Propagation of chaos and corrections to mean-field for interacting particles

Mitia Duerinckx

We consider a system of N classical particles in the torus $\mathbb{T}^d$, interacting via a smooth potential V, in the mean-field regime: trajectories are given by Newton's equations in the phase-space $\mathbb{D} = \mathbb{T}^d \times \mathbb{R}^d$,

$$\partial_t x_j = v_j, \qquad \partial_t v_j = -\tfrac{1}{N}\textstyle\sum_{l:l\neq j}^N \nabla V(x_j - x_l), \qquad \text{for } 1 \le j \le N. \tag{1}$$

For a statistical description, we consider a random ensemble of trajectories: for simplicity, we choose initial data $\{(x_j^\circ, v_j^\circ)\}_{1\le j\le N}$ to be independent and identically distributed (iid) with some smooth law F° on $\mathbb{D}$. In terms of the probability density F_N for the ensemble of particles on the N-particle phase-space $\mathbb{D}^N$, Newton's equations (1) are equivalent to the Liouville equation

$$\partial_t F_N + \textstyle\sum_{j=1}^N v_j \cdot \nabla_{x_j} F_N = \tfrac{1}{N}\textstyle\sum_{j\neq l} \nabla V(x_j - x_l) \cdot \nabla_{v_j} F_N, \tag{2}$$

with chaotic initial data $F_N|_{t=0} = (F^\circ)^{\otimes N}$. Looking for a simplified description of the system, we define the m-particle probability density as the mth marginal $F_N^m(z_1, \ldots, z_m) = \int_{\mathbb{D}^{N-m}} F_N(z_1, \ldots, z_N)\, dz_{m+1} \ldots dz_N$, with the notation $z_j = (x_j, v_j)$. For a large number $N \gg 1$ of particles, the 1-particle density F_N^1 remains close to the solution F of the Vlasov equation

$$\partial_t F + v \cdot \nabla_x F = (\nabla V * F) \cdot \nabla_v F, \tag{3}$$

with $F|_{t=0} = F^\circ$. We refer to [1] for a review of this well-travelled mean-field result. Formally, starting from the BBGKY hierarchy, the Vlasov equation is obtained by neglecting 2-particle correlations, thus replacing F_N^2 by $(F_N^1)^{\otimes 2}$ in the equation for F_N^1. A rigorous proof was obtained in the 1970s following Klimontovich's ideas: starting from the representation $F_N^1 = \mathbb{E}[\mu_N]$ in terms of the empirical measure

$$\mu_N = \tfrac{1}{N}\textstyle\sum_{j=1}^N \delta_{(x_j, v_j)},$$

one notices that μ_N is a distributional solution of the Vlasov equation (3) and that initially $\mu_N|_{t=0}$ converges weakly to F° a.s., hence the stability of the Vlasov equation in weak topology ensures the a.s. weak convergence $\mu_N \rightharpoonup F$. This further entails $F_N^m \rightharpoonup F^{\otimes m}$ for all $m \ge 1$, which is known as propagation of chaos.

Corrections to this mean-field theory are driven by the 2-particle correlation function $G_N^2 = F_N^2 - (F_N^1)^{\otimes 2}$ and are formally obtained by only neglecting 3-particle correlations in the BBGKY hierarchy. A rigorous proof requires to show that the 2-particle correlation function is of order $G_N^2 = O(\frac{1}{N})$, while 3-particle correlations are of higher order $G_N^3 = O(\frac{1}{N^2})$. Such a refined version of propagation of chaos is provided by the following main result.

Theorem A (see [2, Theorem 1]).
For $m \ge 0$, the $(m+1)$-particle correlation function G_N^{m+1} satisfies for all $t \ge 0$,

$$\|G_N^{m+1}(t)\|_{W^{-2m,1}(\mathbb{D}^{m+1})} \le N^{-m}\, C_m e^{C_m t},$$

where C_m only depends on d, m, V, F°.

Due to the loss of derivatives (cf. ∇_v in the right-hand side of (2)), this result cannot be deduced from the BBGKY hierarchy — in stark contrast with e.g. the quantum mean-field setting in [3]. Instead, as for the mean-field result, we develop an approach à la Klimontovich based on the empirical measure μ_N. First, we note that G_N^2 is equivalent to the variance of μ_N; the following representation formula holds more generally for all $m \geq 1$ and $\phi \in C_b(\mathbb{D})$,

$$\int_{\mathbb{D}^m} \phi^{\otimes m} G_N^m \;=\; \kappa_m\big[\int_{\mathbb{D}} \phi\, d\mu_N\big] \;+\; \text{lower-order terms},$$

where $\kappa_m[\cdot]$ stands for the mth cumulant. Next, to estimate the variance of μ_N (and its higher-order cumulants), we appeal to discrete stochastic calculus techniques with respect to iid data. For a random variable Y, we define its Glauber derivative with respect to the initial data of the jth particle as $D_j Y \;=\; Y - \mathbb{E}_j[Y]$, where $\mathbb{E}_j[\cdot]$ stands for the expectation with respect to (x_j°, v_j°) only. In these terms, a variance estimate is given by the following Efron–Stein inequality [4],

$$\mathrm{Var}[Y] \;\leq\; \sum_{j=1}^N \mathbb{E}[(D_j Y)^2].$$

Noting the similarity to Malliavin calculus and arguing as in [5], we prove corresponding cumulant estimates in form of higher-order Poincaré inequalities. We are then reduced to evaluating the multiple Glauber derivatives of μ_N with respect to iid data. Sensitivity estimates for trajectories are easily performed since the mean-field regime corresponds to weak interactions: we find for instance

$$\max_{j\neq l} |D_l(x_j^t, v_j^t)| \;\leq\; N^{-1}\, Ce^{Ct}.$$

Combining these different ingredients yields the conclusion of Theorem A.

As a consequence, the above correlation estimates can be used to rigorously truncate the BBGKY hierarchy to any order, and justify the so-called Bogolyubov corrections to the mean-field Vlasov limit. Alternatively, cumulant estimates also yield an optimal quantitative central limit theorem for μ_N, thus improving on the well-established qualitative result in [6]. We refer to [2, Sections 5–7] for details.

In a spatially homogeneous system $F^\circ(x,v) = f^\circ(v)$, the mean-field force vanishes and the Vlasov solution remains constant $F \;\equiv\; f^\circ$. The evolution of the 1-particle density is then described to leading order by the Bogolyubov correction, which takes on the following guise for the velocity density $f_N^1(v) = \int_{\mathbb{T}^d} F_N^1(x,v)\, dx$,

$$\left\{\begin{array}{l} \partial_t f_N^1 \sim \frac{1}{N} \int_{\mathbb{T}^d} \int_{\mathbb{D}} \nabla V(x-x_*) \cdot \nabla_v (NG_N^2)(x,v,x_*,v_*)\, dx_* dv_* dx, \\ \partial_t (NG_N^2) + iL_{f_N^1}(NG_N^2) \sim \nabla V(x_1 - x_2) \cdot (\nabla_{v_1} - \nabla_{v_2})(f_N^1 \otimes f_N^1), \end{array}\right.$$

where $iL_{f_N^1}$ stands for the linearized Vlasov operator at f_N^1. The effect of particle correlations takes form of a non-Markovian collision process. However, the equations display a timescale separation: f_N^1 evolves on the slow timescale $t = O(N)$ while NG_N^2 evolves on the mean-field timescale $t = O(1)$. In view of linear Landau damping in form of weak relaxation for $iL_{f_N^1}$, we may thus formally replace NG_N^2 in the first equation by its long-time limit as obtained from the second. After tedious computations, as predicted by Guernsey, Balescu, and Lenard independently in 1960 (e.g. [7, Appendix A]), this yields the so-called Lenard–Balescu equation

$$\partial_t f_N^1 \sim \tfrac{1}{N}\,\mathrm{LB}(f_N^1), \qquad \mathrm{LB}(f) = \nabla \cdot \int_{\mathbb{R}^d} B(v, v - v_*; \nabla f)\,(f_* \nabla f - f \nabla_* f_*)\,dv_*,$$

where the collision kernel B brings a strong nonlinearity and is explicit in terms of the potential V. This equation is viewed as a correction to Landau's equation, further taking into account collective screening effects. It satisfies an H-theorem and formally describes relaxation to Maxwellian equilibrium on the slow timescale $t = O(N)$. Due to dynamical screening, even local well-posedness is a reputedly difficult open problem in the Coulomb setting. For a smooth potential V, a work in progress with R. Winter proves global well-posedness close to equilibrium.

Justifying physicists' calculations for the relaxation of NG_N^2, we obtain the following result with L. Saint-Raymond [8, 2]. Note that the exponential time growth in Theorem A requires to restrict to the intermediate timescale $1 \ll t \ll \log N$: although missing the kinetic timescale $t = O(N)$, this constitutes the first rigorous result in this direction starting from particle system (1).

Theorem B (see [2, Corollary 4]).
Given F° spatially homogeneous, compactly supported, and linearly Vlasov-stable, and given V smooth and small enough, there holds for $1 \ll t_N \ll \log N$,

$$N\partial_t f_N^1|_{t=t_N\tau} \;\sim\; \mathrm{LB}(f^\circ), \qquad \textit{in } \mathcal{D}'_{\tau,v}(\mathbb{R}^+ \times \mathbb{R}^d).$$

In order to reach the kinetic timescale $t = O(N)$, the correlation estimates of Theorem A are no longer applicable: propagation of chaos needs to be complemented with some decorrelation mechanism. In [8], with L. Saint-Raymond, we consider a linearized setting: time-uniform estimates on linear correlation functions then follow from an orthogonality argument as in [9]. Yet, due to resonant effects that are reminiscent of plasma echoes, these estimates only allow to extend Theorem B to $1 \ll t_N \ll N^{\frac{1}{4}}$, and all improvements remain open questions.

References

[1] F. Golse. On the dynamics of large particle systems in the mean field limit. In *Macroscopic and large scale phenomena*, pp. 1–144. Springer, 2016.

[2] M. Duerinckx. *On the size of chaos via Glauber calculus in the classical mean-field dynamics.* Commun. Math. Phys. (2020).

[3] T. Paul, M. Pulvireneti, S. Simonella. *On the Size of Chaos in the Mean Field Dynamics.* Arch. Ration. Mech. Anal. **231** (2019), 285–317.

[4] B. Efron, C. Stein. *The jackknife estimate of variance.* Ann. Statist. **9** (1981), 586–596.

[5] I. Nourdin, G. Peccati. *Cumulants on Wiener space.* J. Funct. Anal. **11** (2010), 3775–3791.

[6] W. Braun, K. Hepp. *The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles.* Commun. Math. Phys. **56** (1977), 101–113.

[7] D. R. Nicholson. *Introduction to Plasma Theory.* John Wiley & Sons Inc. (1983).

[8] M. Duerinckx, L. Saint-Raymond. *Lenard–Balescu correction to mean-field theory.* Prob. Math. Phys. (2020).

[9] T. Bodineau, I. Gallagher, L. Saint-Raymond. *From hard sphere dynamics to the Stokes-Fourier equations: an L^2 analysis of the Boltzmann-Grad limit.* Ann. PDE **3** (2017), Art. 2.