions $d \geq 2$.

We first recall the following uniform boundedness result for J_0^ε, establishing the CLT scaling for the fluctuations of the homogenization commutator (cf. Theorem 1(i)). Although essentially contained in the main result of the first contribution [31] of the second and third authors to the field, a short proof with up-to-date tools is included for completeness in Section 3. Note that the norm of the test function is substantially weaker than $\mathrm{L}^1(\mathbb{R}^d)$ in terms of integrability and is thus compatible with the behavior of Helmholtz projections of smooth and compactly supported functions, which is necessary for the pathwise result of Corollary 2.4 below.

Proposition 2.1. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, and set $w_1(z) := 1 + |z|$. For all $\varepsilon > 0$ and all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ we have for all $0 < p - 1 \ll 1$ and all $\alpha > d\frac{p-1}{4p}$,*

$$
\mathbb{E}\big[|J_0^\varepsilon(F)|^2\big]^{\frac{1}{2}} + \mathbb{E}\big[|J_1^\varepsilon(F)|^2\big]^{\frac{1}{2}} + \mathbb{E}\big[|J_2^\varepsilon(F)|^2\big]^{\frac{1}{2}} \lesssim_{\alpha,p} \|w_1^{2\alpha} F\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}.
$$

Above, and in the rest of this article, $\ll$ stands for $\leq$ up to a small enough multiplicative constant $C = C(d) > 0$ that only depends on d. $\diamond$

2.2. **Pathwise structure.** Our first main result establishes the smallness of the rescaled error E^ε in the two-scale expansion of the homogenization commutator (cf. Theorem 1(ii)), which is the key to the pathwise structure (1.12). As for Proposition 2.1, the proof relies on the Poincaré inequality in the probability space that is satisfied for iid coefficients. From a technical point of view, we exploit the large-scale Calderón-Zygmund theory for the random elliptic operator $-\nabla^* \cdot \boldsymbol{a}\nabla$ as developed in [1, 27] (see also [4, Section 7]).

Proposition 2.2. *Let* $d \geq 2$*, let* $\mathbb{P}$ *be a product measure, let* μ_d *be defined in* (1.14)*, and set* $w_1(z) := 1 + |z|$*. For all* $\varepsilon > 0$ *and all* $f, g \in C_c^\infty(\mathbb{R}^d)^d$ *we have for all* $0 < p - 1 \ll 1$ *and all* $\alpha > d\frac{p-1}{4p}$*,*

$$\mathbb{E}\left[|E^\varepsilon(f,g)|^2\right]^{\frac{1}{2}} \lesssim_{\alpha,p} \varepsilon\mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}}\left(\|f\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Dg\|_{\mathrm{L}^{4p}(\mathbb{R}^d)} + \|g\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Df\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}\right). \quad \diamond$$

Remark 2.3. For simplicity, the estimates in Propositions 2.1 and 2.2 above are stated and proved for second moments only, but the same arguments yield similar estimates for all algebraic (and even stretched exponential) moments (cf. [20, 22, 15]). $\quad \diamond$

In view of identity (1.8) (which indeed holds in the discrete setting up to a higher-order discretization error), the above result implies that the large-scale fluctuations of I_1^ε and I_2^ε are driven by those of J_0^ε in a pathwise sense. Identity (1.10) (which again holds up to a discretization error) yields a similar pathwise result for J_1^ε and J_2^ε.

Corollary 2.4. *Let* $d \geq 2$*, let* $\mathbb{P}$ *be a product measure, let* $\bar{\mathcal{P}}_H$*,* $\bar{\mathcal{P}}_H^*$*, and* $\bar{\mathcal{P}}_L^*$ *be defined in* (1.9)*, let* μ_d *be defined in* (1.14)*, and set* $w_1(z) := 1 + |z|$*. For all* $\varepsilon > 0$*, all* $f, g \in C_c^\infty(\mathbb{R}^d)^d$*, and all* $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$*, we have for all* $0 < p - 1 \ll 1$ *and all* $\alpha > d\frac{p-1}{4p}$*,*

$$\mathbb{E}\left[|I_1^\varepsilon(f,g) - J_0^\varepsilon(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g)|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[|I_2^\varepsilon(f,g) + J_0^\varepsilon(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_L^* g)|^2\right]^{\frac{1}{2}}$$

$$\lesssim_{\alpha,p} \varepsilon\mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}}\left(\|f\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Dg\|_{\mathrm{L}^{4p}(\mathbb{R}^d)} + \|g\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Df\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}\right), \quad (2.6)$$

and

$$\mathbb{E}\left[|J_1^\varepsilon(F) + J_0^\varepsilon(\bar{\mathcal{P}}_H^* F)|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[|J_2^\varepsilon(F) - J_0^\varepsilon(\bar{\mathcal{P}}_L^* F)|^2\right]^{\frac{1}{2}} \lesssim_{\alpha,p} \varepsilon\|w_1^{2\alpha}DF\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}, \quad (2.7)$$

where by definition we have $\bar{\mathcal{P}}_H f = -D\bar{u}$ *and* $\bar{\mathcal{P}}_H^* g = -D\bar{v}$*. In particular, we give meaning to* $J_0^\varepsilon(\bar{\mathcal{P}}_H^* F)$ *and* $J_0^\varepsilon(\bar{\mathcal{P}}_L^* F)$ *in* $\mathrm{L}^2(\Omega)$ *for all* $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$*, even when* $\bar{\mathcal{P}}_H^* F$ *and* $\bar{\mathcal{P}}_L^* F$ *do not have integrable decay.* $\quad \diamond$

Remark 2.5. For all $f, g \in C_c^\infty(\mathbb{R}^d)^d$, we may also consider the unique Lax-Milgram solutions u_ε° and v_ε° in $\mathbb{R}^d$ of

$$-\nabla_\varepsilon^* \cdot \boldsymbol{a}_\varepsilon \nabla_\varepsilon u_\varepsilon^\circ = \nabla_\varepsilon^* \cdot \boldsymbol{a}_\varepsilon f, \qquad -\nabla_\varepsilon^* \cdot \boldsymbol{a}_\varepsilon^* \nabla_\varepsilon v_\varepsilon^\circ = \nabla_\varepsilon^* \cdot \boldsymbol{a}_\varepsilon^* g,$$

which, almost surely, converge weakly as $\varepsilon \downarrow 0$ to the unique Lax-Milgram solutions $\bar{u}^\circ$ and $\bar{v}^\circ$ in $\mathbb{R}^d$ of

$$-D \cdot \bar{\boldsymbol{a}}D\bar{u}^\circ = D \cdot \bar{\boldsymbol{a}}f, \qquad -D \cdot \bar{\boldsymbol{a}}^*D\bar{v}^\circ = D \cdot \bar{\boldsymbol{a}}^*g,$$

respectively. Similar considerations as in the proof of Proposition 2.2 and Corollary 2.4 then lead to a pathwise result for the fluctuations of the random linear functionals I_3^ε :

$(f, g) \mapsto I_3^\varepsilon(f, g)$ and $I_4^\varepsilon : (f, g) \mapsto I_4^\varepsilon(f, g)$ defined for all $f, g \in C_c^\infty(\mathbb{R}^d)^d$ by

$$I_3^\varepsilon(f, g) := \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla_\varepsilon(u_\varepsilon^\circ - \mathbb{E}\,[u_\varepsilon^\circ]),$$

$$I_4^\varepsilon(f, g) := \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \left(\boldsymbol{a}_\varepsilon(\nabla_\varepsilon u_\varepsilon^\circ + f) - \mathbb{E}\,[\boldsymbol{a}_\varepsilon(\nabla_\varepsilon u_\varepsilon^\circ + f)]\,\right),$$

that takes the form

$$\mathbb{E}\left[\left|I_3^\varepsilon(f, g) + J_0^\varepsilon\big(\bar{\mathcal{P}}_L f \otimes \bar{\mathcal{P}}_H^* g\big)\right|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[\left|I_4^\varepsilon(f, g) - J_0^\varepsilon\big(\bar{\mathcal{P}}_L f \otimes \bar{\mathcal{P}}_L^* g\big)\right|^2\right]^{\frac{1}{2}} \lesssim_{f,g} \varepsilon \mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}},$$

where by definition $\bar{\mathcal{P}}_H f = -D\bar{u}$, $\bar{\mathcal{P}}_L f = D\bar{u}^\circ + f$, $\bar{\mathcal{P}}_H^* g = -D\bar{v}$, and $\bar{\mathcal{P}}_L^* g = D\bar{v}^\circ + g$. $\quad\diamond$

Incidentally, as a consequence of our analysis, combining the two-scale expansion of the homogenization commutator (1.6) with identity (1.8), we obtain a new (nonlocal) two-scale expansion for the solution $\nabla_\varepsilon u_\varepsilon$ that is not only accurate at order 1 for the strong $\mathrm{L}^2(\mathbb{R}^d)$ topology but also at the order of the CLT scaling for the weak $\mathrm{L}^2(\mathbb{R}^d)$ topology, in contrast to the usual two-scale expansion (1.3) (cf. [36]). (The second estimate below is a reformulation of Proposition 2.2, whereas the first estimate is a corollary of [29, Theorem 3].)

Corollary 2.6. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, and let μ_d be defined in (1.14). For all $\varepsilon > 0$ and all $f \in C_c^\infty(\mathbb{R}^d)^d$, we set*

$$r_\varepsilon(f) := \nabla_\varepsilon u_\varepsilon - \Big(\underbrace{\mathbb{E}\,[\nabla_\varepsilon u_\varepsilon] + \nabla_\varepsilon(-\nabla_\varepsilon^* \cdot \bar{\boldsymbol{a}} \nabla_\varepsilon)^{-1} \nabla_\varepsilon^* \cdot \big(\Xi_i(\tfrac{\cdot}{\varepsilon}) D_i \bar{u}\big)}_{\text{nonlocal two-scale expansion of } \nabla_\varepsilon u_\varepsilon} \Big).$$

This (nonlocal) two-scale expansion correctly captures:

- *the* spatial *oscillations of $\nabla_\varepsilon u_\varepsilon$ in a strong norm: for all $f \in C_c^\infty(\mathbb{R}^d)^d$,*

$$\mathbb{E}\left[\|r_\varepsilon(f)\|_{\mathrm{L}^2(\mathbb{R}^d)}^2\right]^{\frac{1}{2}} \lesssim_f \varepsilon \mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}};$$

- *the* random *fluctuations of $\nabla_\varepsilon u_\varepsilon$ in the CLT scaling: for all $f, g \in C_c^\infty(\mathbb{R}^d)^d$,*

$$\mathbb{E}\left[\left|\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot r_\varepsilon(f)\right|^2\right]^{\frac{1}{2}} \lesssim_{f,g} \varepsilon \mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}}. \qquad\qquad \diamond$$

2.3. **Approximate normality.** We turn to the normal approximation result for the homogenization commutator (cf. Theorem 1(iii)), which states that the fluctuations of $\varepsilon^{-\frac{d}{2}} \Xi(\tfrac{\cdot}{\varepsilon})$ are asymptotically Gaussian (up to a non-degeneracy condition that is elucidated in Proposition 2.9 below). The approach is inspired by previous works by Nolen [54, 55], based on a second-order Poincaré inequality à la Chatterjee [10, 44], which is key to optimal convergence rates. Such functional inequalities are not easily amenable to the use of large-scale Calderón-Zygmund theory for the random elliptic operator $-\nabla^* \cdot \boldsymbol{a} \nabla$, and we rather have to exploit optimal annealed estimates on mixed gradients of the Green's function [46] (see also [8] and [4, Section 8.5]). The proof exploits the approximate locality of the homogenization commutator Ξ, while the lack of exact locality precisely leads to the additional $\log(2 + \tfrac{1}{\varepsilon})$ factor in the convergence rate (we believe that this is optimal).

Proposition 2.7. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, let μ_d and $\delta_{\mathcal{N}}$ be defined in (1.14) and (1.15), and set $w_1(z) := 1 + |z|$. For all $\varepsilon > 0$ and all $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$ with $\mathrm{Var}\,[J_0^\varepsilon(F)] > 0$, we have for all $\alpha > 0$,*

$$
\delta_{\mathcal{N}}(J_0^\varepsilon(F)) \lesssim_\alpha \varepsilon^{\frac{d}{2}} \frac{\|F\|_{\mathrm{L}^3(\mathbb{R}^d)}^3 + \|w_1^\alpha DF\|_{\mathrm{L}^3(\mathbb{R}^d)}^3}{\mathrm{Var}\,[J_0^\varepsilon(F)]^{\frac{3}{2}}}
$$
$$
+ \varepsilon^{\frac{d}{2}} \log(2 + \tfrac{1}{\varepsilon}) \frac{\|w_1^\alpha F\|_{\mathrm{L}^4(\mathbb{R}^d)}^2 + \|w_1^\alpha DF\|_{\mathrm{L}^4(\mathbb{R}^d)}^2}{\mathrm{Var}\,[J_0^\varepsilon(F)]}. \qquad \Diamond
$$

Remark 2.8. In the case of iid conductances that are (smooth local transformations of) Gaussian random variables, a stronger version of a second-order Poincaré inequality is available (cf. [11, Theorem 2.2]), which in addition controls the total variation distance and allows to avoid Green's function estimates (see [22, 14] for such an argument in the continuum setting). $\qquad \Diamond$

2.4. **Covariance structure.** Since J_0^ε is asymptotically Gaussian, it remains to identify the limit of its covariance structure (cf. Theorem 1(iv)). The following shows that the limiting covariance is that of a (matrix-valued) white noise with some non-degenerate covariance tensor $\mathcal{Q}$. The convergence rate in (2.8) below is new in any dimension and is expected to be optimal. The proof crucially relies on the approximate locality of the homogenization commutator and on (an iid version of) the Helffer-Sjöstrand representation formula for the variance [38, 58, 52], which is a stronger tool than the Poincaré inequality. As for the pathwise result, the proof exploits the large-scale Calderón-Zygmund theory for the random elliptic operator $-\nabla^* \cdot \boldsymbol{a} \nabla$.

Proposition 2.9. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, let μ_d be defined in (1.14), and set $w_1(z) := 1 + |z|$. Then the following hold,*

(i) *There exists a symmetric[3] 4-tensor $\mathcal{Q}$ such that for all $\varepsilon > 0$ and all $F, G \in C_c^\infty(\mathbb{R}^d)^{d \times d}$ we have for all $0 < p - 1 \ll 1$ and all $\alpha > d\frac{p-1}{4p}$,*

$$
\left| \mathrm{Cov}\,[J_0^\varepsilon(F); J_0^\varepsilon(G)] - \int_{\mathbb{R}^d} F(x) : \mathcal{Q}\,G(x)dx \right| \lesssim_{\alpha,p} \varepsilon \mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}}
$$
$$
\times \big(\|F\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|w_1^{2\alpha} DF\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \big)\big(\|G\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|w_1^{2\alpha} DG\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \big). \quad (2.8)
$$

Moreover, for all $1 \leq i, j, k, l \leq d$ and all $\delta > 0$, we have for all $L \geq 1$,

$$
\left| \mathcal{Q}_{ijkl} - \int_{Q_{2L}} \frac{|Q_L \cap (x + Q_L)|}{|Q_L|} \mathrm{Cov}\,[\Xi_{ij}(x); \Xi_{kl}(0)]\,dx \right| \lesssim_\delta L^{\delta - \frac{1}{2}}, \qquad (2.9)
$$

where $Q_L := [\frac{L}{2}, \frac{L}{2})^d$ denotes the cube of sidelength L centered at the origin.

(ii) *If in addition $\mathbb{P}$ is nontrivial, then this effective fluctuation tensor $\mathcal{Q}$ is non-degenerate in the sense that $(e \otimes e) : \mathcal{Q}\,(e \otimes e) > 0$ for all $e \in \mathbb{R}^d \setminus \{0\}$.* $\qquad \Diamond$

---

[3]Since $\mathcal{Q}$ is a (limiting) covariance, it is of course symmetric in the sense of $\mathcal{Q}_{ijkl} = \mathcal{Q}_{klij}$. If the coefficients $\boldsymbol{a}$ are symmetric, then it has the additional symmetry $\mathcal{Q}_{ijkl} = \mathcal{Q}_{jikl}$ (hence also $\mathcal{Q}_{ijkl} = \mathcal{Q}_{ijlk}$).

Remarks 2.10.

- When applying a covariance inequality (cf. Lemma 5.1 below) to the argument of the limit in the Green-Kubo formula (2.9), we end up with the bound

$$\int_{Q_{2L}} \frac{|Q_L \cap (x + Q_L)|}{|Q_L|} \, |\operatorname{Cov}\left[\Xi_{ij}(x); \Xi_{kl}(0)\right]| \, dx \lesssim \log L,$$

which is sharp. The main difficulty to characterize the limiting covariance structure is that, as usual for Green-Kubo formulas, the covariance of the homogenization commutator Ξ is not integrable and cancellations have to be unravelled.

- The optimal rate (2.8) for the convergence of the covariance structure of J_0^ε owes to the very local structure of the homogenization commutator Ξ. Combined with the pathwise result of Corollary 2.4, it carries over to I_1^ε, I_2^ε, I_3^ε, I_4^ε, J_1^ε, and J_2^ε. In [49, 36], the Gaussian Helffer-Sjöstrand representation formula for the variance [38, 58, 52] was already used in order to prove the convergence of the covariance structure of I_1^ε and J_1^ε for $d > 2$, but the obtained convergence rate was suboptimal in every dimension.

- The non-degeneracy property (ii) already follows from [30, Proposition 2.1] (modulo the identification (1.17)); see also [49, Remark 2.3] in the Gaussian case.   ◇

The combination of Propositions 2.7 and 2.9 leads to a complete scaling limit result for J_0^ε, and proves the convergence in law to a Gaussian white noise.

Corollary 2.11. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, and let μ_d be defined as in (1.14). Let $\mathcal{Q}$ be the 4-tensor defined in Proposition 2.9(i), and let Γ denote the 2-tensor Gaussian white noise with covariance tensor $\mathcal{Q}$, that is, Γ is the Gaussian random linear functional with zero expectation $\mathbb{E}\left[\Gamma(F)\right] = 0$ and with covariance structure $\operatorname{Cov}\left[\Gamma(F); \Gamma(G)\right] = \int_{\mathbb{R}^d} F : \mathcal{Q} \, G$ for all $F, G \in C_c^\infty(\mathbb{R}^d)^{d \times d}$. Then for all $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$ the random variable $J_0^\varepsilon(F)$ converges in law to $\Gamma(F)$, and for $\int_{\mathbb{R}^d} F : \mathcal{Q} \, F \neq 0$ there holds*

$$(\mathrm{d_W} + \mathrm{d_K})(J_0^\varepsilon(F), \Gamma(F)) \ \lesssim_F \ \varepsilon \mu_d(\tfrac{1}{\varepsilon}). \qquad \qquad ◇$$

Remarks 2.12.

- Combined with the pathwise result of Corollary 2.4, this result leads to a proof of the joint convergence (1.12) and implies quantitative versions of the known scaling limit results for I_1^ε and J_1^ε: For all $f, g \in C_c^\infty(\mathbb{R}^d)^d$ and all $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$ the random variables $I_1^\varepsilon(f, g)$ and $J_1^\varepsilon(F)$ converge in law to $\Gamma(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g)$ and $-\Gamma(\bar{\mathcal{P}}_H^* F)$, respectively. Moreover, whenever $\int_{\mathbb{R}^d} (\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g) : \mathcal{Q} \, (\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g) \neq 0$ and $\int_{\mathbb{R}^d} \bar{\mathcal{P}}_H^* F : \mathcal{Q} \bar{\mathcal{P}}_H^* F \neq 0$, we have

$$\begin{aligned}
(\mathrm{d_W} + \mathrm{d_K})\big(I_1^\varepsilon(f, g), \, \Gamma(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g)\big) \ &\lesssim_{f,g} \ \varepsilon \mu_d(\tfrac{1}{\varepsilon}), \\
(\mathrm{d_W} + \mathrm{d_K})\big(J_1^\varepsilon(F), \, -\Gamma(\bar{\mathcal{P}}_H^* F)\big) \ &\lesssim_F \ \varepsilon \mu_d(\tfrac{1}{\varepsilon}).
\end{aligned}$$

This extends and unifies [30, 49, 48, 36], and yields the first scaling limit results in the critical dimension $d = 2$. Convergence rates are new in any dimension and are expected to be optimal.

- *SPDE representation for the scaling limit of the solution operator.* The scaling limit result for I_1^ε above indicates that $\varepsilon^{-\frac{d}{2}} \nabla_\varepsilon (u_\varepsilon - \mathbb{E}[u_\varepsilon])$ (seen as a random linear functional) converges in law to the solution DU in $\mathbb{R}^d$ of

$$-D \cdot \bar{\boldsymbol{a}} DU = D \cdot (\Gamma_i D_i \bar{u}).$$

  This justifies a posteriori the conclusion (although not the strategy) of the heuristics by Armstrong, Gu, and Mourrat [36, 47] in dimensions $d \geq 2$. (See also [34] for a rigorous treatment of the easier case of dimension $d = 1$.)

- *Scaling limit of the corrector.* The scaling limit result for J_1^ε above shows that the rescaled corrector field $\varepsilon^{-\frac{d}{2}} D\phi(\frac{\cdot}{\varepsilon})$ (seen as a random linear functional) converges in law to $D(-D \cdot \bar{\boldsymbol{a}} D)^{-1} D \cdot \Gamma$, that is, to the gradient of a variant of the so-called Gaussian free field. This variant involves both $\bar{\boldsymbol{a}}$ and $\mathcal{Q}$. As pointed out in [49], it is easily checked in Fourier space that this variant does not coincide in general with the standard Gaussian free field (unless the compatibility condition $\mathcal{Q}_{ijkl} = \eta_{ik} \bar{\boldsymbol{a}}_{lj}$ is satisfied for some matrix η, which however does not hold in elementary examples, see e.g. [37, Section 3] and [13, equation (5.35)]). This variant of the Gaussian free field is studied in [37], where it is shown to be Markovian only in the standard case. In the critical dimension $d = 2$, since the whole-space Gaussian free field is not well-defined (only its gradient is), this implies the non-existence of stationary correctors.

- *Gu and Mourrat's observation.* With the above results at hand, we recover the observation by Gu and Mourrat [36] that the usual two-scale expansion (1.3) of u_ε is not accurate in the CLT scaling. The above indeed shows that the fluctuations of $\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla_\varepsilon (u_\varepsilon - \mathbb{E}[u_\varepsilon])$ and of $\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla \phi_i(\frac{\cdot}{\varepsilon}) \nabla_{\varepsilon,i} \bar{u}$ are asymptotically given by $\Gamma(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g)$ and by $\Gamma(\bar{\mathcal{P}}_H^* ((\bar{\mathcal{P}}_H f) \otimes g))$, respectively, and therefore do not coincide in general. $\diamondsuit$

2.5. **Approximation of the fluctuation tensor.** We finally turn to the representative volume element (RVE) method for approximation of $\mathcal{Q}$ (cf. Theorem 2). Indeed, the Green-Kubo formula (2.9) for the fluctuation tensor $\mathcal{Q}$ is of no practical use in applications since it requires to solve the corrector equation on the whole space and for every realization of the random coefficient field. It is therefore natural to seek a suitable RVE approximation. It consists in introducing an artificial period $L > 0$ and in considering an L-periodized coefficient field $\boldsymbol{a}_L$, typically given by a suitable periodization in law (cf. [23]). In the present iid setting, we simply define $\boldsymbol{a}_L(x + Ly) := \boldsymbol{a}(x)$ for all $y \in \mathbb{Z}^d$ and $x \in Q_L := [-\frac{L}{2}, \frac{L}{2})^d$. Note that the map $\boldsymbol{a} \mapsto \boldsymbol{a}_L$ on Ω pushes forward the measure $\mathbb{P}$ to a measure $\mathbb{P}_L$ concentrated on L-periodic coefficients, so that we may view $\boldsymbol{a}_L$ as an element of $\Omega_L = [\lambda, 1]^{\mathcal{B}_L}$ equipped with the product measure $\mathbb{P}_L := \pi^{\otimes \mathcal{B}_L}$, where $\mathcal{B}_L := \{(x, x + e_i) : x \in Q_L \cap \mathbb{Z}^d, 1 \leq i \leq d\}$. We then define the L-periodized corrector $\phi_{L,i}$ in the direction e_i as the unique L-periodic solution in $Q_L \cap \mathbb{Z}^d$ of

$$-\nabla^* \cdot \boldsymbol{a}_L (\nabla \phi_{L,i} + e_i) = 0, \tag{2.10}$$

satisfying $\sum_{z \in Q_L \cap \mathbb{Z}^d} \phi_{L,i}(z) = 0$, and we set $\phi_L := (\phi_{L,i})_{i=1}^d$ (which we implicitly extend as usual into a periodic piecewise constant map on $\mathbb{R}^d$). The spatial average of the flux,

$$\bar{\boldsymbol{a}}_L e_i := \fint_{Q_L} \boldsymbol{a}_L (\nabla \phi_{L,i} + e_i),$$

is then an RVE approximation for the homogenized coefficient $\bar{\boldsymbol{a}}e_i = \mathbb{E}\left[\boldsymbol{a}(\nabla\phi_i + e_i)\right]$. The optimal numerical analysis of this approximation was originally performed in [31, 32, 28], where it was established that for all $L \geq 2$ there holds

$$|\text{Var}\,[\bar{\boldsymbol{a}}_L]|^{\frac{1}{2}} \lesssim L^{-\frac{d}{2}}, \qquad |\mathbb{E}\,[\bar{\boldsymbol{a}}_L] - \bar{\boldsymbol{a}}| \lesssim L^{-d}\log^d L. \tag{2.11}$$

In Theorem 2, we claim that the fluctuation tensor $\mathcal{Q}$ coincides with the limit of the rescaled variance of $\bar{\boldsymbol{a}}_L^*$. In addition, this characterization naturally leads to an RVE approximation $\mathcal{Q}_{L,N}$ for $\mathcal{Q}$, for which we obtain the optimal error estimate.

Remarks 2.13.

- Definition (1.18) for $\mathcal{Q}_{L,N}$ is equivalent to

$$\mathcal{Q}_{L,N} = \frac{L^d}{N-1}\sum_{n=1}^{N}\Big(\fint_{Q_L}\Xi_{L,N}^{(n)}\Big)\otimes\Big(\fint_{Q_L}\Xi_{L,N}^{(n)}\Big), \tag{2.12}$$

  where

$$\Xi_{L,N,i}^{(n)} := \boldsymbol{a}_L^{(n)}(\nabla\phi_{L,i}^{(n)} + e_i) - \bar{\boldsymbol{a}}_{L,N}(\nabla\phi_{L,i}^{(n)} + e_i),$$

  with the obvious notation $\nabla\phi_L^{(n)} := \nabla\phi_L(\boldsymbol{a}_L^{(n)})$. Since by stationarity

$$\int_{Q_L}\text{Cov}\,[\Xi_{L,N}(x);\Xi_{L,N}(0)]\,dx = L^d\,\text{Var}\left[\fint_{Q_L}\Xi_{L,N}\right],$$

  formula (2.12) is in the spirit of the Green-Kubo formula (2.9).

- In (2.11) the standard deviation $|\text{Var}\,[\bar{\boldsymbol{a}}_L]|^{\frac{1}{2}}$ of the RVE approximation for $\bar{\boldsymbol{a}}$ is seen to be $O(L^{\frac{d}{2}})$ times larger than the systematic error $|\mathbb{E}\,[\bar{\boldsymbol{a}}_L] - \bar{\boldsymbol{a}}|$ (up to a logarithmic correction). In practice, we rather use $\bar{\boldsymbol{a}}_{L,N}$ as an approximation for $\bar{\boldsymbol{a}}$,

$$|\text{Var}\,[\bar{\boldsymbol{a}}_{L,N}]|^{\frac{1}{2}} \lesssim N^{-\frac{1}{2}}L^{-\frac{d}{2}}, \qquad |\mathbb{E}\,[\bar{\boldsymbol{a}}_{L,N}] - \bar{\boldsymbol{a}}| \lesssim L^{-d}\log^d L,$$

  since in the regime $N \simeq L^d$ the standard deviation becomes of the same order as the systematic error $O(L^{-d})$. Combining this with the estimates in Theorem 2, since $\mathcal{Q}_{L,N}$ is extracted at no further cost than $\bar{\boldsymbol{a}}_{L,N}$ itself, we may infer that an RVE approximation for $\mathcal{Q}$ with accuracy $O(L^{-\frac{d}{2}})$ is extracted at the same cost as an RVE approximation for $\bar{\boldsymbol{a}}$ with accuracy $O(L^{-d})$.

- In [54, 55, 30] (see also [57, 9]), the fluctuations of the RVE approximation $\bar{\boldsymbol{a}}_L$ for the homogenized coefficient $\bar{\boldsymbol{a}}$ was investigated. Combined with the characterization (1.17) of the limit of the rescaled variance, the main result in [30] takes on the following guise, for all $L \geq 2$ and all $N \geq 1$,

$$\sup_{e\in\mathbb{R}^d\setminus\{0\}}(\mathrm{d_W} + \mathrm{d_K})\left(N^{\frac{1}{2}}L^{\frac{d}{2}}\frac{e\cdot(\bar{\boldsymbol{a}}_{L,N} - \bar{\boldsymbol{a}})e}{(e\otimes e : \mathcal{Q} : e\otimes e)^{\frac{1}{2}}}, \mathcal{N}\right) \lesssim N^{-\frac{1}{2}}L^{-\frac{d}{2}}\log^d L. \qquad \Diamond$$

3. Pathwise structure

Henceforth we place ourselves in the discrete setting of Section 2. In the present section, we establish the pathwise result stated in Proposition 2.2, that is, the main novelty of this contribution. Similar estimates also lead to the CLT scaling result stated in Proposition 2.1, and we further deduce Corollary 2.4.

3.1. **Structure of the proof and auxiliary results.** The main tool that we use to prove Propositions 2.1 and 2.2 is the following Poincaré inequality (or spectral gap estimate) in the probability space, which holds for any product measure $\mathbb{P}$ on Ω (see e.g. [31, Lemma 2.3] for a proof). Let us first fix some notation. Let $X = X(\boldsymbol{a})$ be a random variable on Ω, that is, a measurable function of $(a(b))_{b \in \mathcal{B}}$. We choose an iid copy $\boldsymbol{a}'$ of $\boldsymbol{a}$,[4] and for all $b \in \mathcal{B}$ we denote by $\boldsymbol{a}^b$ the random field that coincides with $\boldsymbol{a}$ on all edges $b' \neq b$ and with $\boldsymbol{a}'$ on edge b. In particular, $\boldsymbol{a}$ and $\boldsymbol{a}^b$ have the same law. We use the abbreviation $X^b = X(\boldsymbol{a}^b)$ and define the difference operator $\Delta_b X := X - X^b$, which we call the (Glauber or discrete) *vertical derivative* at edge b.

Lemma 3.1 (e.g. [31]). *Let $\mathbb{P}$ be a product measure. For all $X = X(\boldsymbol{a}) \in \mathrm{L}^2(\Omega)$ we have*

$$\mathrm{Var}\,[X] \;\leq\; \frac{1}{2}\,\mathbb{E}\left[\sum_{b \in \mathcal{B}} |\Delta_b X|^2\right]. \qquad\qquad \diamond$$

Next to the corrector ϕ, we need to recall the notion of flux corrector σ, which was recently introduced in [27] in the continuum stochastic setting (see also [36, Lemma 4.4] and [53, Proposition III.2.2] for its subsequent introduction in the discrete setting) and was crucially used in [29, 33]. It allows to put the equation for the two-scale homogenization error in divergence form (cf. (3.29)). Let $\sigma = (\sigma_{ijk})_{i,j,k=1}^d$ be the 3-tensor defined as the unique solution in $\mathbb{Z}^d$ of

$$-\triangle \sigma_{ijk} \;:=\; -\nabla^* \cdot \nabla \sigma_{ijk} \;=\; \nabla_j q_{ik} - \nabla_k q_{ij}, \qquad\qquad (3.1)$$

with $\nabla\sigma$ stationary and having finite second moment, and with $\sigma(0) = 0$, where q_i denotes the flux of the corrector

$$q_i = \boldsymbol{a}(\nabla\phi_i + e_i) - \bar{\boldsymbol{a}}e_i, \qquad q_{ij} := (q_i)_j. \qquad\qquad (3.2)$$

Note that for all i the 2-tensor field $\sigma_i := (\sigma_{ijk})_{j,k=1}^d$ is skew-symmetric, that is,

$$\sigma_{ijk} = -\sigma_{ikj}, \qquad\qquad (3.3)$$

and is shown to satisfy

$$\nabla^* \cdot \sigma_i \;:=\; e_j \nabla_k^* \sigma_{ijk} \;=\; q_i. \qquad\qquad (3.4)$$

Although considering a symmetric coefficient field, we use non-symmetric notation in view of the extension to the continuum setting, and we denote by ϕ^* and σ^* the corrector and flux corrector associated with the pointwise transpose coefficient field $\boldsymbol{a}^*$. For symmetric coefficient fields, $\phi^* = \phi$ and $\sigma^* = \sigma$.

We now describe the string of arguments that leads to Proposition 2.2. We start with a suitable decomposition of the vertical derivative of $E^\varepsilon(f, g)$, which is key to the proof. Note that we rather consider a suitable version $E_0^\varepsilon(f, g)$ of $E^\varepsilon(f, g)$, which only coincides with $E^\varepsilon(f, g)$ up to some minor discretization error (in the continuum setting $\bar{u}_\varepsilon$ and $\bar{v}_\varepsilon$ would simply coincide with $\bar{u}(\varepsilon\cdot)$ and $\bar{v}(\varepsilon\cdot)$). In the proofs, it is convenient to rescale all quantities down to scale 1.

---

[4]Although we are then working on a product probability space $\Omega \times \Omega$, we use for simplicity the same notation $\mathbb{P}$ (and $\mathbb{E}$) for the product probability measure (and expectation), that is, with respect to both $\boldsymbol{a}$ and $\boldsymbol{a}'$.

Lemma 3.2. *For all $\varepsilon > 0$ and all $f, g \in C_c^\infty(\mathbb{R}^d)^d$, setting $f_\varepsilon := f(\varepsilon \cdot)$ and $g_\varepsilon := g(\varepsilon \cdot)$, we denote by $\bar{u}_\varepsilon$ and $\bar{v}_\varepsilon$ the unique Lax-Milgram solutions in $\mathbb{R}^d$ of*

$$-\nabla^* \cdot \bar{\boldsymbol{a}} \nabla \bar{u}_\varepsilon = \nabla^* \cdot (\varepsilon f_\varepsilon), \qquad -\nabla^* \cdot \bar{\boldsymbol{a}}^* \nabla \bar{v}_\varepsilon = \nabla^* \cdot (\varepsilon g_\varepsilon), \qquad (3.5)$$

and we define

$$E_0^\varepsilon(f, g) := \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} g_\varepsilon \cdot \left(\boldsymbol{a}\nabla(u_\varepsilon(\varepsilon\cdot)) - \bar{\boldsymbol{a}}\nabla(u_\varepsilon(\varepsilon\cdot)) - \mathbb{E}\left[\boldsymbol{a}\nabla(u_\varepsilon(\varepsilon\cdot)) - \bar{\boldsymbol{a}}\nabla(u_\varepsilon(\varepsilon\cdot))\right]\right)$$

$$- \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} g_\varepsilon \cdot \Xi_i \nabla_i \bar{u}_\varepsilon, \quad (3.6)$$

as well as the two-scale expansion error $w_{f,\varepsilon} := u_\varepsilon(\varepsilon\cdot) - (1 + \phi_i \nabla_i)\bar{u}_\varepsilon$. Then we have for all $b \in \mathcal{B}$,

$$\Delta_b E_0^\varepsilon(f, g) = \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} g_{\varepsilon,j}(\nabla \phi_j^* + e_j) \cdot \Delta_b \boldsymbol{a}(\nabla w_{f,\varepsilon}^b + \phi_i^b \nabla \nabla_i \bar{u}_\varepsilon)$$

$$+ \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} \phi_j^*(\cdot + e_k) \nabla_k g_{\varepsilon,j} e_k \cdot \Delta_b \boldsymbol{a}\nabla(u_\varepsilon^b(\varepsilon\cdot))$$

$$- \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} \phi_j^*(\cdot + e_k) \nabla_k(g_{\varepsilon,j} \nabla_i \bar{u}_\varepsilon) e_k \cdot \Delta_b \boldsymbol{a}(\nabla \phi_i^b + e_i)$$

$$+ \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} \nabla r_\varepsilon \cdot \Delta_b \boldsymbol{a}\nabla(u_\varepsilon^b(\varepsilon\cdot)) - \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} \nabla R_{\varepsilon,i} \cdot \Delta_b \boldsymbol{a}(\nabla \phi_i^b + e_i), \quad (3.7)$$

where the auxiliary fields r_ε and $R_\varepsilon = (R_{\varepsilon,i})_{i=1}^d$ are the unique Lax-Milgram solutions in $\mathbb{R}^d$ of

$$-\nabla^* \cdot \boldsymbol{a}^* \nabla r_\varepsilon = \nabla_l^*\left(\phi_j^*(\cdot + e_k)\boldsymbol{a}_{kl}\nabla_k g_{\varepsilon,j} + \sigma_{jkl}^*(\cdot - e_k)\nabla_k^* g_{\varepsilon,j}\right), \qquad (3.8)$$

$$-\nabla^* \cdot \boldsymbol{a}^* \nabla R_{\varepsilon,i} = \nabla_l^*\left(\phi_j^*(\cdot + e_k)\boldsymbol{a}_{kl}\nabla_k(g_{\varepsilon,j}\nabla_i \bar{u}_\varepsilon) + \sigma_{jkl}^*(\cdot - e_k)\nabla_k^*(g_{\varepsilon,j}\nabla_i \bar{u}_\varepsilon)\right). \quad (3.9)$$

$$\diamond$$

By the spectral gap estimate of Lemma 3.1, the desired pathwise result (2.6) follows from a suitable estimate of the sum over $\mathcal{B}$ of the squares of the right-hand side terms in (3.7). For that purpose, we make crucial use of the following moment bounds for the extended corrector (ϕ, σ) and its gradient. These bounds are a variation of [31, 32, 28] and are the discrete versions of [27, Theorem 2] and [29, Theorem 2], the proof of which extends to the discrete setting considered here. Similar bounds hold under the assumption of finite range of dependence, cf. [3, 4, 33].

Lemma 3.3 ([28, 29]). *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, and let μ_d be defined in (1.14). For all $q < \infty$ and all $z \in \mathbb{Z}^d$ we have*

$$\mathbb{E}\left[|\phi(z)|^q\right]^{\frac{1}{q}} + \mathbb{E}\left[|\sigma(z)|^q\right]^{\frac{1}{q}} \lesssim_q \mu_d(|z|)^{\frac{1}{2}},$$

and

$$\mathbb{E}\left[|\nabla \phi(z)|^q\right]^{\frac{1}{q}} + \mathbb{E}\left[|\nabla \sigma(z)|^q\right]^{\frac{1}{q}} \lesssim_q 1. \qquad \diamond$$

An additional key ingredient consists of large-scale regularity theory as originating in the work of Armstrong and Smart [6] (see also the prior annealed regularity in [46]), which has recently been fully developed in [5, 2, 3, 4] and in [27, 33]. More precisely, we make use of the following large-scale weighted Calderón-Zygmund estimates for the random elliptic operator $-\nabla^* \cdot \boldsymbol{a}\nabla$ as established in [27] (see also [1] and [4, Section 7]). A proof in the

continuum setting was originally included in the first version of this article, see now [27, Corollary 5]; the adaptation to the discrete setting is straightforward since the argument is solely based on the energy and Caccioppoli estimates. Note that the following holds under mere stationarity and ergodicity of the coefficients if the moment bounds on r_* are replaced by almost sure finiteness.

Lemma 3.4. [27, Corollary 5 & Theorem 2] *Let $d \geq 1$ and let $\mathbb{P}$ be a product measure. There exists a $\frac{1}{8}$-Lipschitz stationary random field $r_* \geq 1$ on $\mathbb{R}^d$ with $\mathbb{E}[r_*^q] \lesssim_q 1$ for all $q < \infty$, such that the following holds almost surely: For $\varepsilon > 0$, $2 \leq p < \infty$, and $0 \leq \gamma < d(p-1)$, for any (sufficiently fast) decaying scalar field w and vector field h related in $\mathbb{R}^d$ by*

$$-\nabla^* \cdot \boldsymbol{a}\nabla w = \nabla^* \cdot h,$$

we have

$$\int_{\mathbb{R}^d} \left(1 + \varepsilon(|x| + r_*(0))\right)^\gamma \left(\fint_{B_*(x)} |\nabla w|^2\right)^{\frac{p}{2}} dx \lesssim_{\gamma,p} \int_{\mathbb{R}^d} \left(1 + \varepsilon(|x| + r_*(0))\right)^\gamma |h(x)|^p dx$$

with the short-hand notation $B_*(x) := B_{r_*(x)}(x)$.   ◇

3.2. Proof of Proposition 2.1.

We focus on J_0^ε, while the proof is similar for J_1^ε and J_2^ε. Let $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$, and set $F_\varepsilon := F(\varepsilon\cdot)$. We split the proof into two steps: we start by giving a suitable representation formula for the vertical derivative $\Delta_b J_0^\varepsilon(F)$, and then apply the spectral gap estimate.

Step 1. We prove the following representation formula for the vertical derivative $\Delta_b J_0^\varepsilon(F)$,

$$\Delta_b J_0^\varepsilon(F) = \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij} e_j \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i) + \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} \nabla s_{\varepsilon,i} \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i), \qquad (3.10)$$

where the auxiliary field s_ε is the unique Lax-Milgram solution in $\mathbb{R}^d$ of

$$-\nabla^* \cdot \boldsymbol{a}^* \nabla s_{\varepsilon,i} = \nabla^* \cdot \left(F_{\varepsilon,ij}(\boldsymbol{a} - \bar{\boldsymbol{a}})e_j\right). \qquad (3.11)$$

By definition of the homogenization commutator,

$$\Delta_b J_0^\varepsilon(F) = \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij} e_j \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i) + \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij} e_j \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}})\nabla\Delta_b\phi_i.$$

By definition (3.11) of $s_{\varepsilon,i}$, we find

$$\Delta_b J_0^\varepsilon(F) = \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij} e_j \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i) - \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} \nabla s_{\varepsilon,i} \cdot \boldsymbol{a}\nabla\Delta_b\phi_i.$$

Using then the vertical derivative of the corrector equation (2.4) in the form

$$-\nabla^* \cdot \boldsymbol{a}\nabla\Delta_b\phi_i = \nabla^* \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i), \qquad (3.12)$$

the claim (3.10) follows.

Step 2. Conclusion.
For $b \in \mathcal{B}$ we use the notation $b = (z_b, z_b + e_b)$. Inserting the representation formula (3.10)

in the spectral gap estimate of Lemma 3.1, and noting that $|\Delta_b \boldsymbol{a}(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$, we obtain

$$\operatorname{Var}\left[J_0^\varepsilon(F)\right] \lesssim \varepsilon^d \sum_{b \in \mathcal{B}} \mathbb{E}\left[|\nabla \phi^b(z_b) + \operatorname{Id}|^2\right] \int_{Q(z_b)} |F_\varepsilon|^2$$

$$+ \varepsilon^d \mathbb{E}\left[\sum_{b \in \mathcal{B}} |\nabla \phi^b(z_b) + \operatorname{Id}|^2 \int_{Q(z_b)} |\nabla s_\varepsilon|^2\right],$$

and hence, by Lemma 3.3 in the form $\mathbb{E}\left[|\nabla \phi^b|^2\right] = \mathbb{E}\left[|\nabla \phi|^2\right] \lesssim 1$,

$$\operatorname{Var}\left[J_0^\varepsilon(F)\right] \lesssim \varepsilon^d \|F_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 + \varepsilon^d \mathbb{E}\left[\sum_{b \in \mathcal{B}} |\nabla \phi^b(z_b) + \operatorname{Id}|^2 \int_{Q(z_b)} |\nabla s_\varepsilon|^2\right]. \qquad (3.13)$$

It remains to estimate the last right-hand side term. Using equation (3.12) in the form $-\nabla^* \cdot \boldsymbol{a}^b \nabla(\phi^b - \phi) = \nabla^* \cdot (\boldsymbol{a}^b - \boldsymbol{a})(\nabla \phi + \operatorname{Id})$, an energy estimate yields

$$|\nabla(\phi^b - \phi)(z_b)|^2 \le \int_{\mathbb{R}^d} |\nabla(\phi^b - \phi)|^2 \lesssim \int_{\mathbb{R}^d} |\boldsymbol{a}^b - \boldsymbol{a}|^2 |\nabla \phi + \operatorname{Id}|^2 \lesssim |\nabla \phi(z_b) + \operatorname{Id}|^2, \quad (3.14)$$

so that $|\nabla \phi^b(z_b) + \operatorname{Id}| \lesssim |\nabla \phi(z_b) + \operatorname{Id}|$. Further estimating in (3.13) integrals over unit cubes by integrals over balls at scale r_* (cf. Lemma 3.4), smuggling in a power $\alpha \frac{p-1}{p}$ of the weight $w_\varepsilon(z) := 1 + \varepsilon|z|$, and applying Hölder's inequality in space with exponent p, we deduce for all $p > 1$,

$$\varepsilon^d \mathbb{E}\left[\sum_{b \in \mathcal{B}} |\nabla \phi^b(z_b) + \operatorname{Id}|^2 \int_{Q(z_b)} |\nabla s_\varepsilon|^2\right] \lesssim \varepsilon^d \mathbb{E}\left[\int_{\mathbb{R}^d} |\nabla \phi(z) + \operatorname{Id}|^2 \Big(\int_{Q_2(z)} |\nabla s_\varepsilon|^2\Big) dz\right]$$

$$\lesssim \varepsilon^d \mathbb{E}\left[\left(\int_{\mathbb{R}^d} |\nabla \phi(z) + \operatorname{Id}|^{\frac{2p}{p-1}} r_*(z)^{\frac{dp}{p-1}} w_\varepsilon(z)^{-\alpha} dz\right)^{\frac{p-1}{p}} \right.$$

$$\left. \times \left(\int_{\mathbb{R}^d} w_\varepsilon(z)^{\alpha(p-1)} \Big(\fint_{B_*(z)} |\nabla s_\varepsilon|^2\Big)^p dz\right)^{\frac{1}{p}}\right].$$

Applying Hölder's inequality in the probability space and Fubini's theorem, using Lemmas 3.3 and 3.4 in the form $\mathbb{E}\left[|\nabla \phi + \operatorname{Id}|^q + r_*^q\right] \lesssim_q 1$ for all $q < \infty$, and noting that $\int_{\mathbb{R}^d} w_\varepsilon(z)^{-\alpha} dz \lesssim_\alpha \varepsilon^{-d}$ provided $\alpha > d$, we obtain for all $p > 1$ and all $\alpha > d$,

$$\varepsilon^d \mathbb{E}\left[\sum_{b \in \mathcal{B}} |\nabla \phi^b(z_b) + \operatorname{Id}|^2 \int_{Q(z_b)} |\nabla s_\varepsilon|^2\right]$$

$$\lesssim_{\alpha, p} \varepsilon^{\frac{d}{p}} \mathbb{E}\left[\int_{\mathbb{R}^d} w_\varepsilon(z)^{\alpha(p-1)} \Big(\fint_{B_*(z)} |\nabla s_\varepsilon|^2\Big)^p dz\right]^{\frac{1}{p}}. \qquad (3.15)$$

By large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.4) applied to equation (3.11) for s_ε with $\alpha(p-1) < d(2p-1)$, and using again the moment bounds on r_*, we deduce for

all $0 < p - 1 \ll 1$ and all $0 < \alpha - d \ll 1$,

$$
\varepsilon^d \, \mathbb{E} \left[\sum_{b \in \mathcal{B}} |\nabla \phi^b(z_b) + \mathrm{Id}\,|^2 \int_{Q(z_b)} |\nabla s_\varepsilon|^2 \right] \;\lesssim_{\alpha,p}\; \varepsilon^{\frac{d}{p}} \, \mathbb{E} \left[r_*(0)^{\alpha(p-1)} \int_{\mathbb{R}^d} w_\varepsilon^{\alpha(p-1)} |F_\varepsilon|^{2p} \right]^{\frac{1}{p}}
$$

$$
\lesssim_{\alpha,p} \; \varepsilon^{\frac{d}{p}} \| w_\varepsilon^{\alpha \frac{p-1}{2p}} F_\varepsilon \|^2_{\mathrm{L}^{2p}(\mathbb{R}^d)}. \tag{3.16}
$$

Inserting this into (3.13) and rescaling spatial integrals, we deduce for all $0 < p - 1 \ll 1$ and all $0 < \alpha - d \ll 1$,

$$
\mathrm{Var}\left[J_0^\varepsilon(F) \right] \;\lesssim_{\alpha,p}\; \| F \|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \| w_1^{\alpha \frac{p-1}{2p}} F \|^2_{\mathrm{L}^{2p}(\mathbb{R}^d)}.
$$

Further using Hölder's inequality in the form

$$
\| F \|_{\mathrm{L}^2(\mathbb{R}^d)} \leq \left(\int_{\mathbb{R}^d} w_1^{-\alpha} \right)^{\frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} w_1^{\alpha(p-1)} |F|^{2p} \right)^{\frac{1}{2p}} \lesssim_{\alpha,p} \| w_1^{\alpha \frac{p-1}{2p}} F \|_{\mathrm{L}^{2p}(\mathbb{R}^d)},
$$

the conclusion follows (after replacing the exponent $\alpha \frac{p-1}{2p}$ by 2α).

3.3. **Proof of Lemma 3.2.** We split the proof into two steps. To simplify notation, in this proof (and only in this proof), we write $u := u_\varepsilon(\varepsilon \cdot)$.

Step 1. We prove the following representation formula for $\Delta_b((\boldsymbol{a} - \bar{\boldsymbol{a}})\nabla u)$,

$$
\Delta_b\big(e_j \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}})\nabla u\big) = (\nabla \phi_j^* + e_j) \cdot \Delta_b \boldsymbol{a} \nabla u^b - \nabla_k^* \big(\phi_j^*(\cdot + e_k) e_k \cdot \Delta_b \boldsymbol{a} \nabla u^b\big)
$$
$$
- \nabla_k^* \big(\phi_j^*(\cdot + e_k) e_k \cdot \boldsymbol{a} \nabla \Delta_b u\big) - \nabla_k \big(\sigma_{jkl}^*(\cdot - e_k)\nabla_l \Delta_b u\big). \tag{3.17}
$$

In particular, replacing $x \mapsto u(x)$ by $x \mapsto \phi_i(x) + x_i$, this implies the following discrete version of (1.11),

$$
\Delta_b \Xi_{ij} = (\nabla \phi_j^* + e_j) \cdot \Delta_b \boldsymbol{a} (\nabla \phi_i^b + e_i) - \nabla_k^* \big(\phi_j^*(\cdot + e_k) e_k \cdot \Delta_b \boldsymbol{a} (\nabla \phi_i^b + e_i)\big)
$$
$$
- \nabla_k^* \big(\phi_j^*(\cdot + e_k) \boldsymbol{a}_{kl} \nabla_l \Delta_b \phi_i\big) - \nabla_k \big(\sigma_{jkl}^*(\cdot - e_k)\nabla_l \Delta_b \phi_i\big). \tag{3.18}
$$

Using the definition (3.4) of σ_j^* in the form $(\boldsymbol{a}^* - \bar{\boldsymbol{a}}^*)e_j = -\boldsymbol{a}^* \nabla \phi_j^* + \nabla^* \cdot \sigma_j^*$, we find

$$
\Delta_b\big(e_j \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}})\nabla u\big) = e_j \cdot \Delta_b \boldsymbol{a} \nabla u^b + e_j \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}})\nabla \Delta_b u
$$
$$
= e_j \cdot \Delta_b \boldsymbol{a} \nabla u^b + (\nabla^* \cdot \sigma_j^*) \cdot \nabla \Delta_b u - \nabla \phi_j^* \cdot \boldsymbol{a} \nabla \Delta_b u. \tag{3.19}
$$

On the one hand, using the following discrete version of the Leibniz rule, for all $\chi_1, \chi_2 : \mathbb{Z}^d \to \mathbb{R}$,

$$
\nabla_l^* (e_l \chi_1(\cdot + e_l) \chi_2) = \chi_2 \nabla \chi_1 + \chi_1 \nabla^* \chi_2, \tag{3.20}
$$

we obtain

$$
(\nabla^* \cdot \sigma_j^*) \cdot \nabla \Delta_b u = \nabla_l^* \big(\sigma_{jkl}^* \nabla_k \Delta_b u(\cdot + e_l)\big) - \sigma_{jkl}^* \nabla_k \nabla_l \Delta_b u,
$$

so that the skew-symmetry (3.3) of σ_j leads to

$$
(\nabla^* \cdot \sigma_j^*) \cdot \nabla \Delta_b u = -\nabla_k^* \big(\sigma_{jkl}^* \nabla_l \Delta_b u(\cdot + e_k)\big) = -\nabla_k \big(\sigma_{jkl}^*(\cdot - e_k)\nabla_l \Delta_b u\big). \tag{3.21}
$$

On the other hand, using the vertical derivative of equation (2.1) in the form $-\nabla^* \cdot \boldsymbol{a}\nabla\Delta_b u = \nabla^* \cdot \Delta_b\boldsymbol{a}\nabla u^b$, the discrete Leibniz rule (3.20) yields

$$\begin{aligned}
\nabla\phi_j^* \cdot \boldsymbol{a}\nabla\Delta_b u &=& -\phi_j^*\nabla^* \cdot \boldsymbol{a}\nabla\Delta_b u + \nabla_k^*\big(\phi_j^*(\cdot + e_k)e_k \cdot \boldsymbol{a}\nabla\Delta_b u\big) \\
&=& -\nabla\phi_j^* \cdot \Delta_b\boldsymbol{a}\nabla u^b + \nabla_k^*\big(\phi_j^*(\cdot + e_k)e_k \cdot \Delta_b\boldsymbol{a}\nabla u^b\big) \\
&& +\nabla_k^*\big(\phi_j^*(\cdot + e_k)e_k \cdot \boldsymbol{a}\nabla\Delta_b u\big). \quad (3.22)
\end{aligned}$$

Inserting (3.21) and (3.22) into (3.19), the claim (3.17) follows.

Step 2. Conclusion.
Integrating identities (3.17) and (3.18) with the test functions g_ε and $\nabla\bar{u}_\varepsilon \otimes g_\varepsilon$, respectively, and integrating by parts, we obtain by definition of E_0^ε,

$$\begin{aligned}
\Delta_b E_0^\varepsilon(f,g) &=& \varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} g_{\varepsilon,j}(\nabla\phi_j^* + e_j) \cdot \Delta_b\boldsymbol{a}\big(\nabla u^b - (\nabla\phi_i^b + e_i)\nabla_i\bar{u}_\varepsilon\big) \\
&& +\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \phi_j^*(\cdot + e_k)\nabla_k g_{\varepsilon,j}e_k \cdot \Delta_b\boldsymbol{a}\nabla u^b \\
&& -\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \phi_j^*(\cdot + e_k)\nabla_k(g_{\varepsilon,j}\nabla_i\bar{u}_\varepsilon)e_k \cdot \Delta_b\boldsymbol{a}(\nabla\phi_i^b + e_i) \\
&& +\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \big(\phi_j^*(\cdot + e_k)\boldsymbol{a}_{kl}\nabla_k g_{\varepsilon,j} + \sigma_{jkl}^*(\cdot - e_k)\nabla_k^* g_{\varepsilon,j}\big)\nabla_l\Delta_b u \\
&& -\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \big(\phi_j^*(\cdot + e_k)\boldsymbol{a}_{kl}\nabla_k(g_{\varepsilon,j}\nabla_i\bar{u}_\varepsilon) + \sigma_{jkl}^*(\cdot - e_k)\nabla_k^*(g_{\varepsilon,j}\nabla_j\bar{u}_\varepsilon)\big)\nabla_l\Delta_b\phi_i.
\end{aligned}$$

The first right-hand side term is reformulated using the definition of $w_{f,\varepsilon}$ in the form $\nabla u^b - (\nabla\phi_i^b + e_i)\nabla_i\bar{u}_\varepsilon = \nabla w_{f,\varepsilon}^b + \phi_i^b\nabla\nabla_i\bar{u}_\varepsilon$. It remains to post-process the last two right-hand side terms. Using equation (3.8) for r_ε and using the vertical derivative of equation (2.1) for u_ε in the form $-\nabla^* \cdot \boldsymbol{a}\nabla\Delta_b u = \nabla^* \cdot \Delta_b\boldsymbol{a}\nabla u^b$, we find

$$\begin{aligned}
\int_{\mathbb{R}^d} \big(\phi_j^*(\cdot + e_k)\boldsymbol{a}_{kl}\nabla_k g_{\varepsilon,j} + \sigma_{jkl}^*(\cdot - e_k)\nabla_k^* g_{\varepsilon,j}\big)\nabla_l\Delta_b u \\
= -\int_{\mathbb{R}^d} \nabla r_\varepsilon \cdot \boldsymbol{a}\nabla\Delta_b u = \int_{\mathbb{R}^d} \nabla r_\varepsilon \cdot \Delta_b\boldsymbol{a}\nabla u^b.
\end{aligned}$$

Similarly, equations (3.9) and (3.12) lead to

$$\begin{aligned}
\int_{\mathbb{R}^d} \big(\phi_j^*(\cdot + e_k)\boldsymbol{a}_{kl}\nabla_k(g_{\varepsilon,j}\nabla_i\bar{u}_\varepsilon) + \sigma_{jkl}^*(\cdot - e_k)\nabla_k^*(g_{\varepsilon,j}\nabla_i\bar{u}_\varepsilon)\big)\nabla_l\Delta_b\phi_i \\
= -\int_{\mathbb{R}^d} \nabla R_{\varepsilon,i} \cdot \boldsymbol{a}\nabla\Delta_b\phi_i = \int_{\mathbb{R}^d} \nabla R_{\varepsilon,i} \cdot \Delta_b\boldsymbol{a}(\nabla\phi_i^b + e_i),
\end{aligned}$$

and the conclusion follows.

3.4. **Proof of Proposition 2.2.** Using the representation formula (3.7), and recalling that for symmetric coefficients we have $(\phi^*, \sigma^*) = (\phi, \sigma)$, the spectral gap estimate of Lemma 3.1 leads to

$$\mathrm{Var}\,[E_0^\varepsilon(f,g)] \lesssim T_1^\varepsilon + T_2^\varepsilon + T_3^\varepsilon + T_4^\varepsilon + T_5^\varepsilon, \tag{3.23}$$

where we have set

$$
\begin{aligned}
T_1^\varepsilon &:= \sum_{b\in\mathcal{B}} \mathbb{E}\left[\left(\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} g_{\varepsilon,j}(\nabla\phi_j+e_j)\cdot\Delta_b\boldsymbol{a}(\nabla w_{f,\varepsilon}^b+\phi_i^b\nabla\nabla_i\bar{u}_\varepsilon)\right)^2\right], \\
T_2^\varepsilon &:= \sum_{b\in\mathcal{B}} \mathbb{E}\left[\left(\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \phi_j(\cdot+e_k)\nabla_k g_{\varepsilon,j}e_k\cdot\Delta_b\boldsymbol{a}\nabla(u_\varepsilon^b(\varepsilon\cdot))\right)^2\right], \\
T_3^\varepsilon &:= \sum_{b\in\mathcal{B}} \mathbb{E}\left[\left(\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \phi_j(\cdot+e_k)\nabla_k(g_{\varepsilon,j}\nabla_i\bar{u}_\varepsilon)e_k\cdot\Delta_b\boldsymbol{a}(\nabla\phi_i^b+e_i)\right)^2\right], \\
T_4^\varepsilon &:= \sum_{b\in\mathcal{B}} \mathbb{E}\left[\left(\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \nabla r_\varepsilon\cdot\Delta_b\boldsymbol{a}\nabla(u_\varepsilon^b(\varepsilon\cdot))\right)^2\right], \\
T_5^\varepsilon &:= \sum_{b\in\mathcal{B}} \mathbb{E}\left[\left(\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \nabla R_{\varepsilon,i}\cdot\Delta_b\boldsymbol{a}(\nabla\phi_i^b+e_i)\right)^2\right],
\end{aligned}
$$

with the auxiliary fields r_ε and R_ε defined in (3.8) and in (3.9). The conclusion of Proposition 2.2 is a consequence of the following five estimates: for all $0 < p-1 \ll 1$ and all $0 < \alpha-d \ll 1$,

$$
T_1^\varepsilon \lesssim_{\alpha,p} \varepsilon^2\mu_d(\tfrac{1}{\varepsilon})\|g\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\|w_1^{\alpha\frac{p-1}{4p}}\mu_d(|\cdot|)^{\frac{1}{2}}Df\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}^2, \tag{3.24}
$$

$$
T_2^\varepsilon \lesssim \varepsilon^2\mu_d(\tfrac{1}{\varepsilon})\|f\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\|\mu_d(|\cdot|)^{\frac{1}{2}}Dg\|_{\mathrm{L}^4(\mathbb{R}^d)}^2, \tag{3.25}
$$

$$
T_3^\varepsilon \lesssim \varepsilon^2\mu_d(\tfrac{1}{\varepsilon})\Big(\|f\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\|\mu_d(|\cdot|)^{\frac{1}{2}}Dg\|_{\mathrm{L}^4(\mathbb{R}^d)}^2
$$
$$
+\|g\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\|\mu_d(|\cdot|)^{\frac{1}{2}}Df\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\Big), \tag{3.26}
$$

$$
T_4^\varepsilon \lesssim_{\alpha,p} \varepsilon^2\mu_d(\tfrac{1}{\varepsilon})\|f\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\|w_1^{\alpha\frac{p-1}{4p}}\mu_d(|\cdot|)^{\frac{1}{2}}Dg\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}^2, \tag{3.27}
$$

$$
T_5^\varepsilon \lesssim_{\alpha,p} \varepsilon^2\mu_d(\tfrac{1}{\varepsilon})\Big(\|f\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\|w_1^{\alpha\frac{p-1}{4p}}\mu_d(|\cdot|)^{\frac{1}{2}}Dg\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}^2
$$
$$
+\|g\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\|w_1^{\alpha\frac{p-1}{4p}}\mu_d(|\cdot|)^{\frac{1}{2}}Df\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}^2\Big). \tag{3.28}
$$

We split the proof into three steps: we prove the above five estimates in the first two steps, and we conclude in the last step by controlling the discretization error.

Step 1. We establish the following equation for the two-scale expansion error $w_{f,\varepsilon}$ on $\mathbb{R}^d$,

$$
-\nabla^*\cdot\boldsymbol{a}\nabla w_{f,\varepsilon} = \nabla_l^*\Big(\sigma_{jkl}(\cdot-e_k)\nabla_k^*\nabla_j\bar{u}_\varepsilon+\phi_j(\cdot+e_k)\boldsymbol{a}_{lk}\nabla_k\nabla_j\bar{u}_\varepsilon\Big), \tag{3.29}
$$

which constitutes a discrete counterpart of similar identities in [27, 29].

Using equations (2.1) and (3.5) in the form $-\nabla^*\cdot\boldsymbol{a}\nabla(u_\varepsilon(\varepsilon\cdot)) = -\nabla^*\cdot\bar{\boldsymbol{a}}\nabla\bar{u}_\varepsilon$, and using the following discrete version of the Leibniz rule, for all $\chi_1,\chi_2:\mathbb{Z}^d\to\mathbb{R}$,

$$
\nabla(\chi_1\chi_2) = \chi_1\nabla\chi_2+e_l\chi_2(\cdot+e_l)\nabla_l\chi_1, \tag{3.30}
$$

we obtain

$$
\begin{aligned}
-\nabla^*\cdot\boldsymbol{a}\nabla w_{f,\varepsilon} &= -\nabla^*\cdot\boldsymbol{a}\nabla(u_\varepsilon(\varepsilon\cdot)-\bar{u}_\varepsilon-\phi_j\nabla_j\bar{u}_\varepsilon) \\
&= -\nabla^*\cdot\bar{\boldsymbol{a}}\nabla\bar{u}_\varepsilon+\nabla^*\cdot\boldsymbol{a}\nabla\bar{u}_\varepsilon+\nabla^*\cdot(\boldsymbol{a}\nabla\phi_j\nabla_j\bar{u}_\varepsilon)+\nabla^*\cdot(\boldsymbol{a}e_k\phi_j(\cdot+e_k)\nabla_k\nabla_j\bar{u}_\varepsilon).
\end{aligned}
$$

Rearranging the terms and using the definition (3.4) of σ_j, this turns into

$$
\begin{aligned}
-\nabla^* \cdot \boldsymbol{a} \nabla w_{f,\varepsilon} &= \nabla^* \cdot \big((\boldsymbol{a}(\nabla \phi_j + e_j) - \bar{\boldsymbol{a}} e_j)\nabla_j \bar{u}_\varepsilon\big) + \nabla^* \cdot (\boldsymbol{a} e_k \phi_j(\cdot + e_k)\nabla_k \nabla_j \bar{u}_\varepsilon) \\
&= \nabla^* \cdot \big((\nabla^* \cdot \sigma_j)\nabla_j \bar{u}_\varepsilon\big) + \nabla^* \cdot (\boldsymbol{a} e_k \phi_j(\cdot + e_k)\nabla_k \nabla_j \bar{u}_\varepsilon).
\end{aligned}
$$

Using again the discrete Leibniz rule (3.30) and the skew-symmetry (3.3) of σ_j, we find

$$
\begin{aligned}
\nabla^* \cdot \big((\nabla^* \cdot \sigma_j)\nabla_j \bar{u}_\varepsilon\big) &= \nabla_k^*(\nabla_l^* \sigma_{jkl} \nabla_j \bar{u}_\varepsilon) = \underbrace{\nabla_k^* \nabla_l^* \sigma_{jkl} \nabla_j \bar{u}_\varepsilon}_{=0} + \nabla_l^* \sigma_{jkl}(\cdot - e_k)\nabla_k^* \nabla_j \bar{u}_\varepsilon \\
&= \nabla_l^*(\sigma_{jkl}(\cdot - e_k)\nabla_k^* \nabla_j \bar{u}_\varepsilon) - \underbrace{\sigma_{jkl}(\cdot - e_k - e_l)\nabla_k^* \nabla_l^* \nabla_j \bar{u}_\varepsilon}_{=0},
\end{aligned}
$$

and the conclusion (3.29) follows.

Step 2. Proof of estimates (3.24)–(3.28).
We start with the first term T_1^ε. For $b \in \mathcal{B}$ we use the notation $b = (z_b, z_b + e_b)$. Since $|\Delta_b \boldsymbol{a}(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$, the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
T_1^\varepsilon &\lesssim \varepsilon^{d-2} \sum_{b \in \mathcal{B}} \mathbb{E}\left[|\nabla \phi(z_b) + \mathrm{Id}|^2 \Big(\int_{Q(z_b)} |g_\varepsilon||\nabla w_{f,\varepsilon}^b + \phi_i^b \nabla \nabla_i \bar{u}_\varepsilon|\Big)^2\right] \\
&\lesssim \varepsilon^{d-2} \mathbb{E}\left[\sum_{b \in \mathcal{B}} |\nabla \phi(z_b) + \mathrm{Id}|^4 \Big(\int_{Q(z_b)} |g_\varepsilon|^2\Big)^2\right]^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad \times \mathbb{E}\left[\sum_{b \in \mathcal{B}} \Big(\int_{Q(z_b)} |\nabla w_{f,\varepsilon}^b + \phi_i^b \nabla \nabla_i \bar{u}_\varepsilon|^2\Big)^2\right]^{\frac{1}{2}},
\end{aligned}
$$

and hence, using the moment bounds of Lemma 3.3 and the exchangeability of $(\boldsymbol{a}, \boldsymbol{a}^b)$,

$$
T_1^\varepsilon \lesssim \varepsilon^{d-2} \|g_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2 \, \mathbb{E}\left[\sum_{b \in \mathcal{B}} \Big(\int_{Q(z_b)} |\nabla w_{f,\varepsilon} + \phi_i \nabla \nabla_i \bar{u}_\varepsilon|^2\Big)^2\right]^{\frac{1}{2}}.
$$

We argue as in (3.15): We rewrite the second right-hand side factor as a norm of averages at the scale r_*, smuggle in a suitable power of the weight w_ε, and apply Hölder's inequality, so that for all $p > 1$ and all $\alpha > d$,

$$
T_1^\varepsilon \lesssim_{\alpha,p} \varepsilon^{\frac{d}{2}(1+\frac{1}{p})-2} \|g_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2
$$

$$
\times \mathbb{E}\left[\int_{\mathbb{R}^d} w_\varepsilon(z)^{\alpha(p-1)} \Big(\fint_{B_*(z)} |\nabla w_{f,\varepsilon}|^2\Big)^{2p} dz + \int_{\mathbb{R}^d} w_\varepsilon^{\alpha(p-1)} |\phi|^{4p} |\nabla^2 \bar{u}_\varepsilon|^{4p}\right]^{\frac{1}{2p}}. \quad (3.31)
$$

By large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.4) applied to equation (3.29) for $w_{f,\varepsilon}$, we deduce for all $0 < p - 1 \ll 1$ and all $0 < \alpha - d \ll 1$,

$$
T_1^\varepsilon \lesssim_{\alpha,p} \varepsilon^{\frac{d}{2}(1+\frac{1}{p})-2} \|g_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2
$$

$$
\times \mathbb{E}\left[r_*(0)^{\alpha(p-1)} \int_{\mathbb{R}^d} w_\varepsilon^{\alpha(p-1)} \Big(|\sigma|^{4p} + |\phi|^{4p} + \sum_{k=1}^d |\phi(\cdot + e_k)|^{4p}\Big)|\nabla^2 \bar{u}_\varepsilon|^{4p}\right]^{\frac{1}{2p}}.
$$

By the moment bounds of Lemmas 3.3 and 3.4, this yields

$$T_1^\varepsilon \lesssim_{\alpha,p} \varepsilon^{\frac{d}{2}(1+\frac{1}{p})-2} \|g_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2 \|w_\varepsilon^{\alpha\frac{p-1}{4p}} \mu_d(|\cdot|)^{\frac{1}{2}} \nabla^2 \bar{u}_\varepsilon\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}^2.$$

We then apply the standard weighted Calderón-Zygmund theory (e.g. [59, Section V.4.2]) to the discrete constant-coefficient equation (3.5) for $\bar{u}_\varepsilon$ (cf. Lemma 3.4 with $r_* = 1$), note that for all $\chi, \zeta \in C_c^\infty(\mathbb{R}^d)$ and all $q < \infty$ the inequality $|\nabla(\zeta(\varepsilon\cdot))| \le \varepsilon \int_0^1 |D_k\zeta(\varepsilon(\cdot+te_k))| dt$ leads to

$$\int_{\mathbb{R}^d} \chi|\nabla(\zeta(\varepsilon\cdot))|^q \le \varepsilon^q \int_{\mathbb{R}^d} \Big(\sup_{B(x)} |\chi|\Big)|D\zeta(\varepsilon x)|^q dx \le \varepsilon^{q-d} \int_{\mathbb{R}^d} \Big(\sup_{B(\frac{x}{\varepsilon})} |\chi|\Big)|D\zeta(x)|^q dx, \quad (3.32)$$

rescale the integrals, estimate $\mu_d(|\frac{\cdot}{\varepsilon}|) \le \mu_d(\frac{1}{\varepsilon})\mu_d(|\cdot|)$, and the conclusion (3.24) follows.

We turn to the second term T_2^ε. Since $|\Delta_b\boldsymbol{a}(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$, the Cauchy-Schwarz inequality yields

$$T_2^\varepsilon \lesssim \varepsilon^{d-2} \mathbb{E}\left[\sum_{b\in\mathcal{B}} |\phi(z_b+e_k)|^2 \Big(\int_{Q(z_b)} |\nabla_k g_\varepsilon|^2\Big)\Big(\int_{Q(z_b)} |\nabla(u_\varepsilon^b(\varepsilon\cdot))|^2\Big)\right].$$

We bound the second local integral by an integral at the scale r_*^b, set $B_*^b(z) := B_{r_*^b(z)}(z)$, apply the Cauchy-Schwarz inequality, appeal to the moment bounds of Lemmas 3.3 and 3.4, and obtain

$$T_2^\varepsilon \lesssim \varepsilon^{d-2} \|\mu_d(|\cdot|)^{\frac{1}{2}}\nabla g_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2 \mathbb{E}\left[\sum_{b\in\mathcal{B}} \Big(\fint_{B_*^b(z_b)\cup Q(z_b)} |\nabla(u_\varepsilon^b(\varepsilon\cdot))|^2\Big)^2\right]^{\frac{1}{2}},$$

which, by exchangeability of $(\boldsymbol{a}, \boldsymbol{a}^b)$, takes the form

$$T_2^\varepsilon \lesssim \varepsilon^{d-2} \|\mu_d(|\cdot|)^{\frac{1}{2}}\nabla g_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2 \mathbb{E}\left[\int_{\mathbb{R}^d} \Big(\fint_{B_*(z)} |\nabla(u_\varepsilon(\varepsilon\cdot))|^2\Big)^2 dz\right]^{\frac{1}{2}}.$$

By large-scale (unweighted) Calderón-Zygmund theory (cf. Lemma 3.4) applied to equation (2.1) for u_ε in the form $-\nabla^* \cdot \boldsymbol{a}\nabla(u_\varepsilon(\varepsilon\cdot)) = \nabla^* \cdot (\varepsilon f_\varepsilon)$, we deduce

$$T_2^\varepsilon \lesssim \varepsilon^d \|\mu_d(|\cdot|)^{\frac{1}{2}}\nabla g_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2 \|f_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2,$$

and the conclusion (3.25) follows similarly as above.

The proof of (3.26) for T_3^ε is more direct. Indeed, using the moment bounds of Lemma 3.3, decomposing $\nabla(g_{\varepsilon,i}\nabla u_\varepsilon) = \nabla g_{\varepsilon,i}\otimes\nabla u_\varepsilon + g_{\varepsilon,i}(\cdot+e_k)e_k\otimes\nabla_k\nabla u_\varepsilon$, and applying the Cauchy-Schwarz inequality, we find

$$
\begin{aligned}
T_3^\varepsilon &\lesssim \varepsilon^{d-2}\sum_{k=1}^d \mathbb{E}\left[\sum_{b\in\mathcal{B}} |\phi(z_b+e_k)|^2|\nabla\phi^b(z_b)+\mathrm{Id}|^2\Big(\int_{Q(z_b)} |\nabla(g_\varepsilon\nabla\bar{u}_\varepsilon)|\Big)^2\right]\\
&\lesssim \varepsilon^{d-2}\int_{\mathbb{R}^d} \mu_d(|\cdot|)|\nabla(g_\varepsilon\nabla\bar{u}_\varepsilon)|^2\\
&\lesssim \varepsilon^{d-2}\Big(\|\nabla\bar{u}_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\|\mu_d(|\cdot|)\nabla g_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2 + \|g_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\|\mu_d(|\cdot|)\nabla^2\bar{u}_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2\Big),
\end{aligned}
$$

and the conclusion (3.26) follows as above.

We turn to the fourth term T_4^ε: We smuggle in a power $\alpha\frac{p-1}{2p}$ of the weight w_ε, apply Hölder's inequality with exponents $(\frac{2p}{p-1}, 2p, 2)$, appeal to the moment bounds of Lemma 3.4, use the exchangeability of $(\boldsymbol{a}, \boldsymbol{a}^b)$, and obtain for all $p > 1$ and all $\alpha > d$,

$$
T_4^\varepsilon \lesssim \varepsilon^{d-2}\,\mathbb{E}\left[\sum_{b\in\mathcal{B}} r_*(z_b)^d r_*^b(z_b)^d \Big(\fint_{B_*(z_b)\cup Q(z_b)} |\nabla r_\varepsilon|^2\Big)\Big(\fint_{B_*^b(z_b)\cup Q(z_b)} |\nabla(u_\varepsilon^b(\varepsilon\cdot))|^2\Big)dz\right]
$$

$$
\lesssim \varepsilon^{\frac{d}{2}(1+\frac{1}{p})-2}\,\mathbb{E}\left[\int_{\mathbb{R}^d} w_\varepsilon(z)^{\alpha(p-1)}\Big(\fint_{B_*(z)} |\nabla r_\varepsilon|^2\Big)^{2p} dz\right]^{\frac{1}{2p}}
$$

$$
\times \mathbb{E}\left[\int_{\mathbb{R}^d}\Big(\fint_{B_*(z)} |\nabla(u_\varepsilon(\varepsilon\cdot))|^2\Big)^2 dz\right]^{\frac{1}{2}}.
$$

By the large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.4) applied to equation (3.8) for r_ε and to equation (2.1) for u_ε, and the moment bounds of Lemma 3.3, we deduce for all $0 < p - 1 \ll 1$ and all $0 < \alpha - d \ll 1$,

$$
T_4^\varepsilon \lesssim_{\alpha,p} \varepsilon^{\frac{d}{2}(1+\frac{1}{p})}\,\|w_\varepsilon^{\alpha\frac{p-1}{4p}}\mu_d(|\cdot|)^{\frac{1}{2}}\nabla g_\varepsilon\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}^2 \|f_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2,
$$

and the conclusion (3.27) follows as before.

Finally, we turn to the last term T_5^ε: We use (3.14) in form of $|\nabla\phi^b(z_b)+\mathrm{Id}| \lesssim |\nabla\phi(z_b)+\mathrm{Id}|$, smuggle in a power $\alpha\frac{p-1}{2p}$ of the weight w_ε, apply Hölder's inequality with exponents $(\frac{2p}{p-1}, \frac{2p}{p+1})$, appeal to the moment bounds of Lemmas 3.3 and 3.4, and therefore obtain for all $p > 1$ and all $\alpha > d$,

$$
T_5^\varepsilon \lesssim \varepsilon^{d-2}\,\mathbb{E}\left[\int_{\mathbb{R}^d} |\nabla\phi(z)+\mathrm{Id}|^2\Big(\fint_{Q_2(z)} |\nabla R_\varepsilon|^2\Big)dz\right]
$$

$$
\lesssim_{\alpha,p} \varepsilon^{\frac{d}{2}(1+\frac{1}{p})-2}\,\mathbb{E}\left[\int_{\mathbb{R}^d} w_\varepsilon(z)^{\alpha\frac{p-1}{p+1}}\Big(\fint_{B_*(z)} |\nabla R_\varepsilon|^2\Big)^{\frac{2p}{p+1}} dz\right]^{\frac{p+1}{2p}}.
$$

By the large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.4) applied to equation (3.9) for R_ε, and the moment bounds of Lemma 3.3, we deduce for all $0 < p - 1 \ll 1$ and all $0 < \alpha - d \ll 1$,

$$
T_5^\varepsilon \lesssim_{\alpha,p} \varepsilon^{\frac{d}{2}(1+\frac{1}{p})-2}\,\|w_\varepsilon^{\alpha\frac{p-1}{4p}}\mu_d(|\cdot|)^{\frac{1}{2}}\nabla(g_\varepsilon\nabla\bar{u}_\varepsilon)\|_{\mathrm{L}^{\frac{4p}{p+1}}(\mathbb{R}^d)}^2.
$$

Decomposing $\nabla(g_{\varepsilon,i}\nabla u_\varepsilon) = \nabla g_{\varepsilon,i} \otimes \nabla u_\varepsilon + g_{\varepsilon,i}(\cdot + e_k)e_k \otimes \nabla_k\nabla u_\varepsilon$ and suitably applying Hölder's inequality with exponents $(\frac{p+1}{p}, p+1)$, the conclusion (3.28) follows as before.

Step 3. Conclusion.
Inserting estimates (3.24)–(3.28) into (3.23) yields for all $0 < p - 1 \ll 1$ and all $\alpha > d\frac{p-1}{4p}$,

$$
\|E_0^\varepsilon(f,g)\|_{\mathrm{L}^2(\Omega)} \lesssim_{\alpha,p} \varepsilon\mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}}
$$

$$
\times \Big(\|f\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Dg\|_{\mathrm{L}^{4p}(\mathbb{R}^d)} + \|g\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Df\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}\Big). \tag{3.33}
$$

It remains to deduce the corresponding result for $E^\varepsilon(f,g)$, and deal with the discretization error. In terms of $\tilde{u}_\varepsilon := \bar{u}_\varepsilon(\frac{\cdot}{\varepsilon})$ and $\tilde{v}_\varepsilon := \bar{v}_\varepsilon(\frac{\cdot}{\varepsilon})$, equations (3.5) take the form

$$-\nabla_\varepsilon^* \cdot \bar{\boldsymbol{a}}\nabla_\varepsilon\tilde{u}_\varepsilon = \nabla_\varepsilon^* \cdot f, \qquad -\nabla_\varepsilon^* \cdot \bar{\boldsymbol{a}}^*\nabla_\varepsilon\tilde{v}_\varepsilon = \nabla_\varepsilon^* \cdot g. \qquad (3.34)$$

The definitions of E^ε and E_0^ε then lead to the relation

$$E^\varepsilon(f,g) = E_0^\varepsilon(f,g) + J_0^\varepsilon\big((\nabla_\varepsilon\tilde{u}_\varepsilon - D\bar{u}) \otimes g\big), \qquad (3.35)$$

where $J_0^\varepsilon((\nabla_\varepsilon\tilde{u}_\varepsilon - D\bar{u}) \otimes g)$ is a discretization error. By Proposition 2.1 (and Cauchy-Schwarz' inequality), it is enough to establish for all $1 < p < \infty$ and all $0 \leq \alpha < d\frac{p-1}{p}$,

$$\|w_1^\alpha(\nabla_\varepsilon\tilde{u}_\varepsilon - D\bar{u})\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_{\alpha,p} \varepsilon\|w_1^\alpha Df\|_{\mathrm{L}^p(\mathbb{R}^d)}. \qquad (3.36)$$

For that purpose, we note that $\bar{u}$ is an approximate solution of the discrete equation (3.34). Indeed, integrating equation (2.2) for $\bar{u}$ on a unit cube yields for all $x \in \mathbb{R}^d$

$$0 = \int_{[-1,0)^d} D \cdot (\bar{\boldsymbol{a}}D\bar{u} + f)(x + \varepsilon y)dy = \nabla_\varepsilon^* \cdot (\bar{\boldsymbol{a}}D\bar{u} + f)(x) + \nabla_\varepsilon^* \cdot T_\varepsilon(x), \qquad (3.37)$$

where the error term T_ε is given by $T_\varepsilon(x) := e_i \int_{S_i}((\bar{\boldsymbol{a}}D\bar{u} + f)_i(x + \varepsilon y) - (\bar{\boldsymbol{a}}D\bar{u} + f)_i(x))dy$ in terms of $S_i := \{y \in [-1,0]^d : y_i = 0\}$, and satisfies for all $1 \leq p < \infty$ and all $0 \leq \alpha < \infty$,

$$\|w_1^\alpha T_\varepsilon\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_\alpha \varepsilon\|w_1^\alpha D(\bar{\boldsymbol{a}}D\bar{u} + f)\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim \varepsilon\|w_1^\alpha Df\|_{\mathrm{L}^p(\mathbb{R}^d)} + \varepsilon\|w_1^\alpha D^2\bar{u}\|_{\mathrm{L}^p(\mathbb{R}^d)}. \quad (3.38)$$

Comparing equations (3.34) and (3.37), the difference $\bar{u} - \tilde{u}_\varepsilon$ satisfies

$$-\nabla_\varepsilon^* \cdot \bar{\boldsymbol{a}}\nabla_\varepsilon(\bar{u} - \tilde{u}_\varepsilon) = \nabla_\varepsilon^* \cdot T_\varepsilon - \nabla_\varepsilon^* \cdot \bar{\boldsymbol{a}}(\nabla_\varepsilon\bar{u} - D\bar{u}).$$

Hence, using the standard weighted Calderón-Zygmund theory (e.g. [59, Section V.4.2]) applied to this discrete constant-coefficient equation, we obtain for all $1 < p < \infty$ and all $0 \leq \alpha < d\frac{p-1}{p}$,

$$\|w_1^\alpha\nabla_\varepsilon(\bar{u} - \tilde{u}_\varepsilon)\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_{\alpha,p} \|w_1^\alpha T_\varepsilon\|_{\mathrm{L}^p(\mathbb{R}^d)} + \|w_1^\alpha(\nabla_\varepsilon\bar{u} - D\bar{u})\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

Since the second right-hand side term is bounded by $\varepsilon\|w_1^\alpha D^2\bar{u}\|_{\mathrm{L}^p(\mathbb{R}^d)}$, estimate (3.38) yields

$$\|w_1^\alpha(\nabla_\varepsilon\tilde{u}_\varepsilon - D\bar{u})\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_{\alpha,p} \varepsilon\|w_1^\alpha Df\|_{\mathrm{L}^p(\mathbb{R}^d)} + \varepsilon\|w_1^\alpha D^2\bar{u}\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

The claim (3.36) then follows from the standard weighted Calderón-Zygmund theory applied to the constant-coefficient equation (2.2) for $\bar{u}$.

3.5. **Proof of Corollary 2.4.** We start with the proof of (2.6) for I_1^ε. By integration by parts, equations (3.34) and (2.1) for $\tilde{v}_\varepsilon$, $\tilde{u}_\varepsilon$, and u_ε lead to

$$\int g \cdot \nabla_\varepsilon(u_\varepsilon - \tilde{u}_\varepsilon) \stackrel{(3.34)}{=} -\int \nabla_\varepsilon\tilde{v}_\varepsilon \cdot \bar{\boldsymbol{a}}\nabla_\varepsilon(u_\varepsilon - \tilde{u}_\varepsilon) \stackrel{(3.34)}{=} -\int \nabla_\varepsilon\tilde{v}_\varepsilon \cdot f - \int \nabla_\varepsilon\tilde{v}_\varepsilon \cdot \bar{\boldsymbol{a}}\nabla_\varepsilon u_\varepsilon$$

$$\stackrel{(2.1)}{=} \int \nabla_\varepsilon\tilde{v}_\varepsilon \cdot (\boldsymbol{a}_\varepsilon\nabla_\varepsilon u_\varepsilon - \bar{\boldsymbol{a}}\nabla_\varepsilon u_\varepsilon).$$

Subtracting the expectation of both sides yields a discrete version of identity (1.8). In terms of J_0^ε, I_1^ε, and E_0^ε (cf. Section 2.1 and (3.6)), this takes on the following guise,

$$I_1^\varepsilon(f,g) - J_0^\varepsilon(D\bar{u} \otimes D\bar{v}) = J_0^\varepsilon(\nabla_\varepsilon\tilde{u}_\varepsilon \otimes \nabla_\varepsilon\tilde{v}_\varepsilon - D\bar{u} \otimes D\bar{v}) + E_0^\varepsilon(f, \nabla_\varepsilon\tilde{v}_\varepsilon). \qquad (3.39)$$

Using (3.36) and the standard weighted Calderón-Zygmund theory (e.g. [59, Section V.4.2]) applied to the constant-coefficient equations (2.2) and (3.34), the conclusion (2.6) for I_1^ε follows from (3.33) together with Proposition 2.1.

We turn to the proof of (2.6) for I_2^ε. By definition of J_0^ε, I_1^ε, I_2^ε, and E^ε (cf. Section 2.1), we find

$$I_2^\varepsilon(f,g) = E^\varepsilon(f,g) + I_1^\varepsilon(f, \bar{\boldsymbol{a}}^* g) + J_0^\varepsilon(D\bar{u} \otimes g).$$

Inserting identities (3.35) and (3.39) (with g replaced by $\bar{\boldsymbol{a}}^* g$ and thus $\bar{v}$ replaced by the solution $\bar{v}^\circ$ of $-D \cdot \bar{\boldsymbol{a}}^* D \bar{v}^\circ = D \cdot \bar{\boldsymbol{a}}^* g$, so that $\bar{\mathcal{P}}_L^* g = D \bar{v}^\circ + g$), the conclusion (2.6) follows similarly as for I_1^ε.

We now turn to the proof of (2.7). Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decaying functions, and consider the subspace $\mathcal{K}_\varepsilon := \{g \in \mathcal{S}(\mathbb{R}^d)^d : \bar{v}_\varepsilon \in \mathcal{S}(\mathbb{R}^d)\}$, cf. (3.5). Given some fixed $\chi \in C_c^\infty(\mathbb{R}^d)$, set $\chi_L := \chi(L\cdot)$ for $L \geq 1$. For $g \in \mathcal{K}_\varepsilon$, we compute by integration by parts, using equation (2.4) for ϕ_j and equation (3.5) for $\bar{v}_\varepsilon$, together with the discrete Leibniz rule (3.30),

$$\int_{\mathbb{R}^d} \chi_L \nabla \bar{v}_\varepsilon \cdot \Xi_i = \int_{\mathbb{R}^d} \chi_L \nabla \bar{v}_\varepsilon \cdot \big(\boldsymbol{a}(\nabla \phi_i + e_i) - \bar{\boldsymbol{a}}(\nabla \phi_i + e_i) \big)$$

$$\overset{(2.4)}{=} -\int_{\mathbb{R}^d} \nabla(\bar{v}_\varepsilon \chi_L) \cdot \bar{\boldsymbol{a}} \nabla \phi_i - \int_{\mathbb{R}^d} \bar{v}_\varepsilon(\cdot + e_j) \nabla_j \chi_L \, \Xi_{ij}$$

$$\overset{(3.5)}{=} \varepsilon \int_{\mathbb{R}^d} \chi_L g_\varepsilon \cdot \nabla \phi_i + \varepsilon \int_{\mathbb{R}^d} \phi_i(\cdot + e_j) g_{\varepsilon,j} \nabla_j \chi_L + \int_{\mathbb{R}^d} \phi_i(\cdot + e_j) \nabla_j \chi_L e_j \cdot \bar{\boldsymbol{a}} \nabla \bar{v}_\varepsilon$$

$$- \int_{\mathbb{R}^d} \bar{v}_\varepsilon(\cdot + e_j) \nabla_j \chi_L e_j \cdot \bar{\boldsymbol{a}} \nabla \phi_i - \int_{\mathbb{R}^d} \bar{v}_\varepsilon(\cdot + e_j) \nabla_j \chi_L \, \Xi_{ij}.$$

For fixed ε and $g \in \mathcal{K}_\varepsilon$, using the moment bounds of Lemma 3.3 and the rapid decay at infinity of g and $\bar{v}_\varepsilon$, we may pass to the limit $L \uparrow \infty$ in both sides in $\mathrm{L}^2(\Omega)$, and we deduce almost surely

$$\int_{\mathbb{R}^d} \nabla \bar{v}_\varepsilon : \Xi_j = \varepsilon \int_{\mathbb{R}^d} g_\varepsilon \cdot \nabla \phi_j,$$

that is, after rescaling,

$$J_1^\varepsilon(e_j \otimes g) = J_0^\varepsilon(e_j \otimes \nabla_\varepsilon \tilde{v}_\varepsilon). \tag{3.40}$$

We now argue that for all $\varepsilon > 0$ this almost sure identity can be extended to hold in $\mathrm{L}^2(\Omega)$ for all $g \in C_c^\infty(\mathbb{R}^d)^d$ (even though $\nabla_\varepsilon \tilde{v}_\varepsilon$ is not integrable). First note that Proposition 2.1 combined with the standard weighted Calderón-Zygmund theory (e.g. [59, Section V.4.2]) for the constant-coefficient equation (3.34) yields for all $0 < p - 1 \ll 1$ and all $d\frac{p-1}{4p} < \alpha < d\frac{2p-1}{4p}$,

$$\mathbb{E}\left[|J_0^\varepsilon(e_j \otimes \nabla_\varepsilon \tilde{v}_\varepsilon)|^2 \right]^{\frac{1}{2}} \lesssim_{\alpha,p} \|w_1^{2\alpha} \nabla_\varepsilon \tilde{v}_\varepsilon\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \lesssim_{\alpha,p} \|w_1^{2\alpha} g\|_{\mathrm{L}^{2p}(\mathbb{R}^d)},$$

and in addition,

$$\mathbb{E}\left[|J_1^\varepsilon(e_j \otimes g)|^2 \right]^{\frac{1}{2}} \lesssim \|w_1^{2\alpha} g\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}.$$

Hence, it suffices to check the following density result: for all test functions $g \in C_c^\infty(\mathbb{R}^d)^d$ there exist a sequence $(g_n)_n$ of elements of $\mathcal{K}_\varepsilon$ such that $\|w_1^{2\alpha}(g_n - g)\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \to 0$ holds for some $0 < p - 1 \ll 1$ and some $\alpha > d\frac{p-1}{4p}$. Let $g \in C_c^\infty(\mathbb{R}^d)^d$ be fixed. Up to a convolution argument on large scales, we may already assume that the Fourier transform $\hat{g}$ has compact support, say contained in B_R. Since the (continuum) Fourier symbol of the discrete Helmholtz projection $\nabla_\varepsilon(\nabla_\varepsilon^* \cdot \bar{a} \nabla_\varepsilon)^{-1} \nabla_\varepsilon^* \cdot$ is bounded and smooth outside of the dual lattice $(\frac{2\pi}{\varepsilon}\mathbb{Z})^d$, a function $g_n \in \mathcal{S}(\mathbb{R}^d)^d$ actually belongs to $\mathcal{K}_\varepsilon$ whenever its Fourier

transform $\hat{g}_n$ vanishes in a neighborhood of $(\frac{2\pi}{\varepsilon}\mathbb{Z})^d$. Choosing $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi = 1$ in a neighborhood of 0, and defining

$$\chi_n := 1 - \sum_{z \in (\frac{2\pi}{\varepsilon}\mathbb{Z})^d} \chi(n(\cdot - z)),$$

the function $g_n \in \mathcal{S}(\mathbb{R}^d)^d$ defined by $\hat{g}_n := \chi_n \hat{g}$ thus belongs to $\mathcal{K}_\varepsilon$. For $p \geq 1$, setting $q := \frac{2p}{2p-1}$, the Hausdorff-Young inequality yields

$$\|w_1^{2\alpha}(g_n - g)\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \lesssim_p \|\mathcal{F}(w_1^{2\alpha}(g_n - g))\|_{\mathrm{L}^q(\mathbb{R}^d)}$$
$$\lesssim \|\hat{g}_n - \hat{g}\|_{W^{2\alpha,q}(\mathbb{R}^d)} = \|(\chi_n - 1)\hat{g}\|_{W^{2\alpha,q}(\mathbb{R}^d)},$$

where $\mathcal{F}(h) = \hat{h}$ denotes the Fourier transform. Hence, since $\hat{g}$ is supported in B_R,

$$\|w_1^{2\alpha}(g_n - g)\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \lesssim_\alpha \|\chi_n - 1\|_{W^{2\alpha,q}(B_R)}\|\hat{g}\|_{W^{2\alpha,\infty}(\mathbb{R}^d)}$$
$$\lesssim \|\chi_n - 1\|_{W^{2\alpha,q}(B_R)}\|w_1^{2\alpha}g\|_{\mathrm{L}^1(\mathbb{R}^d)}.$$

For $2\alpha < \frac{d}{q} = d\frac{2p-1}{2p}$, reflecting the fact that the Sobolev space $W^{2\alpha,q}(\mathbb{R}^d)$ fails to embed into the space of continuous functions, there holds $\chi_n \to 1$ in $W_{\mathrm{loc}}^{2\alpha,q}(\mathbb{R}^d)$ as $n \uparrow \infty$, and hence $\|w_1^{2\alpha}(g_n - g)\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \to 0$. This establishes the claimed density result, and we conclude that identity (3.40) can be extended in $\mathrm{L}^2(\Omega)$ to all $g \in C_c^\infty(\mathbb{R}^d)^d$. The estimate (2.7) for J_1^ε then follows from the discretization error estimate (3.36) together with Proposition 2.1. The estimate (2.7) for J_2^ε is obtained in a similar way.

4. Asymptotic normality

We turn to the normal approximation result for the homogenization commutator Ξ as stated in Proposition 2.7.

4.1. **Structure of the proof and auxiliary results.** The main tool to prove Proposition 2.7 is the following suitable form of a second-order Poincaré inequality à la Chatterjee [10, 44]. Based on Stein's method, it can be shown to hold for any product measure $\mathbb{P}$ on Ω. (The proof follows from [10, Theorem 2.2] and [44, Theorem 4.2] in the case of the Wasserstein and of the Kolmogorov metric, respectively, combined with the spectral gap estimate of Lemma 3.1.) The first use of such functional inequalities in stochastic homogenization is due to Nolen [54, 55]. Let us first fix some more notation. Let $X = X(\boldsymbol{a})$ be a random variable on Ω, that is, a measurable function of $(a(b))_{b \in \mathcal{B}}$. For all $E \subset \mathcal{B}$ we denote by $\boldsymbol{a}^E$ the random field that coincides with $\boldsymbol{a}$ on all edges $b \notin E$ and with the iid copy $\boldsymbol{a}'$ on all edges $b \in E$. In particular, $\boldsymbol{a}$ and $\boldsymbol{a}^E$ always have the same law. We use the abbreviation $X^E := X(\boldsymbol{a}^E)$ and define $\Delta_b X^E := X^E - X^{E \cup \{b\}}$. As before, we write for simplicity $X^b := X^{\{b\}}$, and similarly $X^{b,b'} := X^{\{b,b'\}}$. In particular, $\Delta_b \Delta_{b'} X = X - X^b - X^{b'} + X^{b,b'}$.

Lemma 4.1 ([10, 44]). *Let $\mathbb{P}$ be a product measure and let $\delta_\mathcal{N}$ be defined in (1.15). For all $X = X(\boldsymbol{a}) \in \mathrm{L}^2(\Omega)$, we have*

$$\delta_\mathcal{N}(X) \lesssim \frac{1}{\mathrm{Var}\,[X]^{\frac{3}{2}}} \sum_{b \in \mathcal{B}_L} \mathbb{E}\left[|\Delta_b X|^6\right]^{\frac{1}{2}}$$

$$+ \frac{1}{\mathrm{Var}\,[X]}\left(\sum_{b \in \mathcal{B}}\left(\sum_{e' \in \mathcal{B}} \mathbb{E}\left[|\Delta_{b'} X|^4\right]^{\frac{1}{4}} \mathbb{E}\left[|\Delta_b \Delta_{b'} X|^4\right]^{\frac{1}{4}}\right)^2\right)^{\frac{1}{2}}. \quad \diamond$$

In addition, we make crucial use of the following optimal annealed estimate on the mixed gradient of the Green's function, first proved by Marahrens and the third author [46]. Note that the product space assumption can be substantially relaxed (cf. [8] and [4, Section 8.5]).

Lemma 4.2 ([46])**.** *Let $d \geq 2$ and let $\mathbb{P}$ be a product measure. For all $y \in \mathbb{Z}^d$ there exists a function $\nabla G(\cdot, y)$ that is the unique decaying solution in $\mathbb{Z}^d$ of*

$$-\nabla^* \cdot \boldsymbol{a} \nabla G(\cdot, y) = \delta(\cdot - y).$$

It satisfies the following moment bound: for all $q < \infty$ and all $x, y \in \mathbb{Z}^d$,

$$\mathbb{E}\left[|\nabla \nabla G(x, y)|^q\right]^{\frac{1}{q}} \lesssim_q (1 + |x - y|)^{-d},$$

where $\nabla \nabla$ denotes the mixed second gradient. $\diamondsuit$

4.2. **Proof of Proposition 2.7.** Let $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$, and set $F_\varepsilon := F(\varepsilon \cdot)$. By Lemma 4.1, it is enough to estimate the following two contributions,

$$K_1^\varepsilon := \sum_{b \in \mathcal{B}} \mathbb{E}\left[|\Delta_b J_0^\varepsilon(F)|^6\right]^{\frac{1}{2}}, \quad K_2^\varepsilon := \sum_{b \in \mathcal{B}} \left(\sum_{b' \in \mathcal{B}} \mathbb{E}\left[|\Delta_{b'} J_0^\varepsilon(F)|^4\right]^{\frac{1}{4}} \mathbb{E}\left[|\Delta_b \Delta_{b'} J_0^\varepsilon(F)|^4\right]^{\frac{1}{4}}\right)^2.$$

We split the proof into three steps: we start with an auxiliary estimate, and then estimate K_1^ε and K_2^ε separately.

Step 1. Auxiliary estimate: for all $\zeta \in C_c^\infty(\mathbb{R}^d)$, all $1 \leq p < \infty$, and all $r \geq 0$,

$$\int_{\mathbb{R}^d} \log^r(2 + |z|) \left(\int_{\mathbb{R}^d} \frac{|\zeta(x)|}{(1 + |x - z|)^d} dx\right)^p dz \lesssim_{p,r} \int_{\mathbb{R}^d} \log^{p+r}(2 + |x|) |\zeta(x)|^p dx. \quad (4.1)$$

Let $\alpha > 0$ be fixed. Smuggling in a power $\alpha \frac{p-1}{p}$ of the weight $1 + |x|$, and applying Hölder's inequality with exponent p, we find

$$\int_{\mathbb{R}^d} \log^r(2 + |z|) \left(\int_{\mathbb{R}^d} \frac{|\zeta(x)|}{(1 + |x - z|)^d} dx\right)^p dz$$

$$\leq \int_{\mathbb{R}^d} \log^r(2 + |z|) \left(\int_{\mathbb{R}^d} \frac{(1 + |x|)^{\alpha(p-1)} |\zeta(x)|^p}{(1 + |x - z|)^d} dx\right) \left(\int_{\mathbb{R}^d} \frac{dx}{(1 + |x - z|)^d (1 + |x|)^\alpha}\right)^{p-1} dz.$$

The last integral is controlled by $C(d, \alpha) \frac{\log(2 + |z|)}{(1 + |z|)^\alpha}$, hence by Fubini's theorem

$$\int_{\mathbb{R}^d} \log^r(2 + |z|) \left(\int_{\mathbb{R}^d} \frac{|\zeta(x)|}{(1 + |x - z|)^d} dx\right)^p dz$$

$$\lesssim_{\alpha,p} \int_{\mathbb{R}^d} \frac{\log^{p+r-1}(2 + |z|)}{(1 + |z|)^{\alpha(p-1)}} \left(\int_{\mathbb{R}^d} \frac{(1 + |x|)^{\alpha(p-1)} |\zeta(x)|^p}{(1 + |x - z|)^d} dx\right) dz$$

$$= \int_{\mathbb{R}^d} (1 + |x|)^{\alpha(p-1)} |\zeta(x)|^p \left(\int_{\mathbb{R}^d} \frac{\log^{p+r-1}(2 + |z|)}{(1 + |x - z|)^d (1 + |z|)^{\alpha(p-1)}} dz\right) dx.$$

Since the last integral is controlled by $C(d, p, r, \alpha) \frac{\log^{p+r}(2 + |x|)}{(1 + |x|)^{\alpha(p-1)}}$, the conclusion (4.1) follows.

Step 2. Proof of

$$K_1^\varepsilon \lesssim \varepsilon^{\frac{d}{2}} \left(\|F\|_{L^3(\mathbb{R}^d)}^3 + \|\log(2 + |\cdot|) \mu_d(|\cdot|)^{\frac{1}{2}} DF\|_{L^3(\mathbb{R}^d)}^3\right).$$

After integration by parts, the representation formula for the vertical derivative $\Delta_b\Xi$ in (3.18) leads to

$$\Delta_b J_0^\varepsilon(F) = \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij}(\nabla\phi_j^* + e_j) \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i)$$

$$+ \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} \phi_j^*(\cdot + e_k)\nabla_k F_{\varepsilon,ij}e_k \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i)$$

$$+ \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} \left(\phi_j^*(\cdot + e_k)\boldsymbol{a}_{kl}\nabla_k F_{\varepsilon,ij} + \sigma_{jkl}^*(\cdot - e_k)\nabla_k^* F_{\varepsilon,ij}\right)\nabla_l\Delta_b\phi_i.$$

For $b = (z_b, z_b + e_b)$, the Green representation formula applied to equation (3.12) takes the following form, for all $x \in \mathbb{Z}^d$,

$$\nabla\Delta_b\phi_i(x) = -\nabla\nabla G(x, z_b)\Delta_b\boldsymbol{a}(z_b)(\nabla\phi_i^b(z_b) + e_i). \tag{4.2}$$

Inserted into the above representation formula for $\Delta_b J_0^\varepsilon(F)$, and combined with $|\Delta_b\boldsymbol{a}(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$ and the moment bounds of Lemmas 3.3 and 4.2, it yields for all $q < \infty$,

$$\mathbb{E}\left[|\Delta_b J_0^\varepsilon(F)|^q\right]^{\frac{1}{q}} \lesssim_q \varepsilon^{\frac{d}{2}} \int_{Q(z_b)} |F_\varepsilon| + \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} \frac{\mu_d(|x|)^{\frac{1}{2}}|\nabla F_\varepsilon(x)|}{(1 + |x - z_b|)^d}dx. \tag{4.3}$$

Summing the cube of this estimate over $b \in \mathcal{B}$ for $q = 6$, and using (4.1) with $p = 3$, $r = 0$, and $\zeta = \mu_d(|\cdot|)^{\frac{1}{2}}\nabla F_\varepsilon$, we obtain

$$K_1^\varepsilon \lesssim \varepsilon^{\frac{3d}{2}} \int_{\mathbb{R}^d} |F_\varepsilon|^3 + \varepsilon^{\frac{3d}{2}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{\mu_d(|x|)^{\frac{1}{2}}|\nabla F_\varepsilon(x)|}{(1 + |x - z|^d)}dx\right)^3 dz$$

$$\lesssim \varepsilon^{\frac{3d}{2}} \int_{\mathbb{R}^d} |F_\varepsilon|^3 + \varepsilon^{\frac{3d}{2}} \int_{\mathbb{R}^d} \log^3(2 + |\cdot|)\,\mu_d(|\cdot|)^{\frac{3}{2}}|\nabla F_\varepsilon|^3.$$

Rescaling the integrals and using (3.32), the conclusion follows.

Step 3. Proof of

$$K_2^\varepsilon \lesssim \varepsilon^d \log^2(2 + \tfrac{1}{\varepsilon})\left(\|F\|_{\mathrm{L}^4(\mathbb{R}^d)}^4 + \|\log(2 + |\cdot|)F\|_{\mathrm{L}^4(\mathbb{R}^d)}^4\right)$$

$$+ \varepsilon^{d+2}\log^4(2 + \tfrac{1}{\varepsilon})\mu_d(\tfrac{1}{\varepsilon})\left(\|\log(2 + |\cdot|)F\|_{\mathrm{L}^4(\mathbb{R}^d)}^4 + \|\log^2(2 + |\cdot|)\mu_d(|\cdot|)^{\frac{1}{2}}DF\|_{\mathrm{L}^4(\mathbb{R}^d)}^4\right)$$

$$+ \varepsilon^{d+4}\log^6(2 + \tfrac{1}{\varepsilon})\mu_d(\tfrac{1}{\varepsilon})^2\|\log^2(2 + |\cdot|)\mu_d(|\cdot|)^{\frac{1}{2}}DF\|_{\mathrm{L}^4(\mathbb{R}^d)}^4.$$

We need to iterate the vertical derivative and estimate $\Delta_b\Delta_{b'}I_0^\varepsilon(F)$. By definition of the homogenization commutator, we find

$$\Delta_b\Delta_{b'}J_0^\varepsilon(F) = \varepsilon^{\frac{d}{2}}\Delta_b\int_{\mathbb{R}^d} F_{\varepsilon,ij}e_j \cdot \Delta_{b'}\boldsymbol{a}(\nabla\phi_i^{b'} + e_i) + \varepsilon^{\frac{d}{2}}\Delta_b\int_{\mathbb{R}^d} F_{\varepsilon,ij}e_j \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}})\nabla\Delta_{b'}\phi_i$$

$$= \varepsilon^{\frac{d}{2}}\int_{\mathbb{R}^d} F_{\varepsilon,ij}e_j \cdot \Delta_b\Delta_{b'}\boldsymbol{a}(\nabla\phi_i^{b,b'} + e_i) + \varepsilon^{\frac{d}{2}}\int_{\mathbb{R}^d} F_{\varepsilon,ij}e_j \cdot \left(\Delta_b\boldsymbol{a}\nabla\Delta_{b'}\phi_i^b + \Delta_{b'}\boldsymbol{a}\nabla\Delta_b\phi_i^{b'}\right)$$

$$+ \varepsilon^{\frac{d}{2}}\int_{\mathbb{R}^d} F_{\varepsilon,ij}e_j \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}})\nabla\Delta_b\Delta_{b'}\phi_i. \tag{4.4}$$

In order to avoid additional logarithmic factors, we need to suitably rewrite the last right-hand side term, and we argue similarly as in the proof of (3.18). Using the definition (3.4)

of σ_j^* in the form $(\boldsymbol{a}^* - \bar{\boldsymbol{a}}^*)e_j = -\boldsymbol{a}^* \nabla \phi_j^* + \nabla^* \cdot \sigma_j^*$, applying the discrete Leibniz rule (3.20), and using the skew-symmetry (3.3) of σ_i, we obtain

$$
\begin{aligned}
e_j \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}}) \nabla \Delta_b \Delta_{b'} \phi_i &= (\nabla^* \cdot \sigma_j^*) \cdot \nabla \Delta_b \Delta_{b'} \phi_i - \nabla \phi_j^* \cdot \boldsymbol{a} \nabla \Delta_b \Delta_{b'} \phi_i \\
&= -\nabla_k \big(\sigma_{jkl}^* (\cdot - e_k) \nabla_l \Delta_b \Delta_{b'} \phi_i \big) - \nabla_k^* \big(\phi_j^* (\cdot + e_k) e_k \cdot \boldsymbol{a} \nabla \Delta_b \Delta_{b'} \phi_i \big) + \phi_j^* \nabla^* \cdot \boldsymbol{a} \nabla \Delta_b \Delta_{b'} \phi_i.
\end{aligned}
$$

The vertical derivative of equation (3.12) takes the form

$$
-\nabla^* \cdot \boldsymbol{a} \nabla \Delta_b \Delta_{b'} \phi_i = \nabla^* \cdot \Delta_{b'} \boldsymbol{a} \nabla \Delta_b \phi_i^{b'} + \nabla^* \cdot \Delta_b \Delta_{b'} \boldsymbol{a} (\nabla \phi_i^{b,b'} + e_i) + \nabla^* \cdot \Delta_b \boldsymbol{a} \nabla \Delta_{b'} \phi_i^b, \tag{4.5}
$$

which, combined with the above, yields

$$
\begin{aligned}
e_j \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}}) \nabla \Delta_b \Delta_{b'} \phi_i ={}& -\nabla_k \big(\sigma_{jkl}^* (\cdot - e_k) \nabla_l \Delta_b \Delta_{b'} \phi_i \big) - \nabla_k^* \big(\phi_j^* (\cdot + e_k) e_k \cdot \boldsymbol{a} \nabla \Delta_b \Delta_{b'} \phi_i \big) \\
&- \phi_j^* \nabla^* \cdot \Delta_{b'} \boldsymbol{a} \nabla \Delta_b \phi_i^{b'} - \phi_j^* \nabla^* \cdot \Delta_b \Delta_{b'} \boldsymbol{a} (\nabla \phi_i^{b,b'} + e_i) - \phi_j^* \nabla^* \cdot \Delta_b \boldsymbol{a} \nabla \Delta_{b'} \phi_i^b.
\end{aligned}
$$

Inserting this into (4.4), integrating by parts, and applying the discrete Leibniz rule (3.30), we obtain the following representation formula

$$
\begin{aligned}
\Delta_b \Delta_{b'} J_0^\varepsilon(F) ={}& \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij} (\nabla \phi_j^* + e_j) \cdot \Delta_b \Delta_{b'} \boldsymbol{a} (\nabla \phi_i^{b,b'} + e_i) \\
&+ \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij} (\nabla \phi_j^* + e_j) \cdot \big(\Delta_b \boldsymbol{a} \nabla \Delta_{b'} \phi_i^b + \Delta_{b'} \boldsymbol{a} \nabla \Delta_b \phi_i^{b'} \big) \\
&+ \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} \big(\sigma_{jkl}^* (\cdot - e_k) \nabla_k^* F_{\varepsilon,ij} + \phi_j^* (\cdot + e_k) \boldsymbol{a}_{kl} \nabla_k F_{\varepsilon,ij} \big) \nabla_l \Delta_b \Delta_{b'} \phi_i \\
&+ \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} \phi_j^* (\cdot + e_k) \nabla_k F_{\varepsilon,ij} e_k \cdot \big(\Delta_b \boldsymbol{a} \nabla \Delta_{b'} \phi_i^b + \Delta_{b'} \boldsymbol{a} \nabla \Delta_b \phi_i^{b'} \big) \\
&\hspace{2cm} + \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} \phi_j^* (\cdot + e_k) \nabla_k F_{\varepsilon,ij} e_k \cdot \Delta_b \Delta_{b'} \boldsymbol{a} (\nabla \phi_i^{b,b'} + e_i). \tag{4.6}
\end{aligned}
$$

We need to estimate the moment of each right-hand side term. Fix momentarily $b = (z_b, z_b + e_b)$ and $b' = (z_{b'}, z_{b'} + e_{b'})$. Applying Lemmas 3.3 and 4.2 to the Green representation formula (4.2) for $\nabla \Delta_b \phi$, we find for all $q < \infty$,

$$
\mathbb{E}\left[|\nabla \Delta_b \phi(x)|^q \right]^{\frac{1}{q}} \lesssim_q (1 + |x - z_b|)^{-d}.
$$

We then turn to the second vertical derivatives. We obviously have $\Delta_b \Delta_{b'} \boldsymbol{a} = \mathbb{1}_{b=b'} \Delta_b \boldsymbol{a}$. Next, the Green representation formula applied to equation (4.5) yields

$$
\begin{aligned}
\nabla \Delta_b \Delta_{b'} \phi_j(x) ={}& -\nabla \nabla G(x, z_{b'}) \cdot \Delta_{b'} \boldsymbol{a}(z_{b'}) \nabla \Delta_b \phi_j^{b'}(z_{b'}) - \nabla \nabla G(x, z_b) \cdot \Delta_b \boldsymbol{a}(z_b) \nabla \Delta_{b'} \phi_j^b(z_b) \\
&- \mathbb{1}_{b=b'} \nabla \nabla G(x, z_b) \cdot \Delta_b \boldsymbol{a}(z_b) (\nabla \phi_j^b + e_j),
\end{aligned}
$$

so that, for all $q < \infty$, Lemmas 3.3 and 4.2 lead to

$$
\mathbb{E}\left[|\nabla \Delta_b \Delta_{b'} \phi(x)|^q \right]^{\frac{1}{q}} \lesssim_q (1 + |z_b - z_{b'}|)^{-d} \big((1 + |x - z_{b'}|)^{-d} + (1 + |x - z_b|)^{-d} \big).
$$

Inserting these estimates into (4.6), we obtain

$$
\mathbb{E}\left[|\Delta_b \Delta_{b'} J_0^\varepsilon(F)|^4\right]^{\frac{1}{4}} \lesssim \frac{\varepsilon^{\frac{d}{2}}}{(1+|z_b - z_{b'}|)^d} \bigg(\int_{Q(z_b)} |F_\varepsilon| + \int_{Q(z_{b'})} |F_\varepsilon|
$$
$$
+ \int_{\mathbb{R}^d} \frac{\mu_d(|x|)^{\frac{1}{2}} |\nabla F_\varepsilon(x)|}{(1+|x-z_{b'}|)^d} dx + \int_{\mathbb{R}^d} \frac{\mu_d(|x|)^{\frac{1}{2}} |\nabla F_\varepsilon(x)|}{(1+|x-z_b|)^d} dx \bigg).
$$

Combining this with (4.3) and with the definition of K_2^ε, with the short-hand notation

$$
I(\zeta)(z) := \int_{\mathbb{R}^d} \frac{|\zeta(x)|}{(1+|x-z|)^d} dx,
$$

and $G_\varepsilon := \mu_d(|\cdot|)^{\frac{1}{2}} \nabla F_\varepsilon$, we deduce

$$
K_2^\varepsilon \lesssim \varepsilon^{2d} \int_{\mathbb{R}^d} \Big(|F_\varepsilon|^2 |I(F_\varepsilon)|^2 + |I(|F_\varepsilon|^2)|^2 + |I(F_\varepsilon)|^2 |I(G_\varepsilon)|^2 + |I(F_\varepsilon I(G_\varepsilon))|^2
$$
$$
+ |F_\varepsilon|^2 |I(I(G_\varepsilon))|^2 + |I(|I(G_\varepsilon)|^2)|^2 + |I(G_\varepsilon)|^2 |I(I(G_\varepsilon))|^2 \Big).
$$

By the Cauchy-Schwarz inequality and a multiple use of (4.1) in the form

$$
\| \log^r(2+|\cdot|) \, I(\zeta) \|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_{p,r} \| \log^{r+1}(2+|\cdot|) \, \zeta \|_{\mathrm{L}^p(\mathbb{R}^d)},
$$

we are led to

$$
K_2^\varepsilon \lesssim \varepsilon^{2d} \Big(\|F_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2 \| \log(2+|\cdot|) F_\varepsilon \|_{\mathrm{L}^4(\mathbb{R}^d)}^2 + \| \log^{\frac{1}{2}}(2+|\cdot|) F_\varepsilon \|_{\mathrm{L}^4(\mathbb{R}^d)}^4
$$
$$
+ \| \log(2+|\cdot|) F_\varepsilon \|_{\mathrm{L}^4(\mathbb{R}^d)}^2 \| \log(2+|\cdot|) G_\varepsilon \|_{\mathrm{L}^4(\mathbb{R}^d)}^2 + \|F_\varepsilon\|_{\mathrm{L}^4(\mathbb{R}^d)}^2 \| \log^2(2+|\cdot|) G_\varepsilon \|_{\mathrm{L}^4(\mathbb{R}^d)}^2
$$
$$
+ \| \log^{\frac{3}{2}}(2+|\cdot|) G_\varepsilon \|_{\mathrm{L}^4(\mathbb{R}^d)}^4 + \| \log(2+|\cdot|) G_\varepsilon \|_{\mathrm{L}^4(\mathbb{R}^d)}^2 \| \log^2(2+|\cdot|) G_\varepsilon \|_{\mathrm{L}^4(\mathbb{R}^d)}^2 \Big).
$$

Inserting the definition of G_ε, rescaling the integrals, and using (3.32), the conclusion follows.

5. COVARIANCE STRUCTURE

In this section, we turn to the limiting covariance structure of the homogenization commutator, as stated in Proposition 2.9.

5.1. **Structure of the proof and auxiliary results.** The main tool to prove Proposition 2.9 is the following stronger version of the spectral gap estimate of Lemma 3.1, which gives an identity (rather than a bound) for the variance of a random variable in terms of its variations. This is an iid version of the so-called Helffer-Sjöstrand representation formula [38, 58] (see also [52, 49]), which holds for any product measure $\mathbb{P}$ on Ω. A proof is included for completeness in Subsection 5.2 below. Instead of using the difference operator $\triangle_b$ as in Lemma 3.1, this result is more conveniently formulated in terms of $\tilde{\triangle}_b X := X - \mathbb{E}_{a(b)}[X]$, with the notation $\mathbb{E}_{a(b)}[\cdot] := \mathbb{E}\big[\cdot \, \big| (a(b'))_{b' \neq b} \big]$. Note that $\tilde{\triangle}_b X = \mathbb{E}[\triangle_b X | \boldsymbol{a}]$ and $\mathbb{E}\big[|\tilde{\triangle}_b X|^2\big] = \frac{1}{2}\mathbb{E}\big[|\triangle_b X|^2\big]$.

Lemma 5.1. *Let $\mathbb{P}$ be a product measure. For all $X = X(\boldsymbol{a}) \in \mathrm{L}^2(\Omega)$ we have*

$$
\mathrm{Var}\,[X] = \sum_{b \in \mathcal{B}} \mathbb{E}\left[(\tilde{\triangle}_b X) \, \mathcal{T} \, (\tilde{\triangle}_b X) \right],
$$

where $\mathcal{T} := (\sum_{b \in \mathcal{B}} \tilde{\Delta}_b \tilde{\Delta}_b)^{-1}$ is a self-adjoint positive operator on $\mathrm{L}^2(\Omega)/\mathbb{R} := \{X \in \mathrm{L}^2(\Omega) : \mathbb{E}[X] = 0\}$ with operator norm bounded by 1. In particular, it implies the following covariance inequality: for all $X, Y \in \mathrm{L}^2(\Omega)$ we have

$$\mathrm{Cov}\,[X;Y] \leq \frac{1}{2} \sum_{b \in \mathcal{B}} \mathbb{E}\left[|\Delta_b X|^2\right]^{\frac{1}{2}} \mathbb{E}\left[|\Delta_b Y|^2\right]^{\frac{1}{2}}. \qquad \diamond$$

The proof of Proposition 2.9(i) below further implies that the effective fluctuation tensor $\mathcal{Q}$ is given by the following formula, with the notation $b_n := (0, e_n)$,

$$\mathcal{Q}_{ijkl} := \sum_{n=1}^{d} \mathbb{E}\left[\left(M_{ij}^n - \mathbb{E}[M_{ij}^n]\right) \mathcal{T}\left(M_{kl}^n - \mathbb{E}[M_{kl}^n]\right)\right], \tag{5.1}$$

$$M_{ij}^n := \mathbb{E}\left[\left(a(b_n) - a^{b_n}(b_n)\right)\left(e_n \cdot (\nabla \phi_j^*(0) + e_j)\right)\left(e_n \cdot (\nabla \phi_i^{b_n}(0) + e_i)\right) \Big| \boldsymbol{a}\right],$$

in terms of the abstract operator $\mathcal{T}$ defined above. Although not convenient for numerical approximation of $\mathcal{Q}$, this formula allows to easily deduce the non-degeneracy result contained in Proposition 2.9(ii). In addition, this is key to the proof of Theorem 2 on the RVE method.

5.2. **Proof of Lemma 5.1.** We start with some observations on the difference operator $\tilde{\Delta}_b$ on $\mathrm{L}^2(\Omega)$. For all $X, Y \in \mathrm{L}^2(\Omega)$, by exchangeability of $(\boldsymbol{a}, \boldsymbol{a}^b)$, we find

$$\mathbb{E}[X\tilde{\Delta}_b Y] = \mathbb{E}[XY] - \mathbb{E}[X\mathbb{E}_{a(b)}[Y]] = \mathbb{E}[XY] - \mathbb{E}[\mathbb{E}_{a(b)}[X]\mathbb{E}_{a(b)}[Y]]$$
$$= \mathbb{E}[XY] - \mathbb{E}[Y\mathbb{E}_{a(b)}[X]] = \mathbb{E}[Y\tilde{\Delta}_b X],$$

so that $\tilde{\Delta}_b$ is symmetric on $\mathrm{L}^2(\Omega)$. In addition, we easily compute, for all $b, b' \in \mathcal{B}$,

$$\tilde{\Delta}_b \tilde{\Delta}_b = \tilde{\Delta}_b, \qquad \tilde{\Delta}_b \tilde{\Delta}_{b'} = \tilde{\Delta}_{b'} \tilde{\Delta}_b. \tag{5.2}$$

With these observations at hand, we now turn to the study of the (densely defined) operator $\mathcal{S} := \sum_{b \in \mathcal{B}} \tilde{\Delta}_b \tilde{\Delta}_b$ on $\mathrm{L}^2(\Omega)$. More precisely, we consider the space $\mathrm{L}^2(\Omega)/\mathbb{R} := \{X \in \mathrm{L}^2(\Omega) : \mathbb{E}[X] = 0\}$ of mean-zero square-integrable random variables, and we show that $\mathcal{S}$ is an essentially self-adjoint, non-negative operator on $\mathrm{L}^2(\Omega)/\mathbb{R}$ with dense image. First, since $\mathbb{E}[\tilde{\Delta}_b X] = 0$ for all $b \in \mathcal{B}$ and $X \in \mathrm{L}^2(\Omega)$, the image $\mathrm{Im}\,\mathcal{S}$ is clearly contained in $\mathrm{L}^2(\Omega)/\mathbb{R}$. Second, for all $X \in \mathrm{L}^2(\Omega)$ in the domain of $\mathcal{S}$, we compute

$$\mathbb{E}[X\mathcal{S}X] = \sum_{b \in \mathcal{B}} \mathbb{E}[|\tilde{\Delta}_b X|^2] \geq 0,$$

which shows that $\mathcal{S}$ is non-negative. Third, if $X \in \mathrm{L}^2(\Omega)/\mathbb{R}$ in the domain of $\mathcal{S}$ is orthogonal to the image $\mathrm{Im}\,\mathcal{S}$, then we deduce

$$0 = \mathbb{E}[X\mathcal{S}X] = \sum_{b \in \mathcal{B}} \mathbb{E}[|\tilde{\Delta}_b X|^2],$$

so that $\tilde{\Delta}_b X = 0$ almost surely for all $b \in \mathcal{B}$, which implies that X is constant.

These properties of $\mathcal{S}$ allow us to define (densely) the inverse $\mathcal{T} := \mathcal{S}^{-1}$ as an essentially self-adjoint, non-negative operator on $\mathrm{L}^2(\Omega)/\mathbb{R}$. Finally, the spectral gap of Lemma 3.1 implies, for all $X \in \mathrm{L}^2(\Omega)/\mathbb{R}$ in the domain of $\mathcal{S}$,

$$\|X\|_{\mathrm{L}^2(\Omega)}^2 = \mathrm{Var}\,[X] \leq \sum_{b \in \mathcal{B}} \mathbb{E}[|\tilde{\Delta}_b X|^2] = \mathbb{E}[X\mathcal{S}X] \leq \|X\|_{\mathrm{L}^2(\Omega)} \|\mathcal{S}X\|_{\mathrm{L}^2(\Omega)},$$

and hence $\|X\|_{L^2(\Omega)} \leq \|\mathcal{S}X\|_{L^2(\Omega)}$, which implies that $\mathcal{T} = \mathcal{S}^{-1}$ on $L^2(\Omega)/\mathbb{R}$ has operator norm bounded by 1.

It remains to establish the representation formula for the variance. By density, it suffices to prove it for all $X \in \operatorname{Im} \mathcal{S}$. Writing $X = \mathcal{S}Y$ for some $Y \in L^2(\Omega)/\mathbb{R}$, we decompose

$$\operatorname{Var}[X] = \mathbb{E}[X\mathcal{S}Y] = \sum_{b \in \mathcal{B}} \mathbb{E}[\tilde{\Delta}_b X \tilde{\Delta}_b Y] = \sum_{b \in \mathcal{B}} \mathbb{E}[(\tilde{\Delta}_b X)(\tilde{\Delta}_b \mathcal{T} X)].$$

Since the commutation relations (5.2) ensure that $\tilde{\Delta}_b \mathcal{S} = \mathcal{S}\tilde{\Delta}_b$ holds on the domain of $\mathcal{S}$ in $L^2(\Omega)$, we deduce $\tilde{\Delta}_b \mathcal{T} = \mathcal{T}\tilde{\Delta}_b$ on $L^2(\Omega)/\mathbb{R}$, and the above then leads to the desired representation

$$\operatorname{Var}[X] = \sum_{b \in \mathcal{B}} \mathbb{E}[(\tilde{\Delta}_b X)\mathcal{T}(\tilde{\Delta}_b X)].$$

5.3. **Proof of Proposition 2.9(i).** By polarization and linearity, it is enough to prove (2.8) with $F = G \in C_c^\infty(\mathbb{R}^d)^{d \times d}$. We thus need to establish the convergence of the variance

$$\nu_\varepsilon := \operatorname{Var}\left[\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F : \Xi(\tfrac{\cdot}{\varepsilon})\right] = \operatorname{Var}\left[\varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_\varepsilon : \Xi\right],$$

where we have set $F_\varepsilon := F(\varepsilon\cdot)$. We split the proof into two steps.

Step 1. Proof of (2.8).

The Helffer-Sjöstrand representation of Lemma 5.1 applied to the variance ν_ε yields

$$\nu_\varepsilon = \varepsilon^d \sum_{b \in \mathcal{B}} \mathbb{E}\left[\left(\tilde{\Delta}_b \int_{\mathbb{R}^d} F_\varepsilon : \Xi\right) \mathcal{T}\left(\tilde{\Delta}_b \int_{\mathbb{R}^d} F_\varepsilon : \Xi\right)\right]. \tag{5.3}$$

We now appeal to (3.18) in the form

$$\Delta_b \int_{\mathbb{R}^d} F_\varepsilon : \Xi = \int_{\mathbb{R}^d} F_{\varepsilon,ij}(\nabla\phi_j^* + e_j) \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i)$$
$$+ \int_{\mathbb{R}^d} \phi_j^*(\cdot + e_k)\nabla_k F_{\varepsilon,ij} e_k \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i) + \int_{\mathbb{R}^d} \nabla h_{\varepsilon,i} \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i),$$

where the auxiliary field $h_{\varepsilon,i}$ is the unique Lax-Milgram solution in $\mathbb{R}^d$ of

$$-\nabla^* \cdot \boldsymbol{a}^* \nabla h_{\varepsilon,i} = \nabla_l^*\left(\phi_j^*(\cdot + e_k)\boldsymbol{a}_{kl}\nabla_k F_{\varepsilon,ij} + \sigma_{jkl}^*(\cdot - e_k)\nabla_k^* F_{\varepsilon,ij}\right). \tag{5.4}$$

Recalling that $\tilde{\Delta}_b X = \mathbb{E}_{a^b(b)}[\Delta_b X]$, inserting this representation formula into (5.3), extracting the first term U_ε defined below, and using that $\mathcal{T}$ on $L^2(\Omega)/\mathbb{R}$ has operator norm bounded by 1, we find

$$|\nu_\varepsilon - \varepsilon^d U_\varepsilon| \leq \varepsilon^d \sum_{b \in \mathcal{B}} \left(S_\varepsilon^b T_\varepsilon^b + \frac{1}{2}(T_\varepsilon^b)^2\right), \tag{5.5}$$

where for convenience we define

$$U_\varepsilon := \sum_{b \in \mathcal{B}} \mathbb{E}\left[(V_\varepsilon^b - \mathbb{E}[V_\varepsilon^b])\mathcal{T}(V_\varepsilon^b - \mathbb{E}[V_\varepsilon^b])\right],$$
$$V_\varepsilon^b := \mathbb{E}_{a^b(b)}\left[\int_{\mathbb{R}^d} F_{\varepsilon,ij}(\nabla\phi_j^* + e_j) \cdot \Delta_b \boldsymbol{a}(\nabla\phi_i^b + e_i)\right],$$

while for all $b \in \mathcal{B}$ the error terms are given by

$$
S_\varepsilon^b \; := \; \mathbb{E}\left[\left(\int_{\mathbb{R}^d} |\Delta_b \boldsymbol{a}||\nabla \phi^* + \mathrm{Id}\,||\nabla \phi^b + \mathrm{Id}\,||F_\varepsilon|\right)^2\right]^{\frac{1}{2}},
$$

and by $T_\varepsilon^b := T_{\varepsilon,1}^b + T_{\varepsilon,2}^b$ with

$$
T_{\varepsilon,1}^b \; := \; \sum_{k=1}^d \mathbb{E}\left[\left(\int_{\mathbb{R}^d} |\Delta_b \boldsymbol{a}||\phi^*(\cdot + e_k)||\nabla \phi^b + \mathrm{Id}\,||\nabla F_\varepsilon|\right)^2\right]^{\frac{1}{2}},
$$

$$
T_{\varepsilon,2}^b \; := \; \mathbb{E}\left[\left(\int_{\mathbb{R}^d} |\Delta_b \boldsymbol{a}||\nabla \phi^b + \mathrm{Id}\,||\nabla h_\varepsilon|\right)^2\right]^{\frac{1}{2}}.
$$

We start with the analysis of U_ε. Writing $\Delta_b \boldsymbol{a}(x) = (a(b) - a^b(b))\mathbb{1}_{Q(z_b)}(x)e_b \otimes e_b$ for $b = (z_b, z_b + e_b)$, we may compute

$$
V_\varepsilon^b = \left(\int_{Q(z_b)} F_{\varepsilon,ij}\right)\mathbb{E}_{a^b(b)}\left[(a(b) - a^b(b))(e_b \cdot (\nabla \phi_j^*(z_b) + e_j))(e_b \cdot (\nabla \phi_i^b(z_b) + e_i))\right],
$$

so that, by stationarity,

$$
\varepsilon^d U_\varepsilon = \mathcal{Q}_{ijkl}\, \varepsilon^d \sum_{z \in \mathbb{Z}^d} \left(\int_{Q(z)} F_{\varepsilon,ij}\right)\left(\int_{Q(z)} F_{\varepsilon,kl}\right),
$$

where the coefficient $\mathcal{Q}_{ijkl}$ is defined in (5.1) above. Since $\mathcal{T}$ on $\mathrm{L}^2(\Omega)/\mathbb{R}$ has operator norm bounded by 1, the moment bounds of Lemma 3.3 yield

$$
|\mathcal{Q}_{ijkl}| \lesssim \sum_{n=1}^d \mathbb{E}\left[|\nabla \phi^* + \mathrm{Id}\,|^2|\nabla \phi^{b_n} + \mathrm{Id}\,|^2\right] \lesssim 1.
$$

We may then estimate the discretization error

$$
\begin{aligned}
\left|\varepsilon^d U_\varepsilon - \mathcal{Q}_{ijkl} \int_{\mathbb{R}^d} F_{ij}F_{kl}\right| &= \left|\varepsilon^d U_\varepsilon - \mathcal{Q}_{ijkl}\,\varepsilon^d \int_{\mathbb{R}^d} F_{\varepsilon,ij}F_{\varepsilon,kl}\right| \\
&\lesssim \varepsilon^d \sum_{z \in \mathbb{Z}^d} \int_{Q(z)} \left|F_\varepsilon(x) - \int_{Q(z)} F_\varepsilon\right|^2 dx \\
&\lesssim \varepsilon^d \int_{\mathbb{R}^d} |DF_\varepsilon|^2 = \varepsilon^2 \int_{\mathbb{R}^d} |DF|^2. \qquad (5.6)
\end{aligned}
$$

We now turn to the estimate of the right-hand side of (5.5). Using $|\Delta_b \boldsymbol{a}(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$ and the moment bounds of Lemma 3.3, we obtain

$$
S_\varepsilon^b \lesssim \mathbb{E}\left[|\nabla \phi^* + \mathrm{Id}\,|^2|\nabla \phi^b + \mathrm{Id}\,|^2\right]^{\frac{1}{2}} \int_{Q(z_b)} |F_\varepsilon| \lesssim \int_{Q(z_b)} |F_\varepsilon|.
$$

Hence, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\sum_{b \in \mathcal{B}} S_\varepsilon^b T_\varepsilon^b &\lesssim \sum_{b \in \mathcal{B}} T_\varepsilon^b \int_{Q(z_b)} |F_\varepsilon| \lesssim \|F_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}^d)}\left(\sum_{b \in \mathcal{B}}(T_\varepsilon^b)^2\right)^{\frac{1}{2}} \\
&\lesssim \varepsilon^{-\frac{d}{2}}\|F\|_{\mathrm{L}^2(\mathbb{R}^d)}\left(\sum_{b \in \mathcal{B}}(T_\varepsilon^b)^2\right)^{\frac{1}{2}}, \qquad (5.7)
\end{aligned}
$$

and it remains to estimate

$$\sum_{b \in \mathcal{B}} (T_\varepsilon^b)^2 \le 2 \sum_{b \in \mathcal{B}} (T_{\varepsilon,1}^b)^2 + 2 \sum_{b \in \mathcal{B}} (T_{\varepsilon,2}^b)^2.$$

First, using $|\Delta_b \boldsymbol{a}(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$ and the moment bounds of Lemma 3.3, we find

$$\varepsilon^d \sum_{b \in \mathcal{B}} (T_{\varepsilon,1}^b)^2 \lesssim_{\alpha,p} \varepsilon^d \|\mu_d(|\cdot|)^{\frac{1}{2}} \nabla F_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}^d)}^2. \tag{5.8}$$

Second, arguing as in the proof of Proposition 2.1 (cf. (3.16)), using the large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.4) applied to equation (5.4) for h_ε, we obtain for all $0 < p - 1 \ll 1$ and all $\alpha > d$,

$$\varepsilon^d \sum_{b \in \mathcal{B}} (T_{\varepsilon,2}^b)^2 \lesssim_{\alpha,p} \varepsilon^{\frac{d}{p}} \|w_\varepsilon^{\alpha \frac{p-1}{2p}} \mu_d(|\cdot|)^{\frac{1}{2}} \nabla F_\varepsilon\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^2. \tag{5.9}$$

Rescaling the integrals and using (3.32) and Hölder's inequality, we find

$$\varepsilon^d \sum_{b \in \mathcal{B}} (T_\varepsilon^b)^2 \lesssim_{\alpha,p} \varepsilon^2 \mu_d(\tfrac{1}{\varepsilon}) \|w_1^{\alpha \frac{p-1}{2p}} \mu_d(|\cdot|)^{\frac{1}{2}} DF\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^2,$$

and the conclusion (2.8) follows.

Step 2. Proof of the Green-Kubo formula (2.9).
In order to establish (2.9), it suffices to repeat the argument of Step 1 with the test function $F = \mathbb{1}_Q e_i \otimes e_j$ (hence $F_\varepsilon = \mathbb{1}_{\frac{1}{\varepsilon}Q} e_i \otimes e_j$), for some fixed $1 \le i, j \le d$. Lemma 5.1 again leads to (5.5), and we briefly indicate how to analyze the different terms in the present setting. First, the estimate (5.6) is replaced by the following (no summation over repeated indices),

$$\begin{aligned}
|\varepsilon^d U_\varepsilon - \mathcal{Q}_{ijij}| &\lesssim \varepsilon^d \sum_{z \in \mathbb{Z}^d} \int_Q \left(\mathbb{1}_{\frac{1}{\varepsilon}Q}(z+x) - \int_Q \mathbb{1}_{\frac{1}{\varepsilon}Q}(z+y) dy \right)^2 dx \\
&\le \varepsilon^d \sum_{z \in \mathbb{Z}^d} \mathbb{1}_{(z+Q) \cap \partial(\frac{1}{\varepsilon}Q) \ne \varnothing} \lesssim \varepsilon.
\end{aligned}$$

Second, the estimate (5.7) remains unchanged. Third, using estimates (5.8) and (5.9), and noting that $|\nabla F_\varepsilon| \lesssim \mathbb{1}_{A_\varepsilon}$ with $A_\varepsilon := B + \partial Q_{\frac{1}{\varepsilon}}$ and that $w_\varepsilon \lesssim 1$ and $\mu_d(|\cdot|) \lesssim \mu_d(\tfrac{1}{\varepsilon})$ on A_ε, we deduce

$$\begin{aligned}
\sum_{b \in \mathcal{B}} (T_{\varepsilon,1}^b)^2 &\lesssim \mu_d(\tfrac{1}{\varepsilon}) |A_\varepsilon| \lesssim \varepsilon^{1-d} \mu_d(\tfrac{1}{\varepsilon}), \\
\sum_{b \in \mathcal{B}} (T_{\varepsilon,2}^b)^2 &\lesssim \varepsilon^{-d \frac{p-1}{p}} \mu_d(\tfrac{1}{\varepsilon}) |A_\varepsilon|^{\frac{1}{p}} \lesssim \varepsilon^{\frac{1}{p}-d} \mu_d(\tfrac{1}{\varepsilon}),
\end{aligned}$$

and the conclusion (2.9) follows.

5.4. **Proof of Proposition 2.9(ii).** The following proof of the non-degeneracy of $\mathcal{Q}$ is based on the Helffer-Sjöstrand representation formula (see also [49, Remark 2.3]), and constitutes a shorter alternative to the corresponding proof in [30]. Given a fixed direction

$e \in \mathbb{R}^d \setminus \{0\}$, and letting ϕ_e denote the corrector in this direction, we may write, in view of formula (5.1) (with $\phi_e^* = \phi_e$ by symmetry of the coefficients),

$$(e \otimes e) : \mathcal{Q}(e \otimes e) = \sum_{n=1}^d \mathbb{E}\big[(e \cdot M^n e)\mathcal{T}(e \cdot M^n e)\big], \qquad (5.10)$$

$$e \cdot M^n e := \mathbb{E}\Big[(a(b_n) - a^{b_n}(b_n))\big(e_n \cdot (\nabla\phi_e(0) + e)\big)\big(e_n \cdot (\nabla\phi_e^{b_n}(0) + e)\big) \,\Big|\, \boldsymbol{a}\Big],$$

since the exchangeability of $(\boldsymbol{a}, \boldsymbol{a}^{b_n})$ indeed yields $\mathbb{E}\big[e \cdot M^n e\big] = 0$ for all n. We start with a suitable reformulation of $e \cdot M^n e$. Considering the difference of the corrector equation (2.4) for ϕ_e and $\phi_e^{b_n}$ in the form $-\nabla^* \cdot \boldsymbol{a}^{b_n}\nabla(\phi_e^{b_n} - \phi_e) = \nabla^* \cdot (\boldsymbol{a}^{b_n} - \boldsymbol{a})(\nabla\phi_e + e)$, an integration by parts yields

$$\int_{\mathbb{R}^d} \nabla(\phi_e^{b_n} - \phi_e) \cdot \boldsymbol{a}^{b_n}\nabla(\phi_e^{b_n} - \phi_e) = -\int_{\mathbb{R}^d} \nabla(\phi_e^{b_n} - \phi_e) \cdot (\boldsymbol{a}^{b_n} - \boldsymbol{a})(\nabla\phi_e + e)$$
$$= (a(b_n) - a^{b_n}(b_n))(e_n \cdot \nabla(\phi_e^{b_n} - \phi_e)(0))(e_n \cdot (\nabla\phi_e(0) + e)).$$

Hence, by definition of $e \cdot M^n e$,

$$e \cdot M^n e = \mathbb{E}\bigg[\int_{\mathbb{R}^d} \nabla(\phi_e^{b_n} - \phi_e) \cdot \boldsymbol{a}^{b_n}\nabla(\phi_e^{b_n} - \phi_e) \,\bigg|\, \boldsymbol{a}\bigg]$$
$$+ (a(b_n) - \mathbb{E}[a(b_n)])(e_n \cdot (\nabla\phi_e(0) + e))^2. \quad (5.11)$$

We now argue by contradiction. If $(e \otimes e) : \mathcal{Q}(e \otimes e) = 0$, then by formula (5.10) and by the non-negativity of $\mathcal{T}$ we would have $\mathbb{E}\big[(e \cdot M^n e)\mathcal{T}(e \cdot M^n e)\big] = 0$ for all n. Let $1 \le n \le d$ be momentarily fixed. Recalling that $\mathcal{T} = \mathcal{S}^{-1}$ with $\mathcal{S} = \sum_{b \in \mathcal{B}} \tilde{\Delta}_b \tilde{\Delta}_b$, this would imply

$$0 = \mathbb{E}\left[(\mathcal{T}(e \cdot M^n e))\mathcal{S}(\mathcal{T}(e \cdot M^n e))\right] = \sum_{b \in \mathcal{B}} \mathbb{E}\left[\left|\tilde{\Delta}_b \mathcal{T}(e \cdot M^n e)\right|^2\right],$$

hence $\mathcal{T}(e \cdot M^n e) = 0$, and thus $e \cdot M^n e = 0$ almost surely. Formula (5.11) would then imply

$$(a(b_n) - \mathbb{E}[a(b_n)])(e_n \cdot (\nabla\phi_e(0) + e))^2$$
$$= -\mathbb{E}\bigg[\int_{\mathbb{R}^d} \nabla(\phi_e^{b_n} - \phi_e) \cdot \boldsymbol{a}^{b_n}\nabla(\phi_e^{b_n} - \phi_e) \,\bigg|\, \boldsymbol{a}\bigg], \quad (5.12)$$

almost surely. Since the law of $a(b_n)$ is non-degenerate, the event $a(b_n) > \mathbb{E}[a(b_n)]$ occurs with a positive probability. Conditioning on this event, the left-hand side in (5.12) is non-negative, and the non-positivity of the right-hand side would then imply that both sides vanish, that is,

$$e_n \cdot (\nabla\phi_e(0) + e) = 0 \qquad \text{and} \qquad \mathbb{E}\bigg[\int_{\mathbb{R}^d} \nabla(\phi_e^{b_n} - \phi_e) \cdot \boldsymbol{a}^{b_n}\nabla(\phi_e^{b_n} - \phi_e) \,\bigg|\, \boldsymbol{a}\bigg] = 0,$$

almost surely. Since the integrand in this last expectation is non-negative, we would deduce that the event $a(b_n) > \mathbb{E}[a(b_n)]$ entails $e_n \cdot (\nabla\phi_e(0) + e) = 0$ and $\nabla\phi_e(0) = \nabla\phi_e^{b_n}(0)$, and thus also $e_n \cdot (\nabla\phi_e^{b_n}(0) + e) = 0$ almost surely. Since this last event is independent of $a(b_n)$, hence of the conditioning event, we would deduce unconditionally $e_n \cdot (\nabla\phi_e^{b_n}(0) + e) = 0$ almost surely. By exchangeability of $(\boldsymbol{a}, \boldsymbol{a}^{b_n})$, this means $e_n \cdot (\nabla\phi_e(0) + e) = 0$ almost surely. As this holds for any n, we would conclude $\nabla\phi_e(0) + e = 0$ almost surely, and taking the expectation would lead to a contradiction.

6. Approximation of the fluctuation tensor

In this section, we analyze the RVE method for the approximation of the fluctuation tensor $\mathcal{Q}$ as stated in Theorem 2.

6.1. **Structure of the proof and auxiliary results.** The estimate on the standard deviation is obtained similarly as the CLT scaling in Proposition 2.1, noting that the large-scale Calderón-Zygmund result of Lemma 3.4 also holds for the periodized operator $-\nabla^* \cdot \boldsymbol{a}_L \nabla$ on Q_L.[5] The characterization (1.17) of $\mathcal{Q}$ and the estimate of the systematic error are deduced as corollaries of formula (5.1) for the fluctuation tensor $\mathcal{Q}$, together with the following crucial estimates on the periodized corrector ϕ_L. The first estimate on $\nabla\phi_L$ is stated as such in [28, Proposition 1], and the second one follows from a decomposition of the difference $\nabla\phi_L - \nabla\phi$ via massive approximation of the corrector and Richardson extrapolation, applying [30, Lemma 2.8 and estimate (2.68)], and optimizing the mass.

Lemma 6.1 ([28, 30]). *Let $d \geq 2$ and let $\mathbb{P}$ be a product measure. For all $L \geq 2$ and all $q < \infty$ we have*

$$\mathbb{E}\left[|\nabla\phi_L|^q\right]^{\frac{1}{q}} \lesssim_q 1, \qquad and \qquad \mathbb{E}\left[|\nabla(\phi_L - \phi)(0)|^q\right]^{\frac{1}{q}} \lesssim_q L^{-\frac{d}{2}} \log^{\frac{d}{2}} L. \qquad \diamond$$

6.2. **Proof of Theorem 2.** We split the proof into two steps: we first estimate the variance of the RVE approximation, and then we turn to the characterization (1.17) of $\mathcal{Q}$ and to the systematic error of the RVE approximation.

Step 1. Proof of the random error estimate $|\mathrm{Var}\left[\mathcal{Q}_{L,N}\right]|^{\frac{1}{2}} \lesssim N^{-\frac{1}{2}}$.

Since the realizations $\bar{\boldsymbol{a}}_L^{(n)}$ are iid copies of $\bar{\boldsymbol{a}}_L$, the definition (1.18) of $\mathcal{Q}_{L,N}$ leads after straightforward computations to

$$\mathrm{Var}\left[\mathcal{Q}_{L,N}\right] = N^{-1}\mathrm{Var}\left[\left(L^{\frac{d}{2}}\bar{\boldsymbol{a}}_L^* - \mathbb{E}\left[L^{\frac{d}{2}}\bar{\boldsymbol{a}}_L^*\right]\right)^{\otimes 2}\right],$$

and hence,

$$|\mathrm{Var}\left[\mathcal{Q}_{L,N}\right]| \lesssim N^{-1}\mathbb{E}\left[\left|L^{\frac{d}{2}}(\bar{\boldsymbol{a}}_L - \mathbb{E}\left[\bar{\boldsymbol{a}}_L\right])\right|^4\right].$$

Arguing as in [28, Lemma 2], the spectral gap estimate of Lemma 3.1 is seen to imply the following inequality: for all $X = X(\boldsymbol{a}) \in \mathrm{L}^4(\Omega)$,

$$\mathbb{E}\left[(X - \mathbb{E}\left[X\right])^4\right] \leq 4\,\mathbb{E}\left[\left(\sum_{b \in \mathcal{B}} |\Delta_b X|^2\right)^2\right].$$

Applying this inequality to (each component of) $X = \bar{\boldsymbol{a}}_L$, we deduce

$$|\mathrm{Var}\left[\mathcal{Q}_{L,N}\right]| \lesssim N^{-1}\mathbb{E}\left[\left(\sum_{b \in \mathcal{B}_L}\left(L^{-\frac{d}{2}}\int_{Q_L}\Delta_b\big(\boldsymbol{a}_L(\nabla\phi_L + \mathrm{Id})\big)\right)^2\right)^2\right].$$

Arguing as in the proof of Proposition 2.1 (with ε replaced by $\frac{1}{L}$ and F_ε replaced by Id), using the periodized version of Lemma 3.4 and the moment bounds of Lemma 6.1, the conclusion follows.

---

[5]The only issue concerns the moment bounds for the corresponding minimal radius $r_{*,L}$ associated with the periodized operator, $\mathbb{E}\left[r_{*,L}^q\right] \lesssim_q 1$ for all $q < \infty$. By definition of $r_{*,L}$ in [27], this is a consequence of a sup-bound based on the version of Lemma 3.3 for the periodized correctors (ϕ_L, σ_L) (cf. [28]).

Step 2. Proof of (1.17) and of the systematic error estimate $|\mathbb{E}[\mathcal{Q}_{L,N}] - \mathcal{Q}| \lesssim L^{-\frac{d}{2}} \log^{\frac{d}{2}} L$. Since the realizations $\bar{\boldsymbol{a}}_L^{(n)}$ are iid copies of $\bar{\boldsymbol{a}}_L$, the definition (1.18) of $\mathcal{Q}_{L,N}$ yields after straightforward computations $\mathbb{E}[\mathcal{Q}_{L,N}] = \mathrm{Var}\big[L^{\frac{d}{2}}\bar{\boldsymbol{a}}_L^*\big]$, that is,

$$
\begin{aligned}
\mathbb{E}[(\mathcal{Q}_{L,N})_{ijkl}] &= \mathrm{Cov}\left[L^{\frac{d}{2}}\bar{\boldsymbol{a}}_{L,ji}; L^{\frac{d}{2}}\bar{\boldsymbol{a}}_{L,lk}\right] \\
&= L^{-d}\,\mathrm{Cov}\left[\int_{Q_L} e_j\cdot\boldsymbol{a}_L(\nabla\phi_{L,i}+e_i); \int_{Q_L} e_l\cdot\boldsymbol{a}_L(\nabla\phi_{L,k}+e_k)\right]. \quad (6.1)
\end{aligned}
$$

For $b\in\mathcal{B}$, we write $b=(z_b, z_b+e_b)$. Using the periodized corrector equation (2.10) and its vertical derivative, and recalling that $\Delta_b\boldsymbol{a}_L(x) = (a(b)-a^b(b))\mathbb{1}_{Q(z_b)}(x)e_b\otimes e_b$ for $b\in\mathcal{B}_L$ and $x\in Q_L$, we find

$$
\begin{aligned}
\Delta_b\int_{Q_L} e_j\cdot\boldsymbol{a}_L(\nabla\phi_{L,i}+e_i) &= \int_{Q_L} e_j\cdot\Delta_b\boldsymbol{a}_L(\nabla\phi_{L,i}^b+e_i) + \int_{Q_L} e_j\cdot\boldsymbol{a}_L\nabla\Delta_b\phi_{L,i} \\
&= \int_{Q_L} e_j\cdot\Delta_b\boldsymbol{a}_L(\nabla\phi_{L,i}^b+e_i) - \int_{Q_L}\nabla\phi_{L,j}^*\cdot\boldsymbol{a}_L\nabla\Delta_b\phi_{L,i} \\
&= \int_{Q_L}(\nabla\phi_{L,j}^*+e_j)\cdot\Delta_b\boldsymbol{a}_L(\nabla\phi_{L,i}^b+e_i) \\
&= (a(b)-a^b(b))(e_b\cdot(\nabla\phi_{L,j}^*(z_b)+e_j))(e_b\cdot(\nabla\phi_{L,i}^b(z_b)+e_i)). \quad (6.2)
\end{aligned}
$$

Applying the Helffer-Sjöstrand representation formula of Lemma 5.1 to the covariance in (6.1), we obtain by stationarity, as in the proof of Proposition 2.9(i),

$$
\mathbb{E}[(\mathcal{Q}_{L,N})_{ijkl}] = \sum_{n=1}^{d}\mathbb{E}\left[M_{ij,L}^n\,\mathcal{T}\,M_{kl,L}^n\right],
$$

where we have set

$$
M_{ij,L}^n := \mathbb{E}\left[(a(b_n)-a^{b_n}(b_n))(e_n\cdot(\nabla\phi_{L,j}^*(0)+e_j))(e_n\cdot(\nabla\phi_{L,i}^{b_n}(0)+e_i))\,\Big|\,\boldsymbol{a}\right].
$$

Noting that (6.2) implies $\mathbb{E}[M_{ij,L}^n] = 0$, comparing the above identity for $\mathbb{E}[(\mathcal{Q}_{L,N})_{ijkl}]$ with formula (5.1) for $\mathcal{Q}$, and using that the operator $\mathcal{T}$ on $\mathrm{L}^2(\Omega)/\mathbb{R}$ has operator norm bounded by 1, we deduce

$$
\left|\mathbb{E}[(\mathcal{Q}_{L,N})_{ijkl}] - \mathcal{Q}_{ijkl}\right| \lesssim \mathbb{E}\left[|\nabla(\phi_L-\phi)(0)|^4\right]^{\frac{1}{4}}\left(\mathbb{E}\left[|\nabla\phi_L|^4\right] + \mathbb{E}\left[|\nabla\phi|^4\right]\right)^{\frac{3}{4}},
$$

and the conclusion follows from Lemmas 3.3 and 6.1.

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