Bourgain's surprising result in stochastic homogenization

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Let A be a uniformly elliptic, stationary and ergodic random coefficient field, constructed on a probability space $(\Omega, \mathbb{P})$. Homogenization theory has been focussing on the fine description of the solution ∇u_f^ε of the rescaled elliptic problem

$$-\nabla \cdot A(\tfrac{\cdot}{\varepsilon})\nabla u_f^\varepsilon = \nabla \cdot f, \qquad \text{in } \mathbb{R}^d,$$

in the limit of fast oscillating coefficients $\varepsilon \downarrow 0$, for a given deterministic force field $f \in L^2(\mathbb{R}^d)^d$. A different perspective on the topic has been recently initiated by Sigal [6], based on the following observation (see [4]).

Lemma. *There exist a bounded convolution operator $\bar{\mathcal{A}}(\nabla)$ on $L^2(\mathbb{R}^d)$ and a bounded pseudo-differential operator $\mathcal{F}(\cdot, \nabla) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times \Omega)$ with centered stationary random symbol, related by $\bar{\mathcal{A}}(\nabla) = \mathbb{E}[A(\mathrm{Id} + \mathcal{F}(\cdot, \nabla))]$, such that:*

- *the averaged solution $\mathbb{E}[\nabla u_f^\varepsilon]$ satisfies*

$$-\nabla \cdot \bar{\mathcal{A}}(\varepsilon\nabla)\mathbb{E}[\nabla u_f^\varepsilon] = \nabla \cdot f, \qquad \text{in } \mathbb{R}^d; \tag{1}$$

- *the deviation is described by*

$$\nabla u_f^\varepsilon - \mathbb{E}[\nabla u_f^\varepsilon] = \mathcal{F}(\tfrac{\cdot}{\varepsilon}, \varepsilon\nabla)\mathbb{E}[\nabla u_f^\varepsilon]. \tag{2}$$

This result is obtained as a simple consequence of stationarity and of the Schur complement formula, starting from the block decomposition of the operator $L = -\nabla \cdot A\nabla$ on $L^2(\mathbb{R}^d \times \Omega)$ with respect to projections $P := \mathbb{E}$ and $P^\perp := \mathrm{Id} - P$,

$$L = \begin{pmatrix} PLP & PLP^\perp \\ P^\perp LP & P^\perp LP^\perp \end{pmatrix}. \tag{3}$$

Homogenization can be reformulated in these terms as the regularity of the symbols $\mathbb{R}^d \to \mathbb{R}^{d\times d} : i\xi \mapsto \bar{\mathcal{A}}(i\xi)$ and $\mathbb{R}^d \to L^2(\Omega)^{d\times d} : i\xi \mapsto \mathcal{F}(\cdot, i\xi)$ at $i\xi = 0$. Indeed, this regularity allows to transform the equation (1) for the averaged solution perturbatively into the usual form of (higher-order) effective PDEs, and to transform the relation (2) into (higher-order) two-scale expansions; see [3] for a precise equivalence. Unlike the case of periodic homogenization, we recall that two-scale expansions in the random setting cannot be pursued to arbitrary order, corresponding to the problem of existence of higher-order correctors: under the strongest mixing conditions, the two-scale expansion of the solution ∇u_f^ε is only possible to accuracy $O(\varepsilon^{d/2})$ in $L^2(\mathbb{R}^d \times \Omega)$, which implies that the symbols $\bar{\mathcal{A}}$ and $\mathcal{F}$ are a priori only (almost) of class $\mathcal{C}^{d/2}$ at 0. This regularity is optimal for $\mathcal{F}$, but an improvement can be expected for $\bar{\mathcal{A}}$ as it is an averaged quantity. By refined homogenization techniques, the regularity of $\bar{\mathcal{A}}$ has been shown in [2] to be indeed at least twice better, that is, (almost) of class $\mathcal{C}^d$. Very surprisingly, Bourgain [1] and Kim and Lemm [5] proved in a perturbative regime that it is actually *four times better*, that is, (almost) of class $\mathcal{C}^{2d}$, thus yielding an effective description of the averaged solution by an effective PDE to accuracy (almost) $O(\varepsilon^{2d})$. Still no understanding of this result is available by non-perturbative homogenization techniques, and it has led to formulate the following conjecture.

Conjecture (Bourgain–Spencer)**.** If A satisfies strong enough mixing conditions, then the symbol $\bar{\mathcal{A}}$ is of class (almost) $\mathcal{C}^{2d}$ in a neighborhood of 0.

The rest of this note is devoted to a brief description of Bourgain's perturbative argument. As in [1, 5], we focus on the discrete iid setting for simplicity (see [4] for more general results): we consider the operator $L = \nabla^* A \nabla$ on $L^2(\mathbb{Z}^d)$, where ∇ stands now for the discrete gradient and where $A = \{A_x\}_{x\in\mathbb{Z}^d}$ is a sequence of iid uniformly elliptic random conductivities. The description of the averaged solution still holds as above in that case, and we use the same notation $\bar{\mathcal{A}}(\nabla)$ for the corresponding convolution operator. The perturbative regularity result by Bourgain [1] and Kim and Lemm [5] takes on the following guise.

Theorem (Bourgain [1], Kim–Lemm [5])**.** *Let $d \ge 2$ and assume $A_x = 1 + \delta B_x$ with ellipticity ratio $\delta \ll 1$ and with $|B_x| \le 1$ and $\mathbb{E}[B_x] = 0$. Then we have $\bar{\mathcal{A}}(\nabla) = \mathrm{Id} + \mathbb{L}_\delta$ where $\mathbb{L}_\delta$ is a convolution operator on $\mathbb{Z}^d$ with kernel satisfying*

$$|\mathbb{L}_\delta(x,y)| \le C\delta^2 \langle x-y\rangle^{C\delta-3d}, \qquad (\langle x\rangle := 1+|x|) \tag{4}$$

for some universal constant $C > 0$, meaning that its symbol is of class $\mathcal{C}^{2d-C\delta}$.

This result is essentially optimal in the sense that the decay of the kernel cannot be improved beyond $\langle\cdot\rangle^{-3d}$, as will be clear from the proof, but we emphasize that it does not solve the above conjecture even in the perturbative regime due to the loss $C\delta$ in the exponent. This indicates that the conjecture might, in fact, be false and that the non-perturbative $\mathcal{C}^d$ regularity in [2] might be optimal in general.

The proof starts from the Schur complement formula for the block decomposition (3), combined with a Neumann expansion, which allows to represent $(PL^{-1}P)^{-1} = \nabla^*(\mathrm{Id} + \mathbb{L}_\delta)\nabla$ with

$$\mathbb{L}_\delta = \delta \textstyle\sum_{n=1}^\infty \delta^n \mathbb{L}^{(n)}, \qquad \mathbb{L}^{(n)} := PB(\mathbb{K}P^\perp B)^n P, \tag{5}$$

with the short-hand notation $\mathbb{K} := \nabla\triangle^{-1}\nabla^*$, and we then proceed by analyzing this perturbation series. Expanding the composition of operators, the kernel for the nth term in the series takes the form

$$\mathbb{L}^{(n)}(x_0,x_n) = \sum_{\underline{x}\in(\mathbb{Z}^d)^{n-1}} PB_{x_0}\mathbb{K}(x_0-x_1)P^\perp B_{x_1}\ldots\mathbb{K}(x_{n-1}-x_n)P^\perp B_{x_n}, \tag{6}$$

where the sum runs over all 'paths' $\underline{x} = (x_1,\ldots,x_{n-1})$ in $\mathbb{Z}^d$ connecting x_0 to x_n. A direct estimate of the series, using the pointwise decay $|\mathbb{K}(x,y)| \lesssim \langle x-y\rangle^{-d}$, would yield

$$|\mathbb{L}^{(n)}(x,y)| \le C^n \langle x-y\rangle^{-d}\log(2+|x-y|)^n,$$

where the logarithms come from estimating integrals with borderline decay. For all $\eta > 0$, using $\log t \le \eta^{-1}t^\eta$ for $t \ge 1$, this bound translates into

$$|\mathbb{L}^{(n)}(x,y)| \le n^n (\tfrac{C}{\eta})^n \langle x-y\rangle^{\eta-d}. \tag{7}$$

The combinatorial factor n^n destroys any possible use of this direct estimate in the perturbation series. In [1], Bourgain made a more clever use of the global structure of the paths, together with Calderón–Zygmund theory in form of the L^p-boundedness of $\mathbb{K}$, to show that this factor can, in fact, be removed.

Lemma (Bourgain's deterministic lemma)**.** *For all* $\eta \in (0,1)$ *and* $x \neq y$*,*

$$|\mathbb{L}^{(n)}(x,y)| \lesssim \eta(\tfrac{C}{\eta})^n \langle x-y\rangle^{\eta-d}.$$

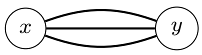
Choosing $\eta = 2C\delta$, this bound can now be used to estimate the perturbation series (5), to the effect of

$$|\mathbb{L}_\delta(x,y)| \le \delta\eta \sum_{n=1}^{\infty} (\tfrac{C\delta}{\eta})^n \langle x-y\rangle^{\eta-d} \le 2C\delta^2\langle x-y\rangle^{2C\delta-d}.$$

To prove the stated decay (4), this naive bound needs to be improved by taking advantage of stochastic cancellations. We indeed easily realize that many paths do not contribute in the sum (6): for instance,

$$PB(x_0)P^\perp B(x_1)\dots P^\perp B(x_n) = 0$$
$$\text{whenever } \{x_0,\dots,x_j\} \cap \{x_{j+1},\dots,x_n\} = \varnothing, \text{ for some } 0 \le j \le n.$$

The sum in (6) can thus be restricted to the so-called 'irreducible' paths that do not satisfy this condition. Further cancellations exist but are not needed in the analysis. For $x \neq y$, we note for instance that there is no irreducible path with $n \le 2$ edges from x to y, and that for $n = 3$ the only irreducible path is (x,y,x,y), that is,



A simple combinatorial argument shows that an irreducible path from x to y can always be decomposed into three disjoint paths from x to y. Evaluating the sum (6) by summing separately over these three paths, a direct estimate as in (7) would then yield the following, for all $n \ge 1$ and $\eta > 0$,

$$|\mathbb{L}^{(n)}(x_0,x_n)| \le n^n(\tfrac{C}{\eta})^n\langle x-y\rangle^{\eta-3d}.$$

This captures the optimal decay $\langle\cdot\rangle^{-3d}$ as stated in (4), but the factor n^n again makes this direct estimate useless in the perturbation series. Since the restriction to irreducible paths breaks the special oscillatory structure of the composition of Calderón–Zygmund kernels in (6), it is a priori unclear how to improve on such direct estimates. In a nutshell, the main contribution of Bourgain's work in [1] is to show how simple enough restrictions on the summations still allow to appeal to the Calderón–Zygmund theory.

References

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