

DERIVATION OF 2D VLASOV-POISSON FOR CLASSICAL PARTICLES WITH COULOMB INTERACTIONS

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ABSTRACT. We obtain the first derivation of the 2D Vlasov-Poisson equation for classical particles with Coulomb interactions without any microscopic cutoff. The proof relies on our recent dual hierarchical approach to mean-field limits, together with a refined analysis of dual BBGKY hierarchies on linearized correlations, based on kinetic regularization effects. The result holds more generally for arbitrary singular interaction forces $K \in L_{loc}^{2-\eta}$ with $\eta > 0$ small enough, in any dimension, and it extends to Brownian particles. It holds globally in time as long as the mean-field solution is regular enough.

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1. INTRODUCTION

1.1. General overview. We consider a system of $N \geq 2$ exchangeable classical particles in phase space $\mathbb{D} := \mathbb{R}^d \times \mathbb{R}^d$, interacting through a pairwise force kernel $K : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Denoting by $(X_{i,N}, V_{i,N})$ the position and velocity of particle i , the dynamics is given by

$$\begin{cases} dX_{i,N} = V_{i,N} dt, \\ dV_{i,N} = \frac{1}{N-1} \sum_{j:j \neq i}^N K(X_{i,N} - X_{j,N}) dt + \sqrt{2\alpha} dB_{i,N}, \end{cases}$$

where $\alpha \geq 0$ and the $B_{i,N}$'s are independent Brownian motions. The deterministic Newton dynamics is recovered for $\alpha = 0$. Throughout, we assume that K is odd, in accordance with the action-reaction principle. Turning to a statistical description of the system, let F_N denote the joint density on the N -particle phase space $\mathbb{D}^N$. Formally, the above dynamics then leads to the Liouville equation

$$\partial_t F_N + \sum_{i=1}^N (v_i \cdot \nabla_{x_i} - \alpha \Delta_{v_i}) F_N + \frac{1}{N-1} \sum_{i \neq j}^N K(x_i - x_j) \cdot \nabla_{v_i} F_N = 0. \quad (1.1)$$

Exchangeability of the particles amounts to the symmetry of F_N under permutations of the particle variables $z_i := (x_i, v_i) \in \mathbb{D}$. In the macroscopic limit $N \uparrow \infty$, we aim at an

averaged description of finite subsets of typical particles, as encoded by the marginals

$$F_{N,k}(z_1, \dots, z_k) := \int_{\mathbb{D}^{N-k}} F_N(z_1, \dots, z_N) dz_{k+1} \dots dz_N, \quad 1 \leq k \leq N.$$

If the particles are initially asymptotically independent with one-particle density $f_\circ$, one expects propagation of chaos,

$$F_{N,k}(t) \rightarrow f(t)^{\otimes k}, \quad \text{for every fixed } k \geq 0,$$

where f solves the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f - \alpha \Delta_v f + (K * f) \cdot \nabla_v f = 0, \quad f|_{t=0} = f_\circ, \quad (1.2)$$

with $K * f(x) := \int_{\mathbb{D}} K(x - x') f(x', v') dx' dv'$. For Lipschitz interaction forces, this mean-field approximation goes back to the classical works of Neunzert and Wick [20], Braun and Hepp [4], and Dobrushin [9] in the deterministic setting, and to the general theory of propagation of chaos for McKean-Vlasov diffusion in the stochastic setting, e.g. [24]. For singular interactions, however, the problem is considerably more delicate: the singularity destroys the stability of the particle dynamics, while close particle encounters may generate large microscopic forces. In particular, the derivation of the Vlasov-Poisson equation from particle dynamics without microscopic regularization had remained open so far in dimensions $d \geq 2$.

In [5], together with D. Bresch, we introduced a dual hierarchical approach to mean-field limits. Rather than estimating the BBGKY hierarchy of marginals directly, which leads to a problematic loss of velocity derivatives, the method focuses on the backward evolution of dual observables and on the associated hierarchy of dual correlations. This allowed us to treat arbitrary forces $K \in L^2_{\text{loc}}(\mathbb{R}^d)$, at both positive and zero temperature $\alpha \geq 0$, without requiring any sign condition or potential structure of the interaction. While this substantially enlarged the admissible class of singular kernels, it still fell short of the 2D Coulomb force, which belongs only to the weak space $L^{2,\infty}$.

The main purpose of the present work is to go beyond this L^2_{loc} threshold. Building on the same dual hierarchical framework, we develop a refined analysis of the dual BBGKY hierarchy that exploits kinetic regularization effects. In the diffusive case $\alpha > 0$, this relies on hypoelliptic regularity estimates, whereas in the non-diffusive case $\alpha = 0$ it relies on velocity averaging; see Propositions 3.3 and 4.4. These estimates provide fractional spatial regularity that allows to pass to the limit in the singular interaction terms of the dual hierarchy. To our knowledge, this is the first hierarchical use of kinetic regularization. Its implementation raises some specific technical difficulties; see in particular Section 4.

The final ingredient is uniqueness for the limiting dual hierarchy, which is needed to identify the limit and recover the Vlasov equation (1.2). Such uniqueness cannot be obtained through direct PDE estimates on the hierarchy, since the available kinetic estimates provide regularity in only one particle variable at a time, whereas a direct argument would require mixed regularity simultaneously in all particle variables. To remedy this, we instead combine kinetic regularization with the explicit triangular structure of the limiting hierarchy; see Section 5.2.

1.2. Main result. Our main result applies to interaction forces $K \in L^{2-\eta}_{\text{loc}}(\mathbb{R}^d)$ for sufficiently small $\eta > 0$, thus getting strictly beyond the L^2_{loc} threshold of [5]. In particular, it includes the 2D Coulomb force $K(x) \simeq \pm \frac{x}{|x|^2}$, which only belongs to $L^{2,\infty}$. For $\alpha = 0$, this

yields the first derivation of the 2D Vlasov-Poisson equation from particle dynamics without microscopic regularization. For $\alpha > 0$, it complements the recent short-time derivation of the 2D Vlasov-Poisson-Fokker-Planck equation in [6], by providing a longer-time result under weaker assumptions on initial data.

The argument does not use repulsiveness and applies to forces that need not derive from a potential. It holds globally in time as long as the mean-field solution remains sufficiently regular. In contrast with our previous work [5], also note that the initial data need not be exactly factorized: it is enough to assume a uniform relative L^2 bound, cf. (1.3) below, which is the natural condition for the duality framework used in the proof.

Theorem 1.1. *Let $\alpha \geq 0$, let $f_\circ \in \mathcal{P} \cap L^\infty(\mathbb{D})$, and let*

$$K \in L_{\text{loc}}^{2d/(d+2\gamma)}(\mathbb{R}^d), \quad \sup_{|x| \geq 1} |K(x)| < \infty,$$

where γ satisfies

$$0 \leq \gamma < \gamma_* := \begin{cases} \frac{1}{6} & : \alpha > 0, \\ \frac{1}{16}(5d+4 - \sqrt{(5d+4)^2 - 32d}) = \frac{1}{5} + O(\frac{1}{d}) & : \alpha = 0, d \geq 2, \\ \frac{1}{2}(\sqrt{5} - 2) & : \alpha = 0, d = 1. \end{cases}$$

For every $N \geq 2$, let $F_N^\circ \in \mathcal{P} \cap L^\infty(\mathbb{D}^N)$ be exchangeable and assume the following L^2 form of $f_\circ$ -chaoticity:

$$\limsup_{N \uparrow \infty} \int_{\mathbb{D}^N} \frac{|F_N^\circ|^2}{f_\circ^{\otimes N}} < \infty. \quad (1.3)$$

Let $F_N \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{D}^N))$ be an exchangeable global weak duality solution of (1.1), with initial data F_N° , in the sense of [5, Appendix]. Let f be a weak solution of (1.2) with initial data $f_\circ$. Assume that, for some $T_0 > 0$, the latter satisfies on $[0, T_0]$ the regularity requirements stated below, namely (1.5), (1.6), together with (1.7)–(1.8) if $\alpha > 0$, and with (1.9) if $\alpha = 0$. Then, for all $k \geq 1$ and $t \in [0, T_0]$,

$$F_{N,k}(t) \rightarrow f(t)^{\otimes k} \quad \text{in } \mathcal{D}'(\mathbb{D}^k). \quad (1.4)$$

We now display the precise regularity assumptions needed on the mean-field solution f . In our previous work [5], which treated the less singular regime $K \in L_{\text{loc}}^2(\mathbb{R}^d)$, it was enough to assume

$$f \in L^\infty(0, T_0; \mathcal{P} \cap L^\infty(\mathbb{D})), \quad K * f \in L^\infty(0, T_0; W^{1,\infty}(\mathbb{R}^d)), \quad (1.5)$$

together with the time-integrability of the Fisher information,

$$\int_0^{T_0} \left(\int_{\mathbb{D}} |\nabla_v \log f|^2 f \right)^{\frac{1}{2}} < \infty.$$

The more singular interactions considered here require stronger regularity assumptions. More precisely, on top of (1.5), we assume throughout

$$f^{-1} \in L_{\text{loc}}^\infty([0, T_0] \times \mathbb{D}), \quad (1.6)$$

as well as the following weighted regularity for logarithmic derivatives:

— in the diffusive case $\alpha > 0$, we assume that, for some $\delta \in (0, \frac{1}{2}]$,

$$\begin{aligned} f^\delta, f^\delta \nabla_v \log f &\in L^\infty(0, T_0; L^2(\mathbb{D}) \cap L_x^\infty L_v^2(\mathbb{D})), \\ f^{(\frac{1}{2}-\delta) \wedge \frac{\delta}{2}} (|\nabla_v \log f|^2 + |\Delta_v \log f|) &\in L^\infty([0, T_0] \times \mathbb{D}), \end{aligned} \quad (1.7)$$

and

$$\Delta_v \log f + \delta |\nabla_v \log f|^2 \geq -C \quad \text{a.e. on } [0, T_0] \times \mathbb{D}; \quad (1.8)$$

— in the non-diffusive case $\alpha = 0$, we assume that, for some $\delta \in (0, \frac{1}{2}]$,

$$\begin{aligned} f^\delta, f^\delta \nabla_v \log f &\in L^\infty(0, T_0; L_x^2 H_v^{3/2}(\mathbb{D}) \cap L_x^\infty H_v^{3/2}(\mathbb{D})), \\ f^{\frac{1}{2}-\delta} \nabla_v \log f &\in L^\infty([0, T_0] \times \mathbb{D}). \end{aligned} \quad (1.9)$$

For the choice $\delta = \frac{1}{2}$, the lower bound (1.8) follows from the boundedness assumptions in (1.7), and the assumptions thus simplify substantially — at the price of requiring the boundedness of $\nabla_v \log f$ itself, which is incompatible with Maxwellian velocity tails. Retaining $\delta < \frac{1}{2}$ allows for a more flexible weighted condition.

Remarks 1.2. A few comments are in order:

- (a) *Initial chaos assumption:* Assumption (1.3) is a uniform relative L^2 -bound, or equivalently a uniform bound on the χ^2 -divergence of F_N° from the tensorized law $f_\circ^{\otimes N}$. It is of course weaker than the exact tensorization assumed in our previous work [5], but it is stronger than the usual notions of Kac or entropic chaos. Indeed, in terms of the Hoeffding decomposition $(H_{N,m}^\circ)_m$ of F_N° introduced in Section 2.1 below, we can bound, for all $1 \leq k \leq N$,

$$\begin{aligned} \int_{\mathbb{D}^k} \left| \frac{F_{N,k}^\circ}{f_\circ^{\otimes k}} - 1 \right|^2 f_\circ^{\otimes k} &= \sum_{m=1}^k \binom{k}{m} \int_{\mathbb{D}^m} |H_{N,m}^\circ|^2 f_\circ^{\otimes m} \\ &\leq \frac{k}{N} \sum_{m=1}^N \binom{N}{m} \int_{\mathbb{D}^m} |H_{N,m}^\circ|^2 f_\circ^{\otimes m} = \frac{k}{N} \left(\int_{\mathbb{D}^N} \frac{|F_N^\circ|^2}{f_\circ^{\otimes N}} - 1 \right), \end{aligned}$$

so that (1.3) implies quantitative $f_\circ$ -chaoticity of every fixed marginal, with an $O(N^{-1/2})$ error bound in L^2 . It also implies, by Jensen's inequality,

$$H(F_N^\circ | f_\circ^{\otimes N}) := \int_{\mathbb{D}^N} F_N^\circ \log \left(\frac{F_N^\circ}{f_\circ^{\otimes N}} \right) \leq \log \int_{\mathbb{D}^N} \frac{|F_N^\circ|^2}{f_\circ^{\otimes N}} = O(1),$$

while relative-entropy approaches generally work under the substantially weaker condition $\frac{1}{N} H(F_N^\circ | f_\circ^{\otimes N}) \rightarrow 0$; see e.g. [18]. Assumption (1.3) is nevertheless the natural condition for our L^2 -duality method as it is equivalent to uniform square summability of the initial Hoeffding components; see Section 2 below.

- (b) *Range of γ :* The restriction on γ in Theorem 1.1 stems from the uniform-in- N regularity estimates of Propositions 3.3 and 4.4, rather than from any apparent criticality of the limiting dual hierarchy. Indeed, as discussed in Remark 5.3, the limiting hierarchy is expected to be well-posed for $\gamma \leq \frac{5}{6}$ when $\alpha > 0$ and for $\gamma \leq \frac{1}{2}$ when $\alpha = 0$. At present, however, we do not know how to establish the required uniform-in- N regularity estimates throughout this larger range.

In the diffusive case $\alpha > 0$, we rely on hypoellipticity for simplicity; this yields a particularly clean argument, at the cost of the restriction $\gamma < \frac{1}{6}$. Using an averaging argument, along the lines of the one used when $\alpha = 0$, together with the additional regularization in v provided by the diffusion, would extend the admissible range to

$$\gamma < \gamma_* := \frac{1}{4} (2d + 1 - \sqrt{4d^2 + 1}) = \frac{1}{4} + O\left(\frac{1}{d}\right).$$

Since this refinement would not bring any additional physically-relevant cases within the scope of the result, we do not pursue it here.

- (c) *Singularity class*: Since the kinetic estimates yield genuine fractional Sobolev regularity, rather than merely improved integrability, a direct inspection of the proof shows that the assumption on the interaction kernel can be relaxed to

$$K \in H_{\text{loc}}^{-\gamma}(\mathbb{R}^d), \quad \sup_{|x| \geq 1} |K(x)| < \infty,$$

with the same range of γ , at the cost of strengthening the regularity assumptions on the mean-field solution f . By the Sobolev embedding $L_{\text{loc}}^{2d/(d+2\gamma)}(\mathbb{R}^d) \subset H_{\text{loc}}^{-\gamma}(\mathbb{R}^d)$, this improves the above statement.

1.3. Comparison to previous results. For Lipschitz interaction forces, the mean-field limit goes back to the classical works [20, 4, 9, 24]. Derivative assumptions on the force were later removed in [18], where arbitrary bounded interaction forces were treated by a relative entropy method at the level of the Liouville equation.

For unbounded forces, substantially less was previously known. For long, in dimensions $d \geq 2$, the only results have remained those of [14, 15], which cover mildly singular forces with $|K(x)| \lesssim |x|^{-c}$ and $|\nabla K(x)| \lesssim |x|^{-c-1}$ for some $c < 1$, for vanishing temperature $\alpha = 0$, thus missing the Coulomb singularity even in 2D. The dual hierarchical method that we introduced in [5] with D. Bresch allowed to cover arbitrary forces $K \in L_{\text{loc}}^2$, at both positive and vanishing temperature, without any repulsiveness or potential structure of the interactions. This substantially enlarged the admissible class of singularities, but still excluded the 2D Coulomb force, which belongs to $L^{2,\infty}$.

In the diffusive setting $\alpha > 0$, a direct hierarchical approach was developed in [6] for repulsive forces deriving from a nonnegative potential with suitable exponential integrability. By exploiting the velocity diffusion to compensate for the derivative loss in the BBGKY hierarchy, this yielded the first derivation of the 2D Vlasov-Poisson-Fokker-Planck equation, together with a partial 3D result, but it is restricted to a short time interval and requires strong structural conditions on initial data. The present work follows a different, dual route; it requires no repulsiveness assumption, nor a potential structure of the interactions, it applies globally in time as long as the mean-field solution remains regular enough, and, crucially, it also covers the non-diffusive case $\alpha = 0$.

A large complementary literature considers Coulomb or nearly Coulomb forces regularized at an N -dependent microscopic scale. In 3D, forces arbitrarily close to Coulomb were treated in [2] with cutoff radius $N^{-1/3}$, while the Coulomb force with cutoff radius $N^{-1/3+\varepsilon}$ was reached in [19]. Corresponding results in the diffusive case $\alpha > 0$ were also obtained in [8, 16]. Probabilistic trajectory methods have now reached the cutoff radius $N^{-5/12+\varepsilon}$ for the 3D Coulomb force [12], and the considerably smaller cutoff radius N^{-2} in 2D [13]. These results approximate the exact microscopic force remarkably closely, but retain a nonzero cutoff for every finite N .

Finally, we emphasize that more singular interactions can be handled without cutoff for simpler dynamics or regimes. For first-order dynamics, in particular, the modulated energy method yields the mean-field limit for Coulomb and Riesz interactions in arbitrary dimension, and also applies to second-order dynamics in the monokinetic regime [23, 7]. The recent multiscale mollification method of [21] further applies to a broad class of attractive or repulsive first-order interactions, not necessarily deriving from a potential. In a different direction, for second-order Hamiltonian dynamics in a perturbative regime near

Gibbs equilibrium, we have derived in [10] the mean-field limit toward the linearized Vlasov equation for a broad class of singular forces including Coulomb in dimensions $d \leq 3$. The present work addresses instead the fully nonlinear mean-field limit for second-order kinetic dynamics away from equilibrium, where neither the first-order structure nor the perturbative control around equilibrium is available.

2. DUAL HIERARCHICAL METHOD

We start by recalling the dual hierarchical approach to mean-field limits, which we introduced in [5] with D. Bresch. We further refine it here to account for the weaker initial chaos assumption (1.3); see in particular Lemma 2.1 below. The method proceeds by reformulating the mean-field limit in terms of the dual Liouville evolution on the space of test functions, then applying the Hoeffding chaos decomposition with respect to the mean-field distribution $f^{\otimes N}$, and finally noting that the obstruction related to losses of velocity derivatives in the BBGKY hierarchy is compensated by the fluctuation scaling.

2.1. Hoeffding chaos decomposition. Given a symmetric function $G_N \in L^2(f^{\otimes N})$, we consider its Hoeffding components with respect to the product mean-field measure $f^{\otimes N}$: for $0 \leq m \leq N$,

$$\Pi_{N,m}[G_N] := ((1 - \pi_f)^{\otimes m} \otimes \pi_f^{\otimes N-m}) G_N,$$

in terms of the projection

$$\pi_f h := \int_{\mathbb{D}} h f.$$

By definition, $\Pi_{N,m}[G_N]$ is a symmetric function on $\mathbb{D}^m$ and satisfies

$$\int_{\mathbb{D}} \Pi_{N,m}[G_N](z_1, \dots, z_m) f(z_j) dz_j = 0 \quad \text{for all } 1 \leq j \leq m.$$

The Hoeffding decomposition of G_N then reads

$$G_N(z_1, \dots, z_N) = \sum_{m=0}^N \sum_{\substack{\sigma \subset [N] \\ \sharp \sigma = m}} \Pi_{N,m}[G_N](z_\sigma),$$

with the short-hand notation $[N] = \{1, \dots, N\}$ and $z_\sigma = (z_{i_1}, \dots, z_{i_\ell})$ for $\sigma = \{i_1, \dots, i_\ell\}$. By orthogonality between Hoeffding components, we deduce

$$\int_{\mathbb{D}^N} |G_N|^2 f^{\otimes N} = \sum_{m=0}^N \binom{N}{m} \int_{\mathbb{D}^m} |\Pi_{N,m}[G_N]|^2 f^{\otimes m}. \quad (2.1)$$

Applied to the relative density $F_N/f^{\otimes N}$ of the N -particle distribution with respect to the mean-field distribution $f^{\otimes N}$, the Hoeffding components

$$H_{N,m} := \Pi_{N,m} \left[\frac{F_N}{f^{\otimes N}} \right]$$

coincide with the linearized cumulants first considered in [22] to characterize corrections to mean field. In terms of marginals, they read

$$H_{N,0} = 1, \quad H_{N,1} f = F_{N,1} - f, \quad H_{N,2} f^{\otimes 2} = F_{N,2} - 2F_{N,1} \otimes_\sigma f + f \otimes f,$$

and so on, where $\otimes_\sigma$ stands for symmetric tensor product. In these terms, the mean-field limit simply amounts to showing $H_{N,m} \rightarrow 0$ for all $m \geq 1$. As the norm $\int_{\mathbb{D}^N} |F_N|^2 / f^{\otimes N}$ is not conserved by the Liouville flow, however, no useful estimate on these correlation

functions can be deduced from identity (2.1). In contrast, in case of bounded observables, since the L^∞ norm is formally conserved by the Liouville flow, we can use (2.1) in the form

$$\int_{\mathbb{D}^m} |\Pi_{N,m}[G_N]|^2 f^{\otimes m} \leq \binom{N}{m}^{-1} \|G_N\|_{L^\infty}^2 \leq C_m N^{-m} \|G_N\|_{L^\infty}^2.$$

This is what motivates the need for a duality argument: passing to the dual, we can focus on bounded observables and the above then provides nontrivial a priori estimates.

2.2. Dual reformulation. We show that the mean-field limit follows from the weak convergence of the second Hoeffding component of bounded backward solutions of the dual Liouville equation. This provides a slight refinement of our dual reformulation of mean-field limits in [5], as we allow here for the weaker initial chaos assumption (1.3). To this aim, the proof proceeds by examining exponential, instead of polynomial, observables: we consider the characteristic function of the empirical measure and characterize its strong convergence.

Lemma 2.1. *Let $\alpha \geq 0$ and $K \in L^1_{\text{loc}}(\mathbb{R}^d)^d$. Let $F_N \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{D}^N))$ be an exchangeable global weak duality solution of the Liouville equation (1.1), in the sense of [5, Appendix], with exchangeable initial data $F_N^\circ \in \mathcal{P} \cap L^\infty(\mathbb{D}^N)$ satisfying the L^2 chaos assumption (1.3) for some $f_\circ \in \mathcal{P} \cap L^\infty(\mathbb{D})$. Let f be a bounded weak solution of the Vlasov equation (1.2) with initial data f° , and assume that, for some $T_0 > 0$,*

$$f \in L^\infty([0, T_0] \times \mathbb{D}), \quad K * f \in L^\infty([0, T_0] \times \mathbb{R}^d), \quad \nabla_v f \in L^1([0, T_0] \times \mathbb{D}).$$

For any $T \in (0, T_0]$ and any symmetric $\Phi_N^T \in L^\infty(\mathbb{D}^N)$ such that

$$\|\Phi_N^T\|_{L^\infty(\mathbb{D}^N)} \leq 1, \quad \lim_{N \uparrow \infty} \|\Phi_N^T - 1\|_{L^2(f^{\otimes N})} = 0, \quad (2.2)$$

consider a global bounded weak solution $\Phi_N \in L^\infty([0, T] \times \mathbb{D}^N)$ of the backward dual Liouville equation on $[0, T]$,

$$\partial_t \Phi_N + \sum_{i=1}^N (v_i \cdot \nabla_{x_i} + \alpha \Delta_{v_i}) \Phi_N + \frac{1}{N-1} \sum_{i \neq j}^N K(x_i - x_j) \cdot \nabla_{v_i} \Phi_N = 0, \quad (2.3)$$

with final data $\Phi_N|_{t=T} = \Phi_N^T$ and with

$$\sup_{[0, T]} \|\Phi_N\|_{L^\infty(\mathbb{D}^N)} \leq 1. \quad (2.4)$$

Assume that for any such Φ_N we have

$$N \int_0^T \left| \int_{\mathbb{D}^2} K_f \Pi_{N,2}[\Phi_N] f^{\otimes 2} \right| \rightarrow 0 \quad \text{as } N \uparrow \infty, \quad (2.5)$$

where we have defined

$$K_f(z, z') := (K(x - x') - (K * f)(x)) \cdot \nabla_v \log f(z). \quad (2.6)$$

Then we have, for all $k \geq 1$ and $t \in [0, T_0]$,

$$F_{N,k}(t) \rightarrow f(t)^{\otimes k} \quad \text{in } \mathcal{D}'(\mathbb{D}^k). \quad (2.7)$$

Proof. Fix $T \in (0, T_0]$, $h \in C_c^\infty(\mathbb{D})$, $\lambda \in \mathbb{R}$, and let

$$\Phi_N^T(z_1, \dots, z_N) := \exp \left(\frac{i\lambda}{N} \sum_{j=1}^N \left(h(z_j) - \int_{\mathbb{D}} h f(T) \right) \right). \quad (2.8)$$

A direct calculation yields

$$\begin{aligned} \|\Phi_N^T - 1\|_{L^2(f(T)^{\otimes N})} &\leq \left\| \frac{\lambda}{N} \sum_{j=1}^N \left(h(z_j) - \int_{\mathbb{D}} h f(T) \right) \right\|_{L^2(f(T)^{\otimes N})} \\ &= \frac{\lambda}{\sqrt{N}} \left(\int_{\mathbb{D}} \left(h - \int_{\mathbb{D}} h f(T) \right)^2 f(T) \right)^{\frac{1}{2}} \rightarrow 0, \end{aligned}$$

that is, (2.2). Consider a global bounded weak solution $\Phi_N \in L^\infty([0, T] \times \mathbb{D}^N)$ of the dual Liouville equation (2.3) with final data $\Phi_N|_{t=T} = \Phi_N^T$ and with the conservation property (2.4). Note that the equation ensures $\Phi_N \in C_{w*}(0, T; L^\infty(\mathbb{D}^N))$. Comparing the weak formulations of the equations for Φ_N and for $f^{\otimes N}$, as in [5, Proposition 4], we find for all $0 \leq t \leq T$,

$$\int_{\mathbb{D}^N} \Phi_N^T f(T)^{\otimes N} - \int_{\mathbb{D}^N} \Phi_N(t) f(t)^{\otimes N} = \frac{1}{N-1} \sum_{i \neq j}^N \int_t^T \left(\int_{\mathbb{D}^N} K_f(z_i, z_j) \Phi_N f^{\otimes N} \right),$$

where K_f is defined in the statement. Since Φ_N is symmetric in its N entries and since the cancellations $\int_{\mathbb{D}} K_f(z_1, z_2) f(z_j) dz_j = 0$ for $j = 1, 2$ allow to replace Φ_N by $\Pi_{N,2}[\Phi_N]$, we get

$$\int_{\mathbb{D}^N} \Phi_N^T f(T)^{\otimes N} - \int_{\mathbb{D}^N} \Phi_N(t) f(t)^{\otimes N} = N \int_t^T \left(\int_{\mathbb{D}^N} K_f \Pi_{N,2}[\Phi_N] f^{\otimes N} \right).$$

The right-hand side tends to 0 by assumption (2.5), while $\int_{\mathbb{D}^N} \Phi_N^T f(T)^{\otimes N} \rightarrow 1$ by (2.2). Hence, for all $0 \leq t \leq T$,

$$\Pi_{N,0}[\Phi_N](t) = \int_{\mathbb{D}^N} \Phi_N(t) f(t)^{\otimes N} \rightarrow 1.$$

Combined with (2.4), this implies the strong convergence

$$\begin{aligned} \|\Phi_N(t) - 1\|_{L^2(f(t)^{\otimes N})}^2 &= \|\Phi_N(t)\|_{L^2(f(t)^{\otimes N})}^2 + 1 - 2\Re \Pi_{N,0}[\Phi_N](t) \\ &\leq 2\Re(1 - \Pi_{N,0}[\Phi_N](t)) \rightarrow 0. \end{aligned}$$

In particular, at $t = 0$, by the weak chaos assumption (1.3),

$$\left| \int_{\mathbb{D}^N} \Phi_N(0) F_N^\circ - 1 \right| \leq \|\Phi_N(0) - 1\|_{L^2(f^\circ)^{\otimes N}} \left(\int_{\mathbb{D}^N} \frac{|F_N^\circ|^2}{f^\circ^{\otimes N}} \right)^{\frac{1}{2}} \rightarrow 0. \quad (2.9)$$

Since F_N is a weak duality solution of the Liouville equation in the sense of [5, Appendix], we can choose the weak dual solution Φ_N to be in duality with F_N , that is,

$$\int_{\mathbb{D}^N} \Phi_N^T F_N(T) = \int_{\mathbb{D}^N} \Phi_N(0) F_N^\circ, \quad (2.10)$$

so that (2.9) entails

$$\int_{\mathbb{D}^N} \Phi_N^T F_N(T) \rightarrow 1.$$

By the choice (2.8) of Φ_N^T , with arbitrary $h \in C_c^\infty(\mathbb{D})$ and $\lambda \in \mathbb{R}$, this easily implies for any $k \geq 1$,

$$\int_{\mathbb{D}^N} \left(\frac{1}{N} \sum_{j=1}^N \left(h(z_j) - \int_{\mathbb{D}} h f(T) \right) \right)^k F_N(T, z_1, \dots, z_N) dz_1 \dots dz_N \rightarrow 0.$$

Expanding the power in the left-hand side, this yields the conclusion (2.7). $\square$

2.3. A priori estimates. Henceforth, we assume that F_N, f are as in Lemma 2.1, we fix $T \in (0, T_0]$ and a symmetric $\Phi_N^T \in L^\infty(\mathbb{D}^N)$ satisfying (2.2), and we consider a global bounded weak solution $\Phi_N \in L^\infty([0, T] \times \mathbb{D}^N)$ of the dual Liouville equation (2.3) with final data $\Phi_N|_{t=T} = \Phi_N^T$ and with the conservation property (2.4). We shall study the associated Hoeffding components, or dual correlations,

$$C_{N,m} := \Pi_{N,m}[\Phi_N], \quad 0 \leq m \leq N,$$

and we aim to prove the convergence (2.5), that is,

$$N \int_0^T \left| \int_{\mathbb{D}^2} K_f C_{N,2} f^{\otimes 2} \right| \rightarrow 0 \quad \text{as } N \uparrow \infty, \quad (2.11)$$

which will then imply the mean-field limit by Lemma 2.1. Due to the orthogonality property (2.1) and to the conservation property (2.4), we have

$$\sup_{[0,T]} \|C_{N,m}\|_{L^2(f^{\otimes m})} \leq \binom{N}{m}^{-\frac{1}{2}} \|\Phi_N^T\|_{L^\infty(\mathbb{D}^N)}, \quad 0 \leq m \leq N.$$

Note in particular that $C_{N,2} = O(N^{-1})$, which is just not enough to prove (2.11) even if $K \in L_{\text{loc}}^2(\mathbb{R}^d)$. In view of these a priori bounds, we define the rescaled correlations

$$\bar{C}_{N,m} = \binom{N}{m}^{\frac{1}{2}} C_{N,m}, \quad 0 \leq m \leq N. \quad (2.12)$$

Since Φ_N is bounded in L^∞ , cf. (2.4), we can interpolate and deduce the following more general bounds; see [5, Lemma 12].

Lemma 2.2 (A priori estimates; see [5]). *For all $0 \leq k \leq m \leq N$ and $q \geq 2$,*

$$\sup_{[0,T]} \|\bar{C}_{N,m}\|_{L^q(f^{\otimes k}, L^2(f^{\otimes m-k}))} \leq C^k \left(\frac{N}{m} \right)^{k(\frac{1}{2} - \frac{1}{q})}.$$

2.4. Hierarchy for dual correlations. As the above a priori bounds do not suffice for the desired convergence (2.11), we further study the dynamics of dual correlations. To this end, we consider the BBGKY-type hierarchy that they satisfy, which can be obtained directly from the dual Liouville equation (2.3) for Φ_N ; see [5, Lemma 11]. We state it here for the rescaled dual correlations (2.12).

We emphasize that most of the collision operators appearing in the hierarchy contain velocity derivatives, and their iteration would thus ordinarily generate an uncontrollable cascade of derivative losses, which is the usual obstruction for classical kinetic hierarchies. Yet, our central observation in [5] was that the fluctuation scaling (2.12) exactly compensates for derivative losses as $N \uparrow \infty$: after rescaling, velocity derivatives are confined to the remainder term $R_{N,m}$ in (2.13) below and are formally $O(N^{-1/2})$.

Lemma 2.3 (Hierarchy for dual correlations; see [5]). *For all $0 \leq m \leq N$, we have in the weak sense on $[0, T]$,*

$$(\partial_t + L_f^m) \bar{C}_{N,m} = R_{N,m} + \sqrt{(m+1)(m+2)} \left(\frac{(N-m)(N-m-1)}{(N-1)^2} \right)^{\frac{1}{2}} K_f[\bar{C}_{N,m+2}], \quad (2.13)$$

where L_f^m stands for the following linearized mean-field operator on the m -particle space,

$$\begin{aligned} L_f^m &:= \sum_{j=1}^m \text{Id}^{\otimes j-1} \otimes L_f \otimes \text{Id}^{m-j}, \\ L_f h &:= v \cdot \nabla_x h + \alpha \Delta_v h + (K * f)(x) \cdot \nabla_v h - \int_{\mathbb{D}} K_f(z_*, z) h(z_*) f(z_*) dz_*, \end{aligned}$$

where $K_f[\bar{C}_{N,m+2}]$ is defined as

$$K_f[\bar{C}_{N,m+2}] = \int_{\mathbb{D}^2} K_f(z_*, z'_*) \bar{C}_{N,m+2}(\cdot, z_*, z'_*) f(z_*) f(z'_*) dz_* dz'_*, \quad (2.14)$$

where the remainder term $R_{N,m}$ is given explicitly by

$$\begin{aligned} R_{N,m} &= \left(\frac{N-m+1}{m(N-1)^2} \right)^{\frac{1}{2}} \sum_{i \neq j}^m (K * f)(x_i) \cdot \nabla_{v_i} \bar{C}_{N,[m] \setminus \{j\}} \\ &\quad - \left(\frac{N-m+1}{m(N-1)^2} \right)^{\frac{1}{2}} \sum_{i \neq j}^m \int_{\mathbb{D}} K_f(z_*, z_j) \bar{C}_{N,[m] \cup \{*\} \setminus \{i,j\}} f(z_*) dz_* \\ &\quad - \left(\frac{N-m+1}{m(N-1)^2} \right)^{\frac{1}{2}} \sum_{i \neq j}^m K(x_i - x_j) \cdot \nabla_{v_i} \bar{C}_{N,[m] \setminus \{j\}} \\ &\quad + \frac{m-1}{N-1} \sum_{i=1}^m (K * f)(x_i) \cdot \nabla_{v_i} \bar{C}_{N,m} - \frac{1}{N-1} \sum_{i \neq j}^m K(x_i - x_j) \cdot \nabla_{v_i} \bar{C}_{N,m} \\ &\quad - \frac{m-1}{N-1} \sum_{j=1}^m \int_{\mathbb{D}} K_f(z_*, z_j) \bar{C}_{N,[m] \cup \{*\} \setminus \{j\}} f(z_*) dz_* \\ &\quad + \frac{1}{N-1} \sum_{i \neq j}^m \int_{\mathbb{D}} K(x_i - x_*) \cdot \nabla_{v_i} \bar{C}_{N,[m] \cup \{*\} \setminus \{j\}} f(z_*) dz_* \\ &\quad - \frac{1}{N-1} \sum_{i \neq j}^m \int_{\mathbb{D}} K_f(z_*, z_j) \bar{C}_{N,[m] \cup \{*\} \setminus \{i\}} f(z_*) dz_* \\ &\quad + \frac{1}{N-1} \sum_{i \neq j}^m \int_{\mathbb{D}^2} K_f(z_*, z'_*) \bar{C}_{N,[m] \cup \{*,*\} \setminus \{i,j\}} f(z_*) f(z'_*) dz_* dz'_* \\ &\quad + \left(\frac{(m+1)(N-m)}{(N-1)^2} \right)^{\frac{1}{2}} \sum_{j=1}^m \int_{\mathbb{D}} K_f(z_*, z_j) \bar{C}_{N,[m] \cup \{*\}} f(z_*) dz_* \\ &\quad - \left(\frac{(m+1)(N-m)}{(N-1)^2} \right)^{\frac{1}{2}} \sum_{i=1}^m \int_{\mathbb{D}} K(x_i - x_*) \cdot \nabla_{v_i} \bar{C}_{N,[m] \cup \{*\}} f(z_*) dz_* \end{aligned}$$

$$-2 \left(\frac{(m+1)(N-m)}{(N-1)^2} \right)^{\frac{1}{2}} \sum_{i=1}^m \int_{\mathbb{D}^2} K_f(z_*, z'_*) \bar{C}_{N, [m] \cup \{*, *'\} \setminus \{i\}} f(z_*) f(z'_*) dz_* dz'_*,$$

and where for notational convenience we have set $\bar{C}_{N,m} \equiv 0$ for $m < 0$ and $m > N$.

3. DIFFUSIVE CASE: HIERARCHICAL ESTIMATES VIA HYPOELLIPTICITY

In order to obtain improved a priori estimates on dual correlations, in the diffusive case $\alpha > 0$, we apply hypoelliptic regularity to the hierarchy of equations derived in Lemma 2.3. More precisely, we appeal to the following classical hypoelliptic estimate. This essentially follows from [3], but we include a short proof in Appendix A for completeness, based on a direct Fourier calculation. We emphasize that the multiplicative constant is independent of the phase-space dimension $2dm$.

Lemma 3.1 (Hypoelliptic regularity). *Given $\alpha > 0$ and $m \geq 1$, let g_m, e_m satisfy in the weak sense on $[0, T] \times \mathbb{D}^m$,*

$$\partial_t g_m + \sum_{i=1}^m (v_i \cdot \nabla_{x_i} - \alpha \Delta_{v_i}) g_m = e_m.$$

Then, for all $0 \leq r \leq \frac{1}{3}$,

$$\int_0^T \|\nabla_{x_{[m]}}|^{\frac{1}{3}-r} g_m\|_{L^2}^2 \leq C(T+1) \sup_{[0,T]} \|g_m\|_{L^2}^2 + C \int_0^T \|\langle \nabla_{x_{[m]}} \rangle^{-r} \langle \nabla_{v_{[m]}} \rangle^{-1} e_m\|_{L^2}^2, \quad (3.1)$$

for some constant C only depending on α .

Remark 3.2. For later purposes, we note that, by a straightforward inspection of the proof in Appendix A, the above can be strengthened as follows in the case $e_m = e_m^0 + \operatorname{div}_v(e_m^1)$: for all $0 \leq r \leq \frac{1}{3}$ and $0 \leq s \leq \frac{2}{3}$,

$$\begin{aligned} & \int_0^T \|\nabla_{x_{[m]}}|^{(\frac{2}{3}-s) \wedge (\frac{1}{3}-r)} g_m\|_{L^2}^2 \\ & \leq C(T+1) \sup_{[0,T]} \|g_m\|_{L^2}^2 + C \int_0^T \|\langle \nabla_{x_{[m]}} \rangle^{-s} e_m^0\|_{L^2}^2 + C \int_0^T \|\langle \nabla_{x_{[m]}} \rangle^{-r} e_m^1\|_{L^2}^2. \end{aligned}$$

Applying this regularity result to the hierarchy (2.13) for dual correlations, we establish the following uniform-in- N a priori estimate. Note that we use the weight $f^{1-\delta}$ instead of $\sqrt{f}$, with $1-\delta \geq \frac{1}{2}$, to avoid boundedness requirements on $\nabla_v \log f$ itself, in line with the weighted regularity requirement (1.7).

Proposition 3.3. *Let $\alpha > 0$, let*

$$K \in L_{\text{loc}}^{2d/(d+2\gamma)}(\mathbb{R}^d), \quad \sup_{|x| \geq 1} |K(x)| < \infty,$$

for some $0 \leq \gamma < \frac{1}{6}$. Also assume that the mean-field solution f satisfies the regularity conditions (1.5), (1.6), and (1.7) for some $\delta \in (0, \frac{1}{2}]$. Then, for all $0 \leq m \leq N$,

$$\int_0^T \|\nabla_{x_1} |^\gamma ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^2 \leq C(T+1)(m+1)^c,$$

for some constant c only depending on γ and some constant C further depending on α, δ and on assumptions on K, f .

Proof. Smuggling the weight $(f^{\otimes m})^{1-\delta}$, with δ as in the regularity condition (1.7), and using the Vlasov equation for f , the equation (2.13) for $\bar{C}_{N,m}$ can be reformulated as

$$\left(\partial_t + \sum_{i=1}^m v_i \cdot \nabla_{x_i} + \alpha \sum_{i=1}^m \Delta_{v_i}\right)((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) = A_m =: \sum_{j=1}^4 A_m^j, \quad (3.2)$$

where we have set

$$\begin{aligned} A_m^1 &:= (f^{\otimes m})^{1-\delta} R_{N,m}, \\ A_m^2 &:= \sqrt{(m+1)(m+2)} \left(\frac{(N-m)(N-m-1)}{(N-1)^2} \right)^{\frac{1}{2}} (f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}] \\ &\quad + (f^{\otimes m})^{1-\delta} \sum_{i=1}^m \int_{\mathbb{D}} K_f(z_*, z_i) \bar{C}_{N,m}(z_{[m] \setminus i}, z_*) f(z_*) dz_*, \\ A_m^3 &:= - \sum_{i=1}^m \operatorname{div}_{v_i} \left((K * f)(x_i) ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) - \alpha (\nabla_v \log f)(z_i) ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \right), \\ A_m^4 &:= -\alpha \left(\sum_{i=1}^m h_\delta(z_i) \right) ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}), \end{aligned}$$

and for abbreviation

$$h_\delta := (1-\delta) \left((2-\delta) |\nabla_v \log f|^2 + 2 \Delta_v \log f \right).$$

Appealing to hypoelliptic regularity in the form of Lemma 3.1 (but backward in time), recalling the a priori L^2 -estimate of Lemma 2.2, and noting that the boundedness of f ensures $f^{1-\delta} \lesssim f^{1/2}$, we get for all $0 \leq r \leq \frac{1}{3}$,

$$\int_0^T \|\langle \nabla_{x_1} \rangle^{\frac{1}{3}-r} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^2 \leq C(T+1) + C \sum_{i=1}^4 \int_0^T \|\langle \nabla_X \rangle^{-r} \langle \nabla_V \rangle^{-1} A_m^i\|_{L^2}^2, \quad (3.3)$$

where we set for abbreviation $Z = (X, V) := (x_{[m]}, v_{[m]}) \in \mathbb{D}^m$. From here, we split the proof into three steps: in the first two, we examine the different right-hand side terms in (3.3), and we conclude in the last step by buckling. In the sequel, C stands for arbitrary constants only depending on γ, α, δ , and on assumptions on K, f , and we write $\lesssim$ for $\leq C \times$.

Step 1: Proof that

$$\begin{aligned} &\sum_{i=2}^4 \|\langle \nabla_V \rangle^{-1} A_m^i\|_{L^2} \\ &\lesssim m + (m+1) \|\langle \nabla_{x_1} \rangle^\gamma ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2})\|_{L^2} + m \|\langle \nabla_{x_1} \rangle^\gamma ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}. \end{aligned}$$

We start with the contribution of A_m^2 . By definition, we can bound

$$\begin{aligned} \|A_m^2\|_{L^2} &\lesssim (m+1) \|(f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}]\|_{L^2} \\ &\quad + \left\| (f^{\otimes m})^{1-\delta} \sum_{i=1}^m \int_{\mathbb{D}} K_f(z_*, z_i) \bar{C}_{N,m}(z_{[m] \setminus i}, z_*) f(z_*) dz_* \right\|_{L^2}. \quad (3.4) \end{aligned}$$

Recalling the definition of the operator $K_f[\cdot]$, cf. (2.14), we can decompose the first term as follows,

$$\begin{aligned}
& \| (f^{\otimes m})^{1-\delta} K_f [\bar{C}_{N,m+2}] \|_{L^2} \\
& \leq \left\| \int_{\mathbb{D}^2} K(x' - x'') \cdot \nabla_v \log f(z') f(z')^\delta f(z'')^\delta ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2})(\cdot, z', z'') dz' dz'' \right\|_{L^2} \\
& \quad + \| K * f \|_{L^\infty} \left\| \int_{\mathbb{D}^2} \nabla_v \log f(z') f(z')^\delta f(z'')^\delta ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2})(\cdot, z', z'') dz' dz'' \right\|_{L^2},
\end{aligned}$$

and thus, by Hölder's inequality and the Sobolev embedding $H^\gamma \subset L^{2d/(d-2\gamma)}$, splitting the kernel $K(x' - x'')$ for $|x' - x''| \leq 1$ and $|x' - x''| > 1$, and using the exchangeability,

$$\begin{aligned}
& \| (f^{\otimes m})^{1-\delta} K_f [\bar{C}_{N,m+2}] \|_{L^2} \\
& \leq \left(\| K \|_{L^{2d/(d+2\gamma)}(B)} \| f^\delta \|_{L_x^\infty L_v^2} + (\| K \|_{L^\infty(B^c)} + \| K * f \|_{L^\infty}) \| f^\delta \|_{L^2} \right) \\
& \quad \times \| f^\delta \nabla_v \log f \|_{L^2} \| \langle \nabla_{x_1} \rangle^\gamma ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2}) \|_{L^2}.
\end{aligned}$$

The different factors are precisely controlled by the assumptions on K and f , hence

$$\| (f^{\otimes m})^{1-\delta} K_f [\bar{C}_{N,m+2}] \|_{L^2} \lesssim \| \langle \nabla_{x_1} \rangle^\gamma ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2}) \|_{L^2}.$$

Similarly estimating the second term in (3.4), this time using Young's convolution inequality to bound the convolution with the singular part of K , we get

$$\begin{aligned}
& \left\| (f^{\otimes m})^{1-\delta} \sum_{i=1}^m \int_{\mathbb{D}} K_f(z_*, z_i) \bar{C}_{N,m}(z_{[m] \setminus i}, z_*) f(z_*) dz_* \right\|_{L^2} \\
& \leq \left(\| K \|_{L^{2d/(d+2\gamma)}(B)} \| f^{1-\delta} \|_{L_x^\infty L_v^2} + (\| K \|_{L^\infty(B^c)} + \| K * f \|_{L^\infty}) \| f^{1-\delta} \|_{L^2} \right) \\
& \quad \times \| f^\delta \nabla_v \log f \|_{L^2} \| \langle \nabla_{x_1} \rangle^\gamma ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \|_{L^2}.
\end{aligned}$$

We are thus led to

$$\| A_m^2 \|_{L^2} \lesssim (m+1) \| \langle \nabla_{x_1} \rangle^\gamma ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2}) \|_{L^2} + m \| \langle \nabla_{x_1} \rangle^\gamma ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \|_{L^2}.$$

Finally, for A_m^3 and A_m^4 , the a priori L^2 -estimates of Lemma 2.2 and the assumptions on f immediately ensure

$$\| \langle \nabla_V \rangle^{-1} A_m^3 \|_{L^2} \lesssim m, \quad \| A_m^4 \|_{L^2} \lesssim m.$$

Step 2: Proof that for all $\gamma \leq r \leq 1$,

$$\| \langle \nabla_X \rangle^{-r} \langle \nabla_V \rangle^{-1} A_m^1 \|_{L^2} \lesssim m^2. \tag{3.5}$$

Recall $A_m^1 = (f^{\otimes m})^{1-\delta} R_{N,m}$, where $R_{N,m}$ is the remainder term defined in Lemma 2.3. The claim (3.5) follows from a careful examination of all the terms in the definition of $R_{N,m}$, using the assumptions on K , f , and the a priori estimates of Lemma 2.2. For shortness, let us focus on the following three typical terms that are part of $R_{N,m}$,

$$\begin{aligned}
R_{N,m}^1 &:= - \left(\frac{N-m+1}{m(N-1)^2} \right)^{\frac{1}{2}} \sum_{i \neq j}^m K(x_i - x_j) \cdot \nabla_{v_i} \bar{C}_{N,[m] \setminus j}, \\
R_{N,m}^2 &:= - \frac{1}{N-1} \sum_{i \neq j}^m K(x_i - x_j) \cdot \nabla_{v_i} \bar{C}_{N,m}, \\
R_{N,m}^3 &:= \left(\frac{(m+1)(N-m)}{(N-1)^2} \right)^{\frac{1}{2}} \sum_{j=1}^m \int_{\mathbb{D}} K_f(z_*, z_j) \bar{C}_{N,[m] \cup \{*\}} f(z_*) dz_*.
\end{aligned} \tag{3.6}$$

For the first one, commuting the velocity derivative with the weight, we have by symmetry

$$\begin{aligned} & \|\langle \nabla_X \rangle^{-r} \langle \nabla_V \rangle^{-1} ((f^{\otimes m})^{1-\delta} R_{N,m}^1)\|_{L^2} \\ & \lesssim m^{\frac{3}{2}} N^{-\frac{1}{2}} \left\| \langle \nabla_{x_1} \rangle^{-r} \left(K(x_1 - x_2) (f^{\otimes m})^{1-\delta} \bar{C}_{N,[2,m]} \right) \right\|_{L^2} \\ & \quad + m^{\frac{3}{2}} N^{-\frac{1}{2}} \left\| \langle \nabla_{x_1} \rangle^{-r} \left(K(x_1 - x_2) \bar{C}_{N,[2,m]} \cdot \nabla_{v_1} (f^{\otimes m})^{1-\delta} \right) \right\|_{L^2}, \end{aligned}$$

and thus, for all $r \geq \gamma$, appealing to the Sobolev embedding $L_x^{2d/(d+2r)} \subset H_x^{-r}$, the assumptions on K, f , and the a priori L^2 -estimate of Lemma 2.2,

$$\|\langle \nabla_X \rangle^{-r} \langle \nabla_V \rangle^{-1} ((f^{\otimes m})^{1-\delta} R_{N,m}^1)\|_{L^2} \lesssim m^{\frac{3}{2}} N^{-\frac{1}{2}}.$$

We turn to the second typical term,

$$\begin{aligned} & \|\langle \nabla_X \rangle^{-r} \langle \nabla_V \rangle^{-1} ((f^{\otimes m})^{1-\delta} R_{N,m}^2)\|_{L^2} \\ & \lesssim m^2 N^{-1} \left\| \langle \nabla_{x_1} \rangle^{-r} \left(K(x_1 - x_2) (f^{\otimes m})^{1-\delta} \bar{C}_{N,m} \right) \right\|_{L^2} \\ & \quad + m^2 N^{-1} \left\| \langle \nabla_{x_1} \rangle^{-r} \left(K(x_1 - x_2) \bar{C}_{N,m} \cdot \nabla_{v_1} (f^{\otimes m})^{1-\delta} \right) \right\|_{L^2}. \quad (3.7) \end{aligned}$$

For all $r \geq \gamma$, appealing again to the Sobolev embedding and using the assumptions on K, f , we now find

$$\begin{aligned} & \left\| \langle \nabla_{x_1} \rangle^{-r} \left(K(x_1 - x_2) (f^{\otimes m})^{1-\delta} \bar{C}_{N,m} \right) \right\|_{L^2} \\ & \lesssim \left\| (f^{\otimes m})^{1-\delta} \bar{C}_{N,m} \right\|_{L^2(\mathbb{D}; L_x^\infty L_v^2 \cap L^2(\mathbb{D}; L^2(\mathbb{D}^{m-2})))} \\ & \lesssim \|\bar{C}_{N,m}\|_{L^\infty(\mathbb{D}^2; L^2(f^{\otimes m-2}))}. \end{aligned}$$

By the mixed a priori estimate of Lemma 2.2, this is $\lesssim N/m$. Similarly estimating the second term in (3.7), using the assumptions on $\nabla_v f^{1-\delta} = (1-\delta)f^{1-\delta}\nabla_v \log f$, we are led to

$$\|\langle \nabla_X \rangle^{-r} \langle \nabla_V \rangle^{-1} ((f^{\otimes m})^{1-\delta} R_{N,m}^2)\|_{L^2} \lesssim m.$$

Finally, for the third typical term $R_{N,m}^3$, we can similarly bound, using again the mixed a priori estimate of Lemma 2.2,

$$\begin{aligned} \|(f^{\otimes m})^{1-\delta} R_{N,m}^3\|_{L^2} & \lesssim m^{\frac{3}{2}} N^{-\frac{1}{2}} \left\| (f^{\otimes m})^{1-\delta} \int_{\mathbb{D}} K_f(z_*, z_1) \bar{C}_{N,[m] \cup \{*\}} f(z_*) dz_* \right\|_{L^2} \\ & \lesssim m^{\frac{3}{2}} N^{-\frac{1}{2}} \|\bar{C}_{N,m+1}\|_{L^{2d/(d-2\gamma)}(f^{\otimes 2}; L^2(f^{\otimes m-1}))} \\ & \lesssim m^{\frac{3}{2}} N^{\frac{2\gamma}{d} - \frac{1}{2}}. \end{aligned}$$

As by assumption $\gamma \leq \frac{d}{4}$, this is bounded by $m^{\frac{3}{2}}$. Arguing similarly for all the other terms in the definition of $R_{N,m}$, the claim (3.5) follows.

Step 3: Conclusion.

Inserting the estimates of Steps 1 and 2 into (3.3), we get for all $\gamma \leq r \leq \frac{1}{3}$,

$$\begin{aligned} & \int_0^T \|\langle \nabla_{x_1} \rangle^{\frac{1}{3}-r} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^2 \lesssim (T+1)(m+1)^4 \\ & \quad + (m+1)^2 \int_0^T \|\langle \nabla_{x_1} \rangle^\gamma ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2})\|_{L^2}^2 \end{aligned}$$

$$+ m^2 \int_0^T \|\langle \nabla_{x_1} \rangle^\gamma ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^2,$$

and thus, by interpolation with the a priori L^2 -estimate of Lemma 2.2, for all $\theta \in (0, 1)$,

$$\begin{aligned} \int_0^T \|\langle \nabla_{x_1} \rangle^{\frac{1}{3}-r} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^2 &\lesssim (T+1)(m+1)^4 \\ &+ (m+1)^2 \int_0^T \|\langle \nabla_{x_1} \rangle^{\frac{\gamma}{1-\theta}} ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2})\|_{L^2}^{2(1-\theta)} \\ &+ m^2 \int_0^T \|\langle \nabla_{x_1} \rangle^{\frac{\gamma}{1-\theta}} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^{2(1-\theta)}. \end{aligned}$$

Provided $\frac{\gamma}{1-\theta} \leq \frac{1}{3}-r$, Young's inequality allows to absorb the last right-hand side term into the left-hand side. In addition, we can restore the power 2 in the second right-hand side term and gain an arbitrarily small prefactor. We obtain in this way, for all $\gamma < s \leq \frac{1}{3}-\gamma$,

$$\begin{aligned} \int_0^T \|\langle \nabla_{x_1} \rangle^s ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^2 \\ \leq C(T+1)(m+1)^{4\vee \frac{2s}{s-\gamma}} + \frac{1}{2} \int_0^T \|\langle \nabla_{x_1} \rangle^s ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2})\|_{L^2}^2, \end{aligned}$$

and the conclusion follows from a direct iteration. $\square$

4. NON-DIFFUSIVE CASE: HIERARCHICAL ESTIMATES VIA AVERAGING

We turn to the case without velocity diffusion, $\alpha = 0$. A priori estimates on dual correlations will now be obtained by applying averaging lemmas to the hierarchy, instead of hypoelliptic regularity. More precisely, we appeal to the following slight variant of the L^2 averaging lemma of [17]. For completeness, a short proof is given in appendix. As for hypoelliptic estimates, we emphasize that the multiplicative constant is independent of the phase-space dimension $2dm$.

Lemma 4.1 (Averaging lemma). *Given $m \geq 1$, let g_m, e_m satisfy in the weak sense on $[0, T] \times \mathbb{D}^m$,*

$$\partial_t g_m + \sum_{i=1}^m v_i \cdot \nabla_{x_i} g_m = e_m.$$

Then, for all $0 \leq r \leq 1$,

$$\begin{aligned} \int_0^T \|\langle \nabla_{x_{[m]}} \rangle^{\frac{1-r}{4}} \langle \nabla_{v_{[m]}} \rangle^{-\frac{3}{2}} g_m\|_{L^2}^2 \\ \leq C \|g_m(0)\|_{L^2}^2 + C \int_0^T \|g_m\|_{L^2}^2 + C \int_0^T \|\langle \nabla_{x_{[m]}} \rangle^{-r} \langle \nabla_{v_{[m]}} \rangle^{-1} e_m\|_{L^2}^2, \end{aligned}$$

for some universal constant C .

Remark 4.2. For later purposes, we note that, by a straightforward inspection of the proof in Appendix A, the above can be strengthened as follows in the case $e_m = e_m^0 + \operatorname{div}_v(e_m^1)$: for all $0 \leq r, s \leq 1$,

$$\int_0^T \|\langle \nabla_{x_{[m]}} \rangle^{\frac{1-s}{2} \wedge \frac{1-r}{4}} \langle \nabla_{v_{[m]}} \rangle^{-\frac{3}{2}} g_m\|_{L^2}^2$$

$$\leq C(T+1) \sup_{[0,T]} \|g_m\|_{L^2}^2 + C \int_0^T \|\langle \nabla_{x_{[m]}} \rangle^{-s} e_m^0\|_{L^2}^2 + C \int_0^T \|\langle \nabla_{x_{[m]}} \rangle^{-r} e_m^1\|_{L^2}^2.$$

Applying this lemma directly to the m -particle equation would distribute the loss of velocity regularity over all particle variables, which would be problematic to adapt the hierarchical iteration in the proof of Proposition 3.3. To remedy this, we appeal to the following convenient Hilbert-valued version. The idea is to view the last $m-2$ particle variables instead as a Hilbert-valued parameter and to keep their mean-field transport on the left-hand side: since that transport is skew-adjoint in L^2 and acts only on the Hilbert-valued variables, it can be removed by a unitary conjugation, after which the two-particle version of the above averaging lemma applies with regularity loss only in (v_1, v_2) .

Lemma 4.3. *Let $\mathfrak{h}$ be a fixed, separable Hilbert space and let L_t be a time-dependent skew-adjoint operator on $\mathfrak{h}$ generating a unitary propagator. Let g, e be $\mathfrak{h}$ -valued functions satisfying in the weak sense on $[0, T] \times \mathbb{D}^2$,*

$$(\partial_t + v_1 \cdot \nabla_{x_1} + v_2 \cdot \nabla_{x_2} + L_t)g = e.$$

Then, for all $0 \leq r \leq 1$,

$$\begin{aligned} \int_0^T \||\nabla_{x_{[2]}}|^{\frac{1-r}{4}} \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} g\|_{L^2(\mathbb{D}^2; \mathfrak{h})}^2 dt \\ \leq C(T+1) \sup_{[0,T]} \|g\|_{L^2(\mathbb{D}^2; \mathfrak{h})}^2 + C \int_0^T \|\langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} e\|_{L^2(\mathbb{D}^2; \mathfrak{h})}^2 dt. \end{aligned}$$

Proof. Let $(\bar{u}_n)_{n \in \mathbb{N}}$ denote an orthonormal basis of $\mathfrak{h}$, and, for each n , let u_n solve

$$\partial_t u_n - L_t^* u_n = 0, \quad u_n|_{t=0} = \bar{u}_n.$$

From the assumptions, $(u_n(t))_{n \in \mathbb{N}}$ form another orthonormal basis of $\mathfrak{h}$ for every $t \in [0, T]$. We now consider $\langle u_n, g \rangle_{\mathfrak{h}}$, which are only functions of t, z_1, z_2 . From the equations on g and u_n , we obtain

$$(\partial_t + v_1 \cdot \nabla_{x_1} + v_2 \cdot \nabla_{x_2}) \langle u_n, g \rangle_{\mathfrak{h}} = \langle u_n, e \rangle_{\mathfrak{h}}.$$

We can hence directly apply Lemma 4.1 to find that

$$\begin{aligned} \int_0^T \||\nabla_{x_{[2]}}|^{\frac{1-r}{4}} \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} \langle u_n, g \rangle_{\mathfrak{h}}\|_{L^2}^2 dt \\ \leq C \|\langle \bar{u}_n, g(0) \rangle_{\mathfrak{h}}\|_{L^2}^2 + C \int_0^T \|\langle u_n, g \rangle_{\mathfrak{h}}\|_{L^2}^2 + C \int_0^T \|\langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} \langle u_n, e \rangle_{\mathfrak{h}}\|_{L^2}^2 dt. \end{aligned}$$

Summing over n and using that $(u_n(t))_n$ is an orthonormal basis for all t , we have

$$\begin{aligned} \sum_n \int_0^T \||\nabla_{x_{[2]}}|^{\frac{1-r}{4}} \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} \langle u_n, g \rangle_{\mathfrak{h}}\|_{L^2}^2 \\ = \int_0^T \int_{\mathbb{D}^2} \sum_n |\langle u_n, |\nabla_{x_{[2]}}|^{\frac{1-r}{4}} \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} g \rangle_{\mathfrak{h}}|^2 \\ = \int_0^T \||\nabla_{x_{[2]}}|^{\frac{1-r}{4}} \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} g\|_{L^2(\mathbb{D}^2; \mathfrak{h})}^2 dt, \end{aligned}$$

and similarly for the other terms. The conclusion follows. $\square$

As singular terms appear in the m -particle equation in all variables, we will need to combine the above with a mollification argument in the background variables. For $\varepsilon \in (0, 1]$ to be chosen below, let $M_\varepsilon(z) := \varepsilon^{-2d} M(z/\varepsilon)$ be a family of mollifiers where M is smooth, exponentially decaying, and normalized by $\int_{\mathbb{D}} M = 1$. For any $P \subset [m]$, denote by M_ε^P the convolution operator acting only in the variables $(z_j)_{j \in P}$,

$$M_\varepsilon^P h_m(z_{[m]}) := \int_{\mathbb{D}^{\#P}} \left(\prod_{j \in P} M_\varepsilon(z_j - z'_j) \right) h_m(z_{[m] \setminus P}, z'_P) dz'_P.$$

In these terms, we establish the following uniform-in- N a priori estimate.

Proposition 4.4. *Let $\alpha = 0$, let*

$$K \in L_{\text{loc}}^{2d/(d+2\gamma)}(\mathbb{R}^d), \quad \sup_{|x| \geq 1} |K(x)| < \infty,$$

for some

$$0 \leq \gamma < \gamma_* := \begin{cases} \frac{1}{16}(5d+4 - \sqrt{(5d+4)^2 - 32d}) & : d \geq 2, \\ \frac{1}{2}(\sqrt{5} - 2) & : d = 1. \end{cases}$$

Also assume that the mean-field solution f satisfies the regularity conditions (1.5), (1.6), and (1.9) for some $\delta \in (0, \frac{1}{2}]$. Then there is $\theta > 0$ such that for the choice $\varepsilon = N^{-\theta}$ we have for all $0 \leq m \leq N$,

$$\int_0^T \|\nabla_{x_{[2]}} |\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^2 \leq C(T+1)(m+1)^c,$$

for some constant c only depending on γ and some constant C further depending on δ and on assumptions on K, f .

Proof. For $\alpha = 0$, equation (3.2) can be reorganized as

$$\begin{aligned} & \left(\partial_t + v_1 \cdot \nabla_{x_1} + v_2 \cdot \nabla_{x_2} + \sum_{i=3}^m (v_i \cdot \nabla_{x_i} + K * f(x_i) \cdot \nabla_{v_i}) \right) ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \\ & = A_m^1 + A_m^2 - \sum_{i=1}^2 \text{div}_{v_i} \left((K * f)(x_i) (f^{\otimes m})^{1-\delta} \bar{C}_{N,m} \right), \end{aligned}$$

where A_m^1, A_m^2 are defined as in (3.2). Applying the mollification operator $M_\varepsilon^{[3,m]}$ to both sides of this equation, only acting in the variables $z_3, \dots, z_m$, we obtain

$$\begin{aligned} & \left(\partial_t + v_1 \cdot \nabla_{x_1} + v_2 \cdot \nabla_{x_2} + \sum_{i=3}^m (v_i \cdot \nabla_{x_i} + K * f(x_i) \cdot \nabla_{v_i}) \right) M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \\ & = R_{\varepsilon,m} + M_\varepsilon^{[3,m]} A_m^1 + M_\varepsilon^{[3,m]} A_m^2 - \sum_{i=1}^2 \text{div}_{v_i} \left((K * f)(x_i) M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \right), \quad (4.1) \end{aligned}$$

in terms of the commutator

$$R_{\varepsilon,m} := \left[\sum_{i=3}^m (v_i \cdot \nabla_{x_i} + K * f(x_i) \cdot \nabla_{v_i}), M_\varepsilon^{[3,m]} \right] (f^{\otimes m})^{1-\delta} \bar{C}_{N,m}.$$

For one particle variable, this commutator is explicitly

$$[v \cdot \nabla_x + K * f(x) \cdot \nabla_v, M_\varepsilon] h(z) = \int_{\mathbb{D}} \left((v - v') \cdot \nabla_x M_\varepsilon(z - z') \right)$$

$$+ (K * f(x) - K * f(x')) \cdot \nabla_v M_\varepsilon(z - z') \Big) h(z') dz'.$$

Recalling $M_\varepsilon(z) = \varepsilon^{-2d} M(z/\varepsilon)$ with M exponentially decaying, and using the Lipschitz bound on $K * f$, we deduce, uniformly in ε ,

$$\| [v \cdot \nabla_x + K * f(x) \cdot \nabla_v, M_\varepsilon] h \|_{L^2} \lesssim \|h\|_{L^2}.$$

Applying this to $R_{\varepsilon,m}$, together with the a priori L^2 -estimate of Lemma 2.2 and the boundedness of f , we get

$$\|R_{\varepsilon,m}\|_{L^2} \lesssim m.$$

Now applying Lemma 4.3 to equation (4.1) with $\mathfrak{h} = L^2(\mathbb{D}^{m-2})$ and with L_t given by mean-field transport in the variables $z_3, \dots, z_m$, using this bound on the commutator term and again the a priori L^2 -estimate of Lemma 2.2, we deduce for any $0 \leq r \leq 1$,

$$\begin{aligned} \int_0^T \| |\nabla_{x_{[2]}}|^{\frac{1-r}{4}} \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \|_{L^2}^2 \\ \lesssim T + 1 + \sum_{j=1}^2 \int_0^T \| \langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} A_m^j \|_{L^2}^2. \end{aligned} \quad (4.2)$$

It remains to estimate these norms of A_m^1 and A_m^2 , which we separately do in the following two steps.

Step 1: We control A_m^2 by showing that for all $0 \leq r \leq 1$,

$$\begin{aligned} \| \langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} A_m^2 \|_{L^2} &\lesssim 1 + (m+1) \varepsilon^r N^{\frac{2\gamma}{d}} \\ &\quad + (m+1) \| \langle \nabla_{x_{[2]}} \rangle^\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m+2]} ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2}) \|_{L^2} \\ &\quad + m \| \langle \nabla_{x_{[2]}} \rangle^\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \|_{L^2}. \end{aligned} \quad (4.3)$$

By definition of A_m^2 ,

$$\begin{aligned} \| \langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} A_m^2 \|_{L^2} \\ \lesssim (m+1) \| \langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}]) \|_{L^2} \\ + \sum_{i=1}^m \left\| \langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} \right. \\ \left. \times \left((f^{\otimes m})^{1-\delta} \int_{\mathbb{D}} K_f(z_*, z_i) \bar{C}_{N,m}(z_{[m] \setminus \{i\}}, z_*) f(z_*) dz_* \right) \right\|_{L^2}. \end{aligned} \quad (4.4)$$

Both right-hand side terms can be treated in the same way, so we only discuss the first one for example. As the mollification kernel M has exponential decay and satisfies $\int_{\mathbb{D}} M = 1$, we have for any function h and any $0 \leq r \leq 1$, uniformly in ε ,

$$\| \langle \nabla_x \rangle^{-r} \langle \nabla_v \rangle^{-1} h \|_{L^2} \lesssim \|M_\varepsilon * h\|_{L^2} + \varepsilon^r \|h\|_{L^2}.$$

Applying this in the first two variables gives a fully mollified contribution and an $O(\varepsilon^r)$ error,

$$\begin{aligned} \| \langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}]) \|_{L^2} \\ \lesssim \|M_\varepsilon^{[m]} ((f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}]) \|_{L^2} + \varepsilon^r \| (f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}] \|_{L^2}. \end{aligned} \quad (4.5)$$

For the first term, recalling the definition of the operator $K_f[\cdot]$, cf. (2.14), and using Hölder's inequality and the Sobolev embedding $H^\gamma \subset L^{2d/(d-2\gamma)}$, similarly as in Step 1 of the proof of Proposition 3.3, we can bound

$$\begin{aligned} & \|M_\varepsilon^{[m]}((f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}])\|_{L^2} \\ & \leq \left(\|K\|_{L^{2d/(d+2\gamma)}(B)} \|f^\delta\|_{L_x^\infty H_v^{3/2}} + (\|K\|_{L^\infty(B^c)} + \|K * f\|_{L^\infty}) \|f^\delta\|_{L_x^2 H_v^{3/2}} \right) \\ & \quad \times \|f^\delta \nabla_v \log f\|_{L_x^2 H_v^{3/2}} \|\langle \nabla_{x_{[m+1,m+2]}} \rangle^\gamma \langle \nabla_{v_{[m+1,m+2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[m]}((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2})\|_{L^2}. \end{aligned}$$

Thus, by the assumptions on K, f and the symmetry of $\bar{C}_{N,m+2}$,

$$\|M_\varepsilon^{[m]}((f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}])\|_{L^2} \lesssim \|\langle \nabla_{x_{[2]}} \rangle^\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m+2]}((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2})\|_{L^2}.$$

For the second term in (4.5), we can simply bound

$$\begin{aligned} & \|(f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}]\|_{L^2} \\ & \leq \left(\|K\|_{L^{2d/(d+2\gamma)}(B)} \|f^\delta\|_{L_x^\infty L_v^2} + (\|K\|_{L^\infty(B^c)} + \|K * f\|_{L^\infty}) \|f^\delta\|_{L^2} \right) \\ & \quad \times \|f^\delta \nabla_v \log f\|_{L^2} \left(\|(f^{\otimes m+1})^{1-\delta} \bar{C}_{N,m+2}\|_{L^2(\mathbb{D}; L_x^{2d/(d-2\gamma)} L_v^2(\mathbb{D}; L^2(\mathbb{D}^m)))} \right. \\ & \quad \left. + \|(f^{\otimes m+1})^{1-\delta} \bar{C}_{N,m+2}\|_{L^2} \right), \end{aligned}$$

and thus, by the assumptions on K, f , Jensen's inequality with weight f , and the mixed a priori estimates of Lemma 2.2,

$$\|(f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}]\|_{L^2} \lesssim \|\bar{C}_{N,m+2}\|_{L^{2d/(d-2\gamma)}(f^{\otimes 2}; L^2(\mathbb{D}^m))} \lesssim N^{\frac{2\gamma}{d}}.$$

Combining these estimates into (4.5), and arguing similarly for the second term in (4.4), the claim (4.3) follows.

Step 2: We control the remainder A_m^1 by showing that for all $\gamma \leq r \leq 1$,

$$\|\langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} A_m^1\|_{L^2} \lesssim m^2 \left(1 + N^{-\frac{1}{2}} \varepsilon^{-1-\gamma} + N^{\frac{2\gamma}{d}-1} \varepsilon^{-1-\frac{d}{2}} + N^{\frac{2\gamma}{d}-\frac{1}{2}} \right). \quad (4.6)$$

Recall $A_m^1 = (f^{\otimes m})^{1-\delta} R_{N,m}$, where $R_{N,m}$ is the remainder term defined in Lemma 2.3. The estimate (4.6) follows from a careful examination of all the terms entering the definition of $R_{N,m}$, using the assumptions on K, f and the mixed a priori estimates of Lemma 2.2. For shortness, we focus again on the three typical contributions $R_{N,m}^1, R_{N,m}^2, R_{N,m}^3$ in (3.6). For $R_{N,m}^1$, distinguishing the four cases $i, j \leq 2$, $i \leq 2 < j$, $j \leq 2 < i$, and $i, j > 2$, and commuting the velocity derivative with the weight, we get by symmetry

$$\|\langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]}((f^{\otimes m})^{1-\delta} R_{N,m}^1)\|_{L^2} \lesssim \sum_{j=1}^4 T_m^j,$$

in terms of

$$\begin{aligned} T_m^1 &:= m^{-\frac{1}{2}} N^{-\frac{1}{2}} \left\| \langle \nabla_{x_1} \rangle^{-r} \left((f^{\otimes m})^{1-\delta} K(x_1 - x_2) \bar{C}_{N,[2,m]} \right) \right\|_{L^2} \\ &\quad + m^{-\frac{1}{2}} N^{-\frac{1}{2}} \left\| \langle \nabla_{x_1} \rangle^{-r} \left(K(x_1 - x_2) \bar{C}_{N,[2,m]} \cdot \nabla_{v_2} (f^{\otimes m})^{1-\delta} \right) \right\|_{L^2}, \\ T_m^2 &:= m^{\frac{1}{2}} N^{-\frac{1}{2}} \left\| M_\varepsilon^{[3,m]} \left((f^{\otimes m})^{1-\delta} K(x_1 - x_3) \bar{C}_{N,[m] \setminus \{3\}} \right) \right\|_{L^2} \end{aligned}$$

$$\begin{aligned}
& +m^{\frac{1}{2}}N^{-\frac{1}{2}}\left\|M_{\varepsilon}^{[3,m]}\left(K(x_1-x_3)\bar{C}_{N,[m]\setminus\{3\}}\cdot\nabla_{v_1}(f^{\otimes m})^{1-\delta}\right)\right\|_{L^2}, \\
T_m^3 &:= m^{\frac{1}{2}}N^{-\frac{1}{2}}\left\|\langle\nabla_{x_1}\rangle^{-r}M_{\varepsilon}^{[3,m]}\operatorname{div}_{v_3}\left((f^{\otimes m})^{1-\delta}K(x_1-x_3)\bar{C}_{N,[m]\setminus\{1\}}\right)\right\|_{L^2} \\
& +m^{\frac{1}{2}}N^{-\frac{1}{2}}\left\|\langle\nabla_{x_1}\rangle^{-r}M_{\varepsilon}^{[3,m]}\left(K(x_1-x_3)\bar{C}_{N,[m]\setminus\{1\}}\cdot\nabla_{v_3}(f^{\otimes m})^{1-\delta}\right)\right\|_{L^2}, \\
T_m^4 &:= m^{\frac{3}{2}}N^{-\frac{1}{2}}\left\|M_{\varepsilon}^{[3,m]}\operatorname{div}_{v_4}\left((f^{\otimes m})^{1-\delta}K(x_3-x_4)\bar{C}_{N,[m]\setminus\{3\}}\right)\right\|_{L^2} \\
& +m^{\frac{3}{2}}N^{-\frac{1}{2}}\left\|M_{\varepsilon}^{[3,m]}\left(K(x_3-x_4)\bar{C}_{N,[m]\setminus\{3\}}\cdot\nabla_{v_4}(f^{\otimes m})^{1-\delta}\right)\right\|_{L^2}.
\end{aligned}$$

Note that the convolution with M_{ε} can be bounded as follows by Young's convolution inequality, for all $0 \leq q \leq \frac{d}{2}$,

$$\|M_{\varepsilon} * h\|_{L^2} \leq \|M_{\varepsilon}\|_{L_x^{d/(d-q)}L_v^1} \|h\|_{L_x^{2d/(d+2q)}L_v^2} \lesssim \varepsilon^{-q} \|h\|_{L_x^{2d/(d+2q)}L_v^2}. \quad (4.7)$$

Using this with $q = \gamma$, appealing to the Sobolev embedding $L_x^{2d/(d+2r)} \subset H_x^{-r}$, using the assumptions on K, f and the a priori estimates of Lemma 2.2, we get for all $r \geq \gamma$,

$$\|\langle\nabla_{x_{[2]}}\rangle^{-r}\langle\nabla_{v_{[2]}}\rangle^{-1}M_{\varepsilon}^{[3,m]}((f^{\otimes m})^{1-\delta}R_{N,m}^1)\|_{L^2} \lesssim m^{\frac{3}{2}}N^{-\frac{1}{2}}\varepsilon^{-1-\gamma}.$$

For $R_{N,m}^2$, distinguishing again the cases $i, j \leq 2$, $i \leq 2 < j$, $j \leq 2 < i$, and $i, j > 2$, we get by symmetry

$$\|\langle\nabla_{x_{[2]}}\rangle^{-r}\langle\nabla_{v_{[2]}}\rangle^{-1}M_{\varepsilon}^{[3,m]}((f^{\otimes m})^{1-\delta}R_{N,m}^2)\|_{L^2} \lesssim \sum_{j=1}^4 S_m^j,$$

in terms of

$$\begin{aligned}
S_m^1 &:= N^{-1}\left\|\langle\nabla_{x_1}\rangle^{-r}\left((f^{\otimes m})^{1-\delta}K(x_1-x_2)\bar{C}_{N,m}\right)\right\|_{L^2} \\
& +N^{-1}\left\|\langle\nabla_{x_1}\rangle^{-r}\left(K(x_1-x_2)\bar{C}_{N,m}\cdot\nabla_{v_2}(f^{\otimes m})^{1-\delta}\right)\right\|_{L^2}, \\
S_m^2 &:= mN^{-1}\left\|M_{\varepsilon}^{[3,m]}\left((f^{\otimes m})^{1-\delta}K(x_1-x_3)\bar{C}_{N,m}\right)\right\|_{L^2} \\
& +mN^{-1}\left\|M_{\varepsilon}^{[3,m]}\left(K(x_1-x_3)\bar{C}_{N,m}\cdot\nabla_{v_1}(f^{\otimes m})^{1-\delta}\right)\right\|_{L^2}, \\
S_m^3 &:= mN^{-1}\left\|\langle\nabla_{x_1}\rangle^{-r}M_{\varepsilon}^{[3,m]}\operatorname{div}_{v_3}\left((f^{\otimes m})^{1-\delta}K(x_1-x_3)\bar{C}_{N,m}\right)\right\|_{L^2} \\
& +mN^{-1}\left\|\langle\nabla_{x_1}\rangle^{-r}M_{\varepsilon}^{[3,m]}\left(K(x_1-x_3)\bar{C}_{N,m}\cdot\nabla_{v_3}(f^{\otimes m})^{1-\delta}\right)\right\|_{L^2}, \\
S_m^4 &:= m^2N^{-1}\left\|M_{\varepsilon}^{[3,m]}\operatorname{div}_{v_4}\left((f^{\otimes m})^{1-\delta}K(x_3-x_4)\bar{C}_{N,m}\right)\right\|_{L^2} \\
& +m^2N^{-1}\left\|M_{\varepsilon}^{[3,m]}\left(K(x_3-x_4)\bar{C}_{N,m}\cdot\nabla_{v_4}(f^{\otimes m})^{1-\delta}\right)\right\|_{L^2}.
\end{aligned}$$

Using again (4.7) and the Sobolev embedding $L_x^{2d/(d+2r)} \subset H_x^{-r}$, we can bound for all $r \geq \gamma$ and $0 \leq q \leq \frac{d}{2}$,

$$\begin{aligned}
& \|\langle\nabla_{x_{[2]}}\rangle^{-r}\langle\nabla_{v_{[2]}}\rangle^{-1}M_{\varepsilon}^{[3,m]}((f^{\otimes m})^{1-\delta}R_{N,m}^2)\|_{L^2} \\
& \lesssim N^{-1}\left\|(f^{\otimes m})^{1-\delta}K(x_1-x_2)\bar{C}_{N,m}\right\|_{L_{x_2}^2L_{x_1}^{2d/(d+2r)}L_{x_1 \neq x_2}^2}
\end{aligned}$$

$$\begin{aligned}
& + mN^{-1} \left\| K(x_1 - x_2) \bar{C}_{N,m} \cdot \nabla_{v_2} (f^{\otimes m})^{1-\delta} \right\|_{L_{x_2}^2 L_{x_1}^{2d/(d+2r)} L_{\neq x_1, x_2}^2} \\
& + m^2 N^{-1} \varepsilon^{-1-q} \left\| (f^{\otimes m})^{1-\delta} K(x_1 - x_2) \bar{C}_{N,m} \right\|_{L_{x_2}^2 L_{x_1}^{2d/(d+2q)} L_{\neq x_1, x_2}^2} \\
& + m^2 N^{-1} \varepsilon^{-q} \left\| K(x_1 - x_2) \bar{C}_{N,m} \cdot \nabla_{v_2} (f^{\otimes m})^{1-\delta} \right\|_{L_{x_2}^2 L_{x_1}^{2d/(d+2q)} L_{\neq x_1, x_2}^2},
\end{aligned}$$

and thus, using the assumptions on K, f , Jensen's inequality with weight f , and the mixed a priori estimates of Lemma 2.2,

$$\begin{aligned}
& \|\langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} R_{N,m}^2) \|_{L^2} \\
& \lesssim mN^{-1} \|\bar{C}_{N,m}\|_{L^\infty(\mathbb{D}^2; L^2(f^{\otimes m-2}))} + m^2 N^{-1} \varepsilon^{-1-q} \|\bar{C}_{N,m}\|_{L^{d/(q-\gamma)}(f^{\otimes 2}; L^2(f^{\otimes m-2}))} \\
& \lesssim 1 + m^2 \varepsilon^{-1-q} N^{-\frac{2}{d}(q-\gamma)}.
\end{aligned}$$

Choosing $q = \frac{d}{2}$ gives

$$\|\langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} R_{N,m}^2) \|_{L^2} \lesssim 1 + m^2 \varepsilon^{-1-\frac{d}{2}} N^{\frac{2\gamma}{d}-1}.$$

Finally, for $R_{N,m}^3$, we can simply bound

$$\begin{aligned}
& \|\langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} R_{N,m}^3) \|_{L^2} \\
& \lesssim m^{\frac{3}{2}} N^{-\frac{1}{2}} \left\| (f^{\otimes m})^{1-\delta} \int_{\mathbb{D}} K_f(z_*, z_1) \bar{C}_{N, [m] \cup \{*\}} f(z_*) dz_* \right\|_{L^2}.
\end{aligned}$$

Thus, using again the assumptions on K, f , Jensen's inequality with weight f , and the mixed a priori estimates of Lemma 2.2, we obtain that

$$\begin{aligned}
& \|\langle \nabla_{x_{[2]}} \rangle^{-r} \langle \nabla_{v_{[2]}} \rangle^{-1} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} R_{N,m}^3) \|_{L^2} \\
& \lesssim m^{\frac{3}{2}} N^{-\frac{1}{2}} \|\bar{C}_{N,m+1}\|_{L^{2d/(d-2\gamma)}(f^{\otimes 2}; L^2(f^{\otimes m-1}))} \lesssim m^{\frac{3}{2}} N^{\frac{2\gamma}{d}-\frac{1}{2}}.
\end{aligned}$$

The other terms in $R_{N,m}$ are handled in the same way, proving (4.6).

Step 3: Conclusion.

Inserting the estimates of Steps 1 and 2 into (4.2), we deduce for all $\gamma \leq r \leq 1$,

$$\begin{aligned}
& \left(\int_0^T \|\nabla_{x_{[2]}}\|^{\frac{1-r}{4}} \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \|_{L^2}^2 dt \right)^{\frac{1}{2}} \\
& \lesssim (T+1)^{\frac{1}{2}} (m+1)^2 \left(1 + \varepsilon^r N^{\frac{2\gamma}{d}} + N^{-\frac{1}{2}} \varepsilon^{-1-\gamma} + N^{\frac{2\gamma}{d}-1} \varepsilon^{-1-\frac{d}{2}} + N^{\frac{2\gamma}{d}-\frac{1}{2}} \right) \\
& + (m+1) \left(\int_0^T \|\langle \nabla_{x_{[2]}} \rangle^\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m+2]} ((f^{\otimes m+2})^{1-\delta} \bar{C}_{N,m+2}) \|_{L^2}^2 dt \right)^{\frac{1}{2}} \\
& + m \left(\int_0^T \|\langle \nabla_{x_{[2]}} \rangle^\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \|_{L^2}^2 dt \right)^{\frac{1}{2}}. \quad (4.8)
\end{aligned}$$

Provided

$$\gamma \leq \frac{d}{4} \wedge 1 \quad \text{and} \quad \frac{2\gamma}{rd} \leq \frac{1}{2(1+\gamma)} \wedge \frac{1-2\gamma/d}{1+d/2}, \quad (4.9)$$

we can choose $\varepsilon = N^{-\theta}$ for some $\theta > 0$ so that the first right-hand side term is uniformly bounded by $(T+1)(m+1)^2$ uniformly in N . It remains to absorb the last two terms. To this aim, we assume in addition

$$\gamma < \frac{1-r}{4}. \quad (4.10)$$

By interpolation and by the a priori L^2 -estimate of Lemma 2.2, we can then bound

$$\begin{aligned} & \left(\int_0^T \|\langle \nabla_{x_{[2]}} \rangle^\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^2 \right)^{\frac{1}{2}} \\ & \leq \left(\int_0^T \|\langle \nabla_{x_{[2]}} \rangle^{\frac{1-r}{4}} \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^{2\beta} \right)^{\frac{1}{2}} \\ & \leq T^{\frac{1}{2}(1-\beta)} \left(\int_0^T \|\langle \nabla_{x_{[2]}} \rangle^{\frac{1-r}{4}} \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^2 \right)^{\frac{\beta}{2}}, \end{aligned}$$

with $\beta := \frac{4\gamma}{1-r} < 1$. Using this to bound the last two terms in (4.8), and setting for abbreviation

$$X_{N,m} := \left(\int_0^T \|\langle \nabla_{x_{[2]}} \rangle^\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} M_\varepsilon^{[3,m]} ((f^{\otimes m})^{1-\delta} \bar{C}_{N,m})\|_{L^2}^2 \right)^{\frac{1}{2}},$$

we obtain

$$X_{N,m} \lesssim (T+1)^{\frac{1}{2}}(m+1)^2 + (m+1)T^{\frac{1}{2}(1-\beta)}X_{N,m+2}^\beta + mT^{\frac{1}{2}(1-\beta)}X_{N,m}^\beta.$$

Hence, by Young's inequality,

$$X_{N,m} \leq C_\beta(T+1)^{\frac{1}{2}}(m+1)^{C_\beta} + \frac{1}{4}X_{N,m+2} + \frac{1}{2}X_{N,m}.$$

Absorbing the last term and iterating the estimate for $m \leq N$, we are led to

$$X_{N,m} \leq C_\beta(T+1)^{\frac{1}{2}}(m+1)^{C_\beta},$$

and the conclusion follows. Note that the choice $\gamma < \gamma_*$ precisely ensures that (4.9) and (4.10) both hold for some $0 \leq r < 1$. $\square$

5. MEAN-FIELD LIMIT

The kinetic regularity estimates established in the previous two sections allow us to pass to the limit in the hierarchy for dual correlations. To conclude the mean-field limit, it then remains to show that the resulting limiting hierarchy uniquely determines the extracted subsequential limit.

5.1. Weak compactness. We start by passing to the limit in the hierarchy for dual correlations, using the kinetic regularity estimates established in Propositions 3.3 and 4.4. For $K \in L_{\text{loc}}^{2d/(d+2\gamma)}(\mathbb{R}^d)$, we emphasize that the interaction term $K_f[\bar{C}_{m+2}]$ in the limiting hierarchy (5.3) is not a priori well-defined: it only makes sense thanks to (5.2).

Lemma 5.1. *Assume the hypotheses of Proposition 3.3 if $\alpha > 0$, those of Proposition 4.4 if $\alpha = 0$, and further assume that the Vlasov solution f satisfies the positivity condition (1.6). Up to a subsequence, as $N \uparrow \infty$, we have for all $m \geq 0$,*

$$\bar{C}_{N,m} \xrightarrow{*} \bar{C}_m \quad \text{weakly-}^* \text{ in } L^\infty(0, T; L^2(f^{\otimes m})),$$

for some limit $(\bar{C}_m)_m$ that satisfies the following estimates:

$$\sup_{[0,T]} \|\bar{C}_m\|_{L^2(f^{\otimes m})} \leq C, \quad (5.1)$$

as well as

$$\begin{aligned} \int_0^T \| |\nabla_{x_1}|^\gamma ((f^{\otimes m})^{1-\delta} \bar{C}_m) \|_{L^2}^2 &\leq C(T+1)(m+1)^c, & \text{if } \alpha > 0, \\ \int_0^T \| |\nabla_{x_1}|^\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} ((f^{\otimes m})^{1-\delta} \bar{C}_m) \|_{L^2}^2 &\leq C(T+1)(m+1)^c, & \text{if } \alpha = 0. \end{aligned} \quad (5.2)$$

Moreover, the limit satisfies the following limiting hierarchy in the weak sense on $[0, T]$: for all $m \geq 0$,

$$\partial_t \bar{C}_m + L_f^m \bar{C}_m = \sqrt{(m+1)(m+2)} K_f[\bar{C}_{m+2}], \quad (5.3)$$

where $L_f^m, K_f[\cdot]$ are defined in Lemma 2.3, with final data

$$\bar{C}_m|_{t=T} = \mathbb{1}_{m=0}. \quad (5.4)$$

Proof. From the a priori L^2 -estimates of Lemma 2.2, weak compactness ensures that, up to a subsequence as $N \uparrow \infty$, we indeed have for all $m \geq 0$,

$$\bar{C}_{N,m} \xrightarrow{*} \bar{C}_m \quad \text{in } L^\infty(0, T; L^2(f^{\otimes m})), \quad (5.5)$$

for some limit $(\bar{C}_m)_m$. As all our estimates hold with weight f , we shall first show that the limit satisfies the limiting hierarchy (5.3)–(5.4) in the following *weighted* weak sense: for all $m \geq 0$ and $H_m \in C_c^\infty((0, T] \times \mathbb{D}^m)$,

$$\begin{aligned} \int_0^T \int_{\mathbb{D}^m} \bar{C}_m f^{\otimes m} (\partial_t + \hat{L}_f^m) H_m \\ + \sqrt{(m+1)(m+2)} \int_0^T \int_{\mathbb{D}^{m+2}} H_m K_f[\bar{C}_{m+2}] f^{\otimes m} = H_0(T) \mathbb{1}_{m=0}, \end{aligned} \quad (5.6)$$

where $\hat{L}_f^m$ stands for the following adjoint operator on the m -particle space,

$$\begin{aligned} \hat{L}_f^m &:= \sum_{j=1}^m \text{Id}^{\otimes j-1} \otimes \hat{L}_f \otimes \text{Id}^{\otimes m-j}, \\ \hat{L}_f h &:= v \cdot \nabla_x h - \alpha \Delta_v h + (K * f)(x) \cdot \nabla_v h - 2\alpha \nabla_v \log f(z) \cdot \nabla_v h \\ &\quad + \int_{\mathbb{D}} K_f(z, z_*) h(z_*) f(z_*) dz_*. \end{aligned} \quad (5.7)$$

Using that f satisfies the Vlasov equation and using the positivity requirement (1.6) for f , this easily implies that $(\bar{C}_m)_m$ satisfies (5.3)–(5.4) in the usual weak sense.

We split the proof of (5.6) into three steps: we start by showing that the remainder term $R_{N,m}$ in the hierarchy of equations for dual correlations derived in Lemma 2.3 tends to 0 in the sense of distributions, and then we conclude the proof, separately considering the cases $\alpha > 0$ and $\alpha = 0$.

Step 1: Remainder estimate: for all $m \geq 0$,

$$f^{\otimes m} R_{N,m} \rightarrow 0 \quad \text{in } \mathcal{D}'([0, T] \times \mathbb{D}^m). \quad (5.8)$$

For shortness, we focus again on the three typical terms $R_{N,m}^1, R_{N,m}^2, R_{N,m}^3$ considered in (3.6). Integrating by parts and using the assumptions on K, f , we easily find for any smooth symmetric test function $H_m \in C_c^\infty([0, T] \times \mathbb{D}^m)$,

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{D}^m} H_m R_{N,m}^1 f^{\otimes m} \right| &\lesssim_m N^{-\frac{1}{2}} \int_0^T \|\bar{C}_{N,m-1}\|_{L^2(f^{\otimes m-1})}, \\ \left| \int_0^T \int_{\mathbb{D}^m} H_m R_{N,m}^2 f^{\otimes m} \right| &\lesssim_m N^{-1} \int_0^T \|\bar{C}_{N,m}\|_{L^{2d/(d-2\gamma)}(f^{\otimes 2}; L^2(f^{\otimes m-2}))}, \\ \left| \int_0^T \int_{\mathbb{D}^m} H_m R_{N,m}^3 f^{\otimes m} \right| &\lesssim_m N^{-\frac{1}{2}} \int_0^T \|\bar{C}_{N,m+1}\|_{L^{2d/(d-2\gamma)}(f^{\otimes 2}; L^2(f^{\otimes m-1}))}. \end{aligned}$$

By the mixed a priori estimates of Lemma 2.2, all three right-hand sides tend to 0 as $\gamma < \frac{d}{4}$, thus proving (5.8).

Step 2: Case $\alpha > 0$.

Proposition 3.3 further implies, along the extracted subsequence (5.5), for all $m \geq 0$,

$$(f^{\otimes m})^{1-\delta} \bar{C}_{N,m} \rightharpoonup (f^{\otimes m})^{1-\delta} \bar{C}_m \quad \text{in } L^2(0, T; H_x^\gamma L_v^2(\mathbb{D}; L^2(\mathbb{D}^{m-1}))).$$

By definition of the operator $K_f[\cdot]$, cf. (2.14), and by the assumptions on K, f , this entails for all $m \geq 0$,

$$(f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}] \rightharpoonup (f^{\otimes m})^{1-\delta} K_f[\bar{C}_{m+2}] \quad \text{in } L^2(0, T; L^2(\mathbb{D}^m)). \quad (5.9)$$

This allows us to pass to the limit in the hierarchy of equations for dual correlations. Indeed, as f satisfies the Vlasov equation, the weak formulation of the hierarchy derived in Lemma 2.3 can be written as follows, for all $0 \leq m \leq N$ and $H_m \in C_c^\infty((0, T] \times \mathbb{D}^m)$,

$$\begin{aligned} \int_0^T \int_{\mathbb{D}^m} \bar{C}_{N,m} f^{\otimes m} (\partial_t + \hat{L}_f^m) H_m + \int_0^T \int_{\mathbb{D}^m} H_m R_{N,m} f^{\otimes m} \\ + \sqrt{(m+1)(m+2)} \left(\frac{(N-m)(N-m-1)}{(N-1)^2} \right)^{\frac{1}{2}} \int_0^T \int_{\mathbb{D}^m} H_m K_f[\bar{C}_{N,m+2}] f^{\otimes m} \\ = \int H_m(T) \bar{C}_{N,m}^T f(T)^{\otimes m}. \end{aligned} \quad (5.10)$$

Using (5.5), (5.8), (5.9), and recalling that the choice (2.2) of final data ensures

$$\bar{C}_{N,m}^T \rightarrow \mathbf{1}_{m=0} \quad \text{in } L^2(\mathbb{D}^m),$$

we conclude that the limit $(\bar{C}_m)_m$ satisfies the weighted weak formulation (5.6) of the limiting dual hierarchy (5.3)–(5.4). Finally, the kinetic regularity estimate (5.2) follows from Proposition 3.3 by lower semicontinuity.

Step 3: Case $\alpha = 0$.

As $M_\varepsilon \rightarrow \text{Id}$ in the strong operator topology on $L^2(\mathbb{D})$ as $\varepsilon \downarrow 0$, it follows from (5.5) and from the boundedness of f that, along the extracted subsequence,

$$M_\varepsilon^{[3,m]}((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \xrightarrow{*} (f^{\otimes m})^{1-\delta} \bar{C}_m \quad \text{in } L^\infty(0, T; L^2(\mathbb{D}^m)),$$

and Proposition 4.4 then further implies

$$M_\varepsilon^{[3,m]}((f^{\otimes m})^{1-\delta} \bar{C}_{N,m}) \rightharpoonup (f^{\otimes m})^{1-\delta} \bar{C}_m \quad \text{in } L^2(0, T; H_x^\gamma H_v^{-3/2}(\mathbb{D}^2; L^2(\mathbb{D}^{m-2}))). \quad (5.11)$$

By definition of the operator $K_f[\cdot]$, cf. (2.14), and by the assumptions on K, f , this entails for all $m \geq 0$,

$$M_\varepsilon^{[m]}((f^{\otimes m})^{1-\delta} K_f[\bar{C}_{N,m+2}]) \rightharpoonup (f^{\otimes m})^{1-\delta} K_f[\bar{C}_{m+2}] \quad \text{in } L^2(0, T; L^2(f^{\otimes m})). \quad (5.12)$$

Together with (5.5), (5.8), and with the choice (2.2) of final data, this allows to pass to the limit in (5.10) with test function H_m replaced by $M_\varepsilon^{[m]} H_m$, which then implies that the limit $(\bar{C}_m)_m$ satisfies the weighted weak formulation (5.6) of the limiting hierarchy (5.3)–(5.4). Finally, the kinetic regularity estimate (5.2) follows from Proposition 4.4 together with (5.11) by lower semicontinuity. $\square$

5.2. Uniqueness for the limiting dual hierarchy. A direct uniqueness argument for the limiting dual hierarchy (5.3) would be straightforward if $\bar{C}_m$ were controlled in the space $L^{2d/(d-2\gamma)}(f^{\otimes m})$. The kinetic regularity estimate (5.2), however, only yields a fixed gain of spatial regularity in one particle variable at a time. Combining these one-particle estimates into an isotropic Sobolev estimate in the m -particle space would only give an $O(\frac{1}{m})$ gain of spatial integrability for $\bar{C}_m$, which degenerates as $m \uparrow \infty$ and is thus a priori insufficient to control the singular interaction operator uniformly along the hierarchy.

This is reminiscent of the hierarchical derivation of the quantum mean-field limit for bosons with Coulomb interactions in [1, 11]. The central ingredient in [11] is the propagation of an iterative Sobolev norm controlling one derivative in every particle variable simultaneously. Such tensorized regularity can be derived from energy estimates based on the conservation of the many-body Hamiltonian and is strong enough to close a direct uniqueness argument for the limiting quantum hierarchy. No analogous tensorized estimate appears to be available in the present classical kinetic setting, where regularity is produced by hypoellipticity or velocity averaging.

We instead exploit the explicit triangular structure of the limiting hierarchy through its Duhamel expansion. In each iterated term, every occurrence of the singular interaction kernel appears to be integrated in time along the linearized mean-field propagator. Kinetic regularization can therefore be applied to this integrated kernel, while all remaining particle variables are propagated merely in L^2 . This replaces the tensorized regularity that is unavailable for the classical hierarchy. Note that for this uniqueness result the restriction on γ , cf. (5.13), is much weaker than for the uniform-in- N regularity estimates of the previous sections; see also Remark 5.3 below.

Proposition 5.2. *Let $\alpha \geq 0$, and let*

$$K \in L_{\text{loc}}^{2d/(d+2\gamma)}(\mathbb{R}^d), \quad \sup_{|x| \geq 1} |K(x)| < \infty,$$

for some

$$0 \leq \gamma \leq \begin{cases} 1/3 & : \alpha > 0, \\ 1/4 & : \alpha = 0. \end{cases} \quad (5.13)$$

Also assume that the mean-field solution f satisfies the regularity conditions (1.5), (1.6), together with (1.7)–(1.8) if $\alpha > 0$, and with (1.9) if $\alpha = 0$, for some $\delta \in (0, \frac{1}{2}]$. Let then $(C_m)_m$ satisfy the limiting dual hierarchy (5.3) in the weak sense on $[0, T]$,

$$\partial_t C_m + L_f^m C_m = \sqrt{(m+1)(m+2)} K_f[C_{m+2}], \quad (5.14)$$

with trivial final data

$$C_m|_{t=T} = 0 \quad \text{for all } m \geq 0,$$

with C_m symmetric, and with the following a priori estimates: for some $L \geq 1$,

$$\sup_{[0,T]} \|C_m\|_{L^2(f^{\otimes m})} \leq L, \quad (5.15)$$

as well as

$$\begin{aligned} \int_0^T \|\nabla_{x_1} |^\gamma ((f^{\otimes m})^{1-\delta} C_m)\|_{L^2}^2 &\leq L^{m+1}, & \text{if } \alpha > 0, \\ \int_0^T \|\nabla_{x_1} |^\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} ((f^{\otimes m})^{1-\delta} C_m)\|_{L^2}^2 &\leq L^{m+1}, & \text{if } \alpha = 0. \end{aligned} \quad (5.16)$$

Then

$$C_m \equiv 0 \quad \text{on } [0, T], \quad \text{for all } m \geq 0.$$

Remark 5.3. Using sharper kinetic regularity estimates in the proof of Lemma 5.5 below would extend the admissible range of γ in this uniqueness result to

$$0 \leq \gamma \leq \begin{cases} 5/6 & : \alpha > 0, \\ 1/2 & : \alpha = 0, \end{cases} \quad (5.17)$$

up to strengthening the regularity requirements on f . We do not pursue this refinement here, since the regularity estimates of Propositions 3.3 and 4.4 impose substantially stronger restrictions on γ in any case.

Before we proceed to the proof, we start with two preliminary results. First, we construct the (non-autonomous) propagator $(B_{t,s})_{0 \leq s \leq t \leq T}$ generated by the operator $\hat{L}_f$ dual to L_f , cf. (5.7). Some care is needed to get estimates for this propagator in $L^2(f^{2\delta})$ instead of $L^2(f)$, which will indeed be needed later on to allow pairing with the regularity estimates (5.16) carrying the weight $f^{1-\delta}$.

Lemma 5.4. Assume the hypotheses of Proposition 5.2. For all $0 \leq s \leq t \leq T$, there is a bounded operator $B_{t,s} : L^2(f_s^{2\delta}) \rightarrow L^2(f_t^{2\delta})$ with

$$\|B_{t,s}\|_{\mathcal{L}(L^2(f_s^{2\delta}), L^2(f_t^{2\delta}))} \leq e^{C(t-s)}, \quad (5.18)$$

with the semigroup property

$$B_{t,r} B_{r,s} = B_{t,s}, \quad 0 \leq s \leq r \leq t \leq T, \quad (5.19)$$

such that in the weak sense

$$\begin{cases} (\partial_t + \hat{L}_{f_t}) B_{t,s} = 0, & 0 \leq s \leq t \leq T, \\ B_{s,s} = \text{Id}. \end{cases} \quad (5.20)$$

Proof. By definition (5.7), we can decompose the operator as

$$\hat{L}_f := \hat{L}_f^0 + A_f, \quad (5.21)$$

in terms of

$$\begin{aligned} \hat{L}_f^0 h &:= v \cdot \nabla_x h - \alpha \Delta_v h + (K * f)(x) \cdot \nabla_v h - 2\alpha \nabla_v \log f(z) \cdot \nabla_v h, \\ A_f h &:= \int_{\mathbb{D}} K_f(z, z_*) h(z_*) f(z_*) dz_*. \end{aligned}$$

We argue by a perturbation argument, starting from the propagator $(P_{t,s})_{0 \leq s \leq t \leq T}$ generated by $\hat{L}_f^0$. We split the proof into three steps.

Step 1: Construction of $(P_{t,s})_{0 \leq s \leq t \leq T}$.

To construct this propagator, we appeal to the underlying Markov process. Since by

assumption $K * f \in L^1(0, T; W^{1,\infty}(\mathbb{R}^d))$, the time-inhomogeneous stochastic differential equation

$$dX_t = V_t dt, \quad dV_t = (K * f_t)(X_t) dt + \sqrt{2\alpha} dB_t \quad (5.22)$$

is well-posed. We take initial data $Z_0 = (X_0, V_0)$ with law f_0 . Since f solves the corresponding Vlasov equation, uniqueness ensures that Z_t has law f_t for all $0 \leq t \leq T$. We then define the backward transition operator, for all $0 \leq s \leq t \leq T$,

$$(P_{t,s}\varphi)(z) := \mathbb{E}[\varphi(Z_s) | Z_t = z], \quad (5.23)$$

which is defined f_t -almost everywhere. The Markov property gives

$$P_{t,r}P_{r,s} = P_{t,s}, \quad 0 \leq s \leq r \leq t \leq T. \quad (5.24)$$

Moreover, Jensen's inequality yields

$$\|P_{t,s}\|_{\mathcal{L}(L^2(f_s), L^2(f_t))} \leq 1, \quad 0 \leq s \leq t \leq T. \quad (5.25)$$

By duality with the forward transition kernel, $(P_{t,s})_{0 \leq s \leq t \leq T}$ satisfies in the weak sense

$$\begin{cases} (\partial_t + \hat{L}_{f_t}^0)P_{t,s} = 0, & 0 \leq s \leq t \leq T, \\ P_{s,s} = \text{Id}. \end{cases} \quad (5.26)$$

Step 2: Proof that

$$\|P_{t,s}\|_{\mathcal{L}(L^2(f_s^{2\delta}), L^2(f_t^{2\delta}))} \leq e^{C(t-s)}, \quad 0 \leq s \leq t \leq T. \quad (5.27)$$

Since the boundedness of f ensures $L^2(f^{2\delta}) \subset L^2(f)$, the operator $P_{t,s}$ is well-defined on $L^2(f_s^{2\delta})$, and it remains to estimate its norm. Given $0 \leq s \leq T$ and $h_s \in L^2(f_s)$, let $h_t := P_{t,s}h_s$ for $s \leq t \leq T$.

We start with the diffusive case $\alpha > 0$. Smuggling the weight f^δ and using the Vlasov equation for f , the equation (5.26) for $h_t = P_{t,s}h_s$ yields

$$\begin{aligned} & \left(\partial_t + v \cdot \nabla_x - \alpha \Delta_v + (K * f)(x) \cdot \nabla_v \right) f^\delta h \\ &= 2\alpha(1 - \delta) \nabla_v \log f \cdot \nabla_v (f^\delta h) - \alpha\delta(1 - \delta) |\nabla_v \log f|^2 f^\delta h, \end{aligned}$$

from which we can deduce the energy identity

$$\partial_t \|f^\delta h\|_{L^2}^2 + 2\alpha \|\nabla_v (f^\delta h)\|_{L^2}^2 + 2\alpha(1 - \delta) \int_{\mathbb{D}} (\Delta_v \log f + \delta |\nabla_v \log f|^2) |f^\delta h|^2 = 0. \quad (5.28)$$

By the assumption (1.8) on f and by Gronwall's inequality, we deduce for all $s \leq t \leq T$,

$$\|f_t^\delta h_t\|_{L^2} \leq e^{C(t-s)} \|f_s^\delta h_s\|_{L^2},$$

thus proving (5.27).

We turn to the non-diffusive case $\alpha = 0$. As in this setting the characteristic flow associated with (5.22),

$$\dot{X}_t = V_t, \quad \dot{V}_t = (K * f_t)(X_t),$$

is volume-preserving and also transports f , we deduce for all $0 \leq s \leq t \leq T$,

$$\|f_t^\delta P_{t,s}h_s\|_{L^2} = \|f_s^\delta h_s\|_{L^2},$$

which again proves (5.27).

Step 3: Conclusion.

It remains to construct the propagator $(B_{t,s})_{0 \leq s \leq t \leq T}$ by perturbation. To this aim, first note that the operator A_f in (5.21) is bounded: indeed,

$$\begin{aligned} \|A_f h\|_{L^2(f^{2\delta})} &\leq \left(\|K\|_{L^1(B)} \|f^{1-\delta}\|_{L_x^\infty L_v^2} \|f^\delta \nabla_v \log f\|_{L_x^\infty L_v^2} \right. \\ &\quad \left. + (\|K\|_{L^\infty(B^c)} + \|K * f\|_{L^\infty}) \|f^{1-\delta}\|_{L^2} \|f^\delta \nabla_v \log f\|_{L^2} \right) \|h\|_{L^2(f^{2\delta})}, \end{aligned} \quad (5.29)$$

hence, by the assumptions on K, f , on $[0, T]$,

$$\|A_f\|_{\mathcal{L}(L^2(f^{2\delta}))} \leq C. \quad (5.30)$$

We can now construct the full propagator by the convergent Dyson series

$$B_{t,s} = P_{t,s} + \sum_{n=1}^{\infty} (-1)^n \int_{s < r_n < \dots < r_1 < t} P_{t,r_1} A_{f_{r_1}} P_{r_1,r_2} \dots A_{f_{r_n}} P_{r_n,s} dr_1 \dots dr_n.$$

Indeed, by (5.27) and (5.30), the norm of the n th term is bounded by

$$e^{C(t-s)} \frac{(C(t-s))^n}{n!},$$

thus proving (5.18) after summation. The semigroup property (5.19) is easily checked from the above series using (5.24). The series also implies the Duhamel formula

$$B_{t,s} = P_{t,s} - \int_s^t P_{t,r} A_{f_r} B_{r,s} dr.$$

Since $P_{t,s}$ satisfies (5.26), this entails that $B_{t,s}$ satisfies (5.20) in the weak sense. $\square$

Next, we appeal to kinetic regularization effects to establish the following key estimate on the time integral of the singular interaction along the linearized mean-field propagator.

Lemma 5.5. *Assume the hypotheses of Proposition 5.2. For all $0 \leq s \leq t \leq T$,*

$$\left\| \int_s^t B_{t,r}^{\otimes 2} K_{f_r} dr \right\|_{L^2((f_t^{\otimes 2})^{2\delta})} \leq C e^{C(t-s)} \sqrt{t-s}. \quad (5.31)$$

Proof. Since K_f needs not belong to the weighted space $L^2((f^{\otimes 2})^{2\delta})$, we first regularize the singular part of K_f and then show that the estimate is uniform in the regularization. Applying the same argument to the difference of two regularizations gives convergence and defines the left-hand side. Let $s = 0$ without loss of generality, and set for abbreviation

$$S_t(z_1, z_2) := \int_0^t B_{t,r}^{\otimes 2} K_{f_r}(z_1, z_2) dr,$$

viewed as an element of $C_w(0, T; L^2((f^{\otimes 2})^{2\delta}))$ under regularization of K , cf. Lemma 5.4. By definition of the propagator, it satisfies in the weak sense,

$$(\partial_t + \hat{L}_f^2) S = K_f, \quad S|_{t=0} = 0. \quad (5.32)$$

Applying the energy inequality (5.28) in each variable, we obtain

$$\partial_t \|(f^{\otimes 2})^\delta S\|_{L^2}^2 + 2\alpha(1-\delta) \sum_{j=1}^2 \int_{\mathbb{D}^2} \left(\Delta_v \log f(z_j) + \delta |\nabla_v \log f(z_j)|^2 \right) |(f^{\otimes 2})^\delta S|^2$$

$$\leq 2 \int_{\mathbb{D}^2} SK_f(f^{\otimes 2})^{2\delta} - 2 \int_{\mathbb{D}^2} SA_f^2(S)(f^{\otimes 2})^{2\delta},$$

in terms of

$$A_f^2(S) := \int_{\mathbb{D}} K_f(z_1, z_*) S(z_*, z_2) f(z_*) dz_* + \int_{\mathbb{D}} K_f(z_2, z_*) S(z_1, z_*) f(z_*) dz_*.$$

By the assumption (1.8) on f in case $\alpha > 0$, and by the operator bound (5.30) on A_f , we deduce

$$\partial_t \|(f^{\otimes 2})^\delta S\|_{L^2}^2 \leq 2 \int_{\mathbb{D}^2} SK_f(f^{\otimes 2})^{2\delta} + C \|(f^{\otimes 2})^\delta S\|_{L^2}^2,$$

and thus, by Gronwall's inequality,

$$\|(f_t^{\otimes 2})^\delta S_t\|_{L^2}^2 \leq 2e^{Ct} \int_0^t \left| \int_{\mathbb{D}^2} SK_f(f^{\otimes 2})^{2\delta} \right|. \quad (5.33)$$

In order to estimate the right-hand side, we appeal to kinetic regularization. We split the proof into two steps, separately considering the cases $\alpha > 0$ and $\alpha = 0$.

Step 1: Diffusive case $\alpha > 0$.

We do not apply hypoelliptic regularity directly to $(f^{\otimes 2})^\delta S$: instead, we use a slightly stronger velocity weight $(f^{\otimes 2})^\beta$ with $\beta = \frac{3}{2}\delta > \delta$. Smuggling this weight and using the equation for f , a direct computation allows to rewrite the equation (5.32) for S as

$$\partial_t ((f^{\otimes 2})^\beta S) + \sum_{j=1}^2 (v_j \cdot \nabla_{x_j} - \alpha \Delta_{v_j}) ((f^{\otimes 2})^\beta S) = E + \sum_{j=1}^2 \operatorname{div}_{v_j} (F_j), \quad (5.34)$$

with

$$\begin{aligned} E &:= (f^{\otimes 2})^\beta K_f - (f^{\otimes 2})^\beta A_f^2(S) \\ &\quad - \alpha(1 - \beta)(f^{\otimes 2})^\beta S \sum_{j=1}^2 \left(2\Delta_v \log f(z_j) + \beta |\nabla_v \log f(z_j)|^2 \right) \\ F_j &:= - \left(K * f(x_j) - 2\alpha(1 - \beta) \nabla_v \log f(z_j) \right) (f^{\otimes 2})^\beta S. \end{aligned}$$

Since $f^\beta = f^{\delta/2} f^\delta$, we can take advantage of the additional factor $f^{\delta/2}$ to bound the terms in the last two lines. By the assumptions on K, f , and recalling the bound (5.30) on A_f , we get

$$\|\langle \nabla_X \rangle^{-\gamma} E\|_{L^2} + \|F_j\|_{L^2} \lesssim 1 + \|(f^{\otimes 2})^\delta S\|_{L^2},$$

where we set for abbreviation $Z = (X, V) = (z_1, z_2)$. Hence, by the refined hypoelliptic estimate of Remark 3.2,

$$\int_0^t \|\langle \nabla_X \rangle^{(\frac{2}{3}-\gamma) \wedge \frac{1}{3}} ((f^{\otimes 2})^\beta S)\|_{L^2}^2 \lesssim t + (t+1) \sup_{[0,t]} \|(f^{\otimes 2})^\delta S\|_{L^2}^2.$$

By the assumptions on K, f , decomposing $(f^{\otimes 2})^{2\delta} = (f^{\otimes 2})^\beta (f^{\otimes 2})^{\delta/2}$, and noting that the restriction $\gamma \leq \frac{1}{3}$ ensures $(\frac{2}{3} - \gamma) \wedge \frac{1}{3} \geq \gamma$, we thus obtain

$$\int_0^t \left| \int_{\mathbb{D}^2} SK_f(f^{\otimes 2})^{2\delta} \right| \lesssim \int_0^t \|\langle \nabla_X \rangle^\gamma ((f^{\otimes 2})^\beta S)\|_{L^2} \lesssim t + \sqrt{t}(\sqrt{t} + 1) \sup_{[0,t]} \|(f^{\otimes 2})^\delta S\|_{L^2}.$$

Combined with the energy estimate (5.33), this leads us to

$$\|(f_t^{\otimes 2})^\delta S_t\|_{L^2}^2 \lesssim e^{Ct} \left(t + \sqrt{t} \sup_{[0,t]} \|(f^{\otimes 2})^\delta S\|_{L^2} \right),$$

and the conclusion (5.31) follows.

Step 2: Non-diffusive case $\alpha = 0$.

In this case, we can directly apply the averaging lemma to $(f^{\otimes 2})^\delta S$, without needing to modify the weight. Equation (5.34) for $\alpha = 0$, with β replaced by δ , reads

$$\partial_t((f^{\otimes 2})^\delta S) + \sum_{j=1}^2 v_j \cdot \nabla_{x_j}((f^{\otimes 2})^\delta S) = E + \sum_{j=1}^2 \operatorname{div}_{v_j} F_j,$$

with

$$\begin{aligned} E &:= (f^{\otimes 2})^\delta K_f - (f^{\otimes 2})^\delta A_f^2(S), \\ F_j &:= -K * f(x_j)(f^{\otimes 2})^\delta S. \end{aligned}$$

By the assumptions on K, f , we find

$$\|\langle \nabla_X \rangle^{-\gamma} E\|_{L^2} + \|F_j\|_{L^2} \lesssim 1 + \|(f^{\otimes 2})^\delta S\|_{L^2}.$$

Hence, by the refined averaging lemma of Remark 4.2,

$$\int_0^t \|\langle \nabla_X \rangle^{\frac{1-\gamma}{2} \wedge \frac{1}{4}} \langle \nabla_V \rangle^{-\frac{3}{2}} ((f^{\otimes 2})^\delta S)\|_{L^2}^2 ds \lesssim t + (t+1) \sup_{[0,t]} \|(f^{\otimes 2})^\delta S\|_{L^2}^2.$$

By the assumptions on K, f , noting that the restriction $\gamma \leq \frac{1}{4}$ yields $\frac{1-\gamma}{2} \wedge \frac{1}{4} \geq \gamma$, we deduce

$$\begin{aligned} \int_0^t \left| \int_{\mathbb{D}^2} S K_f (f^{\otimes 2})^{2\delta} \right| &\lesssim \int_0^t \|\langle \nabla_X \rangle^\gamma \langle \nabla_V \rangle^{-\frac{3}{2}} ((f^{\otimes 2})^\delta S)\|_{L^2} \\ &\lesssim t + \sqrt{t}(\sqrt{t} + 1) \sup_{[0,t]} \|(f^{\otimes 2})^\delta S\|_{L^2}. \end{aligned}$$

Combined with (5.33), this leads us to

$$\|(f_t^{\otimes 2})^\delta S_t\|_{L^2}^2 \lesssim e^{Ct} \left(t + \sqrt{t} \sup_{[0,t]} \|(f^{\otimes 2})^\delta S\|_{L^2} \right),$$

and the conclusion (5.31) follows. $\square$

We are now in a position to conclude the proof of Proposition 5.2. We proceed by an explicit examination of the Duhamel expansion.

Proof of Proposition 5.2. By the regularity estimate (5.16) and the assumptions on K, f , we find that the interaction term in the dual hierarchy (5.14) is well-defined, with

$$\|(f^{\otimes m})^{1-\delta} K_f[C_m]\|_{L^2} \leq \begin{cases} \|\langle \nabla_{x_1} \rangle^\gamma ((f^{\otimes m})^{1-\delta} C_m)\|_{L^2}, & \text{if } \alpha > 0, \\ \|\langle \nabla_{x_1} \rangle^\gamma \langle \nabla_{v_{[2]}} \rangle^{-\frac{3}{2}} ((f^{\otimes m})^{1-\delta} C_m)\|_{L^2}, & \text{if } \alpha = 0, \end{cases}$$

and thus, by (5.16),

$$\int_t^T \|(f^{\otimes m})^{1-\delta} K_f[C_m]\|_{L^2} \leq (T-t)^{\frac{1}{2}} L^{\frac{1}{2}(m+1)}. \quad (5.35)$$

In terms of the propagator $(B_{t,s})_{0 \leq s \leq t \leq T}$ constructed in Lemma 5.5, with $C_m^T = 0$, Duhamel's formula for the weak solution of (5.14) yields for all $m \geq 0$,

$$C_m^t = -\sqrt{(m+1)(m+2)} \int_t^T (B_{s,t}^{\otimes m})^* K_{f_s} [C_{m+2}^s] ds.$$

Iterating this identity, we find for all $m \geq 0$ and $\ell \geq 1$,

$$C_m^t = (-1)^\ell \sqrt{(m+1) \dots (m+2\ell)} \times \int_t^T (B_{s,t}^{\otimes m})^* \left\langle \left(\int_{t < s_1 < \dots < s_{\ell-1} < s} \bigotimes_{j=1}^{\ell-1} (B_{s,s_j}^{\otimes 2} K_{f_{s_j}}) \right), K_{f_s} [C_{m+2\ell}^s] \right\rangle_{L^2(f_s^{\otimes 2(\ell-1)})} ds,$$

that is, by symmetry,

$$C_m^t = (-1)^\ell \frac{\sqrt{(m+2\ell)!}}{(\ell-1)! \sqrt{m!}} \int_t^T (B_{s,t}^{\otimes m})^* \left\langle \left(\int_t^s B_{s,r}^{\otimes 2} K_{f_r} dr \right)^{\otimes \ell-1}, K_{f_s} [C_{m+2\ell}^s] \right\rangle_{L^2(f_s^{\otimes 2(\ell-1)})} ds.$$

Taking the norm, decomposing the weight as $f = f^\delta f^{1-\delta}$, and noting that, by duality, (5.18) yields

$$\|B_{s,t}^*\|_{\mathcal{L}(L^2(f_s^{2(1-\delta)}), L^2(f_t^{2(1-\delta)}))} \leq e^{C(s-t)}, \quad 0 \leq t \leq s \leq T,$$

we obtain

$$\begin{aligned} \|C_m^t\|_{L^2((f_t^{\otimes m})^{2(1-\delta)})} &\leq \frac{\sqrt{(m+2\ell)!}}{(\ell-1)! \sqrt{m!}} \int_t^T e^{Cm(s-t)} \left\| \int_t^s B_{s,r}^{\otimes 2} K_{f_r} dr \right\|_{L^2((f_s^{\otimes 2})^{2\delta})}^{\ell-1} \\ &\quad \times \|(f_s^{\otimes m+2(\ell-1)})^{1-\delta} K_{f_s} [C_{m+2\ell}^s]\|_{L^2(f_s^{\otimes m+2(\ell-1)})} ds. \end{aligned}$$

Hence, by (5.31) and (5.35),

$$\|C_m^t\|_{L^2((f_t^{\otimes m})^{2(1-\delta)})} \leq C_{T,L} \frac{\sqrt{(m+2\ell)!}}{(\ell-1)! \sqrt{m!}} (C L e^{C(T-t)} \sqrt{T-t})^{\ell-1}. \quad (5.36)$$

Using the elementary bound

$$\frac{\sqrt{(m+2\ell)!}}{(\ell-1)! \sqrt{m!}} \leq C^\ell \left(1 + \frac{m^\ell}{\ell!}\right),$$

we find that there is some $\tau_* > 0$ such that for any fixed $m \geq 0$ the right-hand side in (5.36) tends to 0 as $\ell \uparrow \infty$ provided $T-t \leq \tau_*$. Hence $C_m \equiv 0$ on $[T-\tau_*, T]$ for all $m \geq 0$. By iteration, the conclusion follows on the whole interval $[0, T]$. $\square$

5.3. Proof of Theorem 1.1. By Lemma 5.1, up to a subsequence, dual correlations converge to a solution of the limiting dual hierarchy (5.3)–(5.4) satisfying (5.2). This problem has a unique solution by Proposition 5.2. Since a trivial solution of this hierarchy is given by $C_m \equiv \mathbb{1}_{m=0}$ on $[0, T]$, this is necessarily the unique solution. In particular, we can get rid of the extraction of a subsequence and conclude $\bar{C}_{N,m} \xrightarrow{*} 0$ in $L^\infty(0, T; L^2(f^{\otimes m}))$ for all $m > 0$. In addition, by (5.9) and (5.12) in the proof of Lemma 5.1, for $m = 0$, we further obtain $K_f[\bar{C}_{N,2}] \rightharpoonup 0$ weakly in $L^1(0, T)$. By definition of rescaled correlations, cf. (2.12), this proves (2.11), and the conclusion then follows from Lemma 2.1. $\square$

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APPENDIX A. KINETIC REGULARITY

For completeness, in this appendix, we include short self-contained proofs of the specific kinetic regularity statements of Lemmas 3.1 and 4.1.

A.1. Proof of Lemma 3.1. The result essentially follows from [3]. Yet, to account for non-zero initial data and to emphasize that multiplicative constants are uniform in m , we include for completeness a direct proof based on explicit Fourier calculation. Denote by $\xi = (\xi_1, \dots, \xi_m)$ and $\eta = (\eta_1, \dots, \eta_m)$ the Fourier variables associated with $(x_1, \dots, x_m)$ and $(v_1, \dots, v_m)$, respectively. Let P denote the Fourier projection onto $|\xi| > 1$. The low-frequency contribution is simply bounded by

$$\int_0^T \| |\nabla_{x_{[m]}}|^r (1 - P)g_m(t) \|_{L^2}^2 dt \leq \int_0^T \|g_m(t)\|_{L^2}^2 dt \quad \text{for } r \geq 0.$$

It remains to prove the following for the high-frequency part, for $0 \leq r \leq \frac{1}{3}$,

$$\int_0^T \| |\nabla_{x_{[m]}}|^{\frac{1}{3}-r} P g_m(t) \|_{L^2}^2 dt \leq C \|g_m(0)\|_{L^2}^2 + C \int_0^T \| \langle \nabla_{x_{[m]}} \rangle^{-r} \langle \nabla_{v_{[m]}} \rangle^{-1} e_m \|_{L^2}^2 dt,$$

for some constant C only depending on α . Applying $\langle \nabla_{x_{[m]}} \rangle^{-r}$ to both sides of the equation for g_m , we find that it suffices to prove the above with $r = 0$. By Duhamel's formula, the solution g_m reads

$$g_m(t) = S_t^m g_m(0) + \int_0^t S_{t-s}^m e_m(s) ds,$$

where $(S_t^m)_{t \geq 0}$ is the semigroup generated by $\sum_{j=1}^m (v_j \cdot \nabla_{x_j} - \alpha \Delta_{v_j})$. We split the proof into two steps, separately estimating the contribution of initial data and the contribution of the source term.

Step 1: Contribution of initial data.

We claim that for all $r \geq 0$,

$$\int_0^T \| |\nabla_{x_{[m]}}|^r P S_t^m g_m(0) \|_{L^2}^2 dt \leq C \| \langle \nabla_{x_{[m]}} \rangle^{r-\frac{1}{3}} g_m(0) \|_{L^2(\mathbb{D}^m)}^2. \quad (\text{A.1})$$

Taking Fourier transform, a classical calculation of the semigroup S_t^m yields explicitly

$$\widehat{S_t^m f}(\xi, \eta) = e^{-\alpha \int_0^t |\eta + s\xi|^2 ds} \hat{f}(\xi, \eta + t\xi), \quad (\text{A.2})$$

with the shorthand notation $|\xi|^2 = \sum_{j=1}^m |\xi_j|^2$. Changing variables $\zeta = \eta + t\xi$, we then get

$$\| |\nabla_{x_{[m]}}|^r P S_t^m g_m(0) \|_{L^2}^2 = \int_{\mathbb{D}^m} \mathbf{1}_{|\xi| > 1} |\xi|^{2r} e^{-2\alpha \int_0^t |\zeta - s\xi|^2 ds} |\hat{g}_m(0, \xi, \zeta)|^2 d\xi d\zeta.$$

¹Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Since

$$\int_0^t |\zeta - s\xi|^2 ds = t \left| \zeta - \frac{t}{2}\xi \right|^2 + \frac{t^3}{12} |\xi|^2 \geq \frac{t^3}{12} |\xi|^2,$$

we obtain

$$\begin{aligned} \int_0^T \|\nabla_{x_{[m]}}\|^r S_t^m g_m(0) \|_{L^2}^2 dt &\leq \int_{\mathbb{D}^m} \mathbf{1}_{|\xi|>1} |\xi|^{2r} \left(\int_0^T e^{-\frac{1}{6}\alpha t^3 |\xi|^2} dt \right) |\hat{g}_m(0, \xi, \zeta)|^2 d\xi d\zeta \\ &\leq C \|\nabla_{x_{[m]}}\|^{r-\frac{1}{3}} P g_m(0) \|_{L^2(\mathbb{D}^m)}^2, \end{aligned}$$

for some constant C only depending on α . The claim (A.1) follows.

Step 2: Contribution of the source term.

We claim that

$$\int_0^T \left\| |\nabla_{x_{[m]}}|^{\frac{1}{3}} P \left(\int_0^t S_{t-s}^m e_m(s) ds \right) \right\|_{L^2}^2 dt \leq C \int_0^T \|\langle \nabla_{v_{[m]}} \rangle^{-1} e_m(t)\|_{L^2}^2 dt, \quad (\text{A.3})$$

for some constant C only depending on α . Taking Fourier transform, inserting the explicit form (A.2) of the semigroup, and again changing variables $\zeta = \eta + t\xi$, it suffices to prove

$$\int_0^T \int_{\mathbb{D}^m} \mathbf{1}_{|\xi|>1} \left| \int_0^T K_{\xi, \zeta}(t, s) \hat{E}_m(s, \xi, \zeta) ds \right| d\xi d\zeta dt \leq C \int_0^T \int_{\mathbb{D}^m} |\hat{E}_m(t, \xi, \zeta)|^2 d\xi d\zeta dt.$$

where we have set for abbreviation

$$\begin{aligned} \hat{E}_m(s, \xi, \zeta) &:= \langle \zeta - s\xi \rangle^{-1} \hat{e}_m(s, \xi, \zeta - s\xi), \\ K_{\xi, \zeta}(t, s) &:= \mathbf{1}_{0 \leq s \leq t \leq T} |\xi|^{\frac{1}{3}} \langle \zeta - s\xi \rangle^{-\alpha} e^{-\alpha \int_s^t |\zeta - u\xi|^2 du}. \end{aligned}$$

Hence, it suffices to show that, uniformly for all ξ, ζ with $|\xi| > 1$, the kernel $K_{\xi, \zeta}(t, s)$ defines an operator that is bounded on $L^2(0, T)$, with norm bounded by a constant C only depending on α . To this aim, by Schur's test, it suffices to show

$$\sup_{s \in [0, T]} \int_s^T K_{\xi, \zeta}(t, s) dt \leq C, \quad \sup_{t \in [0, T]} \int_0^t K_{\xi, \zeta}(t, s) ds \leq C. \quad (\text{A.4})$$

We focus on the first bound, while the second one can be checked similarly. Let ξ, ζ be fixed with $|\xi| > 1$. Using that

$$\int_s^t |\zeta - u\xi|^2 du = (t-s)|\zeta_{t,s}|^2 + \frac{1}{12}(t-s)^3 |\xi|^2, \quad \zeta_{t,s} := \zeta - s\xi - \frac{1}{2}(t-s)\xi,$$

and bounding

$$\langle \zeta - s\xi \rangle \leq 1 + |\zeta_{t,s}| + (t-s)|\xi|,$$

we get

$$\langle \zeta - s\xi \rangle \int_s^\infty e^{-\alpha \int_s^t |\zeta - u\xi|^2 du} dt \leq \int_s^\infty (1 + |\zeta_{t,s}| + (t-s)|\xi|) e^{-\alpha(t-s)|\zeta_{t,s}|^2 - \frac{\alpha}{12}(t-s)^3 |\xi|^2} dt.$$

Using the elementary bound $|\zeta_{t,s}| e^{-\alpha(t-s)|\zeta_{t,s}|^2} \leq C(t-s)^{-\frac{1}{2}}$ and estimating the integrals, we obtain

$$\begin{aligned} \langle \zeta - s\xi \rangle \int_0^\infty e^{-\alpha \int_s^t |\zeta - u\xi|^2 du} dt &\leq \int_s^\infty \left(1 + C(t-s)^{-\frac{1}{2}} + |\xi|(t-s) \right) e^{-\frac{\alpha}{12}(t-s)^3 |\xi|^2} dt \\ &\leq C(|\xi|^{-\frac{2}{3}} + |\xi|^{-\frac{1}{3}}). \end{aligned}$$

For $|\xi| > 1$, this is controlled by $|\xi|^{-\frac{1}{3}}$, and the first bound in (A.4) follows. This concludes the proof of (A.3). $\square$

A.2. Proof of Lemma 4.1. We follow the commutator approach of [17]. Denote again by $\xi = (\xi_1, \dots, \xi_m)$ and $\eta = (\eta_1, \dots, \eta_m)$ the Fourier variables associated with $(x_1, \dots, x_m)$ and $(v_1, \dots, v_m)$, respectively. We shall show for any $r \in [0, 1]$,

$$\begin{aligned} & \int_0^T \int_{\mathbb{D}^m} |\xi|^{\frac{1-r}{2}} \langle \eta \rangle^{-3} |\hat{g}_m(\xi, \eta)|^2 d\xi d\eta \\ & \lesssim \|g_m(0)\|_{L^2}^2 + \int_0^T \|g_m\|_{L^2}^2 + \int_0^T \int_{\mathbb{D}^m} \langle \xi \rangle^{-2r} \langle \eta \rangle^{-2} |\hat{e}_m(\xi, \eta)|^2 d\xi d\eta. \end{aligned} \quad (\text{A.5})$$

First note that we can focus on estimating the high-frequency part in physical space. More precisely, given a cut-off function $\chi \in C_c^\infty(\mathbb{R}^+)$ with $\chi(a) = 1$ for $a \leq 1$, $\chi(a) = 0$ for $a \geq 2$, and $0 \leq \chi \leq 1$, the low-frequency part of the norm is simply controlled by

$$\int_0^T \int_{\mathbb{D}^m} \chi(|\xi|) |\xi|^{\frac{1-r}{2}} \langle \eta \rangle^{-3} |\hat{g}_m(\xi, \eta)|^2 d\xi d\eta \lesssim \int_0^T \|g_m\|_{L^2}^2. \quad (\text{A.6})$$

Next, let us regularize in velocity at the scale $\langle \xi \rangle^{-\frac{1-r}{2}}$. More precisely, consider the cut-off function $\gamma(\xi, \eta) := \chi(\langle \xi \rangle^{-\frac{1-r}{2}} |\eta|)$. Noting that

$$(1 - \gamma(\xi, \eta))^2 |\xi|^{\frac{1-r}{2}} \langle \eta \rangle^{-3} \leq |\xi|^{\frac{1-r}{2}} \langle \eta \rangle^{-3} \mathbf{1}_{|\eta| \geq \langle \xi \rangle^{(1-r)/2}} \leq \langle \eta \rangle^{-2} \leq 1,$$

we find

$$\int_0^T \int_{\mathbb{D}^m} (1 - \gamma(\xi, \eta))^2 |\xi|^{\frac{1-r}{2}} \langle \eta \rangle^{-3} |\hat{g}_m(\xi, \eta)|^2 d\xi d\eta \lesssim \int_0^T \|g_m\|_{L^2}^2.$$

From this together with (A.6), we deduce that it remains to prove the following estimate, instead of (A.5), for any $r \in [0, 1]$,

$$\begin{aligned} & \int_0^T \int_{\mathbb{D}^m} (1 - \chi(|\xi|)) |\xi|^{\frac{1-r}{2}} \langle \eta \rangle^{-3} |(\gamma \hat{g}_m)(\xi, \eta)|^2 d\xi d\eta \\ & \lesssim \|g_m(0)\|_{L^2}^2 + \int_0^T \|g_m\|_{L^2}^2 + \int_0^T \int_{\mathbb{D}^m} \langle \xi \rangle^{-2r} \langle \eta \rangle^{-2} |\hat{e}_m(\xi, \eta)|^2 d\xi d\eta. \end{aligned} \quad (\text{A.7})$$

Taking Fourier transform, the equation for g_m reads

$$\partial_t \hat{g}_m - \xi \cdot \nabla_\eta \hat{g}_m = \hat{e}_m,$$

and thus, smuggling the cut-off γ ,

$$\partial_t (\gamma \hat{g}_m) - \xi \cdot \nabla_\eta (\gamma \hat{g}_m) = \gamma \hat{e}_m - (\xi \cdot \nabla_\eta \gamma) \hat{g}_m.$$

As in [17], consider the Fourier multiplier

$$M(\xi, \eta) = 1 + \frac{\xi \cdot \eta}{|\xi| \langle \eta \rangle},$$

which satisfies $0 \leq M(\xi, \eta) \leq 2$ and

$$\xi \cdot \nabla_\eta M(\xi, \eta) = \frac{|\xi|}{\langle \eta \rangle} - \frac{(\xi \cdot \eta)^2}{|\xi| \langle \eta \rangle^3} \geq \frac{|\xi|}{\langle \eta \rangle} - \frac{|\xi| |\eta|^2}{\langle \eta \rangle^3} = \frac{|\xi|}{\langle \eta \rangle^3}. \quad (\text{A.8})$$

Multiplying the above equation for $\gamma \hat{g}_m$ with $2(1 - \chi(|\xi|)) |\xi|^{\frac{1-r}{2}-1} M(\xi, \eta) \overline{(\gamma \hat{g}_m)}(\xi, \eta)$, taking the real part, and using $0 \leq M \leq 2$, we get

$$\begin{aligned}
& \int_0^T \int_{\mathbb{D}^m} (1 - \chi(|\xi|)) |\xi|^{\frac{1-r}{2}-1} (\xi \cdot \nabla_\eta M)(\xi, \eta) |(\gamma \hat{g}_m)(\xi, \eta)|^2 d\xi d\eta \\
& \leq 2 \int_{\mathbb{D}^m} (1 - \chi(|\xi|)) |\xi|^{\frac{1-r}{2}-1} |\gamma \hat{g}_m(0, \xi, \eta)|^2 \\
& + 4 \int_0^T \int_{\mathbb{D}^m} (1 - \chi(|\xi|)) |\xi|^{\frac{1-r}{2}-1} |(\gamma \hat{g}_m)(\xi, \eta)| |(\xi \cdot \nabla_\eta \gamma) \hat{g}_m(\xi, \eta)| d\xi d\eta \\
& + 4 \int_0^T \int_{\mathbb{D}^m} (1 - \chi(|\xi|)) |\xi|^{\frac{1-r}{2}-1} |(\gamma \hat{g}_m)(\xi, \eta)| |(\gamma \hat{e}_m)(\xi, \eta)| d\xi d\eta.
\end{aligned}$$

Using (A.8) to bound the left-hand side from below, noting that $|\xi| \geq 1$ on the support of $1 - \chi(|\xi|)$, that $|\xi \cdot \nabla_\eta \gamma| \lesssim \langle \xi \rangle^{\frac{1+r}{2}}$, and that $|\eta| \leq 2\langle \xi \rangle^{\frac{1-r}{2}}$ on the support of $\gamma(\xi, \eta)$, we deduce

$$\begin{aligned}
& \int_0^T \int_{\mathbb{D}^m} (1 - \chi(|\xi|)) |\xi|^{\frac{1-r}{2}} \langle \eta \rangle^{-3} |(\gamma \hat{g}_m)(\xi, \eta)|^2 d\xi d\eta \\
& \lesssim \|g_m(0)\|_{L^2}^2 + \int_0^T \|g_m\|_{L^2}^2 + \int_0^T \int_{\mathbb{D}^m} \langle \xi \rangle^{-r} \langle \eta \rangle^{-1} |\hat{g}_m(\xi, \eta)| |\hat{e}_m(\xi, \eta)| d\xi d\eta,
\end{aligned}$$

and the claim (A.7) follows. $\square$

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