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Approximate spectral theory and wave propagation in quasi-periodic media

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Abstract

In this article we make specific in the quasi-periodic setting the general Floquet-Bloch theory we have introduced for stationary ergodic operators together with the associated approximate spectral theory. As an application we consider the long-time behavior of the Schrödinger flow with a quasi-periodic potential (in the regime of small intensity of the discorder), and the long-time behavior of the wave equation with quasi-periodic coefficients (in the homogenization regime).

Contents

1. General overview

We consider the elliptic operator $\mathcal{L}_\lambda = -\nabla \cdot \mathbf{a}\nabla + \lambda V$ on $\mathrm{L}^2(\mathbb{R}^d)$, where $V : \mathbb{R}^d \to \mathbb{R}$ and $\mathbf{a} : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ are a quasi-periodic potential and a quasi-periodic coefficient field (see definition below), and $\lambda > 0$ is the intensity of the potential.

In particular, we study the long-time behavior of two prototypical time-dependent equations: The classical wave equation (for $\lambda = 0$)

$$\partial_{tt}^2 u_\varepsilon = \nabla \cdot \mathbf{a}(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon, \qquad u_\varepsilon|_{t=0} = u^\circ, \qquad \partial_t u_\varepsilon|_{t=0} = 0, \tag{1.1}$$

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and the Schrödinger equation (for $\mathbf{a} \equiv \mathrm{Id}$ for simplicity)

$$i\partial_t u_\lambda = (-\triangle + \lambda V) u_\lambda, \qquad u_\lambda|_{t=0} = u^\circ. \tag{1.2}$$

The long-standing conjecture for the Schrödinger operator states that there is localization for large ε or λ (see for instance [4, 5, 6] for $d \geq 1$) and ballistic transport for all times as soon as ε and λ are small enough (see the survey [11, 7] for $d = 1$). The existence of delocalized states (understood as ballistic transport) is a difficult problem, which, in dimensions $d > 1$, has only been solved in very specific situations ($d = 2$ for two frequencies in [12]). Rather than proving ballistic transport at all times, we aim at proving that waves are transported in a ballistic regime in large times depending on ε or λ. Roughly speaking, this will owe to the fact that in a *strong quantitative sense*, for $0 < \varepsilon \ll 1$, the equation (1.1) homogenizes and behaves like a constant-coefficient wave equation, whereas for $0 < \lambda \ll 1$, the equation (1.2) is a perturbation of the Laplacian. In this paper, we go beyond the first order homogenization limit for (1.1) and first order perturbation limit for (1.2), and derive families of effective equations valid up to arbitrary long time scales of the form $\varepsilon^{-\ell}T$ and $\lambda^{-\ell}T$ for all $\ell \geq 0$. These results constitute a particular case of a more general theory developed for (1.1) by the first and third authors in [3], and for (1.2) by the last three authors in [8]. We refer the reader to [3] and [8] for a thorough discussion of the literature, for details, and for the proofs of the results reported on here.

The rest of the paper is divided into three parts: The introduction of an approximate spectral theory for quasi-periodic operators (in the form of Taylor-Bloch waves), the analysis of quantum waves, and finally the analysis of classical waves.

To conclude this introduction, we make precise the hypotheses on $\mathbf{a}$ and V, and shall assume that

$$V(x) := \tilde{V}(Fx) \quad \text{and} \quad \mathbf{a}(x) := \tilde{\mathbf{a}}(Fx)$$

for some (winding) matrix $F \in \mathbb{R}^{M\times d}$ and some lifted maps $\tilde{V} \in C(\mathbb{T}^M)$ and $\tilde{\mathbf{a}} \in C^\infty(\mathbb{T}^M, \mathcal{M})$, where $\mathbb{T}^M = [0, 2\pi)^M$ is the M-dimensional torus with $M \geq d$, and $\mathcal{M}$ denotes the set of uniformly elliptic symmetric matrices of fixed ellipticity constant $C > 0$ (say, such that for all $M \in \mathcal{M}$ and $\xi \in \mathbb{R}^d$, $\frac{1}{C}|\xi|^2 \leq (M\xi, \xi) \leq C|\xi|^2$).

Notation. We denote by $\hat{f}(k) = \mathcal{F}[f](k) = \int_{\mathbb{R}^d} e^{-ik\cdot x} f(x)dx$ the usual Fourier transform of a smooth function f on $\mathbb{R}^d$. The inverse Fourier transform is then given by

$$f(x) = \mathcal{F}^{-1}[\hat{f}\,](x) = \int e^{ik\cdot x} \hat{f}(k) d^*k,$$

in terms of the rescaled Lebesgue measure $d^*k := (2\pi)^{-d}dk$ on the momentum space $\mathbb{R}^d$. Likewise, for all $M \in \mathbb{N}$, when dealing with periodic functions on the torus $\mathbb{T}^M$, we denote by $\hat{f}(k) = \mathcal{F}[f](k) = \int_{\mathbb{T}^M} e^{-ik\cdot x} f(x)dx$ the associated Fourier series on $\mathbb{Z}^M$.

2. Approximate Floquet-Bloch theory

In this section we introduce a notion of approximate Bloch decompositions in a quasi-periodic framework (for the general stationary setting, see [8]). We let $M \geq d$, consider a given Diophantine winding matrix $F \in \mathbb{R}^{M\times d}$ (which ensures ergodicity), and denote by $\mathbb{E}[\cdot]$ the average on the torus $\mathbb{T}^M$. The periodic setting is recovered by taking $M = d$ and F invertible.

2.1. Quasi-periodic Floquet-Bloch theory

We start by adapting the usual periodic Bloch-Floquet theory to the quasi-periodic setting.

2.1.1. Preliminary: quasi-periodicity

A measurable function $f : \mathbb{R}^d \times \mathbb{T}^M \to \mathbb{R}$ is said to be *quasi-periodic* if it satisfies $f(x, z) = f(0, z + Fx)$ for all x, z. In particular, this implies $\mathbb{E}[f(x, \cdot)] = \mathbb{E}[f(0, \cdot)]$ for all x. Setting $\tilde{f}(z) := f(0, z)$, quasi-periodicity obviously yields a bijection between periodic functions $\tilde{f} : \mathbb{T}^M \to \mathbb{R}$ and quasi-periodic measurable functions $f : \mathbb{R}^d \times \mathbb{T}^M \to \mathbb{R}$. The function f is then called the *quasi-periodic extension* of the periodic function $\tilde{f}$. In particular, the space of quasi-periodic functions

$f : \mathbb{R}^d \times \mathbb{T}^M \to \mathbb{R}$ in $\mathrm{L}^2(\mathbb{T}^M, \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d))$ is identified with the Hilbert space $\mathrm{L}^2(\mathbb{T}^M)$, and the weak gradient ∇ on locally square integrable functions on $\mathbb{R}^d$ then turns by quasi-periodicity into a linear operator on $\mathrm{L}^2(\mathbb{T}^M)$. For all $l \geq 0$, we may further define the (Hilbert) space $\mathcal{H}^l(\mathbb{T}^M)$ as the space of all periodic functions $\tilde{f} \in \mathrm{L}^2(\mathbb{T}^M)$ the quasi-periodic extension f of which belongs to $\mathrm{L}^2(\mathbb{T}^M; H^l_{\mathrm{loc}}(\mathbb{R}^d))$ — note that $\mathcal{H}^l(\mathbb{T}^M)$ does not coincide with the usual Sobolev space $H^l(\mathbb{T}^M)$ on the torus $\mathbb{T}^M$ since the gradient used here is the degenerate gradient $D = F^T\nabla_x$ (wrt $x \in \mathbb{R}^d$), as opposed to the gradient ∇_z (wrt $z \in \mathbb{R}^M$).

2.1.2. Quasi-periodic Floquet transform

In this paragraph, we extend the standard definition of the Floquet transform (see e.g. [14]) from the periodic setting to the quasi-periodic setting. For $f \in \mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$, we define the Floquet transform $\mathcal{U}f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}^M \to \mathbb{R}$ by

$$\mathcal{U}f(k, x, z) = \mathcal{F}\left[\mathcal{O}_x f(\cdot, z)\right](k), \qquad \mathcal{O}_x f(y, z) = f(x + y, z - Fy). \tag{2.1}$$

The following properties directly follow from this definition.

Lemma 2.1. *Writing $e_k(x) := e^{ik\cdot x}$, we have*

(i) the map $\mathcal{O}_x$ (and therefore also the map $f \mapsto \mathcal{U}f(\cdot, x, \cdot)$) is unitary on $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$ for all x;

(ii) $\mathcal{U}f(k, \cdot, \cdot)$ is e_k-quasi-periodic in the sense that $\mathcal{U}f(k, x + y, z) = e_k(y)\mathcal{U}f(k, x, z + Fy)$;

(iii) $f(x, z) = \mathcal{F}^{-1}\left[\mathcal{U}f(\cdot, x, z)\right](0)$, where the rhs is well-defined as an element of $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$. $\diamondsuit$

For $f \in \mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$, it is thus natural to define

$$\mathcal{V}f(k, x, z) := e^{-ik\cdot x}\mathcal{U}f(k, x, z), \tag{2.2}$$

which, for any fixed $k \in \mathbb{R}^d$, is quasi-periodic by the above properties. Also, for all $x \in \mathbb{R}^d$ the map $f \mapsto \mathcal{V}f(\cdot, x, \cdot)$ is unitary on $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$. With the usual identification of $\mathcal{V}f$ and its restriction $\mathcal{V}f(\cdot, 0, \cdot)$, we obtain the following.

Lemma 2.2. *The* quasi-periodic Floquet transform *$\mathcal{V}$ defines a unitary operator on $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$, and satisfies*

(i) $f(x, z) = \mathcal{F}^{-1}\left[e_\cdot(x)\mathcal{V}f(\cdot, 0, z + Fx)\right](0)$, where the rhs is well-defined as an element of $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$;

(ii) denoting by $\iota : \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$ the canonical injection, we have $\mathcal{V} \circ \iota = \mathcal{F}$ on $L^2(\mathbb{R}^d)$;

(iii) for all $f \in \mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$ and $g \in \mathrm{L}^2(\mathbb{T}^M)$ with $gf \in \mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$, we have $\mathcal{V}(gf) = g\mathcal{V}f$. $\diamondsuit$

2.1.3. Quasi-periodic Bloch-Floquet theory

We start by extending the differential operator $\mathcal{L}_\lambda$ to the space $L^2(\mathbb{R}^d \times \mathbb{T}^M)$ in the following way: For all maps $f \in H^2(\mathbb{R}^d, \mathrm{L}^2(\mathbb{T}^M))$, we set

$$\mathcal{L}_\lambda(x, z) \quad := \quad -\nabla_x \cdot \tilde{\mathbf{a}}(Fx + z)\nabla_x f(z, x) + \lambda\tilde{V}(Fx + z)f(x, z).$$

The main observation (and motivation) is that the quasi-periodic Floquet transform $\mathcal{V}$ decomposes the differential operator $\mathcal{L}_\lambda$ into direct integrals of elementary operators on the simpler space $\mathrm{L}^2(\mathbb{T}^M)$ of periodic functions.

On the one hand, the second-order operator $\mathcal{L}_0 = -\nabla \cdot \mathbf{a}\nabla$ on $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$ is mapped for all $f \in H^2(\mathbb{R}^d, \mathrm{L}^2(\mathbb{T}^M))$ by the quasi-periodic Floquet transform $\mathcal{V}$ into

$$\mathcal{V}[\mathcal{L}_0 f](k, x, z) = \mathcal{L}_{0,k}\mathcal{V}f(k, 0, z + Fx) \tag{2.3}$$

in terms of the fibered second-order operator

$$\mathcal{L}_{0,k} := -(D + ik) \cdot \tilde{\mathbf{a}}(D + ik),$$

where we recall that the differential operator D acts on the z-variable as $D = F^T\nabla_z$ (a degenerate gradient), and the action of the operator $\mathcal{L}_{0,k}$ is considered in (2.3) on $\mathrm{L}^2(\mathbb{T}^M)$. Its domain is then clearly $D(\mathcal{L}_{0,k}) = \mathcal{H}^2(\mathbb{T}^M)$.

On the other hand, since the potential V is quasi-periodic (and bounded), it defines a multiplicative operator on $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$. Hence, the operator $\mathcal{L}_\lambda = -\nabla\cdot\mathbf{a}\nabla + \lambda V$ on $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$ is self-adjoint on $\mathrm{L}^2(\mathbb{T}^M; H^2(\mathbb{R}^d))$. As in (2.3), we obtain for all $f \in D(\mathcal{L}_\lambda)$ using Lemma 2.2(iii)

$$\mathcal{V}[\mathcal{L}_\lambda f](k,x,z) = \mathcal{L}_{\lambda,k}\mathcal{V}f(k,0,z+Fx),$$

in terms of the fibered operator

$$\mathcal{L}_{\lambda,k} := \mathcal{L}_{0,k} + \lambda\tilde{V}.$$

Again, for fixed k we regard the fibered operator $\mathcal{L}_{\lambda,k}$ as a self-adjoint operator on $\mathcal{H}^2(\mathbb{T}^M)$. Using direct integral representation (see e.g. [16, p.280]), we may reformulate Lemma 2.2(i) as

$$\mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M) = \int_\oplus e_k\,\mathrm{L}^2(\mathbb{T}^M)\,dk,\ \mathcal{L}_0 = \int_\oplus e_k\mathcal{L}_{0,k}dk,\ \mathcal{L}_\lambda = \int_\oplus e_k\mathcal{L}_{\lambda,k}dk. \tag{2.4}$$

In view of (1.1) and (1.2), we rescale the space variable in the second-order operator, set $\mathcal{L}_{\varepsilon,\lambda} := -\nabla\cdot\mathbf{a}(\frac{\cdot}{\varepsilon})\nabla + \lambda V$, and denote by $\mathcal{L}_{\varepsilon,\lambda,k}$ the associated fibered operator.

We start with the Schrödinger flow

$$i\partial_t u_{\varepsilon,\lambda} = \mathcal{L}_{\varepsilon,\lambda}u_{\varepsilon,\lambda}, \qquad u_{\varepsilon,\lambda}|_{t=0} = u^\circ \tag{2.5}$$

in the slightly more general form than (1.2). The quasi-periodic version of the so-called Bloch wave decomposition of the Schrödinger flow then takes the form: given an initial condition $u^\circ \in L^2(\mathbb{R}^d)$, and denoting as before by $\iota : \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d \times \mathbb{T}^M)$ the canonical injection, we obtain for the solution $u_{\varepsilon,\lambda}$ of (2.5) by Lemma 2.2 (i) & (ii) and (2.4),

$$\begin{aligned}
u^t_{\varepsilon,\lambda}(x) = \big[e^{-it\mathcal{L}_{\varepsilon,\lambda}}\iota u^\circ\big](x,0) &\overset{(i)}{=} \mathcal{F}^{-1}\Big[k \mapsto e^{ik\cdot x}\mathcal{V}(e^{-it\mathcal{L}_{\varepsilon,\lambda}}\iota u^\circ)(k,x,0)\Big](0)\\
&= \mathcal{F}^{-1}\Big[k \mapsto e^{ik\cdot x}\mathcal{V}(e^{-it\mathcal{L}_{\varepsilon,\lambda}}\iota u^\circ)(k,0,Fx)\Big](0)\\
&\overset{(2.4)}{=} \mathcal{F}^{-1}\Big[k \mapsto e^{ik\cdot x}\big[e^{-it\mathcal{L}_{\varepsilon,\lambda,k}}\mathcal{V}\iota u^\circ\big](k,0,Fx)\Big](0)\\
&\overset{(ii)}{=} \mathcal{F}^{-1}\Big[k \mapsto \hat{u}^\circ(k)e^{ik\cdot x}\big[e^{-it\mathcal{L}_{\varepsilon,\lambda,k}}1\big](Fx)\Big](0)\\
&= \mathcal{F}^{-1}\Big[k \mapsto \hat{u}^\circ(k)e^{ik\cdot x}\int_{\mathbb{R}} e^{-it\kappa}d\mu_{\varepsilon,\lambda,k}(\kappa)(Fx)\Big](0),
\end{aligned}$$

in terms of the $\mathrm{L}^2(\mathbb{T}^M)$-valued spectral measure $\mu_{\varepsilon,\lambda,k}$ of $\mathcal{L}_{\varepsilon,\lambda,k}$ associated with the constant function 1. Provided we have enough integrability with respect to the k-variable, this takes the simpler form

$$u^t_{\varepsilon,\lambda}(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}} \hat{u}^\circ(k)e^{-it\kappa}e^{ik\cdot x}d\mu_{\varepsilon,\lambda,k}(\kappa)(Fx)d^*k. \tag{2.6}$$

For $\lambda = 0$ and $\varepsilon = 0$ (to be understood in the sense that $\mathbf{a}(\frac{\cdot}{\varepsilon})$ is replaced by the constant homogenized coefficients $\mathbf{a}_{\mathrm{hom}}$: $\mathcal{L}_{0,0} = -\nabla\cdot\mathbf{a}_{\mathrm{hom}}\nabla$), we simply have $d\mu_{0,0,k}(\kappa) = d\delta_{k\cdot\mathbf{a}_{\mathrm{hom}}k}(\kappa)$, while for $\lambda > 0$ or $\varepsilon > 0$ the planar wave e_k is corrected into a (potentially non-atomic) *Bloch measure* $e_k d\mu_{\varepsilon,\lambda,k}(\kappa)$, which is adapted to the potential V and coefficient field $\mathbf{a}$. If $\mu_{\varepsilon,\lambda,k}$ admits an atom at κ_*, the function $e_k\mu_{\varepsilon,\lambda,k}(\{\kappa_*\}) \in \mathrm{L}^2(\mathbb{T}^M; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d))$ is called a *Bloch wave*, and $x \mapsto e_k(x)\mu_{\varepsilon,\lambda,k}(\{\kappa_*\})(Fx)$ is in particular a "generalized eigenfunction" of $\mathcal{L}_{\varepsilon,\lambda}$ associated with the "generalized eigenvalue" κ_*.

A similar analysis can be performed at the level of the classical wave equation, in which case we assume that $V \geq 0$ (so that $\mathcal{L}_{\varepsilon,\lambda}$ is non-negative) and consider the flow

$$\partial^2_{tt}u_{\varepsilon,\lambda} = -\mathcal{L}_{\varepsilon,\lambda}u_{\varepsilon,\lambda}, \qquad u_{\varepsilon,\lambda}|_{t=0} = u^\circ, \qquad \partial_t u_{\varepsilon,\lambda}|_{t=0} = 0. \tag{2.7}$$

We then have

$$
\begin{aligned}
u^t_{\varepsilon,\lambda}(x) = \Re\big[e^{it\sqrt{\mathcal{L}_{\varepsilon,\lambda}}}\iota u^\circ\big](x,0) \quad &\overset{(i)}{=} \quad \Re\mathcal{F}^{-1}\Big[k \mapsto e^{ik\cdot x}\mathcal{V}(e^{it\sqrt{\mathcal{L}_{\varepsilon,\lambda}}}\iota u^\circ)(k,x,0)\Big](0)\\
&\overset{(2.4)}{=} \quad \Re\mathcal{F}^{-1}\Big[k \mapsto e^{ik\cdot x}\big[e^{it\sqrt{\mathcal{L}_{\varepsilon,\lambda,k}}}\mathcal{V}\iota u^\circ\big](k,0,Fx)\Big](0)\\
&\overset{(ii)}{=} \quad \Re\mathcal{F}^{-1}\Big[k \mapsto \hat u^\circ(k)e^{ik\cdot x}\big[e^{it\sqrt{\mathcal{L}_{\varepsilon,\lambda,k}}}1\big](Fx)\Big](0)\\
&= \quad \Re\mathcal{F}^{-1}\Big[k \mapsto \hat u^\circ(k)e^{ik\cdot x}\int_{\mathbb{R}_+} e^{it\sqrt{\kappa}}d\mu_{\varepsilon,\lambda,k}(\kappa)(Fx)\Big](0)
\end{aligned}
$$

in terms of the (same) $\mathrm{L}^2(\mathbb{T}^M)$-valued spectral measure $\mu_{\varepsilon,\lambda,k}$ of $\mathcal{L}_{\varepsilon,\lambda,k}$ associated with the constant function 1. As above, provided we have enough integrability with respect to the k-variable, this takes the simpler form

$$
u^t_{\varepsilon,\lambda}(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} \hat u^\circ(k)e^{ik\cdot x}\cos(t\sqrt{\kappa})d\mu_{\varepsilon,\lambda,k}(\kappa)(Fx)d^*k. \tag{2.8}
$$

2.1.4. Difficulties and limitations

The study of the operator $\mathcal{L}_{\varepsilon,\lambda}$ is equivalent to the understanding of all fibered operators $\mathcal{L}_{\varepsilon,\lambda,k}$ for $k\in\mathbb{R}^d$, and in particular to the understanding of the spectral measures $\mu_{\varepsilon,\lambda,k}$ for $k\in\mathbb{R}^d$.

First of all, the spectral properties of the fibered operators strongly depend on the heterogeneity of the potential and of the coefficients. Indeed, the periodic and the quasi-periodic settings are of completely different natures. Whereas in the periodic setting the (unperturbed) fibered operator $-(\nabla+ik)\cdot(\nabla+ik)$ on $\mathrm{L}^2(\mathbb{T}^d)$ has a purely discrete spectrum since its resolvent operator is compact, in the quasi-periodic setting $-(\nabla+ik)\cdot(\nabla+ik)$ has a dense pure point spectrum on $\mathrm{L}^2(\mathbb{T}^M)$.

The main difficulty is now clear. In the periodic setting the spectrum is discrete, and the regular perturbation theory by Rellich [17] and Kato [13] essentially allows to compute the spectral measure $\mu_{\varepsilon,\lambda,k}$. In the quasi-periodic setting however, the spectrum is dense pure point and no general perturbation theory is available. More precisely, in the periodic setting, the spectral measures $\mu_{\varepsilon,\lambda,k}$ are discrete and take the form

$$
d\mu_{\varepsilon,\lambda,k}(\kappa) = \sum_{n=0}^\infty (P_{\varepsilon,\lambda,k}(n)1)\ d\delta_{\kappa_{\varepsilon,\lambda,k}(n)}(\kappa),
$$

where $(\kappa_{\varepsilon,\lambda,k}(n))_n$ are the eigenvalues of $\mathcal{L}_{\varepsilon,\lambda,k}$, and where, for all n, $P_{\varepsilon,\lambda,k}(n)$ is the orthogonal projector onto the eigenspace associated with $\kappa_{\varepsilon,\lambda,k}(n)$. In particular, the Bloch decomposition (2.6) for the Schrödinger flow writes

$$
u^t_\lambda(x) = \sum_{n=0}^\infty\int_{\mathbb{R}^d}\hat u^\circ(k)e^{-it\kappa_{\varepsilon,\lambda,k}(n)}e^{ik\cdot x}(P_{\varepsilon,\lambda,k}(n)1)(x)d^*k,
$$

and the Bloch decomposition (2.8) for the wave equation

$$
u^t_{\varepsilon,\lambda}(x) = \sum_{n=0}^\infty\int_{\mathbb{R}^d}\hat u^\circ(k)\cos(t\sqrt{\kappa_{\varepsilon,\lambda,k}(n)})e^{ik\cdot x}(P_{\varepsilon,\lambda,k}(n)1)(x)d^*k;
$$

where for all k,n the Bloch wave $e_kP_{\varepsilon,\lambda,k}(n)1\in\mathrm{L}^2(\mathbb{T}^d,\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d))$ is such that $x\mapsto e_k(x)(P_{\varepsilon,\lambda,k}(n)1)(x)$ is an extended state associated with the generalized eigenvalue $\kappa_{\varepsilon,\lambda,k}(n)$. Using such a decomposition, it is shown in [2] that ballistic transport takes place for every localized initial condition at an asymptotic velocity characterized in terms of the spectral measure.

In the rest of this article, we consider separately the Schrödinger flow (1.2) and the classical wave equation (1.1).

In the first case (1.2), if we are only interested in the small disorder limit $\lambda\downarrow 0$, we may expect $\mu_{\lambda,k}$ to be a "perturbation" in some sense of $\mu_{0,k}$. Since for the choice $\mathbf{a}=1$, $\mathcal{L}_{0,k}=-(D+ik)\cdot(D+ik)$, we have $d\mu_{0,k}(\kappa)=d\delta_{|k|^2}(\kappa)$ with eigenfunction $\equiv 1$. We thus expect most of the mass of $\mu_{\lambda,k}$ to be concentrated close to $|k|^2$ for λ close to 0. Recall that no general perturbation theory is available for quasi-periodic potentials since the spectrum is dense pure point (so that crossings of eigenvalues destroy local analyticity). We shall first slightly modify the definition of the fibered operator to make it centered (this is not necessary, but turns out to be simpler later on), and define

$$\tilde{\mathcal{L}}_{\lambda,k} := -\triangle_k + \lambda\tilde{V} = -(D+ik)\cdot(D+ik) - |k|^2 + \lambda\tilde{V},$$

in which case 0 is the eigenvalue associated with the eigenfunction 1. Formula (2.6) (for $\mathbf{a} = 1$) then takes the form

$$u_\lambda^t(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}} \hat{u}^\circ(k)e^{-it(|k|^2+\kappa)}e^{ik\cdot x}d\tilde{\mu}_{\lambda,k}(\kappa)(Fx)d^*k,$$

where $\tilde{\mu}_{\lambda,k}$ denotes the spectral measure of $\tilde{\mathcal{L}}_{\lambda,k}$ projected onto 1. A naïve approach consists in postulating that for $\lambda > 0$ the eigenvalue 0 (resp. the eigenfunction 1) of $\tilde{\mathcal{L}}_{0,k}$ is perturbed into an eigenvalue $\kappa_{\lambda,k}$ (resp. an eigenfunction $\psi_{\lambda,k}$), and in trying to identify it as the sum of Rayleigh-Schrödinger power series

$$\kappa_{\lambda,k} = \lambda\sum_{n=0}^{\infty}\lambda^n\nu_{k,n}, \qquad \psi_{\lambda,k} = 1 + \sum_{n=1}^{\infty}\lambda^n\varphi_{k,n}. \tag{2.9}$$

The eigenvalue equation

$$\tilde{\mathcal{L}}_{\lambda,k}\psi_{\lambda,k} = \kappa_{\lambda,k}\psi_{\lambda,k}, \qquad \kappa_{\lambda,k}\in\mathbb{R}, \qquad \psi_{\lambda,k}\in \mathrm{L}^2(\mathbb{T}^M), \tag{2.10}$$

then splits into a hierarchy of equations for the coefficients $\nu_{k,n}\in\mathbb{R}$, $\varphi_{k,n}\in\mathrm{L}^2(\mathbb{T}^M)$. This approach however quickly fails: Although we (essentially) manage to construct all these coefficients, the Rayleigh-Schrödinger series (2.9) should not converge in general. This does not come as a surprise: the perturbation of an eigenvalue lying in a dense pure point spectrum is expected to experience dense crossings of other eigenvalues, which should destroy local analyticity.

In the second case (1.1), we are interested in the homogenization limit $\varepsilon\downarrow 0$. In particular, since $\lambda = 0$, we may rescale the formula (2.8) in the form

$$u_\varepsilon^t(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} \hat{u}^\circ(k)e^{ik\cdot x}\cos(t\varepsilon^{-1}\sqrt{\kappa})d\mu_{1,\varepsilon k}(\kappa)(\frac{1}{\varepsilon}Fx)d^*k,$$

and we are led to the understanding of $\mu_{1,k}$ for $|k|\ll 1$. For $k=0$, $\mu_{1,0}$ is the projection of the spectral measure of the fibered operator $\mathcal{L}_0 = -D\cdot\tilde{\mathbf{a}}D$ onto 1, a Dirac mass at 0. Again, we thus expect the mass of $\mu_{1,k}$ to be concentrated close to zero for $|k|\ll 1$. As above, a naïve approach consists in postulating that for $k=\gamma e$ with $e\in\mathcal{S}^{d-1}$ and $0<\gamma\ll 1$ the eigenvalue 0 (resp. the eigenfunction 1) of $\mathcal{L}_0$ is perturbed into an eigenvalue κ_k (resp. an eigenfunction ψ_k) for $\mathcal{L}_k = -(D+ik)\cdot\tilde{\mathbf{a}}(D+ik)$, and in trying to identify them as the sum of Rayleigh-type power series

$$\kappa_k = \gamma^2\sum_{n=0}^{\infty}(i\gamma)^n\nu_{e,n}, \qquad \psi_k = 1+\sum_{n=1}^{\infty}(i\gamma)^n\varphi_{e,n}. \tag{2.11}$$

The eigenvalue equation

$$\mathcal{L}_k\psi_k = \kappa_k\psi_k, \qquad \kappa_k\in\mathbb{R}, \qquad \psi_k\in\mathrm{L}^2(\mathbb{T}^M), \tag{2.12}$$

then splits into a hierarchy of equations for the coefficients $\nu_{e,n}\in\mathbb{R}$, $\varphi_{e,n}\in\mathrm{L}^2(\mathbb{T}^M)$.

2.2. Taylor-Bloch waves

Even for small disorder $0<\lambda\ll 1$ for the Schrödinger flow and for the homogenization regime $0<\varepsilon\ll 1$ for the wave equation, we do not expect to be able to solve the eigenvalue equations beyond the periodic setting. Nevertheless, in the small disorder and homogenization regimes, we might be able to construct approximate solutions of the eigenvalue equations, and the associated approximate Bloch waves. In the quasi-periodic setting it suffices to truncate the formal power series (2.9) and (2.11). This will allow us to control the quantum and classical flows for large times (depending on λ and ε).

2.2.1. Taylor-Bloch waves for the Schrödinger operator [8]

In the quasi-periodic setting, we are able to solve the equations for a finite jet of λ-derivatives of a formal branch of solutions of (2.10) at $\lambda = 0$. More precisely, we are interested in a k-continuous family of such jets.

Definition 2.3. Given $1 \le \ell < \infty$ and a nontrivial open set $O \subset \mathbb{R}^d$, a family $(\varphi_{k,n}, \nu_{k,n} : k \in O,\, 0 \le n \le \ell) \subset \mathrm{L}^2(\mathbb{T}^M) \times \mathbb{R}$ is called a *field of ℓ-jets of Bloch waves* if

(i) for all n, the map $O \to \mathrm{L}^2(\mathbb{T}^M) \times \mathbb{R} : k \mapsto (\varphi_{k,n}, \nu_{k,n})$ is continuous;

(ii) for all $k \in O$, we have $\varphi_{k,0} = 1$, and for all n the function $\varphi_{k,n+1}$ satisfies $\mathbb{E}[\varphi_{k,n+1}] = 0$ and
$$-\triangle_k \varphi_{k,n+1} = -\Pi\tilde{V}\varphi_{k,n} + \sum_{l=0}^{n-1} \mathbb{E}\left[\tilde{V}\varphi_{k,l}\right] \varphi_{k,n-l}, \tag{2.13}$$
where Π denotes the orthogonal projection onto $\{1\}^\perp$, that is $\Pi f := f - \mathbb{E}[f]$;

(iii) for all $n \ge 0$, we have $\nu_{k,n} = \mathbb{E}\left[\tilde{V}\varphi_{k,n}\right]$.

The $\varphi_{k,n}$'s are called the *correctors*, and the corresponding family $(\psi^\ell_{k,\lambda}, \kappa^\ell_{k,\lambda} : \lambda \ge 0,\, k \in O)$ of partial sums,
$$\psi^\ell_{k,\lambda} := \sum_{n=0}^{\ell} \lambda^n \varphi_{k,n}, \qquad \kappa^\ell_{k,\lambda} := \lambda\,\mathbb{E}\left[\tilde{V}\psi^\ell_{k,\lambda}\right] = \lambda \sum_{n=0}^{\ell} \lambda^n \nu_{k,n},$$
is called the *sheet of Taylor-Bloch waves* of order ℓ. Note that $\nu_{k,0} = 0$ if $\mathbb{E}\left[\tilde{V}\right] = 0$. $\lozenge$

This definition is motivated as follows: if for some fixed k there exists a branch $(\psi_{k,\lambda}, \kappa_{k,\lambda})_\lambda$ of Bloch waves that is analytic in a neighborhood of $\lambda = 0$, then we have for $|\lambda| \ll 1$,
$$\psi_{k,\lambda} = \sum_{n=0}^{\infty} \lambda^n \varphi_{k,n}, \qquad \text{and} \qquad \kappa_{k,\lambda} = \lambda\,\mathbb{E}[V\psi_{k,\lambda}] = \lambda \sum_{n=0}^{\infty} \lambda^n \nu_{k,n}, \tag{2.14}$$
where the couples $(\varphi_{k,n}, \nu_{k,n})$'s precisely satisfy the equations in the above definition. In the present situation, we only assume that the ℓ-jet $(\varphi_{k,n}, \nu_{k,n})_{0 \le n \le \ell}$ can be defined (and, if ℓ can be taken arbitrarily large, that the above series are not known to converge — otherwise we are back to the framework of exact Bloch waves as in the periodic setting), so that we may only consider the partial sums $(\psi^\ell_{k,\lambda}, \kappa^\ell_{k,\lambda})_{k,\lambda}$. As the following shows, for small λ, these Taylor-Bloch waves almost satisfy the eigenvalue equation (2.10).

Lemma 2.4. *Let $\ell \ge 1$, let $(\varphi_{k,n}, \nu_{k,n})_{k,n}$ be a field of ℓ-jets of Bloch waves, and let $(\psi^\ell_{k,\lambda}, \kappa^\ell_{k,\lambda})_{k,\lambda}$ be the corresponding sheet of Taylor-Bloch waves. Then we have*
$$(-\triangle_k + \lambda\tilde{V})\psi^\ell_{k,\lambda} = \kappa^\ell_{k,\lambda}\psi^\ell_{k,\lambda} + \lambda^{\ell+1}\mathfrak{d}^\ell_{k,\lambda},$$
in terms of the Taylor-Bloch eigendefect
$$\mathfrak{d}^\ell_{k,\lambda} := \Big(\Pi\tilde{V}\varphi_{k,\ell} - \sum_{l=0}^{\ell-1} \nu_{k,l}\varphi_{k,\ell-l}\Big) - \lambda \sum_{n=1}^{\ell} \sum_{l=\ell-n}^{\ell-1} \lambda^{n+l-\ell}\nu_{k,l+1}\varphi_{k,n}.$$
$\lozenge$

2.2.2. Taylor-Bloch waves for the wave operator [3]

The definition of Taylor-Bloch waves for the classical wave operator has a more involved structure than for the Schrödinger operator, and invokes an extension of the classical notion of correctors in the homogenization theory of elliptic operators.

Definition 2.5. For all $\ell \ge 0$, we say that $(\varphi_n, \sigma_n, \chi_n)_{0 \le n \le \ell} \in H^1(\mathbb{T}^M)$ are the first ℓ extended correctors in direction e if these functions are square-integrable, if $\mathbb{E}[(\varphi_n, \sigma_n, \nabla\chi_n)] = 0$ for all $0 < n \le \ell$, and if the following extended corrector equations are satisfied:

- $\varphi_0 \equiv 1$, and for all $n \geq 1$, φ_n is a scalar field that satisfies
$$-D \cdot \tilde{\mathbf{a}} D\varphi_n = D \cdot (-\sigma_{n-1} e + \tilde{\mathbf{a}} e \varphi_{n-1} + D\chi_{n-1});$$

- for all $n \geq 0$, the symmetric matrix $\check{\mathbf{a}}_n$, the symmetric $(n+2)$-th order tensor $\bar{\mathbf{a}}_n$, and the scalar ν_n are given by
$$\bar{\mathbf{a}}_n e^{\otimes(n+1)} = \check{\mathbf{a}}_n e := \mathbb{E}\left[\tilde{\mathbf{a}}(D\varphi_{n+1} + e\varphi_n)\right], \quad \nu_n := e \cdot \check{\mathbf{a}}_n e, \quad e^{\otimes(n+1)} := \underbrace{e \otimes \cdots \otimes e}_{n+1 \text{ times}};$$

- $\chi_0 \equiv 0$, $\chi_1 \equiv 0$, and for all $n \geq 2$, χ_n is a scalar field that satisfies
$$-D \cdot D\chi_n = D\chi_{n-1} \cdot e + \sum_{p=1}^{n-1} \nu_{n-1-p} \varphi_p;$$

- for all $n \geq 1$, q_n is a vector field (a higher-order flux) given by
$$q_n := \tilde{\mathbf{a}}(D\varphi_n + e\varphi_{n-1}) - \check{\mathbf{a}}_{n-1} e + D\chi_{n-1} - \sigma_{n-1} e, \quad \mathbb{E}\left[q_n\right] = 0;$$

- $\sigma_0 \equiv 0$, and for all $n \geq 1$, σ_n is a skew-symmetric matrix field (a higher-order flux corrector), i.e. $\sigma_{nkl} = -\sigma_{nlk}$, that satisfies
$$-D \cdot D\sigma_n = D \times q_n, \quad D \cdot \sigma_n = q_n,$$
with the three-dimensional notation: $[D \times q_n]_{lm} = D_l [q_n]_m - D_m [q_n]_l$, and where the divergence is taken with respect to the second index, i. e. $(D \cdot \sigma_n)_l := \sum_{m=1}^{d} D_m \sigma_{nlm}$.

$\Diamond$

Let us make a few comments on this definition.

- The correctors φ_n are related but do not coincide (for $n > 2$) with the higher-order correctors classically used in the multiscale expansion, and we refer the reader to [1] for a discussion of these differences in the periodic setting.

- The higher-order flux q_n is chosen to be divergence-free, so that it is an exact $(d-1)$-form and hence admits a "vector potential", that is a $(d-2)$-form, which can be represented by the skew-symmetric tensor σ_n (the equation for σ_n is the natural choice of gauge). These definitions are natural generalizations to any order of the extended correctors (φ, σ) considered in [10] (see below).

- The correctors φ_n are variants of those defined in [1]. They are however not normalized the same way.

- Let us quickly show that the first extended correctors (of order $n = 1$) are indeed the standard correctors in elliptic homogenization. The equation satisfied by φ_1 takes the form
$$-D \cdot \tilde{\mathbf{a}}(D\varphi_1 + e) \;=\; 0$$
so that φ_1 is the classical corrector in quasi-periodic homogenization. Thus $\bar{\mathbf{a}}_0 = \check{\mathbf{a}}_0 = \mathbf{a}_{\text{hom}}$ (the homogenized coefficients), $\nu_0 = e \cdot \mathbf{a}_{\text{hom}} e$. Hence, $q_1 = \mathbf{a}(D\varphi_1 + e) - \mathbf{a}_{\text{hom}} e$ (the flux of the corrector minus the homogenized flux), so that σ_1 is nothing but the flux corrector (the existence of which is proved in [10] for stationary ergodic coefficients $\mathbf{a}$).

The higher-order homogenized coefficients ν_n satisfy the following important properties:

Lemma 2.6. *Let $\ell \geq 2$ and consider the correctors $(\varphi_n, \sigma_n, \chi_n)_{0 \leq n \leq \ell+1}$ of Definition 2.5, and the well-defined higher-order homogenized coefficients $\{\nu_n = \mathbb{E}\left[e \cdot \tilde{\mathbf{a}}(D\varphi_{n+1} + e\varphi_n)\right]\}_{0 \leq n \leq \ell}$. Then:*

(i) *if $0 \leq n \leq \ell$ is odd, then $\nu_n = 0$;*

(ii) *$\nu_0 > 0$ and $\nu_2 \geq 0$.*

$\lozenge$

We are in the position to define Taylor-Bloch waves.

Definition 2.7. Let $k := \gamma e$ with $\gamma \in \mathbb{R}$ and let $(\varphi_n, \nu_n, \sigma_n, \chi_n)_{0\le n\le \ell}$ be as in Definition 2.5. The Taylor-Bloch wave ψ_k^ℓ, Taylor-Bloch eigenvalue κ_k^ℓ, and Taylor-Bloch eigendefect $\mathfrak{d}_k^\ell$ of order ℓ in direction k are defined by

$$\psi_k^\ell := \sum_{n=0}^{\ell} (i\gamma)^j \varphi_n, \quad \kappa_k^\ell := \gamma^2 \sum_{n=0}^{\ell-1} (i\gamma)^j \nu_n,$$

$$\mathfrak{d}_k^\ell = \nabla \cdot (-\sigma_\ell e + \tilde{\mathbf{a}} e \varphi_\ell + \nabla \chi_\ell) + i\gamma\Big(e \cdot \tilde{\mathbf{a}} e \varphi_\ell - \sum_{n=1}^{\ell} \sum_{l=\ell-n}^{\ell-1} (i\gamma)^{n+l-\ell} \nu_l \varphi_n \Big).$$

$\lozenge$

Note that by Lemma 2.6, κ_k^ℓ is real-valued since $\nu_{2n+1} = 0$ for all n. The interest of this definition is the following proposition, which establishes that the Taylor-Bloch wave ψ_k^ℓ is an eigenvector of the fibered operator $-(\nabla + ik) \cdot \tilde{\mathbf{a}}(\nabla + ik)$ on $\mathbb{T}^M$ for the eigenvalue κ_k^ℓ up to the eigendefect $\mathfrak{d}_k^\ell$.

Lemma 2.8. *Let $k = \gamma e$, and $\psi_k^\ell, \kappa_k^\ell, \mathfrak{d}_k^\ell$ be as in Definition 2.7 for some $\ell \geq 1$. Then we have*

$$-(\nabla + ik) \cdot \tilde{\mathbf{a}}(\nabla + ik)\psi_k^\ell = \kappa_k^\ell \psi_k^\ell - (i\gamma)^{\ell+1} \mathfrak{d}_k^\ell. \tag{2.15}$$

$\lozenge$

Compared to Lemma 2.4, the eigendefect in Lemma 2.8 has a specific structure: It is the sum of a term in conservative form (which allows to proceed by integration by parts for energy estimates) and of a term of higher order in γ.

3. Quantum waves

While the definition of the correctors is simpler for quantum waves, their existence and uniqueness theory is slightly more involved than for classical waves. To go from approximate spectral theory to long-time behavior, we then appeal to energy/semi-group estimates on the Schrödinger equation (we refer to [4] for details).

3.1. Existence of correctors

The existence and uniqueness theory of correctors follows from the following suitable Diophantine condition, which is expressed with respect to spheres rather than to points (as opposed to the usual situation in [9]).

Lemma 3.1 (Diophantine condition wrt spheres)**.** *Let $s_0 > M + d - 1$. For almost all $F \in \mathbb{R}^{d\times M}$, there exists a collection $(\mathcal{O}_R)_{R\geq 1}$ of open subsets $\mathcal{O}_R \subset \mathbb{R}^d$ (the so-called resonant sets) and a constant $C = C_{F,M,s_0} > 0$ such that*

(i) for all $R \geq 1$, we have for all $k \in \mathbb{R}^d \setminus \mathcal{O}_R$ and all $\xi \in \mathbb{Z}^M \setminus \{0\}$,

$$\big||F\xi + k|^2 - |k|^2\big| \geq R^{-1}|\xi|^{-s_0}; \tag{3.1}$$

(ii) for all $R \geq 1$, and all $\kappa > 0$, we have

$$|\mathcal{O}_R \cap B_\kappa| \leq CR^{-1}\kappa^{d-1};$$

(iii) the collection $(\mathcal{O}_R)_{R\geq 1}$ is decreasing with respect to R (that is, $\mathcal{O}_{R_2} \subset \mathcal{O}_{R_1}$ for all $R_2 \geq R_1 \geq 1$).

$\lozenge$

This Diophantine condition allows to invert the fibered operators $-\triangle_k$, and we easily deduce the following.

Proposition 3.2. *Assume that the Fourier transform (in the sense of Fourier series) of $\tilde V$ has compact support on $\mathbb{Z}^M$, and that the winding matrix F satisfies the Diophantine condition of Lemma 3.1. Then for all $R \geq 1$, $n \geq 1$, $k \in \mathbb{R}^d \setminus \mathcal{O}_R$, there exists a unique smooth quasi-periodic corrector $\varphi_{k,n} \in H^\infty(\mathbb{T}^M)$.* $\Diamond$

3.2. Long-time behavior of the Schrödinger flow

Using approximate spectral theory, one can show that the long time behavior of the solution of the Schrödinger equation is described by the solution of a higher-order pseudo-differential equation. For $\ell \geq 0$, we define the pseudo-differential operator

$$\mathcal{L}^0_{\lambda,\ell} := -\triangle + \kappa^\ell_{-i\nabla,\lambda} = -\triangle + \lambda \sum_{n=0}^{\ell} \lambda^n \mathbb{E}\left[V\varphi_{-i\nabla,n}\right],$$

where we use usual pseudo-differential notation and where $k \mapsto \kappa^\ell_{k,\lambda}$ and $k \mapsto \varphi_{k,n}$ are defined in Definition 2.3. Then, the following holds:

Theorem 3.3. *Let $\ell \geq 1$ and let $u^\circ \in \mathrm{L}^2(\mathbb{R}^d)$. For all $\lambda > 0$, let u_λ denote the solution of*

$$i\partial_t u_\lambda = (-\triangle + \lambda V)u_\lambda, \qquad u_\lambda(0,\cdot) = u^\circ,$$

and let $w_{\lambda,\ell}$ denote the (unique) solution of the pseudo-differential equation

$$i\partial_t w_{\lambda,\ell} = \mathcal{L}^0_{\lambda,\ell} w_{\lambda,\ell}, \qquad w_{\lambda,\ell}(0,\cdot) = u^\circ.$$

Then we have for all $T \geq 0$

$$\sup_{0\leq t\leq T} \|u_\lambda - w_{\lambda,\ell}\|_{\mathrm{L}^2(\mathbb{R}^d)} + \sup_{0\leq t\leq T} (1+t)^{-1} \||x|(u_\lambda - w_{\lambda,\ell})\|_{\mathrm{L}^2(\mathbb{R}^d)} \leq C_\ell(u^\circ)(\lambda + T\lambda^\ell),$$

where $C_\ell(u^\circ)$ is a generic Sobolev norm of u° which only depends on ℓ and d, and is finite provided $u^\circ \in \mathcal{S}(\mathbb{R}^d)$. $\Diamond$

Since the approximate solutions $w_{\lambda,\ell}$ obviously display ballistic transport, we deduce asymptotic ballistic transport for u_λ up to superalgebraic times in λ^{-1}. In [4], a very subtle combinatorial argument is used to obtain optimal estimates on the correctors, which allow to determine the ℓ-dependence in the constant $C_\ell(u^\circ)$ above and to conclude with asymptotic ballistic transport in stretched exponential times in λ^{-1}.

4. Classical waves

For classical waves, the subtle part is the definition of the correctors, whereas their existence and uniqueness follows from standard techniques (in the quasi-periodic setting). To go from approximate spectral theory to long-time behavior, we then appeal to energy estimates on the wave equation, and exploit in particular the very specific structure of the eigendefect (we refer to [3] for details).

4.1. Bounds on the correctors

Assume $\mathbf{a}$ is a symmetric matrix field that satisfies $\mathbf{a} \geq \frac{1}{C}\,\mathrm{Id}$ (in the sense of symmetric matrices) for some $C < \infty$. The existence and uniqueness of correctors for smooth quasi-periodic coefficients essentially follow from the argument by Kozlov [15], which relies on a Diophantine condition in the form of a weak Poincaré inequality, on Garding's inequality, and elliptic regularity (see also [9, Theorem 4] for details). This approach allows to prove:

Proposition 4.1. *Let $\tilde{\mathbf{a}}$ be a smooth coefficient field on a higher-dimensional torus $\mathbb{T}^M$. For almost every winding F, set $\mathbf{a} : \mathbb{R}^d \to \mathbb{R}^{d\times d}, x \mapsto \tilde{\mathbf{a}}(Fx)$. Then for all $n \geq 1$ there exist unique smooth quasi-periodic extended correctors $\varphi_n, \sigma_n, \chi_n \in H^\infty(\mathbb{T}^M)$ with zero average (which are in particular bounded).* $\Diamond$

4.2. Long-time behavior of the wave equation

Using approximate spectral theory, one can show that the long time behavior of the solution of the wave equation is described by the solution of a higher-order operator, which we presently define. Let $\ell \geq 0$. We choose some $\gamma_\ell \geq 0$ depending only on $\bar{\Gamma}_\ell = \max_{0\leq j\leq \ell-1} |\bar{\mathbf{a}}_j| < \infty$, ℓ, and $\frac{1}{C}$, such that the higher-order elliptic operator

$$\mathcal{L}_{\mathrm{hom},\varepsilon,\ell} := -\sum_{j=0}^{\ell-1} \varepsilon^j \bar{\mathbf{a}}_j \cdot \nabla^{j+2} - \gamma_\ell (i\varepsilon)^{2([\frac{\ell-1}{2}]+1)} \,\mathrm{Id} \cdot \nabla^{2([\frac{\ell-1}{2}]+2)}, \tag{4.1}$$

satisfies for all $v \in H^{[\frac{\ell-1}{2}]+2}(\mathbb{R}^d)$ and all $\varepsilon > 0$

$$(\mathcal{L}_{\mathrm{hom},\varepsilon,\ell} v, v)_{(H^{-([\frac{\ell-1}{2}]+2)}, H^{[\frac{\ell-1}{2}]+2})(\mathbb{R}^d)} \geq \frac{1}{2C} (\|\nabla v\|^2_{L^2(\mathbb{R}^d)} + \varepsilon^{2[\frac{\ell-1}{2}]+2} \|\nabla^{[\frac{\ell-1}{2}]+2} v\|^2_{L^2(\mathbb{R}^d)}).$$

For $1 \leq \ell \leq 2$, we may choose $\gamma_\ell = 0$. Then the following long-time behavior of the solution of the wave equation holds:

Theorem 4.2. *Let $\ell \geq 1$, let $\gamma_\ell \geq 0$ be as above, and let $u^\circ \in \mathcal{S}(\mathbb{R}^d)$. For all $\varepsilon > 0$, let u_ε and $w_{\varepsilon,\ell}$ denote the solutions of*

$$\left\{ \begin{array}{rcl} \partial^2_{tt} u_\varepsilon - \nabla \cdot \mathbf{a}(\frac{\cdot}{\varepsilon}) \nabla u_\varepsilon &=& 0, \\ u_\varepsilon(0,\cdot) &=& u^\circ, \\ \partial_t u_\varepsilon(0,\cdot) &=& 0, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} \partial^2_{tt} w_{\varepsilon,\ell} + \mathcal{L}_{\mathrm{hom},\varepsilon,\ell} w_{\varepsilon,\ell} &=& 0, \\ w_{\varepsilon,\ell}(0,\cdot) &=& u^\circ, \\ \partial_t w_{\varepsilon,\ell}(0,\cdot) &=& 0. \end{array} \right.$$

Then we have for all $T \geq 0$

$$\sup_{0\leq t\leq T} \|u_\varepsilon - w_{\varepsilon,\ell}\|_{L^2(\mathbb{R}^d)} \leq C_\ell(u^\circ) \big(\varepsilon + \varepsilon^\ell T\big),$$

where $C_\ell(u^\circ)$ is a generic norm of u° which only depends on ℓ and d, and is finite for $u^\circ \in \mathcal{S}(\mathbb{R}^d)$. $\Diamond$

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