TOPICS IN THE MATHEMATICS OF DISORDERED MEDIA

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Foreword

« Dites ! quels temps versés au gouffre des années,
Et quelle angoisse ou quel espoir des destinées,
Et quels cerveaux chargés de noble lassitude
A-t-il fallu pour faire un peu de certitude ? »

Émile Verhaeren

This thesis was carried out from October 2014 onwards during a three-year PhD program in the Département de Mathématique at the Université Libre de Bruxelles (ULB, Belgium) and in the Laboratoire Jacques-Louis Lions (LJLL) at the Université Pierre et Marie Curie (UPMC, France), under the co-supervision of Prof. Antoine Gloria and Prof. Sylvia Serfaty. It was supported financially by the Fonds de la Recherche Scientifique F.R.S.-FNRS through a Research Fellowship, while additional financial support was provided by the European Research Council under the European Community's Seventh Framework Programme (FP7/2014-2019 Grant Agreement QUANTHOM 335410). The present manuscript collects different results obtained during this PhD, aiming at a better mathematical understanding of the effects of disorder in various physical systems. Starting with some classical stochastic homogenization questions, we next investigate fluctuations around the homogenization limit, and in a last part we focus on the interplay between interactions and disorder in the context of the Ginzburg-Landau superconducting vortices.

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Chapter 1

Introduction

« L'analyse mathématique [...] n'est-elle donc qu'un vain jeu de l'esprit ? Elle ne peut donner au physicien qu'un langage commode ; n'est-ce pas là un médiocre service, dont on aurait pu se passer à la rigueur ; et même, n'est-il pas à craindre que ce langage artificiel ne soit un voile interposé entre la réalité et l'œil du physicien ? Loin de là, sans ce langage, la plupart des analogies intimes des choses nous seraient demeurées à jamais inconnues ; et nous aurions toujours ignoré l'harmonie interne du monde, qui est [...] la seule véritable réalité objective. »

Henri Poincaré, *La valeur de la science.*

Though often regarded as an imperfection that must be driven away from experiments wherever possible in order to get closer to predictions of "ideal" theories, disorder appears to be intrinsic to various physical systems and can often not be realistically avoided. Moreover, disorder may sometimes lead to radically new phenomena that are not predicted by classical theories. Crucial in an ever-increasing number of applications, the physics of disordered media has only started in recent decades to emerge consciously as a new domain in its own right, full of challenges both for theoretical and for experimental physicists [192, 194].

The first studies on the effects of disorder are related to the notion of effective behavior, which was gradually developed in the course of the 19th century. This started around 1820 with the works by Navier and Cauchy, who viewed matter as an assemblage of material molecules and formally derived from a discrete "molecular" model suitable "effective" equations describing elastic continua. A similar micro-mechanical perspective was at the basis of Poisson's theory of induced magnetism [364] and of Faraday's theory of dielectric materials [179], both derived from a model of small conducting particles distributed in a nonconducting matrix. This heterogeneous model was subsequently studied by Mossotti [324, 325], Maxwell [317], Clausius [119], Lorentz [308], Lorenz [309], Rayleigh [368], and others. The main focus at that time was thus on two-phase dispersed media composed of a main homogeneous material with small foreign inclusions, and on the definition of their "effective" or "homogenized" properties, which typically differ from the properties of the constituents. On large scales the microstructural detail is somehow averaged out due to a kind of law of large numbers, and heterogeneous physical properties are replaced by homogeneous ones. This process justifies the importance of the study of constant-coefficient equations, although in an intrinsically random and composite world. Formulas were further soon predicted for the deviations caused by the disorder, a topic on which we shall come back in the course of this thesis. A clear parallel is to be noted between these scientific developments and the pictorial experiments of neoimpressionist painters of the last quarter of the 19th century, such as Seurat or Signac, which led to pointillism: at a certain distance the tones are recomposed by the viewer's eye from the small dots of pure color that make up the painting.

The first steps towards a mathematical theory of homogenization are traced back to the first half of the 20th century, with the development of averaging methods for ODEs in nonlinear dynamics by Poincaré [363], von Zeipel [421], van der Pol [416], Krylov, Bogolyubov, and Mitropolsky [276, 71], and others. The general theory of homogenization of PDEs, as a mathematically rigorous approach to composite materials, emerged only in the 1970s, at the crossroads between probability and analysis of PDEs. Not surprisingly, this relatively new field has been heavily fueled by modern technological applications and by the need for improving our knowledge of composites. A further motivation stems from shape optimization and optimal structural design problems in modern engineering: given two materials, we aim at finding the best arrangement that maximizes some overall physical property (like conductivity or elastic stiffness) under e.g. some volume constraint on the "best" of the two materials (which is typically more expensive or heavier). Optimality can usually not be achieved by any given mixture: it is more advantageous to split inclusions into many tinier ones, so that the optimization naturally leads to composite materials with fine microstructure. The use of homogenization theory in this context was initiated by Murat and Tartar already in the late 1970s (see e.g. [12] and references therein).

In many examples, the physical properties of heterogeneous media remain of the same type on large scales: models with heterogeneous coefficients are replaced by the same ones with homogeneous *effective* coefficients, and homogenization is then really a matter of defining and computing the latter. A complete turn was prompted when it became apparent that disorder could sometimes lead to radically new phenomena. The first discovery in this respect is the kinetic theory of Brownian motion developed at the dawn of the 20th century by Sutherland [406], Einstein [175], and von Smoluchowski [420], later theorized by Wiener [423]. The erratic motion of a pollen suspended in water is caused by continuous kicks by lighter water molecules, which lead to an overall diffusive motion, that is, the averaged position of the pollen grows as the square root of time rather than linearly. More precisely, the averaged position of the pollen satisfies an irreversible diffusion equation. Note that this leads to a classical apparent contradiction with the reversibility of the microscopic Hamiltonian mechanics describing the underlying collision process with water molecules. The key to this contradiction is the loss of information as the macroscopic diffusive motion is obtained by neglecting or integrating out many degrees of freedom on small scales [68]. This kinetic theory for Brownian motion truly revolutionized the understanding of the importance of disorder, and led to the emergence of a new paradigm in physics [192]: more than being omnipresent in nature, disorder can have new, non-classical effects.

Another important discovery in this respect concerns conducting materials, where the typically disordered ionic lattice constitutes obstacles to the flow of electrons and transforms their free ballistic motion into a diffusive motion. This precisely creates resistance, that is, the partial conversion of the energy of electrons into heat. The more regular the ionic lattice is, the less the flow of electrons can be disturbed, and consequently, the resistance decreases. Resistance is indeed another crucial example of an effect of disorder. In some cases, random impurities in the metal can actually not only slow down electrons, but even completely stop their flow, leading to an insulating material. This surprising electron localization phenomenon was first predicted by Anderson [22] in 1958, but its full mathematical understanding still remains very challenging.

These two discoveries — the kinetic theory of Brownian motion and the Anderson localization — really marked the emergence of the physics of disordered media as a new domain in its own right. One of the main current challenges for physicists in this area consists in the understanding of mixed effects of disorder and interactions. While interactions tend to make particles behave as an ordered, coherent whole, disorder competes with this order and typically leads to a large number of nearly degenerate energy states separated by huge energy barriers. Interactions thus modify the effects of disorder drastically and lead to a new "glassy" physics, with remarkable static and dynamic properties that are still largely ununderstood [193]. This competition between interactions and disorder is generically realized by elastic-like systems in disordered media, which can have various microscopic origins, ranging from Ginzburg-Landau vortices in type-II superconductors [195, 369] to domain walls in magnetic or ferroelectric systems [355]. This last example is particularly important in applications, as it is found in any computer hard drive.

In this thesis, we start with considering various stochastic homogenization problems in connection with physics questions from the 19th century (Chapters 2–5). In Chapter 2, motivated by the rigorous derivation of rubber elasticity from the statistical physics of polymer-chain networks, we establish the existence of (and we provide ways to compute) the effective properties of heterogeneous hyperelastic materials under quite general assumptions. In Chapter 3, we propose to go beyond the existence and convergence of effective properties and discuss fluctuations. Considering for simplicity the easiest linearized model, we establish the first complete pathwise theory of fluctuations. In Chapter 5, again for simple linear models, we investigate the explicit first-order formulas developed by Mossotti, Maxwell, and Clausius for the effective properties of two-phase dispersed media. We provide the first general and rigorous proof of the so-called Clausius-Mossotti formula, as well as an extension to higher orders. In a second part (Chapters 6–8), we focus on more complicated systems and study the dynamical behavior of Ginzburg-Landau vortices in type-II superconductors in the presence of impurities. Although a complete mathematical understanding of the glassy properties of such systems seems out of reach, we establish the mean-field limit of a large number of vortices, and subsequently investigate the homogenization of these mean-field equations and their peculiar properties.

Whether in the context of rubber elasticity or of glassy properties of vortex systems, a leitmotiv in this work is the rigorous derivation of phenomenological physics equations from first principles by justifying the needed successive limits, for which homogenization theory is in some cases a crucial tool. This is precisely the object of Hilbert's Sixth Problem, asked in the occasion of the International Congress of Mathematicians in Paris in 1900, which concerns the mathematical treatment of the Axioms of Physics, "developing mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua" [239]. One can indeed only be astonished at the incredible diversity of physical phenomena that one observes, all supposed to be deductible from very few fundamental principles governing physics.

In the sequel of this main introduction, we shortly describe and contextualize the content of each chapter. Precise statements, complete references, full details, and many perspectives and open problems are however postponed to the introductions of the chapters themselves.

Contents

1.1 Topics in stochastic homogenization (Part I)

The first part of this thesis is devoted to several topics in stochastic homogenization — the study of macroscopic effective properties of heterogeneous media. Let $\varepsilon > 0$ denote the typical scale of microstructures, that is, the ratio between typical microscopic and macroscopic scales, so that all physical characteristics of the considered material are described as functions of the form $A(\frac{\cdot}{\varepsilon})$. Depending on the nature of heterogeneities, the function A can be periodic, almost periodic, or more generally a typical realization of a stationary random field. Stationarity is a natural assumption here, as it means that microstructures look about the same everywhere, in the sense that their law is translation-invariant. At a technical level, we shall systematically express stationarity in terms of equivariance under a measurable action of the additive group $(\mathbb{R}^d, +)$ on the probability space (cf. Section 2.A.2 for detail), which conveniently places us in the realm of ergodic theory. Ergodicity of this group action — seen as a minimality condition on the probability space — is always assumed in the sequel. Depending on the physical system in consideration, the fast oscillating coefficients $A(\frac{\cdot}{\varepsilon})$ enter the corresponding PDEs. Due to small-scale variations, such PDEs are typically impossible to solve in practice and we rather aim at a suitable asymptotic analysis in the limit $\varepsilon \downarrow 0$. In other words, we are interested in the large-scale effective properties obtained after averaging out over smaller scales. Understanding this averaging process may be difficult, depending on the considered PDE, and is the purpose of the homogenization theory (see e.g. [50, 265] for general references). In this first part of the thesis, we focus on stationary problems. We start with very general variational functionals associated with second-order nonlinear operators motivated by nonlinear elasticity, for which we establish some new *qualitative* results on existence and definition of effective properties. Subsequently, we focus on the simpler linearized setting and turn to more involved *quantitative* homogenization questions.

1.1.1 Homogenization of nonconvex unbounded integral functionals (Chapter 2)

In Chapter 2 we consider the well-travelled problem of stochastic homogenization of a nonlinearly hyperelastic material. Assume that the reference configuration of some sample is given by a bounded open subset $O \subset \mathbb{R}^d$ and that the material is heterogeneous with microstructures at the small scale $\varepsilon > 0$. The elastic energy of the sample, subject to a displacement $u : O \to \mathbb{R}^d$, then takes the form

$$I_\varepsilon(u) := \int_O W(\tfrac{x}{\varepsilon}, \nabla u(x))dx, \tag{1.1}$$

in terms of the (heterogeneous) energy density $W : \mathbb{R}^d \times \mathbb{R}^{d\times d} \to [0, \infty]$. As $\varepsilon \downarrow 0$, the fine structure of the material becomes irrelevant, and overall properties are expected to be described by a simpler *homogenized* energy functional of the form

$$I(u) := \int_O W_{\text{hom}}(\nabla u(x))dx, \tag{1.2}$$

in terms of a so-called homogenized energy density $W_{\text{hom}} : \mathbb{R}^{d\times d} \to [0, \infty]$ that does no longer depend on the microscopic space variable. More precisely we mean the following: for any external force $f : O \to \mathbb{R}^d$, any subset $E \subset \partial O$, and any Dirichlet boundary data $h : E \to \mathbb{R}^d$ on E, the minimizers and the minimal value of the minimization problem

$$\min_{u\,:\,u|_E = h} \int_O \big(W(\tfrac{x}{\varepsilon}, \nabla u(x)) - f(x)\cdot u(x)\big)dx \tag{1.3}$$

should converge in some sense to the minimizers and the minimal value of the constant-coefficient problem

$$\min_{u\,:\,u|_E = h} \int_O \big(W_{\text{hom}}(\nabla u(x)) - f(x)\cdot u(x)\big)dx.$$

Due to the violent oscillations expected for minimizers at the scale ε, convergence is naturally expected here in the weak sense of Sobolev spaces. The natural framework for this asymptotic analysis is De Giorgi's notion of Γ-convergence [139], which is a variational convergence that ensures, under some equicoercivity assumption, the convergence of minimizers and of minimal values, as well as the stability with respect to continuous perturbations (in this case, with respect to the external force f). We refer to [133, 78] for general references.

Usual properties of Γ-limits with Dirichlet boundary data ensure that the homogenized integrand W_{hom} in (1.2) is quasiconvex in the sense of

$$W_{\text{hom}}(\Lambda) \;=\; \min_{u\,:\,u|_{\partial O}=0} \fint_O W_{\text{hom}}(\Lambda + \nabla u),$$

and this leads to the following asymptotic homogenization formula, for all $\Lambda \in \mathbb{R}^{d\times d}$,

$$W_{\text{hom}}(\Lambda) \;=\; \lim_{\varepsilon\downarrow 0} \min_{u\,:\,u|_{\partial O}=0} \fint_O W(\tfrac{x}{\varepsilon}, \Lambda + \nabla u(x))dx \;=\; \lim_{R\uparrow\infty} \min_{u\,:\,u|_{\partial Q_R}=0} \fint_{Q_R} W(y, \Lambda + \nabla u(y))dy, \tag{1.4}$$

which characterizes W_{hom} in terms of W, where $Q_R := [-\frac{R}{2}, \frac{R}{2})^d$ is the centered cube of side-length R. In the periodic convex setting, that is, when $W(y,\cdot)$ is convex on $\mathbb{R}^{d\times d}$ for all $y \in \mathbb{R}^d$ and when $W(\cdot,\Lambda)$ is 1-periodic for all $\Lambda \in \mathbb{R}^{d\times d}$, the asymptotic formula (1.4) is well-known to reduce to a minimization problem on a unit cell with periodic boundary conditions [314, 331], that is,

$$W_{\text{hom}}(\Lambda) \;=\; \min_{\phi \text{ periodic}} \int_Q W(y, \Lambda + \nabla\phi(y))dy. \tag{1.5}$$

The minimizer ϕ_Λ for this cell problem is called the *corrector* in the direction Λ. Note that this single-cell formula (1.5) is false in the nonconvex case due to a possible buckling phenomenon [331, 44]. In the random convex setting, an abstract version of the single-cell formula (1.5) holds as well (cf. Lemma 2.2.7),

$$W_{\text{hom}}(\Lambda) \;=\; \min_{\nabla\phi \text{ stationary}} \mathbb{E}\left[W(0, \Lambda + \nabla\phi(0))\right], \tag{1.6}$$

where minimization is on the set of all gradient-like stationary fields. Due to the failure of the usual Poincaré inequality on the probability space (as it would correspond to a Poincaré inequality in infinite volume), the main issue here is that the corresponding corrector ϕ_Λ (that is, the solution of this abstract cell problem (1.6)) should not be stationary. Only its gradient $\nabla\phi_\Lambda$ is stationary so that ϕ_Λ may actually grow at infinity, in contrast with the periodic case. Stationarity of $\nabla\phi_\Lambda$ nevertheless implies sublinearity of ϕ_Λ at infinity (cf. Lemma 2.2.4), which will happen to be sufficient for our purposes here.

The main tasks in qualitative homogenization theory consist in proving the desired Γ-convergence result for I_ε with any boundary condition, in justifying a homogenization formula to characterize the homogenized energy density W_{hom}, and in establishing qualitative properties of W_{hom} (such as coercivity, quasiconvexity, and upper bounds). For simplicity in this introduction we focus on the periodic case. The main homogenization results previously known for integral functionals of the form (1.1) are due to Marcellini [314], Braides [77], and Müller [331], and cover the following two situations,

— $\frac{1}{C}|\Lambda|^p \le W(y,\Lambda) \le C(1+|\Lambda|^p)$ for all y,Λ, with $p>1$ and with $W(y,\cdot)$ nonconvex and locally Lipschitz for all y;

— $\frac{1}{C}|\Lambda|^p \le W(y,\Lambda)$ and $\sup_z W(z,\Lambda) < \infty$ for all y,Λ, with $p>d$ and with $W(y,\cdot)$ convex for all y.

These results have recently been generalized by Anza Hafsa and Mandallena [25], who considered the following,

— $\frac{1}{C}|\Lambda|^p \le M(\Lambda) \le W(y,\Lambda) \le C(1+M(\Lambda))$ for all y,Λ, with $p>d$, with M convex, and with $W(y,\cdot)$ nonconvex and radially uniformly upper semicontinuous (ru-usc in short, cf. Definition 2.1.5) for all y.

Note that in each of these situations the domain of the map $\Lambda \mapsto W(y,\Lambda)$ is convex and does not depend on the space variable $y \in \mathbb{R}^d$. To our knowledge, only one work has gone beyond this setting [79], but it focuses on stiff inclusions and exploits a very precise control of the geometry.

As motivated by the derivation of nonlinear elasticity from the statistical physics of polymer-chain networks [9, 200, 138], much weaker assumptions on the energy density W should be considered. Indeed, the free energy of the polymer-chain network is given by two contributions: the sum of the free energies of the deformed chains, and a nonconvex steric effect prohibiting the interpenetration of matter. The free energy of a single chain is a convex increasing function of the square of the length of the deformed polymer-chain, which blows up at a finite deformation depending on the number of monomers in the considered chain. The corresponding problem in a continuum setting is the homogenization of a nonconvex density W of the form

$$W(y,\Lambda) \;=\; V(y,\Lambda) + a(y)\, g(\det \Lambda),$$

where V is an unbounded convex density such that the domain of $\Lambda \mapsto V(y,\Lambda)$ can strongly vary with respect to the space variable $y \in \mathbb{R}^d$, where a is uniformly bounded, and where g is a nonnegative convex function with $g(t) \uparrow \infty$ as $t \downarrow 0$. However, proving homogenization for the steric effect $a(y)\, g(\det \Lambda)$ is one of the most important open problems in the field (cf. Section 2.1.7), so that we decide here to truncate this effect for simplicity in order to focus on the first contribution only. More precisely, we consider a ru-usc nonconvex density W satisfying a two-sided estimate by a convex integrand, that is,

$$\frac{1}{C}|\Lambda|^p \le V(y,\Lambda) \le W(y,\Lambda) \le C(1+V(y,\Lambda)), \tag{1.7}$$

where V is an unbounded convex integrand as above and where $p>d$.

Two major difficulties appear for this homogenization problem. First, the domain of the homogenized density W_{hom} is unknown a priori, since the domain of V is no longer fixed, in stark contrast with all previously known results. Second, boundary data yielding a finite homogenized energy may not be adapted to the energy at any fixed value of the parameter $\varepsilon > 0$, so that we do not expect in general any homogenization result to hold with Dirichlet conditions. As a consequence, the usual asymptotic homogenization formula (1.4) with Dirichlet boundary conditions can no longer hold. The results that we obtain in Chapter 2 are summarized as follows, further assuming that the interior of the domain of $\Lambda \mapsto \sup_y V(y,\Lambda)$ is nonempty,

— *Convex result.* The convex energy functional $J_\varepsilon(u) := \int_O V(y/\varepsilon, \nabla u(y))dy$ Γ-converges to the homogenized functional $J(u) := \int_O V_{\mathrm{hom}}(\nabla u(y))dy$ with respect to the weak $W^{1,p}(O)^d$ topology (with Neumann boundary data), where V_{hom} is given e.g. by the usual single-cell formula (1.5)–(1.6).

— *Nonconvex result.* The nonconvex energy functional I_ε Γ-converges to the homogenized functional I with respect to the weak $W^{1,p}(O)^d$ topology (that is, with Neumann boundary data), where W_{hom} is given by the following formula,

$$W_{\mathrm{hom}}(\Lambda) = \liminf_{t\uparrow 1} \lim_{R\uparrow\infty} \inf_{u\in W_0^{1,p}(Q_R)} \fint_{Q_R} W(y, t\Lambda + t\nabla\phi_\Lambda(y) + \nabla u(y))dy, \tag{1.8}$$

where ϕ_Λ denotes the corrector for the (single-cell) convex problem in the direction Λ.

Note that the two-sided estimate (1.7) indeed ensures that the convex corrector is well-adapted as a Dirichlet boundary data in the asymptotic formula for W_{hom}: this is the idea behind formula (1.8), while the first $\liminf$ corresponds to some further needed relaxation.

We briefly comment on the proof of these Γ-convergence results, and start with the convex functional J_ε. The most original parts of the argument are a quantitative use of the sublinearity of correctors at infinity in the random setting, and a particularly careful gluing construction. The main difficulty for the Γ-convergence result comes from the construction of recovery sequences: for all $u \in W^{1,p}(O)^d$ we need to construct a sequence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O)^d$ such that $\limsup_\varepsilon J_\varepsilon(u_\varepsilon) \le J(u)$. For that purpose, we argue à la Müller [331], by first reducing by approximation to the case when the limiting test function u is piecewise affine. If $u = \Lambda x$ is affine on the whole of O, then a recovery sequence is given by a rescaling of the corrector, $u_\varepsilon := u + \varepsilon\phi_\Lambda(\frac{\cdot}{\varepsilon})$. For a piecewise affine u, it thus remains to understand how to glue such recovery sequences for each of the affine pieces. Assume that $O = O_1 \biguplus O_2$ and that $u|_{O_i} = \Lambda_i x + c_i$ for $i = 1, 2$. Choosing a cut-off χ_η for O_1 inside $O_1 + B_\eta$, with η to be later optimized as a function of ε, we are led to consider the following gluing,

$$u_{\varepsilon,\eta}(x) := u(x) + \chi_\eta(x)\varepsilon\phi_{\Lambda_1}(\tfrac{x}{\varepsilon}) + (1 - \chi_\eta(x))\varepsilon\phi_{\Lambda_2}(\tfrac{x}{\varepsilon}).$$

As the energy functional J_ε only involves the gradient, we compute

$$\begin{aligned}\nabla u_{\varepsilon,\eta} = \chi_\eta(\Lambda_1 + \nabla\phi_{\Lambda_1}(\tfrac{\cdot}{\varepsilon})) + (1 - \chi_\eta)(\Lambda_2 + \nabla\phi_{\Lambda_2}(\tfrac{\cdot}{\varepsilon}))& \\ + O(|\Lambda_1 - \Lambda_2|)\mathbb{1}_{(O_1+B_\eta)\cap O_2} + O(\tfrac{\varepsilon}{\eta})\|(\phi_{\Lambda_1}, \phi_{\Lambda_2})\|_{\mathrm{L}^\infty}.&\qquad (1.9)\end{aligned}$$

If the last two terms can be treated as errors, then by convexity of V we would conclude

$$\limsup J_\varepsilon(u_{\varepsilon,\eta}) \le |O_1|V_{\text{hom}}(\Lambda_1) + |O_2|V_{\text{hom}}(\Lambda_2) = J(u),$$

as $\varepsilon, \eta \downarrow 0$. In the periodic setting, the correctors ϕ_{Λ_i} are periodic and uniformly bounded if $p > d$, hence the last term in (1.9) is pointwise small in the regime $\varepsilon \ll \eta$. In the random setting, correctors may grow at infinity, but their sublinearity is still enough to conclude similarly (with η going sufficiently slowly to 0). The penultimate term in (1.9) is concentrated in space on the η-neighborhood of the boundary ∂O_1, and hence, if the integrand V satisfies $\sup_y V(y, \Lambda) < \infty$ for all Λ, this term would vanish in the limit $\eta \downarrow 0$. However, in the present case, no such bound is available. Our idea is rather to refine the gradient jump $|\Lambda_1 - \Lambda_2|$ and make it pointwise small: we set up an additional approximation argument and replace u by a sequence of piecewise affine functions with vanishing gradient jumps.

The Γ-convergence result for the nonconvex energy functional I_ε is deduced by adapting relatively standard methods together with the same gluing construction as that needed for the convex functional J_ε. To treat all error terms, we make a strong use of the two-sided estimate (1.7) and reduce to a convex situation.

1.1.2 Quantitative stochastic homogenization in the linear setting

Beyond these purely qualitative questions of existence and characterization of an effective energy functional, quantitative aspects are naturally of interest as well, namely rates of convergence for minimizers, rates of convergence for the infinite volume limit in the asymptotic homogenization formula (1.4), as well as fluctuations around these convergences. While for the qualitative theory we tried to consider the most general assumptions possible on the nonlinearity, the quantitative theory is mainly developed for linearized equations (see however e.g. [95, 35, 36, 34, 341]) and the focus is then on establishing optimal convergence rates. Until recently, most quantitative results were confined to the periodic setting, since in that case the problem is reduced to one on the torus and optimal estimates can then be deduced by compactness methods [50, 38]. In the following, we rather focus on the random case, and we restrict attention to linear equations.

Formally linearizing the minimization problem (1.3), we are led to the following second-order elliptic system describing linear elasticity,

$$-\nabla\cdot A(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon \,=\, f, \quad \text{in } O, \qquad \text{and} \qquad u_\varepsilon|_{\partial O}=h, \quad \text{on } \partial O, \tag{1.10}$$

where the domain $O\subset\mathbb{R}^d$ corresponds to a reference configuration, where $A(\frac{\cdot}{\varepsilon})$ is the (fourth-order) stiffness tensor field with heterogeneities at the scale $\varepsilon>0$, where $f:\mathbb{R}^d\to\mathbb{R}^d$ is an applied force, where $h:\partial O\to\mathbb{R}^d$ is a Dirichlet boundary data, and where the solution $u_\varepsilon:O\to\mathbb{R}^d$ describes the elastic displacement. In the sequel, we shall also consider the corresponding scalar PDE, that is, equation (1.10) for $u_\varepsilon:O\to\mathbb{R}$, with some domain $O\subset\mathbb{R}^d$, some matrix coefficient field $A(\frac{\cdot}{\varepsilon})$, some $f:O\to\mathbb{R}$, and some boundary data $h:\partial O\to\mathbb{R}$. This heterogeneous scalar Laplace equation arises e.g. in the context of Poisson's law in electrostatics, of Fourier's law for stationary temperature distribution, of Fick's law for stationary diffusion, etc. Henceforth, we always use scalar notation for simplicity, and the coefficient field A is assumed to be an ergodic stationary random field on the ambient space $\mathbb{R}^d$ satisfying the following boundedness and ellipticity properties,

$$\begin{gathered}|A(x)\xi|\le|\xi|, \quad \text{for all } \xi,x\in\mathbb{R}^d,\\ \int_{\mathbb{R}^d}\nabla\zeta\cdot A\nabla\zeta\ge\lambda\int_{\mathbb{R}^d}|\nabla\zeta|^2, \quad \text{for all } \zeta\in C_c^\infty(\mathbb{R}^d),\end{gathered}\tag{1.11}$$

for some $\lambda>0$. For suitable $f\in\mathrm{L}^2(O)$, we consider the weak solution $u_\varepsilon\in H^1_h(O)$ of (1.10). Since the pioneering works by Kozlov [273] and by Papanicolaou and Varadhan [354], we know that, almost surely, the solution u_ε converges weakly in $H^1(O)$ as $\varepsilon\downarrow0$ to the unique weak solution $\bar u\in H^1_h(O)$ of

$$-\nabla\cdot A_{\mathrm{hom}}\nabla\bar u \,=\, f, \quad \text{in } O, \qquad \text{and} \qquad \bar u|_{\partial O}=h, \quad \text{on } \partial O,$$

where A_{hom} is a deterministic constant coefficient that depends only on the law of A and satisfies the ellipticity condition (1.11). For all directions $\xi\in\mathbb{R}^d$, the projections $A_{\mathrm{hom}}\xi$ are given by the expectation of the flux of the corrector,

$$A_{\mathrm{hom}}\xi \,=\, \mathbb{E}\left[A(\nabla\phi_\xi+\xi)\right], \tag{1.12}$$

where ϕ_ξ is the corrector in the direction ξ, that is, the unique (up to additive constant) almost sure solution of the corrector equation on $\mathbb{R}^d$,

$$-\nabla\cdot A(\nabla\phi_\xi+\xi) \,=\, 0, \tag{1.13}$$

in the class of functions with stationary gradient and finite second moment. We denote by $\phi=(\phi_i)_{i=1}^d$ the vector field whose entries $\phi_i:=\phi_{e_i}$ are the correctors in the canonical directions e_i of $\mathbb{R}^d$. Note that the cell problem (1.13) is the linearization of (1.6), and again the possible non-stationarity of the corrector ϕ_ξ itself is related to the failure of the Poincaré inequality in the probability space. As formula (1.12) shows, the corrector field $\nabla\phi_\xi$ precisely makes the link between the microstructure A and the macrostructure A_{hom}, somehow correcting the arithmetic mean along A-harmonic coordinates. In the scalar 1D case, we simply compute $A_{\mathrm{hom}}=\mathbb{E}[A^{-1}]^{-1}$, so that homogenization corresponds to taking the harmonic average of coefficients, but in higher dimensions the correction is more subtle and, in general, not explicit.

In this linear setting, the quantitative questions mainly concern the homogenization error $\|u_\varepsilon-\bar u\|_{\mathrm{L}^2(O)}$. More precisely, while the convergence of u_ε to $\bar u$ in $H^1(O)$ is only weak since ∇u_ε typically displays spatial oscillations at scale ε, which are not captured by the limit $\nabla\bar u$, these oscillations are expected to be well-described by those of the corrector field $\nabla\phi(\frac{\cdot}{\varepsilon})$ through the two-scale expansion

$$u_\varepsilon \,\approx\, \bar u+\varepsilon\phi_i(\tfrac{\cdot}{\varepsilon})\nabla_i\bar u, \tag{1.14}$$

in the sense of strong convergence in $H^1(O)$ up to boundary layers. In the periodic case such an expansion is well-known to hold, but in the stochastic setting a first difficulty originates in the potentially bad behavior of the corrector ϕ. Another quantitative question concerns the numerical approximation of the homogenized coefficients A_{hom}: in contrast with the periodic case, the representation (1.12) is indeed of no direct use for numerical methods since the corrector equation (1.13) needs to be solved for every realization of the random coefficients and in the whole space $\mathbb{R}^d$. This difficulty is typically overcome by the representative volume element scheme: a large but finite sample volume of the random medium is chosen, then an approximation for the exact effective coefficient is obtained by using the cell formula on this sample volume, which leads to random finite-volume approximations of A_{hom}, and a crucial question concerns the accuracy of these. Naturally, for such quantitative considerations, ergodicity of the coefficient field A must be strengthened into suitable quantitative ergodicity assumptions. The main underlying question in quantitative stochastic homogenization is how ergodicity properties of the coefficient field are transmitted to the solution operator $(-\nabla\cdot A\nabla)^{-1}$, which is a particularly nontrivial question since the solution operator is a nonlinear nonlocal function of the coefficients.

The first suboptimal quantitative convergence result for u_ε is due to Yurinskii [428] in the late 1980s, but the theory literally exploded in the last decade with the emergence of a completely optimal quantitative theory. Everything started in 2009 with the works by Gloria and Otto [209, 210] in the simpler discrete setting with independent and identically distributed (i.i.d.) coefficients, where they established an optimal analysis of the representative volume element scheme and show in passing that the corrector ϕ is stationary and has all bounded moments in dimension $d > 2$. This was inspired by an unpublished work by Naddaf and Spencer [334], and in particular strongly relied on the use of a spectral gap in the probability space — seen as a quantification of ergodicity, and somehow curing the lack of the usual Poincaré inequality. These results were continued by Gloria, Neukamm, and Otto [206, 205], including a justification of the two-scale expansion, and further extended to the continuum setting [212]. The need for a large-scale regularity theory for the random elliptic operator $-\nabla\cdot A\nabla$ was quickly recognized [142, 313, 201]. A striking contribution by Armstrong and Smart [36], inspired by previous works by Dal Maso and Modica [134] and by Avellaneda and Lin [38, 39], then paved the way for a quenched large-scale regularity theory of A-harmonic functions and for quantitative homogenization results avoiding any use of functional inequalities in the probability space (but rather based on milder mixing-type assumptions, thus allowing for a greater generality). The main insight is that one should separate error estimates, which require strong ergodicity assumptions, from the large-scale regularity theory, which should hold under milder assumptions. This program was further developed by Armstrong, Kuusi, and Mourrat in a variational framework [34, 33, 32] (see also [30]), and found a different, intrinsic formulation in the works by Gloria, Neukamm, and Otto [204, 203, 208].

1.1.3 Pathwise structure of fluctuations (Chapter 3)

As opposed to periodic homogenization, which essentially boils down to the understanding of the spatial oscillations of the solution ∇u_ε of (1.10) in form of a suitable two-scale expansion (3.3), the random setting involves the random fluctuations of ∇u_ε on top of its oscillations. Whereas oscillations are concerned with the almost sure lack of strong compactness for ∇u_ε in L^2, fluctuations are concerned with the leading-order probabilistic behavior of weak-type expressions of the form $\int_{\mathbb{R}^d} g\cdot\nabla u_\varepsilon$. Henceforth, in order to avoid boundary layers, we consider the following version of equation (1.10) on the whole space $\mathbb{R}^d$: given $f\in\mathrm{L}^2(\mathbb{R}^d)^d$, we let u_ε denote the unique solution in $\dot H^1(\mathbb{R}^d)$ of

$$-\nabla\cdot A(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon \;=\; \nabla\cdot f.$$

We are mainly interested in the fluctuations of the field ∇u_ε and flux $A(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon$ of the solution, and of the field $\nabla\phi(\frac{\cdot}{\varepsilon})$ and flux $A(\frac{\cdot}{\varepsilon})\nabla\phi(\frac{\cdot}{\varepsilon})$ of the corrector. It follows from the usual quantitative theory of stochastic homogenization that the fluctuations of these quantities have (generically) the same scaling as the random coefficient field $A(\frac{\cdot}{\varepsilon})$ itself, hence they display the central limit theorem (CLT) scaling $\varepsilon^{\frac{d}{2}}$ in the case of weakly dependent coefficient fields. CLT results for these quantities were first established in dimension $d > 2$ in the discrete case with i.i.d. Gaussian coefficients [1] by Mourrat and Otto [329], Mourrat and Nolen [328], and Gu and Mourrat [225]. These first results indicated some intriguing link between the different limiting laws: the limiting fluctuations of $\nabla\phi(\frac{\cdot}{\varepsilon})$ is the Helmholtz projection of a Gaussian white noise with some particular covariance tensor, and the same tensor appears in the limiting fluctuations of ∇u_ε. A natural question that occurs is then to understand the origin of this relation between limiting laws. As observed by Gu and Mourrat [225], such a relation is however quite surprising, since the fluctuations of the solution operator cannot be inferred from those of the corrector via the usual two-scale expansion (3.3): this expansion (as well as its higher-order versions) is indeed not accurate in the CLT scaling in dimension $d \geq 2$, and the corrector field $\nabla\phi$ is therefore a priori not the driving quantity for fluctuations.

In Chapter 3, we provide a complete theory of fluctuations, and our main achievement is the identification of the suitable driving quantity. The key in our theory consists in focusing on the *homogenization commutator* $A(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon - A_{\text{hom}}\nabla u_\varepsilon$ of the solution, and in studying its relation to the *standard homogenization commutator* $\Xi = (\Xi_i)_{i=1}^d$ defined by

$$\Xi_i := A(\nabla\phi_i + e_i) - A_{\text{hom}}(\nabla\phi_i + e_i).$$

This stationary random 2-tensor field Ξ finds a natural motivation in terms of H-convergence (cf. Section 3.1.1), and was simultaneously independently introduced by Armstrong, Kuusi, and Mourrat [32] formalizing previous ideas initiated in [36]. We establish that the homogenization commutator satisfies the following three key principles, which lead to our complete theory of fluctuations:

(I) First and most importantly, the two-scale expansion of the homogenization commutator of the solution,
$$A(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon - A_{\text{hom}}\nabla u_\varepsilon - \mathbb{E}\left[A(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon - A_{\text{hom}}\nabla u_\varepsilon\right] \approx \Xi_i(\tfrac{\cdot}{\varepsilon})\nabla_i\bar{u},$$
is (generically) accurate in the fluctuation scaling in the sense of
$$\mathbb{E}\left[\Big|\int_{\mathbb{R}^d} g\cdot\big(A(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon - A_{\text{hom}}\nabla u_\varepsilon - \mathbb{E}\left[A(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon - A_{\text{hom}}\nabla u_\varepsilon\right]\big) - \int_{\mathbb{R}^d} g\cdot\Xi_i(\tfrac{\cdot}{\varepsilon})\nabla_i\bar{u}\Big|^2\right]^{\frac{1}{2}} \\ \leq\ o(1)\,\mathbb{E}\left[\Big|\int_{\mathbb{R}^d} g\cdot\Xi_i(\tfrac{\cdot}{\varepsilon})\nabla_i\bar{u}\Big|^2\right]^{\frac{1}{2}}, \quad (1.15)$$
where $o(1)\downarrow 0$ as $\varepsilon\downarrow 0$, for all $g\in C_c^\infty(\mathbb{R}^d)^d$. Let us emphasize that this property is nontrivial and is due to the special form of the commutator.

(II) Second, both the fluctuations of the field ∇u_ε and of the flux $A(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon$ can be recovered through *deterministic* projections of those of the homogenization commutator $A(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon - A_{\text{hom}}\nabla u_\varepsilon$ of the solution, which shows that no information is lost by passing to the homogenization commutator. In addition, the fluctuations of the field $\nabla\phi(\frac{\cdot}{\varepsilon})$ and of the flux $A(\frac{\cdot}{\varepsilon})\nabla\phi(\frac{\cdot}{\varepsilon})$ of the

1. More precisely, for coefficients that are C_b^2-functions of i.i.d. Gaussian random variables, so as to satisfy the uniform ellipticity assumption (1.11).

corrector are determined by those of the standard commutator $\Xi(\frac{\cdot}{\varepsilon})$. For instance,

$$\int_{\mathbb{R}^d} g\cdot\nabla(u_\varepsilon-\mathbb{E}\left[u_\varepsilon\right]) \\ = -\int_{\mathbb{R}^d}(\bar{\mathcal{P}}_H^* g)\cdot\big(A(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon - A_{\mathrm{hom}}\nabla u_\varepsilon - \mathbb{E}\big[A(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon - A_{\mathrm{hom}}\nabla u_\varepsilon\big]\big), \qquad (1.16)$$

$$\int_{\mathbb{R}^d} F : \nabla\phi(\tfrac{\cdot}{\varepsilon}) = -\int_{\mathbb{R}^d}\bar{\mathcal{P}}_H^* F : \Xi(\tfrac{\cdot}{\varepsilon}), \qquad (1.17)$$

in terms of the Helmholtz projection $\bar{\mathcal{P}}_H^* := \nabla(\nabla\cdot A_{\mathrm{hom}}^*\nabla)^{-1}\nabla\cdot$, where A_{hom}^* denotes the transpose matrix.

(III) Third, the standard homogenization commutator Ξ is an approximately local function of the coefficients A, which allows to infer the large-scale behavior of Ξ from the large-scale behavior of A itself.

On the one hand, items (I)–(II) reveal the *pathwise structure* of fluctuations in stochastic homogenization. Indeed, combined with identities of the form (1.16)–(1.17), the accuracy (1.15) of the two-scale expansion of the homogenization commutator implies that the fluctuations of ∇u_ε, $A(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon$, $\nabla\phi(\frac{\cdot}{\varepsilon})$, and $A(\frac{\cdot}{\varepsilon})\nabla\phi(\frac{\cdot}{\varepsilon})$ are determined at leading order by those of $\Xi(\frac{\cdot}{\varepsilon})$ in a strong norm in probability. This almost sure ("pathwise" in the language of SPDE) relation thus reduces the leading-order fluctuations of all quantities of interest to those of the sole homogenization commutator Ξ in a pathwise sense. Besides its theoretical importance, this *pathwise structure* is bound to affect multi-scale computing and uncertainty quantification in an essential way. Independently of the present work, Armstrong, Gu, and Mourrat [225] have proposed an interesting formal heuristics that also suggests the pathwise character of fluctuations but from which no rigorous proof has yet been extracted at the level of the solution operator.

On the other hand, item (III) is the key to the understanding of the limiting fluctuations of the standard homogenization commutator Ξ. In the case of a weakly dependent coefficient field, it implies that Ξ is itself an (approximately) weakly dependent random field, so that its rescaling $\varepsilon^{-\frac{d}{2}}\Xi(\frac{\cdot}{\varepsilon})$ (seen as a random Schwartz distribution) must converge in law to a Gaussian white noise (see also [32, 208]). Item (III) further opens the way to determine the limiting fluctuations in the case of coefficient fields with strong correlations, for which, as well-understood in 1D [42, 224, 291], Ξ may display different (not CLT) scalings, and different (not white, and potentially not even Gaussian) limiting laws.

In this very first work on the pathwise structure of fluctuations, we focus on the model framework of a discrete linear elliptic equation with i.i.d. coefficients. Using a spectral gap and various other functional inequalities in the probability space, we establish optimal quantitative estimates in any dimension $d\ge 2$. As described in Section 3.1.4, various generalizations are postponed to forthcoming work. More precisely, we prove the following:

— *CLT scaling:* for all $\varepsilon>0$ and all $F\in C_c^\infty(\mathbb{R}^d)^{d\times d}$,

$$\mathrm{Var}\left[\varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d}F:\Xi(\tfrac{\cdot}{\varepsilon})\right]\lesssim_F 1.$$

— *Pathwise structure (with optimal error estimates):* for all $f,g\in C_c^\infty(\mathbb{R}^d)^d$, the accuracy of the two-scale expansion of the homogenization commutator holds in the form (1.15) with $o(1)\simeq_{f,g}\varepsilon\mu_d(\frac{1}{\varepsilon})^{\frac12}$, where $\mu_d(\varepsilon):=1$ for $d>2$ and $\mu_d(\frac{1}{\varepsilon}):=\log(2+\frac{1}{\varepsilon})$ for $d=2$. In particular, identity (1.16) implies for all $\varepsilon>0$ and all $f,g\in C_c^\infty(\mathbb{R}^d)^d$,

$$\mathbb{E}\left[\left(\varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d}g\cdot\nabla(u_\varepsilon-\mathbb{E}\left[u_\varepsilon\right])+\varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d}(\bar{\mathcal{P}}_H^* g)\cdot\Xi_i(\tfrac{\cdot}{\varepsilon})\nabla_i\bar{u}\right)^2\right]^{\frac12}\lesssim_{f,g}\varepsilon\mu_d(\tfrac{1}{\varepsilon})^{\frac12}. \qquad (1.18)$$

— *Approximate normality (with nearly optimal rate):* for all $\varepsilon > 0$ and all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$,

$$\delta_{\mathcal{N}}\bigg(\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F : \Xi(\tfrac{\cdot}{\varepsilon})\bigg) \lesssim_F \varepsilon^{\frac{d}{2}} \log(2 + \tfrac{1}{\varepsilon}), \tag{1.19}$$

where for a random variable $X \in \mathrm{L}^2(\Omega)$ with variance $\sigma^2 = \mathrm{Var}[X]$ its distance to normality is defined by $\delta_{\mathcal{N}}(X) := \mathrm{d_W}\left(\frac{1}{\sigma}X, \mathcal{N}\right) + \mathrm{d_K}\left(\frac{1}{\sigma}X, \mathcal{N}\right)$, with $\mathcal{N}$ a standard Gaussian random variable and with $\mathrm{d_W}(\cdot,\cdot)$ and $\mathrm{d_K}(\cdot,\cdot)$ the Wasserstein and Kolmogorov metrics.

— *Convergence of the covariance structure (with optimal rate):* there exists a non-degenerate symmetric 4-tensor $\mathcal{Q}$ such that for all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$,

$$\bigg|\mathrm{Var}\bigg[\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F : \Xi(\tfrac{\cdot}{\varepsilon})\bigg] - \int_{\mathbb{R}^d} F : \mathcal{Q}\, F\bigg| \lesssim_F \varepsilon \mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}}. \tag{1.20}$$

Combined with (1.19), this yields the convergence of $\varepsilon^{-\frac{d}{2}}\Xi(\frac{\cdot}{\varepsilon})$ in law to a (2-tensor) Gaussian white noise with covariance tensor $\mathcal{Q}$. Together with identity (1.17) and with the pathwise result (1.18), this leads in particular to the first (nearly) optimal quantitative version of the known scaling limit results for ∇u_ε and $\nabla\phi(\frac{\cdot}{\varepsilon})$.

In addition, we complement this fluctuation theory with the study of the accuracy of the random volume element scheme for the numerical approximation of the so-called *effective fluctuation tensor* $\mathcal{Q}$.

1.1.4 Weighted functional inequalities for correlated random fields (Chapter 4)

Let A be an ergodic stationary random coefficient field on $\mathbb{R}^d$. We say that it satisfies the standard spectral gap (∂-SG) if for all $\sigma(A)$-measurable random variables $X(A)$ there holds

$$\mathrm{Var}[X(A)] \le C \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\partial_{A,B_R(x)} X(A)\Big)^2\right] dx, \tag{1.21}$$

where the "derivative" $\partial_{A,B_R(x)}X(A)$ measures the local dependence of $X(A)$ with respect to the restriction $A|_{B_R(x)}$ of the coefficient field on the ball $B_R(x)$ of radius R centered at x. Note that in the continuum setting there is no canonical choice of a vertical derivative ∂, which can be chosen as e.g. the usual Glauber derivative ∂^{G}, the oscillation ∂^{osc}, or the functional derivative ∂^{fct} (see Section 4.1.2 for definitions), and these different choices are not at all equivalent. We also consider covariance and logarithmic Sobolev inequalities (∂-CI) and (∂-LSI), which are useful variants of the above spectral gap. As illustrated in Chapter 3 in the proof of the pathwise result for fluctuations in stochastic homogenization, these functional inequalities on the probability space lead to a powerful and very convenient sensitivity calculus for nonlinear functions $X(A)$ of A. The use of such inequalities in quantitative stochastic homogenization originates in the inspiring unpublished work by Naddaf and Spencer [334]. In addition, these inequalities are well-known in mathematical physics as powerful tools to establish strong nonlinear concentration of measure properties for functions of A. Unfortunately, they are extremely restrictive in the context of coefficient fields of interest to homogenization in practice, as they are essentially only known to hold in the following situations,

— any local transformation A of a product (i.i.d.) structure (e.g. Poisson inclusions with bounded radius) satisfies (∂^{G}-SG), (∂^{G}-CI), and (∂^{osc}-LSI);

— a stationary Gaussian random field A satisfies (∂^{fct}-SG) if and only if $\int_{\mathbb{R}^d} |\mathrm{Cov}[A(x);A(0)]|dx < \infty$;

— a stationary Gaussian random field A satisfies (∂^{fct}-CI) if and only if it has finite range of dependence.

In Chapter 4, we introduce a new hierarchy of generalized versions of these functional inequalities, which still imply strong concentration properties but are satisfied for various examples of random fields with strong long-range dependences. More precisely, the idea is to modify (1.21) by explicitly taking into account dependences at all scales $r > 0$ according to some weight: given an integrable function $\pi : \mathbb{R}^+ \to \mathbb{R}^+$, we say that A satisfies the weighted spectral gap (∂-WSG) with weight π if for all $\sigma(A)$-measurable random variable $X(A)$ there holds

$$\operatorname{Var}[X(A)] \le \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \Big(\partial_{A,B_{\ell+1}(x)} X(A)\Big)^2 dx\, (\ell+1)^{-d} \pi(\ell)\, d\ell\right], \tag{1.22}$$

and we similarly define weighted covariance and logarithmic Sobolev inequalities (∂-WCI) and (∂-WLSI). The standard spectral gap (∂-SG) is naturally recovered for compactly supported weights π, and we note that (∂-WSG) with weight π implies the ergodicity of A whenever $\int_0^\infty \pi(r)dr < \infty$. In this sense, this hierarchy of weighted functional inequalities provides a quantification of ergodicity.

As we show, these weighted functional inequalities still imply strong concentration properties, which crucially depend both on the decay of the weight and on the choice of the derivative, and which are generally stronger than those implied by the corresponding α-mixing. In addition, we develop a ready-to-use criterion to produce random fields that satisfy such weighted inequalities, based on transformations of higher-dimensional product structures, and relying on approximate chain rules for nonlinear and random changes of variables for random fields. This approach allows us to treat all the models of heterogeneous materials encountered in the applied sciences [413], including for instance the following typical examples,

— Gaussian random fields: if the covariance function $\mathcal{C}$ satisfies $\sup_{B(x)} |\mathcal{C}| \le c(|x|)$ for some non-increasing $c : \mathbb{R}^+ \to \mathbb{R}^+$, then A satisfies (∂^{fct}-WSG) and (∂^{fct}-WLSI) with weight $\pi(\ell) \simeq |c'(\ell)|$;

— Poisson random inclusions with i.i.d. random radii: if the radius law is given by some random variable V, then A satisfies (∂^{osc}-WSG) with weight $\pi(\ell) \simeq (\ell+1)^d\, \mathbb{P}[\ell - 4 \le V < \ell + 4]$;

— Poisson random tessellations (Voronoi or Delaunay): A satisfies (∂^{osc}-WSG) and (∂^{osc}-WLSI) with weight $\pi(\ell) \simeq e^{-\frac{1}{C}\ell^d}$;

— random parking process: A satisfies (∂^{osc}-WSG) and (∂^{osc}-WLSI) with weight $\pi(\ell) \simeq e^{-\frac{1}{C}\ell}$.

We further extend this weighted approach to the case of second-order Poincaré inequalities à la Chatterjee [112, 113]. While “first-order” functional inequalities, such as spectral gap, quantify the distance to constants for nonlinear functions $X(A)$ in terms of their local dependence on the random field A, these second-order inequalities quantify their distance to normality.

We then turn to two applications of this theory. The first application is in quantitative stochastic homogenization. More precisely, we consider the quenched large-scale regularity theory for A-harmonic functions [32, 204]: the integrability properties of the minimal radius beyond which the regularity theory holds are known to follow from the concentration properties of spatial averages of approximately local functions of the random field A. For random fields that satisfy weighted functional inequalities, such concentration properties can be optimally determined, resulting in optimal integrability properties for the minimal radius, which are in general stronger than those obtained from the corresponding α-mixing. For instance, for Poisson random tessellations, the minimal radius r_* is shown to satisfy $\mathbb{E}\big[\exp(\frac{1}{C} r_*^d)\big] < \infty$ instead of $\mathbb{E}\big[\exp(\frac{1}{C} r_*^{d/2})\big] < \infty$.

The second application concerns random sequential adsorption models in stochastic geometry, and more precisely fluctuations of the jamming limit. Using our weighted first- and second-order functional inequalities, we revisit and complete previous works pioneered by Penrose and Yukich [357, 359, 358, 360, 391, 286].

1.1.5 Clausius-Mossotti formulas and beyond (Chapter 5)

As described in Section 1.1.2 above, for second-order linear elliptic PDEs in divergence form, cf. (1.10), the qualitative theory of stochastic homogenization is well-understood and leads to the

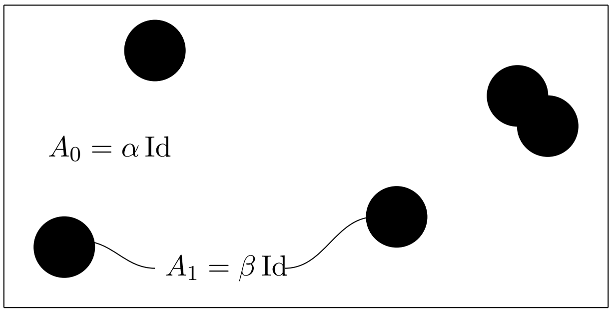


Figure 1.1 – Clausius-Mossotti model of dispersed spherical inclusions.

existence of a homogenized coefficient A_{hom}, defined in terms of an abstract cell problem (1.12)–(1.13) on the probability space. For some very particular two-dimensional geometries, exact solutions of the cell problem are possible and lead to explicit expressions for A_{hom} (see e.g. [265, Sections 1.5 and 7.3]). In general no explicit formula is however possible and we can only try to develop numerical approximations. In contrast with the simpler periodic case, the definition of A_{hom} has no direct use for numerical methods in practice, since it requires to solve the corrector equation (1.13) for every realization of the random coefficient field A and in the whole space $\mathbb{R}^d$. Numerical approximations of A_{hom} follow from the representative volume element method, the efficiency of which is well understood since the pioneering works by Gloria and Otto [209, 210]. Estimating A_{hom} nevertheless remains computationally demanding. In Chapter 5, we focus on a particular situation of interest, that is, when A is a "small random perturbation" of some better-known coefficient field A_0, which can be e.g. constant or periodic (cf. [20, 21]). It is then expected that the perturbed homogenized coefficient A_{hom} is a small perturbation of the unperturbed one $(A_0)_{\mathrm{hom}}$, and the main interest lies in finding a simpler formula for the first-order deviation, which can be used as a good proxy for A_{hom} that is less demanding to compute, and can also be used to reduce the variance in some numerical approximation methods for A_{hom} [296]. More precisely, given two reference coefficient fields A_0 and A_1, we consider an inclusion process (e.g. Poisson spherical random inclusions) with small volume fraction $v \ll 1$, and we consider the random field A equal to A_0 outside inclusions and equal to A_1 inside. In other words, the considered perturbation is only small in an L^1 sense. This encompasses the example of errors occurring with small probability in the construction process of some material with engineered microstructures: errors result in small foreign inclusions and a strong practical interest resides in determining the deviation in the effective properties of the constructed material, with respect to the expected ones.

The study of effective properties of such two-phase dispersed media finds its origin in the prehistory of stochastic homogenization in the 19th century, motivated by the works of Poisson [364] and Faraday [179] on induced magnetism and on dielectric materials. Mossotti [324, 325] was the first to investigate the effective large-scale properties of two-phase conducting materials composed of a homogeneous matrix with dilute foreign spherical inclusions. Continuing this study, Clausius [119] came up with an explicit formula for the first-order deviation due to the foreign inclusions — the so-called Clausius-Mossotti formula: for spherical inclusions with $A_0 = \alpha\,\mathrm{Id}$ and $A_1 = \beta\,\mathrm{Id}$ (cf. Figure 1.1), it is predicted at first order in the volume fraction $v \ll 1$,

$$A_{\mathrm{hom}} = \underbrace{\alpha\,\mathrm{Id}}_{A_{0,\mathrm{hom}}} + v\frac{\alpha d(\beta-\alpha)}{\beta+\alpha(d-1)}\,\mathrm{Id} + O(v^2). \tag{1.23}$$

This formula was independently also derived by Maxwell [317], followed by Rayleigh [368]. These considerations were adapted to a refractivity context in optics by Lorentz [308] and Lorenz [309], while Einstein [176] used similar ideas to compute the effective shear viscosity of dilute suspensions of rigid particles in a fluid. Several decades later, Bruggeman [89] established similar formulas in linear elasticity in the form of an explicit expansion of Lamé's coefficients. While these works focused on first-order deviations, higher-order terms were first computed by Ross [374] and Jeffrey [259, Section 5] more than a century after Mossotti. Interactions between inclusions must then be taken into account, and spatial statistics of the distribution of inclusions enter the result. For precise historical detail, we refer to [283, 315].

A more convenient mathematical framework consists in viewing A as a Bernoulli perturbation of a given coefficient field A_0. Let A_0 and A_1 be reference ergodic stationary random coefficient fields (in particular, not necessarily constant), and let $(q_n)_n$ be an ergodic stationary point process. For simplicity in this introduction we focus on unit spherical inclusions (although random shapes could be considered as well in our analysis), and we consider the associated inclusion process $\bigcup_n B(q_n)$, which is regarded as a collection of *possible* inclusions. We now vary the volume fraction of this inclusion process by choosing each inclusion independently of the others only with a small fixed probability $p \in [0,1]$. For that purpose, we independently choose a Bernoulli process $(b_n^{(p)})_n$, that is, a sequence of i.i.d. Bernoulli random variables such that $b_n^{(p)}$ equals 1 with probability p and 0 otherwise. Denoting by $E^{(p)} := \{n : b_n^{(p)} = 1\}$ the set of chosen indices, we consider the corresponding (ergodic stationary) perturbed coefficient field,

$$A^{(p)} = A_0 + (A_1 - A_0)\mathbb{1}_{\bigcup_{n\in E^{(p)}} B(q_n)}.$$

and we study the associated homogenized coefficient $A_{\mathrm{hom}}^{(p)}$. Since the volume fraction associated with this inclusion process satisfies $v^{(p)} \simeq p$, expansions of the form (1.23) are naturally reformulated as expansions with respect to $p \ll 1$: in the case of isotropic constant reference coefficients $A_0 = \alpha\,\mathrm{Id}$ and $A_1 = \beta\,\mathrm{Id}$,

$$A_{\mathrm{hom}}^{(p)} = \alpha\,\mathrm{Id} + v^{(p)}\frac{\alpha d(\beta-\alpha)}{\beta+\alpha(d-1)}\,\mathrm{Id} + O(p^2). \tag{1.24}$$

The first rigorous proofs of this formula are due to Almog [14, 15, 13], who establishes (1.24) in dimension $d > 2$ with an error of size $o(p)$. Another contribution is due to Mourrat [327], who proves (1.24) in dimension $d \ge 2$ with almost optimal error $O(p^{2-})$, under strong mixing assumptions. These results are both limited to the scalar case. The question is thus threefold. First, can we establish (1.24) with the optimal error $O(p^2)$? If so, can the expansion be pursued to higher orders? And what is the optimal regularity of the map $p \mapsto A_{\mathrm{hom}}^{(p)}$ in terms of assumptions on A? Second, are mixing assumptions on A really needed for such expansions to hold? Third, can similar results be proven in the case of linear elasticity?

In Chapter 5 we answer these questions by proving under the sole assumptions of stationarity and ergodicity, both in the scalar case and in the case of linear elasticity, that the map $p \mapsto A_{\mathrm{hom}}^{(p)}$ is analytic whenever the inclusion process $\bigcup_n B(q_n)$ has finite penetrability, meaning, whenever $\sharp\{n : 0 \in B(q_n)\}$ is a.s. bounded by a deterministic constant. In addition, we provide semi-explicit formulas for all derivatives. As discussed in Section 5.1.5, the finite penetrability assumption is expected to be necessary for the analyticity result, and prohibits the example of Poisson random inclusions. We further investigate the corresponding expansions for the perturbed effective fluctuation tensor $\mathcal{Q}^{(p)}$ introduced in (1.20).

This new analyticity result is not so surprising, as it is very much in line with the cluster expansions used in this setting in the physics literature [413]. Let us briefly describe this heuristics. At leading order for $p \ll 1$ no inclusion is seen in a given finite sample, while at first order at most one inclusion is seen in the sample, etc. Hence there holds at leading order $A_{\mathrm{hom}}^{(p)} \sim (A_0)_{\mathrm{hom}}$, while the first-order correction should correspond to comparing values if we put or not one inclusion at each possible location, etc. This formally justifies the following expansion,

$$A^{(p)}_{\mathrm{hom}} = \mathbb{E}\big[A^{(p)}(\nabla\phi^{(p)}+\mathrm{Id})\big] = \underbrace{\mathbb{E}\left[A_0(\nabla\phi_0+\mathrm{Id})\right]}_{=(A_0)_{\mathrm{hom}}} + p\sum_n \mathbb{E}\left[A^{\{n\}}(\nabla\phi^{\{n\}}+\mathrm{Id}) - A_0(\nabla\phi_0+\mathrm{Id})\right]$$

$$+p^2\sum_{n<m}\mathbb{E}\left[A^{\{n,m\}}(\nabla\phi^{\{n,m\}}+\mathrm{Id}) - A^{\{n\}}(\nabla\phi^{\{n\}}+\mathrm{Id}) - A^{\{m\}}(\nabla\phi^{\{m\}}+\mathrm{Id}) + A_0(\nabla\phi_0+\mathrm{Id})\right]+\ldots$$

where $\nabla\phi_0$ is the corrector gradient associated with the reference coefficient A_0, and where for all subset $F\subset\mathbb{N}$ we set $A^F := A_0 + (A_1-A_0)\mathbb{1}_{\bigcup_{n\in F}B(q_n)}$ and we let $\nabla\phi^F$ denote the corresponding corrector gradient. The meaning of the above expansion is however unclear, as even the series defining the first-order term does not converge absolutely. In Chapter 5 we show that we can make sense of these cluster formulas by using massive approximations of the corrector gradients, computing the sums, and then passing to the limit of a vanishing mass, and that in this precise sense these formulas are correct for the derivatives of $p\mapsto A^{(p)}_{\mathrm{hom}}$ at $p=0$.

We now briefly recapitulate the main ideas of the proof. A natural strategy to establish the desired result consists in considering the cluster expansion of the corrector gradient $\nabla\phi^{(p)} = \nabla\phi + \sum_{n\in E^{(p)}}\nabla(\phi^{\{n\}}-\phi)+\ldots$ and justifying it in $\mathrm{L}^2(\Omega)$. Formally injecting this expansion into the definition of the perturbed homogenized coefficient $A^{(p)}_{\mathrm{hom}}$ then easily leads to the expected cluster expansion for $A^{(p)}_{\mathrm{hom}}$. This was done at first order by Mourrat [327], and can be iterated to any order (up to technicalities, cf. Appendix 5.A). This argument however requires a strong use of the quantitative theory of stochastic homogenization, hence of superfluous strong mixing assumptions. In addition, such an argument does a priori not lead to the analyticity of $p\mapsto A^{(p)}_{\mathrm{hom}}$ (but only to a C^∞ result). This gap is not so surprising as we are only interested in the averaged quantity $A^{(p)}_{\mathrm{hom}}$, which should be much easier to analyze than pointwise quantities such as the corrector gradient.

With this in mind, we rather need to focus exclusively on the perturbed homogenized coefficient $A^{(p)}_{\mathrm{hom}}$ and to understand cancellations at that level, for which we drew a crucial inspiration from some ingenious computations by Anantharaman and Le Bris (see in particular [19, Proposition 3.4]). A careful investigation of the algebraic and combinatorial structure of the perturbed homogenized coefficient leads to natural expansions in powers of p that can be pursued up to any order $n\geq 0$,

$$A^{(p)}_{\mathrm{hom}} = (A_0)_{\mathrm{hom}} + \sum_{k=1}^n p^k\Delta_k + p^{n+1}E^{(p)}_n,$$

where Δ_k and $E^{(p)}_n$ are given by explicit formulas (cf. Lemmas 5.1.10 and 5.3.1). Again, this expansion should rather be seen at the level of massive term approximations, which are omitted here for simplicity. The analyticity result then follows if we manage to prove bounds of the form $|\Delta_k|\leq C^k$ and $|E^{(p)}_n|\leq C^n$ for all $k,n\geq 0$. A quick look at the formulas indicates that these bounds follow from the following hierarchy of a priori estimates: for all $k\geq j\geq 1$,

$$\mathbb{E}\left[\sum_{|G|=j}\Big|\sum_{\substack{|F|=k-j\\ F\cap G=\varnothing}}\nabla\delta^F\phi\Big|^2\right]\leq C^k, \tag{1.25}$$

where we define the difference operators $\delta^{\{n\}}\phi = \phi^{\{n\}}-\phi_0$ and $\delta^F := \prod_{n\in F}\delta^{\{n\}}$. These estimates are the core of the analyticity result. The proof is obtained by an intricate triangular induction argument. In order to illustrate the structure, let us briefly schematize the argument at the level $k=2$. We write down the equation for $\delta^{\{n,m\}}\phi$ in three different ways and deduce different energy estimates from each of them,

— the first equation,

$$-\nabla\cdot A^{\{n,m\}}\nabla\delta^{\{n,m\}}\phi=\nabla\cdot(A^{\{m\}}-A_0)\nabla\delta^{\{n\}}\phi+\nabla\cdot(A^{\{n\}}-A_0)\nabla\delta^{\{m\}}\phi,$$

leads to

$$\mathbb{E}\left[\sum_{n\neq m}|\nabla\delta^{\{n,m\}}\phi|^2\right]\lesssim\mathbb{E}\left[\sum_{n}|\nabla\delta^{\{n\}}\phi|^2\right];$$

— the second equation,

$$-\nabla\cdot A^{\{n\}}\nabla\delta^{\{n,m\}}\phi=\nabla\cdot(A^{\{m\}}-A_0)\nabla\delta^{\{n\}}\phi^{\{m\}}+\nabla\cdot(A^{\{n\}}-A_0)\nabla\delta^{\{m\}}\phi,$$

leads to

$$\mathbb{E}\left[\sum_{n}\Big|\sum_{m:m\neq n}\nabla\delta^{\{n,m\}}\phi\Big|^2\right]\lesssim\mathbb{E}\left[\sum_{n\neq m}|\nabla\delta^{\{n,m\}}\phi|^2\right]+\mathbb{E}\left[\Big|\sum_{n}\nabla\delta^{\{n\}}\phi\Big|^2\right]+\mathbb{E}\left[\sum_{n}|\nabla\delta^{\{n\}}\phi|^2\right];$$

— the third equation,

$$-\nabla\cdot A_0\nabla\delta^{\{n,m\}}\phi=\nabla\cdot(A^{\{m\}}-A_0)\nabla\delta^{\{n\}}\phi^{\{m\}}+\nabla\cdot(A^{\{n\}}-A_0)\nabla\delta^{\{m\}}\phi^{\{n\}},$$

leads to

$$\mathbb{E}\left[\Big|\sum_{n\neq m}\nabla\delta^{\{n,m\}}\phi\Big|^2\right]\lesssim\mathbb{E}\left[\sum_{n}\Big|\sum_{m:m\neq n}\nabla\delta^{\{n,m\}}\phi\Big|^2\right]+\mathbb{E}\left[\Big|\sum_{n}\nabla\delta^{\{n\}}\phi\Big|^2\right]+\mathbb{E}\left[\sum_{n}|\nabla\delta^{\{n\}}\phi|^2\right].$$

Combining these three different energy estimates yields

$$\begin{aligned}\mathbb{E}\left[\Big|\sum_{n\neq m}\nabla\delta^{\{n,m\}}\phi\Big|^2\right]+\mathbb{E}\left[\sum_{n}\Big|\sum_{m:m\neq n}\nabla\delta^{\{n,m\}}\phi\Big|^2\right]+\mathbb{E}\left[\sum_{n\neq m}|\nabla\delta^{\{n,m\}}\phi|^2\right]&\\ \lesssim\mathbb{E}\left[\Big|\sum_{n}\nabla\delta^{\{n\}}\phi\Big|^2\right]+\mathbb{E}\left[\sum_{n}|\nabla\delta^{\{n\}}\phi|^2\right],&\end{aligned}$$

which proves that the case $k=2$ is, indeed, reduced to the easier case $k=1$. As we show in Chapter 5, such induction arguments can be pursued up to any order to establish the key a priori estimates (1.25) (cf. Section 5.2.3).

1.2 Ginzburg-Landau vortices in disordered media (Part II)

Superconductors are materials that lose their resistivity at sufficiently low temperature (or low pressure), which allows them to carry electric currents without energy dissipation. An important property of these materials is the so-called Meissner effect: moderate external magnetic fields are completely expelled from the sample. If the external field is too strong, however, the superconducting material returns to a normal state. In the case of a type-II superconductor, an intermediate regime is possible between these two critical values of the external field. The material is then in a mixed state, allowing a partial penetration of the external field through "vortex filaments". This mixed state presents however a major drawback: when an electric current is applied, it flows through the sample, inducing a Lorentz-like force that sets the vortices in motion, and hence, since vortices are flux filaments, their movement generates an electric field in the direction of the electric current, which dissipates energy and destroys the superconductivity property.

While ordinary superconductors need extreme cooling to achieve superconductivity, the discovery of high-temperature superconductors from the 1980s onwards has given a major boost to technological applications, as the critical temperature of such materials is reached by simple liquid nitrogen. These high-temperature superconductors happen to be in practice strongly of type II and, as such, they show vortices for a very wide range of values of the applied magnetic field. Most technological applications of superconductors therefore occur in this mixed state, and methods must be designed to prevent vortices from moving and recover the crucial property of dissipation-free current flow. For that purpose a common attempt consists in introducing normal impurities in the material, which are meant to destroy superconductivity locally and therefore "pin down" the vortices to their locations, if the applied current is not too strong. With these applications in mind, there is a strong interest in the physics community in understanding the precise effect of such impurities, typically randomly scattered around the sample, on the statics and dynamics of vortices.

In Part II of this thesis, we study the collective dynamics of many vortices in a 2D section of a type-II superconductor with impurities and applied current, based on their mesoscopic description by the 2D Ginzburg-Landau model, and we aim at establishing in various regimes the correct mean-field equations describing the macroscopic evolution of the vortex matter. Note that in the asymptotic limit of point-like vortices the Ginzburg-Landau vortex dynamics is subjected to three forces: the mutual repulsive Coulomb interaction between the vortices, the Lorentz-like force due to the applied current and pushing them in a given direction, and the pinning force attracting them towards the random impurities. This is therefore an example of a physical system where interactions compete with disorder: interactions tend to favor some kind of order in the system, making vortices behave as a coherent elastic whole, which in turn strongly modifies the effect of disorder on the vortices. This leads to a glassy physics with remarkable static and dynamical properties that are still largely unundersto od [195, 369]. We aim at writing correct PDEs for the evolution of the vortex matter, thus settling a first mathematically rigorous basis for the study of such systems in the mean-field limit, and allowing to ask some relevant questions on their expected glassy behavior.

1.2.1 Mean-field limit of Coulomb-like interaction gradient flows (Chapter 6)

A main task in this part of the thesis is to understand the mean-field evolution of 2D Ginzburg-Landau vortices in the presence of impurities and applied current. For that purpose, we shall follow Serfaty [395] and make use of a modulated energy method. Before going into technical details of vortex analysis, we study in Chapter 6 how this modulated energy strategy is applied to the mean-field limit of the simplified example of a Coulomb interaction gradient flow (without impurities and forcing). On the one hand, in the asymptotic limit of point-like vortices, the 2D Ginzburg-Landau vortex dynamics is known to coincide with the gradient flow evolution of Coulomb particles, so that both problems are indeed physically related. On the other hand, many mean-field limit questions for Coulomb-like interaction gradient flows are still open, and we aim at understanding what new can be established with modulated energy methods.

More precisely, we consider the gradient flow evolution of a system of N identical interacting particles with interaction potential g, that is, for all $1 \leq i \leq N$,

$$\partial_t x_{i,N}^t = -\frac{1}{N} \sum_{j:j\neq i}^{N} \nabla g(x_{i,N}^t - x_{j,N}^t),$$

where $\{t \mapsto x_{j,N}^t\}_{j=1}^N$ denotes the collection of particle trajectories. When the number N of particles is large, we are led to a large system of coupled ODEs, which is impossible to solve or describe exactly in practice. For various purposes, we are however only interested in the overall flow of particles, that is, in the overall evolution of the empirical (probability) measure

$$\mu_N^t \ := \ \frac{1}{N}\sum_i \delta_{x_{i,N}^t}.$$

Formally, if at initial time $\mu_N^\circ \xrightharpoonup{*} \mu^\circ$ holds for some smooth probability measure μ°, then we expect $\mu_N^t \xrightharpoonup{*} \mu^t$ for all $t \geq 0$, where μ^t satisfies

$$\partial_t \mu = \operatorname{div}(\mu \nabla g * \mu), \qquad \mu|_{t=0} = \mu^\circ. \tag{1.26}$$

Such a mean-field limit result is well-known if g is smooth (in the sense $g \in C_b^1(\mathbb{R}^d)$) or if g is convex (cf. Section 6.1.2). We shall therefore rather focus on the singular case of Riesz potentials,

$$g(x) = g_s(x) := c_{d,s}^{-1} \begin{cases} |x|^{-s}, & \text{if } 0 < s < d; \\ -\log(|x|), & \text{if } s = 0. \end{cases}$$

Note that for this choice $g = g_s$ the convolution $g_s *$ coincides with the fractional Laplacian $(-\triangle)^{-\frac{d-s}{2}}$, and the limiting mean-field equation (1.26) is then the so-called fractional porous medium equation. The particular choice $s = d-2$, $d \geq 2$ is the Coulomb case. The corresponding mean-field limit result was proven in the following cases (cf. Section 6.1.2 for more detail on these approaches),

— by Schochet [390] in any dimension $d \geq 1$ in the logarithmic case $s = 0$;

— by Hauray and Jabin [234, 233] in dimension $d \geq 3$ for $0 \leq s < d-2$;

— by Berman and Önnheim [54] in the very particular case of dimension $d = 1$ for all $0 \leq s < 1$;

but it remains an open question for all other parameter values. In Chapter 6, using a modulated energy approach inspired by Serfaty [395], we manage to establish the mean-field limit result in dimensions $d = 1$ and 2 for all $0 \leq s < 1$, hence leading in particular to a new result in dimension $d = 2$.

We briefly describe some ideas of the proof. The key observation is that the limiting equation (1.26) satisfies a weak-strong stability estimate in the modulated energy metric. More precisely, this equation can be seen as a Wasserstein gradient flow for the energy functional $\mathcal{E}(\mu) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} g_s(x-y) d\mu(x) d\mu(y)$, and the "modulation" of this energy structure (that is, the associated Bregman divergence [84]) leads to the following metric,

$$\begin{aligned}\mathcal{E}(\mu_1|\mu_2) &:= \mathcal{E}(\mu_1) - \mathcal{E}(\mu_2) - \Big\langle \frac{\delta \mathcal{E}}{\delta \mu}(\mu_2)\,,\, \mu_1 - \mu_2 \Big\rangle \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} g_s(x-y) d(\mu_1 - \mu_2)(x) d(\mu_1 - \mu_2)(y) = \|\mu_1 - \mu_2\|^2_{\dot H^{-\frac{d-s}{2}}}.\end{aligned}$$

Then, for $0 \vee (d-2) \leq s < d$, the following weak-strong stability result holds (cf. Lemmas 6.1.7 and 6.2.1): for any two solutions μ_1 and μ_2 of the limiting equation (1.26),

$$\mathcal{E}(\mu_1^t|\mu_2^t) \leq \mathcal{E}(\mu_1^\circ|\mu_2^\circ) \exp\Big(C \int_0^t \|\nabla^2 g_s * \mu_2^u\|_{\mathrm{L}^\infty} du\Big). \tag{1.27}$$

The idea of modulated energy methods, originating in the relative entropy method first designed by DiPerna [145] and Dafermos [131, 132] (see e.g. [378] and references therein for more recent developments), consists in devising an adapted metric modeled on the available energy (or entropy) structure and in expecting that this new metric is much better behaved along the flow than arbitrary metrics like W_2 and that it indeed leads to stronger stability results. In the Coulomb case $s = d-2$,

$d \geq 2$, the above result (1.27) follows from a simple Grönwall argument: using equation (1.26) for μ_1 and μ_2, and setting $h_i := g_{d-2} * \mu_i = (-\triangle)^{-1}\mu_i$ for $i = 1, 2$, we compute

$$\begin{aligned}\partial_t \mathcal{E}(\mu_1^t|\mu_2^t) &= 2\int_{\mathbb{R}^d} (h_1^t - h_2^t)(\partial_t \mu_1^t - \partial_t \mu_2^t) = -2\int_{\mathbb{R}^d} \nabla(h_1^t - h_2^t)(\mu_1^t \nabla h_1^t - \mu_2^t \nabla h_2^t)\\ &= -2\int_{\mathbb{R}^d} |\nabla(h_1^t - h_2^t)|^2 \mu_1^t - 2\int_{\mathbb{R}^d} \nabla h_2^t \cdot \nabla(h_1^t - h_2^t)\,(\mu_1^t - \mu_2^t),\end{aligned}$$

where the product $\nabla(h_1^t - h_2^t)\,(\mu_1^t - \mu_2^t)$ in the last term can be rewritten à la Delort using the modulated stress-energy tensor,

$$\begin{aligned}-2\nabla(h_1^t - h_2^t)\,(\mu_1^t - \mu_2^t) = 2\nabla(h_1^t - h_2^t)\,\Delta(h_1^t - h_2^t) = \operatorname{div} T(\mu_1^t|\mu_2^t),\\ T(\mu_1^t|\mu_2^t) := 2\nabla(h_1^t - h_2^t) \otimes \nabla(h_1^t - h_2^t) - \mathrm{Id}\,|\nabla(h_1^t - h_2^t)|^2,\end{aligned} \tag{1.28}$$

and we easily conclude, integrating by parts and using that $\mathcal{E}(\mu_1^t|\mu_2^t) = \int_{\mathbb{R}^d} |\nabla(h_1^t - h_2^t)|^2$,

$$\partial_t \mathcal{E}(\mu_1^t|\mu_2^t) \leq -\int_{\mathbb{R}^d} \nabla^2 h_2^t : T(\mu_1^t|\mu_2^t) \lesssim \|\nabla^2 h_2^t\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^d} |\nabla(h_1^t - h_2^t)|^2 = \|\nabla^2 h_2^t\|_{\mathrm{L}^\infty}\, \mathcal{E}(\mu_1^t|\mu_2^t),$$

from which the stability result (1.27) follows. In other Riesz cases the result is more subtle and requires to use the extension representation for the fractional Laplacian popularized by Caffarelli and Silvestre [92] in order to find a suitable proxy for the Delort-type identity (1.28).

The prefactor in the right-hand side of the stability result (1.27) only depends on the regularity of μ_2, hence the naming "weak-strong". A natural idea to prove a mean-field limit result then consists in trying to reproduce the above argument with $\mu_1 := \mu_N$ and $\mu_2 = \mu$. However, μ_N is not a solution of the limiting equation (1.26) and it has in addition infinite continuum energy due to the presence of Dirac masses so that $\mathcal{E}(\mu_N|\mu)$ is not well-defined. It is a matter of removing divergent diagonal terms in the product $\mu_N \nabla g_s * \mu_N$ and in the definition of the continuum energy. We therefore need to rather consider the following "renormalized" modulated energy,

$$\tilde{\mathcal{E}}(\mu_N^t|\mu^t) := \iint_{x \neq y} g_s(x-y)d(\mu_N^t - \mu^t)(x)d(\mu_N^t - \mu^t)(y).$$

Unfortunately the removal of diagonal terms prevents us from repeating the same Grönwall argument as above: in the Coulomb case $s = d-2$, $d \geq 2$, we may again compute

$$\partial_t \tilde{\mathcal{E}}(\mu_N^t|\mu^t) \leq -\int \nabla^2 h^t : \tilde{T}(\mu_N^t|\mu^t), \qquad \text{and} \qquad \tilde{\mathcal{E}}(\mu_N^t|\mu^t) = \int_{\mathbb{R}^d} \tilde{|}\nabla(h_N^t - h^t)\tilde{|}^2,$$

where $\tilde{T}(\mu_N|\mu)$ is the diagonal-free version of $T(\mu_N|\mu)$,

$$\tilde{T}(\mu_N|\mu)(x) := \iint_{y \neq z} \big(2\nabla g(x-y) \otimes \nabla g(x-z) - \mathrm{Id}\,\nabla g(x-y)\cdot\nabla g(x-z)\big)d(\mu_N - \mu)(y)d(\mu_N - \mu)(z),$$

and where the notation $\tilde{|}\nabla(h_N - h)\tilde{|}^2$ stands for the diagonal-free version of the square $|\nabla(h_N - h)|^2$,

$$\tilde{|}\nabla(h_N - h)\tilde{|}^2(x) := \iint_{y \neq z} \nabla g(x-y)\cdot\nabla g(x-z)d(\mu_N - \mu)(y)d(\mu_N - \mu)(z),$$

but it is no longer true that $|\tilde{T}(\mu_N|\mu)| \lesssim \tilde{|}\nabla(h_N - h)\tilde{|}^2$ holds pointwise, so that the above continuum proof indeed fails in that case.

The key idea is to rather use another formulation of $\tilde{\mathcal{E}}(\mu_N|\mu)$, where diagonal contributions are subtracted differently. With 2D Ginzburg-Landau vortex analysis in mind, a natural choice is (cf. Lemma 6.2.10)

$$\tilde{\mathcal{E}}(\mu_N|\mu) = \lim_{\eta\downarrow 0} \Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d} (g_s(\eta) \wedge g_s(x-y))\, d(\mu_N-\mu)(x)\, d(\mu_N-\mu)(y) \; - \frac{g_s(\eta)}{N}\Big).$$

This formulation appears to be much better suited for our purposes here. Mimicking the mean-field limit proof for 2D Ginzburg-Landau vortices [395], the main technical ingredient is given by a suitable version of the so-called ball construction lower bound (cf. Proposition 6.2.15): in the Coulomb case, we show that there exists a collection of disjoint closed balls $\mathcal{B}_N$ with total radius that tends to 0 as $N \uparrow \infty$ and such that

$$\liminf_{N\uparrow\infty} \liminf_{\eta\downarrow 0} \Big(\int_{\mathcal{B}_N} |\nabla h_{N,\eta}|^2 - \frac{1}{N} g_s(\eta)\Big) \geq 0.$$

This is classical for the Coulomb case in the 2D Ginzburg-Landau context [382, Chapter 4], and we show that a suitable version of this lower bound also holds for Riesz potentials if and only if $0 \leq s < 1$, hence our restriction to this parameter range. Interestingly, such arguments are natural for Ginzburg-Landau vortices but were not common in the context of mean-field limits of particle systems. Note that this mean-field limit proof obtained by adapting a weak-strong uniqueness argument makes a particularly strong use of the regularity properties of the limiting equation.

1.2.2 Well-posedness for mean-field evolutions (Chapter 7)

As announced, the ultimate goal of this second part of the thesis is to obtain various mean-field limit results for the evolution of vortex matter in the 2D Ginzburg-Landau model with pinning impurities and with imposed current. Since in certain regimes the fluid-like mean-field evolutions that we shall obtain appear to be new in the literature, we devote Chapter 7 to establishing a global well-posedness theory for these equations, considering general vortex-sheet initial data as well, and further investigating the uniqueness and regularity properties of the solutions. Depending on the considered regime of the vortex density, characterized by some parameter $\lambda \in [0, \infty]$, the mean-field equation takes on the following guise,

$$\partial_t \mathrm{v} = \lambda \nabla(a^{-1} \operatorname{div}(a\mathrm{v})) - \alpha(\Psi + \mathrm{v})\operatorname{curl}\mathrm{v} + \beta(\Psi + \mathrm{v})^\perp \operatorname{curl}\mathrm{v}, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{1.29}$$

where $\mathrm{v} : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is the mean-field supercurrent density associated with a nonnegative vorticity $\operatorname{curl}\mathrm{v} \geq 0$, where $\alpha > 0$, $\beta \in \mathbb{R}$, where $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a given forcing vector field, and where $a := e^h$ is determined by a given "pinning potential" $h : \mathbb{R}^2 \to \mathbb{R}$. The degenerate case $\lambda = 0$ is also of interest and physically corresponds to a high vortex density regime, while the limiting case $\lambda = \infty$ corresponds to a low vortex density and is to be understood as the following incompressible model,

$$\partial_t \mathrm{v} = \nabla \mathrm{p} - \alpha(\Psi + \mathrm{v})\operatorname{curl}\mathrm{v} + \beta(\Psi + \mathrm{v})^\perp \operatorname{curl}\mathrm{v}, \qquad \operatorname{div}(a\mathrm{v}) = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2. \tag{1.30}$$

Note that in the parabolic case $\alpha > 0$, $\beta = 0$, this incompressible model can be seen as a Wasserstein gradient flow for the vorticity $\operatorname{curl}\mathrm{v}$, which coincides with the mean-field limit of the formal Coulomb discrete vortex dynamics, as is natural for low vortex density. However, a common gradient flow structure seems to be missing for the whole family of equations (1.29) with $\lambda \in [0, \infty]$.

Let us briefly describe the physics and history of these models (1.29)–(1.30). In the framework of the (mesoscopic) 2D Ginzburg-Landau model, vortices are known to become point-like in the asymptotic limit of a large Ginzburg-Landau parameter (which is indeed typically the case in real-life superconductors), and to interact with one another according to a Coulomb pair potential. In the mean-field limit of a large number of vortices, the evolution of the (macroscopic) suitably normalized mean-field density $\mathrm{m} : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ of the vortex liquid was then naturally conjectured to satisfy the following Chapman-Rubinstein-Schatzman-E equation [173, 111],

$$\partial_t \mathrm{m} = \operatorname{div}(|\mathrm{m}|\nabla(-\triangle)^{-1}\mathrm{m}), \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2,$$

where $(-\triangle)^{-1}\mathrm{m}$ is indeed the Coulomb potential generated by the vortices. Although the vortex density m is a priori a signed measure, we restrict here (and throughout this part of the thesis) to nonnegative measures, $|\mathrm{m}| = \mathrm{m} \geq 0$, so that the above is replaced by

$$\partial_t \mathrm{m} = \operatorname{div}(\mathrm{m}\nabla(-\triangle)^{-1}\mathrm{m}). \tag{1.31}$$

More precisely, the mean-field supercurrent density $\mathrm{v} : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ (linked to the vortex density through the relation $\mathrm{m} = \operatorname{curl} \mathrm{v}$) was conjectured to satisfy

$$\partial_t \mathrm{v} = \nabla \mathrm{p} - \mathrm{v} \operatorname{curl} \mathrm{v}, \qquad \operatorname{div} \mathrm{v} = 0. \tag{1.32}$$

Taking the curl of this equation indeed formally yields (1.31), noting that the incompressibility constraint $\operatorname{div} \mathrm{v} = 0$ allows to write $\mathrm{v} = \nabla^\perp \triangle^{-1} \mathrm{m}$.

In the context of superfluidity [4, 376], a conservative counterpart of the usual parabolic Ginzburg-Landau equation is used as a mesoscopic model. This counterpart is given by the Gross-Pitaevskii equation, which is a particular instance of a nonlinear Schrödinger equation. At the level of the mean-field evolution of the corresponding vortices, we then need to replace (1.31) by its conservative version, thus replacing $\nabla(-\triangle)^{-1}\mathrm{m}$ by $\nabla^\perp(-\triangle)^{-1}\mathrm{m}$. As argued in [26], there is also physical interest in rather starting from the "mixed-flow" (or "complex") Ginzburg-Landau model, which is a mix between the usual Ginzburg-Landau equation describing superconductivity ($\alpha = 1$, $\beta = 0$), and its conservative counterpart given by the Gross-Pitaevskii equation ($\alpha = 0$, $\beta = 1$). The above mean-field equation for the supercurrent density v is then replaced by the following, for $\alpha \geq 0$, $\beta \in \mathbb{R}$,

$$\partial_t \mathrm{v} = \nabla \mathrm{p} - \alpha \mathrm{v} \operatorname{curl} \mathrm{v} + \beta \mathrm{v}^\perp \operatorname{curl} \mathrm{v}, \qquad \operatorname{div} \mathrm{v} = 0. \tag{1.33}$$

Note that in the conservative case $\alpha = 0$, this equation is equivalent to the 2D Euler equation, as is clear from the identity $\mathrm{v}^\perp \operatorname{curl} \mathrm{v} = (\mathrm{v} \cdot \nabla)\, \mathrm{v} - \frac{1}{2}\nabla|\mathrm{v}|^2$.

The first rigorous deductions of these macroscopic mean-field limit models from the mesoscopic Ginzburg-Landau equation are due to [281, 263, 395]. As discovered by Serfaty [395], in the dissipative case $\alpha > 0$, the limiting equation (1.33) is only correct in a regime of dilute vortices, while for higher vortex density it must be replaced by the following compressible flow,

$$\partial_t \mathrm{v} = \lambda \nabla(\operatorname{div} \mathrm{v}) - \alpha \mathrm{v} \operatorname{curl} \mathrm{v} + \beta \mathrm{v}^\perp \operatorname{curl} \mathrm{v}, \tag{1.34}$$

for some $0 < \lambda < \infty$. In Chapter 8 we show that for even higher vortex density the relevant limiting equation is (1.34) in the degenerate case $\lambda = 0$. In contrast, in the conservative case $\alpha = 0$, the equation (1.33) is always expected to hold in the corresponding mean-field limit; this is further discussed in Section 1.2.3 below. In Chapter 8 we rather consider the 2D Ginzburg-Landau model with pinning impurities and with applied current, in which case the mean-field equations (1.33)–(1.34) are replaced by (1.30)–(1.29), with a pinning weight $a = e^h$ and with a forcing term $\Psi := F^\perp - \nabla^\perp h$ in terms of the pinning force $-\nabla h$ and of some vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ related to the imposed electric current. In the conservative regime $\alpha = 0$, $\beta = 1$, the incompressible model (1.30) takes the form of the following inhomogeneous version of the 2D Euler equation: using the identity $\mathrm{v}^\perp \operatorname{curl} \mathrm{v} = (\mathrm{v} \cdot \nabla)\, \mathrm{v} - \frac{1}{2}\nabla|\mathrm{v}|^2$, and setting $\tilde{\mathrm{p}} := \mathrm{p} - \frac{1}{2}|\mathrm{v}|^2$,

$$\partial_t \mathrm{v} = \nabla \tilde{\mathrm{p}} + \Psi^\perp \operatorname{curl} \mathrm{v} + (\mathrm{v} \cdot \nabla)\, \mathrm{v}, \qquad \operatorname{div}(a\mathrm{v}) = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2. \tag{1.35}$$

In the context of 2D fluid dynamics, this conservative equation is known as the lake equation [217, p.235] (see also [96, 97]): the pinning weight a corresponds to the effect of a varying depth in shallow water [351], while the forcing Ψ is similar to a background flow.

Rather than discussing the proof of global well-posedness results for (1.29)–(1.30), we describe the structure of these equations, which is best understood from their vorticity formulation. Setting $\mathrm{m} := \operatorname{curl} \mathrm{v}$ and $\mathrm{d} := \operatorname{div}(a\mathrm{v})$, these equations take the form of a nonlinear nonlocal transport equation for the vorticity m,

$$\partial_t \mathrm{m} = \operatorname{div}\Big(\mathrm{m}\big(\alpha(\Psi+\mathrm{v})^\perp + \beta(\Psi+\mathrm{v})\big)\Big), \qquad \mathrm{v} = a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\mathrm{m} + \nabla(\operatorname{div} a\nabla)^{-1}\mathrm{d}, \quad (1.36)$$

where for the incompressible model (1.30) we have $\mathrm{d} := 0$, while for the compressible model (1.29) the divergence d is the solution of the following transport-diffusion equation (which is highly degenerate when $\lambda = 0$),

$$\partial_t \mathrm{d} - \lambda \triangle \mathrm{d} + \lambda \operatorname{div}(\mathrm{d}\nabla h) = \operatorname{div}\Big(a\mathrm{m}\big(-\alpha(\Psi+\mathrm{v}) + \beta(\Psi+\mathrm{v})^\perp\big)\Big).$$

In Chapter 7, in the non-degenerate case $\lambda > 0$, we establish a local existence result for all values of the parameters, while global existence is obtained for the incompressible model (1.30), as well as for the compressible model (1.29) in the parabolic case $\alpha > 0$, $\beta = 0$. General vortex-sheet initial data $\mathrm{m}^\circ \in \mathcal{P}(\mathbb{R}^2)$ can be considered in all parabolic cases: the Coulomb repulsion has the effect of spreading the mass, and the vorticity is indeed shown to become instantaneously bounded. In contrast, for the incompressible model both in the mixed-flow and in the conservative cases we may only consider initial data $\mathrm{m}^\circ \in \mathcal{P} \cap \mathrm{L}^q(\mathbb{R}^2)$ with $q > 1$. This result happens to be particularly subtle in the conservative case due to a lack of strong enough a priori estimates, and only a notion of "very weak" solutions is then obtained for such initial data. This is in sharp contrast with the simpler situation of the 2D Euler equation, which corresponds to the choice $a \equiv 1$, and for which existence with vortex-sheet initial data is well-known [143]. Inhomogeneities $a \not\equiv 1$ indeed give rise to important difficulties, as can be inferred from the following observation: in terms of the stress-energy tensor $S_\mathrm{v} := \mathrm{v} \otimes \mathrm{v} - \frac{1}{2}\operatorname{Id}|\mathrm{v}|^2$, for all smooth vector fields v with $\operatorname{div}(a\mathrm{v}) = 0$, the nonlinearity $\mathrm{mv} = \mathrm{v}\operatorname{curl}\mathrm{v}$ in equation (1.36) can be written as

$$\mathrm{v}\operatorname{curl}\mathrm{v} = -|\mathrm{v}|^2\frac{\nabla^\perp a}{2a} - a^{-1}(\operatorname{div}(aS_\mathrm{v}))^\perp,$$

where the first right-hand side term involving $|\mathrm{v}|^2$ is clearly not weakly continuous as a function of v (although the second term is, as in the classical theory [143]) and it only vanishes in the homogeneous case $a \equiv 1$. Various weak-strong uniqueness principles are proven for equations (1.29)–(1.30), in addition to a uniqueness result for bounded vorticity for the incompressible model (1.30), in parallel with Yudovich's theorem [427]. Finally, a well-posedness theory is also established for (1.29) in the degenerate regime $\lambda = 0$ in the parabolic case $\alpha > 0$, $\beta = 0$, but this is based on very different arguments, rather exploiting the scalar structure of the corresponding solution v and using ODE type arguments to obtain an explicit representation.

1.2.3 Mean-field dynamics of Ginzburg-Landau vortices (Chapter 8)

As announced, the main task in Chapter 8 consists in rigorously establishing (1.29) as the mean-field evolution of the supercurrent density described by the mesoscopic 2D Ginzburg-Landau model. In order to illustrate the problem, we start by describing this question in the simpler case without pinning impurities and without applied current. Working in the whole space to avoid delicate boundary issues, and omitting the magnetic gauge for simplicity of notation (see Section 8.2.3 otherwise), we start from the following mixed-flow 2D Ginzburg-Landau model,

$$\lambda_\varepsilon(\alpha + i|\log\varepsilon|\beta)\partial_t u_\varepsilon = \triangle u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2), \tag{1.37}$$

where $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}$ is the complex-valued order parameter describing superconductivity, where $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$, where $\lambda_\varepsilon > 0$ is a suitable time rescaling, and where $\varepsilon > 0$ is the inverse Ginzburg-Landau parameter, a characteristic of the material that is typically very small for real-life superconductors. We refer e.g. to [412, 411] for further reference on this model, and to [382] for a mathematical introduction. The order parameter u_ε has the following meaning: the values $|u_\varepsilon| = 1$ and 0 correspond to a superconducting and to a normal phase, respectively, and the vortices are the zeroes of u_ε with non-zero topological degree. A vortex of degree d carries an energy $\pi|d||\log\varepsilon|$. Vortices typically have a core of size of order ε, hence they become point-like in the asymptotic limit $\varepsilon \downarrow 0$. The *supercurrent density* is defined by

$$j_\varepsilon := \langle \nabla u_\varepsilon, iu_\varepsilon \rangle,$$

where $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{C}$ as identified with $\mathbb{R}^2$, that is, $\langle x, y\rangle = \Re(x\bar{y})$ for all $x, y \in \mathbb{C}$. As in fluid mechanics, the *vorticity* μ_ε is then derived from the supercurrent via $\mu_\varepsilon := \operatorname{curl} j_\varepsilon$. Note that this indeed corresponds to the density of vortices, in the sense that

$$\mu_\varepsilon := \operatorname{curl} j_\varepsilon \;\approx\; 2\pi \sum_i d_i \delta_{x_i}, \qquad \text{as } \varepsilon \downarrow 0, \tag{1.38}$$

with $\{x_i\}_i$ the vortex locations and $\{d_i\}_i$ their degrees (this is made precise by the so-called Jacobian estimates [382, Chapter 6]). For a fixed number N of vortices, the asymptotic limit $\varepsilon \downarrow 0$ of equation (1.37) is well-understood and vortices are known to behave like Coulomb particles,

$$\begin{aligned} \partial_t x_{i,N}^t &= -N^{-1}\big(\alpha\nabla_{x_i} W_N - \beta\nabla_{x_i}^\perp W_N\big)(x_{1,N}^t, \ldots, x_{N,N}^t), \\ W_N(x_1,\ldots,x_N) &:= -\sum_{i\neq j}^N \log|x_i - x_j|, \end{aligned} \tag{1.39}$$

with $\{t \mapsto x_{i,N}^t\}_{i=1}^N$ the macroscopic vortex trajectories. In this second part of the thesis, we rather study the situation when the number N_ε of vortices blows up as $\varepsilon \downarrow 0$, which is a physically more realistic situation, and we aim at describing the evolution of the density of the corresponding vortex liquid. For a small enough vortex density, that is, if N_ε does not blow up too quickly with respect to ε, the correct mean-field equation is naturally expected to coincide with the mean-field limit of the discrete Coulomb vortex dynamics (1.39), that is, the incompressible model (1.33) (as justified in Chapter 6). In contrast, the mean-field behavior changes drastically for a higher vortex density in the dissipative case $\alpha > 0$, and rather leads to the compressible model (1.34).

In order to experience the structure of the 2D Ginzburg-Landau model (1.37) and the importance of a careful vortex analysis, we now give a formal derivation of the mean-field equations (1.33)–(1.34). For that purpose, in addition to the supercurrent density j_ε and the vorticity μ_ε, we define the vortex velocity

$$V_\varepsilon := 2\langle \nabla u_\varepsilon, i\partial_t u_\varepsilon\rangle,$$

the energy density

$$e_\varepsilon := \frac{1}{2}\Big(|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big),$$

and the stress-energy tensor

$$(S_\varepsilon)_{kl} := \langle \partial_k u_\varepsilon, \partial_l u_\varepsilon\rangle - \frac{\delta_{kl}}{2}\Big(|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).$$

These definitions easily imply the following algebraic identities,

$$\partial_t j_\varepsilon = V_\varepsilon + \nabla\langle \partial_t u_\varepsilon, iu_\varepsilon\rangle, \qquad \partial_t \mu_\varepsilon = \operatorname{curl} V_\varepsilon. \tag{1.40}$$

Moreover, using equation (1.37) for u_ε, we further find the following identities for the divergence of the supercurrent density

$$\operatorname{div} j_\varepsilon = \langle \triangle u_\varepsilon, iu_\varepsilon\rangle = \lambda_\varepsilon\alpha\langle \partial_t u_\varepsilon, iu_\varepsilon\rangle - \frac{\lambda_\varepsilon\beta|\log\varepsilon|}{2}\partial_t(1-|u_\varepsilon|^2), \tag{1.41}$$

for the divergence of the stress-energy tensor

$$\operatorname{div} S_\varepsilon = \Big\langle \nabla u_\varepsilon, \triangle u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)\Big\rangle = \lambda_\varepsilon\alpha\langle \nabla u_\varepsilon, \partial_t u_\varepsilon\rangle + \frac{\lambda_\varepsilon|\log\varepsilon|\beta}{2}V_\varepsilon, \tag{1.42}$$

and for the time derivative of the energy density

$$\partial_t e_\varepsilon = \operatorname{div}\langle \nabla u_\varepsilon, \partial_t u_\varepsilon\rangle - \lambda_\varepsilon\alpha|\partial_t u_\varepsilon|^2.$$

Using (1.42) to rewrite the quantity $\langle \nabla u_\varepsilon, \partial_t u_\varepsilon\rangle$, this last identity rather takes on the following guise,

$$\lambda_\varepsilon\alpha\partial_t e_\varepsilon = \operatorname{div}\operatorname{div} S_\varepsilon - \frac{\lambda_\varepsilon|\log\varepsilon|\beta}{2}\operatorname{div} V_\varepsilon - \lambda_\varepsilon^2\alpha^2|\partial_t u_\varepsilon|^2. \tag{1.43}$$

If there is no excess energy, the Ginzburg-Landau energy is expected to split into a (concentrated) vortex energy of order $O(N_\varepsilon|\log\varepsilon|)$ and a (diffuse) phase energy of order $O(N_\varepsilon^2)$. Since the quantity $|1-|u_\varepsilon|^2|$ is bounded by $\varepsilon(e_\varepsilon)^{1/2}$, it is therefore formally of order $O(\varepsilon N_\varepsilon + \varepsilon|\log\varepsilon|)$, which is negligible as soon as N_ε remains much smaller than ε^{-1}. Choosing the critical scaling $\lambda_\varepsilon := N_\varepsilon/|\log\varepsilon|$, the above identities (1.41), (1.42), and (1.43) then become

$$\operatorname{div}\frac{j_\varepsilon}{N_\varepsilon} \approx \alpha\frac{\langle \partial_t u_\varepsilon, iu_\varepsilon\rangle}{|\log\varepsilon|}, \tag{1.44}$$

$$2\operatorname{div}\frac{S_\varepsilon}{N_\varepsilon^2} = 2\alpha\frac{\langle \nabla u_\varepsilon, \partial_t u_\varepsilon\rangle}{N_\varepsilon|\log\varepsilon|} + \beta\frac{V_\varepsilon}{N_\varepsilon}, \tag{1.45}$$

$$\alpha\partial_t\frac{2e_\varepsilon}{N_\varepsilon|\log\varepsilon|} = 2\operatorname{div}\operatorname{div}\frac{S_\varepsilon}{N_\varepsilon^2} - \beta\operatorname{div}\frac{V_\varepsilon}{N_\varepsilon} - 2\alpha^2\frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|^2}. \tag{1.46}$$

In order to be able to take weak limits in these equations, a priori bounds on all the terms are needed, and in addition relations between the various weak limits need to be found. In the limit $\varepsilon\downarrow 0$, vortices become point-like and the vorticity μ_ε looks like a sum of N_ε Dirac masses, cf. (1.38). We may thus formally assume that the rescaled vorticity $N_\varepsilon^{-1}\mu_\varepsilon$ converges weakly-* to some probability measure $\mathrm{m} \in \mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}(\mathbb{R}^2))$. Similarly, the vortex velocity V_ε concentrates at vortex locations, and we may assume that its rescaled version $N_\varepsilon^{-1}V_\varepsilon$ converges weakly-* to some measure $\mathrm{V}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{M}(\mathbb{R}^2)^2)$. For all $p<2$ the rescaled supercurrent density $N_\varepsilon^{-1}j_\varepsilon$ may be assumed to be bounded in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^2)$ and thus to converge weakly to some limit $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^2)^2)$, but it cannot converge in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ due to energy concentration. In short,

$$N_\varepsilon^{-1}\mu_\varepsilon \overset{*}{\rightharpoonup} \mathrm{m}, \qquad N_\varepsilon^{-1}V_\varepsilon \overset{*}{\rightharpoonup} \mathrm{V}, \qquad N_\varepsilon^{-1}j_\varepsilon \rightharpoonup \mathrm{v}\,. \tag{1.47}$$

Quadratic quantities such as $e_\varepsilon \approx \frac12|j_\varepsilon|^2$ and $|\partial_t u_\varepsilon|^2$ have a part that concentrates at vortex locations in the limit $\varepsilon\downarrow 0$, and their concentrated and diffuse parts must be analyzed separately. If there is no excess energy, the concentrated part of the energy density $e_\varepsilon \approx \frac12|j_\varepsilon|^2$ should coincide with the vortex self-interaction energy $\frac12|\log\varepsilon|\mu_\varepsilon \approx \frac12 N_\varepsilon|\log\varepsilon|\mathrm{m}$ (this is made precise by the Jerrard-Sandier ball-construction lower bound, see e.g. [382, Chapter 4]), while the diffuse part should be given by

$\frac{1}{2}N_\varepsilon^2|\mathrm{v}|^2$ in terms of the weak limit v of $N_\varepsilon^{-1}j_\varepsilon$. Such properties could be phrased in terms of defect measures for the convergence of $N_\varepsilon^{-1}j_\varepsilon$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ (cf. [381]). Similarly, if there is no excess energy, the concentrated part of $|\partial_t u_\varepsilon|^2$ should coincide with $\frac{1}{2}|\log\varepsilon|\mu_\varepsilon^{-1}|V_\varepsilon|^2 \approx \frac{1}{2}N_\varepsilon|\log\varepsilon|\mathrm{m}^{-1}|\mathrm{V}|^2$ in terms of the vortex velocity and the vorticity (this is made precise by the so-called product estimate [381]), while identity (1.44) in the form $\alpha^2|\partial_t u_\varepsilon|^2 \approx \alpha^2|\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle|^2 \approx \lambda_\varepsilon^{-2}|\operatorname{div} j_\varepsilon|^2$ suggests that the diffuse part of $\alpha^2|\partial_t u_\varepsilon|^2$ should simply be given by $|\log\varepsilon|^2|\operatorname{div}\mathrm{v}|^2$. In short,

$$2e_\varepsilon \approx |j_\varepsilon|^2 \approx N_\varepsilon|\log\varepsilon|\mathrm{m} + N_\varepsilon^2|\mathrm{v}|^2, \tag{1.48}$$

$$2\alpha^2|\partial_t u_\varepsilon|^2 \approx 2|\log\varepsilon|^2|\operatorname{div}\mathrm{v}|^2 + \alpha^2 N_\varepsilon|\log\varepsilon|\mathrm{m}^{-1}|\mathrm{V}|^2. \tag{1.49}$$

Let us now turn to the limit of the stress-energy tensor $S_\varepsilon \approx j_\varepsilon\otimes j_\varepsilon - \frac{\mathrm{Id}}{2}|j_\varepsilon|^2$. Due to the isotropy of the vortex core energy, in link with equipartition properties of the Ginzburg-Landau energy [280], the stress-energy tensor S_ε should not be sensitive to the concentrated part of j_ε in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$, and we simply expect $N_\varepsilon^{-2}S_\varepsilon \approx \mathrm{v}\otimes\mathrm{v} - \frac{\mathrm{Id}}{2}|\mathrm{v}|^2$ in terms of the weak limit v of $N_\varepsilon^{-1}j_\varepsilon$ (see also [382, Chapter 13]). In particular,

$$\operatorname{div}\frac{S_\varepsilon}{N_\varepsilon^2} \approx \operatorname{div}\Big(\mathrm{v}\otimes\mathrm{v} - \frac{\mathrm{Id}}{2}|\mathrm{v}|^2\Big) = \mathrm{v}^\perp\mathrm{m} + \mathrm{v}\operatorname{div}\mathrm{v}\,. \tag{1.50}$$

Injecting the convergences (1.47) and the identifications (1.48), (1.49), and (1.50) into identities (1.44), (1.45), and (1.46), we obtain after straightforward simplifications,

$$\operatorname{div}\mathrm{v} \approx \alpha\frac{\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle}{|\log\varepsilon|}, \tag{1.51}$$

$$2\mathrm{v}^\perp\mathrm{m} + 2\mathrm{v}\operatorname{div}\mathrm{v} \approx 2\alpha\frac{\langle\nabla u_\varepsilon, \partial_t u_\varepsilon\rangle}{N_\varepsilon|\log\varepsilon|} + \beta\mathrm{V}, \tag{1.52}$$

$$\alpha\partial_t\mathrm{m} + 2\alpha\lambda_\varepsilon\mathrm{v}\cdot\partial_t\mathrm{v} \approx 2\operatorname{div}(\mathrm{v}^\perp\mathrm{m}) + 2\mathrm{v}\cdot\nabla\operatorname{div}\mathrm{v} - \beta\operatorname{div}\mathrm{V} - \lambda_\varepsilon\alpha^2\mathrm{m}^{-1}|\mathrm{V}|^2, \tag{1.53}$$

and further injecting (1.51) into (1.40), we obtain

$$\alpha\partial_t\mathrm{v} \approx \alpha\mathrm{V} + \lambda_\varepsilon^{-1}\nabla\operatorname{div}\mathrm{v}, \qquad \partial_t\mathrm{m} = \operatorname{curl}\mathrm{V}\,. \tag{1.54}$$

We now separately consider the Gross-Pitaevskii and the dissipative cases.

- *Gross-Pitaevskii case ($\alpha = 0$, $\beta = 1$).* Identity (1.51) yields $\operatorname{div}\mathrm{v} = 0$, while identity (1.52) takes the form $\mathrm{V} = 2\mathrm{v}^\perp\mathrm{m}$. Injecting this into (1.54) then leads to

  $$\partial_t\mathrm{m} = 2\operatorname{div}(\mathrm{vm}), \qquad \operatorname{curl}\mathrm{v} = \mathrm{m}, \qquad \operatorname{div}\mathrm{v} = 0,$$

  or alternatively,

  $$\partial_t\mathrm{v} = \nabla\mathrm{p} + 2\mathrm{v}^\perp\operatorname{curl}\mathrm{v}, \qquad \operatorname{div}\mathrm{v} = 0.$$

  In the regime $1 \ll N_\varepsilon \ll \varepsilon^{-1}$ with the critical choice $\lambda_\varepsilon = N_\varepsilon/|\log\varepsilon|$, the rescaled supercurrent density $N_\varepsilon^{-1}j_\varepsilon$ is thus expected to converge to the solution v of this incompressible 2D Euler equation.
- *Dissipative case ($\alpha > 0$, $\alpha^2 + \beta^2 = 1$).* Injecting (1.54) into (1.53) yields

  $$\partial_t\mathrm{m} \approx \frac{2}{\alpha}\operatorname{div}(\mathrm{v}^\perp\mathrm{m}) - \frac{\beta}{\alpha}\operatorname{div}\mathrm{V} - \lambda_\varepsilon\mathrm{V}\cdot(2\mathrm{v} + \alpha\mathrm{m}^{-1}\mathrm{V}). \tag{1.55}$$

  Comparing with (1.54) in the form $\partial_t\mathrm{m} = \operatorname{curl}\mathrm{V}$, we deduce in the parabolic case ($\alpha = 1$, $\beta = 0$) that $\mathrm{V} = -2\mathrm{vm}$, while a more careful computation in the general mixed-flow case leads to $\mathrm{V} = -2\alpha\mathrm{vm} + 2\beta\mathrm{v}^\perp\mathrm{m}$. Injecting this into (1.54), we obtain

  $$\partial_t\mathrm{v} \approx (\lambda_\varepsilon\alpha)^{-1}\nabla\operatorname{div}\mathrm{v} + 2(-\alpha\mathrm{v} + \beta\mathrm{v}^\perp)\operatorname{curl}\mathrm{v}\,. \tag{1.56}$$

  We need to distinguish between three regimes:

— for $N_\varepsilon \ll |\log\varepsilon|$ (hence $\lambda_\varepsilon \ll 1$), equation (1.55) and the identification of V yield

$$\partial_t \mathrm{m} = \operatorname{div}\left(2(\alpha \mathrm{v}^\perp + \beta \mathrm{v})\,\mathrm{m}\right),$$

while equation (1.56) together with (1.51) leads to div v = 0, so that we deduce, using the relation div v = 0 in the form $\mathrm{v} = \nabla^\perp \triangle^{-1}\mathrm{m}$, and setting $\mathrm{p} := -2\triangle^{-1}\operatorname{div}((-\alpha\mathrm{v} + \beta\mathrm{v}^\perp)\,\mathrm{m})$,

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + 2(-\alpha \mathrm{v} + \beta \mathrm{v}^\perp)\operatorname{curl}\mathrm{v}, \qquad \operatorname{div}\mathrm{v} = 0;$$

— for $N_\varepsilon \simeq |\log\varepsilon|$ with $\lambda_\varepsilon \to \lambda \in (0,\infty)$, equation (1.56) becomes

$$\lambda \partial_t \mathrm{v} = \alpha^{-1}\nabla \operatorname{div}\mathrm{v} + 2\lambda(-\alpha\mathrm{v} + \beta\mathrm{v}^\perp)\operatorname{curl}\mathrm{v};$$

— for $N_\varepsilon \gg |\log\varepsilon|$ (hence $\lambda_\varepsilon \gg 1$), equation (1.56) becomes

$$\partial_t \mathrm{v} = 2(-\alpha\mathrm{v} + \beta\mathrm{v}^\perp)\operatorname{curl}\mathrm{v}\,.$$

In the regime $1 \ll N_\varepsilon \ll \varepsilon^{-1}$ with the critical choice $\lambda_\varepsilon = N_\varepsilon/|\log\varepsilon|$, the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon$ is thus expected to converge to the solution v of one of these equations depending on the vortex density regime.

This careful heuristic argument therefore allows to predict the whole family of announced mean-field evolutions (1.34), and formally explains the (a priori unexpected) higher variety of possible behaviors in the dissipative case depending on the vortex density regime. Note however that this argument relies on important unproven assumptions such as the absence of energy excess and the equipartition of energy.

In order to establish these mean-field limit results, we follow Serfaty [395] and make use of a modulated energy technique (as illustrated in a much simpler setting in Chapter 6). In the present situation, the method consists in considering the following *modulated Ginzburg-Landau energy*,

$$\mathcal{E}_\varepsilon := \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{1}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2\Big), \tag{1.57}$$

where v denotes the solution of the (postulated) limiting equation. This modulated energy somehow measures the distance between the supercurrent density $j_\varepsilon = \langle \nabla u_\varepsilon, i u_\varepsilon\rangle$ and the postulated limit $N_\varepsilon \mathrm{v}$, in a way that is well adapted to the Ginzburg-Landau energy structure. Since each vortex of degree d carries a self-interaction energy $\pi|d||\log\varepsilon|$, and assuming that all vortices have positive degrees initially, we need to subtract the fixed quantity $\pi N_\varepsilon |\log\varepsilon|$ from (1.57), thus defining the *modulated energy excess* $\mathcal{D}_\varepsilon := \mathcal{E}_\varepsilon - \pi N_\varepsilon|\log\varepsilon|$. In order to prove the desired convergence $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$, showing $\mathcal{D}_\varepsilon = o(N_\varepsilon^2)$ is then sufficient. For that purpose, the strategy is to establish a Grönwall relation for $\mathcal{D}_\varepsilon$, so that if $\mathcal{D}_\varepsilon$ is initially of order $o(N_\varepsilon^2)$ then it remains so, which in turn implies the desired convergence. In order to prove the Grönwall relation for $\mathcal{D}_\varepsilon$, we exploit as in [395] the strong regularity properties of v (as established in Chapter 7), and various similar identities and identifications are needed as in the above formal argument. This requires to use all the by-now standard tools of vortex analysis like vortex-balls constructions, Jacobian estimates, and product estimates (see e.g. [382]). The argument in [395] covers the dissipative case in the regime $1 \ll N_\varepsilon \lesssim |\log\varepsilon|$, and the Gross-Pitaevskii case in the regime $|\log\varepsilon| \ll N_\varepsilon \ll \varepsilon^{-1}$. In Chapter 8, we treat in addition the parabolic case in the regime $|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|\log|\log\varepsilon|$, while other missing regimes remain open questions (except the Gross-Pitaevskii case with $N_\varepsilon \gg 1$ blowing up sufficiently slowly, which is treated in [263] by other methods); see Section 8.1.4.

1.2.4 Ginzburg-Landau vortices in disordered media (Chapter 8, cont'd)

As explained above, when an electric current is applied to a type-II superconductor, it flows through the material, inducing a Lorentz-like force that makes the vortices move, which in turn dissipates energy and destroys the superconductivity property. In order to prevent vortices from moving and therefore reduce this energy dissipation, a common attempt consists in introducing normal impurities in the material, which are meant to destroy superconductivity locally and therefore "pin down" the vortices — at least if the applied current is not too strong. The resulting competition between vortex interactions and disorder leads to glassy effects that are still largely not understood [193]. In Chapter 8 we aim at a rigorous description of the corresponding evolution of the vortex matter in the mean-field regime.

More precisely, normal impurities are usually modeled by correcting the Ginzburg-Landau equations with a non-uniform equilibrium density $a : \mathbb{R}^2 \to [0, 1]$, which locally lowers the energy penalty associated with the vortices [284, 109] (see also [108]). Writing $a = e^h$ in terms of the so-called pinning potential $h : \mathbb{R}^2 \to \mathbb{R}$, and also considering the effect of an imposed current, the 2D Ginzburg-Landau model (1.37) is replaced after some transformations and simplifications by the following (cf. Section 8.2.1),

$$\lambda_\varepsilon(\alpha + i|\log \varepsilon|\beta)\partial_t u_\varepsilon = \triangle u_\varepsilon + \frac{a u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2) + \nabla h \cdot \nabla u_\varepsilon + iF^\perp \cdot \nabla u_\varepsilon + f u_\varepsilon, \tag{1.58}$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is some vector field related to the imposed electric current and where $f : \mathbb{R}^2 \to \mathbb{R}$ is some unimportant zeroth-order term. As formally predicted by Chapman and Richardson [110], the non-uniform density a translates at the level of the vortex dynamics into a pinning force $-\nabla h$, indeed attracting the vortices to the minima of a. For a fixed number N of vortices, the asymptotic limit $\varepsilon \downarrow 0$ of equation (1.58) is rigorously well-understood [264, 397, 261, 279], and vortices are subjected to three forces:

— their mutual Coulomb interaction;
— the Lorentz-like force F due to the applied current;
— the pinning force $-\nabla h$.

Note that the pinning and applied current intensities are parameters which can be tuned, leading to regimes in which one or two forces dominate over the others, or all are of the same order. In the sequel, we rather consider the physically more realistic situation when the number N_ε of vortices blows up as $\varepsilon \downarrow 0$, and we wish to describe the evolution of the density of the corresponding vortex liquid. The most interesting situation from the modeling viewpoint is to further let the pinning weight oscillate quickly at some mesoscale η_ε, which also tends to 0 as $\varepsilon \downarrow 0$. Since impurities are typically randomly scattered in the sample, we consider a pinning weight of the form $a(x) := a^0(x/\eta_\varepsilon)$ for some typical realization a^0 of a stationary random field. For simplicity, we may focus attention on the periodic case. One is thus led to the question of combining the mean-field limit for the Ginzburg-Landau model (1.58) together with a homogenization limit. In Chapter 8, we split the difficulty into two parts: we first establish a rigorous mean-field limit result in the presence of a fixed disorder $\eta_\varepsilon = \eta > 0$, and then we consider the homogenization limit of the obtained mean-field equations as $\eta \downarrow 0$. Since all arguments are quantitative, this yields a complete result at least for diagonal regimes with a large number of particles and with a small pin separation that tends sufficiently slowly to 0.

Let us first briefly describe the mean-field limit result for fixed disorder $\eta_\varepsilon = \eta > 0$, which is about generalizing the results described in Section 1.2.3 above to the case with pinning and forcing: starting from (1.58) instead of (1.37), we show that the mean-field equation (1.34) is replaced by (1.29) with $\Psi := F^\perp - \nabla^\perp h$. Regarding the strategy of the proof, we adapt the modulated energy method first used by Serfaty [395]. In the present context with pinning weight a, the modulated energy (1.57) must naturally be changed into a weighted one,

$$\tilde{\mathcal{E}}_\varepsilon := \frac{1}{2}\int_{\mathbb{R}^2} a\left(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\,|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2\right). \tag{1.59}$$

In addition, a vortex of degree d at a point x now carries a self-interaction energy $\pi|d|a(x)|\log\varepsilon|$, which non-trivially depends on the vortex location x. The vortex self-interaction energy that needs to be subtracted from the modulated energy (1.59) is therefore no longer $\pi N_\varepsilon|\log\varepsilon|$ but rather $\pi\sum_i d_i a(x_i)|\log\varepsilon| \sim \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\mu_\varepsilon$ (cf. (1.38)), so that the modulated energy excess now takes the form

$$\tilde{\mathcal{D}}_\varepsilon := \frac{1}{2}\int_{\mathbb{R}^2} a\left(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\,|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\right).$$

In order to establish a Grönwall relation for $\tilde{\mathcal{D}}_\varepsilon$, even at a formal level, the heavier structure of the weighted Ginzburg-Landau equation must first be well understood. We mention some additional technical difficulties that are encountered along the way.

— In some regimes, the solution v of the limiting equation must crucially be replaced in the modulated energy (1.59) by some suitable ε-dependent map $\mathrm{v}_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$, which is separately shown to converge to v. This amounts to including lower-order terms in the modulated energy.

— In this weighted setting, we need to establish a *localized* version of the so-called ball construction lower bound [382, Chapter 4] with a very precise error estimate $o(N_\varepsilon^2)$ (which gets very small when N_ε diverges slowly). The usual error in the ball construction lower bound is essentially $O(N_\varepsilon|\log r|)$, where r is the total radius of the balls, so that we need to take r large enough (almost as large as $O(1)$ when N_ε diverges slowly), but here the pinning weight a adds an important difficulty since it may vary significantly over the size of the balls of this construction, thus perturbing the lower bound itself. A particularly careful vortex analysis is therefore needed (cf. Section 8.5).

— If ∇h, F, and f in (1.58) are only assumed to be bounded, then the modulated energy $\tilde{\mathcal{E}}_\varepsilon$ does usually not remain finite along the flow, which forces us to truncate it at some scale. As a consequence, the vortex analysis must further be refined to the setting of the infinite plane with no global energy control, hence no a priori finiteness assumption on the total number of vortices (cf. Section 8.5).

Once the mean-field limit equations are rigorously established, we may turn to their homogenization limit for a small pin separation $\eta \downarrow 0$. This happens to be a difficult problem, related to the subtle competition between vortex interactions and pinning and to the possible glassy properties of such systems.

In the regime of negligible vortex interactions, particles are independent and the mean-field equations are reduced to simple linear transport equations, which are much easier to understand. On the one hand, for $F \equiv 0$, the vorticity is simply attracted towards the neighborhood of the local wells of the pinning potential h. On the other hand, a constant applied force $F \not\equiv 0$ can be absorbed into the pinning force $-\nabla h$ by adding to the potential h an affine function, which effectively tilts the potential landscape into a washboard-shaped graph. Beyond some positive critical value of the intensity $|F|$, the tilted potential has no local minimum, leading the vorticity to fall in the direction of F, while below this critical value the vorticity remains pinned. This critical value corresponds to the so-called *depinning current*. Such a system seems to be known as a "washboard" in the physics literature, and the limiting stick-slip dynamics is rigorously justified in Section 8.9.5.

In critical regimes with non-negligible interactions, we formally derive corresponding nonlinear stick-slip homogenized equations (cf. Section 8.9.4). Due to the complexity of the collective effects of the interacting vortices, the rigorous treatment of this homogenization limit is however extremely delicate and is postponed to future works. Note that all glassy properties of vortex matter (such as the value of the depinning exponent) should be enclosed in these formal homogenized equations. Unravelling and describing them is a separate topic and is not pursued in this thesis.

Notation

— d is the dimension of the ambient space $\mathbb{R}^d$.
— $d(\cdot,\cdot)$ denotes the Euclidean distance between subsets of $\mathbb{R}^d$.
— For all $k \in \mathbb{N}$, B^k denotes the Euclidean ball of unit radius centered at 0 in $\mathbb{R}^k$, and for all $x \in \mathbb{R}^k$ and $r > 0$ we set $B^k(x) := x + B^k$, $B^k_r := rB^k$, and $B^k(x,r) := B^k_r(x) := x + rB^k$. When $k = d$ or when there is no confusion possible on the meant dimension, we drop the superscript k.
— We use similar notation for cubes as for balls, replacing B^k by $Q^k := [-\frac{1}{2}, \frac{1}{2})^k$ (the unit cube in dimension k). Note that Q^k is frequently identified with the k-torus $\mathbb{T}^k$.
— $\mathcal{B}(\mathbb{R}^k)$ denotes the Borel σ-algebra on $\mathbb{R}^k$.
— For all subsets A of a reference set E, we let $A^c := E \setminus A$ denote the complement of A in E.
— $\mathbb{1}_A$ denotes the indicator function of a set A.
— $\mathbb{R}^+ := [0,\infty)$ denotes the set of nonnegative real numbers.
— We use the notation $a \wedge b := \min\{a,b\}$, $a \vee b := \max\{a,b\}$, $a_+ := a \vee 0$, and $a_- := (-a) \vee 0$ for all $a, b \in \mathbb{R}$. In particular, given a function $f : \mathbb{R}^k \to \mathbb{R}$, we denote its positive and negative parts by $f_+(x) := 0 \vee f(x)$ and $f_-(x) := 0 \vee (-f)(x)$, respectively.
— $\lfloor a \rfloor$ denotes the largest integer $\le a$, and $\lceil a \rceil$ denotes the smallest integer $\ge a$.
— $\mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^d)$ denotes the convex cone of locally finite non-negative Borel measures on $\mathbb{R}^d$, and $\mathcal{P}(\mathbb{R}^d)$ denotes the convex subset of Borel probability measures, endowed with the usual weak-* topology.
— $\mathrm{Mes}(\mathbb{R}^d) = \mathrm{Mes}(\mathbb{R}^d;\mathbb{R})$ denotes the space of Lebesgue-measurable functions $\mathbb{R}^d \to \mathbb{R}$.
— For all $\sigma > 0$, $C^\sigma_b(\mathbb{R}^d)$ denotes the Banach space $C^{\lfloor\sigma\rfloor,\sigma-\lfloor\sigma\rfloor}_b(\mathbb{R}^d)$ of bounded Hölder functions, while $C^\sigma_c(\mathbb{R}^d)$ denotes the subspace of compactly supported functions, and $C^\sigma(\mathbb{R}^d)$ denotes the space of functions that are locally in $C^\sigma_b(\mathbb{R}^d)$. For $\sigma \in (0,1)$, we denote by $|\cdot|_{C^\sigma}$ the usual Hölder seminorm, and by $\|\cdot\|_{C^\sigma} := |\cdot|_{C^\sigma} + \|\cdot\|_{\mathrm{L}^\infty}$ the corresponding norm.
— $\mathrm{L}^p_{\mathrm{uloc}}(\mathbb{R}^d)$ denotes the Banach space of functions that are uniformly locally L^p-integrable on $\mathbb{R}^d$, endowed with norm $\|f\|_{\mathrm{L}^p_{\mathrm{uloc}}} := \sup_x \|f\|_{\mathrm{L}^p(B(x))}$. We similarly define the corresponding Sobolev spaces $W^{k,p}_{\mathrm{uloc}}(\mathbb{R}^2)$.
— Given a Banach space X and given $t > 0$, we use the notation $\|\cdot\|_{\mathrm{L}^p_t X}$ for the usual norm in $\mathrm{L}^p([0,t];X)$.
— Given a function $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$, the superscript t in the notation u^t indicates that the function u is evaluated at time t, that is, $u^t := u(t,\cdot)$.
— Given an exponent $1 \le p \le \infty$, we denote its Hölder conjugate by $p' := \frac{p}{p-1}$.
— $\mathcal{F}$ denotes Fourier transformation on $\mathbb{R}^d$.
— Given two linear operators A, B on some function space, we denote by $[A,B] := AB - BA$ their commutator.
— For $a \in \mathbb{R}^d$, we set $a^{\otimes 2} := a \otimes a$, and we similarly define $a^{\otimes k}$ for all $k \ge 1$.
— Given a matrix M, we denote by M^t or M^* its transpose matrix.
— For any vector $a = (a_1, a_2) \in \mathbb{R}^2$, we use the notation $a^\perp = (-a_2, a_1)$. We also write $\mathbb{J} : \mathbb{R}^2 \to \mathbb{R}^2$ for the rotation of vectors by angle $\frac{\pi}{2}$ in the plane, so that $\mathbb{J}a = a^\perp$.
— For any vector field $F = (F_1, \ldots, F_d) = \mathbb{R}^d \to \mathbb{R}^d$, we use the notation $\nabla \cdot F = \sum_{i=1}^d \nabla_i F_i$. For any 2D vector field $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$, we use the notation $\operatorname{curl} F = \partial_1 F_2 - \partial_2 F_1$.
— For any k-tensor field T, we denote its components by $(T)_{i_1\ldots i_k}$. Its divergence is defined by $(\operatorname{div} T)_{i_1\ldots i_{k-1}} = \sum_{j=1}^d \nabla_j (T)_{i_1\ldots i_{k-1}j}$, and its rotational is defined by $(\nabla \times T)_{i_1\ldots i_k} = \sum_{j,l=1}^d \varepsilon_{i_k jl} \nabla_j (T)_{i_1\ldots i_{k-1}l}$.
— C denotes various positive constants that only depend on fixed parameters (like the space dimension d, etc.) and possibly on other controlled quantities. Note that these constants may vary from line to line. We denote by C_t any positive constant that further depends on an upper

bound on time $t \geq 0$, while additional subscripts indicate the dependence on other parameters.

— We write $\lesssim$ and $\gtrsim$ for $\leq$ and $\geq$ up to such multiplicative constants C. We use the notation $\simeq$ if both relations $\lesssim$ and $\gtrsim$ hold. Alternatively, we write $a \leq O(b)$ if $a \lesssim b$. We add a subscript in order to indicate the dependence of the multiplicative constants on other parameters (e.g. on time t).

— For $a, b \geq 0$, the notation $a \ll b$ (or equivalently $b \gg a$) stands for $a \leq \frac{1}{C} b$ for some large enough constant $C \simeq 1$.

— When considering sequences $(a_\varepsilon)_\varepsilon, (b_\varepsilon)_\varepsilon \subset \mathbb{R}^+$ indexed by a parameter ε that is sent to 0, we rather write $a_\varepsilon \ll b_\varepsilon$ (or $b_\varepsilon \gg a_\varepsilon$) if $a_\varepsilon / b_\varepsilon$ converges to 0 as the parameter $\varepsilon \downarrow 0$. Alternatively, we write $a_\varepsilon \leq o(b_\varepsilon)$ or $a_\varepsilon \leq o_\varepsilon(b_\varepsilon)$ if $a_\varepsilon \ll b_\varepsilon$.

— $o_r(1)$ denotes a quantity that goes to 0 when the parameter r goes to its limit, uniformly with respect to all other parameters. We write $o_r^{(b)}(1)$ if it converges to 0 only for any *fixed* value of the parameter b.

— Given an underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we write $\mathbb{E}[\cdot]$ for the expectation, $\mathrm{Var}[\cdot]$ for the variance, and $\mathrm{Cov}[\cdot; \cdot]$ for the covariance, and the notation $\mathbb{E}[\cdot \,\|\, \cdot]$ stands for the conditional expectation.

— $\mathcal{N}$ denotes a standard normal random variable (on $\mathbb{R}$).

— $\mathrm{d_{TV}}(\cdot, \cdot)$, $\mathrm{d_W}(\cdot, \cdot)$, and $\mathrm{d_K}(\cdot, \cdot)$ denote the total variation distance, the 1-Wasserstein (or Monge-Kantorovich) distance, and the Kolmogorov distance, respectively: given two random variables X_1, X_2 with laws $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R})$, we recall that

$$\mathrm{d_{TV}}(X_1, X_2) := \sup_{A \in \mathcal{B}(\mathbb{R})} \big|\mu_1(A) - \mu_2(A)\big|, \tag{1.60}$$

$$\mathrm{d_W}(X_1, X_2) := \sup_{\substack{\psi \in C^{0,1}(\mathbb{R}), \\ |\psi|_{C^{0,1}} \leq 1}} \int_{\mathbb{R}} \psi(x) d(\mu_1 - \mu_2)(x), \tag{1.61}$$

$$\mathrm{d_K}(X_1, X_2) := \sup_{x \in \mathbb{R}} \big|\mu_1((-\infty, x]) - \mu_2((-\infty, x])\big|. \tag{1.62}$$

Part I

Topics in stochastic homogenization

Chapter 2

Stochastic homogenization of nonconvex unbounded integral functionals with convex growth

We consider the well-travelled problem of homogenization of random integral functionals. When the integrand satisfies standard growth conditions, the qualitative theory is well-understood. When it comes to unbounded functionals, that is, when the domain of the integrand is not the whole space and may depend on the space variable, there is no satisfactory theory. In this chapter we develop a complete qualitative stochastic homogenization theory for nonconvex unbounded functionals with convex growth. We first prove that if the integrand is convex and has p-growth from below (with $p > d$, the space dimension), then it admits homogenization regardless of growth conditions from above (in particular its domain may depend on the space variable). In the case of nonconvex integrands, we prove that a similar homogenization result holds provided the nonconvex integrand admits a two-sided estimate by a convex integrand that itself admits homogenization. These results are also new in the periodic setting, and are motivated by (and of interest to) the rigorous derivation of rubber elasticity from polymer physics, which indeed involves the stochastic homogenization of such unbounded functionals.

This chapter essentially corresponds to the article [166] jointly written with Antoine Gloria.

Contents

2.1 Introduction

2.1.1 General overview

Let O be a bounded Lipschitz domain of $\mathbb{R}^d$, $d, m \geq 1$. We consider the homogenization of a random integral functional $I_\varepsilon : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ given by

$$I_\varepsilon(u) \;=\; \int_O W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx,$$

where W is a typical realization of a random Borel function that is stationary in its first variable and satisfies for almost every $y \in \mathbb{R}^d$ for all $\Lambda \in \mathbb{R}^{m\times d}$ the two-sided estimate

$$\frac{1}{C}|\Lambda|^p - C \leq V(y,\Lambda) \;\leq\; W(y,\Lambda) \;\leq\; C(1+V(y,\Lambda)), \tag{2.1}$$

for some $C > 0$, $p > 1$, and some typical realization of a random Borel function $V : \mathbb{R}^d \times \mathbb{R}^{m\times d} \to [0,\infty]$ that is stationary in its first variable and convex in the second. The originality of the growth condition that we consider here is that $V(y,\cdot)$ may take infinite values and that its domain may depend on y, so that the domain of the homogenized integrand $\overline{W}$ (if it exists) is unknown a priori.

The motivation for considering such a problem comes from the derivation of nonlinear elasticity from the statistical physics of polymer-chain networks [9, 200, 138]. Indeed, the free energy of the polymer-chain network is given by two contributions: a steric effect (for which proving homogenization is one of the most important open problems in the field, cf. Section 2.1.7), and the sum of the free energies of the deformed chains. The free energy of a single chain is a convex increasing function of the square of the length of the deformed polymer-chain, which blows up at finite deformation but depends on the number of monomers in the considered chain. Truncating the steric effect for simplicity, the corresponding problem in a continuum setting would be the homogenization of the nonconvex integrand

$$W(y,\Lambda) \;=\; V(y,\Lambda) + a(y)\,g(\det\Lambda) \;\leq\; C(1+V(y,\Lambda)), \tag{2.2}$$

where V is an *unbounded convex stationary ergodic integrand* (the domain of which is allowed to depend on the space variable), where a is a uniformly bounded stationary random field, and where g is a nonnegative convex function. Unfortunately, the present work crucially relies on the upper bound in (2.2) (cf. (2.1)), which prevents us from considering any physically relevant model for the steric effect since this upper bound is incompatible with the non-interpenetration of matter (cf. Section 2.1.7).

Homogenization of multiple integrals has a long history, and we start with the state of the art when V and W are periodic in the first variable:

(i) The first contribution (beyond the linear case) is due to Marcellini [314], who addressed the homogenization of *convex periodic* integrands satisfying a *polynomial standard growth condition*, that is, (2.1) for $V(y,\Lambda) = |\Lambda|^p$.

(ii) Marcellini's result was then generalized to *nonconvex periodic* integrands satisfying a *polynomial standard growth condition*, by Braides [77] (which covers in addition the case of almost-periodic coefficients) and independently by Müller [331, Theorem 1.3].

(iii) Müller [331, Theorem 1.5] also addressed the case of a *convex periodic* integrand satisfying a *convex standard growth condition* (2.1) for $V(y,\Lambda) = \tilde V(\Lambda)$ with $\tilde V : \mathbb{R}^{m\times d} \to \mathbb{R}^+$ a (general) convex finite-valued map, and $p > d$.

(iv) Braides and Defranceschi [78, Chapter 21] treated the case of *nonconvex periodic* integrands (see also [126] in the convex case) satisfying (2.1) where V is *convex periodic* and satisfies the *polynomial non-standard growth condition*

$$\frac{1}{C}|\Lambda|^p - C \,\le\, V(y,\Lambda) \,\le\, C(1+|\Lambda|^q) \tag{2.3}$$

for some $q < p^*$ (with p^* the Sobolev-conjugate of $p > 1$), and the *doubling property*

$$V(y, 2\Lambda) \,\le\, C(1 + V(y,\Lambda)).$$

(v) Given a collection of well-separated periodic inclusions, Braides and Garroni [79] treated the case of *nonconvex periodic* integrands satisfying a *polynomial standard growth condition* as well as the (strong) *doubling property*

$$W(y, 2\Lambda) \,\le\, CW(y,\Lambda).$$

outside the inclusions, but only satisfying inside the inclusions a *convex standard growth condition* (2.1) for some possibly unbounded map $V(y,\Lambda)$ that is convex in the Λ-variable.

(vi) More recently, Anza Hafsa and Mandallena [25] studied the homogenization of *quasiconvex periodic* integrands satisfying a *standard (unbounded) convex growth condition*, that is, (2.1) for $V(y,\Lambda) = \tilde V(\Lambda)$ with $\tilde V : \mathbb{R}^{m\times d} \to [0,\infty]$ an unbounded convex map such that $\tilde V(\Lambda) \ge |\Lambda|^p$, and with $p > d$. Note that in this case the domain is fixed.

When W is random, the results are sparser:

(vii) The first contribution beyond the linear case is due to Dal Maso and Modica [134], who addressed the homogenization of *convex random stationary* integrands satisfying a *polynomial standard growth condition*, generalizing Marcellini's result (ii) to the random setting.

(viii) Messaoudi and Michaille [321] later treated the homogenization of *quasiconvex stationary ergodic* integrands satisfying a *polynomial standard growth condition*, following Dal Maso and Modica's approach.

(ix) In their monograph on homogenization, Jikov, Kozlov, and Oleinik [265, Chapter 15] treated the case of *convex stationary ergodic* integrands satisfying a *polynomial non-standard growth condition* (2.3) for some $q < p^*$ (with p^* the Sobolev-conjugate of $p > 1$).

For scalar functionals, that is, when $m = 1$, results are much more precise, and we refer the reader to the monograph [100] by Carbone and De Arcangelis (which is however only concerned with the periodic setting) and to [98, 99]. The main technical tool for *scalar* unbounded functionals is truncation of test-functions (see also [265, Section 15.2]), which is in general no longer available for systems. More precisely, such truncation arguments replace the Sobolev embedding that is used for systems, thus allowing to relax the assumption $p > d$.

In this chapter we give a far-reaching generalization of (i), (ii), (iii), (iv), (vi), (vii), (viii), and (ix) for systems in the random setting, by relaxing the assumption that the domain of $V(y,\cdot)$ in (2.1) is independent of y. Our contribution also generalizes (v) by relaxing all geometric assumptions. In the scalar case $m = 1$, combining our approach with truncation arguments, we may further refine our results by relaxing the condition $p > d$: our approach then improves and extends to the stochastic setting some scalar periodic results of [98, 99, 100].

The argument splits into two main steps. First, for convex integrands, our results show that homogenization holds without any growth condition from above (cf. Theorem 2.1.2 for Neumann boundary conditions, and Corollary 2.1.4 for the more subtle case of Dirichlet boundary conditions), so that we may homogenize the integral functional associated with the bound V itself. For that purpose we proceed by truncation of the energy density (following the approach by Müller [331]), and start by proving in Proposition 2.2.8 that homogenization and truncation commute at the level of the definition of the homogenized energy density. The proof relies on the existence of correctors with stationary gradients for convex problems and exploits quantitatively their sublinearity at infinity (cf.

Lemma 2.2.4), which is a substantial difference between the periodic and the random settings and makes the latter more subtle. The other main technical achievement is the construction of recovery sequences in Proposition 2.2.10 (a gluing argument based on affine boundary data trivially fails since the domain of $V(y,\cdot)$ depends on y — this difficulty is already present in the periodic setting, and prevents us from using the standard homogenization formula with Dirichlet boundary conditions).

Second, in the case of nonconvex integrands with a two-sided convex estimate (2.1), we show in Theorem 2.1.6 that homogenization reduces to the homogenization of the convex bound V. The first obstacle in this program is the definition of the homogenized energy density itself. Indeed, in the absence of correctors (which is a consequence of nonconvexity), one usually defines the homogenized energy density through an asymptotic limit with linear boundary data on increasing cubes, but as already noticed such an approach fails in general when the domain of the integrand is not fixed. Instead, in Lemma 2.3.1, we use the (well-defined) correctors of the associated *convex* problem as boundary data for the *nonconvex* problem on these increasing cubes. Next we argue by blow-up in Proposition 2.3.3 for the Γ-$\liminf$ inequality (following the approach by Fonseca and Müller [183]), and make a systematic use of the correctors of the convex problem to control the nonconvex energy from above. Then for the Γ-$\limsup$ we argue in Proposition 2.3.6 by a relaxation method (following the approach introduced by Fonseca [182] and first used in homogenization by Anza Hafsa and Mandallena [25]), again making use of the corrector of the convex problem in the estimates.

To conclude this overview, let us go back to the initial motivation, that is, the homogenization of (2.2). On the one hand, we have reduced the homogenization for such integrands to the homogenization for the convex integrand V. On the other hand, we have proved homogenization for convex integrands without growth condition from above, and we have therefore established homogenization for (2.2) itself. We believe that a similar approach can be successfully implemented in the discrete setting considered in [9] for the derivation of nonlinear elasticity from polymer physics (with truncated steric effect).

This chapter is organized as follows. The main results are stated in Section 2.1.2. The proof of the results for convex integrands are displayed in Section 2.2, whereas Section 2.3 is dedicated to the proof for nonconvex integrands. In Section 2.4 we turn to various possible improvements of our general results under additional assumptions. In Appendix 2.A we prove several standard and less standard technical results on approximation of functions and on measurability of integral functionals, which are abundantly used in this chapter.

2.1.2 Notation and assumptions

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and let $\tau:=(\tau_y)_{y\in\mathbb{R}^d}$ be a measurable ergodic action of $(\mathbb{R}^d,+)$ on $(\Omega,\mathcal{F},\mathbb{P})$ (cf. Appendix 2.A.2), that are fixed once and for all throughout the chapter. We denote by $\mathbb{E}[\cdot]$ the expectation on Ω with respect to $\mathbb{P}$.

Consider a map $W:\mathbb{R}^d\times\mathbb{R}^{m\times d}\times\Omega\to[0,\infty]$ that is τ-stationary in the sense that, for all $\Lambda\in\mathbb{R}^{m\times d}$, all $\omega\in\Omega$, and all $y,z\in\mathbb{R}^d$,

$$W(y,\Lambda,\tau_{-z}\omega)=W(y+z,\Lambda,\omega), \tag{2.4}$$

and assume that $W(y,\cdot,\omega)$ is lower semicontinuous on $\mathbb{R}^{m\times d}$ for almost all y,ω. We also assume in the rest of this chapter that, for almost all ω, the map $y\mapsto W(y,\Lambda+u(y),\omega)$ is measurable for all $u\in\mathrm{Mes}(\mathbb{R}^d;\mathbb{R}^{m\times d})$, and that, for almost all $y\in\mathbb{R}^d$, the map $\omega\mapsto W(y,\Lambda+v(\omega),\omega)$ is measurable for all $v\in\mathrm{Mes}(\Omega;\mathbb{R}^{m\times d})$. Continuity in the second variable and joint measurability (in which case W is called a *Carathéodory* integrand) would ensure these properties, but weaker sufficient conditions are given in Appendix 2.A.1. Such integrands W will be called *τ-stationary normal random integrands.* We further make the following additional measurability assumption on W.

Hypothesis 2.1.1. *For all jointly measurable functions $f : \mathbb{R}^d \times \Omega \to \mathbb{R}$, all bounded domains $O \subset \mathbb{R}^d$, and all $\varepsilon > 0$, the function*

$$\omega \mapsto \inf_{u \in W_0^{1,1}(O;\mathbb{R}^m)} \int_O W(y/\varepsilon, f(y,\omega) + \nabla u(y), \omega) dy$$

is $\mathcal{F}$-measurable on Ω. $\Diamond$

As discussed in Appendix 2.A.3, this last hypothesis is always satisfied if W is convex in the second variable, and more generally if it is sup-quasiconvex (in the sense of Definition 2.A.9), which includes e.g. the case of a sum of a convex integrand and of a "nice" nonconvex part (cf. Hypothesis 2.1.9).

For any bounded domain $O \subset \mathbb{R}^d$, we define the following family of random integral functionals parametrized by $\varepsilon > 0$,

$$I_\varepsilon(\cdot,\cdot;O) : W^{1,1}(O;\mathbb{R}^m) \times \Omega \to [0,\infty] : \quad (u,\omega) \mapsto I_\varepsilon(u,\omega;O) := \int_O W(y/\varepsilon, \nabla u(y), \omega) dy. \tag{2.5}$$

The aim of this chapter is to prove homogenization for I_ε as $\varepsilon \downarrow 0$ under mild growth conditions on W, which we formulate in terms of Γ-convergence for the weak convergence of $W^{1,p}(O;\mathbb{R}^m)$ (for some $p > 1$). When $\Lambda \mapsto W(y,\Lambda,\omega)$ is convex for almost all y, ω, we say that W is a *τ-stationary convex normal random integrand*, and shall use the notation V and J_ε instead of W and I_ε, that is, for every bounded domain $O \subset \mathbb{R}^d$ and $\varepsilon > 0$,

$$J_\varepsilon(\cdot,\cdot;O) : W^{1,1}(O,\mathbb{R}^m) \times \Omega \to [0,\infty] : \quad (u,\omega) \mapsto J_\varepsilon(u,\omega;O) := \int_O V(y/\varepsilon, \nabla u(y), \omega) dy. \tag{2.6}$$

The notation W and I_ε will be used for nonconvex integrands only. We start our analysis with the case of convex integrands, then turn to nonconvex integrands, discuss several possible improvements of these general results under additional assumptions, and conclude with a description of the application to nonlinear elasticity.

2.1.3 Main results: convex integrands

In this section we state homogenization results for J_ε with (essentially) no growth condition from above. We start with Neumann boundary conditions, and then address the more subtle case of Dirichlet conditions.

Homogenization with Neumann boundary conditions

Our first result is as follows.

Theorem 2.1.2 (Convex integrands with Neumann boundary data)**.** *Let $V : \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \Omega \to [0,\infty]$ be a τ-stationary convex normal random integrand that satisfies the following uniform coercivity condition: there exist $C > 0$ and $p > d$ such that for almost all ω and y, we have for all Λ,*

$$\frac{1}{C}|\Lambda|^p - C \le V(y,\Lambda,\omega). \tag{2.7}$$

Assume that the convex function $M := \sup\mathrm{ess}_{y,\omega} V(y,\cdot,\omega)$ has 0 in the interior of its domain. Then, for almost all $\omega \in \Omega$ and all bounded Lipschitz domains $O \subset \mathbb{R}^d$, the integral functionals $J_\varepsilon(\cdot,\omega;O)$ Γ-converge to the integral functional $J(\cdot;O) : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ defined by

$$J(u;O) = \int_O \overline{V}(\nabla u(y)) dy,$$

for some lower semicontinuous convex function $\overline{V} : \mathbb{R}^{m \times d} \to [0,\infty]$ characterized by the following three equivalent formulas:

(i) Formula in probability: for all $\Lambda \in \mathbb{R}^{m\times d}$,

$$\overline{V}(\Lambda) = \inf_{g\in F^p_{\mathrm{pot}}(\Omega)^m} \mathbb{E}[V(0,\Lambda+g,\cdot)], \tag{2.8}$$

where the definition of the space $F^p_{\mathrm{pot}}(\Omega)$ *of mean-zero potential random variables is recalled in Section 2.2.1 below.*

(ii) Dirichlet formula with truncation: for any increasing sequence $V^k \uparrow V$ *of* τ*-stationary convex random integrands that satisfy the standard* p*-growth condition*

$$\frac{1}{C}|\Lambda|^p - C \le V^k(y,\Lambda,\omega) \le C_k(1+|\Lambda|^p) \tag{2.9}$$

for all y,ω,Λ*, and some sequence* $C_k < \infty$*, we have for almost all* ω*, all* Λ*, and all bounded Lipschitz domains* $O \subset \mathbb{R}^d$*,*

$$\overline{V}(\Lambda) = \lim_{k\uparrow\infty}\lim_{\varepsilon\downarrow 0} \inf_{\phi\in W^{1,p}_0(O/\varepsilon;\mathbb{R}^m)} \fint_{O/\varepsilon} V^k(y,\Lambda+\nabla\phi(y),\omega)dy. \tag{2.10}$$

(iii) Convexification formula: for all Λ *and all bounded Lipschitz domains* $O \subset \mathbb{R}^d$*, we have for almost all* ω*,*

$$\overline{V}(\Lambda) = \lim_{t\uparrow 1}\lim_{\varepsilon\downarrow 0} \inf_{\substack{\phi\in W^{1,p}(O/\varepsilon;\mathbb{R}^m)\\ \fint_{O/\varepsilon}\nabla\phi=0}} \fint_{O/\varepsilon} V(y,t\Lambda+\nabla\phi(y),\omega)dy. \tag{2.11}$$

As a consequence of convexity, the limit $t\uparrow 1$ *can be omitted when* $\Lambda \notin \partial\mathrm{dom}\overline{V}$*.* $\Diamond$

Comments are in order:

— The limit $t \uparrow 1$ cannot be omitted in (2.11) in general for $\Lambda \in \partial\mathrm{dom}\overline{V}$. Indeed, let V coincide with a convex map $\tilde{V} : \mathbb{R}^{m\times d} \to [0,\infty]$ with a closed domain, and which is not lower semicontinuous at the boundary of its domain. In the interior of $\mathrm{dom}\tilde{V}$, $\overline{V}$ coincides with $\tilde{V}$. However, since $\overline{V}$ is necessarily lower semicontinuous on its domain, it cannot coincide with $\tilde{V}$ on $\partial\mathrm{dom}\tilde{V}$.

— In the proof we take (2.10) as the defining formula for $\overline{V}$, following the approach by Müller in [331]. Formula (2.8) is interesting in two respects: first, it is intrinsinc (no approximation is required), and second it is an exact formula (there is no asymptotic limit involved). The equivalence of both formulas, which can be interpreted as the commutation of truncation and homogenization, is the key to the proof of the Γ-convergence result.

We may extend Theorem 2.1.2 in two directions:

— The extension of Theorem 2.1.2 to the case of domains with holes (or more generally to soft inclusions, for which the coercivity assumption (2.7) does not hold everywhere) is straighforward provided we have a suitable extension result at our disposal. When holes are well-separated, such extension results are standard (see e.g. [265, Sections 3.1]). For general situations however, this can become a subtle issue (see in particular [265, Sections 3.1 and 3.5]). In the particular case of the periodic setting, there is a very general extension result [2], which is used e.g. in [79].

— In the generality of Theorem 2.1.2, the assumption $p > d$ is crucially used in the form of the Sobolev embedding of $W^{1,p}(O;\mathbb{R}^m)$ in $\mathrm{L}^\infty(O;\mathbb{R}^m)$. In the case $1 < p \le d$, the conclusions of the theorem still hold true provided that $V(y,\Lambda,\omega) \le M(\Lambda)$ holds for some convex function $M : \mathbb{R}^{m\times d} \to \mathbb{R}$ that satisfies the growth condition $\limsup_{|\Lambda|\uparrow\infty} M(\Lambda)/|\Lambda|^q < \infty$ for $q = dp/(d-p)$ if $p < d$ or for some $q < \infty$ if $p = d$. In the scalar case $m = 1$, the use of the Sobolev embedding can be replaced by a truncation argument, as explained in Corollary 2.1.7 below (see also [100, 98]).

Dirichlet boundary conditions

We now discuss the homogenization result in the case of Dirichlet boundary conditions (the case of mixed boundary data can then be dealt with in a straightforward way, and we leave the detail to the reader). A first remark is that Dirichlet data need to be well-prepared, as the following elementary example shows.

Example 2.1.3. Consider random unit spherical inclusions centered at the points of a Poisson point process, choose the integrand V to be equal to $|\Lambda|^p$ outside the inclusions and to be equal to some fixed convex function with bounded domain $\mathcal{D} \subset \mathbb{R}^{m\times d}$ inside the inclusions. Given a (nonempty) bounded open set $O \subset \mathbb{R}^d$, for almost all ω, the realization of the inclusions corresponding to ω intersects $\partial(O/\varepsilon)$ for infinitely many $\varepsilon > 0$, and hence for $\Lambda \notin \mathcal{D}$ we have $\limsup_\varepsilon J_\varepsilon(u + \Lambda \cdot x, \omega; O) = \infty$ for all $u \in W_0^{1,p}(O;\mathbb{R}^m)$, due to the Dirichlet boundary condition. In contrast, if the intensity of the underlying Poisson process is not too big, it is easily seen that the homogenized integral functional J defined in Theorem 2.1.2 is finite-valued. This proves that, for all $\Lambda \notin \mathcal{D}$ and almost all ω, $J_\varepsilon(\cdot + \Lambda \cdot x, \omega; O)$ cannot Γ-converge to $J(\cdot; O)$ on $W_0^{1,p}(O;\mathbb{R}^m)$, due to the intersection of rigid inclusions with the boundary of the domain where the Dirichlet condition is imposed. ♢

We propose two ways to prepare Dirichlet data:
— by relaxing the boundary data so that the energy remains finite for all $\varepsilon > 0$ while ensuring that the boundary data are recovered at the limit $\varepsilon \downarrow 0$; we call “lifting” this procedure;
— by replacing the integrand V by a softer integrand on a neighborhood of the boundary where the boundary condition is imposed; we call this a “soft buffer zone”.

Corollary 2.1.4 (Convex integrands with Dirichlet boundary data)**.** *Let V, M, J_ε, and J be as in Theorem 2.1.2 for some $p > d$. Then, for almost all $\omega \in \Omega$ and all bounded Lipschitz domains $O \subset \mathbb{R}^d$, the following hold.*

(i) Lifting Dirichlet boundary data:
For all boundary data $u \in W^{1,p}(O;\mathbb{R}^m)$ such that $J(\alpha u; O) < \infty$ for some $\alpha > 1$, there exists a lifted sequence $(u_\varepsilon)_\varepsilon$ with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O;\mathbb{R}^m)$, such that we have on $W_0^{1,p}(O;\mathbb{R}^m)$,

$$\begin{aligned} J(\cdot + u; O) &= \Gamma\text{-}\lim_{t\uparrow 1} \ \Gamma\text{-}\liminf_{\varepsilon\downarrow 0} \ J_\varepsilon(\cdot + tu_\varepsilon, \omega; O) \\ &= \Gamma\text{-}\lim_{t\uparrow 1} \ \Gamma\text{-}\limsup_{\varepsilon\downarrow 0} \ J_\varepsilon(\cdot + tu_\varepsilon, \omega; O). \end{aligned}$$

In particular,

$$\begin{aligned} \inf_{v\in W_0^{1,p}(O)} J(v + u; O) &= \lim_{t\uparrow 1} \liminf_{\varepsilon\downarrow 0} \inf_{v\in W_0^{1,p}(O)} J_\varepsilon(v + tu_\varepsilon; O) \\ &= \lim_{t\uparrow 1} \limsup_{\varepsilon\downarrow 0} \inf_{v\in W_0^{1,p}(O)} J_\varepsilon(v + tu_\varepsilon; O). \end{aligned}$$

If u satisfies the additional condition $\int_O M(\nabla u(y))dy < \infty$, then we may choose $u_\varepsilon \equiv u$, and if this condition is strengthened to $\int_O M(\alpha\nabla u(y))dy < \infty$ for some $\alpha > 1$, then the limit $t \uparrow 1$ can be omitted.

(ii) Soft buffer zone for Dirichlet boundary data:
For all boundary data $u \in W^{1,p}(O;\mathbb{R}^m)$ such that $J(u; O) < \infty$, we have on $W_0^{1,p}(O;\mathbb{R}^m)$,

$$\begin{aligned} J(\cdot + u; O) &= \Gamma\text{-}\lim_{t\uparrow 1,\eta\downarrow 0} \ \Gamma\text{-}\liminf_{\varepsilon\downarrow 0} \ J_\varepsilon^\eta(\cdot + tu, \omega; O) \\ &= \Gamma\text{-}\lim_{t\uparrow 1,\eta\downarrow 0} \ \Gamma\text{-}\limsup_{\varepsilon\downarrow 0} \ J_\varepsilon^\eta(\cdot + tu, \omega; O), \end{aligned}$$

where J_ε^η is the following modification of J_ε on an η-neighborhood of ∂O,

$$J_\varepsilon^\eta(v,\omega;O) := \int_O V_\varepsilon^{O,\eta}(y,\nabla v(y),\omega)dy,$$
$$V_\varepsilon^{O,\eta}(y,\Lambda,\omega) := \begin{cases} V(y/\varepsilon,\Lambda,\omega), & \text{if } d(y,\partial O) > \eta; \\ |\Lambda|^p, & \text{if } d(y,\partial O) < \eta. \end{cases} \tag{2.12}$$

In particular,

$$\begin{aligned}\inf_{v\in W_0^{1,p}(O)} J(v+u;O) &= \lim_{t\uparrow 1,\eta\downarrow 0}\liminf_{\varepsilon\downarrow 0}\inf_{v\in W_0^{1,p}(O)} J_\varepsilon^\eta(v+tu;O)\\ &= \lim_{t\uparrow 1,\eta\downarrow 0}\limsup_{\varepsilon\downarrow 0}\inf_{v\in W_0^{1,p}(O)} J_\varepsilon^\eta(v+tu;O).\end{aligned}$$

If u satisfies the additional condition $J(\alpha u;O)<\infty$ for some $\alpha>1$, then the limit $t\uparrow 1$ can be omitted. ♢

Comments are in order:

— The results of Corollary 2.1.4 above are not completely satisfactory. Indeed, if one makes a diagonal extraction of t and η with respect to ε to obtain a Γ-convergence in ε only, then the extraction for the Γ-limsup depends on the target function $v+u$ and not only on the boundary data u as we would hope for. This dependence is however restricted to a dilation parameter only in the case of lifting. In the case of the buffer zone, the result can be (optimally) improved to $\eta_\varepsilon = \theta\varepsilon$ for any fixed $\theta > 0$, under some additional structural assumption in the form of the existence of stationary quasi-correctors (see Proposition 2.4.1). For specific examples for which this assumption holds, see Corollary 2.1.8 below.

— In the specific situation when $\mathrm{dom}\overline{V} = \mathrm{dom}M$ (this is trivially the case when the domain is fixed, that is, when $\mathrm{dom}V(y,\cdot,\omega) = \mathrm{dom}M$ for almost all y,ω), then the strong assumptions $\int_O M(\nabla u(y))dy < \infty$ and $\int_O M(\alpha\nabla u(y))dy < \infty$ (for some $\alpha > 1$) in part (i) of the statement reduce to the simpler assumptions $J(u;O) < \infty$ and $J(\alpha u;O) < \infty$ (for some $\alpha > 1$), respectively. In particular, in that situation, the lifting can always be chosen to be trivial, $u_\varepsilon := u$ for all ε.

— In [79] (see also [78, Chapter 20]), Braides and Garroni prepare the boundary data in a different way in the specific case of stiff inclusions. In particular they introduce an operator R^ε which acts on functions u as follows: on each stiff inclusion $R^\varepsilon(u)$ has value the average of u on the considered stiff inclusion, away from all inclusions $R^\varepsilon(u)$ coincides with u, and in between $R^\varepsilon(u)$ is an interpolation between u and the average of u on the inclusion. Such a construction can be used here as well, but seem to admit no natural generalization in other settings than stiff inclusions.

2.1.4 Main results: nonconvex integrands with convex growth

In the case when W is nonconvex and admits a two-sided estimate by a convex function (which may itself depend on the space variable), we show that a Γ-convergence result similar to the convex case holds. Before we precisely state this result, let us recall the useful notion of radial uniform upper semicontinuity (which is trivially satisfied by convex maps).

Definition 2.1.5. A map $Z : \mathbb{R}^{m\times d} \to [0,\infty]$ is said to be *ru-usc* (i.e. *radially uniformly upper semicontinuous*) if there is some $\alpha \geq 0$ such that the function

$$\Delta_Z^\alpha(t) = \sup_{\Lambda\in\mathrm{dom}Z} \frac{Z(t\Lambda)-Z(\Lambda)}{\alpha+Z(\Lambda)}$$

satisfies $\limsup_{t\uparrow 1}\Delta_Z^a(t)\le 0$. A τ-stationary normal random integrand W is said to be *ru-usc* if there exists a τ-stationary integrable random field $a:\mathbb{R}^d\times\Omega\to[0,\infty]$ such that the function

$$\Delta_W^a(t):=\operatorname*{sup\,ess}_{y\in\mathbb{R}^d}\operatorname*{sup\,ess}_{\omega\in\Omega}\sup_{\Lambda\in\operatorname{dom}W(y,\cdot,\omega)}\frac{W(y,t\Lambda,\omega)-W(y,\Lambda,\omega)}{a(y,\omega)+W(y,\Lambda,\omega)}$$

satisfies $\limsup_{t\uparrow 1}\Delta_W^a(t)\le 0$. ◊

The following result is a far-reaching generalization of [331, Theorem 1.5] to a wide class of *random and nonconvex* integrands; it is also a substantial extension of [25, Corollary 2.2]. (In general, we do not expect that the limit $t\uparrow 1$ can be dropped in (2.14) below.)

Theorem 2.1.6 (Nonconvex integrands with convex growth)**.** *Let $W:\mathbb{R}^d\times\mathbb{R}^{m\times d}\times\Omega\to[0,\infty]$ be a (nonconvex) ru-usc τ-stationary normal random integrand satisfying Hypothesis 2.1.1. Assume that, for almost all ω, y, and for all Λ,*

$$V(y,\Lambda,\omega)\;\le\;W(y,\Lambda,\omega)\;\le\;C(1+V(y,\Lambda,\omega)),\tag{2.13}$$

for some $C>0$ and some τ-stationary convex random integrand $V:\mathbb{R}^d\times\mathbb{R}^{m\times d}\times\Omega\to[0,\infty]$ that satisfies the assumptions of Theorem 2.1.2 for some $p>d$. Then, for almost all $\omega\in\Omega$ and all bounded Lipschitz domains $O\subset\mathbb{R}^d$, the integral functionals $I_\varepsilon(\cdot,\omega;O)$ Γ-converge to the integral functional $I(\cdot;O):W^{1,p}(O;\mathbb{R}^m)\to[0,\infty]$ defined by

$$I(u;O)=\int_O\overline{W}(\nabla u(y))dy,$$

for some ru-usc lower semicontinuous quasiconvex function $\overline{W}:\mathbb{R}^{m\times d}\to[0,\infty]$ that satisfies $\overline{V}\le\overline{W}\le C(1+\overline{V})$, where $\overline{V}$ is the homogenized integrand associated with V by Theorem 2.1.2. In addition, the results stated in Corollary 2.1.4 for J_ε also hold for I_ε.

For all $\Lambda\in\mathbb{R}^{m\times d}$, let g_Λ be the mean-zero potential field in probability minimizing $\mathbb{E}[V(0,\Lambda+\cdot)]$ (cf. (2.8)), and note that $x\mapsto g_\Lambda(\tau_x\omega)$ is a gradient field on $\mathbb{R}^d$ for almost all $\omega\in\Omega$, which we denote by $x\mapsto\nabla\varphi_\Lambda(x,\omega)$. The homogenized integrand $\overline{W}$ is characterized for all $\Lambda\in\mathbb{R}^{m\times d}$ by

$$\overline{W}(\Lambda)=\liminf_{t\uparrow 1}\lim_{\varepsilon\downarrow 0}\inf_{v\in W_0^{1,p}(O/\varepsilon;\mathbb{R}^m)}\fint_{O/\varepsilon}W(y,t\Lambda+t\nabla\varphi_\Lambda(y,\omega)+\nabla v(y),\omega)dy,\tag{2.14}$$

for any bounded Lipschitz domain $O\subset\mathbb{R}^d$ and almost every $\omega\in\Omega$. ◊

2.1.5 Some improved results

The general results above naturally call for some questions concerning possible improvements:

— What about the subcritical case $1<p\le d$?

— What is the minimal size η_ε of the soft buffer zone needed in the presence of Dirichlet boundary conditions (see Corollary 2.1.4(ii))? Under which conditions can we take $\eta_\varepsilon=\theta\varepsilon$ for some constant $\theta>0$?

These two questions are partially addressed below under various additional assumptions.

Subcritical case $1<p\le d$

The first improvement concerns the growth condition from below in Theorem 2.1.2. It is relaxed here to any $p>1$ in the convex scalar case $m=1$ under the additional assumption that V has fixed domain. The idea is to avoid the use of Sobolev embedding by using suitable truncations (in the spirit of e.g. [100, proof of Lemma 13.1.5]), which are indeed only available in the scalar case with fixed domain. We recover in particular in this way the results of [100, Section 13.4] in Sobolev spaces.

Corollary 2.1.7 (Subcritical case)**.** *Let V and M satisfy the assumptions of Theorem 2.1.2 in the scalar case $m = 1$, for some $p > 1$. Also assume that* $\mathrm{dom}V(y, \cdot, \omega) = \mathrm{dom}M$ *for almost all y, ω. Then the conclusions of Theorem 2.1.2 and Corollary 2.1.4 hold true for this $p > 1$.* $\Diamond$

Minimal soft buffer zone

The second improvement concerns the size of the buffer zone for Dirichlet boundary data, at least for affine target functions. The minimal size $\eta_\varepsilon = \theta\varepsilon$, for any constant $\theta > 0$, is achieved under the technical structural assumption that stationary quasi-correctors exist (cf. Proposition 2.4.1). Understanding the validity of this assumption in general seems to be a difficult question of functional analytic nature. It is trivially satisfied in the periodic case. It is also valid provided that truncations are available, which holds in the scalar case with fixed domain.

Corollary 2.1.8 (Minimal soft buffer zone)**.** *Let V, J_ε, J, M and W, I_ε, I be as in Theorems 2.1.2 and 2.1.6 for some $p > 1$. Also assume that one of the following holds:*

(1) $p > d$, and, for all Λ, ω, $V(\cdot, \Lambda, \omega)$ and $W(\cdot, \Lambda, \omega)$ are Q-periodic;

(2) $m = 1$, and $\mathrm{dom}V(y, \cdot, \omega) = \mathrm{dom}M$ *is open for almost all y, ω.*

Then, for all Λ, for almost all $\omega \in \Omega$,

$$\overline{V}(\Lambda) = \lim_{\varepsilon\downarrow 0} \inf_{v \in W_0^{1,p}(O)} \fint_O V_\varepsilon^{O,\theta\varepsilon}(y, \Lambda + \nabla v(y), \omega) dy,$$

where $V_\varepsilon^{O,\theta\varepsilon}$ is defined as in (2.12) *with $\eta = \theta\varepsilon$. The same result also holds for V, J_ε, J replaced by W, I_ε, I (for $p > d$).* $\Diamond$

2.1.6 Application to nonlinear elasticity

In the example from the statistical physics of polymer-chain networks (with truncated steric effect), the integrand has the specific decomposition (2.2). Moreover, the nonconvex part of the integrand satisfies the following assumption, which in particular implies that W satisfies Hypothesis 2.1.1 (see indeed Lemma 2.A.10), as well as the ru-usc property.

Hypothesis 2.1.9. *There exists a τ-stationary convex map $V : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0, \infty]$ and some $p > 1$ such that*

$$W(y, \Lambda, \omega) = V(y, \Lambda, \omega) + W^{nc}(y, \Lambda, \omega),$$

where $W^{nc} : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0, \infty]$ is a (nonconvex) τ-stationary normal random integrand satisfying the following p-th order upper bound and local Lipschitz condition, for some $C > 0$,

(i) for almost all y, ω, for all Λ,

$$W^{nc}(y, \Lambda, \omega) \le C(|\Lambda|^p + 1);$$

(ii) for almost all y, ω, for all Λ, Λ',

$$|W^{nc}(y, \Lambda, \omega) - W^{nc}(y, \Lambda', \omega)| \le C(1 + |\Lambda|^{p-1} + |\Lambda'|^{p-1})|\Lambda - \Lambda'|. \qquad \Diamond$$

Under this assumption, the conclusions of Theorem 2.1.6 hold true.

Corollary 2.1.10. *Let $W : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0, \infty]$ be a (nonconvex) τ-stationary normal random integrand satisfying Hypothesis 2.1.9. Assume that, for almost all ω, y, and for all Λ,*

$$V(y, \Lambda, \omega) \le W(y, \Lambda, \omega) \le C(1 + V(y, \Lambda, \omega)), \tag{2.15}$$

for some $C > 0$, where $V : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0, \infty]$ satisfies the assumptions of Theorem 2.1.2 for some $p > d$. Then the conclusions of Theorem 2.1.6 hold true. $\Diamond$

2.1.7 Perspectives and open questions

As already explained, in the context of the derivation of nonlinear elasticity from the statistical physics of polymer-chain networks [9, 200, 138], the discrete-to-continuum limit of interest would in a purely continuum setting take the form of the homogenization of a random integral functional $I_\varepsilon(u,\omega) = \int_O W(\frac{x}{\varepsilon}, \nabla u(x), \omega)dx$, where W is a nonconvex integrand of the form

$$W(y,\Lambda,\omega) = V(y,\Lambda,\omega) + a(y,\omega)\, g(\det \Lambda), \tag{2.16}$$

where V is as before an unbounded convex stationary ergodic integrand whose domain depends on the space variable, where a is an ergodic stationary random field, and where $g : \mathbb{R} \to [0,\infty]$ is a continuous convex function with $g(t) = \infty$ for $t \le 0$ and with $g(t) < \infty$ for $t > 0$. With such assumptions, the term $a(y,\omega)\, g(\det \Lambda)$ indeed correctly models the steric effect, since it is compatible both with the non-interpenetration of matter and with the necessity of an infinite amount of energy to compress a non-zero volume into a zero volume [118]. Even for $a \equiv 1$ and for a convex integrand V with polynomial growth, this homogenization question remains a major open problem in the field. Carefully considering the discrete polymer-chain network model, we note that the homogenization of (2.16) is the relevant continuum version of the problem only in dimension $d = 2$, while the continuum steric term $g(\det \Lambda)$ should be understood differently in higher dimensions due to the possibility of reversing some cells with finite energy in the discrete model. (In dimension $d > 2$ the relevant steric effect in the discrete model is quite subtle to describe due to its strong non-locality: sending two polymer chains into a same small volume costs much energy even though the two chains are initially far apart. This non-locality is absent in dimension $d = 2$ since cells cannot be reversed in that case.)

Using a theorem by Dacorogna and coauthors [129, 130] on some partial differential inclusion, Anza Hafsa and Mandallena [24] recently established a homogenization result for (2.16) in the periodic case in any dimension $d \ge 1$ under the additional assumption that $V(y,\Lambda)$ satisfies a polynomial standard growth condition of order $p > 1$, that the periodic functions $y \mapsto V(y,\Lambda)$ and $y \mapsto a(y)$ are continuous (uniformly in Λ), and that for some $T > 0$ the function $g : \mathbb{R} \to [0,\infty]$ is continuous and convex on $(-T,\infty)$, and satisfies $g(t) = \infty$ for $-T \le t \le 0$, $g(t) < \infty$ for $t > 0$, and $g(t) = T$ for $t < -T$. It would be interesting to generalize this result to the stochastic setting and under less stringent assumptions on the convex part V. This result is however not physically relevant since the assumptions on g are not compatible with the non-interpenetration of matter.

When trying to adapt Müller's approach [331] to treat the original problem of homogenizing (2.16), two main issues appear:

— Can any function u with bounded energy $\int_O W(y, \nabla u(y))dy < \infty$ be approached by a sequence of piecewise affine homeomorphisms such that the corresponding energies converge?

— Given two functions u, v with bounded energy on two disjoint sets, can they be glued in such a way that the energy remains controlled?

In dimension $d = 2$, a useful tool is provided by the Rado-Kneser-Choquet theorem (in the form of e.g. [171, 8]). It was used by Iwaniec, Kovalev, and Onninen [253, 252] to approach any function with bounded energy by piecewise affine homeomorphisms in the strong $W^{1,p}$ topology with $1 < p < \infty$. An important step is however missing in order to ensure the convergence of the whole energy. Similarly, devising a general procedure to glue functions with bounded energy into a function with controlled energy remains an open problem that we would like to keep in mind in the future.

Another possible direction of research concerns the adaptation of our main results of this chapter to the case when the integrand is degenerate in the sense that it does not satisfy the coercivity assumption (2.7), — a situation that has captured much attention recently. More precisely, we may consider the case when the lower bound (2.7) holds with $C = C(\omega)$ satisfying some moment condition (see e.g. [342] and references therein).

2.2 Proof of the results for convex integrands

This section is dedicated to the proofs of Theorem 2.1.2 and Corollary 2.1.4. Let V be a convex τ-stationary normal random integrand. Up to the addition of a constant, we may restrict to the following stronger version of (2.7): for almost all ω, y, we have for all Λ,

$$\frac{1}{C}|\Lambda|^p \le V(y,\Lambda,\omega), \tag{2.17}$$

for some $C>0$ and $1<p<\infty$. We assume that 0 belongs to the interior of the domain of the convex function $M := \sup\operatorname{ess}_{y,\omega} V(y,\cdot,\omega)$.

Following the strategy of [331, Theorem 1.5], we proceed by truncation of V. We let $(V^k)_k$ be an increasing sequence of τ-stationary convex normal random integrands $V^k : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to \mathbb{R}^+$ such that, for almost all ω, y, we have for all Λ,

$$\lim_{k\uparrow\infty} V^k(y,\Lambda,\omega) = V(y,\Lambda,\omega), \qquad \text{and} \qquad \frac{1}{C}|\Lambda|^p \le V^k(y,\Lambda,\omega) \le C_k(|\Lambda|^p+1), \tag{2.18}$$

for some $C>0$ and some sequence $C_k \uparrow \infty$ (see [331, Lemma 3.4] for such a construction). Let $\Omega_0 \in \mathcal{F}$ be an event of full probability on which all these assumptions (about V, V^k) are simultaneously pointwise satisfied.

We shall prove the existence of a subset $\Omega' \subset \Omega_0$, $\Omega' \in \mathcal{F}$, of full probability such that for all $\omega \in \Omega'$ and all bounded Lipschitz domains $O \subset \mathbb{R}^d$ the functionals $J_\varepsilon(\cdot,\omega;O)$ Γ-converge to the functional $J(\cdot;O)$ on $W^{1,p}(O;\mathbb{R}^m)$, where we recall the definitions

$$J_\varepsilon(u,\omega;O) := \int_O V(y/\varepsilon, \nabla u(y),\omega)dy, \qquad J(u;O) := \int_O \overline{V}(\nabla u(y))dy.$$

As usual, the proof of Γ-convergence splits into two parts: the proof of a lower bound (Γ-$\liminf$ inequality) and the explicit construction of a recovery sequence which achieves the lower bound (Γ-$\limsup$ inequality).

2.2.1 Preliminaries

We first need to briefly recall the standard stationary differential calculus in probability (first introduced by Papanicolaou and Varadhan [354, Section 2]), as well as some results on ergodic Weyl decompositions.

Stationary differential calculus in probability

Let $1 \le p < \infty$. For all $1 \le i \le d$, consider the partial action $(T_h^i)_{h\in\mathbb{R}}$ of $(\mathbb{R},+)$ on $\mathrm{L}^p(\Omega)$, defined by $(T_h^i f)(\omega) = f(\tau_{-he_i}\omega)$ for $h \in \mathbb{R}$. The actions $(T_h^i)_{h\in\mathbb{R}}$ (for $1\le i\le d$) commute with each other and are unitary and strongly continuous by Lemma 2.A.5. For all i, we may then consider the infinitesimal generator D_i of $(T_h^i)_{h\in\mathbb{R}}$, defined by

$$D_i f = \lim_{h\to 0} \frac{T_h^i f - f}{h}, \quad f \in \mathrm{L}^p(\Omega),$$

whenever the limit exists in the strong sense of $\mathrm{L}^p(\Omega)$. By classical semigroup theory, the generators D_i are closed linear operators with dense domains $\mathcal{D}_i \subset \mathrm{L}^p(\Omega)$, and the intersection $W^{1,p}(\Omega) := \bigcap_{i=1}^d \mathcal{D}_i$ is also dense in $\mathrm{L}^p(\Omega)$. Moreover, $W^{1,p}(\Omega)$ is endowed with a natural Banach space structure.

For $f \in W^{1,p}(\Omega)$, its *stationary gradient* is then defined by $Df := (D_1 f, \ldots, D_d f) \in \mathrm{L}^p(\Omega;\mathbb{R}^d)$. By unitarity of the action T, the operator D is skew-symmetric, so that the following "integration by parts formula" holds, for all $f \in W^{1,p}(\Omega)$ and $g \in W^{1,p'}(\Omega)$, $p' = p/(p-1)$,

$$\mathbb{E}[Df] = 0, \qquad \text{and} \qquad \mathbb{E}[fDg] = -\mathbb{E}[gDf].$$

Through the usual correspondence between random variables and τ-stationary random fields as recalled in Appendix 2.A.2 (for all $g \in \mathrm{L}^p(\Omega)$, writing $g(x,\omega) := g(\tau_{-x}\omega)$, we have $g \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d; \mathrm{L}^p(\Omega))$), we may define $Df(x,\omega) := Df(\tau_{-x}\omega)$ for all x. As explained in Lemma 2.A.7, for almost all ω, the function $Df(\cdot,\omega)$ coincides with the distributional derivative of $f(\cdot,\omega) \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$, and the following identity holds,

$$W^{1,p}(\Omega) = \{f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d; \mathrm{L}^p(\Omega)) : f(x+y,\omega) = f(x, \tau_{-y}\omega), \forall x, y, \omega\}. \tag{2.19}$$

This justifies that in the sequel we simply use the notation $Df = \nabla f$.

Ergodic Weyl decomposition

Ergodicity of the measurable action τ of $(\mathbb{R}^d, +)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is crucial in the sequel. Let $1 < p < \infty$. In analogy with the classical Weyl subspaces of $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$, we define the subspaces of potential and solenoidal random fields with respect to the differential calculus associated with the group action in the following way: for $p' = p/(p-1)$,

$$\begin{aligned} \mathrm{L}^p_{\mathrm{pot}}(\Omega) &= \{f \in \mathrm{L}^p(\Omega; \mathbb{R}^d) \,:\, \mathbb{E}[f \cdot (\nabla \times g)] = 0,\ \forall g \in W^{1,p'}(\Omega; \mathbb{R}^d)\}, \\ \mathrm{L}^p_{\mathrm{sol}}(\Omega) &= \{f \in \mathrm{L}^p(\Omega; \mathbb{R}^d) \,:\, \mathbb{E}[f \cdot \nabla g] = 0,\ \forall g \in W^{1,p'}(\Omega)\}. \end{aligned} \tag{2.20}$$

Reinterpreting these definitions in physical space, we easily obtain the following reformulations in terms of stationary extensions:

$$\begin{aligned} \mathrm{L}^p_{\mathrm{pot}}(\Omega) &= \{f \in \mathrm{L}^p(\Omega; \mathbb{R}^d) \,:\, \text{for almost all } \omega,\, x \mapsto f(\tau_x\omega) \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d) \text{ is potential}\}, \\ \mathrm{L}^p_{\mathrm{sol}}(\Omega) &= \{f \in \mathrm{L}^p(\Omega; \mathbb{R}^d) \,:\, \text{for almost all } \omega,\, x \mapsto f(\tau_x\omega) \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d) \text{ is solenoidal}\}, \end{aligned} \tag{2.21}$$

where a function $h \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$ is said to be potential (resp. solenoidal) if $\nabla \times h = 0$ (resp. $\nabla \cdot f = 0$) in the distributional sense. As constant functions belong to both subspaces, we further define

$$F^p_{\mathrm{pot}}(\Omega) = \{f \in \mathrm{L}^p_{\mathrm{pot}}(\Omega) \,:\, \mathbb{E}[f] = 0\}, \qquad \text{and} \qquad F^p_{\mathrm{sol}}(\Omega) = \{f \in \mathrm{L}^p_{\mathrm{sol}}(\Omega) \,:\, \mathbb{E}[f] = 0\}. \tag{2.22}$$

The spaces $\mathrm{L}^p_{\mathrm{pot}}(\Omega)$, $\mathrm{L}^p_{\mathrm{sol}}(\Omega)$, $F^p_{\mathrm{pot}}(\Omega)$, and $F^p_{\mathrm{sol}}(\Omega)$ are all closed in $\mathrm{L}^p(\Omega; \mathbb{R}^d)$, and the following orthogonality relations hold [265, Lemma 15.1],

$$\begin{aligned} (F^p_{\mathrm{pot}}(\Omega))^\perp &= \mathrm{L}^{p'}_{\mathrm{sol}}(\Omega) = F^{p'}_{\mathrm{sol}}(\Omega) \oplus \mathbb{R}^d, \\ (F^p_{\mathrm{sol}}(\Omega))^\perp &= \mathrm{L}^{p'}_{\mathrm{pot}}(\Omega) = F^{p'}_{\mathrm{pot}}(\Omega) \oplus \mathbb{R}^d, \end{aligned} \tag{2.23}$$

as well as the following density results (for the strong $\mathrm{L}^p(\Omega)$ topology),

$$\begin{aligned} F^p_{\mathrm{pot}}(\Omega) &= \mathrm{adh}_{\mathrm{L}^p(\Omega;\mathbb{R}^d)}\{\nabla g \,:\, g \in W^{1,p}(\Omega)\}, \\ F^p_{\mathrm{sol}}(\Omega) &= \mathrm{adh}_{\mathrm{L}^p(\Omega;\mathbb{R}^d)}\{\nabla \times g \,:\, g \in W^{1,p}(\Omega)\}. \end{aligned} \tag{2.24}$$

2.2.2 Γ-convergence of truncated energies

Since the approximations V^k of V all satisfy standard polynomial growth conditions, we can appeal to the classical stochastic homogenization result of [134] (which could by the way be reproved as a direct adaptation of the (periodic) arguments of [331, Theorem 1.3]). More precisely, there exists a subset $\Omega_1 \subset \Omega_0$, $\Omega_1 \in \mathcal{F}$, of full probability such that, for all $\omega \in \Omega_1$, all k, and all $\Lambda \in \mathbb{R}^{m\times d}$, the following limit exists (as a consequence of the Ackoglu-Krengel subadditive ergodic theorem) and defines the homogenized integrand $\overline{V}^k$,

$$\overline{V}^k(\Lambda) = \lim_{R\uparrow\infty} \inf_{\phi \in W^{1,p}_0(Q_R;\mathbb{R}^m)} \fint_{Q_R} V^k(y, \Lambda + \nabla\phi(y), \omega)dy, \tag{2.25}$$

where $Q_R := [-\frac{R}{2}, \frac{R}{2})^d$. By dominated convergence, this convergence also holds when taking the expectation of the infimum. In addition, for any bounded Lipschitz domain $O \subset \mathbb{R}^d$, and for all $\omega \in \Omega_1$ and all k, the functionals $J_\varepsilon^k(\cdot, \omega; O)$ Γ-converge, as $\varepsilon \downarrow 0$, to the functional $J^k(\cdot; O)$, defined by

$$J_\varepsilon^k(u, \omega; O) := \int_O V^k(y/\varepsilon, \nabla u(y), \omega) dy, \qquad \text{and} \qquad J^k(u; O) := \int_O \overline{V}^k(\nabla u(y)) dy.$$

Since $k \mapsto V^k$ is increasing, $k \mapsto \overline{V}^k$ is increasing as well, and for all $\Lambda \in \mathbb{R}^{m\times d}$ we may define $\overline{V}(\Lambda) := \lim_{k\uparrow\infty} \overline{V}^k(\Lambda)$. In particular,

$$\overline{V}(\Lambda) = \sup_k \overline{V}^k(\Lambda) = \sup_k \lim_{R\uparrow\infty} \inf_{\phi \in W_0^{1,p}(Q_R;\mathbb{R}^m)} \fint_{Q_R} V^k(y, \Lambda + \nabla\phi(y), \omega) dy. \tag{2.26}$$

It remains to pass to the limit $k \uparrow \infty$ in the Γ-convergence result. The key is to prove the commutation of homogenization and truncation, which we do in Subsection 2.2.4 below.

Alternative formulas for $\overline{V}^k$ are obtained in Lemma 2.2.7. Since $\overline{V}^k$ is convex and everywhere finite, it is continuous on $\mathbb{R}^{m\times d}$, and the following is a direct consequence of the definition $\overline{V}(\Lambda) := \sup_k \overline{V}^k(\Lambda)$,

Lemma 2.2.1. *The map $\overline{V} : \mathbb{R}^{m\times d} \to [0, \infty]$ is convex and lower semicontinuous.* $\Diamond$

2.2.3 Γ-$\liminf$ inequality

In view of the definition of $\overline{V}$, the Γ-$\liminf$ inequality is an elementary consequence of the monotone convergence theorem.

Proposition 2.2.2 (Γ-$\liminf$ inequality)**.** *For all $\omega \in \Omega_1$, all bounded domains $O \subset \mathbb{R}^d$, and all sequences $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O; \mathbb{R}^m)$, we have*

$$\liminf_{\varepsilon\downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O) \geq J(u; O). \qquad \Diamond$$

Proof. Let O, $(u_\varepsilon)_\varepsilon$, and u be as in the statement. Then, for all $\omega \in \Omega_1$ and all $k \in \mathbb{N}$, using the Γ-$\liminf$ result for $J_\varepsilon^k(\cdot, \omega; O)$ towards $J^k(\cdot; O)$ (see Section 2.2.2 above), and recalling that $V \geq V^k$,

$$\liminf_{\varepsilon\downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O) \geq \liminf_{\varepsilon\downarrow 0} J_\varepsilon^k(u_\varepsilon, \omega; O) \geq J^k(u; O),$$

so that, by monotone convergence,

$$\liminf_{\varepsilon\downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O) \geq \lim_{k\uparrow\infty} J^k(u; O) = J(u; O). \qquad \square$$

From this Γ-$\liminf$ result, we deduce the locality of recovery sequences, if they exist.

Corollary 2.2.3 (Locality of recovery sequences)**.** *If for some $\omega \in \Omega_1$, some bounded domain $O \subset \mathbb{R}^d$, and some function $u \in W^{1,p}(O; \mathbb{R}^m)$, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(u_\varepsilon, \omega; O) \to J(u; O)$, then we also have $J_\varepsilon(u_\varepsilon, \omega; O') \to J(u; O')$ for any subdomain $O' \subset O$. Hence, by an extension result, the* Γ-$\limsup$ *inequality on a bounded Lipschitz domain O implies the* Γ-$\limsup$ *inequality on any subdomain $O' \subset O$.* $\Diamond$

Proof. Choose a subdomain $O' \subset O$, and define $O'' := \mathrm{int}(O \setminus O')$. We then have by assumption

$$\begin{aligned} J(u; O) = \lim_{\varepsilon\downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O) &= \lim_{\varepsilon\downarrow 0} \big(J_\varepsilon(u_\varepsilon, \omega; O') + J_\varepsilon(u_\varepsilon, \omega; O'')\big) \\ &\geq \liminf_{\varepsilon\downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O') + \liminf_{\varepsilon\downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O''). \end{aligned}$$

Now by Proposition 2.2.2 we have $\liminf_{\varepsilon\downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O') \geq J(u; O')$ and $\liminf_{\varepsilon\downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O'') \geq J(u; O'')$. The conclusion then follows from the identity $J(u; O') + J(u; O'') = J(u; O)$. $\square$

2.2.4 Commutation of truncation and homogenization

The crucial ingredient to prove the commutation of truncation and homogenization is the reformulation of the asymptotic homogenization formula in the probability space. For that purpose, we first introduce the following proxy for $\overline{V}$,

$$\overline{P}(\Lambda) := \inf_{f\in F^p_{\mathrm{pot}}(\Omega)^m} \mathbb{E}[V(0,\Lambda+f,\cdot)]. \tag{2.27}$$

Likewise, for all $k\in\mathbb{N}$, we set

$$\overline{P}^k(\Lambda) := \inf_{f\in F^p_{\mathrm{pot}}(\Omega)^m} \mathbb{E}[V^k(0,\Lambda+f,\cdot)]. \tag{2.28}$$

In this case, due to the growth condition (2.18), we may check (see Lemma 2.2.7 below) that

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\inf_{\phi\in W^{1,p}_0(O/\varepsilon;\mathbb{R}^m)} \fint_{O/\varepsilon} V^k(y,\Lambda+\nabla\phi(y),\cdot)dy\right] = \overline{P}^k(\Lambda).$$

However, for V itself, this equality has no chance to be true if $\Lambda\in\mathrm{dom}\overline{P}\setminus\mathrm{dom}M$ since the left-hand side could be infinite (because of the Dirichlet boundary condition) while the right-hand side is not (cf. Example 2.1.3). We thus rather use a “relaxed version” of the Dirichlet boundary conditions and set for all $\varepsilon>0$,

$$P^\varepsilon(\Lambda,\omega;O) := \inf_{\substack{\phi\in W^{1,p}(O/\varepsilon;\mathbb{R}^m)\\ \fint_{O/\varepsilon}\nabla\phi=0}} \fint_{O/\varepsilon} V(y,\Lambda+\nabla\phi(y),\omega)dy. \tag{2.29}$$

As opposed to the case of Dirichlet boundary conditions, there is no natural subadditive property in this definition (two test-functions on disjoint domains cannot be glued together). This difficulty will be overcome by using a more sophisticated gluing argument that relies quantitatively on the following sublinearity property of the correctors.

Lemma 2.2.4 (Sublinearity of correctors). *For all $\Lambda\in\mathbb{R}^{m\times d}$, there exists a corrector field $\varphi_\Lambda\in\mathrm{Mes}(\Omega;W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)^m)$ such that $\nabla\varphi_\Lambda(0,\cdot)\in F^p_{\mathrm{pot}}(\Omega)^m$, and*

$$\overline{P}(\Lambda) = \mathbb{E}\left[V(0,\Lambda+\nabla\varphi_\Lambda(0,\cdot),\cdot)\right].$$

In addition, φ_Λ is sublinear at infinity in the sense that for almost all $\omega\in\Omega$,

$$\varepsilon\varphi_\Lambda(\cdot/\varepsilon,\omega)\rightharpoonup 0, \tag{2.30}$$

weakly in $W^{1,p}_{\mathrm{loc}}(O;\mathbb{R}^m)$. ◊

Remark 2.2.5. Although the space $\{\nabla g : g\in W^{1,p}(\Omega)\}$ is dense in $F^p_{\mathrm{pot}}(\Omega)$ (cf. (2.24)), the infimum (2.27) defining $\overline{P}(\Lambda)$ cannot be replaced in general by an infimum over this smaller dense subspace because of a possible lack of strong continuity of the functional; see however Proposition 2.4.1 and Lemma 2.4.2. ◊

Proof. Let $\Lambda\in\mathbb{R}^{m\times d}$ be fixed. By convexity and by the lower bound (2.17) on V, the map $\chi\mapsto\mathbb{E}[V(0,\Lambda+\chi,\cdot)]$ is lower semicontinuous and coercive on $F^p_{\mathrm{pot}}(\Omega)^m$, and therefore attains its infimum. Let $g\in F^p_{\mathrm{pot}}(\Omega)^m$ be a minimizer. The τ-stationary extension $(x,\omega)\mapsto g(\tau_{-x}\omega)$ of g is a potential field on $\mathbb{R}^d$ for almost every ω. Hence, there exists a map $\varphi_\Lambda\in\mathrm{Mes}(\Omega;W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m))$ such that $g(\tau_{-x}\omega)=\nabla\varphi_\Lambda(x,\omega)$ for almost all x,ω (see indeed Proposition 2.A.8 for measurability issues). The claim now follows from the combination of the following two applications of the Birkhoff-Khinchin ergodic theorem: for almost all ω,

$$\nabla\varphi_\Lambda(\cdot/\varepsilon,\omega) \rightharpoonup 0, \quad \text{(weakly) in } \mathrm{L}^p(O;\mathbb{R}^m), \tag{2.31}$$

$$\varepsilon\int_O \varphi_\Lambda(y/\varepsilon,\omega)dy \to 0. \tag{2.32}$$

Indeed, by Poincaré's inequality and (2.31), the sequence $y \mapsto \varepsilon\varphi_\Lambda(y/\varepsilon,\omega) - \varepsilon\int_O \varphi_\Lambda(z/\varepsilon,\omega)dz$ converges weakly to zero in $W^{1,p}(O;\mathbb{R}^m)$ for almost every ω. Combined with (2.32), this implies (2.30).

To conclude, we turn to the proofs of (2.31) and (2.32). The weak convergence (2.31) is a direct consequence of the Birkhoff-Khinchin ergodic theorem in the form $\nabla\varphi_\Lambda(\cdot/\varepsilon,\omega) \rightharpoonup \mathbb{E}[\nabla\varphi_\Lambda(0,\cdot)] = 0$ in $\mathrm{L}^p(O;\mathbb{R}^m)$. It remains to prove (2.32). Without loss of generality we may assume $\varphi_\Lambda(0,\cdot) = 0$ almost surely, so that

$$\begin{aligned}\left|\varepsilon \fint_O \varphi_\Lambda(y/\varepsilon,\omega)dy\right| = \left|\varepsilon\fint_{O/\varepsilon}\varphi_\Lambda(\cdot,\omega)\right| &= \left|\varepsilon\fint_{O/\varepsilon}\int_0^1 x\cdot\nabla\varphi_\Lambda(tx,\omega)dtdx\right| \\ &\leq \fint_0^{1/\varepsilon}\left|\fint_O x\cdot\nabla\varphi_\Lambda(tx,\omega)dx\right|dt.\end{aligned} \tag{2.33}$$

For almost all ω, the function $\psi_\omega(t) := \fint_O x\cdot\nabla\varphi_\Lambda(tx,\omega)dx$ is continuous on $(0,\infty)$. By (2.31), $\psi_\omega(t) \to 0$ as $t\uparrow\infty$ for almost all ω. By joint measurability and (local) integrability of $\nabla\varphi_\Lambda$, and by stationarity, 0 is a Lebesgue point of $\nabla\varphi_\Lambda(\cdot,\omega)$ for almost all ω, and hence $\limsup_{t\downarrow 0}|\psi_\omega(t)| < \infty$ for almost all ω. The result (2.32) then follows from (2.33). $\square$

For all $\Lambda \in \mathrm{dom}\overline{P}$, let φ_Λ be defined as in Lemma 2.2.4 above, and let $\Omega_\Lambda \subset \Omega_1$, $\Omega_\Lambda \in \mathcal{F}$, be a subset of full probability such that (2.30) holds on Ω_Λ for all bounded Lipschitz domains. Restricting Ω_Λ further, the Birkhoff-Khinchin ergodic ensures that for all $\omega\in\Omega_\Lambda$ we have for all bounded subsets $O\subset\mathbb{R}^d$ and all $t\in\mathbb{Q}$,

$$\fint_{O/\varepsilon}\nabla\varphi_{t\Lambda}(\cdot,\omega) \xrightarrow{\varepsilon\downarrow 0} 0 \tag{2.34}$$

and

$$\fint_{O/\varepsilon} V(y, t\Lambda + \nabla\varphi_{t\Lambda}(y,\omega),\omega)dy \xrightarrow{\varepsilon\downarrow 0} \mathbb{E}[V(0,t\Lambda+\nabla\varphi_{t\Lambda}(0,\cdot))] = \overline{P}(t\Lambda). \tag{2.35}$$

We now turn to the proof that $\lim_\varepsilon P^\varepsilon(\Lambda,\omega;O) = \overline{P}(\Lambda)$ for all Λ for almost all $\omega\in\Omega$. The following inequality is the most subtle part.

Lemma 2.2.6. *For all $\Lambda \in \mathrm{int}\,\mathrm{dom}\overline{P}$ and all bounded domains $O\subset\mathbb{R}^d$, there exists a sequence $\psi_{\Lambda,O,\varepsilon}\in \mathrm{Mes}(\Omega;W^{1,p}(O/\varepsilon;\mathbb{R}^m))$ such that for all $\omega\in\Omega_\Lambda$ we have $\fint_{O/\varepsilon}\nabla\psi_{\Lambda,O,\varepsilon}(\cdot,\omega) = 0$ and*

$$\varepsilon\psi_{\Lambda,O,\varepsilon}(\cdot/\varepsilon,\omega) \rightharpoonup 0$$

weakly in $W^{1,p}(O;\mathbb{R}^m)$ as $\varepsilon\downarrow 0$, and

$$\overline{P}(\Lambda) \geq \limsup_{\varepsilon\downarrow 0}\fint_{O/\varepsilon} V(y,\Lambda+\nabla\psi_{\Lambda,O,\varepsilon}(y,\omega),\omega)dy \geq \limsup_{\varepsilon\downarrow 0} P^\varepsilon(\Lambda,\omega;O). \tag{2.36}$$

$\lozenge$

Proof. Let $\Lambda \in \operatorname{int}\operatorname{dom}\overline{P}$ be fixed, and let $\omega \in \Omega_\Lambda$. For all $t \in [0,1) \cap \mathbb{Q}$ and $\varepsilon > 0$, set

$$\Lambda^\omega_{O,\varepsilon,t} := -t \fint_{O/\varepsilon} \nabla\varphi_{\Lambda/t}(\cdot,\omega), \quad \text{and} \quad \psi_{\Lambda,O,\varepsilon,t}(x,\omega) := t\varphi_{\Lambda/t}(x,\omega) + \Lambda^\omega_{O,\varepsilon,t}\cdot x.$$

By definition, we have $\fint_{O/\varepsilon} \nabla\psi_{\Lambda,O,\varepsilon,t} = 0$, and by Lemma 2.2.4 we also have $\varepsilon\psi_{\Lambda,O,\varepsilon,t}(\cdot/\varepsilon,\omega) \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$ as $\varepsilon \downarrow 0$. Hence,

$$P^\varepsilon(\Lambda,\omega;O) \leq \fint_{O/\varepsilon} V(y, \Lambda + \nabla\psi_{\Lambda,O,\varepsilon,t}(y,\omega),\omega) =: \widehat{P}^\varepsilon_t(\Lambda,\omega;O).$$

By convexity

$$\begin{aligned}\widehat{P}^\varepsilon_t(\Lambda,\omega;O) &= \fint_{O/\varepsilon} V(y,\Lambda + t\nabla\varphi_{\Lambda/t}(y,\omega) + \Lambda^\omega_{O,\varepsilon,t},\omega)dy \\ &\leq t\fint_{O/\varepsilon} V(y,\Lambda/t + \nabla\varphi_{\Lambda/t}(y,\omega),\omega)dy + (1-t)\fint_{O/\varepsilon} V\left(y, \frac{1}{1-t}\Lambda^\omega_{O,\varepsilon,t},\omega\right)dy.\end{aligned}$$

Since $0 \in \operatorname{int}\operatorname{dom}M$, there exists $\delta > 0$ such that $\operatorname{adh}B_\delta \subset \operatorname{int}\operatorname{dom}M$. As t is rational and $\omega \in \Omega_\Lambda$, we have $\Lambda^\omega_{O,\varepsilon,t} \to 0$ as $\varepsilon \downarrow 0$ by the Birkhoff-Khinchin ergodic theorem in the form of (2.34). Hence there exists $\varepsilon^\omega_{\Lambda,O,t} > 0$ such that for all $0 < \varepsilon < \varepsilon^\omega_{\Lambda,O,t}$ we have

$$\left|\frac{1}{1-t}\Lambda^\omega_{O,\varepsilon,t}\right| < \delta,$$

and therefore,

$$\fint_{O/\varepsilon} V\left(y,\frac{1}{1-t}\Lambda^\omega_{O,\varepsilon,t},\omega\right)dy \leq \sup_{|\Lambda'|<\delta} M(\Lambda') < \infty.$$

This implies that

$$\limsup_{t\uparrow 1, t\in\mathbb{Q}}\limsup_{\varepsilon\downarrow 0} \widehat{P}^\varepsilon_t(\Lambda,\omega;O) \leq \limsup_{t\uparrow 1, t\in\mathbb{Q}}\limsup_{\varepsilon\downarrow 0} \fint_{O/\varepsilon} V(y,\Lambda/t + \nabla\varphi_{\Lambda/t}(y,\omega),\omega)dy.$$

By the Birkhoff-Khinchin ergodic theorem in the form (2.35) and by the continuity of $\overline{P}$ in the interior of its domain (as a consequence of convexity), this yields

$$\limsup_{t\uparrow 1, t\in\mathbb{Q}}\limsup_{\varepsilon\downarrow 0} \widehat{P}^\varepsilon_t(\Lambda,\omega;O) \leq \limsup_{t\uparrow 1, t\in\mathbb{Q}} \mathbb{E}[V(0,\Lambda/t + \nabla\varphi_{\Lambda/t}(0,\cdot),\cdot)] = \limsup_{t\uparrow 1, t\in\mathbb{Q}} \overline{P}(\Lambda/t) = \overline{P}(\Lambda).$$

We have thus proved,

$$\limsup_{t\uparrow 1, t\in\mathbb{Q}}\limsup_{\varepsilon\downarrow 0}\left(\left(\widehat{P}^\varepsilon_t(\Lambda,\omega;O) - \overline{P}(\Lambda)\right)^+ + \|\varepsilon\psi_{\Lambda,O,\varepsilon,t}(\cdot/\varepsilon,\omega)\|_{\mathrm{L}^p(O;\mathbb{R}^m)}\right) = 0.$$

By Attouch's diagonalization lemma [37, Corollary 1.16], this implies the existence of a sequence $(\psi_{\Lambda,O,\varepsilon})_\varepsilon$ with $\psi_{\Lambda,O,\varepsilon} \in W^{1,p}(O/\varepsilon;\mathbb{R}^m)$ such that $\int_{O/\varepsilon}\nabla\psi_{\Lambda,O,\varepsilon} = 0$, $\limsup_\varepsilon \widehat{P}^\varepsilon_t(\Lambda,\omega;O) \leq \overline{P}(\Lambda)$, and $\varepsilon\psi_{\Lambda,O,\varepsilon}(\cdot/\varepsilon,\omega) \to 0$ in $\mathrm{L}^p(O;\mathbb{R}^m)$, for all $\omega \in \Omega_\Lambda$. By the choice of Λ, $\overline{P}(\Lambda) < \infty$, so that the lower bound (2.17) on V implies that the sequence $(\nabla\psi_{\Lambda,O,\varepsilon}(\cdot/\varepsilon,\omega))_\varepsilon$ is bounded in $\mathrm{L}^p(O;\mathbb{R}^m)$. We thus conclude that $\varepsilon\psi_{\Lambda,O,\varepsilon}(\cdot/\varepsilon,\omega) \rightharpoonup 0$ weakly in $W^{1,p}(O;\mathbb{R}^m)$, as claimed. □

In the case of standard growth conditions (thus e.g. for the V^k's), the corresponding inequality (2.36) in Lemma 2.2.6 is indeed an equality. The following lemma gives equivalent definitions for the $\overline{V}^k$'s, which will be crucial in the sequel.

Lemma 2.2.7. *Let O be a bounded Lipschitz domain of $\mathbb{R}^d$. For all $\omega \in \Omega_1$, all k, and all $\Lambda \in \mathbb{R}^{m\times d}$, the following quantities are well-defined,*

$$\overline{V}_1^k(\Lambda) := \lim_{\varepsilon\downarrow 0} \inf_{\phi\in W_0^{1,p}(O/\varepsilon;\mathbb{R}^m)} \fint_{O/\varepsilon} V^k(y,\Lambda+\nabla\phi(y),\omega)dy, \tag{2.37}$$

$$\overline{V}_2^k(\Lambda,\omega) := \lim_{\varepsilon\downarrow 0} \inf_{\substack{\phi\in W^{1,p}(O/\varepsilon;\mathbb{R}^m)\\ \fint_{O/\varepsilon}\nabla\phi=0}} \fint_{O/\varepsilon} V^k(y,\Lambda+\nabla\phi(y),\omega)dy, \tag{2.38}$$

$$\overline{V}_3^k(\Lambda) := \inf_{\tilde f\in F^p_{\mathrm{pot}}(\Omega)^m} \mathbb{E}[V^k(0,\Lambda+\tilde f,\cdot)], \tag{2.39}$$

and we have

$$\overline{V}^k(\Lambda) = \overline{V}_1^k(\Lambda) = \overline{V}_2^k(\Lambda) = \overline{V}_3^k(\Lambda). \qquad \lozenge$$

This result is standard (see for instance [265, Chapter 15]), but we display its proof for completeness. Note that the formulas (2.38) and (2.39) for V will be shown to be equivalent to $\overline{V}$, whereas formula (2.37) is in general larger than $\overline{V}$.

Proof. Let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and $k \in \mathbb{N}$. By the definition of Γ-convergence for J^k on $W_0^{1,p}(O;\mathbb{R}^m)$ and the convergence of infima with Dirichlet boundary conditions, for all Λ and $\omega \in \Omega_1$ we have

$$\begin{aligned}\overline{V}^k(\Lambda) &= \frac{1}{|O|} \inf_{\phi\in W_0^{1,p}(O;\mathbb{R}^m)} \int_O \overline{V}^k(\Lambda+\nabla\phi)\\ &= \frac{1}{|O|} \lim_{\varepsilon\downarrow 0} \inf_{\phi\in W_0^{1,p}(O;\mathbb{R}^m)} \fint_O V^k(y/\varepsilon,\Lambda+\nabla\phi(y),\cdot)dy \;=\; \overline{V}_1^k(\Lambda).\end{aligned} \tag{2.40}$$

Likewise, the Γ-convergence result holds on $\{u \in W^{1,p}(O) : \int_O \nabla u = 0\}$ so that the identity

$$\overline{V}^k(\Lambda) = \overline{V}_2^k(\Lambda)$$

also follows from the convergence of infima. Since Lemma 2.2.6 (applied to V^k instead of V) yields $\overline{V}_2^k(\Lambda) \le \overline{V}_3^k(\Lambda)$, it remains to prove that $\overline{V}_3^k(\Lambda) \le \overline{V}_1^k(\Lambda)$ for all Λ.

Let $O' \subset \mathbb{R}^d$ be a bounded domain. By the coercivity and the lower semicontinuity of the integral functional J^k (which follow from the growth condition (2.18) and the convexity of V^k), there exists a minimizer $\zeta \in \mathrm{L}^\infty(\Omega; W_0^{1,p}(O';\mathbb{R}^m))$ (see Proposition 2.A.13 for measurability issues) such that for almost all ω,

$$\fint_{O'} V^k(y,\Lambda+\nabla\zeta(y,\omega),\omega)dy = \inf_{\phi\in W_0^{1,p}(O';\mathbb{R}^m)} \fint_{O'} V^k(y,\Lambda+\nabla\phi(y),\omega)dy.$$

Set

$$\xi(x,\omega) := \frac{1}{|O'|}\int_{\mathbb{R}^d} \zeta(x+z,\tau_z\omega)dz = \fint_{-x+O'} \zeta(x+z,\tau_z\omega)dz.$$

Clearly, ξ is well-defined and stationary, belongs to $W^{1,p}(\Omega;\mathbb{R}^m)$, and

$$\nabla\xi(x,\omega) = \fint_{-x+O'} \nabla\zeta(x+z,\tau_z\omega)dz.$$

Hence

$$\overline{V}_3^k(\Lambda) \le \mathbb{E}\left[V^k(0,\Lambda+\nabla\xi(0,\cdot),\cdot)\right] = \mathbb{E}\left[V^k\left(0,\Lambda+\fint_{O'} \nabla\zeta(z,\tau_z\cdot)dz,\,\cdot\right)\right],$$

and by convexity of V^k,

$$\overline{V}_3^k(\Lambda) \le \mathbb{E}\left[\fint_{O'} V^k(0, \Lambda + \nabla\zeta(z, \tau_z \cdot), \cdot) dz \right].$$

By stationarity and the Fubini theorem, we may conclude

$$\overline{V}_3^k(\Lambda) \le \fint_{O'} \mathbb{E}[V^k(z, \Lambda + \nabla\zeta(z, \cdot), \cdot)] dz = \mathbb{E}\left[\inf_{\phi \in W_0^{1,p}(O';\mathbb{R}^m)} \fint_{O'} V^k(y, \Lambda + \nabla\phi(y), \cdot) dy \right].$$

With $O' := O/\varepsilon$, the claim $\overline{V}_3^k(\Lambda) \le \overline{V}_1^k(\Lambda)$ follows by the dominated convergence theorem and the growth condition from above (2.18). $\square$

The following result proves the equivalence between formulas (i), (ii) and (iii) in Theorem 2.1.2.

Proposition 2.2.8 (Commutation of limits)**.** *For all bounded Lipschitz domains $O \subset \mathbb{R}^d$, and all $\Lambda \in \mathbb{R}^{m\times d}$, we have for almost all ω*

$$\overline{V}(\Lambda) = \overline{P}(\Lambda) = \lim_{t\uparrow 1} \lim_{\varepsilon\downarrow 0} P^\varepsilon(t\Lambda, \omega; O).$$

By convexity, for all $\Lambda \notin \partial\mathrm{dom}\overline{V}$, this takes the form $\overline{V}(\Lambda) = \overline{P}(\Lambda) = \lim_{\varepsilon\downarrow 0} P^\varepsilon(\Lambda, \omega; O)$. $\Diamond$

Remark 2.2.9. Although not stated explicitly, this result proves the commutation of truncation and homogenization. By monotone convergence (cf. the proof of $\overline{V} = \overline{P}$ below) we have for all $\varepsilon > 0$ and almost all ω,

$$\inf_{\substack{\phi \in W^{1,p}(O/\varepsilon;\mathbb{R}^m) \\ \fint_{O/\varepsilon} \nabla\phi = 0}} \fint_{O/\varepsilon} V(y, \Lambda + \nabla\phi(y), \omega) dy = \sup_k \inf_{\substack{\phi \in W_0^{1,p}(O/\varepsilon;\mathbb{R}^m) \\ \fint_{O/\varepsilon} \nabla\phi = 0}} \fint_{O/\varepsilon} V^k(y, \Lambda + \nabla\phi(y), \omega) dy$$

so that Proposition 2.2.8, combined with (2.38) in Lemma 2.2.7, yields the desired commutation result

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} \sup_k \inf_{\substack{\phi \in W^{1,p}(O/\varepsilon;\mathbb{R}^m) \\ \fint_{O/\varepsilon} \nabla\phi = 0}} \fint_{O/\varepsilon} V^k(y, \Lambda + \nabla\phi(y), \omega) dy \\ &\qquad = \overline{V}(\Lambda) = \sup_k \overline{V}^k(\Lambda) = \sup_k \lim_{\varepsilon\downarrow 0} \inf_{\substack{\phi \in W^{1,p}(O/\varepsilon;\mathbb{R}^m) \\ \fint_{O/\varepsilon} \nabla\phi = 0}} \fint_{O/\varepsilon} V^k(y, \Lambda + \nabla\phi(y), \omega)\, dy. \ \Diamond \end{aligned}$$

Proof of Proposition 2.2.8. We split the proof into two steps.

Step 1. Proof of $\overline{V} \equiv \overline{P}$.

Let $\Lambda \in \mathrm{dom}\overline{V}$. By (2.39) in Lemma 2.2.7, for all k,

$$\overline{V}^k(\Lambda) = \inf_{f \in F_{\mathrm{pot}}^p(\Omega)^m} \mathbb{E}[V^k(0, \Lambda + f)].$$

By convexity, the map $f \mapsto \mathbb{E}[V^k(0, \Lambda + f)]$ is lower semicontinuous on $F_{\mathrm{pot}}^p(\Omega)^m$, hence by coercivity the infimum is attained. Therefore, there exists $g_k \in F_{\mathrm{pot}}^p(\Omega)^m$ such that

$$\overline{V}^k(\Lambda) = \mathbb{E}[V^k(0, \Lambda + g_k)].$$

By the uniform growth condition from below (2.18), $(g_k)_k$ is bounded in $\mathrm{L}^p(\Omega; \mathbb{R}^{m\times d})$,

$$\frac{1}{C} 2^{-p+1} \mathbb{E}[|g_k|^p] - \frac{1}{C}|\Lambda|^p \le \frac{1}{C}\mathbb{E}[|\Lambda + g_k|^p] \le \mathbb{E}[V^k(0, \Lambda + g_k)] = \overline{V}^k(\Lambda) \le \overline{V}(\Lambda).$$

Let $g \in F^p_{\mathrm{pot}}(\Omega)^m$ be a cluster point of the sequence $(g_k)_k$ for the weak convergence of $\mathrm{L}^p(\Omega;\mathbb{R}^{m\times d})$. We have along the subsequence

$$\overline{V}(\Lambda) = \sup_k \overline{V}^k(\Lambda) = \lim_{k\uparrow\infty} \mathbb{E}[V^k(0,\Lambda+g_k)].$$

Since $k \mapsto V^k$ is increasing and since the map $f \mapsto \mathbb{E}[V^k(0,\Lambda+f)]$ is lower semicontinuous for the weak convergence of $\mathrm{L}^p(\Omega;\mathbb{R}^{m\times d})$, this yields for all ℓ,

$$\overline{V}(\Lambda) \geq \liminf_{k\uparrow\infty} \mathbb{E}[V^\ell(0,\Lambda+g_k)] \geq \mathbb{E}[V^\ell(0,\Lambda+g)].$$

We then conclude by monotone convergence that

$$\overline{V}(\Lambda) \geq \mathbb{E}[V(0,\Lambda+g)] \geq \inf_{f\in F^p_{\mathrm{pot}}(\Omega)^m} \mathbb{E}[V(0,\Lambda+f)] = \overline{P}(\Lambda).$$

For $\Lambda \notin \mathrm{dom}\overline{V}$, the above inequality is trivial so that $\overline{V}(\Lambda) \geq \overline{P}(\Lambda)$ holds for all $\Lambda \in \mathbb{R}^{m\times d}$. For the converse inequality, note that for all Λ,

$$\overline{P}(\Lambda) \geq \sup_k \inf_{f\in F^p_{\mathrm{pot}}(\Omega)^m} \mathbb{E}[V^k(0,\Lambda+f)] = \sup_k \overline{V}^k(\Lambda) = \overline{V}(\Lambda).$$

Hence, $\overline{V} \equiv \overline{P}$, as claimed.

Step 2. Proof of $\lim_{t\uparrow 1}\lim_{\varepsilon\downarrow 0} P^\varepsilon(t\Lambda,\omega;O) = \overline{V}(\Lambda)$.

Since for $\Lambda \in \mathrm{dom}\overline{V}$ and $t \in [0,1)$, $t\Lambda \in \mathrm{int\,dom}\overline{V}$, Lemma 2.2.6 and Step 1 yield for almost all ω,

$$\overline{V}(t\Lambda) = \overline{P}(t\Lambda) \geq \limsup_{\varepsilon\downarrow 0} P^\varepsilon(t\Lambda,\omega;O). \tag{2.41}$$

By (2.38) in Lemma 2.2.7, for all $\Lambda \in \mathbb{R}^{m\times d}$ and almost all ω,

$$\liminf_{\varepsilon\downarrow 0} P^\varepsilon(\Lambda,\omega;O) \geq \sup_k \lim_{\varepsilon\downarrow 0} \inf_{\substack{\phi\in W^{1,p}(O/\varepsilon;\mathbb{R}^m)\\ \fint_{O/\varepsilon}\nabla\phi=0}} \fint_{O/\varepsilon} V^k(y,\Lambda+\nabla\phi(y),\omega)dy = \sup_k \overline{V}^k(\Lambda) = \overline{V}(\Lambda). \tag{2.42}$$

Combined with (2.41), this yields $\lim_\varepsilon P^\varepsilon(t\Lambda,\omega;O) = \overline{V}(t\Lambda)$ for almost all ω, for all $\Lambda \in \mathrm{dom}\overline{V}$ and $t \in [0,1)$. By convexity and lower semicontinuity of $\overline{V}$, this implies for all $\Lambda \in \mathrm{dom}\overline{V}$,

$$\lim_{t\uparrow 1}\lim_{\varepsilon\downarrow 0} P^\varepsilon(t\Lambda,\omega;O) = \lim_{t\uparrow 1} \overline{V}(t\Lambda) = \overline{V}(\Lambda), \tag{2.43}$$

and (2.42) ensures that this equality also holds for $\Lambda \notin \mathrm{dom}\overline{V}$. By convexity and by (2.43), the function $\Lambda \mapsto \lim_\varepsilon P^\varepsilon(\Lambda,\omega;O)$ is continuous outside $\partial\mathrm{dom}\overline{V}$, so that the limit $t\uparrow 1$ can be omitted for $\Lambda \notin \partial\mathrm{dom}\overline{V}$. ☐

2.2.5 Γ-convergence with Neumann boundary data

In this section, we conclude the proof of Theorem 2.1.2. It only remains to prove the following Γ-lim sup inequality.

Proposition 2.2.10 (Γ-lim sup inequality with Neumann boundary data)**.** *Assume $p > d$. There exists a subset $\Omega' \subset \Omega_1$, $\Omega' \in \mathcal{F}$, of full probability with the following property: for all $\omega \in \Omega'$, all bounded Lipschitz domain $O \subset \mathbb{R}^d$, and all $u \in W^{1,p}(O;\mathbb{R}^m)$, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O;\mathbb{R}^m)$ and $J_\varepsilon(u_\varepsilon,\omega;O) \to J(u;O)$.* ◊

Proof. We split the proof into three steps. We first treat the case of affine functions, then the case of continuous piecewise affine functions, and finally the general case. The novelty of our approach is the careful gluing argument needed to pass from affine to piecewise affine functions.

Step 1. Recovery sequence for affine functions.

In this step, we consider the case when $u = \Lambda \cdot x$ is an affine function. More precisely, we prove the existence of a subset $\Omega' \subset \Omega_1$, $\Omega' \in \mathcal{F}$, of full probability with the following property: given a bounded Lipschitz domain $O \subset \mathbb{R}^d$, for all $\omega \in \Omega'$ and all $\Lambda \in \operatorname{int}\operatorname{dom}\overline{V}$, there exists a sequence $(u^\omega_{\Lambda,\varepsilon})_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ with $u^\omega_{\Lambda,\varepsilon} \rightharpoonup \Lambda \cdot x$ weakly in $W^{1,p}(O;\mathbb{R}^m)$ such that, for all Lipschitz subdomains $O' \subset O$, we have $J_\varepsilon(u^\omega_{\Lambda,\varepsilon},\omega;O') \to J(\Lambda \cdot x;O')$. By Corollary 2.2.3, it suffices to prove this for $O' = O$.

By Lemma 2.2.4 and Proposition 2.2.8, there exists a sequence $\varphi_\Lambda \in \operatorname{Mes}(\Omega;W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m))$ such that, for all $\omega \in \Omega_\Lambda$, we have $\varepsilon\varphi_\Lambda(\cdot/\varepsilon,\omega) \rightharpoonup 0$ weakly in $W^{1,p}(O;\mathbb{R}^m)$ and, by the Birkhoff-Khinchin ergodic theorem in the form (2.35),

$$\overline{V}(\Lambda) = \overline{P}(\Lambda) = \lim_{\varepsilon\downarrow 0} \fint_{O/\varepsilon} V(y,\Lambda + \nabla\varphi_\Lambda(y,\omega),\omega)dy.$$

In particular, by a change of variables, this yields

$$J(\Lambda \cdot x;O) = |O|\overline{V}(\Lambda) = \lim_{\varepsilon\downarrow 0} J_\varepsilon(\Lambda \cdot x + \varepsilon\varphi_\Lambda(\cdot/\varepsilon,\omega),\omega;O).$$

The function $u^{\Lambda,\omega}_\varepsilon(x) := \Lambda \cdot x + \varepsilon\varphi_\Lambda(x/\varepsilon,\omega)$ thus satisfies $u^{\Lambda,\omega}_\varepsilon \rightharpoonup \Lambda \cdot x$ in $W^{1,p}(O;\mathbb{R}^m)$ and $J_\varepsilon(u^{\Lambda,\omega}_\varepsilon,\omega;O) \to J(\Lambda \cdot x;O)$ as $\varepsilon \downarrow 0$, for all $\omega \in \Omega_\Lambda$.

We then define $\Omega' \subset \Omega_1$, $\Omega' \in \mathcal{F}$, as the (countable) intersection of all Ω_Λ's with $\Lambda \in \mathbb{Q}^{m\times d} \cap \operatorname{int}\operatorname{dom}\overline{V}$, which is still of full probability. Let $\Lambda \in \operatorname{int}\operatorname{dom}\overline{V}$ and $\omega \in \Omega'$ be fixed. Choose a sequence $(\Lambda_n)_n \subset \mathbb{Q}^{m\times d} \cap \operatorname{int}\operatorname{dom}\overline{V}$ such that $\Lambda_n \to \Lambda$. For all n, we have already constructed a sequence $(u^\omega_{\varepsilon,n})_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ such that $u^\omega_{\varepsilon,n} \rightharpoonup \Lambda_n \cdot x$ in $W^{1,p}(O;\mathbb{R}^m)$ and $J_\varepsilon(u^\omega_{\varepsilon,n},\omega;O) \to J(\Lambda_n \cdot x;O)$. Since by convexity $\overline{V}$ is continuous on $\operatorname{int}\operatorname{dom}\overline{V}$, we have

$$\begin{aligned}
&\limsup_{n\uparrow\infty}\limsup_{\varepsilon\downarrow 0} \big(|J_\varepsilon(u^\omega_{\varepsilon,n},\omega;O) - J(\Lambda \cdot x;O)| + \|u^\omega_{\varepsilon,n} - \Lambda \cdot x\|_{\mathrm{L}^p(O;\mathbb{R}^m)}\big)\\
&\quad= \limsup_{n\uparrow\infty} \big(|J(\Lambda_n \cdot x;O) - J(\Lambda \cdot x;O)| + \|\Lambda_n \cdot x - \Lambda \cdot x\|_{\mathrm{L}^p(O;\mathbb{R}^m)}\big)\\
&\quad\le \limsup_{n\uparrow\infty} \big(|O||\overline{V}(\Lambda_n) - \overline{V}(\Lambda)| + C_O|\Lambda_n - \Lambda|\big) = 0.
\end{aligned}$$

By Attouch's diagonalization lemma [37, Corollary 1.16], this implies the existence of a sequence $(v^\omega_\varepsilon)_\varepsilon$ such that $J_\varepsilon(v^\omega_\varepsilon,\omega;O) \to J(\Lambda \cdot x;O)$ and $v^\omega_\varepsilon \to \Lambda \cdot x$ in $\mathrm{L}^p(O;\mathbb{R}^m)$ for all $\omega \in \Omega'$. By the p-th order lower bound for V, we conclude that v^ω_ε converges weakly to $\Lambda \cdot x$ in $W^{1,p}(O;\mathbb{R}^m)$.

Step 2. Recovery sequence for continuous piecewise affine functions.

Let $\omega \in \Omega'$, $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and u be a continuous piecewise affine function on O such that $\nabla u \in \operatorname{int}\operatorname{dom}\overline{V}$ pointwise. We shall prove that there exists a sequence $(u^\omega_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ with $u^\omega_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(O;\mathbb{R}^m)$, such that $J_\varepsilon(u^\omega_\varepsilon,\omega;O) \to J(u;O)$. For that purpose, the major issue consists in gluing the recovery sequences for the different affine parts together, which requires a particularly careful treatment.

Consider the open partition $O = \biguplus_{l=1}^k O_l$ associated with u (note that the O_l's have piecewise flat boundary outside ∂O), and define $c_l + \Lambda_l \cdot x := u|_{O_l}$, with $\Lambda_l \in \operatorname{int}\operatorname{dom}\overline{V}$, for all $1 \le l \le k$. Let $\mathcal{M} := (\bigcup_{l=1}^k \partial O_l) \setminus \partial O$ be the interior boundary of the partition of O, and for all $r > 0$ set $\mathcal{M}_r := (\mathcal{M} + B_r) \cap O = \{x \in O : d(x,\mathcal{M}) < r\}$, the r-neighborhood of this interior boundary. By Proposition 2.A.15, for all $0 < \kappa \le 1$ and $r > 0$, there exists a continuous piecewise affine function $u_{\kappa,r}$ on O with the following properties:

(i) $\nabla u_{\kappa,r}=\nabla u$ pointwise on $O\setminus\mathcal{M}_r$, and

$$\limsup_{r\downarrow 0}\sup_{0<\kappa\le 1}\|u_{\kappa,r}-u\|_{\mathrm{L}^\infty(O)}=0; \tag{2.44}$$

(ii) $\nabla u_{\kappa,r}\in\operatorname{conv}(\{\Lambda_l:1\le l\le k\})\Subset\operatorname{int}\operatorname{dom}\overline{V}$ pointwise (where $\operatorname{conv}(\cdot)$ denotes the convex hull);

(iii) denoting by $O:=\biguplus_{l=1}^{n_{\kappa,r}}O^l_{\kappa,r}$ the open partition associated with $u_{\kappa,r}$, and setting $c^l_{\kappa,r}+\Lambda^l_{\kappa,r}\cdot x:=u_{\kappa,r}|_{O^l_{\kappa,r}}$ for all l, we have $|\Lambda^i_{\kappa,r}-\Lambda^j_{\kappa,r}|\le\kappa$ for all i,j with $\partial O^i_{\kappa,r}\cap\partial O^j_{\kappa,r}\neq\varnothing$.

We shall approximate u with these refined continuous piecewise affine functions $u_{\kappa,r}$ having smoother variations; in the sequel, we shall successively take the limits $\kappa\downarrow 0$ and $r\downarrow 0$.

Since $\omega\in\Omega'$ and $O\subset\mathbb{R}^d$ are fixed in the argument, we drop them from our notation. Fix $\kappa,r>0$. By Step 1, for all $1\le i\le n_{\kappa,r}$ there exists a sequence $(u^i_{\varepsilon,\kappa,r})_\varepsilon\subset W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ with $u^i_{\varepsilon,\kappa,r}\rightharpoonup c^i_{\kappa,r}+\Lambda^i_{\kappa,r}\cdot x$ in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ and such that, for all Lipschitz subdomains $O'\subset O$, we have $J_\varepsilon(u^i_{\varepsilon,\kappa,r},\omega;O')\to J(\Lambda^i_{\kappa,r}\cdot x;O')$. For all $\eta>0$ and all $1\le i\le n_{\kappa,r}$, define the sets

$$O^{i+}_{\kappa,r,\eta}:=\{x\in O:d(x,O^i_{\kappa,r})<\eta\}=O\cap(O^i_{\kappa,r}+B_\eta),$$
$$O^{i-}_{\kappa,r,\eta}:=\{x\in O^i_{\kappa,r}:d(x,\partial O^i_{\kappa,r})>\eta\}.$$

Let then $\sum_{i=1}^{n_{\kappa,r}}\chi^i_{\kappa,r,\eta}=1$ be a partition of unity on O, where for all $1\le i\le n_{\kappa,r}$ the smooth cut-off function $\chi^i_{\kappa,r,\eta}$ has values in $[0,1]$, equals 1 on $O^{i-}_{\kappa,r,\eta}$, vanishes outside $O^{i+}_{\kappa,r,\eta}$, and satisfies the bound $|\nabla\chi^i_{\kappa,r,\eta}|\le C'/\eta$ pointwise for some constant $C'>0$. We now set

$$u_{\varepsilon,\kappa,r,\eta}:=u_{\kappa,r}+\sum_{i=1}^{n_{\kappa,r}}(u^i_{\varepsilon,\kappa,r}-(c^i_{\kappa,r}+\Lambda^i_{\kappa,r}\cdot x))\,\chi^i_{\kappa,r,\eta}.$$

By the Sobolev compact embedding for $p>d$, we have $u^i_{\varepsilon,\kappa,r}\to c^i_{\kappa,r}+\Lambda^i_{\kappa,r}\cdot x$ in $\mathrm{L}^\infty(O;\mathbb{R}^m)$ as $\varepsilon\downarrow 0$, and hence $\limsup_\eta\limsup_\varepsilon\|u_{\varepsilon,\kappa,r,\eta}-u_{\kappa,r}\|_{\mathrm{L}^\infty(O)}=0$, so that (2.44) yields

$$\lim_{t\uparrow 1}\limsup_{r\downarrow 0}\limsup_{\kappa\downarrow 0}\limsup_{\eta\downarrow 0}\limsup_{\varepsilon\downarrow 0}\|tu_{\varepsilon,\kappa,r,\eta}-u\|_{\mathrm{L}^\infty(O)}=0. \tag{2.45}$$

Let us now evaluate the integral functional $J_\varepsilon(\cdot,\omega;O)$ at $tu_{\varepsilon,\kappa,r,\eta}$ for $t\in[0,1)$. Since

$$\begin{aligned}t\nabla u_{\varepsilon,\kappa,r,\eta}&=\sum_{i=1}^{n_{\kappa,r}}t\chi^i_{\kappa,r,\eta}\nabla u^i_{\varepsilon,\kappa,r}\\&\qquad+(1-t)\frac{t}{1-t}\sum_{i=1}^{n_{\kappa,r}}\Big((u^i_{\varepsilon,\kappa,r}-(c^i_{\kappa,r}+\Lambda^i_{\kappa,r}\cdot x))\nabla\chi^i_{\kappa,r,\eta}+(\nabla u_{\kappa,r}-\Lambda^i_{\kappa,r})\chi^i_{\kappa,r,\eta}\Big),\end{aligned}$$

and $(1-t)+\sum_{i=1}^{n_{\kappa,r}}t\chi^i_{\kappa,r,\eta}=1$, we have by convexity and non-negativity of V

$$\begin{aligned}J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta},\omega;O)&\le(1-t)E_{\varepsilon,\kappa,r,\eta,t}+t\sum_{i=1}^{n_{\kappa,r}}\int_{O^{i+}_{\kappa,r,\eta}}\chi^i_{\kappa,r,\eta}(y)V(y/\varepsilon,\nabla u^i_{\varepsilon,\kappa,r}(y),\omega)dy\\&\le(1-t)E_{\varepsilon,\kappa,r,\eta,t}+\sum_{i=1}^{n_{\kappa,r}}J_\varepsilon(u^i_{\varepsilon,\kappa,r},\omega;O^{i+}_{\kappa,r,\eta}),\end{aligned} \tag{2.46}$$

where the error term takes the form

$$\begin{aligned}E_{\varepsilon,\kappa,r,\eta,t}:=\int_O V\bigg(&y/\varepsilon\,,\\&\frac{t}{1-t}\sum_{i=1}^{n_{\kappa,r}}\Big((u^i_{\varepsilon,\kappa,r}(y)-(c^i_{\kappa,r}+\Lambda^i_{\kappa,r}\cdot y))\nabla\chi^i_{\kappa,r,\eta}(y)+(\nabla u_{\kappa,r}(y)-\Lambda^i_{\kappa,r})\chi^i_{\kappa,r,\eta}(y)\Big)\,,\ \omega\bigg)dy.\end{aligned}$$

For all i, set $N^i_{\kappa,r,\eta} := \{j : j \neq i,\ O^{j+}_{\kappa,r,\eta} \cap O^{i+}_{\kappa,r,\eta} \neq \varnothing\}$. We then rewrite the argument of the energy density in the error term as

$$\begin{aligned}S_{\varepsilon,\kappa,r,\eta}(y) &:= \left|\sum_{i=1}^{n_{\kappa,r}} \Big((u^i_{\varepsilon,\kappa,r}(y) - (c^i_{\kappa,r} + \Lambda^i_{\kappa,r} \cdot y))\nabla\chi^i_{\kappa,r,\eta}(y) + (\nabla u_{\kappa,r}(y) - \Lambda^i_{\kappa,r})\chi^i_{\kappa,r,\eta}(y)\Big)\right| \\ &\leq \frac{C'}{\eta}\sum_{i=1}^{n_{\kappa,r}} \|u^i_{\varepsilon,\kappa,r} - (c^i_{\kappa,r} + \Lambda^i_{\kappa,r}\cdot x)\|_{\mathrm{L}^\infty(O)} + \sup_{1\leq i\leq n_{\kappa,r}} \sup_{j\in N^i_{\kappa,r,\eta}} |\Lambda^j_{\kappa,r} - \Lambda^i_{\kappa,r}|.\end{aligned}$$

Since by definition $\limsup_{\eta\downarrow 0} \sup_{j\in N^i_{\kappa,r,\eta}} |\Lambda^j_{\kappa,r} - \Lambda^i_{\kappa,r}| \leq \kappa$ for all i, we have

$$\limsup_{\kappa\downarrow 0}\limsup_{\eta\downarrow 0}\limsup_{\varepsilon\downarrow 0} S_{\varepsilon,\kappa,r,\eta}(y) = 0$$

for all $r,\eta > 0$. By assumption, there exists $\delta > 0$ such that $\mathrm{adh}B_\delta \subset \mathrm{int}\,\mathrm{dom}M$. Hence, for all $r,t > 0$ there exists $\kappa_{r,t} > 0$ such that for all $0 < \kappa < \kappa_{r,t}$ there exists $\eta_{\kappa,r} > 0$ such that for all $0 < \eta < \eta_{\kappa,r}$ there exists $\varepsilon_{\kappa,r,\eta,t} > 0$ with the following property: for all $0 < \varepsilon < \varepsilon_{\kappa,r,\eta,t}$, we have

$$\left\|\frac{t}{1-t} S_{\varepsilon,\kappa,r,\eta}\right\|_{\mathrm{L}^\infty(O)} < \delta.$$

This yields the bound

$$E_{\varepsilon,\kappa,r,\eta,t} \leq |O| \sup_{|\Lambda'|<\delta} M(\Lambda') < \infty,$$

and proves

$$\lim_{t\uparrow 1}\limsup_{r\downarrow 0}\limsup_{\kappa\downarrow 0}\limsup_{\eta\downarrow 0}\limsup_{\varepsilon\downarrow 0}(1-t)E_{\varepsilon,\kappa,r,\eta,t} = 0,$$

so that (2.46) turns into

$$\begin{aligned}&\limsup_{t\uparrow 1}\limsup_{r\downarrow 0}\limsup_{\kappa\downarrow 0}\limsup_{\eta\downarrow 0}\limsup_{\varepsilon\downarrow 0} J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta},\omega;O) \\ &\qquad\leq \limsup_{r\downarrow 0}\limsup_{\kappa\downarrow 0}\sum_{i=1}^{n_{\kappa,r}}\limsup_{\eta\downarrow 0}\limsup_{\varepsilon\downarrow 0} J_\varepsilon(u^i_{\varepsilon,\kappa,r},\omega;O^{i+}_{\kappa,r,\eta}). \qquad (2.47)\end{aligned}$$

For all i, we have by construction

$$\lim_{\varepsilon\downarrow 0} J_\varepsilon(u^i_{\varepsilon,\kappa,r},\omega;O^{i+}_{\kappa,r,\eta}) = |O^{i+}_{\kappa,r,\eta}|\overline{V}(\Lambda^i_{\kappa,r}),$$

so that, by definition of $O^{i+}_{\kappa,r,\eta}$,

$$\lim_{\eta\downarrow 0}\lim_{\varepsilon\downarrow 0} J_\varepsilon(u^i_{\varepsilon,\kappa,r},\omega;O^{i+}_{\kappa,r,\eta}) = |O^i_{\kappa,r}|\overline{V}(\Lambda^i_{\kappa,r}).$$

Hence, summing over i, $1 \leq i \leq n_{\kappa,r}$, yields

$$\sum_{i=1}^{n_{\kappa,r}}\lim_{\eta\downarrow 0}\lim_{\varepsilon\downarrow 0} J_\varepsilon(u^i_{\varepsilon,\kappa,r},\omega;O^{i+}_{\kappa,r,\eta}) = \sum_{i=1}^{n_{\kappa,r}} |O^i_{\kappa,r}|\overline{V}(\Lambda^i_{\kappa,r}).$$

On the one hand, $\nabla u_{\kappa,r} = \nabla u$ holds on $O\backslash\mathcal{M}_r$. On the other hand, for all i,κ,r, we have $\Lambda^i_{\kappa,r} \in K := \mathrm{conv}(\{\Lambda_l : 1 \leq l \leq k\})$, which is a compact subset of $\mathrm{int}\,\mathrm{dom}\overline{V}$. Using in addition the non-negativity of the energy density, one may then turn the above equality into

$$\sum_{i=1}^{n_{\kappa,r}}\lim_{\eta\downarrow 0}\lim_{\varepsilon\downarrow 0} J_\varepsilon(u^i_{\varepsilon,\kappa,r},\omega;O^{i+}_{\kappa,r,\eta}) = J(u_{\kappa,r};O) = J(u;O\setminus\mathcal{M}_r) + J(u_{\kappa,r};\mathcal{M}_r) \leq J(u;O) + |\mathcal{M}_r|\sup_K \overline{V}.$$

Combined with (2.47), this yields

$$\limsup_{t\uparrow 1}\limsup_{r\downarrow 0}\limsup_{\kappa\downarrow 0}\limsup_{\eta\downarrow 0}\limsup_{\varepsilon\downarrow 0} J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta},\omega;O) \le J(u;O). \tag{2.48}$$

We are now in position to conclude. By coercivity of V, the sequence $\nabla(tu_{\varepsilon,\kappa,r,\eta})$ is bounded in $\mathrm{L}^p(O;\mathbb{R}^{m\times d})$. Combined with (2.45) (convergence in $\mathrm{L}^\infty(O;\mathbb{R}^m)$), this shows that any weakly converging subsequence of $(tu_{\varepsilon,\kappa,r,\eta})_{\varepsilon,\eta,\kappa,r,t}$ in $W^{1,p}(O;\mathbb{R}^m)$ converges to u. Hence the Γ-$\liminf$ inequality of Proposition 2.2.2 yields

$$\liminf_{t\uparrow 1}\liminf_{r\downarrow 0}\liminf_{\kappa\downarrow 0}\liminf_{\eta\downarrow 0}\liminf_{\varepsilon\downarrow 0} J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta},\omega;O) \ge J(u;O).$$

These last two inequalities combine to

$$\limsup_{t\uparrow 1}\limsup_{r\downarrow 0}\limsup_{\kappa\downarrow 0}\limsup_{\eta\downarrow 0}\limsup_{\varepsilon\downarrow 0}\big(|J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta},\omega;O)-J(u;O)|+\|tu_{\varepsilon,\kappa,r,\eta}-u\|_{\mathrm{L}^p(O;\mathbb{R}^m)}\big)=0,$$

and we conclude as before by Attouch's diagonalization lemma [37, Corollary 1.16].

Step 3. Recovery sequence for general functions.

We claim that, for all $\omega\in\Omega'$, all bounded Lipschitz domains $O\subset\mathbb{R}^d$, and all $u\in W^{1,p}(O;\mathbb{R}^m)$, there is a sequence $(u_\varepsilon)_\varepsilon\subset W^{1,p}(O;\mathbb{R}^m)$ with $u_\varepsilon\rightharpoonup u$ in $W^{1,p}(O;\mathbb{R}^m)$ and $J_\varepsilon(u_\varepsilon,\omega;O)\to J(u;O)$. By the locality of recovery sequences (cf. Corollary 2.2.3), we may consider that O is a ball of $\mathbb{R}^d$, to which we may then apply the approximation result of Proposition 2.A.14. By the Γ-$\liminf$ inequality of Proposition 2.2.2, we can further assume that $u\in W^{1,p}(O;\mathbb{R}^m)$ satisfies

$$J(u;O)=\int_O \overline{V}(\nabla u(y))dy<\infty,$$

so that $\nabla u\in\mathrm{dom}\overline{V}$ almost everywhere. Let u be such a function and let $\omega\in\Omega'$ be fixed.

Since O is bounded, Lipschitz, and strongly star-shaped, and since $\overline{V}$ is convex with $0\in\mathrm{int}\,\mathrm{dom}\overline{V}$, Proposition 2.A.14(ii) shows that there exists a sequence $(u_n)_n$ of continuous piecewise affine functions with $\nabla u_n\in\mathrm{int}\,\mathrm{dom}\overline{V}$ pointwise such that $u_n\to u$ (strongly) in $W^{1,p}(O;\mathbb{R}^m)$ and $J(u_n;O)\longrightarrow J(u;O)$ as $n\uparrow\infty$. By Step 2, for all n, there exists a sequence $(u_{\varepsilon,n})_\varepsilon\subset W^{1,p}(O;\mathbb{R}^m)$ such that $u_{\varepsilon,n}\rightharpoonup u_n$ in $W^{1,p}(U;\mathbb{R}^m)$ and $J_\varepsilon(u_{\varepsilon,n},\omega;O)\to J(u_n;O)$, as $\varepsilon\downarrow 0$. In particular,

$$\begin{aligned}\lim_{n\uparrow\infty}\lim_{\varepsilon\downarrow 0}&\big(|J_\varepsilon(u_{\varepsilon,n},\omega;O)-J(u;O)|+\|u_{\varepsilon,n}-u\|_{\mathrm{L}^p(O;\mathbb{R}^m)}\big)\\&=\lim_{n\uparrow\infty}\big(|J(u_n;O)-J(u;O)|+\|u_n-u\|_{\mathrm{L}^p(O;\mathbb{R}^m)}\big)=0.\end{aligned}$$

We then conclude as before by Attouch's diagonalization argument [37, Corollary 1.16]. ☐

2.2.6 Lifting Dirichlet boundary data

In this section, we establish Corollary 2.1.4(i). We split the proof into two steps. We first consider the case when $J(\alpha u;O)<\infty$ for some $\alpha>1$, and then turn to the case when in addition $\int_O M(\nabla u)<\infty$ or $\int_O M(\alpha\nabla u)<\infty$ for some $\alpha>1$.

Step 1. Case when $J(\alpha u;O)<\infty$ for some $\alpha>1$.

As $v\in u+W^{1,p}_0(O;\mathbb{R}^m)$ and $J(\alpha u;O)<\infty$, Proposition 2.A.14(ii)(a) yields the existence of a sequence $(v_k)_k\subset u+C^\infty_c(O;\mathbb{R}^m)$ with $v_k\to v$ in $W^{1,p}(O;\mathbb{R}^m)$ and $J(v_k;O)\to J(v;O)$. For all $r>0$, set $O^1_r:=\{x\in O: d(x,\partial O)>2r\}$, $O^2_r:=\{x\in O: d(x,\partial O)>r\}$, and choose smooth cut-off functions χ^1_r,χ^2_r with the following properties: the functions take values in $[0,1]$, χ^1_r equals 1 on O^1_r and 0 on $\mathbb{R}^d\setminus O^2_r$, χ^2_r equals 1 on O^2_r and 0 on $\mathbb{R}^d\setminus O$, and $|\nabla\chi^1_r|,|\nabla\chi^2_r|\le C'/r$ for some constant C'. For all $\omega \in \Omega'$, Proposition 2.2.10 provides sequences $(u_\varepsilon^\omega)_\varepsilon$ and $(v_{\varepsilon,r,k}^\omega)_\varepsilon$ in $W^{1,p}(O;\mathbb{R}^m)$ such that $u_\varepsilon^\omega \rightharpoonup u$ and $v_{\varepsilon,r,k}^\omega \rightharpoonup \chi_r^1 v_k + (1-\chi_r^1)u$ in $W^{1,p}(O;\mathbb{R}^m)$, and $J_\varepsilon(u_\varepsilon^\omega,\omega;O') \to J(u;O')$ and $J_\varepsilon(v_{\varepsilon,r,k}^\omega,\omega;O') \to J(\chi_r^1 v_k + (1-\chi_r^1)u;O')$, for any subdomain $O' \subset O$. We then set $w_{\varepsilon,r,k}^\omega := \chi_r^2 v_{\varepsilon,r,k}^\omega + (1-\chi_r^2)u_\varepsilon^\omega$. Given $t \in [0,1)$, using the decomposition

$$t\nabla w_{\varepsilon,r,k}^\omega = t\chi_r^2 \nabla v_{\varepsilon,r,k}^\omega + t(1-\chi_r^2)\nabla u_\varepsilon^\omega + (1-t)\frac{t}{1-t}\nabla\chi_r^2(v_{\varepsilon,r,k}^\omega - u_\varepsilon^\omega),$$

we obtain by convexity,

$$J_\varepsilon(tw_{\varepsilon,r,k}^\omega,\omega;O) \le (1-t)E_{\varepsilon,r,k,t}^\omega + J_\varepsilon(v_{\varepsilon,r,k}^\omega,\omega;O) + J_\varepsilon(u_\varepsilon^\omega,\omega;O\setminus O_r^2), \tag{2.49}$$

where the error term reads

$$E_{\varepsilon,r,k,t}^\omega := \int_O V\left(y/\varepsilon, \frac{t}{1-t}\nabla\chi_r^2(y)(v_{\varepsilon,r,k}^\omega(y) - u_\varepsilon^\omega(y)),\omega\right)dy.$$

For all $y \in O\setminus O_r^2$, since $\chi_r^1(y) = 0$, we have

$$|v_{\varepsilon,r,k}^\omega(y) - u_\varepsilon^\omega(y)| \le \|v_{\varepsilon,r,k}^\omega - (\chi_r^1 v_k + (1-\chi_r^1)u)\|_{\mathrm{L}^\infty(O)} + \|u_\varepsilon^\omega - u\|_{\mathrm{L}^\infty(O)}.$$

By assumption, there is some $\delta > 0$ with $\operatorname{adh} B_\delta \subset \operatorname{int}\operatorname{dom} M$. Hence, for all fixed r,k,t, there exists $\varepsilon_{r,k,t} > 0$ such that for all $0 < \varepsilon < \varepsilon_{r,k,t}$ we have

$$\left\|\frac{t}{1-t}\nabla\chi_r^2(v_{\varepsilon,r,k}^\omega - u_\varepsilon^\omega)\right\|_{\mathrm{L}^\infty(O)} < \delta,$$

and therefore

$$\limsup_{\varepsilon\downarrow 0} E_{\varepsilon,r,k,t}^\omega \le |O| \sup_{|\Lambda'|<\delta} M(\Lambda') < \infty.$$

Inequality (2.49) then turns into

$$\begin{aligned}&\limsup_{t\uparrow 1}\limsup_{k\uparrow\infty}\limsup_{r\downarrow 0}\limsup_{\varepsilon\downarrow 0} J_\varepsilon(tw_{\varepsilon,r,k}^\omega,\omega;O)\\ &\qquad\le \limsup_{k\uparrow\infty}\limsup_{r\downarrow 0} J(\chi_r^1 v_k + (1-\chi_r^1)u;O) + \limsup_{r\downarrow 0} J(u;O\setminus O_r^2).\end{aligned}$$

The second term in the right-hand side vanishes since $J(u;O) < \infty$, and it only remains to study the first term. By definition, for fixed k, we have $v_k \in u + C_c^\infty(O;\mathbb{R}^m)$, so that for all $r > 0$ small enough there holds $\chi_r^1 v_k + (1-\chi_r^1)u = v_k$ pointwise on O. This implies

$$\limsup_{k\uparrow\infty}\limsup_{r\downarrow 0} J(\chi_r^1 v_k + (1-\chi_r^1)u;O) = \limsup_{k\uparrow\infty} J(v_k;O) = J(v;O),$$

and thus

$$\limsup_{t\uparrow 1}\limsup_{k\uparrow\infty}\limsup_{r\downarrow 0}\limsup_{\varepsilon\downarrow 0} J_\varepsilon(tw_{\varepsilon,r,k}^\omega,\omega;O) \le J(v;O).$$

Combined with the Γ-lim inf inequality of Proposition 2.2.2 and a diagonalization argument, this proves the first part of the statement.

Step 2. Cases when $\int_O M(\nabla u) < \infty$ or $\int_O M(\alpha\nabla u) < \infty$ for some $\alpha > 1$.

If u is chosen in such a way that $\int_O M(\nabla u(y))dy < \infty$, then we can repeat the argument of Step 1 with $u_\varepsilon^\omega := u$, and bound the last term in the right-hand side of (2.49) by

$$\limsup_{r\downarrow 0}\limsup_{\varepsilon\downarrow 0} J_\varepsilon(u,\omega;O\setminus O_r^2) \le \limsup_{r\downarrow 0}\int_{O\setminus O_r^2} M(\nabla u(y))dy = 0.$$

We conclude with the case when u satisfies $\int_O M(\alpha\nabla u(y))dy < \infty$ for some $\alpha > 1$. Let v_k, χ_r^1, χ_r^2 be chosen as in Step 1, and let $\omega \in \Omega'$ be fixed. For any $t \in [0,1)$, Proposition 2.2.10 shows the existence of a sequence $(v_{\varepsilon,r,k,t}^\omega)_\varepsilon$ in $W^{1,p}(O;\mathbb{R}^m)$ such that $v_{\varepsilon,r,k,t}^\omega \rightharpoonup \chi_r^1 v_k + (1-\chi_r^1)u/t$ in $W^{1,p}(O;\mathbb{R}^m)$ and $J_\varepsilon(v_{\varepsilon,r,k,t}^\omega, \omega; O') \to J(\chi_r^1 v_k + (1-\chi_r^1)u/t; O')$, for any subdomain $O' \subset O$. Set $w_{\varepsilon,r,k,t}^\omega := \chi_r^2 v_{\varepsilon,r,k,t}^\omega + (1-\chi_r^2)u/t$. As before, we obtain by convexity

$$
\begin{aligned}
J_\varepsilon(tw_{\varepsilon,r,k,t}^\omega, \omega) \le J_\varepsilon(v_{\varepsilon,r,k,t}^\omega, \omega; O) + J_\varepsilon(u/t, \omega; O \setminus O_r^2) \\
+ (1-t)\int_O M\left(\frac{t}{1-t}\nabla\chi_r^2(y)(v_{\varepsilon,r,k,t}^\omega(y) - u(y)/t)\right)dy,
\end{aligned}
$$

and the conclusion then follows, using the convexity once more in the following form, for $t > 1/\alpha$,

$$
\begin{aligned}
\limsup_{r\downarrow 0}\limsup_{\varepsilon\downarrow 0} J_\varepsilon(u/t, \omega; O \setminus O_r^2) &\le \limsup_{r\downarrow 0}\int_{O\setminus O_r^2} M(\nabla u(y)/t)dy \\
&\le \frac{1}{\alpha t}\limsup_{r\downarrow 0}\int_{O\setminus O_r^2} M(\alpha\nabla u(y))dy + \limsup_{r\downarrow 0}|O\setminus O_r^2|M(0) = 0.
\end{aligned}
$$

This completes the proof. $\square$

Remark 2.2.11. As can be seen in the proof, the assumption that $J(\alpha u; O) < \infty$ can be relaxed to $J(\alpha u; O') < \infty$ for some open neighborhood $O' \subset O$ of ∂O in O. $\lozenge$

2.2.7 Soft buffer zone for Dirichlet boundary data

In this section, we establish Corollary 2.1.4(ii). We split the proof into two steps. For all $s > 0$ and $O \subset \mathbb{R}^d$, we use the notation $O_s := \{x \in O : d(x, \partial O) > s\}$.

Step 1. Γ-$\liminf$ inequality.

Let $\omega \in \Omega'$, let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, let $u \in W^{1,p}(O;\mathbb{R}^m)$ with $J(u;O) < \infty$, and let $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ be a sequence with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O;\mathbb{R}^m)$. By the Γ-$\liminf$ inequality for J_ε in Proposition 2.2.2,

$$
\liminf_{\varepsilon\downarrow 0} J_\varepsilon^\eta(u_\varepsilon, \omega; O) \ge \liminf_{\varepsilon\downarrow 0} J_\varepsilon(u_\varepsilon, \omega; O_\eta) \ge J(u; O_\eta) = \int_{O_\eta} \overline{V}(\nabla u(y))dy,
$$

that is, using that $\int_{O\setminus O_\eta} \overline{V}(\nabla u(y))dy \to 0$ as $\eta \downarrow 0$,

$$
\liminf_{\eta\downarrow 0}\liminf_{\varepsilon\downarrow 0} J_\varepsilon^\eta(u_\varepsilon, \omega; O) \ge J(u; O).
$$

Step 2. Γ-$\limsup$ inequality.

Let $\omega \in \Omega'$, let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $u \in W^{1,p}(O;\mathbb{R}^m)$ with $J(u;O) < \infty$. By Proposition 2.2.10, there exists a sequence $(w_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ such that $w_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$ and $J_\varepsilon(u + w_\varepsilon, \omega; O) \to J(u;O)$. Given $\eta > 0$, choose a cut-off function χ_η with values in $[0,1]$, such that χ_η equals 1 on O_η and 0 outside O, and that satisfies $|\nabla\chi_\eta| \le C'/\eta$ for some constant $C' > 0$. Set $v_{\varepsilon,\eta} := \chi_\eta w_\varepsilon \in W_0^{1,p}(O;\mathbb{R}^m)$. For all $t \in [0,1)$, we have

$$
t\nabla u + t\nabla v_{\varepsilon,\eta} = t\chi_\eta\nabla(u + w_\varepsilon) + t(1-\chi_\eta)\nabla u + (1-t)\frac{t}{1-t}w_\varepsilon\nabla\chi_\eta,
$$

so that by convexity and by definition of $V_\varepsilon^{O,\eta}$,

$$
\begin{aligned}
J_\varepsilon^\eta(tu + tv_{\varepsilon,\eta}, \omega; O) &\le (1-t)E_{\varepsilon,\eta,t} + J_\varepsilon^\eta(u + w_\varepsilon, \omega; O) + \int_O (1-\chi_\eta(y))V_\varepsilon^{O,\eta}(y, \nabla u(y), \omega)dy \\
&\le (1-t)E_{\varepsilon,\eta,t} + J_\varepsilon(u + w_\varepsilon, \omega; O) + \int_{O\setminus O_\eta} |\nabla u(y)|^p dy, \qquad (2.50)
\end{aligned}
$$

where the error is defined by

$$E_{\varepsilon,\eta,t} := \int_O V_\varepsilon^{O,\eta}\left(y, \frac{t}{1-t}w_\varepsilon(y)\nabla\chi_\eta(y), \omega\right) dy \;\le\; |O_\eta|M(0) + \int_{O\setminus O_\eta} \left|\frac{t}{1-t}w_\varepsilon(y)\nabla\chi_\eta(y)\right|^p dy. \tag{2.51}$$

By the Rellich-Kondrachov theorem, $w_\varepsilon \to 0$ (strongly) in $\mathrm{L}^p(O)$, so that $\limsup_\varepsilon E_{\varepsilon,\eta,t} \le |O|M(0)$ for all t, η. Passing to the limit in inequality (2.50) thus yields

$$\limsup_{t\uparrow 1}\limsup_{\eta\downarrow 0}\limsup_{\varepsilon\downarrow 0} J_\varepsilon^\eta(tu + tv_\varepsilon, \omega; O, \eta) \le \limsup_{\varepsilon\downarrow 0} J_\varepsilon(u + w_\varepsilon, \omega; O) = J(u; O).$$

We then conclude by the same diagonalization argument as before and by Step 1. This proves the first part of the statement.

Now consider the case when u satisfies $J(\alpha u; O) < \infty$ for some $\alpha > 1$. Then, for all $t \in [0, 1)$, Proposition 2.2.10 provides a sequence $(w_{\varepsilon,t})_\varepsilon \subset W^{1,p}(O; \mathbb{R}^m)$ such that $w_{\varepsilon,t} \rightharpoonup 0$ in $W^{1,p}(O; \mathbb{R}^m)$ and $J_\varepsilon(u/t + w_{\varepsilon,t}, \omega; O) \to J(u/t; O)$ as $\varepsilon \downarrow 0$. Define $v_{\varepsilon,t,\eta} := \chi_\eta w_{\varepsilon,t}$, where χ_η is the same cut-off function as above. We then have

$$\nabla u + t\nabla v_{\varepsilon,t,\eta} = t\chi_\eta \nabla(u/t + w_{\varepsilon,t}) + t(1 - \chi_\eta)\nabla u/t + (1 - t)\frac{t}{1-t} w_{\varepsilon,t}\nabla\chi_\eta,$$

so that by convexity and definition of $V_\varepsilon^{O,\eta}$,

$$J_\varepsilon^\eta(u + tv_{\varepsilon,\eta}, \omega; O) \le (1 - t)E'_{\varepsilon,\eta,t} + J_\varepsilon(u/t + w_{\varepsilon,t}, \omega; O) + t^{1-p}\int_{O\setminus O_\eta} |\nabla u(y)|^p dy, \tag{2.52}$$

where the error is defined by

$$E'_{\varepsilon,\eta,t} := |O_\eta|M(0) + \int_{O\setminus O_\eta} \left|\frac{t}{1-t} w_{\varepsilon,t}(y)\nabla\chi_\eta(y)\right|^p dy.$$

By the Rellich-Kondrachov theorem, $w_{\varepsilon,t} \to 0$ (strongly) in $\mathrm{L}^p(O)$ for all t, so that $\limsup_\varepsilon E'_{\varepsilon,\eta,t} = |O_\eta|M(0)$ for all t, η. Passing to the limit in inequality (2.52) then yields

$$\limsup_{t\uparrow 1}\limsup_{\eta\downarrow 0}\limsup_{\varepsilon\downarrow 0} J_\varepsilon^\eta(u + tv_{\varepsilon,t,\eta}, \omega; O) \le \limsup_{t\uparrow 1}\limsup_{\varepsilon\downarrow 0} J_\varepsilon(u/t + w_{\varepsilon,t}, \omega; O) = \limsup_{t\uparrow 1} J(u/t; O).$$

Since u satisfies $J(\alpha u; O) < \infty$, we deduce by convexity that the map $t \mapsto J(u/t; O)$ is continuous on $(1/\alpha, 1]$. This implies $\limsup_{t\uparrow 1} J(u/t; O) = J(u; O)$, and the conclusion follows. ☐

Remark 2.2.12. As can be seen in the proof, the assumption that $J(\alpha u; O) < \infty$ can be relaxed to $J(\alpha u; O') < \infty$ for some open neighborhood $O' \subset O$ of ∂O in O. ♢

2.3 Proof of the results for nonconvex integrands

In this section, we study the case when W is nonconvex but admits a two-sided estimate by a convex function (which may depend on the space variable), and we prove Theorem 2.1.6. Let W be a (nonconvex) τ-stationary normal random integrand, which is further assumed to be ru-usc (in the sense of Definition 2.1.5, with respect to some τ-stationary integrable random field a) and to satisfy Hypothesis 2.1.1. Up to a translation, for simplicity of notation, we can restrict to the following stronger version of (2.7) and (2.13): for almost all ω, y, we have for all Λ,

$$\frac{1}{C}|\Lambda|^p \le V(y, \Lambda, \omega) \le W(y, \Lambda, \omega) \le C(1 + V(y, \Lambda, \omega)), \tag{2.53}$$

for some $C > 0$ and $d < p < \infty$, and for some convex τ-stationary normal random integrand V. Also assume that 0 belongs to the interior of the domain of the convex function $M := \sup \mathrm{ess}_{y,\omega} V(y, \cdot, \omega)$. We can then apply Theorem 2.1.2 to V, yielding a homogenized energy density $\overline{V}$ with the following property: defining

$$J_\varepsilon(u, \omega; O) = \int_O V(y/\varepsilon, \nabla u(y), \omega) dy, \qquad J(u; O) = \int_O \overline{V}(\nabla u(y)) dy,$$

for almost all ω, the integral functionals $J_\varepsilon(\cdot, \omega; O)$ Γ-converge to $J(\cdot; O)$ on $W^{1,p}(O; \mathbb{R}^m)$, for any bounded Lipschitz domain $O \subset \mathbb{R}^d$. Let $\Omega_0 \in \mathcal{F}$ be a subset of full probability on which all these assumptions and properties (of V, W) are simultaneously pointwise satisfied.

2.3.1 Definition of the homogenized energy density

We need to define in this section a candidate for the homogenized energy density $\overline{W}$. As before, the standard homogenization formula with Dirichlet boundary conditions does not hold because of the generality of the growth conditions considered here. Instead, we use the corrector for the convex problem as a boundary condition for the nonconvex problem, which is indeed admissible because of the two-sided growth condition (2.53).

More precisely, for all $\Lambda \in \mathbb{R}^{m \times d}$, Lemma 2.2.4 yields a function $\varphi_\Lambda \in \mathrm{Mes}(\Omega; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m))$ such that $\nabla \varphi_\Lambda(0, \cdot) \in F^p_{\mathrm{pot}}(\Omega)^m$ and

$$\overline{V}(\Lambda) = \mathbb{E}[V(0, \Lambda + \nabla \varphi_\Lambda(0, \cdot), \cdot)].$$

Now, for any $t \in [0, 1)$, consider the function μ^t_Λ defined by

$$\mu^t_\Lambda(O, \omega) := \inf_{v \in W^{1,p}_0(O; \mathbb{R}^m)} \int_O W(y, t\Lambda + t\nabla \varphi_\Lambda(y, \omega) + \nabla v(y), \omega) dy.$$

As this quantity is stationary and subadditive, the Ackoglu-Krengel ergodic theorem leads to the following.

Lemma 2.3.1 (Definition of the homogenized energy density). *Let $t \in [0, 1)$ be fixed. Then there exists a function $\overline{W}_t : \mathrm{dom}\overline{V} \to [0, \infty)$ such that, for all $\Lambda \in \mathrm{dom}\overline{V}$, for almost all $\omega \in \Omega_0$, we have for all bounded Lipschitz domain $O \subset \mathbb{R}^d$,*

$$\overline{W}_t(\Lambda) = \lim_{\varepsilon \downarrow 0} \frac{\mu^t_\Lambda(O/\varepsilon, \omega)}{|O/\varepsilon|}, \tag{2.54}$$

where the convergence also holds for expectations. Now define $\overline{W}(\Lambda) := \liminf_{t \uparrow 1} \liminf_{\Lambda' \to \Lambda} \overline{W}_t(\Lambda')$ for $\Lambda \in \mathrm{dom}\overline{V}$, and set $\overline{W}(\Lambda) = \infty$ for $\Lambda \notin \mathrm{dom}\overline{V}$. Then, $\overline{W}$ satisfies $\overline{V} \le \overline{W} \le C(1 + \overline{V})$ on the whole of $\mathbb{R}^{m \times d}$, and for all $\Lambda \in \mathbb{R}^{m \times d}$, for almost all ω, we have for all bounded Lipschitz domain $O \subset \mathbb{R}^d$,

$$\overline{W}(\Lambda) = \liminf_{t \uparrow 1} \liminf_{\Lambda' \to \Lambda} \lim_{\varepsilon \downarrow 0} \frac{\mu^t_{\Lambda'}(O/\varepsilon, \omega)}{|O/\varepsilon|}, \tag{2.55}$$

where the $\liminf$ as $t \uparrow 1$ can further be restricted to $t \in \mathbb{Q}$. Finally, in the particular case when W is convex, then $\overline{W}$ coincides with the various definitions for the homogenized integrand as given by Theorem 2.1.2. $\Diamond$

Proof. We split the proof into three steps.

Step 1. Definition of $\overline{W}_t(\Lambda)$ and proof of (2.54).

First consider the case when $\Lambda \in \mathrm{dom}\overline{V}$. Let $t \in [0,1)$ be fixed. The upper bound in (2.53) then implies $\mathbb{E}[\mu_\Lambda^t(O,\cdot)] \le C|O|(1+t\overline{V}(\Lambda)+(1-t)M(0)) < \infty$. As the function μ_Λ^t is obviously stationary and subadditive, and as $\mu_\Lambda^t(O,\cdot)$ is measurable by Hypothesis 2.1.1, the Ackoglu-Krengel subadditive ergodic theorem (see e.g. [275, Section 6.2]) can be applied and asserts the existence of some $\overline{W}_t(\Lambda) \in [0,\infty)$ such that, for almost all ω, we have

$$\overline{W}_t(\Lambda) = \lim_{n\uparrow\infty} \frac{\mu_\Lambda^t(I_n,\omega)}{|I_n|},$$

for any regular sequence $(I_n)_n \subset \mathcal{I} := \{[a,b) : a,b \in \mathbb{Z}^d\}$ such that $\lim_{n\uparrow\infty} I_n = \mathbb{R}^d$ (in the usual sense of [275, Section 6.2]), and moreover this convergence also holds for expectations. In particular, we easily see that the same result must hold for the choice $I_n = nQ_0$, where Q_0 is any cube aligned with the axes. Further note that, for all bounded Lipschitz subsets $O' \subset O \subset \mathbb{R}^d$, we can estimate, as $W_0^{1,p}(O'/\varepsilon;\mathbb{R}^m) \subset W_0^{1,p}(O/\varepsilon;\mathbb{R}^m)$,

$$\begin{aligned}\varepsilon^d\mu_\Lambda^t(O/\varepsilon,\omega) &\le \varepsilon^d\mu_\Lambda^t(O'/\varepsilon,\omega) + \varepsilon^d\int_{(O\setminus O')/\varepsilon} W(y,t\Lambda+t\nabla\varphi_\Lambda(y,\omega),\omega)dy\\ &\le \varepsilon^d\mu_\Lambda^t(O'/\varepsilon,\omega) + C|O\setminus O'|\left(1+\fint_{(O\setminus O')/\varepsilon} V(y,\Lambda+\nabla\varphi_\Lambda(y,\omega),\omega)dy\right),\end{aligned}$$

where the last expression in brackets converges to $1+\overline{V}(\Lambda) < \infty$ as $\varepsilon \downarrow 0$. Now based on this estimate, an easy approximation argument (see e.g. [213, Step 4 of the proof of Theorem 3.1]) allows us to conclude as follows: for almost all ω (for all $\omega \in \Omega_\Lambda$, for some subset $\Omega_\Lambda \in \mathcal{F}$ of full probability, say), we have for all bounded Lipschitz domains $O \subset \mathbb{R}^d$,

$$\lim_{\varepsilon\downarrow 0} \frac{\mu_\Lambda^t(O/\varepsilon,\omega)}{|O/\varepsilon|} = \overline{W}_t(\Lambda). \tag{2.56}$$

Step 2. Definition of $\overline{W}$ and proof of the bounds $\overline{V}(\Lambda) \le \overline{W}(\Lambda) \le C(1+\overline{V}(\Lambda))$ for $\Lambda \in \mathrm{dom}\overline{V}$.

Let $\Lambda \in \mathrm{dom}\overline{V}$ be fixed, and let $O \subset \mathbb{R}^d$ be some bounded Lipschitz domain. As in the statement, we define $\overline{W}(\Lambda) := \liminf_t \liminf_{\Lambda'\to\Lambda} \overline{W}_t(\Lambda')$. The bounds $\overline{V}(\Lambda) \le \overline{W}(\Lambda) \le C(1+\overline{V}(\Lambda))$ directly follow from the two-sided estimate (2.53) together with the following equality, for almost all ω,

$$\overline{V}(\Lambda) = \overline{V}_0(\Lambda,\omega) := \liminf_{t\uparrow 1}\liminf_{\Lambda'\to\Lambda}\lim_{\varepsilon\downarrow 0}\inf_{v\in W_0^{1,p}(O;\mathbb{R}^m)} \fint_O V(y/\varepsilon, t\Lambda'+t\nabla\varphi_{\Lambda'}(y/\varepsilon,\omega)+\nabla v(y),\omega)dy. \tag{2.57}$$

Let us give the argument for (2.57). On the one hand, we can estimate

$$\overline{V}_0(\Lambda,\omega) \ge \liminf_{t\uparrow 1}\liminf_{\Lambda'\to\Lambda}\liminf_{\varepsilon\downarrow 0}\inf_{\substack{v\in W^{1,p}(O;\mathbb{R}^m)\\ \fint_O \nabla v = 0}} \fint_O V(y/\varepsilon, t\Lambda'+t\Lambda'_\varepsilon(\omega)+\nabla v(y),\omega)dy,$$

where we have set $\Lambda'_\varepsilon(\omega) := \fint_O \nabla\varphi_{\Lambda'}(\cdot/\varepsilon,\omega)$. For almost all ω, since $\Lambda'_\varepsilon(\omega) \to 0$, we can write for any $\kappa > 0$,

$$\overline{V}_0(\Lambda,\omega) \ge \inf_{\Lambda':|\Lambda'-\Lambda|\le\kappa}\liminf_{t\uparrow 1}\liminf_{\varepsilon\downarrow 0}\inf_{\substack{v\in W^{1,p}(O;\mathbb{R}^m)\\ \fint_O \nabla v = 0}} \fint_O V(y/\varepsilon, t\Lambda'+\nabla v(y),\omega)dy,$$

so that formula (2.11) yields $\overline{V}_0(\Lambda,\omega) \ge \inf_{\Lambda':|\Lambda'-\Lambda|\le\kappa} \overline{V}(\Lambda')$. Passing to the limit $\kappa \downarrow 0$, the lower semicontinuity of $\overline{V}$ directly gives $\overline{V}_0(\Lambda,\omega) \ge \overline{V}(\Lambda)$. On the other hand, the convexity of V, the Birkhoff-Khinchin ergodic theorem, and the definition of $\varphi_{\Lambda'}$ give for all $t \in [0,1]$ and all $\Lambda' \in \mathbb{R}^{m\times d}$,

$$\lim_{\varepsilon\downarrow 0} \inf_{v\in W_0^{1,p}(O;\mathbb{R}^m)} \fint_O V(y/\varepsilon, t\Lambda' + t\nabla\varphi_{\Lambda'}(y/\varepsilon,\omega) + \nabla v(y),\omega)dy$$
$$\le \lim_{\varepsilon\downarrow 0} \fint_O V(y/\varepsilon, \Lambda' + \nabla\varphi_{\Lambda'}(y/\varepsilon,\omega),\omega)dy + (1-t)M(0) = \overline{V}(\Lambda') + (1-t)M(0).$$

Passing to the limit $\Lambda' \to \Lambda$ and $t \uparrow 1$, and using the lower semicontinuity of $\overline{V}$ in the form of $\overline{V}(\Lambda) = \liminf_{\Lambda'\to\Lambda} \overline{V}(\Lambda')$, this yields $\overline{V}_0(\Lambda,\omega) \le \overline{V}(\Lambda)$. The desired identity (2.57) is proven.

Step 3. Case when $\Lambda \notin \mathrm{dom}\overline{V}$.

For $\Lambda \notin \mathrm{dom}\overline{V}$, arguing as in Step 2 above, we can estimate, using the pointwise bound $V \le W$,

$$\liminf_{t\uparrow 1} \liminf_{\Lambda'\to\Lambda} \liminf_{\varepsilon\downarrow 0} \inf_{v\in W_0^{1,p}(O;\mathbb{R}^m)} \fint_O W(y/\varepsilon, t\Lambda + t\nabla\varphi_\Lambda(y/\varepsilon,\omega) + \nabla v(y),\omega)dy \ge \overline{V}(\Lambda) = \infty,$$

so that (2.55) trivially holds with $\overline{W}(\Lambda) := \infty$. Moreover, the bounds $\overline{V} \le \overline{W} \le C(1+\overline{V})$ holds as well. $\square$

Although the definition of the homogenized energy density $\overline{W}(\Lambda)$ may a priori depend on the choice of a corrector φ_Λ, it would follow a posteriori from the Γ-convergence result that the value of $\overline{W}(\Lambda)$ is independent of that choice. As this independence will actually be useful in the proof of the Γ-lim sup inequality (cf. proof of Lemma 2.3.4(c) below), we display a direct proof.

Lemma 2.3.2 (Independence upon the choice of a corrector). *Assume $p > d$, and let $t \in [0,1)$, $\Lambda \in \mathrm{dom}\overline{V}$ be fixed. For almost all ω, given a bounded domain $O \subset \mathbb{R}^d$, if $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ satisfies $\|u_\varepsilon\|_{\mathrm{L}^\infty(O)} \to 0$ and $\limsup_\varepsilon \int_D V(y/\varepsilon, \Lambda + \nabla u_\varepsilon(y),\omega)dy \le C_\Lambda|D|$ for all subdomains $D \subset O$ and some constant $C_\Lambda > 0$, then we have for all Lipschitz subdomains $D \subset O$,*

$$\overline{W}_t(\Lambda) = \lim_{\varepsilon\downarrow 0} \inf_{v\in W_0^{1,p}(D;\mathbb{R}^m)} \fint_D W(y/\varepsilon, t\Lambda + t\nabla u_\varepsilon(y) + \nabla v(y),\omega)dy, \tag{2.58}$$

where in particular the limit is well-defined. $\Diamond$

Proof. Let $t \in [0,1)$ and $\Lambda \in \mathrm{dom}\overline{V}$ be fixed. Let $\omega \in \Omega$ be fixed such that (2.54) holds on all bounded Lipschitz domains and such that moreover, for all bounded domains $D \subset \mathbb{R}^d$,

$$\|\varepsilon\varphi_\Lambda(\cdot/\varepsilon,\omega)\|_{\mathrm{L}^\infty(D)} \to 0, \qquad \fint_D V(y/\varepsilon, \Lambda + \nabla\varphi_\Lambda(y/\varepsilon,\omega),\omega)dy \to \overline{V}(\Lambda),$$

which follows from Lemma 2.2.4, the Sobolev embedding, and the Birkhoff-Khinchin ergodic theorem. Let $(u_\varepsilon)_\varepsilon$ be as in the statement of the lemma. Also denote $v_\varepsilon^\omega := \varepsilon\varphi_\Lambda(\cdot/\varepsilon,\omega) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$. By the choice of ω, the sequence $(v_\varepsilon^\omega)_\varepsilon$ satisfies the same properties as u_ε on any bounded domain, with C_Λ replaced by $C'_\Lambda = \overline{V}(\Lambda)$, and moreover, for all bounded Lipschitz domains $D \subset \mathbb{R}^d$,

$$\overline{W}_t(\Lambda) = \lim_{\varepsilon\downarrow 0} \inf_{v\in W_0^{1,p}(D;\mathbb{R}^m)} \fint_D W(y/\varepsilon, t\Lambda + t\nabla v_\varepsilon^\omega(y) + \nabla v(y),\omega)dy. \tag{2.59}$$

Let $D \subset O$ be some fixed Lipschitz subdomain. On the one hand, define

$$\overline{W}'_t(\Lambda,\omega;D) = \limsup_{\varepsilon\downarrow 0} \inf_{v\in W_0^{1,p}(D;\mathbb{R}^m)} \fint_D W(y/\varepsilon, t\Lambda + t\nabla u_\varepsilon(y) + \nabla v(y),\omega)dy. \tag{2.60}$$

Given $\eta > 0$, set $D_\eta := \{x \in D : d(x, \partial D) > \eta\}$ and consider the difference

$$\Delta^\omega_{\varepsilon,t,\eta} := \inf_{v \in W^{1,p}_0(D;\mathbb{R}^m)} \int_D W(y/\varepsilon, t\Lambda + t\nabla u_\varepsilon(y) + \nabla v(y), \omega) dy \\ - \inf_{w \in W^{1,p}_0(D_\eta;\mathbb{R}^m)} \int_{D_\eta} W(y/\varepsilon, t\Lambda + t\nabla v^\omega_\varepsilon(y) + \nabla w(y), \omega) dy. \quad (2.61)$$

Choose a smooth cut-off function χ_η such that χ_η equals 1 on D_η and vanishes outside D, with $|\nabla \chi_\eta| \le C'/\eta$ for some constant $C' > 0$, and define $w^\omega_{\varepsilon,\eta} := \chi_\eta v^\omega_\varepsilon + (1-\chi_\eta) u_\varepsilon$. Restricting the first infimum in (2.61) to those v's that are equal to $t(v_\varepsilon - u_\varepsilon)$ on ∂D_η, we obtain

$$\Delta^\omega_{\varepsilon,t,\eta} \le \inf_{v \in W^{1,p}_0(D \setminus D_\eta;\mathbb{R}^m)} \int_{D\setminus D_\eta} W(y/\varepsilon, t\Lambda + t\nabla w^\omega_{\varepsilon,\eta}(y) + \nabla v(y), \omega) dy.$$

Hence, choosing $v = 0$, using the upper bound $W \le C(1+V)$ and decomposing

$$t\nabla w^\omega_{\varepsilon,\eta} = t\chi_\eta \nabla v^\omega_\varepsilon + t(1-\chi_\eta)\nabla u_\varepsilon + (1-t)\frac{t}{1-t}\nabla \chi_\eta (v^\omega_\varepsilon - u_\varepsilon),$$

we obtain by convexity

$$\Delta^\omega_{\varepsilon,t,\eta} \le C|D \setminus D_\eta| \Big(1 + \fint_{D\setminus D_\eta} V(y/\varepsilon, \Lambda + \nabla u_\varepsilon(y), \omega) dy + \fint_{D\setminus D_\eta} V(y/\varepsilon, \Lambda + \nabla v^\omega_\varepsilon(y), \omega) dy + E^\omega_{\varepsilon,t,\eta}\Big),$$

where the error is given by

$$E^\omega_{\varepsilon,t,\eta} := \fint_{D\setminus D_\eta} V\left(y/\varepsilon, \frac{t}{1-t}\nabla \chi_\eta(y)(v^\omega_\varepsilon(y) - u_\varepsilon(y)), \omega\right) dy.$$

Since v^ω_ε and u_ε go to 0 in $\mathrm{L}^\infty(D;\mathbb{R}^m)$, we can prove that, for any $t \in (0,1)$,

$$\limsup_{\eta\downarrow 0} \limsup_{\varepsilon \downarrow 0} \Delta^\omega_{\varepsilon,t,\eta} \le \limsup_{\eta\downarrow 0} C|D\setminus D_\eta|(1 + C_\Lambda + C'_\Lambda) = 0.$$

In view of equalities (2.59) and (2.60), this implies $\overline{W}'_t(\Lambda, \omega; D) \le \overline{W}_t(\Lambda)$.

On the other hand, define

$$\overline{W}''_t(\Lambda, \omega; D) = \liminf_{\varepsilon\downarrow 0} \inf_{v \in W^{1,p}_0(D;\mathbb{R}^m)} \fint_D W(y/\varepsilon, t\Lambda + t\nabla u_\varepsilon(y) + \nabla v(y), \omega) dy,$$

and repeat the same argument as above with $D^\eta := \{x : d(x, D) < \eta\}$ and

$$\tilde{\Delta}^\omega_{\varepsilon,t,\eta} := \inf_{v \in W^{1,p}_0(D^\eta;\mathbb{R}^m)} \int_{D^\eta} W(y/\varepsilon, t\Lambda + t\nabla v^\omega_\varepsilon(y) + \nabla v(y), \omega) dy \\ - \inf_{w \in W^{1,p}_0(D;\mathbb{R}^m)} \int_D W(y/\varepsilon, t\Lambda + t\nabla u_\varepsilon(y) + \nabla w(y), \omega) dy,$$

which then yields $\overline{W}''_t(\Lambda, \omega; D) \ge \overline{W}_t(\Lambda)$. This shows that $\overline{W}''_t(\Lambda, \omega; D) = \overline{W}'_t(\Lambda, \omega; D) = \overline{W}_t(\Lambda)$, and the result is proven. □

Let $\Omega_1 \subset \Omega_0$, $\Omega_1 \in \mathcal{F}$, a subset of full probability such that (2.54) holds for all $\omega \in \Omega_1$, $t \in \mathbb{Q} \cap [0,1)$, and $\Lambda \in \mathbb{Q}^{m\times d} \cap \mathrm{dom}\overline{V}$, such that (2.55) holds for all $\omega \in \Omega_1$ and $\Lambda \in \mathbb{Q}^{m\times d}$, and such that we further have, for all $\omega \in \Omega_1$, $\Lambda \in \mathbb{Q}^{m\times d}$, and all bounded domains $O \subset \mathbb{R}^d$,

$$\overline{V}(\Lambda) = \lim_{\varepsilon\downarrow 0} \fint_{O/\varepsilon} V(y, \Lambda + \nabla \varphi_\Lambda(y, \omega), \omega) dy.$$

2.3.2 Γ-lim inf **inequality by blow-up**

In this section, we prove the Γ-lim inf inequality for Theorem 2.1.6 by adapting the blow-up method introduced by Fonseca and Müller [183] (see also [25, Section 4.1] and [80]). In the present context, a subtle use of the corrector for the convex problem is further needed.

Proposition 2.3.3 (Γ-lim inf inequality)**.** *For all $\omega \in \Omega_1$, all bounded Lipschitz domain $O \subset \mathbb{R}^d$, and all sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O;\mathbb{R}^m)$, we have*

$$\liminf_{\varepsilon\downarrow 0} I_\varepsilon(u_\varepsilon,\omega;O) \geq I(u;O).$$

In addition, for all $\Lambda \in \mathbb{R}^{m\times d}$, for almost all $\omega \in \Omega_1$, for all bounded Lipschitz domain $O \subset \mathbb{R}^d$, and all sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O;\mathbb{R}^m)$, we have

$$\liminf_{\varepsilon\downarrow 0} I_\varepsilon(u_\varepsilon,\omega;O) \geq |O| \liminf_{t\uparrow 1} \lim_{R\uparrow\infty} \inf_{v\in W_0^{1,p}(O;\mathbb{R}^m)} \fint_{Q_R} W(y,t\Lambda + t\nabla\varphi_\Lambda(y,\omega)+\nabla v(y),\omega)dy \geq I(u;O).$$

$\Diamond$

Proof. For all $r>0$ and $x\in\mathbb{R}^d$, define $Q_r(x) = x + rQ$ and $S_{r,\kappa}(x) = Q_r(x)\setminus Q_{r\kappa}(x)$ for all $\kappa>0$. For all $\varepsilon>0$ and all Λ, ω, define $\chi^\omega_{\varepsilon,\Lambda} = \varepsilon\varphi_\Lambda(\cdot/\varepsilon,\omega)$. For all $\omega\in\Omega_1$ and $\Lambda\in\mathbb{Q}^{m\times d}$, the sequence $(\chi^\omega_{\varepsilon,\Lambda})_\varepsilon$ satisfies $\chi^\omega_{\varepsilon,\Lambda} \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$ and $J_\varepsilon(\chi^\omega_{\varepsilon,\Lambda}+\Lambda\cdot x,\,\omega;\,O') \to J(\Lambda\cdot x;\,O') = |O'|\overline{V}(\Lambda)$ as $\varepsilon\downarrow 0$, for any subdomain $O'\subset O$.

From now on, let $\omega\in\Omega_1$ be *fixed*, let $O\subset\mathbb{R}^d$ be some bounded Lipschitz subset, and let $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ be some fixed sequence with $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(O;\mathbb{R}^m)$. We need to prove

$$\liminf_{\varepsilon\downarrow 0} I_\varepsilon(u_\varepsilon,\omega;O) \geq I(u;O). \tag{2.62}$$

It does not restrict generality to assume $\liminf_\varepsilon I_\varepsilon(u_\varepsilon,\omega;O) = \lim_\varepsilon I_\varepsilon(u_\varepsilon,\omega;O) < \infty$ and also $\sup_\varepsilon I_\varepsilon(u_\varepsilon,\omega;O)<\infty$. Hence, $\nabla u_\varepsilon(x) \in \mathrm{dom}W(x/\varepsilon,\cdot,\omega) = \mathrm{dom}V(x/\varepsilon,\cdot,\omega)$ for almost all x. Furthermore, the Γ-convergence result for V yields $J(u;O) \leq \liminf_\varepsilon J_\varepsilon(u_\varepsilon,\omega;O) \leq \lim_\varepsilon I_\varepsilon(u_\varepsilon,\omega;O) < \infty$, so that $\nabla u(x)\in\mathrm{dom}\overline{V}$ for almost all x.

Step 1. Localization by blow up: we prove that it suffices to show that for almost all x_0,

$$\liminf_{t\uparrow 1}\liminf_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\fint_{Q_r(x_0)} W(y/\varepsilon, t\nabla u_\varepsilon(y),\omega)dy \geq \overline{W}(\nabla u(x_0)). \tag{2.63}$$

For all $\varepsilon>0$, consider the positive Radon measure on O defined by $d\rho_\varepsilon(x) = W(x/\varepsilon,\nabla u_\varepsilon(x),\omega)dx$. As $\sup_\varepsilon \rho_\varepsilon(\mathrm{adh}O)<\infty$ by hypothesis, the Prokhorov theorem asserts the convergence $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho$ up to extraction of a subsequence, for some positive Radon measure ρ on $\mathrm{adh}O$. (The extraction will remain implicit in our notation in the sequel.) By Lebesgue's decomposition theorem, we can consider the absolutely continuous part of the positive measure ρ, and the Radon-Nikodym theorem allows to define the density $f\in \mathrm{L}^1(U)$ of the latter. As O is open, we then have by the portmanteau theorem (see e.g. [62, Theorem 2.1]),

$$\liminf_{\varepsilon} I_\varepsilon(u_\varepsilon,\omega;O) = \liminf_\varepsilon \rho_\varepsilon(O) \geq \rho(O) \geq \int_O f(x)dx.$$

Hence, in order to prove (2.62), it suffices to show that $f(x)\geq\overline{W}(\nabla u(x))$ for almost all x. Since $\rho(\mathrm{adh}O)<\infty$, we have $\rho(\partial Q_r(x))=0$ for all $r\in(0,1)\setminus D_x$, where D_x is at most countable, so that, for almost all x, Lebesgue's differentiation theorem and the portmanteau theorem successively give

$$f(x) = \lim_{\substack{r\downarrow 0\\ r\notin D_x}} \frac{\rho(Q_r(x))}{r^d} = \lim_{\substack{r\downarrow 0\\ r\notin D_x}} \lim_{\varepsilon\downarrow 0} \frac{\rho_\varepsilon(Q_r(x))}{r^d}.$$

Hence, it suffices to show that for almost all x_0,

$$\liminf_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\fint_{Q_r(x_0)} W(y/\varepsilon,\nabla u_\varepsilon(y),\omega)dy \geq \overline{W}(\nabla u(x_0)).$$

Using the ru-usc assumption on W, we easily deduce the following inequality,

$$\limsup_{t\uparrow 1}\liminf_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\fint_{Q_r(x_0)} W(y/\varepsilon,t\nabla u_\varepsilon(y),\omega)dy \leq \liminf_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\fint_{Q_r(x_0)} W(y/\varepsilon,\nabla u_\varepsilon(y),\omega)dy. \quad (2.64)$$

Indeed, as $\nabla u_\varepsilon(y)\in \mathrm{dom} W(y/\varepsilon,\cdot,\omega)$ for almost all y, we can write

$$\fint_{Q_r(x_0)} W(y/\varepsilon,t\nabla u_\varepsilon(y),\omega)dy \\ \leq (1+\Delta_W^a(t))\fint_{Q_r(x_0)} W(y/\varepsilon,\nabla u_\varepsilon(y),\omega)dy + \Delta_W^a(t)\fint_{Q_r(x_0)} a(y/\varepsilon,\omega)dy,$$

and thus, by τ-stationarity of a, the Birkhoff-Khinchin ergodic theorem yields

$$\liminf_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\fint_{Q_r(x_0)} W(y/\varepsilon,t\nabla u_\varepsilon(y),\omega)dy \\ \leq (1+\Delta_W^a(t))\liminf_{r\downarrow 0}\lim_{\varepsilon\downarrow 0}\fint_{Q_r(x_0)} W(y/\varepsilon,\nabla u_\varepsilon(y),\omega)dy + \Delta_W^a(t)\mathbb{E}[a(0,\cdot)],$$

so that inequality (2.64) now directly follows from the ru-usc assumption with respect to a (meaning indeed that $\limsup_{t\uparrow 1}\Delta_W^a(t)\leq 0$). Using (2.64), we finally conclude that it is sufficient to show (2.63) for almost all x_0.

Step 2. Proof of (2.63) by truncation.

The idea is to truncate u_ε at the boundary, in order to make appear in the left-hand side of (2.63) precisely $\overline{W}(t\nabla u(x_0))$, which will then allow us to conclude.

Let $t,\kappa\in(0,1)$ be fixed. Since $p>d$, the Sobolev embedding yields $u_\varepsilon\to u$ in $\mathrm{L}^\infty(O;\mathbb{R}^m)$. Moreover, combining the Lebesgue differentiation theorem for ∇u and the Sobolev embedding for $p>d$, we deduce that for all $x_0\notin\mathcal{N}$ (for some null set $\mathcal{N}\subset\mathbb{R}^d$, $|\mathcal{N}|=0$),

$$\lim_{r\downarrow 0}\frac{1}{r}\|u-u(x_0)-\nabla u(x_0)\cdot(x-x_0)\|_{\mathrm{L}^\infty(Q_r(x_0))}=0. \quad (2.65)$$

Enlarging the null set $\mathcal{N}$, we can also assume that $\nabla u(x_0)\in\mathrm{dom}\overline{V}$ for any $x_0\notin\mathcal{N}$. From now on, let $x_0\in O\setminus\mathcal{N}$ be fixed and write for simplicity $\Lambda:=\nabla u(x_0)$. Since $\overline{V}$ is convex and lower semicontinuous, we have $\overline{V}(\Lambda)=\liminf_{\Lambda'\to\Lambda,\,\Lambda'\in\mathbb{Q}^{m\times d}}\overline{V}(\Lambda')$, and a diagonalization argument then allows us to choose a sequence $(\Lambda_r)_r\subset\mathbb{Q}^{m\times d}$ such that $\Lambda_r\to\Lambda$ and $\overline{V}(\Lambda_r)\to\overline{V}(\Lambda)$ as $r\downarrow 0$, and simultaneously

$$\lim_{r\downarrow 0}\frac{1}{r}\|u-u(x_0)-\Lambda_r\cdot(x-x_0)\|_{\mathrm{L}^\infty(Q_r(x_0))}=0. \quad (2.66)$$

Let $\phi_{r,\kappa}$ be a smooth cut-off function with values in $[0,1]$, such that $\phi_{r,\kappa}$ equals 1 on $Q_{r\kappa}(x_0)$, vanishes outside $Q_r(x_0)$, and satisfies $\|\nabla\phi_{r,\kappa}\|_{\mathrm{L}^\infty}\leq\frac{2}{r(1-\kappa)}$. We then set

$$v_{\varepsilon,r,\kappa}:=\phi_{r,\kappa}u_\varepsilon+(1-\phi_{r,\kappa})(u(x_0)+\Lambda_r\cdot(x-x_0)+\chi^\omega_{\varepsilon,\Lambda_r}(x)).$$

Since $v_{\varepsilon,r,\kappa}$ coincides with u_ε on $Q_{r\kappa}(x_0)$ and $0 \le W \le C(1+V)$, we have

$$\begin{aligned}&\fint_{Q_r(x_0)} W(y/\varepsilon, t\nabla v_{\varepsilon,r,\kappa}(y),\omega)dy\\ &\qquad \le \fint_{Q_r(x_0)} W(y/\varepsilon, t\nabla u_\varepsilon(y),\omega)dy + \frac{C}{r^d}\int_{S_{r,\kappa}(x_0)} V(y/\varepsilon, t\nabla v_{\varepsilon,r,\kappa}(y),\omega)dy + C(1-\kappa^d).\end{aligned} \tag{2.67}$$

Defining $\Psi_{\varepsilon,r,\kappa}(x) := \nabla\phi_{r,\kappa}(x)\otimes(u_\varepsilon(x)-u(x_0)-\Lambda_r\cdot(x-x_0)-\chi^\omega_{\varepsilon,\Lambda_r})$ and decomposing

$$t\nabla v_{\varepsilon,r,\kappa} = t\phi_{r,\kappa}\nabla u_\varepsilon + t(1-\phi_{r,\kappa})(\Lambda_r+\nabla\chi^\omega_{\varepsilon,\Lambda_r}) + (1-t)\frac{t}{1-t}\Psi_{\varepsilon,r,\kappa},$$

we obtain by convexity of V,

$$\begin{aligned}V(y/\varepsilon, t\nabla v_{\varepsilon,r,\kappa}(y),\omega) &\le t\phi_{r,\kappa}V(y/\varepsilon,\nabla u_\varepsilon(y),\omega) + t(1-\phi_{r,\kappa})V(y/\varepsilon,\Lambda_r+\nabla\chi^\omega_{\varepsilon,\Lambda_r}(y),\omega)\\ &\qquad + (1-t)V\left(y/\varepsilon, \frac{t}{1-t}\Psi_{\varepsilon,r,\kappa}(y),\omega\right)\\ &\le W(y/\varepsilon,\nabla u_\varepsilon(y),\omega) + V(y/\varepsilon,\Lambda_r+\nabla\chi^\omega_{\varepsilon,\Lambda_r}(y),\omega)\\ &\qquad + (1-t)V\left(y/\varepsilon, \frac{t}{1-t}\Psi_{\varepsilon,r,\kappa}(y),\omega\right).\end{aligned} \tag{2.68}$$

Combined with

$$\begin{aligned}&\|\Psi_{\varepsilon,r,\kappa}\|_{\mathrm{L}^\infty(S_{r,\kappa}(x_0))}\\ &\qquad \le \frac{2}{r(1-\kappa)}\Big(\|u_\varepsilon-u\|_{\mathrm{L}^\infty(O)} + \|u-u(x_0)-\Lambda_r\cdot(x-x_0)\|_{\mathrm{L}^\infty(Q_r(x_0))} + \|\chi^\omega_{\varepsilon,\Lambda_r}\|_{\mathrm{L}^\infty(O)}\Big),\end{aligned}$$

and with (2.66), the convergences $u_\varepsilon\to u$ and $\chi^\omega_{\varepsilon,\Lambda_r}\to 0$ in $\mathrm{L}^\infty(O;\mathbb{R}^m)$ lead to

$$\limsup_{r\downarrow 0}\limsup_{\varepsilon\downarrow 0}\|\Psi_{\varepsilon,r,\kappa}\|_{\mathrm{L}^\infty(S_{r,\kappa}(x_0))} = 0.$$

By assumption, we can find $\delta>0$ with $\mathrm{adh}B_\delta\subset \mathrm{int\,dom}M$. Hence, for all $t,\kappa\in(0,1)$, there exists $r_{\kappa,t}>0$ such that, for all $0<r<r_{\kappa,t}$, there exists some $\varepsilon_{r,\kappa,t}>0$ such that for all $0<\varepsilon<\varepsilon_{r,\kappa,t}$,

$$\left\|\frac{t}{1-t}\Psi_{\varepsilon,r,\kappa}\right\|_{\mathrm{L}^\infty(S_{r,\kappa}(x_0))} < \delta.$$

This implies

$$\int_{S_{r,\kappa}(x_0)} V\left(y/\varepsilon,\frac{t}{1-t}\Psi_{\varepsilon,r,\kappa}(y),\omega\right)dy \le |S_{r,\kappa}(x_0)|\sup_{|\Lambda'|<\delta}M(\Lambda') = r^d(1-\kappa^d)|\sup_{|\Lambda'|<\delta}M(\Lambda'),$$

where the supremum is finite, by virtue of the convexity of M and our choice of $\delta>0$. Combined with inequality (2.68) and with the definition of the correctors $\chi^\omega_{\varepsilon,\Lambda_r}$ (with $\lim_r\overline{V}(\Lambda_r)=\overline{V}(\Lambda)<\infty$), this yields

$$\begin{aligned}&\liminf_{\kappa\uparrow 1}\liminf_{t\uparrow 1}\liminf_{r\downarrow 0}\liminf_{\varepsilon\downarrow 0}\int_{S_{r,\kappa}(x_0)} V(y/\varepsilon,t\nabla v_{\varepsilon,r,\kappa}(y),\omega)dy\\ &\qquad\qquad \le \liminf_{\kappa\uparrow 1}\liminf_{r\downarrow 0}\liminf_{\varepsilon\downarrow 0}\int_{S_{r,\kappa}(x_0)} W(y/\varepsilon,\nabla u_\varepsilon(y),\omega)dy.\end{aligned}$$

This turns inequality (2.67) into

$$\liminf_{\kappa\uparrow 1}\liminf_{t\uparrow 1}\liminf_{r\downarrow 0}\liminf_{\varepsilon\downarrow 0} \fint_{Q_r(x_0)} W(y/\varepsilon, t\nabla v_{\varepsilon,r,\kappa}(y),\omega)dy$$
$$\leq \liminf_{t\uparrow 1}\liminf_{r\downarrow 0}\liminf_{\varepsilon\downarrow 0} \fint_{Q_r(x_0)} W(y/\varepsilon, t\nabla u_\varepsilon(y),\omega)dy$$
$$+ \liminf_{\kappa\uparrow 1}\liminf_{r\downarrow 0}\liminf_{\varepsilon\downarrow 0} \frac{C}{r^d}\int_{S_{r,\kappa}(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y),\omega)dy. \quad (2.69)$$

Since we have chosen $\omega \in \Omega_1$, $t \in \mathbb{Q}\cap(0,1)$, and $\Lambda_r \in \mathbb{Q}^{m\times d}$, the convergence (2.54) yields

$$\overline{W}_t(\Lambda_r) = \liminf_{\varepsilon\downarrow 0} \inf_{v\in W_0^{1,p}(Q_r(x_0);\mathbb{R}^m)} \fint_{Q_r(x_0)} W(y/\varepsilon, t\Lambda_r + t\nabla\varphi_{\Lambda_r}(\cdot/\varepsilon,\omega) + \nabla v(y),\omega)dy.$$

Hence, since $v_{\varepsilon,r,\kappa} - u(x_0) - \Lambda_r\cdot(x-x_0) \in \chi^\omega_{\varepsilon,\Lambda_r} + W_0^{1,p}(Q_r(x_0);\mathbb{R}^m)$ with $\chi^\omega_{\varepsilon,\Lambda_r} = \varepsilon\varphi_{\Lambda_r}(\cdot/\varepsilon,\omega)$, the inequality (2.69) yields

$$\overline{W}(\nabla u(x_0)) = \overline{W}(\Lambda) \leq \liminf_{t\uparrow 1}\liminf_{r\downarrow 0} \overline{W}_t(\Lambda_r)$$
$$\leq \liminf_{t\uparrow 1}\liminf_{r\downarrow 0}\liminf_{\varepsilon\downarrow 0} \fint_{Q_r(x_0)} W(y/\varepsilon, t\nabla v_{\varepsilon,r,\kappa}(y),\omega)dy$$
$$\leq \liminf_{t\uparrow 1}\liminf_{r\downarrow 0}\liminf_{\varepsilon\downarrow 0} \fint_{Q_r(x_0)} W(y/\varepsilon, t\nabla u_\varepsilon(y),\omega)dy$$
$$+ \limsup_{\kappa\uparrow 1}\limsup_{r\downarrow 0}\limsup_{\varepsilon\downarrow 0} \frac{C}{r^d}\int_{S_{r,\kappa}(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y),\omega)dy. \quad (2.70)$$

It remains to get rid of the second right-hand side term. By the portmanteau theorem,

$$\limsup_{\varepsilon\downarrow 0}\frac{1}{r^d}\int_{S_{r,\kappa}(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y),\omega)dy = \limsup_{\varepsilon\downarrow 0}\frac{\rho_\varepsilon(S_{r,\kappa}(x_0))}{r^d}$$
$$\leq \limsup_{\varepsilon\downarrow 0}\frac{\rho_\varepsilon(\mathrm{adh}S_{r,\kappa}(x_0))}{r^d} \leq \frac{\rho(\mathrm{adh}S_{r,\kappa}(x_0))}{r^d}.$$

Since the singular part of ρ must be supported in a closed subset of adh$O$ of Lebesgue measure 0, we deduce for almost all $x_0 \in O\setminus\mathcal{N}$ the existence of some $r_0 > 0$ sufficiently small such that adh$Q_r(x_0)$ has no intersection with that support for all $0 < r < r_0$. Hence, for all $0 < r < r_0$,

$$\limsup_{\varepsilon\downarrow 0}\frac{1}{r^d}\int_{S_{r,\kappa}(x_0)} W(y/\varepsilon, \nabla u_\varepsilon(y),\omega)dy$$
$$\leq \frac{\rho(\mathrm{adh}S_{r,\kappa}(x_0))}{r^d} = \frac{1}{r^d}\int_{S_{r,\kappa}(x_0)} f(y)dy = \fint_{Q_r(x_0)} f(y)dy - \kappa^d \fint_{Q_{r\kappa}(x_0)} f(y)dy,$$

where for almost all x_0 the right-hand side converges to $(1-\kappa^d)f(x_0)$ as $r\downarrow 0$ by Lebesgue's differentiation theorem. Hence, for almost all x_0, (2.70) turns into

$$\overline{W}(\nabla u(x_0)) \leq \limsup_{t\uparrow 1}\liminf_{r\downarrow 0}\liminf_{\varepsilon\downarrow 0} \fint_{Q_r(x_0)} W(y/\varepsilon, t\nabla u_\varepsilon(y),\omega)dy,$$

and the desired result (2.63) is proven. □

2.3.3 Γ-lim sup inequality with Neumann boundary data

In this section, we prove the Γ-lim sup inequality, first considering the affine case, and then deducing the general case by approximation. For this approximation argument to hold, we would however need to know a priori that the homogenized energy $\overline{W}$ satisfies good regularity properties (i.e. lower semicontinuity on $\mathbb{R}^{m\times d}$ and continuity on $\operatorname{int}\operatorname{dom}\overline{V}$). Since this is not clear at all a priori, our strategy (inspired by Anza Hafsa and Mandallena [25]) consists in introducing some relaxations of $\overline{W}$ that enjoy the required properties, and then in deducing a posteriori from Γ-convergence (or a weaker form of it) the equality of $\overline{W}$ with its relaxations (so that $\overline{W}$ itself has all the desired properties). Motivated by the work of Fonseca [182] (see also [25]), we consider the following relaxation of $\overline{W}$,

$$\mathcal{Z}W(\Lambda) := \inf\left\{ \fint_O \overline{W}(\Lambda+\nabla\phi(y))dy \,:\, \phi \text{ continuous piecewise affine on } O \text{ and } \phi|_{\partial O}=0 \right\},$$

where the definition does clearly not depend on the chosen underlying (nonempty) bounded Lipschitz domain $O\subset\mathbb{R}^d$. Also write $\widehat{\mathcal{Z}}W$ for the lower semicontinuous envelope of $\mathcal{Z}W$ (defined by $\widehat{\mathcal{Z}}W(\Lambda) := \liminf_{\Lambda'\to\Lambda}\mathcal{Z}W(\Lambda')$ for all Λ). Now define the integral functionals corresponding to all these relaxed integrands: for any bounded domain $O\subset\mathbb{R}^d$ and $u\in W^{1,p}(O;\mathbb{R}^m)$,

$$\mathcal{Z}I(u;O) := \int_O \mathcal{Z}W(\nabla u(y))dy, \qquad \widehat{\mathcal{Z}}I(u;O) := \int_O \widehat{\mathcal{Z}}W(\nabla u(y))dy.$$

The following result gives some properties of these relaxations, which will be crucial in the sequel.

Lemma 2.3.4 (Properties of relaxations)**.** *Assume $p>d$. Then the following holds.*

(a) $\mathcal{Z}W$ (and thus also $\widehat{\mathcal{Z}}W$) is continuous on $\operatorname{int}\operatorname{dom}\mathcal{Z}W$.

(b) $\overline{V}\le\widehat{\mathcal{Z}}W\le\overline{W}\le C(1+\overline{V})$.

(c) $\overline{W}$ and $\mathcal{Z}W$ are ru-usc.

(d) For all $t\in(0,1)$ we have $t\,\operatorname{adh}\operatorname{dom}\mathcal{Z}W\subset\operatorname{int}\operatorname{dom}\mathcal{Z}W$, and the following representation result holds,

$$\widehat{\mathcal{Z}}W(\Lambda) = \liminf_{t\to1}\mathcal{Z}W(t\Lambda) = \begin{cases} \mathcal{Z}W(\Lambda), & \text{if } \Lambda\in\operatorname{int}\operatorname{dom}\mathcal{Z}W;\\ \lim_{t\uparrow1}\mathcal{Z}W(t\Lambda), & \text{if } \Lambda\in\partial\operatorname{dom}\mathcal{Z}W;\\ \infty, & \text{otherwise;}\end{cases}$$

where in particular the limit exists.

(e) Let $\Lambda\in\mathbb{R}^{m\times d}$ and let $O\subset\mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists a sequence $(\phi_k)_k\subset W^{1,p}_0(O;\mathbb{R}^m)$ of piecewise affine functions such that $\phi_k\to0$ in $\mathrm{L}^\infty(O;\mathbb{R}^m)$ and

$$\lim_{k\uparrow\infty}\fint_O \overline{W}(\Lambda+\nabla\phi_k(y))dy = \mathcal{Z}W(\Lambda). \qquad \Diamond$$

Proof. Continuity of $\mathcal{Z}W$ on $\operatorname{int}\operatorname{dom}\mathcal{Z}W$ is a result due to Fonseca [182], which yields part (a) (even without any ru-usc assumption on W). The inequalities stated in part (b) directly follow from the definitions of $\overline{V}$, $\overline{W}$ and $\widehat{\mathcal{Z}}W$. Part (e) is standard (see [25, Proposition 3.17] for detail). It remains to prove properties (c) and (d).

Step 1. Proof of (c).

We first prove that $\overline{W}_t$ is ru-usc for all $t\in[0,1)$. Let $s>0$, $\Lambda\in\operatorname{dom}\overline{W}=\operatorname{dom}\overline{V}$, and let $O\subset\mathbb{R}^d$ be some bounded Lipschitz domain. For almost all ω, note that by convexity we have for all subdomains $D \subset O$,

$$\limsup_{\varepsilon\downarrow 0} \int_D V(y/\varepsilon, s\Lambda + s\nabla\varphi_\Lambda(y/\varepsilon,\omega),\omega)dy$$
$$\leq \lim_{\varepsilon\downarrow 0}\int_D V(y/\varepsilon, \Lambda + \nabla\varphi_\Lambda(y/\varepsilon,\omega),\omega)dy + (1-s)M(0) = \overline{V}(\Lambda) + (1-s)M(0) < \infty.$$

Hence for almost all ω we can apply equality (2.58) at $s\Lambda$ with $u_\varepsilon = s\varepsilon\varphi_\Lambda(\cdot/\varepsilon,\omega)$, which yields

$$\overline{W}_t(s\Lambda) = \lim_{\varepsilon\downarrow 0} \inf_{v\in W_0^{1,p}(O;\mathbb{R}^m)} \fint_O W(y/\varepsilon, st\Lambda + st\nabla\varphi_\Lambda(y/\varepsilon,\omega) + \nabla v(y),\omega)dy.$$

Given $\omega \in \Omega$ such that this convergence and (2.54) both hold, and choosing a sequence $(v_\varepsilon^\omega)_\varepsilon \subset W_0^{1,p}(O;\mathbb{R}^d)$ such that

$$\overline{W}_t(\Lambda) = \lim_{\varepsilon\downarrow 0} \fint_O W(y/\varepsilon, t\Lambda + t\nabla\varphi_\Lambda(y/\varepsilon,\omega) + \nabla v_\varepsilon^\omega(y),\omega)dy,$$

we deduce

$$\begin{aligned}\overline{W}_t(s\Lambda) - \overline{W}_t(\Lambda) &\leq \lim_{\varepsilon\downarrow 0} \fint_O \Big(W(y/\varepsilon, s(t\Lambda + t\nabla\varphi_\Lambda(y/\varepsilon,\omega) + \nabla v_\varepsilon^\omega(y)),\omega) \\ &\qquad\qquad - W(y/\varepsilon, t\Lambda + t\nabla\varphi_\Lambda(y/\varepsilon,\omega) + \nabla v_\varepsilon^\omega(y),\omega)\Big)dy \\ &\leq \Delta_W^a(s)\lim_{\varepsilon\downarrow 0}\fint_O (a(y/\varepsilon,\omega) + W(y/\varepsilon, t\Lambda + t\nabla\varphi_\Lambda(y/\varepsilon,\omega) + \nabla v_\varepsilon^\omega(y),\omega))dy \\ &= \Delta_W^a(s)(\mathbb{E}[a(0,\cdot)] + \overline{W}_t(\Lambda)),\end{aligned} \tag{2.71}$$

using the Birkhoff-Khinchin ergodic theorem for the stationary field a. As $\overline{W}_t$ and the field a are nonnegative, as $\alpha := \mathbb{E}[a(0,\cdot)]$ is finite, and as $\limsup_{s\uparrow 1}\Delta_W^a(s) \leq 0$ by assumption, we deduce that $\overline{W}_t$ is also ru-usc. Now rewriting inequality (2.71) in the form

$$\overline{W}_t(s\Lambda) \leq \alpha\Delta_W^a(s) + (1 + (-1)\vee\Delta_W^a(s))\overline{W}_t(\Lambda),$$

and taking the suitable $\liminf$, we directly deduce $\overline{W}(s\Lambda) - \overline{W}(\Lambda) \leq (-1)\vee\Delta_W^a(s)(\alpha + \overline{W}(\Lambda))$ for all $\Lambda \in \mathrm{dom}\overline{V}$ and $s \in [0,1)$, proving that $\overline{W}$ is itself ru-usc with $\Delta_{\overline{W}}^\alpha = (-1)\vee\Delta_W^a(s)$.

We now show that $\mathcal{Z}W$ is also ru-usc. Take $s > 0$ and $\Lambda \in \mathrm{dom}\overline{V}$. By definition, there exists a sequence of piecewise affine functions $(\phi_k)_k \subset W_0^{1,p}(O)$ such that

$$\mathcal{Z}W(\Lambda) = \lim_{k\uparrow\infty}\fint_O \overline{W}(\Lambda + \nabla\phi_k(y))dy.$$

As $\Lambda \in \mathrm{dom}\overline{V}$, the left-hand side is finite, and we can thus assume $\Lambda + \nabla\phi_k \in \mathrm{dom}\overline{W}$ almost everywhere. Hence the ru-usc property satisfied by $\overline{W}$ gives

$$\begin{aligned}\mathcal{Z}W(s\Lambda) - \mathcal{Z}W(\Lambda) &\leq \lim_{k\uparrow\infty}\fint_O (\overline{W}(s(\Lambda + \nabla\phi_k(y))) - \overline{W}(\Lambda + \nabla\phi_k(y)))dy \\ &\leq \Delta_{\overline{W}}^\alpha(s)\lim_{k\uparrow\infty}\fint_O (\alpha + \overline{W}(\Lambda + \nabla\phi_k(y))dy = \Delta_{\overline{W}}^\alpha(s)(\alpha + \mathcal{Z}W(\Lambda)).\end{aligned}$$

Step 2. Proof of (d).

Since $\mathrm{dom}\mathcal{Z}W = \mathrm{dom}\overline{V}$ is a convex set containing 0, it is clear that, for all $t \in [0,1)$, $t\,\mathrm{adhdom}\mathcal{Z}W$ is contained in $\mathrm{int}\,\mathrm{dom}\mathcal{Z}W$. We first show that the limit $\lim_{t\uparrow 1}\mathcal{Z}W(t\Lambda)$ exists for all $\Lambda \in \mathrm{adhdom}\overline{V}$. Given some fixed $\Lambda \in \mathrm{adhdom}\overline{V}$, choose two sequences $s_n \uparrow 1$ and $t_n \uparrow 1$ with $t_n/s_n \uparrow 1$ such that

$$\lim_{n\uparrow\infty}\mathcal{Z}W(s_n\Lambda) = \liminf_{t\uparrow 1}\mathcal{Z}W(t\Lambda) \qquad \text{and} \qquad \lim_{n\uparrow\infty}\mathcal{Z}W(t_n\Lambda) = \limsup_{t\uparrow 1}\mathcal{Z}W(t\Lambda).$$

As $s_n\Lambda, t_n\Lambda \in \mathrm{dom}\overline{V}$ for all n, and as $\mathcal{Z}W$ is ru-usc, we have

$$\begin{aligned}\lim_{n\uparrow\infty} \mathcal{Z}W(t_n\Lambda) &\le \limsup_{n\uparrow\infty}(\alpha + \mathcal{Z}W(s_n\Lambda))\Delta_W^a(t_n/s_n) + \lim_{n\uparrow\infty} \mathcal{Z}W(s_n\Lambda)\\ &\le \lim_{n\uparrow\infty} \mathcal{Z}W(s_n\Lambda) \le \lim_{n\uparrow\infty} \mathcal{Z}W(t_n\Lambda),\end{aligned}$$

which thus proves the existence of the limit $\lim_{t\uparrow 1} \mathcal{Z}W(t\Lambda)$ for all $\Lambda \in \mathrm{adhdom}\overline{V}$.

We now prove the claimed representation result. First, if $\Lambda \in \mathrm{int}\,\mathrm{dom}\overline{V}$, then $\liminf_{t\to 1} \mathcal{Z}W(t\Lambda) = \mathcal{Z}W(\Lambda) = \widehat{\mathcal{Z}}W(\Lambda)$ follows from part (a). Second, if $\Lambda \notin \mathrm{adhdom}\overline{V}$, then $\mathcal{Z}W(t\Lambda) = \infty$ for any t sufficiently close to 1, and thus $\liminf_{t\to 1} \mathcal{Z}W(t\Lambda) = \infty = \widehat{\mathcal{Z}}W(\Lambda)$. Now it only remains to consider $\Lambda \in \partial\,\mathrm{dom}\mathcal{Z}W$. Then $\mathcal{Z}W(t\Lambda) = \infty$ whenever $t > 1$, so that we simply have $\liminf_{t\to 1} \mathcal{Z}W(t\Lambda) = \liminf_{t\uparrow 1} \mathcal{Z}W(t\Lambda) = \lim_{t\uparrow 1} \mathcal{Z}W(t\Lambda)$, since we have already proven the existence of this limit. Hence, it suffices to prove that $\widehat{\mathcal{Z}}W(\Lambda) = \liminf_{t\uparrow 1} \mathcal{Z}W(t\Lambda)$. By definition of the lower semicontinuous envelope $\widehat{\mathcal{Z}}W$ of $\mathcal{Z}W$, this equality would follow if we could show that, for any sequence $\Lambda_n \to \Lambda$, we have

$$\liminf_{n\uparrow\infty} \mathcal{Z}W(\Lambda_n) \ge \liminf_{t\uparrow 1} \mathcal{Z}W(t\Lambda). \tag{2.72}$$

It is of course sufficient to assume $\liminf_n \mathcal{Z}W(\Lambda_n) = \lim_n \mathcal{Z}W(\Lambda_n) < \infty$ and $\sup_n \mathcal{Z}W(\Lambda_n) < \infty$. Hence, $\Lambda_n \in \mathrm{dom}\overline{V}$ for all n, and thus, for all $t \in [0,1)$, $t\Lambda \in \mathrm{int}\,\mathrm{dom}\overline{V}$, so that, using part (a) as well as the ru-usc property satisfied by $\mathcal{Z}W$, we have

$$\mathcal{Z}W(t\Lambda) = \lim_{n\uparrow\infty} \mathcal{Z}W(t\Lambda_n) \le \lim_{n\uparrow\infty} \mathcal{Z}W(\Lambda_n) + \Delta_W^a(t) \lim_{n\uparrow\infty}(\alpha + \mathcal{Z}W(\Lambda_n)).$$

This yields

$$\limsup_{t\uparrow 1} \mathcal{Z}W(t\Lambda) \le \liminf_{n\uparrow\infty} \mathcal{Z}W(\Lambda_n),$$

and proves (2.72). □

Combining the Γ-$\liminf$ inequality for I_ε towards I with a Γ-$\limsup$ argument, we manage to identify $\overline{W}$ with its relaxations.

Lemma 2.3.5 (Regularity of the homogenized energy density)**.** *Assume $p > d$. Then $\overline{W}(\Lambda) = \mathcal{Z}W(\Lambda) = \widehat{\mathcal{Z}}W(\Lambda)$ for all $\Lambda \in \mathbb{R}^{m\times d}$. In particular, $\overline{W}$ is lower semicontinuous on $\mathbb{R}^{m\times d}$ and is continuous on* $\mathrm{int}\,\mathrm{dom}\overline{V}$. ♢

Proof. We split the proof into four steps.

Step 1. Recovery sequence for $I(\Lambda\cdot x; O)$.

Let $\Lambda \in \mathrm{int}\,\mathrm{dom}\overline{V}$ and let $t \in [0,1)$. In this step, for almost all ω, for all bounded Lipschitz domain $O \subset \mathbb{R}^d$, we prove the existence of sequences $t_\varepsilon \uparrow 1$, $\Lambda_\varepsilon \to \Lambda$ and $(w_\varepsilon)_\varepsilon \subset W_0^{1,p}(O;\mathbb{R}^m)$ such that $\varepsilon\varphi_{\Lambda_\varepsilon}(\cdot/\varepsilon,\omega) \rightharpoonup 0$, $w_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$ and $I_\varepsilon(t_\varepsilon\Lambda_\varepsilon\cdot x + \varepsilon t_\varepsilon\varphi_{\Lambda_\varepsilon}(\cdot/\varepsilon,\omega) + w_\varepsilon, \omega; O) \to |O|\overline{W}(\Lambda) = I(\Lambda\cdot x; O)$.

By definition of $\overline{W}$ and a diagonalization argument, for almost all ω and all bounded Lipschitz domains $O \subset \mathbb{R}^d$, it suffices to prove the existence of a sequence $(v_\varepsilon)_\varepsilon \subset W_0^{1,p}(O;\mathbb{R}^m)$ such that $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$ and $I_\varepsilon(t\Lambda\cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon,\omega) + v_\varepsilon, \omega; O) \to |O|\overline{W}_t(\Lambda)$ as $\varepsilon \downarrow 0$.

Let O be some fixed bounded Lipschitz domain. Given $\varepsilon > 0$, consider the cubes of the form $k(z+Q)$, $z \in \mathbb{Z}^d$, that are contained in O/ε, and denote by $z_j \in \mathbb{Z}^d$, $j = 1,\ldots,N_{\varepsilon,k}$, the centers of these cubes (the enumeration of which can be chosen independent of ε, k). Since O is Lipschitz, we have $N_{\varepsilon,k}(\varepsilon k)^d \to |O|$ as $\varepsilon \downarrow 0$, for all k. For all j, ω, we can choose a sequence $(v_k^{j,\omega})_k$ with $v_k^{j,\omega} \in W_0^{1,p}(k(z_j+Q);\mathbb{R}^m)$ such that

$$\fint_{k(z_j+Q)} W(y, t\Lambda + t\nabla\varphi_\Lambda(y,\omega) + \nabla v_k^{j,\omega}(y), \omega) dy \\ \leq \frac{1}{k} + \inf_{v \in W_0^{1,p}(k(z_j+Q);\mathbb{R}^m)} \fint_{k(z_j+Q)} W(y, t\Lambda + t\nabla\varphi_\Lambda(y,\omega) + \nabla v(y), \omega) dy.$$

For all ε, k, ω, we then consider the function $v_{\varepsilon,k}^\omega := \sum_{j=1}^{N_{\varepsilon,k}} v_k^{j,\omega} \mathbb{1}_{k(z_j+Q)} \in W_0^{1,p}(O/\varepsilon;\mathbb{R}^m)$, and we define $w_{\varepsilon,k}^\omega := \varepsilon v_{\varepsilon,k}^\omega(\cdot/\varepsilon) \in W_0^{1,p}(O;\mathbb{R}^m)$. Up to a diagonalization argument, it suffices to show that, for almost all ω (independent of the choice of O, as it is clear in the proof below),

$$\limsup_{k\uparrow\infty} \limsup_{\varepsilon\downarrow 0} \Big(\big| I_\varepsilon(t\Lambda \cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon,\omega) + w_{\varepsilon,k}^\omega, \omega; O) - |O|\overline{W}_t(\Lambda) \big| + \|w_{\varepsilon,k}^\omega\|_{\mathrm{L}^p(O)} \Big) = 0. \tag{2.73}$$

First we argue that for almost all ω,

$$\limsup_{k\uparrow\infty} \limsup_{\varepsilon\downarrow 0} I_\varepsilon(t\Lambda \cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon,\omega) + w_{\varepsilon,k}^\omega, \omega; O) \leq |O|\overline{W}_t(\Lambda). \tag{2.74}$$

Indeed, by definition of $w_{\varepsilon,k}^\omega$,

$$I_\varepsilon(t\Lambda \cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon,\omega) + w_{\varepsilon,k}^\omega, \omega; O) \\ \leq \frac{1}{k}(\varepsilon k)^d N_{\varepsilon,k} + (\varepsilon k)^d \sum_{j=1}^{N_{\varepsilon,k}} \inf_{v \in W_0^{1,p}(k(z_j+Q);\mathbb{R}^m)} \fint_{k(z_j+Q)} W(y, t\Lambda + t\nabla\varphi_\Lambda(y,\omega) + \nabla v(y), \omega) dy \\ + \varepsilon^d \int_{(O/\varepsilon)\setminus \bigcup_{j=1}^{N_{\varepsilon,k}} k(z_j+Q)} W(y, t\Lambda + t\nabla\varphi_\Lambda(y,\omega), \omega) dy.$$

Since $W \leq C(1+V)$, the last term of the right-hand side goes to 0 as $\varepsilon \downarrow 0$ for almost all ω by construction of the cubes $k(z_j+Q)$ and definition of φ_Λ. The Birkhoff-Khinchin ergodic theorem (which we apply to a measurable map by Hypothesis 2.1.1) then gives for almost all ω,

$$\limsup_{\varepsilon\downarrow 0} I_\varepsilon(t\Lambda \cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon,\omega) + w_{\varepsilon,k}^\omega, \omega; O) \\ \leq \frac{|O|}{k} + |O|\,\mathbb{E}\left[\inf_{v \in W_0^{1,p}(kQ;\mathbb{R}^m)} \fint_{kQ} W(y, t\Lambda + t\nabla\varphi_\Lambda(y,\cdot) + \nabla v(y), \cdot) dy \right]. \tag{2.75}$$

Lemma 2.3.1 then yields the desired result (2.74) as $k \uparrow \infty$. On the other hand, by definition (2.54) of $\overline{W}_t$, for all k and almost all ω, we have

$$\liminf_{\varepsilon\downarrow 0} I_\varepsilon(t\Lambda \cdot x + \varepsilon t\varphi_\Lambda(\cdot/\varepsilon,\omega) + w_{\varepsilon,k}(\cdot,\omega), \omega; O) \\ \geq |O| \liminf_{\varepsilon\downarrow 0} \inf_{v \in W_0^{1,p}(O;\mathbb{R}^m)} \fint_{O/\varepsilon} W(y, t\Lambda + t\nabla\varphi_\Lambda(y,\omega) + \nabla v(y), \omega; O) = |O|\overline{W}_t(\Lambda).$$

We now show that $w_{\varepsilon,k}^\omega \to 0$ in $\mathrm{L}^p(O;\mathbb{R}^m)$ as $\varepsilon \downarrow 0$, for almost all ω. Combining inequality (2.75) with the bound $W \leq C(1+V)$, the p-th order lower bound for W and the convexity of V, we indeed have

$$\limsup_{\varepsilon\downarrow 0} \|t\Lambda + t\nabla\varphi_\Lambda(\cdot/\varepsilon,\omega) + \nabla w_{\varepsilon,k}^\omega\|_{\mathrm{L}^p(O)}^p \leq \frac{|O|}{k} + C|O|(1 + \overline{V}(\Lambda) + (1-t)M(0)) < \infty.$$

For almost all ω, the weak L^p convergence of the sequence $(\nabla\varphi_\Lambda(\cdot/\varepsilon,\omega))_\varepsilon$ to 0 implies the boundedness of this sequence in $\mathrm{L}^p(O;\mathbb{R}^{m\times d})$, so that $(\nabla w^\omega_{\varepsilon,k})_\varepsilon$ is also bounded in $\mathrm{L}^p(O;\mathbb{R}^{m\times d})$, for any fixed k. By Poincaré's inequality on cubes of side length $k\varepsilon$, this implies

$$\|w^\omega_{\varepsilon,k}\|_{\mathrm{L}^p(O)} \le C_k(\omega)\varepsilon,$$

for some (random) constant $C_k(\omega)$. Combined with (2.74), this proves (2.73).

Step 2. Recovery sequence for $\mathcal{Z}I(\Lambda\cdot x;O)$.

Let $\Lambda\in\operatorname{int}\operatorname{dom}\overline{V}$ and let $O\subset\mathbb{R}^d$ be a bounded Lipschitz domain. In this step, for almost all ω, we prove the existence of a sequence $(u_\varepsilon)_\varepsilon\subset W^{1,p}(O;\mathbb{R}^m)$ such that $u_\varepsilon\rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$ and $I_\varepsilon(\Lambda\cdot x+u_\varepsilon,\omega;O)\to\mathcal{Z}I(\Lambda\cdot x;O)$.

Lemma 2.3.4(e) gives a sequence $(\phi_k)_k\subset W^{1,p}_0(O;\mathbb{R}^m)$ of piecewise affine functions such that $\phi_k\to 0$ in $\mathrm{L}^p(O;\mathbb{R}^m)$ (and even in $\mathrm{L}^\infty(O;\mathbb{R}^m)$), and

$$I(\phi_k+\Lambda\cdot x;O)=\int_O \overline{W}(\Lambda+\nabla\phi_k(y))dy \xrightarrow{k\uparrow\infty} |O|\mathcal{Z}W(\Lambda).$$

Denote by $(P^i_k)_{i=1}^{n_k}$ the partition of O associated with the piecewise affine function ϕ_k. For all k and $1\le i\le n_k$, considering $\Lambda^i_k:=\Lambda+\nabla\phi_k|_{P^i_k}$ on P^i_k, Step 1 above gives, for almost all ω, a sequence $t_\varepsilon\uparrow 1$, a sequence $\Lambda^i_{k,\varepsilon}\to\Lambda^i_k$ and a sequence $(v^i_{\varepsilon,k})_\varepsilon\subset W^{1,p}_0(P^i_k;\mathbb{R}^m)$ such that $\varepsilon\varphi_{\Lambda^i_{\varepsilon,k}}(\cdot/\varepsilon,\omega), v^i_{\varepsilon,k}\rightharpoonup 0$ in $W^{1,p}(P^i_k;\mathbb{R}^m)$ and $I_\varepsilon(t_\varepsilon\Lambda^i_{\varepsilon,k}\cdot x+\varepsilon t_\varepsilon\varphi_{\Lambda^i_{\varepsilon,k}}(\cdot/\varepsilon,\omega)+v^i_{\varepsilon,k},\omega;P^i_k)\to I(\Lambda^i_k\cdot x;P^i_k)=I(\phi_k+\Lambda\cdot x;P^i_k)$. As the $v^i_{\varepsilon,k}$'s satisfy Dirichlet boundary conditions, they can be directly glued together, while for the φ_Λ's we need to repeat the more complicated gluing argument of Step 2 of the proof of Proposition 2.2.10, with $p>d$. Although the functional I_ε is not convex here, as in the proof of Proposition 2.3.3, the idea is to use the bound $W\le C(1+V)$ at all points where the cut-off functions are different from 1 or 0, then use the convexity of V and estimate the corresponding error terms as before. We leave the detail to the reader.

Step 3. Recovery sequence for $\widehat{\mathcal{Z}}I(\Lambda\cdot x;O)$.

Let $\Lambda\in\operatorname{dom}\overline{V}$ and let $O\subset\mathbb{R}^d$ be a bounded Lipschitz domain. In this step, for almost all ω, we prove the existence of a sequence $(u^\omega_\varepsilon)_\varepsilon\subset W^{1,p}(O;\mathbb{R}^m)$ such that $u^\omega_\varepsilon\rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$ and $I_\varepsilon(\Lambda\cdot x+u^\omega_\varepsilon,\omega;O)\to\widehat{\mathcal{Z}}I(\Lambda\cdot x;O)$.

By Lemma 2.3.4(d), $\mathcal{Z}W$ and $\widehat{\mathcal{Z}}W$ coincide on $\operatorname{int}\operatorname{dom}\overline{V}$, and hence the result on $\operatorname{int}\operatorname{dom}\overline{V}$ already follows from Step 2. Let now $\Lambda\in\partial\operatorname{dom}\mathcal{Z}W$. Lemma 2.3.4(d) then asserts $\widehat{\mathcal{Z}}W(\Lambda)=\lim_{t\uparrow1}\mathcal{Z}W(t\Lambda)$. By convexity of $\operatorname{dom}\overline{V}$, for all $t\in[0,1)$, we have $t\Lambda\in\operatorname{int}\operatorname{dom}\overline{V}$, and hence, for almost all ω, Step 2 above gives a sequence $(u_{\varepsilon,t})_\varepsilon\subset W^{1,p}(O;\mathbb{R}^m)$ such that $u_{\varepsilon,t}\rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$ and $I_\varepsilon(t\Lambda\cdot x+u_{\varepsilon,t},\omega;O)\to\mathcal{Z}I(t\Lambda\cdot x;O)=|O|\mathcal{Z}W(t\Lambda)$. The conclusion then follows from a diagonalization argument.

Step 4. Conclusion.

Let $\Lambda\in\operatorname{dom}\overline{V}$, let $O\subset\mathbb{R}^d$ be a bounded Lipschitz domain, and let $(u^\omega_\varepsilon)_\varepsilon$ be the sequence given by Step 3 above. As $u^\omega_\varepsilon\rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$, the Γ-$\liminf$ inequality (see Proposition 2.3.3) gives, for almost all ω,

$$|O|\widehat{\mathcal{Z}}W(\Lambda)=\lim_{\varepsilon\downarrow0}I_\varepsilon(\Lambda\cdot x+u^\omega_\varepsilon,\omega;O)\ge I(\Lambda\cdot x;O)=|O|\overline{W}(\Lambda).$$

This being true for any $\Lambda\in\operatorname{dom}\overline{V}$, we conclude that $\widehat{\mathcal{Z}}W=\overline{W}$ everywhere. □

With Lemma 2.3.5 at hand, we now prove the Γ-$\limsup$ inequality with Neumann boundary data. (For the adaptation of Corollary 2.1.4 with Dirichlet boundary data, the approach is similar and we leave the detail to the reader.)

Proposition 2.3.6 (Γ-$\limsup$ inequality). *Assume $p > d$. There exists a subset $\Omega' \subset \Omega_1$, $\Omega' \in \mathcal{F}$, of full probability with the following property: for all $\omega \in \Omega'$, all strongly star-shaped (in the sense of Proposition 2.A.14) bounded Lipschitz domains $O \subset \mathbb{R}^d$, and all $u \in W^{1,p}(O;\mathbb{R}^m)$, there exist a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ and a sequence $(v_\varepsilon)_\varepsilon \subset W^{1,p}_0(O;\mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup u$ and $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$, and such that $I_\varepsilon(u_\varepsilon + v_\varepsilon, \omega; O) \to I(u;O)$ and $J_\varepsilon(u_\varepsilon, \omega; O) \to J(u;O)$ as $\varepsilon \downarrow 0$.* $\Diamond$

Proof. Recall that the Γ-$\liminf$ inequality implies the locality of recovery sequences (see the proof of Corollary 2.2.3). Hence, due to Proposition 2.3.3, the Γ-$\limsup$ result on a Lipschitz domain O for Neumann boundary conditions follows from the Γ-$\limsup$ on a ball $B \supset O$. We split the proof into three steps.

Step 1. Recovery sequence for affine functions.

In this step, we consider the case when $u = \Lambda \cdot x$. is an affine function. More precisely, we prove the existence of a subset $\Omega' \subset \Omega_1$, $\Omega' \in \mathcal{F}$, of full probability with the following property: given a bounded Lipschitz domain $O \subset \mathbb{R}^d$, for all $\omega \in \Omega'$ and all $\Lambda \in \operatorname{int}\operatorname{dom}\overline{W}$, there exist sequences $(u_\varepsilon)_\varepsilon \subset W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ and $(v_\varepsilon)_\varepsilon \subset W^{1,p}_0(O;\mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup \Lambda \cdot x$ in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ and $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$, and such that $I_\varepsilon(u_\varepsilon + v_\varepsilon, \omega; O) \to I(\Lambda \cdot x; O)$ and $J_\varepsilon(u_\varepsilon, \omega; O') \to J(\Lambda \cdot x; O')$ for all bounded domains $O' \subset \mathbb{R}^d$.

Let $\Lambda \in \operatorname{int}\operatorname{dom}\overline{V}$. For almost all $\omega \in \Omega_1$, and all bounded domains $O' \subset \mathbb{R}^d$, we have by convexity, the Birkhoff-Khinchin ergodic theorem, definition of $\varphi_{\Lambda'}$, and continuity of $\overline{V}$ at Λ:

$$\begin{aligned}\limsup_{t\uparrow 1, \Lambda' \to \Lambda} \lim_{\varepsilon\downarrow 0} J_\varepsilon(t\Lambda' \cdot x + \varepsilon t \varphi_{\Lambda'}(\cdot/\varepsilon, \omega), \omega; O') &\le \limsup_{\Lambda'\to\Lambda} \lim_{\varepsilon\downarrow 0} J_\varepsilon(\Lambda' \cdot x + \varepsilon \varphi_{\Lambda'}(\cdot/\varepsilon,\omega),\omega;O') \\ &= |O'| \lim_{\Lambda'\to\Lambda} \overline{V}(\Lambda') = |O'|\overline{V}(\Lambda) = J(\Lambda\cdot x; O').\end{aligned}$$

Combined with the Γ-$\liminf$ inequality for $J_\varepsilon(\cdot,\omega;O')$ towards $J(\cdot;O')$ (for $\omega \in \Omega_1$), this yields

$$\lim_{t\uparrow 1, \Lambda'\to\Lambda} \lim_{\varepsilon\downarrow 0} J_\varepsilon(t\Lambda' \cdot x + \varepsilon t\varphi_{\Lambda'}(\cdot/\varepsilon,\omega),\omega;O') = J(\Lambda \cdot x; O'). \tag{2.76}$$

By definition of $\overline{W}$, we may choose sequences $\Lambda_n \to \Lambda$ and $t_n \uparrow 1$ such that $\overline{W}_{t_n}(\Lambda_n) \to \overline{W}(\Lambda)$. For this choice, (2.76) yields for almost all ω and all bounded domain $O' \subset \mathbb{R}^d$

$$\lim_{n\uparrow\infty} \lim_{\varepsilon\downarrow 0} J_\varepsilon(t_n\Lambda_n \cdot x + \varepsilon t_n \varphi_{\Lambda_n}(\cdot/\varepsilon,\omega),\omega;O') = J(\Lambda\cdot x;O').$$

For all n and almost all ω, set $u^\omega_{\varepsilon,n} := t_n\Lambda_n \cdot x + \varepsilon t_n\varphi_{\Lambda_n}(\cdot/\varepsilon,\omega)$. By Step 1 of the proof of Lemma 2.3.5, for any bounded Lipschitz domains $O \subset \mathbb{R}^d$, there exists a sequence $(v^\omega_{\varepsilon,n})_\varepsilon \subset W^{1,p}_0(O;\mathbb{R}^m)$ such that $v^\omega_{\varepsilon,n} \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$ and $I_\varepsilon(u^\omega_{\varepsilon,n} + v^\omega_{\varepsilon,n}, \omega; O) \to |O|\overline{W}_{t_n}(\Lambda_n)$.

By a diagonalization argument, we then conclude that for almost all ω and all bounded Lipschitz domains $O \subset \mathbb{R}^d$ there exist sequences $(u_\varepsilon)_\varepsilon \subset W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ and $(v_\varepsilon)_\varepsilon \subset W^{1,p}_0(O;\mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup \Lambda\cdot x$ and $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}$, and such that $I_\varepsilon(u_\varepsilon + v_\varepsilon,\omega;O) \to I(\Lambda\cdot x;O)$ and $J_\varepsilon(u_\varepsilon,\omega;O') \to J(\Lambda\cdot x;O')$ for all bounded domains $O' \subset \mathbb{R}^d$.

Now define $\Omega' \subset \Omega_1$, $\Omega' \in \mathcal{F}$, as a subset of full probability such that this result holds for all $\Lambda \in \mathbb{Q}^{m\times d} \cap \operatorname{int}\operatorname{dom}\overline{V}$ and all $\omega \in \Omega'$. Arguing as in the end of Step 1 of the proof of Proposition 2.2.10, and using the continuity of both $\overline{W}$ and $\overline{V}$ in the interior of the domain (see Lemma 2.3.5), the conclusion follows.

Step 2. Recovery sequence for continuous piecewise affine functions.

We now show that, for any $\omega \in \Omega'$, any bounded Lipschitz domain $O \subset \mathbb{R}^d$, and any continuous piecewise affine function u on O with $\nabla u \in \operatorname{int}\operatorname{dom}\overline{V}$ pointwise, there exist a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ and a sequence $(v_\varepsilon)_\varepsilon \subset W^{1,p}_0(O;\mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup u$ and $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$, and such that $I_\varepsilon(u_\varepsilon + v_\varepsilon,\omega;O) \to I(u;O)$ and $J_\varepsilon(u_\varepsilon,\omega;O) \to J(\Lambda\cdot x;O)$. This follows from an immediate adaptation of Step 2 of the proof of Proposition 2.2.10. Again, the functional I_ε is not convex, but we may use the bound $W \le C(1+V)$ at all points where the cut-off functions are different from 1 or 0, and use the convexity of V to estimate the corresponding error terms. We leave the detail to the reader.

Step 3. Recovery sequence for general functions.

We show that, for all $\omega \in \Omega'$, all strongly star-shaped bounded Lipschitz domains $O \subset \mathbb{R}^d$, and all $u \in W^{1,p}(O;\mathbb{R}^m)$, there exist a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ and a sequence $(v_\varepsilon)_\varepsilon \subset W^{1,p}_0(O;\mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup u$ and $v_\varepsilon \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$, and such that $I_\varepsilon(u_\varepsilon + v_\varepsilon, \omega; O) \to I(u;O)$ and $J_\varepsilon(u_\varepsilon, \omega; O) \to J(u;O)$. Let $O \subset \mathbb{R}^d$ be some fixed strongly star-shaped bounded Lipschitz domain. By the Γ-lim inf inequality of Proposition 2.3.3, we can restrict attention to those $u \in W^{1,p}(O;\mathbb{R}^m)$ that satisfy

$$I(u;O) = \int_O \overline{W}(\nabla u(y))dy < \infty,$$

so that $\nabla u \in \mathrm{dom}\overline{V}$ almost everywhere. Let u be such a function and let $\omega \in \Omega'$ be fixed.

Let $t \in (0,1)$. Since O is Lipschitz and strongly star-shaped, and since $\overline{W}$ is lower semicontinuous on $\mathbb{R}^{m\times d}$, continuous on $\mathrm{int}\,\mathrm{dom}\overline{V}$, ru-usc, and satisfies $\overline{V} \le \overline{W} \le C(1+\overline{V})$ (see indeed Lemmas 2.3.4 and 2.3.5), the nonconvex approximation result of Proposition 2.A.14(ii)(c) yields a sequence $(u_n)_n$ of continuous piecewise affine functions such that $u_n \to u$ (strongly) in $W^{1,p}(O;\mathbb{R}^m)$, $I(u_n;O) \to I(u;O)$ and $J(u_n;O) \to J(u;O)$ as $n \uparrow \infty$, and such that $\nabla u_n \in \mathrm{int}\,\mathrm{dom}\overline{V}$ pointwise. Now Step 2 above gives, for any n, sequences $(u_{\varepsilon,n})_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ and $(v_{\varepsilon,n})_\varepsilon \subset W^{1,p}_0(O;\mathbb{R}^m)$ such that $u_{\varepsilon,n} \rightharpoonup u_n$ and $v_{\varepsilon,n} \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$, and such that $I_\varepsilon(u_{\varepsilon,n}+v_{\varepsilon,n}, \omega; O) \to I(u_n;O)$ and $J_\varepsilon(u_{\varepsilon,n}, \omega; O) \to J(u_n;O)$ as $\varepsilon \downarrow 0$. The result then follows from a diagonalization argument. ☐

2.4 Proof of the improved results

2.4.1 Subcritical case $1 < p \le d$

In this section, we establish Corollary 2.1.7, using truncations in the scalar case in place of the Sobolev compact embedding. For such truncation arguments to work, we further need to assume that the domain is fixed, i.e. $\mathrm{dom}V(y,\cdot,\omega) = \mathrm{dom}M$ for almost all y, ω.

Proof of Corollary 2.1.7. In the proof of Theorem 2.1.2 and Corollary 2.1.4, the Sobolev compact embedding into bounded functions is used both in Step 2 of the proof of Proposition 2.2.10 and in Step 1 of the proof of Corollary 2.1.4(i) (see Section 2.2.6). We only display the argument for Proposition 2.2.10 (the argument for Step 1 of the proof of Corollary 2.1.4(i) is similar).

We use the notation of Step 2 of the proof of Proposition 2.2.10. For all $s > 0$, define the truncation map $T_s : \mathbb{R} \to \mathbb{R}$ as follows,

$$T_s(x) = \mathrm{sgn}(x)\,|x| \wedge s = \begin{cases} s, & \text{if } x \ge s; \\ x, & \text{if } -s \le x \le s; \\ -s, & \text{if } x \le -s; \end{cases} \tag{2.77}$$

and for all $s > 0$ consider the following s-truncation of $u_{\varepsilon,\kappa,r,\eta}$,

$$u_{\varepsilon,\kappa,r,\eta,s} := T_s(u_{\varepsilon,\kappa,r,\eta} - u_{\kappa,r}) + u_{\kappa,r} \in W^{1,p}(O;\mathbb{R}). \tag{2.78}$$

Since $|tu_{\varepsilon,\kappa,r,\eta,s} - u| \le s + |tu_{\kappa,r} - u|$, we may replace (2.45) by

$$\lim_{t\uparrow 1} \limsup_{r\downarrow 0} \limsup_{\kappa\downarrow 0} \limsup_{\eta\downarrow 0} \limsup_{s\downarrow 0} \limsup_{\varepsilon\downarrow 0} \|tu_{\varepsilon,\kappa,r,\eta,s} - u\|_{\mathrm{L}^\infty(O)} = 0.$$

Since $\nabla u_{\varepsilon,\kappa,r,\eta,s} = T'_s(u_{\varepsilon,\kappa,r,\eta} - u_{\kappa,r})\nabla u_{\varepsilon,\kappa,r,\eta} + (1 - T'_s(u_{\varepsilon,\kappa,r,\eta} - u_{\kappa,r}))\nabla u_{\kappa,r}$, we deduce by convexity, noting that T'_s takes values in $[0,1]$,

$$\begin{aligned}
J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta,s},\omega;O) &\le \int_O T'_s(u_{\varepsilon,\kappa,r,\eta}(y) - u_{\kappa,r}(y))V(y/\varepsilon, t\nabla u_{\varepsilon,\kappa,r,\eta},\omega)dy \\
&\qquad + \int_O (1 - T'_s(u_{\varepsilon,\kappa,r,\eta}(y) - u_{\kappa,r}(y)))V(y/\varepsilon, t\nabla u_{\kappa,r},\omega)dy \\
&\le J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta},\omega;O) + |\{y \in O : |u_{\varepsilon,\kappa,r,\eta}(y) - u_{\kappa,r}(y)| > s\}| \max_{1\le l\le k} M(t\Lambda_l) \\
&\le J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta},\omega;O) + s^{-p}\|u_{\varepsilon,\kappa,r,\eta} - u_{\kappa,r}\|^p_{\mathrm{L}^p(O)} \left((1-t)M(0) + \max_{1\le l\le k} M(\Lambda_l)\right).
\end{aligned}$$

Since by definition (and by the Rellich-Kondrachov theorem) we have $u_{\varepsilon,\kappa,r,\eta} \to u_{\kappa,r}$ (strongly) in $\mathrm{L}^p(O)$ as $\varepsilon \downarrow 0$, since the Λ_l's all belong to $\mathrm{dom}\overline{V}$, and since by assumption $\mathrm{dom}M = \mathrm{dom}\overline{V}$, we deduce, combining this with (2.48), that

$$\lim_{t\uparrow 1} \limsup_{r\downarrow 0} \limsup_{\kappa\downarrow 0} \limsup_{\eta\downarrow 0} \limsup_{s\downarrow 0} \limsup_{\varepsilon\downarrow 0} J_\varepsilon(tu_{\varepsilon,\kappa,r,\eta,s},\omega;O) \le J(u;O).$$

The rest of the proof is unchanged. □

2.4.2 Minimal soft buffer zone for Dirichlet boundary data

In this section, we prove Corollary 2.1.8. In view of the error term (2.51) in the proof of Corollary 2.1.4, it seems that the speed of convergence of η to 0 with respect to ε must depend quantitatively on the speed of convergence of w_ε to 0 in $\mathrm{L}^\infty(O;\mathbb{R}^m)$. In the case when the target function is affine ($\Lambda \cdot x$, say), then $w_\varepsilon := \varepsilon\varphi_\Lambda(\cdot/\varepsilon,\cdot)$ is the rescaling of the corrector and its convergence to zero is strictly related to the sublinearity of φ_Λ at infinity (cf. Lemma 2.2.4). Even in the linear scalar case when $V(y,\Lambda) = \Lambda \cdot A(y)\Lambda$ for some matrix-valued random field A, this sublinear growth can be arbitrary: indeed, it follows from [203] that for all $\gamma < 1$ there exists a (strongly correlated) random field A such that $\mathbb{E}\big[|\varphi_\Lambda(x)|^2\big]^{1/2} \simeq |x|^\gamma$ as $|x| \gg 1$. Yet, if instead of using the corrector φ_Λ itself — which is in general not stationary, nor well-behaved — we may use a stationary proxy, then the size of the buffer zone can be (optimally) reduced, at least for affine target functions, as the following proposition shows.

Proposition 2.4.1. *If for all $\Lambda \in \mathbb{R}^{m\times d}$ we have*

$$\overline{V}(\Lambda) = \inf_{\phi\in W^{1,p}(\Omega;\mathbb{R}^m)} \mathbb{E}[V(0,\Lambda+\nabla\phi,\cdot)], \tag{2.79}$$

then the conclusion of Corollary 2.1.8 holds (and we can further replace $\theta\varepsilon$ by any sequence $\eta_\varepsilon \downarrow 0$ satisfying $\liminf_\varepsilon \eta_\varepsilon/\varepsilon > 0$). ◊

Identity (2.79) is essentially a regularity statement on quasi-minimizers of $f \mapsto \mathbb{E}[V(0,\Lambda+f,\cdot)]$ on $\mathrm{L}^p_{\mathrm{pot}}(\Omega)^m$. By Poincaré's inequality, periodic gradients with mean-value zero are gradients of periodic functions, and hence in that case the space $F^p_{\mathrm{pot}}(\Omega)$ coincide with $\{\nabla\phi : \phi \in W^{1,p}(\Omega)^m\}$, so that (2.79) is trivially satisfied. This already proves Corollary 2.1.8 under assumption (1).

On the other hand, the following result shows that (2.79) is also satisfied in the scalar case $m = 1$ if the domain of V is fixed, in which case truncations are available. This proves Corollary 2.1.8 under the additional assumption (2).

Lemma 2.4.2. *If $m = 1$ and if $\mathrm{dom}V(y,\cdot,\omega) = \mathrm{dom}M$ is open for almost all y,ω, then assumption (2.79) holds true for all $\Lambda \in \mathbb{R}^{m\times d}$.* ◊

We start with the proof of Proposition 2.4.1.

Proof of Proposition 2.4.1. For all $\Lambda \in \mathbb{R}^{m\times d}$, by assumption (2.79), there exists for all $\delta > 0$ a stationary random field $\varphi_{\Lambda,\delta} \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\mathbb{E}[V(0,\Lambda + \nabla\varphi_{\Lambda,\delta}(0,\cdot),\cdot)] \leq \overline{V}(\Lambda) + \delta$. Set $u_\varepsilon^{\Lambda,\delta,\omega} := \varepsilon\varphi_{\Lambda,\delta}(\cdot/\varepsilon,\omega)$. By stationarity, for almost all ω, the Birkhoff-Khinchin ergodic theorem asserts that, for any bounded domain $O \subset \mathbb{R}^d$,

$$\lim_{\varepsilon\downarrow 0}\int_O |u_\varepsilon^{\Lambda,\delta,\omega}/\varepsilon|^p = |O|\mathbb{E}[|\varphi_{\Lambda,\delta}|^p], \qquad \lim_{\varepsilon\downarrow 0}\int_O |\nabla u_\varepsilon^{\Lambda,\delta,\omega}|^p = |O|\mathbb{E}[|\nabla\varphi_{\Lambda,\delta}|^p]. \tag{2.80}$$

Let $O \subset \mathbb{R}^d$ be some fixed bounded domain. For $\eta > 0$, set $O_\eta := \{x \in O : d(x,\partial O) > \eta\}$. For any sequence $\eta_\varepsilon \downarrow 0$, (2.80) yields

$$\lim_{\varepsilon\downarrow 0}\int_{O\setminus O_{\eta_\varepsilon}} |u_\varepsilon^{\Lambda,\delta,\omega}/\varepsilon|^p = 0 = \lim_{\varepsilon\downarrow 0}\int_{O\setminus O_{\eta_\varepsilon}} |\nabla u_\varepsilon^{\Lambda,\delta,\omega}|^p. \tag{2.81}$$

Fix such a sequence $\eta_\varepsilon \downarrow 0$. As in Step 1 of the proof of Proposition 2.2.10, for all $\Lambda \in \mathbb{R}^{m\times d}$, we obtain the following, for some subset $\Omega_\Lambda \in \mathcal{F}$ of full probability: for all $\omega \in \Omega_\Lambda$ and all $\delta > 0$, there is a sequence $(u_\varepsilon^{\Lambda,\delta,\omega})_\varepsilon \subset W^{1,p}(O;\mathbb{R}^m)$ such that $u_\varepsilon^{\Lambda,\delta,\omega} \rightharpoonup 0$ in $W^{1,p}(O;\mathbb{R}^m)$ as $\varepsilon \downarrow 0$, $\limsup_\varepsilon J_\varepsilon(\Lambda\cdot x + u_\varepsilon^{\Lambda,\delta,\omega},\omega;O) \leq J(\Lambda\cdot x;O) + \delta$, and such that (2.81) is satisfied.

Let Λ and $\omega \in \Omega_\Lambda$ be fixed, and let $(u_\varepsilon^{\Lambda,\delta,\omega})_\varepsilon$ be as above. For all $\varepsilon > 0$, choose a smooth cut-off function χ_ε with values in $[0,1]$, equal to 1 on $O_{\eta_\varepsilon} = \{x \in O : d(x,\partial O) > \eta_\varepsilon\}$, vanishing outside O, and with $|\nabla\chi_\varepsilon| \leq C'/\eta_\varepsilon$ for some constant C'. Defining $v_\varepsilon^{\Lambda,\delta,\omega} := \chi_\varepsilon u_\varepsilon^{\Lambda,\delta,\omega} \in W_0^{1,p}(O;\mathbb{R}^m)$, we obtain

$$\begin{aligned} J_\varepsilon^{\eta_\varepsilon}(\Lambda\cdot x + v_\varepsilon^{\Lambda,\delta,\omega},\omega;O) &= J_\varepsilon(\Lambda\cdot x + v_\varepsilon^{\Lambda,\delta,\omega},\omega;O_{\eta_\varepsilon}) + \int_{O\setminus O_{\eta_\varepsilon}} |\Lambda + \nabla v_\varepsilon^{\Lambda,\delta,\omega}|^p \\ &\leq J_\varepsilon(\Lambda\cdot x + v_\varepsilon^{\Lambda,\delta,\omega},\omega;O) + 3^{p-1}|\Lambda|^p|O\setminus O_{\eta_\varepsilon}| \\ &\qquad + 3^{p-1}\int_{O\setminus O_{\eta_\varepsilon}} (|\nabla u_\varepsilon^{\Lambda,\delta,\omega}|^p + |C'u_\varepsilon^{\Lambda,\delta,\omega}/\eta_\varepsilon|^p), \end{aligned}$$

and hence, if the sequence $\eta_\varepsilon \downarrow 0$ is further chosen such that $\liminf_\varepsilon \eta_\varepsilon/\varepsilon > 0$,

$$\limsup_{\varepsilon\downarrow 0} J_\varepsilon^{\eta_\varepsilon}(\Lambda\cdot x + v_\varepsilon^{\Lambda,\delta,\omega},\omega;O) \leq \limsup_{\varepsilon\downarrow 0} J_\varepsilon(\Lambda\cdot x + v_\varepsilon^{\Lambda,\delta,\omega},\omega;O) \leq J(\Lambda\cdot x;O) + \delta.$$

Therefore, $\limsup_\delta\limsup_\varepsilon J_\varepsilon^{\eta_\varepsilon}(\Lambda\cdot x + v_\varepsilon^{\Lambda,\delta,\omega},\omega;O) \leq J(\Lambda\cdot x;O)$. Combined with the Γ-$\liminf$ inequality of Proposition 2.2.2 and a diagonalization argument, this proves the result. ☐

We now establish Lemma 2.4.2.

Proof of Lemma 2.4.2. We split the proof into two steps.

Step 1. Preliminary.

We claim that it suffices to prove that, for all $\Lambda \in \mathrm{dom}\overline{V}$,

$$\limsup_{t\uparrow 1}\inf_{\phi\in W^{1,p}(\Omega)} \mathbb{E}[V(0,t\Lambda + \nabla\phi,\cdot)] \leq \overline{V}(\Lambda). \tag{2.82}$$

Define $\overline{V}'(\Lambda) := \inf_{\phi\in W^{1,p}(\Omega)}\mathbb{E}[V(0,\Lambda + \nabla\phi,\cdot)]$. By definition $\overline{V}'(\Lambda) \geq \overline{V}(\Lambda)$ for all Λ, and hence property (2.82) together with the lower semicontinuity of $\overline{V}$ directly yields $\overline{V}(\Lambda) = \lim_{t\uparrow 1}\overline{V}'(t\Lambda)$ for all Λ (and in particular the limit exists). Since $\overline{V}'$ is obviously convex, it is continuous on the interior of its domain. Since the domain is assumed to be open, this yields $\overline{V}(\Lambda) = \overline{V}'(\Lambda)$ for all Λ.

Step 2. Proof of (2.82).

Let $\Lambda \in \mathrm{dom}\overline{V}$ be fixed. Lemma 2.2.4 gives a measurable corrector $u := \varphi_\Lambda \in \mathrm{Mes}(\Omega; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d))$ such that $\nabla u \in F^p_{\mathrm{pot}}(\Omega)$ and $\overline{V}(\Lambda) = \mathbb{E}[V(0, \Lambda + \nabla u(0,\cdot),\cdot)]$. For all $R > r > 0$, choose a smooth cut-off function $\chi_{R,r}$ taking values in $[0,1]$, equal to 1 on Q_{R-r}, vanishing outside Q_R and satisfying $|\nabla \chi_{R,r}| \le 2/r$. Also recall the definition (2.77) of the truncation T_s. We then set

$$u_R(x,\omega) = u(x,\omega) - \fint_{Q_R} u(\cdot,\omega), \qquad v^s_{R,r}(x,\omega) = \chi_{R,r}(x)\, T_s u_R(x,\omega),$$

and

$$w^s_{R,r}(x,\omega) = \frac{1}{|Q_R|}\int_{\mathbb{R}^d} v^s_{R,r}(x+y, \tau_y\omega)dy = \fint_{-x+Q_R} v^s_{R,r}(x+y,\tau_y\omega)dy.$$

Clearly, $w^s_{R,r}$ is well-defined, stationary, and belongs to $W^{1,p}(\Omega)$, with

$$\nabla w^s_{R,r}(x,\omega) = \fint_{-x+Q_R} \nabla v^s_{R,r}(x+y,\tau_y\omega)dy.$$

Let $t \in [0,1)$. By Jensen's inequality,

$$\begin{aligned} K^s_{R,r}(t) := \mathbb{E}[V(0, t\Lambda + t\nabla w^s_{R,r}(0,\cdot),\cdot)] &= \mathbb{E}\left[V\left(0, t\Lambda + t\fint_{Q_R} \nabla v^s_{R,r}(y,\tau_y\cdot)dy, \cdot\right)\right] \\ &\le \mathbb{E}\left[\fint_{Q_R} V(0, t\Lambda + t\nabla v^s_{R,r}(y,\tau_y\cdot),\cdot)dy\right], \end{aligned}$$

and hence, by stationarity and the Fubini theorem,

$$K^s_{R,r}(t) \le \mathbb{E}\left[\fint_{Q_R} V(y, t\Lambda + t\nabla v^s_{R,r}(y,\cdot),\cdot)dy\right].$$

Decomposing

$$\begin{aligned} t\Lambda + t\nabla v^s_{R,r}(y,\omega) &= t\Lambda + t\chi_{R,r}(y)\nabla T_s u_R(y,\omega) + (1-t)\frac{t}{1-t}\nabla\chi_{R,r}(y) T_s u_R(y,\omega) \\ &= t\chi_{R,r}(y) T_s'(u_R(y,\omega))(\Lambda + \nabla u(y,\omega)) + t(1-\chi_{R,r}(y))T_s'(u_R(y,\omega))\Lambda \\ &\qquad + t(1 - T_s'(u_R(y,\omega)))\Lambda + (1-t)\frac{t}{1-t}\nabla\chi_{R,r}(y)T_s u_R(y,\omega), \end{aligned}$$

with T_s' taking values in $[0,1]$, we may then bound by convexity

$$\begin{aligned} K^s_{R,r}(t) &\le \mathbb{E}\left[\fint_{Q_R} V(y, \Lambda + \nabla u(y,\cdot),\cdot)dy\right] + (1-t)E^s_{R,r}(t) \\ &\qquad + M(\Lambda)\fint_{Q_R}(1-\chi_{R,r}) + M(\Lambda)\mathbb{E}\left[\fint_{Q_R}(1 - T_s'(u_R(y,\cdot)))\right], \end{aligned} \tag{2.83}$$

where the error term reads

$$E^s_{R,r}(t) = \mathbb{E}\left[\fint_{Q_R} M\left(\frac{t}{1-t}\nabla\chi_{R,r}(y)T_s u_R(y,\omega)\right)dy\right].$$

By stationarity of ∇u, note that

$$\mathbb{E}\left[\fint_{Q_R} V(y,\Lambda + \nabla u(y,\cdot),\cdot)dy\right] = \mathbb{E}[V(0,\Lambda + \nabla u(0,\cdot),\cdot)] = \overline{V}(\Lambda). \tag{2.84}$$

For the error term, note that

$$\left\| \frac{t}{1-t} \nabla \chi_{R,r}(\cdot) T_s u_R(\cdot,\omega) \right\|_{\mathrm{L}^\infty(O)} \le \frac{2t}{1-t}\frac{s}{r}. \tag{2.85}$$

It remains to treat the last two terms of (2.83). Noting that $⨍_{Q_R}(1-\chi_{R,r}) = R^{-d}(R^d-(R-r)^d) \le dr/R$, we obtain

$$\begin{aligned} ⨍_{Q_R}(1-\chi_{R,r}) + \mathbb{E}\left[⨍_{Q_R}(1-T_s'(u_R(y,\cdot)))\right] &\le \frac{dr}{R} + \mathbb{E}\left[⨍_{Q_R} \mathbb{1}_{|u_R(y)|\ge s} dy\right] \\ &\le \frac{dr}{R} + \int_Q \mathbb{P}\left[\frac{1}{R}\Big|u(Ry,\cdot) - ⨍_Q u(Rz,\cdot)dz\Big| \ge \frac{s}{R}\right] dy. \end{aligned} \tag{2.86}$$

Lemma 2.2.4 (together with the Rellich-Kondrachov theorem) gives $\frac{1}{R}|u(R\cdot,\omega) - ⨍_Q u(Rz,\omega)dz| \to 0$ (strongly) in $\mathrm{L}^p(Q)$ as $R\uparrow\infty$, for almost all ω. Hence up to an extraction in R (implicit in the sequel) we deduce that, for almost all $y\in Q$, $\frac{1}{R}|u(Ry,\cdot) - ⨍_Q u(Rz,\cdot)dz| \to 0$ almost surely. Since almost sure convergence implies convergence in probability, we deduce by dominated convergence, for all $\varepsilon>0$,

$$\lim_{R\uparrow\infty} \int_Q \mathbb{P}\left[\frac{1}{R}\Big|u(Ry,\cdot) - ⨍_Q u(Rz,\cdot)dz\Big| \ge \varepsilon\right] dy = 0.$$

A diagonalization argument then gives a sequence $\varepsilon_R \downarrow 0$ such that

$$\lim_{R\uparrow\infty} \int_Q \mathbb{P}\left[\frac{1}{R}\Big|u(Ry,\cdot) - ⨍_Q u(Rz,\cdot)dz\Big| \ge \varepsilon_R\right] dy = 0. \tag{2.87}$$

Choose $s = s_R := R\varepsilon_R$ and $r = r_R := R\sqrt{\varepsilon_R}$. By assumption, there exists some $\delta>0$ with $\mathrm{adh}B_\delta \subset \mathrm{int}\,\mathrm{dom}M$. By (2.85), for all $t\in[0,1)$, there is some $R_t>0$ such that for all $R>R_t$

$$\left\| \frac{t}{1-t} \nabla \chi_{R,r_R}(\cdot) T_{s_R} u_R(\cdot,\omega) \right\|_{\mathrm{L}^\infty(O)} \le \frac{2t}{1-t}\sqrt{\varepsilon_R} < \delta. \tag{2.88}$$

Combining this with (2.83), (2.84), (2.86), (2.87), and noting that $M(\Lambda)<\infty$ follows from the choice $\Lambda \in \mathrm{dom}\overline{V}$, we obtain

$$\limsup_{t\uparrow 1} \limsup_{R\uparrow\infty} K^{s_R}_{R,r_R}(t) \le \overline{V}(\Lambda),$$

and the result (2.82) follows. □

2.A Appendix: Some technical results

2.A.1 Normal random integrands

In this appendix, we briefly recall the precise definition of normal random integrands (as defined e.g. in [177, Section VIII.1.3]) and we prove their main properties, mentioned in Section 2.1.2 and used throughout this chapter. Let $(\Omega,\mathcal{F},P)$ be a complete probability space. Recall that we denote by $\mathcal{B}(\mathbb{R}^k)$ the (not completed!) Borel σ-algebra on $\mathbb{R}^k$.

Definition 2.A.1. A *normal random integrand* is a map $W:\mathbb{R}^d\times\mathbb{R}^{m\times d}\times\Omega\to[0,\infty]$ such that

(a) W is jointly measurable (i.e. with respect to the completion of $\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^{m\times d})\times\mathcal{F}$);

(b) for almost all ω, there exists a map $V_\omega:\mathbb{R}^d\times\mathbb{R}^{m\times d}\to[0,\infty]$ that is $\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^{m\times d})$-measurable and such that $W(y,\cdot,\omega) = V_\omega(y,\cdot)$ for almost all y;

(c) for almost all y, there exists a map $V_y : \mathbb{R}^{m\times d} \times \Omega \to [0,\infty]$ that is $\mathcal{B}(\mathbb{R}^{m\times d}) \times \mathcal{F}$-measurable and such that $W(y,\cdot,\omega) = V_y(\cdot,\omega)$ for almost all ω;

(d) for almost all y, ω, the map $W(y,\cdot,\omega)$ is lower semicontinuous on $\mathbb{R}^{m\times d}$.

It is said to be *τ-stationary* if it satisfies (2.4) for all Λ, y, z, ω. ◊

As shown e.g. in [177, Section VIII.1.3], a simple example of normal random integrands is given by the so-called *Carathéodory random integrands*, that are maps $W : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0,\infty]$ such that $W(y,\cdot,\omega)$ is continuous on $\mathbb{R}^{m\times d}$ for almost all y, ω, and such that $W(\cdot,\Lambda,\cdot)$ is jointly measurable on $\mathbb{R}^d \times \Omega$ for all Λ.

As already advertised in Section 2.1.2, the reason for these technical assumptions is that they are particularly weak but still guarantee the following key properties.

Lemma 2.A.2. *Let $W : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0,\infty]$ be a normal random integrand. Then,*

(i) for almost all ω, the map $y \mapsto W(y,u(y),\omega)$ is measurable for all $u \in \mathrm{Mes}(\mathbb{R}^d, \mathbb{R}^{m\times d})$;

(ii) for almost all y, the map $\omega \mapsto W(y,u(\omega),\omega)$ is measurable for all $u \in \mathrm{Mes}(\Omega, \mathbb{R}^{m\times d})$. ◊

Proof. For almost all ω, part (b) of Definition 2.A.1 gives a $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{m\times d})$-measurable map V_ω on $\mathbb{R}^d \times \mathbb{R}^{m\times d}$ such that $W(y,\cdot,\omega) = V_\omega(y,\cdot)$ for almost all y. Hence, for $u \in \mathrm{Mes}(\mathbb{R}^d; \mathbb{R}^{m\times d})$, the map $y \mapsto W(y,u(y),\omega)$ is equal almost everywhere to the map $y \mapsto V_\omega(y,u(y))$, which is necessarily measurable since $\mathrm{Id} \times u : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^{m\times d}$ is measurable and since V_ω is Borel-measurable. This proves (i), and (ii) is similar. □

If W is τ-stationary, we may write $W(y,\Lambda,\omega) = W(0,\Lambda,\tau_{-y}\omega)$, which thus receives a pointwise meaning in the first variable, and Lemma 2.A.2(ii) may obviously be strengthened as follows.

Lemma 2.A.3. *Let $W : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0,\infty]$ be a τ-stationary normal random integrand. For all y and all $u \in \mathrm{Mes}(\Omega; \mathbb{R}^{m\times d})$, the map $\omega \mapsto W(y,u(\omega),\omega)$ is measurable.* ◊

2.A.2 Stationary differential calculus in probability

In this appendix, we precisely define the notion of measurable action that is used throughout this chapter as well as in the sequel of this thesis to induce the stationarity. We further discuss the properties of the stationary derivatives defined in Section 2.2.1, and prove in particular the useful identity (2.19).

Stationary random fields

As usual, the standard notion of stationarity of random fields (defined as the translation invariance of all the finite-dimensional distributions) is strictly equivalent to a formulation of stationarity as the invariance under some (measure-preserving) action of the group of translations $(\mathbb{R}^d, +)$ on the probability space (see e.g. [272, Section 16.1]). This point of view is of great interest, since it puts us into the realm of ergodic theory.

Because we focus on jointly measurable random fields, which is standard in stochastic homogenization theory (see also Remark 2.A.6 below), a further measurability requirement is added in our definition of an action (similarly as e.g. in [265, Section 7.1]).

Definition 2.A.4. A *measurable action* of the group $(\mathbb{R}^d, +)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection $\tau := (\tau_x)_{x\in\mathbb{R}^d}$ of measurable transformations of Ω such that

(i) $\tau_x \circ \tau_y = \tau_{x+y}$ for all $x, y \in \mathbb{R}^d$;

(ii) $\mathbb{P}[\tau_x A] = \mathbb{P}[A]$ for all $x \in \mathbb{R}^d$ and all $A \in \mathcal{F}$;

(iii) the map $\mathbb{R}^d \times \Omega \to \Omega : (x,\omega) \mapsto \tau_x\omega$ is measurable. ◊

For any random variable $f \in \mathrm{Mes}(\Omega;\mathbb{R})$, we may define its τ-stationary extension $f : \mathbb{R}^d \times \Omega \to \mathbb{R}$ by $f(x,\omega) := f(\tau_{-x}\omega)$, which is a τ-stationary random field on $\mathbb{R}^d$ (in the sense that $f(x+y,\omega) = f(x,\tau_{-y}\omega)$ for all x,y,ω) and which is by definition jointly measurable on $\mathbb{R}^d \times \Omega$. For $f \in \mathrm{L}^p(\Omega)$, $1 \le p < \infty$, the τ-stationary extension f belongs to $\mathrm{L}^p(\Omega;\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d))$. In this way, we get a bijection between the random variables (resp. in $\mathrm{L}^p(\Omega)$) and the τ-stationary random fields (resp. in $\mathrm{L}^p(\Omega;\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d))$).

We may also naturally consider the associated action $T := (T_x)_{x\in\mathbb{R}^d}$ of $(\mathbb{R}^d,+)$ on $\mathrm{Mes}(\Omega;\mathbb{R})$, defined by $(T_x f)(\omega) = f(\tau_{-x}\omega)$ for all $\omega \in \Omega$ and $f \in \mathrm{Mes}(\Omega;\mathbb{R})$. Let $1 \le p < \infty$. The following gives elementary properties of this action (see e.g. [265, Section 7.1]).

Lemma 2.A.5. *The action T defined above is unitary and strongly continuous on $\mathrm{L}^p(\Omega)$.* $\Diamond$

In the context of stochastic homogenization theory, the measurability hypotheses made above (as in e.g. [265, Section 7.1]) are sometimes replaced by stochastic continuity hypotheses (see e.g. [354, Section 2]). As the following remark explains, both are actually equivalent.

Remark 2.A.6 (Measurability or continuity)**.** It should be noted that the additional measurability assumption (iii) in Definition 2.A.4 above is not inoffensive at all. Indeed, a stochastic version of the Lusin theorem can easily be proven: a random field h on $\mathbb{R}^d$ is jointly measurable if and only if for almost all $x \in \mathbb{R}^d$ it satisfies for all $\delta > 0$,

$$\lim_{y\to 0} \mathbb{P}[|h(x+y,\omega) - h(x,\omega)| > \delta] = 0.$$

Hence, for a *stationary* random field h on $\mathbb{R}^d$, joint measurability is actually equivalent to stochastic continuity (and even to continuity in the p-th mean, in the case when $h(0,\cdot) \in \mathrm{L}^p(\Omega)$). In the same vein, the measurability property (iii) in Definition 2.A.4 is equivalent to the strong continuity of the action T of $(\mathbb{R}^d,+)$ on $\mathrm{L}^p(\Omega)$, and also to the property that all τ-stationary extensions are stochastically continuous. $\Diamond$

Stationary Sobolev spaces

Let $1 \le p < \infty$, and let the stationary gradient D and the space $W^{1,p}(\Omega)$ be defined as in Section 2.2.1. Now we present another useful vision for derivatives of stationary random fields.

Given a random variable $f \in \mathrm{L}^p(\Omega)$, the τ-stationary extension is an element $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d;\mathrm{L}^p(\Omega)) = \mathrm{L}^p(\Omega;\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d))$ and can thus be seen as an $\mathrm{L}^p(\Omega)$-valued distribution on $\mathbb{R}^d$. We may then define its distributional gradient ∇f in the usual way. Note that by definition, for almost all ω, $\nabla f(\cdot,\omega)$ is nothing but the usual distributional gradient of $f(\cdot,\omega) \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$. As usual, the Sobolev space $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathrm{L}^p(\Omega))$ is defined as the space of functions $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d;\mathrm{L}^p(\Omega))$ such that $\nabla f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d;\mathrm{L}^p(\Omega;\mathbb{R}^d))$, and in that case ∇f is called the *weak gradient*. The following result shows the link with stationary gradients and with the space $W^{1,p}(\Omega)$, in particular proving identity (2.19).

Lemma 2.A.7. *Modulo the correspondence between random variables and τ-stationary random fields, we have*

$$W^{1,p}(\Omega) = \{f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathrm{L}^p(\Omega)) : f(x+y,\omega) = f(x,\tau_{-y}\omega), \forall x,y,\omega\},$$

and moreover $\nabla f = Df$ for all $f \in W^{1,p}(\Omega)$. $\Diamond$

Proof. Denote for simplicity $E_p := \{f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathrm{L}^p(\Omega)) : f(x+y,\omega) = f(x,\tau_{-y}\omega), \forall x,y,\omega\}$. For all i, the stationary derivative $D_i f$ is defined as the strong derivative of the map $\mathbb{R} \to \mathrm{L}^p(\Omega) : h \mapsto T_{-he_i} f$, so that

$$W^{1,p}(\Omega) = \{f \in C^1(\mathbb{R}^d;\mathrm{L}^p(\Omega)) : f(x+y,\omega) = f(x,\tau_{-y}\omega), \forall x,y,\omega\}.$$

Hence, for all $f \in W^{1,p}(\Omega)$, we have $f \in E_p$, and $\nabla f = Df$, since weak derivatives are generalizations of strong derivatives.

We now turn to the converse statement. Let $f \in W^{1,p}(\Omega)$. As in [265, Section 7.2], choose a nonnegative even function $\rho \in C_c^\infty(\mathbb{R}^d)$ with $\int \rho = 1$ and $\operatorname{supp} \rho \subset B_1$, write $\rho_\delta(x) = \delta^{-d}\rho(x/\delta)$ for all $\delta > 0$, and define a regularization $R_\delta[f] \in \mathrm{L}^p(\Omega)$ by

$$R_\delta[f](\omega) = \int_{\mathbb{R}^d} \rho_\delta(y) f(y,\omega) dy, \tag{2.89}$$

or equivalently, as ρ_δ is even,

$$R_\delta[f](x,\omega) = \int_{\mathbb{R}^d} \rho_\delta(y) f(x+y,\omega) dy = \int_{\mathbb{R}^d} \rho_\delta(y-x) f(y,\omega) dy = (\rho_\delta * f(\cdot,\omega))(x).$$

Clearly, $R_\delta[f] \to f$ in $\mathrm{L}^p(\Omega)$, and hence by stationarity $R_\delta[f](x,\cdot) \to f(x,\cdot)$ in $\mathrm{L}^p(\Omega)$ uniformly in x. As by definition $R_\delta[f] \in C^\infty(\mathbb{R}^d;\mathrm{L}^p(\Omega))$, we have $R_\delta[f] \in W^{1,p}(\Omega)$ and the stationary gradient is simply $DR_\delta[f] = R_\delta[\nabla f]$. This proves $DR_\delta[f] \to \nabla f$ in $\mathrm{L}^p(\Omega;\mathbb{R}^d)$, and thus $DR_\delta[f](x,\cdot) \to \nabla f(x,\cdot)$ in $\mathrm{L}^p(\Omega;\mathbb{R}^d)$ uniformly in x. Hence $R_\delta[f] \to f$ in $C^1(\mathbb{R}^d;\mathrm{L}^p(\Omega))$, so $f \in C^1(\mathbb{R}^d;\mathrm{L}^p(\Omega))$, from which we conclude $f \in W^{1,p}(\Omega)$. ☐

2.A.3 Measurability results

This appendix is concerned with various measurability properties.

Measurable potentials for random fields

The following result complements the equivalent definition (2.21) of $\mathrm{L}^p_{\mathrm{pot}}(\Omega)$, and shows that potentials associated with potential random fields may be chosen in a measurable way with respect to the alea.

Proposition 2.A.8. *Let τ be an ergodic measurable action on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, and let $1 < p < \infty$. For all $f \in \mathrm{L}^p_{\mathrm{pot}}(\Omega)$ there exists a random field $\phi \in \mathrm{Mes}(\Omega; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d))$ such that $f(\cdot,\omega) = \nabla\phi(\cdot,\omega)$ for almost all ω.* ◊

Proof. Let $f \in \mathrm{L}^p_{\mathrm{pot}}(\Omega)$ be fixed. For all $n,k \ge 1$, define the space

$$X_{n,k} := \Big\{\phi \in W^{1,p}(B_n) : \|\nabla\phi\|_{\mathrm{L}^p(B_n)} \le k,\ \int_{B_n}\phi = 0\Big\},$$

endowed with the weak topology. By Poincaré's inequality and by the Banach-Alaoglu theorem, this space is metrizable and compact, hence Polish. Consider the multifunction $\Gamma_{n,k} : \Omega \rightrightarrows X_{n,k}$ defined by

$$\Gamma_{n,k}(\omega) := \{\phi \in X_{n,k} \,:\, \nabla\phi|_{B_n} = f(\cdot,\omega)|_{B_n}\}.$$

Clearly $\Gamma_{n,k}(\omega)$ is closed for all ω. We first prove further properties of this multifunction, and the conclusion will then follow by applying the Rokhlin–Kuratowski–Ryll Nardzewski theorem [371, 278].

Step 1. For all $n \ge 1$, we claim the existence of an increasing sequence of events $\Omega_{n,k} \subset \Omega$ such that $\mathbb{P}[\Omega_{n,k}] \uparrow 1$ as $k \uparrow \infty$ for fixed n, and such that $\Gamma_{n,k}(\omega) \ne \varnothing$ for all $\omega \in \Omega_{n,k}$.

By Definition 2.21, there is a set $\Omega' \in \mathcal{F}$ of full probability, such that for all $\omega \in \Omega'$ the function $f(\cdot,\omega)$ is a potential field in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, and hence there exists $\phi^\omega \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ such that $f(\cdot,\omega) = \nabla\phi^\omega$. For all $n \ge 1$, define $\phi^\omega_n := \phi^\omega - \fint_{B_n}\phi^\omega(z)dz \in W^{1,p}(B_n)$. By definition, $\int_{B_n}\phi^\omega_n = 0$ and $\nabla\phi^\omega_n = f(\cdot,\omega)$ on B_n. Moreover, $\|\nabla\phi^\omega_n\|_{\mathrm{L}^p(B_n)} \le M$ holds for all $\omega \in \Omega_{n,k}$, where we define the event

$$\Omega_{n,k} := \{\omega \in \Omega' : \|f(\cdot,\omega)\|_{\mathrm{L}^p(B_n)} \le k\}.$$

Integrability and stationarity of f easily imply that $\mathbb{P}[\Omega_{n,k}] \uparrow 1$ as $k \uparrow \infty$.

Step 2. Proof that $\Gamma_{n,k}$ is measurable, in the sense that $\Gamma_{n,k}^{-1}(O) \in \mathcal{F}$ for all open subset $O \subset X_{n,k}$, where we have set

$$\Gamma_{n,k}^{-1}(O) := \{\omega \in \Omega \,:\, \Gamma_{n,k}(\omega) \cap O \neq \varnothing\}.$$

As $X_{n,k}$ is metrizable, it suffices to check $\Gamma_{n,k}^{-1}(F) \in \mathcal{F}$ for all closed subset $F \subset X_{n,k}$ (see e.g. [10, Lemma 18.2]). Given a closed subset $F \subset X_{n,k}$, we may write, using Poincaré's inequality and the weak lower semicontinuity of the norm,

$$\begin{aligned}\Gamma_{n,k}^{-1}(F) &= \{\omega \in \Omega : \exists \phi \in F, \nabla\phi = f(\cdot,\omega)|_{B_n}\} \\ &= \bigcap_{j=1}^{\infty} \{\omega \in \Omega : \exists \phi \in F, \|\nabla\phi - f(\cdot,\omega)\|_{\mathrm{L}^p(B_n)} \leq 1/j\}.\end{aligned}$$

Separability of the Polish space $X_{n,k}$ implies that F is itself separable, and there exists a countable dense subset $F_0 \subset F$. Hence

$$\Gamma_{n,k}^{-1}(F) = \bigcap_{j=1}^{\infty} \bigcup_{\phi \in F_0} \{\omega \in \Omega : \|\nabla\phi - f(\cdot,\omega)\|_{\mathrm{L}^p(B_n)} \leq 1/j\},$$

and measurability of $\Gamma_{n,k}^{-1}(F)$ then follows from measurability of f.

Step 3. Conclusion.

By steps 1 and 2, for all $n, k \geq 1$, the restricted multifunction $\Gamma_{n,k}|_{\Omega_{n,k}} : \Omega_{n,k} \rightrightarrows X_{n,k}$ is measurable and has nonempty closed values. As $X_{n,k}$ is a Polish space, we may apply the Rokhlin–Kuratowski–Ryll Nardzewski theorem (see e.g. [10, Theorem 18.13]), which gives a measurable function $\phi_{n,k} : \Omega_{n,k} \to X_{n,k}$ such that $\phi_{n,k}(\omega) \in \Gamma_{n,k}(\omega)$, that is $\nabla\phi_{n,k}(\cdot,\omega) = f(\cdot,\omega)|_{B_n}$ for all $\omega \in \Omega_{n,k}$. For all $n > 0$, define a measurable function $\phi_n : \Omega \to W^{1,p}(B_n)$ by

$$\phi_n(\omega) = \mathbb{1}_{\Omega_1}(\omega)\phi_{1,n}(\omega) + \sum_{k=2}^{\infty} \mathbb{1}_{\Omega_{n,k}\setminus\Omega_{n,k-1}}(\omega)\phi_{n,k}(\omega).$$

By definition, we have $\nabla\phi_n(\cdot,\omega) = f(\cdot,\omega)|_{B_n}$ for all $\omega \in \Omega_n$, where $\Omega_n := \bigcup_{k=1}^{\infty} \Omega_{n,k} \in \mathcal{F}$ is a subset of full probability. Denote $\Omega'' := \bigcap_{n=1}^{\infty} \Omega_n$.

Let $n \geq 1$. By definition, $\nabla\phi_n - \nabla\phi_1$ vanishes on B_1, hence the difference $\delta_n(\omega) := \phi_n(\cdot,\omega) - \phi_1(\cdot,\omega)$ is constant on B_1 for all $\omega \in \Omega''$ and defines a measurable function $\delta_n : \Omega'' \to \mathbb{R}$. Then consider the measurable function $\psi_n : \Omega \to W^{1,p}(B_n)$ defined by $\psi_n(x,\omega) := \phi_n(x,\omega) - \delta_n(\omega)$. By construction, for all $m > n \geq 1$, we have $\psi_n = \psi_m$ on B_n, so the ψ_n's can be glued together and yield a measurable function $\psi : \Omega'' \to W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\nabla\psi(\cdot,\omega) = f(\cdot,\omega)$ for all $\omega \in \Omega''$. ☐

Sufficient conditions for Hypothesis 2.1.1

As shown below, the measurability Hypothesis 2.1.1 is automatically satisfied if the integrand is quasiconvex and satisfies following useful approximation property.

Definition 2.A.9. A normal random integrand $W : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0,\infty]$ is said to be *quasiconvex* if $W(y,\cdot,\omega)$ is quasiconvex for almost all y,ω. Given $p \geq 1$, it is further said to be *p-sup-quasiconvex* if there exists a sequence $(W_k)_k$ of quasiconvex normal random integrands such that $W_k(y,\Lambda,\omega) \uparrow W(y,\Lambda,\omega)$ pointwise as $k \uparrow \infty$, and such that, for all k, for almost all y,ω, and for all Λ,Λ',

$$\frac{1}{C}|\Lambda|^p - C \leq W_k(y,\Lambda,\omega) \leq C_k(1 + |\Lambda|^p), \tag{2.90}$$

for some constants $C, C_k > 0$. ◊

Note that Tartar [409] has proven the existence of quasiconvex functions that are not p-sup-quasiconvex for any $p \geq 1$. Before stating our measurability result, let us examine some important particular cases.

Lemma 2.A.10. *Let $W : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0,\infty]$ be a normal random integrand. Assume that there exist $C > 0$ and $p > 1$ such that, for almost all ω, y, for all Λ,*

$$\frac{1}{C}|\Lambda|^p - C \leq W(y,\Lambda,\omega). \tag{2.91}$$

Also assume that one of the following holds:

(1) $W = V$ is a convex normal random integrand, and the convex function $M := \sup\mathrm{ess}_{y,\omega} V(y,\cdot,\omega)$ has 0 in the interior of its domain;

(2) W satisfies Hypothesis 2.1.9, with a convex part V such that $M := \sup\mathrm{ess}_{y,\omega} V(y,\cdot,\omega)$ has 0 in the interior of its domain.

Then, W is a p-sup-quasiconvex normal random integrand. ◊

Proof. Case (2) directly follows from the approximation result of case (1) applied to the convex part V. We may thus focus on case (1). Let $W = V$ be a convex normal random integrand. For all $k \geq 0$, consider the following Yosida transform,

$$V_k(y,\Lambda,\omega) = \inf_{\Lambda'\in\mathbb{R}^{m\times d}} \left(V(y,\Lambda',\omega) + k|\Lambda-\Lambda'|^p\right). \tag{2.92}$$

For almost all y,ω, convexity of $V_k(y,\cdot,\omega)$ easily follows from convexity of $V(y,\cdot,\omega)$. For almost all y,ω, the lower semicontinuity of $V(y,\cdot,\omega)$ ensures that $V_k(y,\cdot,\omega) \uparrow V(y,\cdot,\omega)$ pointwise as $k \uparrow \infty$. Moreover we have by definition $V_k(y,\Lambda,\omega) \leq M(0) + k|\Lambda|^p$, while the lower bound (2.91) implies $V_k(y,\Lambda,\omega) \geq \frac{1}{C}|\Lambda|^p - C$.

It remains to check that the V_k's are normal random integrands. For almost all y,ω, the function $V(y,\cdot,\omega)$ is convex and lower semicontinuous, hence it is continuous on its domain $D_{y,\omega}$ (not only on the interior). As by assumption $0 \in \mathrm{int}\, D_{y,\omega}$, the set $D_{y,\omega}$ is a convex subset of maximal dimension, and hence points with rational coordinates are dense in $D_{y,\omega}$. The infimum (2.92) defining V_k may thus be restricted to $\mathbb{Q}^{m\times d}$. As a countable infimum, the required measurability properties follow. □

We now turn to the validity of the measurability Hypothesis 2.1.1 for p-sup-quasiconvex integrands.

Proposition 2.A.11. *Let $O \subset \mathbb{R}^d$ be a bounded domain, let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space, and let $W : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0,\infty]$ be a p-sup-quasiconvex normal random integrand for some $p > 1$ (in the sense of Definition 2.A.9). Given some fixed function $f \in \mathrm{L}^p(\Omega;\mathrm{L}^p(O;\mathbb{R}^{m\times d}))$, consider the random integral functional $I : W^{1,p}(O;\mathbb{R}^m) \times \Omega \to [0,\infty]$ defined by*

$$I(u,\omega) = \int_O W(y, f(y,\omega) + \nabla u(y), \omega)dy. \tag{2.93}$$

Then, I is weakly lower semicontinuous on $W^{1,p}(O;\mathbb{R}^m)$. Moreover, for all weakly closed subsets $F \subset W^{1,p}_0(O;\mathbb{R}^m)$ or $F \subset \{u \in W^{1,p}(O;\mathbb{R}^m) : \int_O u = 0\}$, the function $\omega \mapsto \inf_{v\in F} I(v,\omega)$ is $\mathcal{F}$-measurable. In particular, Hypothesis 2.1.1 is satisfied. ◊

Proof. For all k, define the approximated random functional $I_k : W^{1,p}(O;\mathbb{R}^m) \times \Omega \to [0,\infty]$ by

$$I_k(u,\omega) = \int_O W_k(y, f(y,\omega) + \nabla u(y), \omega)dy.$$

As the W_k's are nonnegative, monotone convergence yields that $I_k \uparrow I$ pointwise. Moreover, for all k, and almost all ω, the quasiconvexity and the upper bound (2.90) satisfied by $W_k(\cdot,\cdot,\omega)$ imply the weak lower semicontinuity of $I_k(\cdot,\omega)$ on $W^{1,p}(O;\mathbb{R}^m)$ (see [3]). As a pointwise supremum of weakly lower semicontinuous functions, we deduce that $I(\cdot,\omega)$ is itself weakly lower semicontinuous on $W^{1,p}(O;\mathbb{R}^m)$.

Combining the weak lower semicontinuity of I_k with the uniform coercivity assumption (cf. lower bound in (2.90)) and with Poincaré's inequality, we easily conclude, for any weakly closed subset $F \subset W_0^{1,p}(O;\mathbb{R}^m)$ or $F \subset \{u \in W^{1,p}(O;\mathbb{R}^m) : \fint_O u = 0\}$,

$$\lim_{k\uparrow\infty} \inf_{v\in F} I_k(v,\omega) = \inf_{v\in F} I(v,\omega). \tag{2.94}$$

For all k, as W_k is quasiconvex hence rank-1 convex in its second variable (see [43]), the p-growth condition (2.90) implies the following local Lipschitz condition: for almost all y,ω, for all Λ,Λ',

$$|W_k(y,\Lambda,\omega) - W_k(y,\Lambda',\omega)| \le C_k|\Lambda-\Lambda'|(1+|\Lambda|^{p-1}+|\Lambda'|^{p-1}),$$

for some constant $C_k > 0$. The Hölder inequality then gives, for all u,v, for almost all ω,

$$|I_k(u,\omega) - I_k(v,\omega)| \le C_k C\|\nabla(u-v)\|_{\mathrm{L}^p(O)}(1+\|\nabla u\|_{\mathrm{L}^p(O)}^{p-1}+\|\nabla v\|_{\mathrm{L}^p(O)}^{p-1}+\|f(\cdot,\omega)\|_{\mathrm{L}^p(O)}^{p-1}).$$

This proves that the map $I_k(\cdot,\omega)$ is strongly continuous on $W^{1,p}(O;\mathbb{R}^m)$, for almost all ω. Given a weakly (hence strongly) closed subset $F \subset W_0^{1,p}(O;\mathbb{R}^m)$ or $F \subset \{u \in W^{1,p}(O;\mathbb{R}^m) : \fint_O u = 0\}$, strong separability of $W^{1,p}(O;\mathbb{R}^m)$ implies strong separability of F, so there exists a countable strongly dense subset $F_0 \subset F$. Therefore, the map

$$\omega \mapsto \inf_{v\in F} I_k(v,\omega) = \inf_{v\in F_0} I_k(v,\omega)$$

is $\mathcal{F}$-measurable, and the conclusion follows from (2.94). □

Measurable minimizers

We show that in the convex case (or more generally in the p-sup-quasiconvex case) the random functional (2.93) admits a measurable minimizer. For that purpose, we begin with the following useful reformulation of the Rokhlin–Kuratowski–Ryll Nardzewski theorem [371, 278], which essentially asserts that the measurability of the infimum implies the measurability of minimizers.

Lemma 2.A.12. *Let X be a Polish space, let $(\Omega,\mathcal{F},\mu)$ be a complete measure space, and let $I : X\times\Omega \to [0,\infty]$. Assume that*

(i) for all ω, $I(\cdot,\omega)$ is lower semicontinuous on X;

(ii) for all ω, $I(\cdot,\omega)$ is coercive on X (i.e. the sublevel sets $\{u\in X : I(u,\omega)\le c\}$ are compact for all $c>0$);

(iii) for all closed subset $F\subset X$, the map $\phi_F : \Omega\to[0,\infty]$ defined by $\phi_F(\omega) := \min_{v\in F} I(v,\omega)$ is $\mathcal{F}$-measurable.

Then, there exists an $\mathcal{F}$-measurable map $u : \Omega\to X$ such that, for all $\omega\in\Omega$,

$$I(u(\omega),\omega) = \min_{v\in X} I(v,\omega) = \phi_X(\omega). \qquad \Diamond$$

Proof. By coercivity and lower semicontinuity, the minima of $I(\cdot,\omega)$ are always attained on all closed subsets F, so that the function ϕ_F is always well-defined.

Consider the multifunction $\Gamma : \Omega \rightrightarrows X$ defined by $\Gamma(\omega) = \{u\in X : I(u,\omega) = \phi_X(\omega)\}$. By lower semicontinuity, $\Gamma(\omega)\subset X$ is nonempty and closed for all ω. Moreover, we claim that Γ is measurable, in the sense that

$$\Gamma^{-1}(O) := \{\omega\in\Omega : \Gamma(\omega)\cap O \neq \varnothing\}$$

belongs to $\mathcal{F}$ for all open subsets $O \subset X$. As X is metrizable, it actually suffices to check $\Gamma^{-1}(F) \in \mathcal{F}$ for all closed subsets $F \subset X$ (see e.g. [10, Lemma 18.2]). By the coercivity and the lower semicontinuity of I, we may write

$$\begin{aligned}\Gamma^{-1}(F) &= \{\omega \in \Omega : \exists u \in F,\, I(u,\omega) = \phi_X(\omega)\} \\ &= \bigcap_{n=1}^{\infty} \left\{\omega \in \Omega : \exists u \in F,\, I(u,\omega) \leq \phi_X(\omega) + \frac{1}{n}\right\} \\ &= \bigcap_{n=1}^{\infty} \left\{\omega \in \Omega : \phi_F(\omega) \leq \phi_X(\omega) + \frac{1}{n}\right\},\end{aligned}$$

where the right-hand side belongs to $\mathcal{F}$, by measurability of ϕ_X and ϕ_F. Hence, Γ is measurable, and we may thus apply the Rokhlin–Kuratowski–Ryll Nardzewski measurable selection theorem (see e.g. [10, Theorem 18.13]), which states the existence of a $\mathcal{F}$-measurable map $u : \Omega \to X$ such that $u(\omega) \in \Gamma(\omega)$ for all ω. □

Combining this measurable selection lemma with the measurability result of Proposition 2.A.11, we obtain the following.

Proposition 2.A.13. *Let O be some bounded domain, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and let $W : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times \Omega \to [0,\infty]$ be a normal random integrand. Assume that there exists $C > 0$ and $p > 1$ such that, for almost all y, ω and for all Λ,*

$$\frac{1}{C}|\Lambda|^p - C \leq W(y,\Lambda,\omega). \tag{2.95}$$

Also assume that W satisfies Hypothesis 2.1.1 and that $I(\cdot,\omega)$ is weakly lower semicontinuous on $W^{1,p}(O;\mathbb{R}^m)$ for almost all ω (in particular this is the case if W is convex or p-sup-quasiconvex in the sense of Definition 2.A.9). Given some fixed function $f \in \mathrm{L}^p(\Omega; \mathrm{L}^p(O; \mathbb{R}^{m\times d}))$, consider the random integral functional $I : W^{1,p}(O;\mathbb{R}^m) \times \Omega \to [0,\infty]$ defined by

$$I(u,\omega) = \int_O W(y, f(y,\omega) + \nabla u(y), \omega)dy.$$

Then, for all nonempty weakly closed subsets $F \subset W_0^{1,p}(O;\mathbb{R}^m)$ or $F \subset \{u \in W^{1,p}(O;\mathbb{R}^m) : \int_O u = 0\}$, there exists a $\mathcal{F}$-measurable map $u : \Omega \to F$ such that, for almost all ω,

$$I(u(\omega),\omega) = \inf_{v\in F} I(v,\omega). \qquad \Diamond$$

Proof. Let X denote the Banach space $W_0^{1,p}(O;\mathbb{R}^m)$ or $\{u \in W^{1,p}(O;\mathbb{R}^m) : \int_O u = 0\}$, endowed with the weak topology, and, for all $k \geq 1$, consider the subset $X_k := \{u \in X : \|\nabla u\|_{\mathrm{L}^p(O)} \leq k\}$, endowed with the induced weak topology. By Poincaré's inequality and by the Banach-Alaoglu theorem, X_k is easily seen to be metrizable and compact, hence Polish. Let $F \subset X$ be a nonempty (weakly) closed subset.

Let $\Omega' \in \mathcal{F}$ denote a subset of full probability such that $I(\cdot,\omega)$ is weakly lower semicontinuous on $W^{1,p}(O;\mathbb{R}^m)$ for all $\omega \in \Omega'$. Since $X_k \subset X$ is (weakly) closed, the intersection $X_k \cap F$ is also (weakly) closed, and hence Hypothesis 2.1.1 asserts that the map $\omega \to \inf_{v\in F\cap X_k} I(v,\omega)$ is $\mathcal{F}$-measurable. Applying Lemma 2.A.12 on the compact Polish space X_k and on Ω' then yields a $\mathcal{F}$-measurable map $u_k : \Omega' \to X_k$ such that, for all $\omega \in \Omega'$,

$$I(u_k(\omega),\omega) = \min_{v\in F\cap X_k} I(v,\omega).$$

The lower bound (2.95) and the triangle inequality give

$$\int_O |\nabla u|^p \leq C^2 2^{p-1} + C2^{p-1} I(u,\omega) + 2^{p-1} \int_O |f(\cdot,\omega)|^p.$$

Define for all $k \geq 1$,

$$\Omega_k := \left\{ \omega \in \Omega' : C^2 2^{p-1} + C2^{p-1} \inf_{v \in F} I(v,\omega) + 2^{p-1} \int_O |f(\cdot,\omega)|^p \leq k \right\},$$

and note that $\Omega_k \in \mathcal{F}$ by Hypothesis 2.1.1. By definition, for all $\omega \in \Omega_k$ we have

$$I(u_k(\omega),\omega) = \min_{v \in F \cap X_k} I(v,\omega) = \min_{v \in F} I(v,\omega).$$

The sequence $(\Omega_k)_k$ is increasing, $\Omega_k \uparrow \Omega'' := \bigcup_{k=1}^\infty \Omega_k$. By integrability of f, $\Omega'' \subset \Omega'''$ and $\mathbb{P}[\Omega''' \setminus \Omega''] = 0$, where we have defined the event $\Omega''' := \{\omega \in \Omega : \inf_{v \in F} I(v,\omega) < \infty\}$. Given some fixed $w \in F$, the measurable map $u : \Omega \to X$ defined by

$$u(\omega) := \mathbb{1}_{\Omega \setminus \Omega''} w + \mathbb{1}_{\Omega_1} u_1(\omega) + \sum_{k=2}^\infty \mathbb{1}_{\Omega_k \setminus \Omega_{k-1}} u_k(\omega)$$

satisfies by definition $I(u(\omega),\omega) = \inf_{v \in F} I(v,\omega)$ for all $\omega \in \Omega'' \cup (\Omega \setminus \Omega''')$. $\square$

2.A.4 Approximation results

In this appendix, we prove two general approximation results that are crucially needed in this chapter. The first one is an extension of [331, Lemma 3.6] and [177, Proposition 2.6 of Chapter X].

Proposition 2.A.14. *Let $O \subset \mathbb{R}^d$ be a bounded Lipschitz domain, which is also strongly star-shaped, in the sense that there exists $x_0 \in O$ such that*

$$\overline{-x_0 + O} \subset \alpha(-x_0 + O), \quad \textit{for all } \alpha > 1.$$

Let $\Theta : \mathbb{R}^{m \times d} \to [0,\infty]$ be a convex lower semicontinuous function with $0 \in \operatorname{int}\operatorname{dom}\Theta$, and let $u \in W^{1,1}(O;\mathbb{R}^m)$ such that $\int_O \Theta(\nabla u) < \infty$. Then,

(i) there is a sequence $(v_n)_n \subset C^\infty(\mathrm{adh}O;\mathbb{R}^m)$ such that $\nabla v_n \in \operatorname{int}\operatorname{dom}\Theta$ pointwise,

$$v_n \to u \quad \textit{in } W^{1,1}(O;\mathbb{R}^m), \qquad \textit{and} \qquad \int_O \Theta(\nabla v_n(y))dy \to \int_O \Theta(\nabla u(y))dy;$$

(ii) there is a sequence $(w_n)_n$ of (continuous) piecewise affine functions such that $\nabla w_n \in \mathbb{Q}^{m \times d} \cap \operatorname{int}\operatorname{dom}\Theta$ pointwise,

$$w_n \to u \quad \textit{in } W^{1,1}(O;\mathbb{R}^m), \qquad \textit{and} \qquad \int_U \Theta(\nabla w_n(y))dy \to \int_O \Theta(\nabla u(y))dy.$$

If in addition u belongs to $W^{1,p}(O;\mathbb{R}^m)$ for some $1 \leq p < \infty$, then the sequences $(v_n)_n$ and $(w_n)_n$ can be chosen such that $v_n \to u$ and $w_n \to u$ in $W^{1,p}(O;\mathbb{R}^m)$. Moreover,

(a) if u belongs to $W_0^{1,1}(O;\mathbb{R}^m)$, then we can choose $v_n \in C_c^\infty(O;\mathbb{R}^m)$ and $w_n|_{\partial O} = 0$ (and in that case the assumption that O be strongly star-shaped can be relaxed);

(b) if $O = Q$ and $u \in W^{1,1}_{\mathrm{per}}(Q;\mathbb{R}^m)$, then v_n and w_n can be both chosen to be Q-periodic;

(c) if $\Xi : \mathbb{R}^{m \times d} \to [0,\infty]$ is a (nonconvex) ru-usc lower semicontinuous function which is continuous on $\operatorname{int}\operatorname{dom}\Theta$ and satisfies $0 \leq \Xi \leq \Theta$ pointwise, then the sequences $(v_n)_n$ and $(w_n)_n$ can be chosen in such a way that $\int_O \Xi(\nabla v_n) \to \int_O \Xi(\nabla u)$ and $\int_O \Xi(\nabla w_n) \to \int_O \Xi(\nabla u)$. $\Diamond$

Proof. We divide the proof into four steps.

Step 1. Proof of (i).

First, we show that we can assume $\nabla u \in \operatorname{int}\operatorname{dom}\Theta$ almost everywhere. Indeed, assume that (i) is proven for such u's, and let us deduce the general case. Given $u \in W^{1,p}(O;\mathbb{R}^m)$ with $\int_O \Theta(\nabla u) < \infty$, we have $\nabla u \in \operatorname{dom}\Theta$ almost everywhere, and hence by convexity $t\nabla u \in \operatorname{dom}\Theta$ almost everywhere for all $t \in [0,1)$. As convexity also implies $\int_O \Theta(t\nabla u) \le t\int_O \Theta(\nabla u) + (1-t)\Theta(0) < \infty$, we can apply the result (i) to tu, for any $t \in [0,1)$: this gives a sequence $(v_{n,t})_n \subset C^\infty(\operatorname{adh}O;\mathbb{R}^m)$ such that $v_{n,t} \to tu$ in $W^{1,1}(O;\mathbb{R}^m)$ and $\int_O \Theta(\nabla v_{n,t}(y))dy \to \int_O \Theta(t\nabla u(y))dy$. Weak lower semicontinuity of the integral functional $u \mapsto \int_O \Theta(\nabla u)$ on $W^{1,1}(O;\mathbb{R}^m)$ (which follows from convexity and lower semicontinuity of Θ) implies $\liminf_{t\uparrow 1}\int_O \Theta(t\nabla u) \ge \int_O \Theta(\nabla u)$. As the converse inequality follows from convexity, we obtain

$$\begin{aligned}\limsup_{t\uparrow 1}\limsup_{n\uparrow\infty}&\left(\|v_{n,t}-u\|_{W^{1,1}(O)} + \left|\int_O \Theta(\nabla v_{n,t}) - \int_O \Theta(\nabla u)\right|\right)\\ &= \limsup_{t\uparrow 1}\left(\|tu-u\|_{W^{1,1}(O)} + \left|\int_O \Theta(t\nabla u) - \int_O \Theta(\nabla u)\right|\right) = 0,\end{aligned}$$

and hence a standard diagonalization argument gives a sequence $(v_n)_n \subset C^\infty(\operatorname{adh}O;\mathbb{R}^m)$ such that $\nabla v_n \in \operatorname{int}\operatorname{dom}\Theta$ almost everywhere, $v_n \to u$ in $W^{1,1}(O;\mathbb{R}^m)$, and $\int_O \Theta(\nabla v_n) \to \int_O \Theta(\nabla u)$, and proves the general version of (i).

Hence, from now on, we assume $\nabla u \in \operatorname{int}\operatorname{dom}\Theta$ almost everywhere. Moreover, without loss of generality, we may also assume that O is strongly star-shaped with respect to $x_0 = 0$. Choose $(\alpha_k)_k \subset (1,\infty)$ a decreasing sequence of positive numbers converging to 1, sufficiently slowly so that $\frac{1}{k} < \frac{1}{2}d(\operatorname{adh}O, \partial(\alpha_k O))$ for all k, and define

$$u_k : \alpha_k O \to \mathbb{R}^m : x \mapsto u_k(x) = u(x/\alpha_k).$$

Take $\rho \in C_c^\infty(\mathbb{R}^d)$ such that $\int \rho = 1$, $\rho \ge 0$ and $\operatorname{supp}\rho \subset B(0,1)$, and write $\rho_k(x) = k^{-d}\rho(kx)$, for all $k \ge 1$. Consider the sequence $(v_k)_k$ defined by $v_k = \rho_k * (\alpha_k u_k)$ (which is well-defined by virtue of the condition on the α_k's). Note that $v_k \in C^\infty(\operatorname{adh}O;\mathbb{R}^m)$ and $\nabla v_k = \rho_k * (\nabla u)_k$, where we use the notation $(\nabla u)_k(x) = \nabla u(x/\alpha_k)$. Moreover, we then observe $v_k \to u$ in $W^{1,1}(O;\mathbb{R}^m)$ since $u_k \to u$ et $(\nabla u)_k \to \nabla u$ in $\mathrm{L}^1(O;\mathbb{R}^m)$. Now, Jensen's inequality yields

$$0 \le \Theta(\nabla v_k) = \Theta(\rho_k * (\nabla u)_k) \le \rho_k * (\Theta((\nabla u)_k)) = \rho_k * (\Theta(\nabla u))_k.$$

As the sequence $(\rho_k * (\Theta(\nabla u))_k)_k$ converges to $\Theta \circ \nabla u$ in $\mathrm{L}^1(O;\mathbb{R}^m)$, it is uniformly integrable, and the same thus holds for the sequence $(\Theta(\nabla v_k))_k$. As $\nabla v_k \to \nabla u$ in $\mathrm{L}^1(O;\mathbb{R}^m)$, the convergence holds almost everywhere up to an extraction. Since by convexity Θ is continuous on $\operatorname{int}\operatorname{dom}\Theta$, since $\nabla u \in \operatorname{int}\operatorname{dom}\Theta$ almost everywhere, and since $\nabla v_k \to \nabla u$ almost everywhere up to an extraction, we deduce $\Theta(\nabla v_k) \to \Theta(\nabla u)$ almost everywhere up to an extraction. By uniform integrability the latter convergence also holds in $\mathrm{L}^1(O;\mathbb{R}^m)$, which allows us to get rid of the extraction. In particular,

$$\int_O \Theta(\nabla v_k) \to \int_O \Theta(\nabla u).$$

Moreover, $(\nabla u)_k \in \operatorname{int}\operatorname{dom}\Theta$ almost everywhere, and thus $\nabla v_k = \rho_k * (\nabla u)_k \in \operatorname{int}\operatorname{dom}\Theta$ everywhere for all k, since $\operatorname{int}\operatorname{dom}\Theta$ is a convex set containing 0. This proves part (i).

Let us now assume that $u \in W^{1,p}(O;\mathbb{R}^m)$ for some $1 \le p < \infty$, and consider the sequences $(u_k)_k$, $((\nabla u)_k)_k$ and $(v_k)_k$ defined above. First, Lebesgue's dominated convergence theorem implies $u_k \to u$ and $(\nabla u)_k \to \nabla u$ in $\mathrm{L}^p(O;\mathbb{R}^m)$. Further, since the Jensen inequality gives

$$\int_O |v_k - u|^p = \int_O |\alpha_k\rho_k * u_k - u|^p \le \int_O\int_{B_{1/k}} \rho_k(t)|\alpha_k u_k(x-t) - u(x)|^p dt\,dx,$$

and likewise for gradients, we conclude that the sequence $(v_k)_k$ converges to u in $W^{1,p}(O;\mathbb{R}^m)$. This proves part (i) in the case when $u \in W^{1,p}(O;\mathbb{R}^m)$.

Step 2. Proof of (ii).

For all k, since $v_k \in C^\infty(\mathrm{adh}O;\mathbb{R}^m)$, there exists a sequence $(w_{k,j})_j$ of piecewise affine functions such that $w_{k,j} \to v_k$ in $W^{1,\infty}(O;\mathbb{R}^m)$ as $j \uparrow \infty$ and $\|\nabla w_{k,j}\|_{\mathrm{L}^\infty} \le \|\nabla v_k\|_{\mathrm{L}^\infty}$ for all j,k (see e.g. [177, Proposition 2.1 of Chapter X]). Further, these functions $w_{k,j}$ can (simply remember their construction by triangulation) be chosen taking their values in $\mathrm{int}\,\mathrm{dom}\Theta$, since we have constructed $\nabla v_k \in \mathrm{int}\,\mathrm{dom}\Theta$ everywhere for all k. Another approximation argument further allows us to choose $w_{k,j}$ such that $\nabla w_{k,j}$ only takes rational values. The desired result then follows from Step 1 and a diagonalization argument. Finally, the particular case when u belongs to $W^{1,p}(O;\mathbb{R}^m)$ is obtained similarly as in Step 1.

Step 3. Proof of the additional statements.

It remains to address the particular cases (a) and (b). First assume that u belongs to $W_0^{1,1}(O;\mathbb{R}^m)$. For the corresponding result, we refer to [177, Proposition 2.6 of Chapter X]. The only difference is that the argument in [177] requires continuity of Θ. Instead, we replace u by tu for $t<1$ as in Step 1, so that by convexity $t\nabla u \in \mathrm{int}\,\mathrm{dom}\Theta$ almost everywhere, hence all the constructed quantities have almost all their values in $\mathrm{int}\,\mathrm{dom}\Theta$, on which Θ is continuous by convexity. No further continuity assumption is then needed.

Finally, if we assume $O=Q$ with $u \in W^{1,1}_{\mathrm{per}}(Q;\mathbb{R}^m)$, then we can consider the periodic extension of u on $\mathbb{R}^d$ and repeat the arguments in such a way that periodicity is conserved.

Step 4. Proof in the nonconvex case.

Let $u \in W^{1,p}(O;\mathbb{R}^m)$ be such that $\int_U \Theta(\nabla u) < \infty$, which implies $\nabla u \in \mathrm{dom}\Theta$ almost everywhere. Let $t \in (0,1)$. The approximation result given by point (i) gives a sequence $(u_{n,t})_n$ of smooth functions such that $u_{n,t} \to tu$ (strongly) in $W^{1,p}(O;\mathbb{R}^m)$ and $\int_O \Theta(\nabla u_{n,t}) \to \int_O \Theta(t\nabla u)$ as $n \uparrow \infty$, and such that $\nabla u_{n,t} \in \mathrm{int}\,\mathrm{dom}\Theta$ pointwise. Up to an extraction, we have $\nabla u_{n,t} \to t\nabla u$ almost everywhere, and thus $\Xi(\nabla u_{n,t}) \to \Xi(t\nabla u)$ almost everywhere, which follows from continuity of Ξ on the interior of its domain, with indeed $t\nabla u \in \mathrm{int}\,\mathrm{dom}\Xi$ almost everywhere. Then noting that

$$0 \le \Xi(\nabla u_{n,t}) \le C(1+\Theta(\nabla u_{n,t})),$$

and invoking both Lebesgue's dominated convergence theorem (for $\Xi(\nabla u_{n,t})$) and its converse (for $\Theta(\nabla u_{n,t})$), we deduce convergence $\int_O \Xi(\nabla u_{n,t}) \to \int_O \Xi(t\nabla u)$ as $n \uparrow \infty$. As Ξ is lower semi-continuous and also ru-usc, we compute, by Fatou's lemma,

$$\begin{aligned}\int_O \Xi(\nabla u(y))dy \le \int_O \liminf_{t\uparrow 1} \Xi(t\nabla u(y))dy \le \liminf_{t\uparrow 1} \int_O \Xi(t\nabla u(y))dy \\ \le \limsup_{t\uparrow 1} \int_O \Xi(t\nabla u(y))dy \le \int_O \Xi(\nabla u(y))dy.\end{aligned}$$

Hence,

$$\lim_{t\uparrow 1}\lim_{n\uparrow\infty} \int_O \Xi(\nabla u_{n,t}) = \lim_{t\uparrow 1} \int_O \Xi(t\nabla u) = \int_O \Xi(\nabla u),$$

and similarly

$$\lim_{t\uparrow 1}\lim_{n\uparrow\infty} \int_O \Theta(\nabla u_{n,t}) = \int_O \Theta(\nabla u),$$

so that the conclusion follows from a standard diagonalization argument. The other properties are deduced in a similar way. □

For technical reasons, we need in the proof of Proposition 2.2.10 to further approximate piecewise affine functions by refined ones with smoother variations. The precise approximation result that we need is the following.

Proposition 2.A.15. *Let u be an $\mathbb{R}^m$-valued continuous piecewise affine function on a bounded Lipschitz domain $O \subset \mathbb{R}^d$. Consider the open partition $O = \biguplus_{l=1}^k O^l$ associated with u (i.e. u is affine on each piece O^l). Define $M := (\bigcup_{l=1}^k \partial O^l) \setminus \partial O$ the interior boundary of this partition of O, and, for fixed $r > 0$, also define $M_r := (M + B_r) \cap O$ the r-neighborhood of this interior boundary. Then, for all $\kappa > 0$, there exists a continuous piecewise affine function $u_{\kappa,r}$ on O with the following properties:*

(i) $\nabla u_{\kappa,r} = \nabla u$ *pointwise on* $O \setminus M_r$, *and* $\limsup_{r\downarrow 0} \sup_{0<\kappa\leq 1} \|u_{\kappa,r} - u\|_{\mathrm{L}^\infty(O)} = 0$;

(ii) $\nabla u_{\kappa,r} \in \mathrm{conv}(\{\nabla u(x) : x \in O\})$ *pointwise (where* $\mathrm{conv}(\cdot)$ *denotes the convex hull);*

(iii) denoting by $O := \biguplus_{l=1}^{n_{\kappa,r}} O_{\kappa,r}^l$ *the open partition associated with* $u_{\kappa,r}$*, and* $\Lambda_{\kappa,r}^l := \nabla u_{\kappa,r}|_{O_{\kappa,r}^l}$ *for all* l*, we have* $|\Lambda_{\kappa,r}^i - \Lambda_{\kappa,r}^j| \leq \kappa$ *for all* i, j *with* $\partial O_{\kappa,r}^i \cap \partial O_{\kappa,r}^j \neq \varnothing$. ◊

Proof. Let u, O and r be fixed. Without loss of generality, we can assume $0 \in O$. Denote $r_0 := d(0, \partial O)$ and $R_0 := \max_{x\in\partial O} |x|$, and define $\alpha_r > 1$ by $(\alpha_r - 1)R_0 = r/2$. Choose a nonnegative smooth function ρ_r supported in $B_{(\alpha_r-1)r_0}$ with $\int \rho_r = 1$, and consider the smooth function u_r on O defined by $u_r = \rho_r * [\alpha_r u(\cdot/\alpha_r)]$. Since by definition inequality $|\alpha_r x - x| \leq (\alpha_r - 1)R_0 = r/2$ holds for any $x \in O$, we easily check $\nabla u_r = \nabla u$ on the set $O \setminus M_r$. As u_r is smooth, we can consider $L_r := \max_{x\in\mathrm{adh}O} |\nabla\nabla u_r(x)| < \infty$. Choose a triangulation $(O_{\kappa,r}^l)_{l=1}^{n_{\kappa,r}}$ of O which is a refinement of the partition $\{O^l \setminus M_r : l = 1, \ldots, k\} \cup \{M_r\}$, such that the diameter of each of the $O_{\kappa,r}^l$'s is at most $\kappa/(2L_r)$. Now construct as usual the piecewise affine approximation $u_{\kappa,r}$ of u_r with respect to the triangulation $(O_{\kappa,r}^l)_l$ (see e.g. [177, Proposition 2.1 of Chapter X]). By construction, $\nabla u_{\kappa,r} = \nabla u_r = \nabla u$ on $O\setminus M_r$, since u_r is affine on each connected component of $O\setminus M_r$. Also note that the construction ensures that $\nabla u_{\kappa,r}$ belongs pointwise to the set $\{\nabla u_r(x) : x \in O\}$ (as a consequence of the mean value theorem), which is by definition included in the convex hull $\mathrm{conv}(\{\nabla u(x) : x \in O\})$. Moreover, if $\partial O_{\kappa,r}^i \cap \partial O_{\kappa,r}^j \neq \varnothing$, it implies that $O_{\kappa,r}^i \cup O_{\kappa,r}^j$ has diameter bounded by κ/L_r. The construction of $u_{\kappa,r}$ then gives

$$\left|\nabla u_{\kappa,r}|_{O_{\kappa,r}^i} - \nabla u_{\kappa,r}|_{O_{\kappa,r}^j}\right| \leq \sup_{x\in O_{\kappa,r}^i} \sup_{y\in O_{\kappa,r}^j} |\nabla u_r(x) - \nabla u_r(y)| \leq \frac{\kappa}{L_r} \sup_{x\in O} |\nabla\nabla u_r(x)| \leq \kappa,$$

which proves property (iii). Finally, the last property of (i) directly follows from the construction. □

Chapter 3

Pathwise structure of fluctuations in stochastic homogenization

Four quantities are fundamental in homogenization of elliptic systems in divergence form and in its applications: the field and the flux of the solution operator (applied to a general deterministic right-hand side), and the field and the flux of the corrector. Homogenization is the study of the large-scale properties of these objects. For random coefficients, these quantities fluctuate and their fluctuations are a priori unrelated. Depending on the law of the coefficient field, and in particular on the decay of its correlations on large scales, these fluctuations may display different scalings and different limiting laws (if any). In this chapter, we identify a fifth and crucial intrinsic quantity, a random 2-tensor field, which we refer to as the *homogenization commutator* and which is related to variational quantities first considered by Armstrong and Smart. In the model framework of discrete linear elliptic equations in divergence form with independent and identically distributed coefficients, we show what we believe to be a general principle, namely that the homogenization commutator drives at leading order the fluctuations of each of the four other quantities in a strong norm in probability, which reveals the *pathwise structure* of fluctuations in stochastic homogenization. In addition, we show in this framework that the (rescaled) fluctuations of the homogenization commutator converge in law to a (2-tensor) Gaussian white noise, the distribution of which is thus characterized by some 4-tensor, and we analyze to which precision this tensor can be extracted from the representative volume element method. All these results are optimally quantified and hold in any dimension. This constitutes the first complete theory of fluctuations in stochastic homogenization. As a consequence, we retrieve (optimal quantitative versions of) all the previously known results according to which the solution operator satisfies a functional central limit theorem, and the field and flux of the corrector converge to the Helmholtz and Leray projections of a Gaussian white noise, respectively.

This chapter corresponds to a thoroughly revised version of the paper [167] jointly written with Antoine Gloria and Felix Otto.

Contents

3.1 Introduction

This chapter constitutes the first part of a series of joint works with Antoine Gloria and Felix Otto that develops a theory of fluctuations in stochastic homogenization of elliptic (non-necessarily symmetric) systems. In this first part, using an elementary approach, we provide a complete picture of our theory (with optimal error estimates and convergence rates) in the model framework of discrete elliptic equations with independent and identically distributed (i.i.d.) conductances. Links to the literature are discussed in Section 3.1.3 below, while the extension to more general situations is postponed to forthcoming work and is shortly described in Section 3.1.4.

3.1.1 General overview

Although in the sequel we shall focus on the case of discrete equations, we use non-symmetric continuum notation in this introduction. Let A be a stationary and ergodic random coefficient field on $\mathbb{R}^d$ that satisfies the boundedness and ellipticity properties

$$|A(x)\xi| \le |\xi|, \qquad \xi \cdot A(x)\xi \ge \lambda|\xi|^2, \qquad \text{for all } x, \xi \in \mathbb{R}^d,$$

for some $\lambda > 0$. For all $\varepsilon > 0$ we set $A_\varepsilon := A(\frac{\cdot}{\varepsilon})$, and for all deterministic vector fields $f \in C_c^\infty(\mathbb{R}^d)^d$, we consider the random family $(u_\varepsilon)_{\varepsilon>0}$ of unique Lax-Milgram solutions in $\mathbb{R}^d$ (which, in the rest of this chapter, means the unique weak solutions in $\dot H^1(\mathbb{R}^d)$) of the rescaled problems

$$-D \cdot A_\varepsilon D u_\varepsilon \;=\; D \cdot f, \tag{3.1}$$

where D denotes the continuum gradient (while the notation ∇ is reserved in this chapter for the discrete gradient). It is known since the pioneering works by Papanicolaou and Varadhan [354] and by Kozlov [273] that, almost surely, u_ε converges weakly (in $\dot H^1(\mathbb{R}^d)$) as $\varepsilon \downarrow 0$ to the unique Lax-Milgram solution $\bar u$ in $\mathbb{R}^d$ of

$$-D \cdot A_{\hom} D\bar u \;=\; D \cdot f, \tag{3.2}$$

where $A_{\hom}$ is a deterministic and constant matrix that only depends on A. More precisely, for any direction $\xi \in \mathbb{R}^d$, the projection $A_{\hom}\xi$ is the expectation of the flux of the corrector in the direction ξ,

$$A_{\hom}\xi \;=\; \mathbb{E}\left[A(D\phi_\xi + \xi)\right],$$

where the corrector ϕ_e is the unique (up to a random additive constant) almost-sure solution of the corrector equation in $\mathbb{R}^d$,

$$-D \cdot A(D\phi_\xi + \xi) \;=\; 0,$$

in the class of functions the gradient of which is stationary and has finite second moment. We denote by $\phi = (\phi_i)_{i=1}^d$ the vector field the entries of which are the correctors ϕ_i in the canonical directions e_i of $\mathbb{R}^d$. Note that the convergence of Du_ε to $D\bar u$ in $\mathrm{L}^2(\mathbb{R}^d)^d$ is only weak since Du_ε typically displays spatial oscillations at scale ε, which are not captured by the limit $D\bar u$. These oscillations are however well-described by those of the corrector field $D\phi(\frac{\cdot}{\varepsilon})$ through the two-scale expansion (we systematically use Einstein's summation rule on repeated indices)

$$Du_\varepsilon \;\approx\; (D\phi_i(\tfrac{\cdot}{\varepsilon}) + e_i) D_i \bar u, \tag{3.3}$$

in the sense that $Du_\varepsilon - (D\phi_i(\frac{\cdot}{\varepsilon}) + e_i)D_i\bar u$ converges strongly to zero in $\mathrm{L}^2(\mathbb{R}^d)^d$. In the random setting, this theory of oscillations was recently optimally quantified in [205], [203, Theorem 3], [208, Corollary 3], and [30, Chapter 6].

As opposed to periodic homogenization, which boils down to the sole understanding of the (spatial) oscillations of Du_ε, the stochastic setting involves the (random) fluctuations of Du_ε on top of its oscillations. More precisely, whereas oscillations are concerned with the (almost sure) lack of strong compactness for Du_ε in $\mathrm{L}^2(\mathbb{R}^d)^d$, fluctuations are concerned with the leading-order probabilistic behavior of weak-type expressions of the form $\int_{\mathbb{R}^d} g \cdot Du_\varepsilon$ for $g \in C_c^\infty(\mathbb{R}^d)^d$. Let us emphasize that in the case of a weakly correlated coefficient field A the error in the two-scale expansion (3.3) is of order ε in $\mathrm{L}^2(\mathbb{R}^d)^d$ (or $\varepsilon|\log \varepsilon|^{\frac12}$ for $d = 2$) while fluctuations of Du_ε display the central limit theorem (CLT) scaling $\varepsilon^{\frac{d}{2}}$, so that (3.3) is not expected to be accurate in that scaling. This was indeed first checked in dimension $d \geq 3$ by Gu and Mourrat [225, Section 3.2] (see also the last item in Remarks 3.2.12 below for $d = 2$), who further argue that accuracy in (3.3) in the fluctuation scaling cannot even be reached by the use of higher-order correctors. The corrector field $D\phi$ is therefore the driving quantity for oscillations but a priori not for fluctuations.

In the present chapter, we develop a complete theory of fluctuations in stochastic homogenization in line with the known theory of oscillations, and our main achievement is the identification of the driving quantity for fluctuations. The key in our theory consists in focusing on the *homogenization commutator of the solution* $A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}} Du_\varepsilon$ and in studying its relation to the *(standard) homogenization commutator* $\Xi := (\Xi_i)_{i=1}^d$ defined by

$$\Xi_i \;:=\; A(D\phi_i + e_i) - A_{\mathrm{hom}}(D\phi_i + e_i), \qquad \Xi_{ij} := (\Xi_i)_j. \tag{3.4}$$

This stationary random (non-symmetric) 2-tensor field Ξ enjoys the following three crucial properties, which lead to our complete theory of fluctuations:

(I) First and most importantly, the two-scale expansion of the homogenization commutator of the solution

$$A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}} Du_\varepsilon - \mathbb{E}\left[A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}} Du_\varepsilon\right] \;\approx\; \Xi_i(\tfrac{\cdot}{\varepsilon}) D_i \bar u \tag{3.5}$$

is (generically) accurate in the fluctuation scaling in the sense of

$$\mathbb{E}\left[\Big|\int_{\mathbb{R}^d} g \cdot \big(A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}} Du_\varepsilon - \mathbb{E}\left[A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}} Du_\varepsilon\right]\big) - \int_{\mathbb{R}^d} g \cdot \Xi_i(\tfrac{\cdot}{\varepsilon}) D_i \bar u\Big|^2\right]^{\frac12} \\ \leq\; o(1)\, \mathbb{E}\left[\Big|\int_{\mathbb{R}^d} g \cdot \Xi_i(\tfrac{\cdot}{\varepsilon}) D_i \bar u\Big|^2\right]^{\frac12}, \tag{3.6}$$

where $o(1) \downarrow 0$ as $\varepsilon \downarrow 0$, for all $g \in C_c^\infty(\mathbb{R}^d)^d$. Let us emphasize again that this property is nontrivial and is due to the form of the commutator.

(II) Second, both the fluctuations of the field Du_ε and of the flux $A_\varepsilon Du_\varepsilon$ can be recovered through *deterministic* projections of those of the homogenization commutator $A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}} Du_\varepsilon$ of the solution, which shows that no information is lost by passing to the homogenization commutator. More precisely,

$$\begin{aligned} &\int_{\mathbb{R}^d} g \cdot D(u_\varepsilon - \mathbb{E}\left[u_\varepsilon\right]) \\ &\qquad = -\int_{\mathbb{R}^d} (\bar{\mathcal{P}}_H^* g) \cdot \big(A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}} Du_\varepsilon - \mathbb{E}\big[A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}} Du_\varepsilon\big]\big), \\ &\int_{\mathbb{R}^d} g \cdot (A_\varepsilon Du_\varepsilon - \mathbb{E}\left[A_\varepsilon Du_\varepsilon\right]) \\ &\qquad = \int_{\mathbb{R}^d} (\bar{\mathcal{P}}_L^* g) \cdot \big(A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}} Du_\varepsilon - \mathbb{E}\big[A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}} Du_\varepsilon\big]\big), \end{aligned} \tag{3.7}$$

in terms of the Helmholtz and Leray projections in $\mathrm{L}^2(\mathbb{R}^d)^d$,

$$\begin{aligned} \bar{\mathcal{P}}_H &:= D(D \cdot A_{\mathrm{hom}} D)^{-1} D\cdot, \qquad & \bar{\mathcal{P}}_L &:= \mathrm{Id} - \bar{\mathcal{P}}_H A_{\mathrm{hom}}, \\ \bar{\mathcal{P}}_H^* &:= D(D \cdot A_{\mathrm{hom}}^* D)^{-1} D\cdot, \qquad & \bar{\mathcal{P}}_L^* &:= \mathrm{Id} - \bar{\mathcal{P}}_H A_{\mathrm{hom}}^*, \end{aligned} \tag{3.8}$$

where A_{hom}^* denotes the transpose of A_{hom}. In addition, the fluctuations of the field $D\phi(\frac{\cdot}{\varepsilon})$ and of the flux $A_\varepsilon D\phi(\frac{\cdot}{\varepsilon})$ of the corrector are clearly determined by those of the standard commutator $\Xi(\frac{\cdot}{\varepsilon})$. Indeed, the definition of Ξ leads to $-D\cdot A_{\mathrm{hom}}D\phi_i = D\cdot\Xi_i$ and $A(D\phi_i+e_i)-A_{\mathrm{hom}}e_i = \Xi_i + A_{\mathrm{hom}}D\phi_i$, to the effect of $D\phi_i = -\bar{\mathcal{P}}_H e_i$ and $A(D\phi_i+e_i)-A_{\mathrm{hom}}e_i = (\mathrm{Id}-A_{\mathrm{hom}}\bar{\mathcal{P}}_H)\Xi_i$ in the stationary sense, and hence, formally,

$$\begin{aligned}\int_{\mathbb{R}^d} F : D\phi(\tfrac{\cdot}{\varepsilon}) \;&=\; -\int_{\mathbb{R}^d} \bar{\mathcal{P}}_H^* F : \Xi(\tfrac{\cdot}{\varepsilon}), \\ \int_{\mathbb{R}^d} F : \big(A_\varepsilon(D\phi(\tfrac{\cdot}{\varepsilon})+\mathrm{Id})-A_{\mathrm{hom}}\big) \;&=\; \int_{\mathbb{R}^d} \mathcal{P}_L^* F : \Xi(\tfrac{\cdot}{\varepsilon}),\end{aligned} \tag{3.9}$$

where $\bar{\mathcal{P}}_H^*$ and $\bar{\mathcal{P}}_L^*$ act on the second index of the tensor field $\mathcal{F}$. A suitable sense to these identities is given as part of Corollary 3.2.4 below.

Let us highlight the *pathwise structure* of fluctuations revealed here. Combined with (3.6), identities (3.7) and (3.9) imply that the fluctuations of Du_ε, $A_\varepsilon Du_\varepsilon$, $D\phi(\frac{\cdot}{\varepsilon})$, and $A_\varepsilon D\phi(\frac{\cdot}{\varepsilon})$ are determined at leading order by those of $\Xi(\frac{\cdot}{\varepsilon})$ in a strong norm in probability. This almost sure ("pathwise" in the language of SPDE) relation thus reduces the leading-order fluctuations of all quantities of interest to those of the sole homogenization commutator Ξ in a pathwise sense. Besides its theoretical importance, this *pathwise structure* is bound to affect multi-scale computing and uncertainty quantification in an essential way.

(III) Third, the standard homogenization commutator Ξ is an approximately local function of the coefficients A, which allows to infer the large-scale behavior of Ξ from the large-scale behavior of A itself. This locality is best seen when formally computing partial derivatives of Ξ with respect to A: letting ϕ^* denote the corrector associated with the pointwise transpose coefficient field A^*, and letting σ^* denote the corresponding flux corrector (cf. (3.30) below), we obtain (cf. (3.47))

$$\begin{aligned}\frac{\partial}{\partial A(x)}\Xi_{ij} = (D\phi_j^*+e_j)\cdot\frac{\partial A}{\partial A(x)}(D\phi_i+e_i)\qquad\qquad\qquad\qquad\\ -D\cdot\bigg(\phi_j^*\frac{\partial A}{\partial A(x)}(D\phi_i+e_i)\bigg)-D\cdot\bigg((\phi_j^*A+\sigma_j^*)\frac{\partial D\phi_i}{\partial A(x)}\bigg).\end{aligned} \tag{3.10}$$

The first right-hand side term reveals an exactly local dependence upon A. The second term is exactly local as well, but since it is written in divergence form its contribution is negligible when integrating on large scales. The only non-local effect comes from the last term due to $\frac{\partial D\phi}{\partial A}$, which is given by the mixed derivative of the Green's function for $-D\cdot AD$ and thus is expected to have only borderline integrable decay. However, it appears inside a divergence, hence is also negligible on large scales. (In fact, in this paper, the accuracy in (3.5) is established relying on a similar representation of $\frac{\partial}{\partial A}(A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}}Du_\varepsilon - \Xi_i(\frac{\cdot}{\varepsilon})D_i\bar u)$, cf. Lemma 3.3.2.)

We quickly comment on the form of the homogenization commutator, which was simultaneously independently introduced by Armstrong, Kuusi, and Mourrat [32] formalizing previous ideas by Armstrong and Smart [36]. As well-known in applications, homogenization is the rigorous version of averaging fields and fluxes in a consistent way. This is best seen in the very definition of H-convergence by Murat and Tartar [333], which requires both weak convergence of the fields $Du_\varepsilon \rightharpoonup D\bar u$ and of the fluxes $A_\varepsilon Du_\varepsilon \rightharpoonup A_{\mathrm{hom}}D\bar u$ in $\mathrm{L}^2(\mathbb{R}^d)^d$, to the effect of

$$A_\varepsilon Du_\varepsilon - A_{\mathrm{hom}}Du_\varepsilon \rightharpoonup 0.$$

This weak convergence of the homogenization commutator is the mathematical formulation of the so-called Hill-Mandel relation in mechanics [240, 242]. Applied to the corrector, this justifies the definition of the (standard) homogenization commutator Ξ, cf. (3.4), which is seen as a natural and intrinsic measure of the accuracy of homogenization for large-scale averages.

Remark 3.1.1. If a suitable rescaling $\varepsilon^{-\frac{\beta}{2}}\Xi(\frac{\cdot}{\varepsilon})$ of the homogenization commutator converges in law to some random functional Γ, then combining (3.6) with identities (3.7) and (3.9) leads to the joint convergence in law

$$\Big(\varepsilon^{-\frac{\beta}{2}}\int_{\mathbb{R}^d} F:\Xi(\tfrac{\cdot}{\varepsilon})\ ,\ \varepsilon^{-\frac{\beta}{2}}\int_{\mathbb{R}^d} g\cdot(Du_\varepsilon-\mathbb{E}\left[Du_\varepsilon\right])\ ,\ \varepsilon^{-\frac{\beta}{2}}\int_{\mathbb{R}^d} g\cdot(A_\varepsilon Du_\varepsilon-\mathbb{E}\left[A_\varepsilon Du_\varepsilon\right])\,,$$
$$\varepsilon^{-\frac{\beta}{2}}\int_{\mathbb{R}^d} F:D\phi(\tfrac{\cdot}{\varepsilon})\ ,\ \varepsilon^{-\frac{\beta}{2}}\int_{\mathbb{R}^d} F:\big(A_\varepsilon(D\phi(\tfrac{\cdot}{\varepsilon})+\mathrm{Id})-A_{\mathrm{hom}}\big)\Big)$$
$$\to\Big(\Gamma(F)\ ,\ \Gamma\big(\bar{\mathcal{P}}_H f\otimes\bar{\mathcal{P}}_H^* g\big)\ ,\ -\Gamma\big(\bar{\mathcal{P}}_H f\otimes\bar{\mathcal{P}}_L^* g\big)\ ,\ -\Gamma\big(\bar{\mathcal{P}}_H^*\mathcal{F}\big)\ ,\ \Gamma\big(\bar{\mathcal{P}}_L^* F\big)\Big), \tag{3.11}$$

thus establishing the non-trivial fact that the limiting joint law is degenerate. This *almost sure relation* between the marginals is precisely the manifestation of the *pathwise structure* (I)–(II) above. For a weakly correlated coefficient field A we expect from property (III) that $\Xi(\frac{\cdot}{\varepsilon})$ displays the CLT scaling $\beta=d$ and that its rescaling converges in law to a Gaussian white noise Γ, so that the convergence in (3.11) then leads to the known (or expected) scaling limit results for the different quantities of interest in stochastic homogenization. We however emphasize that the main novelty of the present contribution does not rely in such convergence results in themselves, but rather in the mechanism behind, which is summarized in items (I)–(III) above. $\Diamond$

3.1.2 Main results

In order to present our complete theory of fluctuations and address items (I)–(III), we place ourselves in the simplest setting possible and consider discrete elliptic equations with i.i.d. conductances, which we think of as the prototype for weakly correlated coefficient fields. Although conceptually simpler on the stochastic side, the discrete setting has some technical inconveniences on the deterministic side, including a discretization error.

Our main results take on the following guise. Precise notation and assumptions are postponed to Section 3.2, as well as many remarks and corollaries. While items (i) and (ii) below (together with the non-degeneracy in (iv)) imply property (I) in the form (3.6) with the optimal rate $o(1)\simeq_{f,g}\varepsilon\mu_d(\frac{1}{\varepsilon})^{\frac{1}{2}}$, items (iii) and (iv) are manifestations of property (III). Regarding the proofs, items (i) and (ii) are established using a Poincaré inequality in the probability space, which holds in the i.i.d. setting (cf. Lemma 3.3.1), item (iii) using a second-order Poincaré inequality due to Chatterjee [112, 113] (cf. Lemma 3.4.1), and item (iv) using (an i.i.d. version of) the so-called Helffer-Sjöstrand representation formula for covariances [238, 399, 335] (cf. Lemma 3.5.1). Apart from these simplifying tools (which can be either extended or avoided in more general contexts, cf. Section 3.1.4), the proofs only rely on arguments that extend to the continuum setting and to the case of systems. At a technical level, we make strong use of the (quenched) large-scale weighted Calderón-Zygmund theory for the operator $-\nabla\cdot A\nabla$ (cf. [29, 204]).

Theorem 3.1.2. *Consider the discrete i.i.d. setting, and assume that the law is non-degenerate. Then the following hold for all $\varepsilon>0$,*

(i) CLT scaling: *for all $F\in C_c^\infty(\mathbb{R}^d)^{d\times d}$,*

$$\mathbb{E}\left[\Big|\varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d}F:\Xi(\tfrac{\cdot}{\varepsilon})\Big|^2\right]^{\frac{1}{2}}\lesssim_F 1.$$

(ii) Pathwise structure (with optimal error estimates): *for all $f,g\in C_c^\infty(\mathbb{R}^d)^d$, letting u_ε and $\bar{u}$*

denote the solutions of (the discrete version of) (3.1) *and of* (3.2),

$$\mathbb{E}\left[\left|\varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d} g\cdot\big(A_\varepsilon\nabla_\varepsilon u_\varepsilon - A_{\mathrm{hom}}\nabla_\varepsilon u_\varepsilon - \mathbb{E}\left[A_\varepsilon\nabla_\varepsilon u_\varepsilon - A_{\mathrm{hom}}\nabla_\varepsilon u_\varepsilon\right]\big) - \varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d} g\cdot\Xi_i(\tfrac{\cdot}{\varepsilon})D_i\bar u\right|^2\right]^{\frac12} \lesssim_{f,g} \varepsilon\mu_d(\tfrac{1}{\varepsilon})^{\frac12}, \tag{3.12}$$

where we set for all $r>0$,

$$\mu_d(r) := \begin{cases} r & : \quad d=1,\\ \log(2+r) & : \quad d=2,\\ 1 & : \quad d>2. \end{cases} \tag{3.13}$$

(iii) Asymptotic normality (with nearly optimal rate): *for all $F\in C_c^\infty(\mathbb{R}^d)^{d\times d}$,*

$$\delta_{\mathcal{N}}\bigg(\varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d} F:\Xi(\tfrac{\cdot}{\varepsilon})\bigg) \lesssim_F \varepsilon^{\frac{d}{2}}\log(2+\tfrac{1}{\varepsilon}),$$

where for a random variable $X\in \mathrm{L}^2(\Omega)$ its distance to normality is defined by

$$\delta_{\mathcal{N}}(X) := \mathrm{d_W}\left(\frac{X}{\mathrm{Var}\left[X\right]^{\frac12}},\mathcal{N}\right) + \mathrm{d_K}\left(\frac{X}{\mathrm{Var}\left[X\right]^{\frac12}},\mathcal{N}\right), \tag{3.14}$$

with $\mathcal{N}$ a standard Gaussian random variable and with $\mathrm{d_W}(\cdot,\cdot)$ and $\mathrm{d_K}(\cdot,\cdot)$ the Wasserstein and Kolmogorov metrics.

(iv) Convergence of the covariance structure (with optimal rate): *there exists a non-degenerate symmetric 4-tensor $\mathcal{Q}$ such that for all $F\in C_c^\infty(\mathbb{R}^d)^{d\times d}$,*

$$\left|\mathrm{Var}\left[\varepsilon^{-\frac{d}{2}}\int_{\mathbb{R}^d} F:\Xi(\tfrac{\cdot}{\varepsilon})\right] - \int_{\mathbb{R}^d} F:\mathcal{Q}\,F\right| \lesssim_F \varepsilon\mu_d(\tfrac{1}{\varepsilon})^{\frac12}.$$

In particular, combined with item (iii), this yields the convergence of $\varepsilon^{-\frac{d}{2}}\Xi(\frac{\cdot}{\varepsilon})$ in law to a 2-tensor Gaussian white noise Γ with covariance structure $\mathcal{Q}$, and the (discrete version of the) joint convergence result (3.11) *follows.* ◊

This fluctuation theory is complemented by the following characterization of the fluctuation tensor $\mathcal{Q}$ by periodization in law. This characterization comes in form of a representative volume element (RVE) method, of which we give the optimal error estimate. In particular, comparing with the results for the RVE approximation $A_{\mathrm{hom},L,N}$ of the homogenized coefficients A_{hom} (cf. [209, 210, 206]), and choosing $N\simeq L^d$ below, we may conclude that an RVE approximation for $\mathcal{Q}$ with accuracy $O(L^{-\frac{d}{2}})$ (up to logarithmic corrections) is extracted at the same cost as an RVE approximation for A_{hom} with accuracy $O(L^{-d})$. Precise assumptions and notation are again postponed to Section 3.2.

Theorem 3.1.3. *Consider the discrete i.i.d. setting. Define*

$$A_{\mathrm{hom},L}e_i := \fint_{Q_L} A_L(\nabla\phi_{L,i}+e_i), \tag{3.15}$$

in terms of the L-periodized coefficient field A_L and corrector ϕ_L. Then the fluctuation tensor $\mathcal{Q}$ defined in Theorem 3.1.2(iv) satisfies

$$\mathcal{Q} = \lim_{L\uparrow\infty}\mathrm{Var}\left[L^{\frac{d}{2}}A_{\mathrm{hom},L}^*\right]. \tag{3.16}$$

In addition, considering i.i.d. realizations $(A_L^{(n)})_{n=1}^N$ *of* A_L *and setting* $A_{\mathrm{hom},L}^{(n)} := A_{\mathrm{hom},L}(A_L^{(n)})$, *we define the RVE approximation as the square of the sample standard deviation*

$$\mathcal{Q}_{L,N} := \frac{L^d}{N-1}\sum_{n=1}^N \big(A_{\mathrm{hom},L}^{(n)} - A_{\mathrm{hom},L,N}\big)^* \otimes \big(A_{\mathrm{hom},L}^{(n)} - A_{\mathrm{hom},L,N}\big)^*, \tag{3.17}$$

$$A_{\mathrm{hom},L,N} := \frac{1}{N}\sum_{n=1}^N A_{\mathrm{hom},L}^{(n)},$$

and for all $L, N \geq 2$ *there holds*

$$|\mathrm{Var}[\mathcal{Q}_{L,N}]|^{\frac12} \lesssim N^{-\frac12}, \qquad |\mathbb{E}[\mathcal{Q}_{L,N}] - \mathcal{Q}| \lesssim L^{-\frac{d}{2}} \log^{\frac{d}{2}} L. \qquad \Diamond$$

3.1.3 Relation to previous works

Uniform moment bounds on $\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla_\varepsilon u_\varepsilon$ were first obtained by Conlon and Naddaf [124] and by Gloria [198] in weaker forms, and were established in their optimal form in any dimension $d \geq 2$ by Marahrens and Otto [313] for discrete elliptic equations with i.i.d. conductances; see also [201] for continuum scalar equations. The proof of a CLT result classically splits into two parts: the approximate normality of fluctuations and the convergence of the rescaled variance towards some limiting variance. The latter part requires to go beyond the scaling of the variance by identifying the limiting "prefactor", which is a finer quantity. The first CLT result in stochastic homogenization concerned the fluctuations of $A_{\mathrm{hom},L}$ (cf. (3.15)) by Biskup, Salvi, and Wolff [63] in the discrete setting with i.i.d. conductances in the regime of small ellipticity contrast. The corresponding asymptotic normality result without restriction on the ellipticity contrast was obtained by Rossignol [375] and by Nolen [346, 347] (who was the first in stochastic homogenization to make use of Chatterjee's second-order Poincaré inequalities [112, 113]), while the convergence of the rescaled variance was later established by Gloria and Nolen [207]. CLT results for $\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla_\varepsilon u_\varepsilon$ and $\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F : \nabla\phi(\frac{\cdot}{\varepsilon})$ were subsequently established in dimensions $d > 2$ for discrete elliptic equations with i.i.d. Gaussian conductances by Mourrat and Otto [329], Mourrat and Nolen [328], and Gu and Mourrat [225], — based on the Helffer-Sjöstrand representation formula [238, 399, 335] and on tools introduced by Gloria, Marahrens, Neukamm, and Otto [209, 210, 313, 206, 205] (inspired by the unpublished work by Naddaf and Spencer [334]).

These various results indicated some intriguing link between the different limiting laws: the scaling limit of the gradient of the corrector is the Helmholtz projection of a Gaussian white noise with some particular covariance tensor, and the same tensor appears in the scaling limit of the solution operator. The pathwise convergence result in the form (3.11) was then partially discovered by Gu and Mourrat [225]. The mechanism behind this relation was initially quite mysterious: as emphasized in [225], such a relation between limiting laws is quite surprising since the fluctuations of the solution operator cannot be inferred from those of the corrector via the usual two-scale expansion (3.3). The pathwise character of fluctuations nevertheless appeared to be consistent with the variational approach to quantitative stochastic homogenization initiated by Armstrong and Smart [36], which indeed consists in extracting all information from energy-based quantities by deterministic arguments. In this spirit, together with a renormalization perspective in line with [33, 32, 31], Armstrong, Gu, and Mourrat [225, 326] proposed an interesting heuristics which suggests the validity of a pathwise theory of fluctuations. However, this heuristics — which was proposed shortly before the present contribution [167] appeared online — has not yet led to any rigorous proof of the pathwise link between the fluctuations of the solution operator and those of the corrector.

A variational quantity related to the homogenization commutator can be traced back to the work [36] by Armstrong and Smart. The stationary version of this quantity, which is key to study fluctuations and essentially coincides with the standard homogenization commutator Ξ that we define here, was independently introduced by Armstrong, Kuusi, and Mourrat in [32, Definition 5.3 and Paragraph 8.1] around the same time as the first version of the present work [167]. In their work, they established a functional CLT result for the homogenization commutator and for the corrector field under the assumption of a finite range of dependence of the coefficient field A (which could be further relaxed into weaker mixing-type assumptions) rather than using functional inequalities in the probability space. The same result was recovered shortly after by different methods by Gloria and Otto [208]. In contrast, the use of functional inequalities for the pathwise link between the solution operator and the corrector is more challenging to bypass, and functional inequalities are moreover particularly convenient if one wishes to obtain sharp error estimates. We refer to Section 3.1.4 for future perspectives on these questions.

Although this list of works is already quite long, the pathwise structure of fluctuations revealed in this contribution and the underlying mechanism described in (I)–(III) are made precise and rigorous here for the first time — in any setting.

3.1.4 Perspectives

In the case of (non-symmetric) continuum systems, we may extend our fluctuation theory in two different directions, which are postponed to the forthcoming works [168, 161, 160].

A first extension concerns the case of a coefficient field with strong correlations. In [168, 161], we consider the model framework of a coefficient field given by (the image by a Lipschitz function of) a Gaussian field that has algebraically decaying (but not necessarily integrable) covariance function c, say at some fixed (yet arbitrary) rate $c(x) \simeq (1+|x|)^{-\beta}$ parametrized by $\beta > 0$. For such coefficient fields, we establish in [168] the accuracy (3.12) of the two-scale expansion of the homogenization commutator in the suitable fluctuation scaling. The proof relies on a suitable weighted version of a Poincaré inequality in the probability space (cf. [162, 163]), (quenched) large-scale Calderón-Zygmund theory for $-\nabla \cdot A\nabla$ (cf. [204]), and moment bounds on the corrector (cf. [203]). This illustrates the surprising robustness of the pathwise structure with respect to the large-scale behavior of the homogenization commutator. Indeed, in dimension $d = 1$ (in which case the quantities under investigation are simpler and explicit), two typical behaviors have been identified in terms of scaling limit of the homogenization commutator Ξ, depending on the parameter β (cf. [42]):

— For $\beta > d = 1$: The commutator Ξ displays the CLT scaling and $\varepsilon^{-\frac{d}{2}}\Xi(\frac{\cdot}{\varepsilon})$ converges to a Gaussian white noise (Gaussian fluctuations, local limiting covariance structure).

— For $0 < \beta < d = 1$: The suitable rescaling $\varepsilon^{-\frac{\beta}{2}}\Xi(\frac{\cdot}{\varepsilon})$ converges up to a subsequence to a fractional Gaussian field (Gaussian fluctuations, nonlocal limiting covariance structure, potentially no uniqueness of the limit).

(Note that a different, non-Gaussian behavior may also occur, cf. [224, 291].) In particular, the pathwise result holds in these two examples whereas the rescaled homogenization commutator Ξ does not necessarily converge to white noise or may even not converge at all. The identification of the scaling limit of the homogenization commutator in higher dimensions is addressed in [161] for the whole range of values of $\beta > 0$, where we investigate the consequences of the locality of Ξ with respect to the coefficient field, combining techniques developed in [204, 203] with Malliavin calculus versions of the Helffer-Sjöstrand representation formula and of a second-order Poincaré inequality. This work extends [42] to dimensions $d \geq 2$.

A second extension concerns the case of a coefficient field with finite range of dependence, for which no functional inequality is in general satisfied. The convergence in law of $\varepsilon^{-\frac{d}{2}}\Xi(\frac{\cdot}{\varepsilon})$ to a Gaussian white noise (albeit without convergence rate) was already obtained in that setting in [32, 208]. In [208], it was achieved by combining a semi-group approach with the approximate locality of Ξ with respect to the coefficient field and with the assumption of finite range of dependence of the coefficients. The proof of the accuracy (3.12) of the two-scale expansion of the homogenization commutator in the CLT scaling is more involved and will be presented in [160] based on this semi-group approach. Optimal rates will also be considered.

Another direction of research that we plan to investigate in the future is the possibility of pursuing the 2-scale expansion of the homogenization commutator (3.5) to higher orders.

3.2 Main results

In this section, we introduce notation and assumptions, we state more precise versions of the main results (and make in particular explicit the norms of the test functions in the estimates), and we discuss various corollaries.

3.2.1 Notation and assumptions

We start by introducing the discrete i.i.d. framework in which our main results are established. We consider a random conductance problem on the integer lattice $\mathbb{Z}^d$, and denote by $\{e_i\}_{i=1}^d$ the canonical basis of $\mathbb{R}^d$. We regard $\mathbb{Z}^d$ as a graph with (unoriented) edge set $\mathcal{B} = \{(x,z) \in \mathbb{Z}^d \times \mathbb{Z}^d : |x-z| = 1\}$. For edges $(x,z) \in \mathcal{B}$, we also write $x \sim z$. We define the set of conductances $\{a(b)\}_{b\in\mathcal{B}}$ by $\Omega = [\lambda, 1]^{\mathcal{B}}$ for some fixed $0 < \lambda \leq 1$. We endow Ω with the σ-algebra generated by cylinder sets and with a probability measure $\mathbb{P}$. We denote by $\mathbb{E}[\cdot]$, $\mathrm{Var}[\cdot]$, and $\mathrm{Cov}[\cdot;\cdot]$ the associated expectation, variance, and covariance. A random field $u : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is said to be stationary if it is shift-covariant, in the sense of $u(x, a(\cdot - z)) = u(x-z, a)$ for all $x, z \in \mathbb{R}^d$ and $a \in \Omega$. In this contribution, we focus on the case when the probability measure $\mathbb{P}$ is a product measure, that is, when the conductances $\{a(b)\}_{b\in\mathcal{B}}$ are i.i.d. random variables, and we shall make use of available functional inequalities in this product probability space.

A realization $a \in \Omega$ is by definition a countable set $\{a(b)\}_{b\in\mathcal{B}}$ of conductances and is called an environment. Let ∇ denote (in this chapter only) the forward discrete gradient $(u : \mathbb{Z}^d \to \mathbb{R}) \mapsto (\nabla u : \mathbb{Z}^d \to \mathbb{R}^d)$ defined component-wise by $\nabla_i u(x) = u(x+e_i) - u(x)$ for $1 \leq i \leq d$, and let ∇^* denote the backward discrete gradient $(u : \mathbb{Z}^d \to \mathbb{R}) \mapsto (\nabla^* u : \mathbb{Z}^d \to \mathbb{R}^d)$ defined component-wise by $\nabla_i^* u(x) = u(x) - u(x-e_i)$ for all $1 \leq i \leq d$. The operator $-\nabla^* \cdot$ is thus the adjoint of ∇ on $\ell^2(\mathbb{Z}^d)$, and we consider the elliptic operator $-\nabla^* \cdot A\nabla$ with coefficients

$$A : x \mapsto A(x) := \mathrm{diag}\left[a(x, x+\xi_1), \ldots, a(x, x+\xi_d)\right],$$

acting on functions $u : \mathbb{Z}^d \to \mathbb{R}$ as

$$-\nabla^* \cdot A\nabla u(x) := \sum_{z:z\sim x} a(x,z)(u(x) - u(z)).$$

In order to state the standard qualitative homogenization result [277, 274] for the corresponding discrete elliptic equation, we consider for all $\varepsilon > 0$ the rescaled operator $-\nabla_\varepsilon^* \cdot A_\varepsilon \nabla_\varepsilon$, where $A_\varepsilon(\cdot) := A(\frac{\cdot}{\varepsilon})$, and where ∇_ε and ∇_ε^* act on functions $u_\varepsilon : \mathbb{Z}_\varepsilon^d := \varepsilon\mathbb{Z}^d \to \mathbb{R}$ and are defined componentwise by $\nabla_{\varepsilon,i} u_\varepsilon(x) = \varepsilon^{-1}(u_\varepsilon(x+\varepsilon e_i) - u_\varepsilon(x))$ and $\nabla_{\varepsilon,i}^* u_\varepsilon(x) = \varepsilon^{-1}(u_\varepsilon(x) - u_\varepsilon(x-\varepsilon e_i))$ for all i. We shall also let ∇_ε and ∇_ε^* act on continuous functions $u : \mathbb{R}^d \to \mathbb{R}$, so that $\nabla_\varepsilon u$ and $\nabla_\varepsilon^* u$ are continuous functions as well. If $u \in C^1(\mathbb{R}^d)$, then $\nabla_\varepsilon u(x)$ and $\nabla_\varepsilon^* u(x)$ converge to the continuum gradient $Du(x)$ for all $x \in \mathbb{R}^d$ as $\varepsilon \downarrow 0$. In what follows, for all $m \geq 1$, we systematically extend maps $v : \mathbb{Z}^d \to \mathbb{R}^m$ to piecewise constant maps $\mathbb{R}^d \to \mathbb{R}^m$ (still denoted by v) by setting $v|_{Q(x)} := v(x)$ for all $x \in \mathbb{Z}^d$ (where $Q(x)$ is the unit cube centered at x), and we use this notation e.g. for $A : \mathbb{Z}^d \to \mathbb{R}^{d\times d}$ (but also for $\phi : \mathbb{Z}^d \to \mathbb{R}^d$ and $\Xi : \mathbb{Z}^d \to \mathbb{R}^{d\times d}$ defined below). This systematic extension of functions on the lattice $\mathbb{Z}^d$ allows to state all discrete results in a form that would hold mutatis mutandis in the continuum setting. In addition, although in the discrete setting it is more natural to consider a symmetric coefficient field A, we use non-symmetric notation in the statement of the results in view of the non-symmetric continuum setting, and we denote by A^* the pointwise transpose field associated with A.

Qualitative stochastic homogenization [277, 274] ensures that for all $f, g \in C_c^\infty(\mathbb{R}^d)^d$ the unique Lax-Milgram solutions u_ε and v_ε in $\mathbb{R}^d$ of[1]

$$-\nabla_\varepsilon^* \cdot A_\varepsilon \nabla_\varepsilon u_\varepsilon \;=\; \nabla_\varepsilon^* \cdot f, \qquad -\nabla_\varepsilon^* \cdot A_\varepsilon^* \nabla_\varepsilon v_\varepsilon \;=\; \nabla_\varepsilon^* \cdot f, \tag{3.18}$$

almost surely converges weakly as $\varepsilon \downarrow 0$ to the unique Lax-Milgram solutions $\bar{u}$ and $\bar{v}$ in $\mathbb{R}^d$ of the (continuum) elliptic equations

$$-D \cdot A_{\mathrm{hom}} D\bar{u} \;=\; D \cdot f, \qquad -D \cdot A_{\mathrm{hom}}^* D\bar{u} \;=\; D \cdot g, \tag{3.19}$$

respectively, where A_{hom} is the homogenized matrix characterized by

$$A_{\mathrm{hom}} e_i \;=\; \mathbb{E}\left[A(\nabla\phi_i + e_i)\right], \tag{3.20}$$

for all $1 \le i \le d$, where ϕ_i is the so-called corrector in direction e_i. It is defined, for almost every realization A, as the unique solution in $\mathbb{Z}^d$ of

$$-\nabla^* \cdot A(\nabla\phi_i + e_i) \;=\; 0, \tag{3.21}$$

with $\nabla\phi_i$ stationary and $\phi_i(0) = 0$. We then set $\phi := (\phi_i)_{i=1}^d$. We denote by ϕ^* the corrector associated with the coefficient field A^*, and note that $(A^*)_{\mathrm{hom}} = (A_{\mathrm{hom}})^*$. For symmetric coefficient fields, $A^* = A$, $\phi^* = \phi$, and $A_{\mathrm{hom}}^* = A_{\mathrm{hom}}$.

We consider the fluctuations of the field ∇u_ε and of the flux $A_\varepsilon \nabla u_\varepsilon$, as encoded in the random functionals $I_1^\varepsilon : (f,g) \mapsto I_1^\varepsilon(f,g)$ and $I_2^\varepsilon : (f,g) \mapsto I_2^\varepsilon(f,g)$ defined for all $f, g \in C_c^\infty(\mathbb{R}^d)$ by

$$\begin{aligned} I_1^\varepsilon(f,g) \;&:=\; \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla_\varepsilon (u_\varepsilon - \mathbb{E}\left[u_\varepsilon\right]), \\ I_2^\varepsilon(f,g) \;&:=\; \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \big(A_\varepsilon \nabla_\varepsilon u_\varepsilon - \mathbb{E}\left[A_\varepsilon \nabla_\varepsilon u_\varepsilon\right]\big). \end{aligned}$$

We further encode the fluctuations of the corrector field $\nabla\phi$ and flux $A(\nabla\phi + \mathrm{Id})$ in the random funcitonals $J_1^\varepsilon : F \mapsto J_1^\varepsilon(F)$ and $J_2^\varepsilon : F \mapsto J_2^\varepsilon(F)$ defined for all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ by

$$\begin{aligned} J_1^\varepsilon(F) \;&:=\; \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F(x) : \nabla\phi(\tfrac{x}{\varepsilon})\, dx, \\ J_2^\varepsilon(F) \;&:=\; \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F(x) : \big(A_\varepsilon(x)(\nabla\phi(\tfrac{x}{\varepsilon}) + \mathrm{Id}) - A_{\mathrm{hom}}\big)\, dx. \end{aligned}$$

As explained above, a crucial role is played by the (standard) homogenization commutator, which in the present discrete setting takes the form $\Xi := (\Xi_i)_{i=1}^d$ with

$$\Xi_i \;:=\; A(\nabla\phi_i + e_i) - A_{\mathrm{hom}}(\nabla\phi_i + e_i), \qquad \Xi_{ij} := (\Xi_i)_j, \tag{3.22}$$

and by the error in the two-scale expansion of the homogenization commutator of the solution. These quantities are encoded in the random functionals $I_0^\varepsilon : \mathcal{F} \mapsto I_0^\varepsilon(F)$ and $E^\varepsilon : (f,g) \mapsto E^\varepsilon(f,g)$ defined for all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ and all $f, g \in C_c^\infty(\mathbb{R}^d)^d$ by

$$\begin{aligned} I_0^\varepsilon(F) \;&:=\; \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F(x) : \Xi(\tfrac{x}{\varepsilon})\, dx, \\ E^\varepsilon(f,g) \;&:=\; \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \big(A_\varepsilon \nabla_\varepsilon u_\varepsilon - A_{\mathrm{hom}} \nabla_\varepsilon u_\varepsilon - \mathbb{E}\left[A_\varepsilon \nabla_\varepsilon u_\varepsilon - A_{\mathrm{hom}} \nabla_\varepsilon u_\varepsilon\right]\big) - \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot e_i(\tfrac{\cdot}{\varepsilon}) D_i \bar{u}. \end{aligned}$$

1. These equations are understood as follows: for all $x \in Q$ the function $u_\varepsilon(\varepsilon x + \cdot)$ on $\mathbb{Z}_\varepsilon^d$ is the solution of the discrete elliptic equation with coefficient A_ε and with right-hand side $\nabla_\varepsilon^* \cdot f(\varepsilon x + \cdot)$. This definition allows to state results in a form that holds in the continuum setting, in terms of norms of the right-hand side that do not need to embed into the space of continuous functions.

Since the case $d = 1$ is much simpler and well-understood [223], we shall only focus in the sequel on dimensions $d \geq 2$.

We first recall the following uniform boundedness result for I_0^ε, establishing the CLT scaling for the fluctuations of the homogenization commutator (cf. Theorem 3.1.2(i)). Although essentially contained in the main result of the first contribution of Gloria and Otto to the field [209], a short proof is included for completeness in Section 3.3.

Proposition 3.2.1. *Let $d \geq 2$ and let $\mathbb{P}$ be a product measure, and let $w_1(z) := 1 + |z|$. For all $\varepsilon > 0$ and all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ we have for all $0 < p - 1 \ll 1$ and all $\alpha > d\frac{p-1}{4p}$,*

$$\mathbb{E}\left[|I_0^\varepsilon(F)|^2\right]^{\frac{1}{2}} \lesssim_{\alpha,p} \|w_1^{2\alpha} F\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}. \qquad \Diamond$$

3.2.2 Pathwise structure

Our first main result establishes the smallness of the rescaled error E^ε in the two-scale expansion of the homogenization commutator (cf. Theorem 3.1.2(ii)), which is the key to the pathwise structure (3.11). As for Proposition 3.2.1, the proof relies on the Poincaré inequality in the probability space that is satisfied for i.i.d. coefficients. From a technical point of view, we exploit the large-scale Calderón-Zygmund theory for the operator $-\nabla^* \cdot A\nabla$ in the form developed in [204].

Proposition 3.2.2. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, let μ_d be defined in (3.13), and let $w_1(z) := 1 + |z|$. For all $\varepsilon > 0$ and all $f, g \in C_c^\infty(\mathbb{R}^d)^d$ we have for all $0 < p - 1 \ll 1$ and all $\alpha > d\frac{p-1}{4p}$,*

$$\mathbb{E}\left[|E^\varepsilon(f,g)|^2\right]^{\frac{1}{2}} \lesssim_{\alpha,p} \varepsilon\mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}}\Big(\|f\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Dg\|_{\mathrm{L}^{4p}(\mathbb{R}^d)} + \|g\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Df\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}\Big). \qquad \Diamond$$

Remark 3.2.3. For simplicity the estimates in Propositions 3.2.1 and 3.2.2 above are stated and proved for second moments only, but the same arguments yield similar estimates for all algebraic (and even stretched exponential) moments (cf. [168, 161]). $\Diamond$

In view of identity (3.7) (which indeed holds in the discrete setting up to a higher-order discretization error), the above result implies that the large-scale fluctuations of I_1^ε and I_2^ε are driven by those of I_0^ε in a pathwise sense. Identity (3.9) (which again holds up to a discretization error) yields a similar pathwise result for J_1^ε and J_2^ε.

Corollary 3.2.4. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, let $\bar{\mathcal{P}}_H$, $\bar{\mathcal{P}}_H^*$, and $\bar{\mathcal{P}}_L^*$ be defined in (3.8), let μ_d be defined in (3.13), and let $w_1(z) := 1 + |z|$. For all $\varepsilon > 0$, all $f, g \in C_c^\infty(\mathbb{R}^d)^d$, and all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$, we have for all $0 < p - 1 \ll 1$ and all $\alpha > d\frac{p-1}{4p}$,*

$$\begin{aligned}\mathbb{E}\left[|I_1^\varepsilon(f,g) - I_0^\varepsilon(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g)|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[|I_2^\varepsilon(f,g) + I_0^\varepsilon(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_L^* g)|^2\right]^{\frac{1}{2}} \\ \lesssim_{\alpha,p} \varepsilon\mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}}\Big(\|f\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Dg\|_{\mathrm{L}^{4p}(\mathbb{R}^d)} + \|g\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Df\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}\Big),\end{aligned} \tag{3.23}$$

and also

$$\mathbb{E}\left[|J_1^\varepsilon(F) + I_0^\varepsilon(\bar{\mathcal{P}}_H^* F)|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[|J_2^\varepsilon(F) - I_0^\varepsilon(\bar{\mathcal{P}}_L^* F)|^2\right]^{\frac{1}{2}} \lesssim_{\alpha,p} \varepsilon\|w_1^{2\alpha} DF\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}, \tag{3.24}$$

where by definition we have $\bar{\mathcal{P}}_H f = -D\bar{u}$ and $\bar{\mathcal{P}}_H^* g = -D\bar{v}$. In particular, we give a sense to $I_0^\varepsilon(\bar{\mathcal{P}}_H^* F)$ and $I_0^\varepsilon(\bar{\mathcal{P}}_L^* F)$ in $\mathrm{L}^2(\Omega)$ for all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$, even when $\bar{\mathcal{P}}_H^* F$ and $\bar{\mathcal{P}}_L^* F$ do not have integrable decay. $\Diamond$

Remark 3.2.5. For all $f, g \in C_c^\infty(\mathbb{R}^d)^d$, we may also consider the unique Lax-Milgram solutions u_ε° and v_ε° in $\mathbb{R}^d$ of

$$-\nabla_\varepsilon^* \cdot A_\varepsilon \nabla_\varepsilon u_\varepsilon^\circ = \nabla_\varepsilon^* \cdot A_\varepsilon f, \qquad -\nabla_\varepsilon^* \cdot A_\varepsilon^* \nabla_\varepsilon v_\varepsilon^\circ = \nabla_\varepsilon^* \cdot A_\varepsilon^* g,$$

which, almost surely, converge weakly as $\varepsilon \downarrow 0$ to the unique Lax-Milgram solutions $\bar{u}^\circ$ and $\bar{v}^\circ$ in $\mathbb{R}^d$ of

$$-D \cdot A_{\mathrm{hom}} D\bar{u}^\circ = D \cdot A_{\mathrm{hom}} f, \qquad -D \cdot A_{\mathrm{hom}}^* D\bar{v}^\circ = D \cdot A_{\mathrm{hom}}^* g,$$

respectively. Similar considerations as in the proof of Proposition 3.2.2 and Corollary 3.2.4 then lead to a pathwise result for the fluctuations of the random functionals $I_3^\varepsilon : (f,g) \mapsto I_3^\varepsilon(f,g)$ and $I_4^\varepsilon : (f,g) \mapsto I_4^\varepsilon(f,g)$ defined for all $f, g \in C_c^\infty(\mathbb{R}^d)^d$ by

$$\begin{aligned} I_3^\varepsilon(f,g) &:= \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla_\varepsilon (u_\varepsilon^\circ - \mathbb{E}[u_\varepsilon^\circ]), \\ I_4^\varepsilon(f,g) &:= \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \big(A_\varepsilon(\nabla_\varepsilon u_\varepsilon^\circ + f) - \mathbb{E}[A_\varepsilon(\nabla_\varepsilon u_\varepsilon^\circ + f)] \big), \end{aligned}$$

and it takes the form

$$\mathbb{E}\Big[\big|I_3^\varepsilon(f,g) + I_0^\varepsilon\big(\bar{\mathcal{P}}_L f \otimes \bar{\mathcal{P}}_H^* g\big)\big|^2\Big]^{\frac12} + \mathbb{E}\Big[\big|I_4^\varepsilon(f,g) - I_0^\varepsilon\big(\bar{\mathcal{P}}_L f \otimes \bar{\mathcal{P}}_L^* g\big)\big|^2\Big]^{\frac12} \lesssim_{f,g} \varepsilon \mu_d(\tfrac{1}{\varepsilon})^{\frac12},$$

where by definition $\bar{\mathcal{P}}_H f = -D\bar{u}$, $\bar{\mathcal{P}}_L f = D\bar{u}^\circ + f$, $\bar{\mathcal{P}}_H^* g = -D\bar{v}$, and $\bar{\mathcal{P}}_L^* g = D\bar{v}^\circ + g$. $\Diamond$

Incidentally, as a consequence of our analysis, combining the two-scale expansion of the homogenization commutator (3.5) with identity (3.7), we obtain a new (nonlocal) two-scale expansion for the solution u_ε that is not only accurate in the strong $\mathrm{L}^2(\mathbb{R}^d)$-norm but also in the fluctuation scaling, in contrast with the usual two-scale expansion (3.3) (cf. [225]). (The second estimate below is a reformulation of Proposition 3.2.2, whereas the first estimate is a corollary of [203, Theorem 3].)

Corollary 3.2.6. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, and let μ_d be defined in* (3.13). *For all $\varepsilon > 0$ and all $f \in C_c^\infty(\mathbb{R}^d)^d$, we set*

$$r_\varepsilon(f) := u_\varepsilon - \Big(\underbrace{\mathbb{E}[u_\varepsilon] + (-\nabla_\varepsilon^* \cdot A_{\mathrm{hom}} \nabla_\varepsilon)^{-1} (\nabla_\varepsilon^* \cdot \Xi(\tfrac{\cdot}{\varepsilon}) \nabla_\varepsilon \bar{u})}_{\text{nonlocal two-scale expansion of } u_\varepsilon} \Big).$$

Then, this (nonlocal) two-scale expansion correctly captures

— *the* spatial *oscillations of $\nabla_\varepsilon u_\varepsilon$ in a strong norm: for all $f \in C_c^\infty(\mathbb{R}^d)^d$,*

$$\mathbb{E}\Big[\|\nabla_\varepsilon r_\varepsilon(f)\|_{\mathrm{L}^2(\mathbb{R}^d)}^2\Big]^{\frac12} \lesssim_f \varepsilon \mu_d(\tfrac{1}{\varepsilon})^{\frac12};$$

— *the* random *fluctuations of $\nabla_\varepsilon u_\varepsilon$ in the CLT scaling: for all $f, g \in C_c^\infty(\mathbb{R}^d)^d$,*

$$\mathbb{E}\Big[\Big|\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla_\varepsilon r_\varepsilon(f)\Big|^2\Big]^{\frac12} \lesssim_{f,g} \varepsilon \mu_d(\tfrac{1}{\varepsilon})^{\frac12}. \qquad \Diamond$$

3.2.3 Approximate normality

We turn to the normal approximation result for the homogenization commutator (cf. Theorem 3.1.2(iii)), which states that the fluctuations of $\varepsilon^{-\frac{d}{2}} \Xi(\frac{\cdot}{\varepsilon})$ are asymptotically Gaussian (up to a non-degeneracy condition that is elucidated in Proposition 3.2.9 below). The approach is inspired by previous works by Nolen [346, 347], based on a second-order Poincaré inequality à la Chatterjee [112, 282], which is key to optimal convergence rates. Since such functional inequalities are not easily amenable to the use of large-scale Calderón-Zygmund theory for the operator $-\nabla^* \cdot A\nabla$, we rather have to exploit optimal annealed estimates on mixed gradients of the Green's function [313], which leads to an additional $\log(2+\frac{1}{\varepsilon})$ factor in the rate below (we do not know whether this is optimal). The proof exploits the approximate locality of the homogenization commutator Ξ. Note that the norms of the test function below are substantially weaker than $\mathrm{L}^1(\mathbb{R}^d)$ in terms of integrability and are thus compatible with the behavior of Helmholtz projections of smooth and compactly supported functions, which is crucial in view of the combination with the pathwise result of Corollary 3.2.4.

Proposition 3.2.7. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, and let μ_d and $\delta_{\mathcal{N}}$ be defined in* (3.13) *and* (3.14)*. For all $\varepsilon > 0$ and all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ with $\mathrm{Var}\,[I_0^\varepsilon(F)] > 0$, we have for all $\alpha > 0$,*

$$\delta_{\mathcal{N}}(I_0^\varepsilon(F)) \lesssim \varepsilon^{\frac{d}{2}} \frac{\|F\|^3_{\mathrm{L}^3(\mathbb{R}^d)} + \|w_1^\alpha DF\|^3_{\mathrm{L}^3(\mathbb{R}^d)}}{\mathrm{Var}\,[I_0^\varepsilon(F)]^{\frac{3}{2}}} + \varepsilon^{\frac{d}{2}} \log(2+\tfrac{1}{\varepsilon}) \frac{\|w_1^\alpha F\|^2_{\mathrm{L}^4(\mathbb{R}^d)} + \|w_1^\alpha DF\|^2_{\mathrm{L}^4(\mathbb{R}^d)}}{\mathrm{Var}\,[I_0^\varepsilon(F)]}. \quad \Diamond$$

Remark 3.2.8. In the case of i.i.d. conductances that are (smooth transformations of) Gaussian random variables, a nicer version of a second-order Poincaré inequality is available (cf. [113, Theorem 2.2]), which further gives a control on the total variation distance and is amenable to the use of large-scale Calderón-Zygmund theory, thus avoiding the use of Green's functions for the operator $-\nabla^* \cdot A\nabla$ and leading to the optimal rate $\varepsilon^{\frac{d}{2}}$ (without the spurious logarithmic factor). Such an argument is detailed in the continuum case in the forthcoming work [161]. $\Diamond$

3.2.4 Covariance structure

Since I_0^ε is asymptotically Gaussian, it remains to identify the limit of its covariance structure (cf. Theorem 3.1.2(iv)). The following shows that the limiting covariance is that of a (tensorial) white noise with some non-degenerate covariance tensor $\mathcal{Q}$. The convergence rate in (3.25) below is new in any dimension and is expected to be optimal. The proof crucially relies on the approximate locality of the homogenization commutator and on (an i.i.d. version of) the Helffer-Sjöstrand representation formula for the variance [238, 399, 335], which is a stronger tool than the Poincaré inequality in the probability space. As for the pathwise result, the proof exploits the large-scale Calderón-Zygmund theory for the operator $-\nabla^* \cdot A\nabla$. Again note that the norms of the test functions below are substantially weaker than $\mathrm{L}^1(\mathbb{R}^d)$ in terms of integrability, as is required in view of the combination with the pathwise result of Corollary 3.2.4.

Proposition 3.2.9. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, let μ_d be defined in* (3.13)*, and let $w_1(z) := 1 + |z|$.*

(i) There exists a symmetric[2] 4-tensor $\mathcal{Q}$ such that for all $\varepsilon > 0$ and all $F, G \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ we have for all $0 < p-1 \ll 1$ and all $\alpha > d\frac{p-1}{4p}$,

$$\left| \mathrm{Cov}\,[I_0^\varepsilon(F); I_0^\varepsilon(G)] - \int_{\mathbb{R}^d} F(x) : \mathcal{Q}\, G(x)dx \right| \lesssim_{\alpha,p} \varepsilon \mu_d(\tfrac{1}{\varepsilon})^{\frac{1}{2}} \times \big(\|F\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|w_1^{2\alpha} DF\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\big)\big(\|G\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|w_1^{2\alpha} DG\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\big). \quad (3.25)$$

Moreover, for all $1 \leq i,j,k,l \leq d$ and all $\delta > 0$, we have for all $L \geq 1$,

$$\left| \mathcal{Q}_{ijkl} - \int_{Q_{2L}} \frac{|Q_L \cap (x+Q_L)|}{|Q_L|} \mathrm{Cov}\,[\Xi_{ij}(x); \Xi_{kl}(0)]\, dx \right| \lesssim_\delta L^{\delta - \frac{1}{2}}, \quad (3.26)$$

where Q_L denotes the cube of sidelength L centered at the origin.

2. Since $\mathcal{Q}$ is a (limiting) covariance, it is of course symmetric in the sense of $\mathcal{Q}_{ijkl} = \mathcal{Q}_{klij}$. If the coefficients A are symmetric, then it has the additional symmetry $\mathcal{Q}_{ijkl} = \mathcal{Q}_{jikl}$ (hence also $\mathcal{Q}_{ijkl} = \mathcal{Q}_{ijlk}$).

(ii) If in addition $\mathbb{P} = \pi^{\otimes \mathcal{B}}$ *with* π *a nontrivial probability measure on* $[\lambda, 1]$*, then this* effective fluctuation tensor $\mathcal{Q}$ *is non-degenerate in the sense that* $(\xi \otimes \xi) : \mathcal{Q}\,(\xi \otimes \xi) > 0$ *for all* $\xi \in \mathbb{R}^d \setminus \{0\}$. ◊

Remarks 3.2.10. Comments are in order.

— When applying a covariance inequality (cf. Lemma 3.5.1 below) to the argument of the limit in the Green-Kubo formula (3.26), we end up with the bound

$$\int_{Q_{2L}} \frac{|Q_L \cap (x + Q_L)|}{|Q_L|} |\operatorname{Cov}[\Xi_{ij}(x); \Xi_{kl}(0)]|\, dx \lesssim \log L,$$

which is sharp. The main difficulty to characterize the limiting covariance structure is that, as usual for Green-Kubo formulas, the covariance of the homogenization commutator Ξ is not an integrable function, and cancellations have to be unravelled.

— The optimal rate (3.25) for the convergence of the covariance structure of I_0^ε owes to the very local structure of the homogenization commutator Ξ, and, combined with the pathwise result of Corollary 3.2.4, it carries over to I_1^ε, I_2^ε, J_1^ε, and J_2^ε. In [329, 225, 328], the usual Gaussian Helffer-Sjöstrand representation formula for the variance [238, 399, 335] was already used in order to prove the convergence of the covariance structure of I_1^ε and J_1^ε for $d > 2$, but the obtained convergence rate was suboptimal in every dimension. ◊

The combination of Propositions 3.2.7 and 3.2.9 leads to a complete scaling limit result for I_0^ε, which thus converges in law to a Gaussian white noise. As in Proposition 3.2.7, we do not know whether the full logarithmic factor is optimal for $d = 2$.

Corollary 3.2.11. *Let* $d \geq 2$*, let* $\mathbb{P}$ *be a product measure, and let* μ_d *be defined as in* (3.13)*. Let* $\mathcal{Q}$ *be the* 4*-tensor defined in Proposition 3.2.9(i), and let* Γ *denote the* 2*-tensor Gaussian white noise with covariance tensor* $\mathcal{Q}$*, that is, the Gaussian random Schwartz distribution with zero expectation* $\mathbb{E}[\Gamma(F)] = 0$ *and with covariance structure* $\operatorname{Cov}[\Gamma(F); \Gamma(G)] = \int_{\mathbb{R}^d} F : \mathcal{Q}\, G$ *for all* $F, G \in C_c^\infty(\mathbb{R}^d)^{d \times d}$*. Then for all* $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ *the random variable* $I_0^\varepsilon(F)$ *converges in law to* $\Gamma(F)$*, and for* $\int_{\mathbb{R}^d} F : \mathcal{Q}\, F \neq 0$ *there holds*

$$(\mathrm{d_W} + \mathrm{d_K})\,(I_0^\varepsilon(F), \Gamma(F)) \;\lesssim_F\; \varepsilon \mu_d(\tfrac{1}{\varepsilon}). \qquad ◊$$

Remarks 3.2.12. Comments are in order.

— Combined with the pathwise result of Corollary 3.2.4, this result further leads to a proof of the joint convergence (3.11) and implies in particular quantitative versions of the known scaling limit results for I_1^ε and J_1^ε: For all $f, g \in C_c^\infty(\mathbb{R}^d)^d$ and all $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ the random variables $I_1^\varepsilon(f, g)$ and $J_1^\varepsilon(F)$ converge in law to $\Gamma(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g)$ and $-\Gamma(\bar{\mathcal{P}}_H^* F)$, respectively, and moreover for $\int_{\mathbb{R}^d} (\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g) : \mathcal{Q}\,(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g) \neq 0$ and $\int_{\mathbb{R}^d} \bar{\mathcal{P}}_H^* F : \mathcal{Q}\, \bar{\mathcal{P}}_H^* F \neq 0$ there hold

$$\begin{aligned} (\mathrm{d_W} + \mathrm{d_K})\left(I_1^\varepsilon(f,g), \Gamma(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g)\right) &\;\lesssim_{f,g}\; \varepsilon \mu_d(\tfrac{1}{\varepsilon}), \\ (\mathrm{d_W} + \mathrm{d_K})\left(J_1^\varepsilon(F), -\Gamma(\bar{\mathcal{P}}_H^* F)\right) &\;\lesssim_{\mathcal{F}}\; \varepsilon \mu_d(\tfrac{1}{\varepsilon}). \end{aligned}$$

This extends and unifies [207, 329, 328, 225], and yields the first scaling limit results in the critical dimension $d = 2$. Convergence rates are new in any dimension, and are optimal at least for $d > 2$.

— *SPDE representation for the scaling limit of the solution operator.* The scaling limit result for I_1^ε above indicates that $\varepsilon^{-\frac{d}{2}} \nabla_\varepsilon (u_\varepsilon - \mathbb{E}[u_\varepsilon])$ (seen as a random functional) converges in law to the formal solution DU in $\mathbb{R}^d$ of

$$-D \cdot A_{\mathrm{hom}} DU = D \cdot (\Gamma_i D_i \bar{u}).$$

This justifies a posteriori the conclusion (although not the strategy) of the heuristics due to Armstrong, Gu, and Mourrat [225] in dimensions $d \geq 2$. (See also [223] for a rigorous treatment of the easier case of dimension $d = 1$.)

— *Scaling limit of the corrector.* The scaling limit result for J_1^ε above shows that the rescaled corrector field $\varepsilon^{-\frac{d}{2}} D\phi(\frac{\cdot}{\varepsilon})$ (seen as a random functional) converges in law to $D(-D \cdot A_{\mathrm{hom}} D)^{-1} D \cdot \Gamma$, that is, to the gradient of a variant of the so-called Gaussian free field. This variant involves both A_{hom} and $\mathcal{Q}$. As pointed out in [329], it is easily checked in Fourier space that this variant does in general not coincide with the standard Gaussian free field (unless the compatibility condition $\mathcal{Q}_{ijkl} = \eta_{ik} A_{\mathrm{hom},lj}$ is satisfied for some matrix η, which however does not hold in general, see e.g. [226, Section 3] and (5.35)). This variant of the Gaussian free field is studied in [226], where it is in particular shown that it is Markovian only in the standard case. In the critical dimension $d = 2$, since the whole-space Gaussian free field is not well-defined (only its gradient is), this implies the non-existence of stationary correctors.

— *Gu and Mourrat's observation.* With the above results at hand, we recover the observation by Gu and Mourrat [225] that the usual two-scale expansion (3.3) of u_ε is not accurate in the fluctuation scaling. The above indeed shows that the fluctuations of $\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla_\varepsilon (u_\varepsilon - \mathbb{E}[u_\varepsilon])$ and of $\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \nabla\phi_i(\frac{\cdot}{\varepsilon}) \nabla_{\varepsilon,i} \bar{u}$ are asymptotically given by $\Gamma(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g)$ and by $\Gamma(\bar{\mathcal{P}}_H^*((\bar{\mathcal{P}}_H f) \otimes g))$, respectively, and therefore do not coincide. ◊

3.2.5 Approximation of the fluctuation tensor

We finally turn to the representative volume element (RVE) approximation of $\mathcal{Q}$ (cf. Theorem 3.1.3). Indeed, the Green-Kubo formula (3.26) for the fluctuation tensor $\mathcal{Q}$ is of no practical use in applications since it requires to solve the corrector equation on the whole space and for every realization of the random coefficient field. It is then natural to look for a suitable RVE approximation. This consists in introducing an artificial period $L > 0$ and in considering an L-periodized coefficient field A_L, typically given by a suitable periodization in law (cf. [174]). In the present i.i.d. setting, we simply define $A_L(x + Ly) := A(x)$ for all $y \in \mathbb{Z}^d$ and $x \in Q_L$. Note that the map $\Omega \to \Omega : A \mapsto A_L$ pushes forward the measure $\mathbb{P}$ to a measure $\mathbb{P}_L$ concentrated on L-periodic coefficients, so that we may view A_L as an element of $\Omega_L = [\lambda, 1]^{\mathcal{B}_L}$ equipped with the product measure $\mathbb{P}_L = \pi^{\otimes \mathcal{B}_L}$, where $\mathcal{B}_L := \{(x, x + e_i) : x \in Q_L \cap \mathbb{Z}^d, 1 \leq i \leq d\}$. We then define the L-periodized corrector $\phi_{L,i}$ in the direction e_i as the unique L-periodic solution in $Q_L \cap \mathbb{Z}^d$ of

$$-\nabla^* \cdot A_L(\nabla\phi_{L,i} + e_i) = 0, \tag{3.27}$$

satisfying $\sum_{z \in Q_L \cap \mathbb{Z}^d} \phi_{L,i}(z) = 0$, and we set $\phi_L := (\phi_{L,i})_{i=1}^d$ (which we implicitly extend as usual into a periodic piecewise constant map on $\mathbb{R}^d$). The spatial average of the flux,

$$A_{\mathrm{hom},L} e_i := \fint_{Q_L} A_L(\nabla\phi_{L,i} + e_i),$$

is then an RVE approximation for the homogenized coefficient $A_{\mathrm{hom}} e_i = \mathbb{E}[A(\nabla\phi_i + e_i)]$. The optimal numerical analysis of this approximation was originally performed in [209, 210, 206], where it was established that for all $L \geq 2$ there holds

$$|\mathrm{Var}[A_{\mathrm{hom},L}]|^{\frac{1}{2}} \lesssim L^{-\frac{d}{2}}, \qquad |\mathbb{E}[A_{\mathrm{hom},L}] - A_{\mathrm{hom}}| \lesssim L^{-d} \log^d L. \tag{3.28}$$

In Theorem 3.1.3, we claim that the fluctuation tensor $\mathcal{Q}$ coincides with the limit of the rescaled variance of $A^*_{\mathrm{hom},L}$. In addition, this characterization naturally leads to an RVE approximation $\mathcal{Q}_{L,N}$ for $\mathcal{Q}$, cf. (3.17), of which we obtain the optimal error estimate.

Remarks 3.2.13. Comments are in order.

— Definition (3.17) for $\mathcal{Q}_{L,N}$ is equivalent to

$$\mathcal{Q}_{L,N} = \frac{L^d}{N-1} \sum_{n=1}^{N} \Big(\fint_{Q_L} \Xi_{L,N}^{(n)} \Big) \otimes \Big(\fint_{Q_L} \Xi_{L,N}^{(n)} \Big), \tag{3.29}$$

in terms of

$$\Xi_{L,N,i}^{(n)} := A_L^{(n)} (\nabla \phi_{L,i}^{(n)} + e_i) - A_{\mathrm{hom},L,N} (\nabla \phi_{L,i}^{(n)} + e_i),$$

with the obvious notation $\nabla \phi_L^{(n)} := \nabla \phi_L(A_L^{(n)})$. Since by stationarity

$$\int_{Q_L} \mathrm{Cov}\left[\Xi_{L,N}(x); \Xi_{L,N}(0)\right] dx = L^d \,\mathrm{Var}\left[\fint_{Q_L} \Xi_{L,N}\right],$$

formula (3.29) is in the spirit of the Green-Kubo formula (3.26).

— In (3.28) the standard deviation $|\mathrm{Var}\,[A_{\mathrm{hom},L}]|^{\frac12}$ of the RVE approximation for A_{hom} is seen to be $O(L^{\frac d2})$ times larger than the systematic error $|\mathbb{E}\,[A_{\mathrm{hom},L}] - A_{\mathrm{hom}}|$ (up to a logarithmic correction). In practice, we rather use $A_{\mathrm{hom},L,N}$ as an approximation for A_{hom},

$$|\mathrm{Var}\,[A_{\mathrm{hom},L,N}]|^{\frac12} \lesssim N^{-\frac12} L^{-\frac d2}, \qquad |\mathbb{E}\,[A_{\mathrm{hom},L,N}] - A_{\mathrm{hom}}| \lesssim L^{-d} \log^d L,$$

since in the regime $N \simeq L^d$ the standard deviation becomes of the same order as the systematic error $O(L^{-d})$. Combining this with the estimates in Theorem 3.1.3, since $\mathcal{Q}_{L,N}$ is extracted at no further cost than $A_{\mathrm{hom},L,N}$ itself, we may infer that an RVE approximation for $\mathcal{Q}$ with accuracy $O(L^{-\frac d2})$ is extracted at the same cost as an RVE approximation for A_{hom} with accuracy $O(L^{-d})$.

— In [346, 347, 207] (see also [63, 375]), the fluctuations of the RVE approximation $A_{\mathrm{hom},L}$ for the homogenized coefficient A_{hom} was investigated. Combined with the characterization (3.16) of the limit of the rescaled variance, the main result in [207] takes on the following guise, for all $L \geq 2$ and all $N \geq 1$,

$$\sup_{\xi \in \mathbb{R}^d \setminus \{0\}} (\mathrm{d_W} + \mathrm{d_K}) \left(N^{\frac12} L^{\frac d2} \frac{\xi \cdot (A_{\mathrm{hom},L,N} - A_{\mathrm{hom}})\xi}{(\xi \otimes \xi : \mathcal{Q} : \xi \otimes \xi)^{\frac12}}, \mathcal{N} \right) \lesssim N^{-\frac12} L^{-\frac d2} \log^d L. \qquad \Diamond$$

3.3 Pathwise structure

Henceforth we place ourselves in the discrete setting of Section 3.2. In the present section, we establish the pathwise result stated in Proposition 3.2.2, that is, the main novelty of this contribution. Similar estimates also lead to the CLT scaling result stated in Proposition 3.2.1, and we further deduce Corollary 3.2.4.

3.3.1 Structure of the proof and auxiliary results

The main tool that we use to prove Propositions 3.2.1 and 3.2.2 is the following Poincaré inequality (or spectral gap estimate) in the probability space, which holds for any product measure $\mathbb{P}$ on Ω (see e.g. [209, Lemma 2.3] or Chapter 4 for a proof). Let us first fix some notation. Let $X = X(A)$ be a random variable on Ω, that is, a measurable function of $(a(b))_{b \in \mathcal{B}}$. We choose an i.i.d. copy A' of A,[3] and for all $b \in \mathcal{B}$ we denote by A^b the random field that coincides with A on all edges $b' \neq b$ and

3. Although we are then working on a product probability space, we use for simplicity the same notation $\mathbb{P}$ (and $\mathbb{E}$) for the product probability measure (and expectation), that is, with respect to both A and A'.

with A' on edge b. In particular, A and A^b have the same law. We use the abbreviation $X^b = X(A^b)$ and define the difference operator $\Delta_b X := X - X^b$, which we call the *Glauber vertical derivative* at edge b (with a slight abuse of terminology).

Lemma 3.3.1 (e.g. [209])**.** *Let $\mathbb{P}$ be a product measure. For all $X = X(A) \in \mathrm{L}^2(\Omega)$ we have*

$$\operatorname{Var}[X] \le \frac{1}{2}\,\mathbb{E}\left[\sum_{b\in\mathcal{B}} |\Delta_b X|^2\right]. \qquad \Diamond$$

Next to the corrector ϕ, we need to recall the notion of flux corrector σ, which was recently introduced in [204] in the continuum stochastic setting (see also [345, Proposition III.2.2] for the discrete case) and was crucially used in [203, 208]. It allows to put the equation for the two-scale homogenization error in divergence form (cf. (3.58)). Let $\sigma = (\sigma_{ijk})_{i,j,k=1}^d$ be the 3-tensor defined as the unique solution in $\mathbb{Z}^d$ of

$$-\triangle\sigma_{ijk} := -\nabla^*\cdot\nabla\sigma_{ijk} = \nabla_j q_{ik} - \nabla_k q_{ij}, \tag{3.30}$$

with $\nabla\sigma$ stationary and $\sigma(0) = 0$, where q_i denotes the flux of the corrector

$$q_i = A(\nabla\phi_i + e_i) - A_{\mathrm{hom}} e_i, \qquad q_{ij} := (q_i)_j. \tag{3.31}$$

Note that for all i the 2-tensor field $\sigma_i := (\sigma_{ijk})_{j,k=1}^d$ is skew-symmetric, that is,

$$\sigma_{ijk} = -\sigma_{ikj}, \tag{3.32}$$

and it satisfies

$$\nabla^*\cdot\sigma_i := e_j \nabla_k^* \sigma_{ijk} = q_i. \tag{3.33}$$

(Although considering a symmetric coefficient field, we use non-symmetric notation in view of the extension to the continuum case, and we denote by ϕ^* and σ^* the corrector and flux corrector associated with the pointwise transpose coefficient field A^*.)

We now describe the string of arguments that leads to Proposition 3.2.2. We start with a suitable decomposition of the vertical derivative of $E^\varepsilon(f,g)$, which is key to the proof. Note that we rather consider a suitable version $E_0^\varepsilon(f,g)$ of $E^\varepsilon(f,g)$, which only coincides up to some minor discretization error (in the continuum setting $\bar u_\varepsilon$ and $\bar v_\varepsilon$ would simply coincide with $\bar u(\varepsilon\cdot)$ and $\bar v(\varepsilon\cdot)$). Note that as usual it is convenient in the proofs to rescale all quantities down to scale 1.

Lemma 3.3.2. *For all $\varepsilon > 0$ and all $f, g \in C_c^\infty(\mathbb{R}^d)^d$, setting $f_\varepsilon := f(\varepsilon\cdot)$ and $g_\varepsilon := g(\varepsilon\cdot)$, we denote by $\bar u_\varepsilon$ and $\bar v_\varepsilon$ the unique Lax-Milgram solutions in $\mathbb{R}^d$ of*

$$-\nabla^*\cdot A_{\mathrm{hom}}\nabla\bar u_\varepsilon = \nabla^*\cdot(\varepsilon f_\varepsilon), \qquad -\nabla^*\cdot A_{\mathrm{hom}}^*\nabla\bar v_\varepsilon = \nabla^*\cdot(\varepsilon g_\varepsilon), \tag{3.34}$$

and we define

$$\begin{aligned} E_0^\varepsilon(f,g) := \varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} g_\varepsilon\cdot\big(A\nabla(u_\varepsilon(\varepsilon\cdot)) - A_{\mathrm{hom}}\nabla(u_\varepsilon(\varepsilon\cdot)) - \mathbb{E}\left[A\nabla(u_\varepsilon(\varepsilon\cdot)) - A_{\mathrm{hom}}\nabla(u_\varepsilon(\varepsilon\cdot))\right]\big)& \\ - \varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} g_\varepsilon\cdot e_i\nabla_i\bar u_\varepsilon,& \end{aligned} \tag{3.35}$$

as well as the two-scale expansion error $w_{f,\varepsilon} := u_\varepsilon(\varepsilon\cdot) - (1+\phi_i\nabla_i)\bar u_\varepsilon$. Then we have

$$
\begin{aligned}
\Delta_b E_0^\varepsilon(f,g) = \varepsilon^{\frac d2 -1}\int_{\mathbb{R}^d} & g_{\varepsilon,j}(\nabla\phi_j^* + e_j)\cdot \Delta_b A(\nabla w_{f,\varepsilon}^b + \phi_i^b\nabla\nabla_i\bar u_\varepsilon)\\
&+ \varepsilon^{\frac d2 -1}\int_{\mathbb{R}^d}\phi_j^*(\cdot+e_k)\nabla_k g_{\varepsilon,j}e_k\cdot\Delta_b A\nabla(u_\varepsilon^b(\varepsilon\cdot))\\
&- \varepsilon^{\frac d2 -1}\int_{\mathbb{R}^d}\phi_j^*(\cdot+e_k)\nabla_k(g_{\varepsilon,j}\nabla_i\bar u_\varepsilon)e_k\cdot\Delta_b A(\nabla\phi_i^b+e_i)\\
&+ \varepsilon^{\frac d2 -1}\int_{\mathbb{R}^d}\nabla r_\varepsilon\cdot\Delta_b A\nabla(u_\varepsilon^b(\varepsilon\cdot)) - \varepsilon^{\frac d2 -1}\int_{\mathbb{R}^d}\nabla R_{\varepsilon,i}\cdot\Delta_b A(\nabla\phi_i^b+e_i), \quad (3.36)
\end{aligned}
$$

where the auxiliary fields r_ε and R_ε are the unique Lax-Milgram solutions in $\mathbb{R}^d$ of

$$
\begin{aligned}
-\nabla^*\cdot A^*\nabla r_\varepsilon &= \nabla_l^*\big(\phi_j^*(\cdot+e_k)A_{kl}\nabla_k g_{\varepsilon,j} + \sigma_{jkl}^*(\cdot-e_k)\nabla_k^* g_{\varepsilon,j}\big), && (3.37)\\
-\nabla^*\cdot A^*\nabla R_{\varepsilon,i} &= \nabla_l^*\big(\phi_j^*(\cdot+e_k)A_{kl}\nabla_k(g_{\varepsilon,j}\nabla_i\bar u_\varepsilon) + \sigma_{jkl}^*(\cdot-e_k)\nabla_k^*(g_{\varepsilon,j}\nabla_i\bar u_\varepsilon)\big). && (3.38)
\end{aligned}
$$

◊

By the spectral gap estimate of Lemma 3.3.1, the desired pathwise result (3.23) would follow from a suitable estimate of the sum over $\mathcal{B}$ of the squares of the right-hand side terms in (3.36). For that purpose, we make crucial use of the following moment bounds for the extended corrector (ϕ,σ) and its gradient. (These bounds are a variation of [209] and are the discrete versions of a result in [203], the proof of which extends to the discrete setting considered here.)

Lemma 3.3.3 ([209, 203])**.** *Let $d\ge 2$, let $\mathbb{P}$ be a product measure, and let μ_d be defined in* (3.13)*. For all $q<\infty$ and all $z\in\mathbb{Z}^d$ we have*

$$
\mathbb{E}\left[|\phi(z)|^q\right]^{\frac1q} + \mathbb{E}\left[|\sigma(z)|^q\right]^{\frac1q} \lesssim_q \mu_d(|z|)^{\frac12},
$$

and

$$
\mathbb{E}\left[|\nabla\phi(z)|^q\right]^{\frac1q} + \mathbb{E}\left[|\nabla\sigma(z)|^q\right]^{\frac1q} \lesssim_q 1.
$$

◊

An additional crucial ingredient is the following large-scale weighted Calderón-Zygmund estimate for the operator $-\nabla^*\cdot A\nabla$. (A proof in the continuum setting was originally given in the first version of this article, see now in [204], and the adaptation to the discrete setting is straightforward since it is solely based on the energy and Caccioppoli estimates.)

Lemma 3.3.4 ([204])**.** *Let $d\ge 1$, let $\mathbb{P}$ be a product measure, and let $w_\varepsilon(x) := 1+\varepsilon|x|$. There exists a $\frac12$-Lipschitz stationary random field $r_*\ge 1$ on $\mathbb{R}^d$ with $\mathbb{E}[r_*^q]\lesssim_q 1$ for all $q<\infty$, such that the following holds almost surely: For $1\le p<\infty$ and $0\le\gamma<d(2p-1)$, for any (sufficiently fast) decaying scalar field u and vector field f related in $\mathbb{R}^d$ by*

$$
-\nabla^*\cdot A\nabla u = \nabla^*\cdot f,
$$

we have

$$
\int_{\mathbb{R}^d} w_\varepsilon(x)^\gamma\Big(\fint_{B_*(x)}|\nabla u|^2\Big)^p dx \lesssim_{\gamma,p} r_*(0)^\gamma\int_{\mathbb{R}^d} w_\varepsilon(x)^\gamma|f(x)|^{2p}dx,
$$

where we use the short-hand notation $B_*(x) := B_{r_*(x)}(x)$. ◊

3.3.2 Proof of Proposition 3.2.1

Let $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$, and set $F_\varepsilon := F(\varepsilon\cdot)$. We split the proof into two steps: we start by giving a suitable representation formula for the vertical derivative $\Delta_b I_0^\varepsilon(F)$, and then apply the spectral gap estimate.

Step 1. Representation formula for $\Delta_b I_0^\varepsilon(F)$:

$$\Delta_b I_0^\varepsilon(F) = \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij} e_j \cdot \Delta_b A(\nabla\phi_i^b + e_i) + \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} \nabla s_{\varepsilon,i} \cdot \Delta_b A(\nabla \phi_i^b + e_i), \tag{3.39}$$

where the auxiliary field s_ε is the unique Lax-Milgram solution in $\mathbb{R}^d$ of

$$-\nabla^* \cdot A^* \nabla s_{\varepsilon,i} = \nabla^* \cdot \big(F_{\varepsilon,ij}(A - A_{\mathrm{hom}})e_j\big). \tag{3.40}$$

By definition of the homogenization commutator,

$$\Delta_b I_0^\varepsilon(F) = \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij} e_j \cdot \Delta_b A(\nabla\phi_i^b + e_i) + \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij} e_j \cdot (A - A_{\mathrm{hom}}) \nabla \Delta_b \phi_i.$$

By definition (3.40) of $s_{\varepsilon,i}$, we find

$$\Delta_b I_0^\varepsilon(F) = \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_{\varepsilon,ij} e_j \cdot \Delta_b A(\nabla\phi_i^b + e_i) - \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} \nabla s_{\varepsilon,i} \cdot A \nabla \Delta_b \phi_i.$$

Then using the vertical derivative of the corrector equation (3.21) in the form

$$-\nabla^* \cdot A \nabla \Delta_b \phi_i = \nabla^* \cdot \Delta_b A(\nabla \phi_i^b + e_i), \tag{3.41}$$

the claim (3.39) follows.

Step 2. Conclusion.

For $b \in \mathcal{B}$ we use the notation $b = (z_b, z_b + \xi_b)$. Inserting the representation formula (3.39) in the spectral gap estimate of Lemma 3.3.1, and noting that $|\Delta_b A(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$, we obtain

$$\mathrm{Var}\left[I_0^\varepsilon(F)\right] \lesssim \varepsilon^d \sum_{b\in\mathcal{B}} \mathbb{E}\left[|\nabla\phi^b(z_b) + \mathrm{Id}\,|^2\right] \int_{Q(z_b)} |F_\varepsilon|^2 + \varepsilon^d\, \mathbb{E}\left[\sum_{b\in\mathcal{B}} |\nabla\phi^b(z_b) + \mathrm{Id}\,|^2 \int_{Q(z_b)} |\nabla s_\varepsilon|^2\right],$$

and hence, appealing to Lemma 3.3.3 in the form $\mathbb{E}\left[|\nabla\phi^b|^2\right] = \mathbb{E}\left[|\nabla\phi|^2\right] \lesssim 1$,

$$\mathrm{Var}\left[I_0^\varepsilon(F)\right] \lesssim \varepsilon^d \|F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \varepsilon^d\, \mathbb{E}\left[\sum_{b\in\mathcal{B}} |\nabla\phi^b(z_b) + \mathrm{Id}\,|^2 \int_{Q(z_b)} |\nabla s_\varepsilon|^2\right]. \tag{3.42}$$

It remains to estimate the last right-hand side term. Using equation (3.41) in the form $-\nabla^* \cdot A^b \nabla(\phi^b - \phi) = \nabla^* \cdot (A^b - A)(\nabla\phi + \mathrm{Id})$, an energy estimate yields

$$|\nabla(\phi^b - \phi)(z_b)|^2 \le \int_{\mathbb{R}^d} |\nabla(\phi^b - \phi)|^2 \lesssim \int_{\mathbb{R}^d} |A^b - A|^2 |\nabla\phi + \mathrm{Id}\,|^2 \lesssim |\nabla\phi(z_b) + \mathrm{Id}\,|^2, \tag{3.43}$$

hence $|\nabla\phi^b(z_b) + \mathrm{Id}\,| \lesssim |\nabla\phi(z_b) + \mathrm{Id}\,|$. Further estimating in (3.42) integrals over unit cubes by integrals over balls at the scale r_* (cf. Lemma 3.3.4), smuggling in a power $\alpha\frac{p-1}{p}$ of the weight $w_\varepsilon(z) := 1 + \varepsilon|z|$, and applying Hölder's inequality in space with exponent p, we deduce for all $p > 1$,

$$\begin{aligned}
&\varepsilon^d\, \mathbb{E}\left[\sum_{b\in\mathcal{B}} |\nabla\phi^b(z_b) + \mathrm{Id}\,|^2 \int_{Q(z_b)} |\nabla s_\varepsilon|^2\right] \lesssim \varepsilon^d\, \mathbb{E}\left[\int_{\mathbb{R}^d} |\nabla\phi(z) + \mathrm{Id}\,|^2 \Big(\int_{Q_2(z)} |\nabla s_\varepsilon|^2\Big) dz\right] \\
&\lesssim \varepsilon^d\, \mathbb{E}\left[\Big(\int_{\mathbb{R}^d} |\nabla\phi(z) + \mathrm{Id}\,|^{\frac{2p}{p-1}} r_*(z)^{\frac{dp}{p-1}} w_\varepsilon(z)^{-\alpha} dz\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^d} w_\varepsilon(z)^{\alpha(p-1)} \Big(\fint_{B_*(z)} |\nabla s_\varepsilon|^2\Big)^p dz\Big)^{\frac{1}{p}}\right].
\end{aligned}$$

Applying Hölder's inequality in the probability space, using Lemmas 3.3.3 and 3.3.4 in the form $\mathbb{E}\big[|\nabla\phi+\mathrm{Id}\,|^q+r_*^q\big]\lesssim_q 1$ for all $q<\infty$, and noting that $\int_{\mathbb{R}^d} w_\varepsilon(z)^{-\alpha}dz\lesssim_\alpha \varepsilon^{-d}$ provided $\alpha>d$, we obtain for all $p>1$ and all $\alpha>d$,

$$\varepsilon^d\,\mathbb{E}\left[\sum_{b\in\mathcal{B}}|\nabla\phi^b(z_b)+\mathrm{Id}\,|^2\int_{Q(z_b)}|\nabla s_\varepsilon|^2\right]\lesssim_{\alpha,p}\,\varepsilon^{\frac{d}{p}}\,\mathbb{E}\left[\int_{\mathbb{R}^d}w_\varepsilon(z)^{\alpha(p-1)}\Big(\fint_{B_*(z)}|\nabla s_\varepsilon|^2\Big)^p dz\right]^{\frac1p}. \tag{3.44}$$

By large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.3.4) applied to equation (3.40) for s_ε with $\alpha(p-1)<d(2p-1)$, using again the moment bounds on r_*, and rescaling spatial integrals, we deduce for all $0<p-1\ll 1$ and all $0<\alpha-d\ll1$,

$$\begin{aligned}\varepsilon^d\mathbb{E}\left[\sum_{b\in\mathcal{B}}|\nabla\phi^b(z_b)+\mathrm{Id}\,|^2\int_{Q(z_b)}|\nabla s_\varepsilon|^2\right]\quad&\lesssim_{\alpha,p}\ \varepsilon^{\frac{d}{p}}\,\mathbb{E}\left[r_*(0)^{\alpha(p-1)}\int_{\mathbb{R}^d}w_\varepsilon^{\alpha(p-1)}|F_\varepsilon|^{2p}\right]^{\frac1p}\\&\lesssim_{\alpha,p}\ \varepsilon^{\frac{d}{p}}\|w_\varepsilon^{\alpha\frac{p-1}{2p}}F_\varepsilon\|^2_{\mathrm{L}^{2p}(\mathbb{R}^d)}.\end{aligned}\tag{3.45}$$

Inserting this into (3.42) and rescaling spatial integrals, we deduce for all $0<p-1\ll1$ and all $0<\alpha-d\ll1$,

$$\mathrm{Var}\,[I_0^\varepsilon(F)]\lesssim_{\alpha,p}\|F\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\|w_1^{\alpha\frac{p-1}{2p}}F\|^2_{\mathrm{L}^{2p}(\mathbb{R}^d)}.$$

Further using Hölder's inequality in the form

$$\|F\|_{\mathrm{L}^2(\mathbb{R}^d)}\le\Big(\int_{\mathbb{R}^d}w_1^{-\alpha}\Big)^{\frac{p-1}{2p}}\Big(\int_{\mathbb{R}^d}w_1^{\alpha(p-1)}|F|^{2p}\Big)^{\frac1{2p}}\lesssim_{\alpha,p}\|w_1^{\alpha\frac{p-1}{2p}}F\|_{\mathrm{L}^{2p}(\mathbb{R}^d)},$$

the conclusion follows (after replacing the exponent $\alpha\frac{p-1}{2p}$ with 2α).

3.3.3 Proof of Lemma 3.3.2

We split the proof into two steps. To simplify notation, in this proof only, we write $u:=u_\varepsilon(\varepsilon\cdot)$.
Step 1. Representation formula for $\Delta_b((A-A_{\mathrm{hom}})\nabla u)$:

$$\begin{aligned}\Delta_b\big(e_j\cdot(A-A_{\mathrm{hom}})\nabla u\big)=(\nabla\phi_j^*+e_j)\cdot\Delta_bA\nabla u^b-\nabla_k^*\big(\phi_j^*(\cdot+e_k)e_k\cdot\Delta_bA\nabla u^b\big)\\-\nabla_k^*\big(\phi_j^*(\cdot+e_k)e_k\cdot A\nabla\Delta_bu\big)-\nabla_k\big(\sigma_{jkl}^*(\cdot-e_k)\nabla_l\Delta_bu\big).\end{aligned}\tag{3.46}$$

In particular, replacing $x\mapsto u(x)$ by $x\mapsto\phi_i(x)+x_i$, we deduce the following discrete version of (3.10),

$$\begin{aligned}\Delta_b\Xi_{ij}=(\nabla\phi_j^*+e_j)\cdot\Delta_bA(\nabla\phi_i^b+e_i)-\nabla_k^*\big(\phi_j^*(\cdot+e_k)e_k\cdot\Delta_bA(\nabla\phi_i^b+e_i)\big)\\-\nabla_k^*\big(\phi_j^*(\cdot+e_k)A_{kl}\nabla_l\Delta_b\phi_i\big)-\nabla_k\big(\sigma_{jkl}^*(\cdot-e_k)\nabla_l\Delta_b\phi_i\big)\end{aligned}\tag{3.47}$$

Using the definition (3.33) of σ_j in the form $(A^*-A^*_{\mathrm{hom}})e_j=-A^*\nabla\phi_j^*+\nabla^*\cdot\sigma_j^*$, we find

$$\begin{aligned}\Delta_b\big(e_j\cdot(A-A_{\mathrm{hom}})\nabla u\big)\ &=\ e_j\cdot\Delta_bA\nabla u^b+e_j\cdot(A-A_{\mathrm{hom}})\nabla\Delta_bu\\&=\ e_j\cdot\Delta_bA\nabla u^b+(\nabla^*\cdot\sigma_j^*)\cdot\nabla\Delta_bu-\nabla\phi_j^*\cdot A\nabla\Delta_bu.\end{aligned}\tag{3.48}$$

On the one hand, using the following discrete version of the Leibniz rule, for all $\chi_1,\chi_2:\mathbb{Z}^d\to\mathbb{R}$,

$$\nabla(\chi_1\chi_2)=\chi_1\nabla\chi_2+e_l\,\chi_2(\cdot+e_l)\,\nabla_l\chi_1,\tag{3.49}$$

we obtain

$$(\nabla^*\cdot\sigma_j^*)\cdot\nabla\Delta_bu\ =\ \nabla_l^*\big(\sigma_{jkl}^*\nabla_k\Delta_bu(\cdot+e_l)\big)-\sigma_{jkl}^*\nabla_k\nabla_l\Delta_bu,$$

so that the skew-symmetry (3.32) of σ_j leads to

$$(\nabla^* \cdot \sigma_j^*) \cdot \nabla \Delta_b u \;=\; -\nabla_k^* \big(\sigma_{jkl}^* \nabla_l \Delta_b u(\cdot + e_k)\big) \;=\; -\nabla_k \big(\sigma_{jkl}^*(\cdot - e_k) \nabla_l \Delta_b u\big). \tag{3.50}$$

On the other hand, using the vertical derivative of equation (3.18) in the form $-\nabla^* \cdot A \nabla \Delta_b u = \nabla^* \cdot \Delta_b A \nabla u^b$, the discrete Leibniz rule (3.49) yields

$$\begin{aligned}
\nabla \phi_j^* \cdot A \nabla \Delta_b u \;&=\; -\phi_j^* \nabla^* \cdot A \nabla \Delta_b u + \nabla_k^* \big(\phi_j^*(\cdot + e_k) e_k \cdot A \nabla \Delta_b u\big) \\
&=\; -\nabla \phi_j^* \cdot \Delta_b A \nabla u^b + \nabla_k^* \big(\phi_j^*(\cdot + e_k) e_k \cdot \Delta_b A \nabla u^b\big) \\
&\qquad + \nabla_k^* \big(\phi_j^*(\cdot + e_k) e_k \cdot A \nabla \Delta_b u\big).
\end{aligned} \tag{3.51}$$

Inserting (3.50) and (3.51) into (3.48), the claim (3.46) follows.

Step 2. Conclusion.

Integrating identities (3.46) and (3.47) with the test functions g_ε and $\nabla \bar{u}_\varepsilon \otimes g_\varepsilon$, respectively, and integrating by parts, we obtain by definition of E_0^ε,

$$\begin{aligned}
\Delta_b E_0^\varepsilon(f, g) &= \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} g_{\varepsilon,j} (\nabla \phi_j^* + e_j) \cdot \Delta_b A \big(\nabla u^b - (\nabla \phi_i^b + e_i) \nabla_i \bar{u}_\varepsilon\big) \\
&+ \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} \phi_j^*(\cdot + e_k) \nabla_k g_{\varepsilon,j} e_k \cdot \Delta_b A \nabla u^b - \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} \phi_j^*(\cdot + e_k) \nabla_k (g_{\varepsilon,j} \nabla_i \bar{u}_\varepsilon) e_k \cdot \Delta_b A (\nabla \phi_i^b + e_i) \\
&\qquad + \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} \big(\phi_j^*(\cdot + e_k) A_{kl} \nabla_k g_{\varepsilon,j} + \sigma_{jkl}^*(\cdot - e_k) \nabla_k^* g_{\varepsilon,j}\big) \nabla_l \Delta_b u \\
&\qquad\quad - \varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} \big(\phi_j^*(\cdot + e_k) A_{kl} \nabla_k (g_{\varepsilon,j} \nabla_i \bar{u}_\varepsilon) + \sigma_{jkl}^*(\cdot - e_k) \nabla_k^* (g_{\varepsilon,j} \nabla_j \bar{u}_\varepsilon)\big) \nabla_l \Delta_b \phi_i.
\end{aligned}$$

The first right-hand side term is reformulated using the definition of $w_{f,\varepsilon}$ in the form $\nabla u^b - (\nabla \phi_i^b + e_i) \nabla_i \bar{u}_\varepsilon = \nabla w_{f,\varepsilon}^b + \phi_i^b \nabla \nabla_i \bar{u}_\varepsilon$. It remains to post-process the last two right-hand side terms. Using equation (3.37) for r_ε and using the vertical derivative of equation (3.18) for u_ε in the form $-\nabla^* \cdot A \nabla \Delta_b u = \nabla^* \cdot \Delta_b A \nabla u^b$, we find

$$\int_{\mathbb{R}^d} \big(\phi_j^*(\cdot + e_k) A_{kl} \nabla_k g_{\varepsilon,j} + \sigma_{jkl}^*(\cdot - e_k) \nabla_k^* g_{\varepsilon,j}\big) \nabla_l \Delta_b u \;=\; -\int_{\mathbb{R}^d} \nabla r_\varepsilon \cdot A \nabla \Delta_b u \;=\; \int_{\mathbb{R}^d} \nabla r_\varepsilon \cdot \Delta_b A \nabla u^b.$$

Similarly, equations (3.38) and (3.41) lead to

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big(\phi_j^*(\cdot + e_k) A_{kl} \nabla_k (g_{\varepsilon,j} \nabla_i \bar{u}_\varepsilon) + \sigma_{jkl}^*(\cdot - e_k) \nabla_k^* (g_{\varepsilon,j} \nabla_i \bar{u}_\varepsilon)\big) \nabla_l \Delta_b \phi_i \\
&\qquad\qquad = -\int_{\mathbb{R}^d} \nabla R_{\varepsilon,i} \cdot A \nabla \Delta_b \phi_i = \int_{\mathbb{R}^d} \nabla R_{\varepsilon,i} \cdot \Delta_b A (\nabla \phi_i^b + e_i),
\end{aligned}$$

and the conclusion follows.

3.3.4 Proof of Proposition 3.2.2

Using the representation formula (3.36), and recalling that for symmetric coefficients we have $(\phi^*, \sigma^*) = (\phi, \sigma)$, the spectral gap estimate of Lemma 3.3.1 leads to

$$\mathrm{Var}\left[E_0^\varepsilon(f, g)\right] \lesssim T_1^\varepsilon + T_2^\varepsilon + T_3^\varepsilon + T_4^\varepsilon + T_5^\varepsilon, \tag{3.52}$$

where we have set

$$T_1^\varepsilon \;:=\; \sum_{b \in \mathcal{B}} \mathbb{E}\left[\Big(\varepsilon^{\frac{d}{2}-1} \int_{\mathbb{R}^d} g_{\varepsilon,j} (\nabla \phi_j + e_j) \cdot \Delta_b A (\nabla w_{f,\varepsilon}^b + \phi_i^b \nabla \nabla_i \bar{u}_\varepsilon)\Big)^2\right],$$

$$
\begin{aligned}
T_2^\varepsilon &:= \sum_{b\in\mathcal{B}} \mathbb{E}\left[\Big(\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \phi_j(\cdot+e_k)\nabla_k g_{\varepsilon,j} e_k\cdot \Delta_b A\nabla(u_\varepsilon^b(\varepsilon\cdot))\Big)^2\right],\\
T_3^\varepsilon &:= \sum_{b\in\mathcal{B}} \mathbb{E}\left[\Big(\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \phi_j(\cdot+e_k)\nabla_k (g_{\varepsilon,j}\nabla_i \bar u_\varepsilon) e_k\cdot \Delta_b A(\nabla\phi_i^b+e_i)\Big)^2\right],\\
T_4^\varepsilon &:= \sum_{b\in\mathcal{B}} \mathbb{E}\left[\Big(\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \nabla r_\varepsilon\cdot \Delta_b A\nabla(u_\varepsilon^b(\varepsilon\cdot))\Big)^2\right],\\
T_5^\varepsilon &:= \sum_{b\in\mathcal{B}} \mathbb{E}\left[\Big(\varepsilon^{\frac{d}{2}-1}\int_{\mathbb{R}^d} \nabla R_{\varepsilon,i}\cdot \Delta_b A(\nabla\phi_i^b+e_i)\Big)^2\right],
\end{aligned}
$$

with the auxiliary fields r_ε and R_ε defined in (3.37) and in (3.38). The conclusion of Proposition 3.2.2 is a consequence of the following five estimates: for all $0<p-1\ll 1$ and all $0<\alpha-d\ll 1$,

$$
\begin{aligned}
T_1^\varepsilon &\lesssim_{\alpha,p} \varepsilon^2\mu_d(\tfrac{1}{\varepsilon})\,\|g\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^{\alpha\frac{p-1}{4p}}\,\mu_d(|\cdot|)^{\frac12}Df\|^2_{\mathrm{L}^{4p}(\mathbb{R}^d)}, && (3.53)\\
T_2^\varepsilon &\lesssim_{\alpha,p} \varepsilon^2\mu_d(\tfrac{1}{\varepsilon})\,\|f\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|\mu_d(|\cdot|)^{\frac12}Dg\|^2_{\mathrm{L}^4(\mathbb{R}^d)}, && (3.54)\\
T_3^\varepsilon &\lesssim_{\alpha,p} \varepsilon^2\mu_d(\tfrac{1}{\varepsilon})\Big(\|f\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|\mu_d(|\cdot|)^{\frac12}Dg\|^2_{\mathrm{L}^4(\mathbb{R}^d)}+\|g\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|\mu_d(|\cdot|)^{\frac12}Df\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\Big), && (3.55)\\
T_4^\varepsilon &\lesssim_{\alpha,p} \varepsilon^2\mu_d(\tfrac{1}{\varepsilon})\,\|f\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^{\alpha\frac{p-1}{4p}}\,\mu_d(|\cdot|)^{\frac12}Dg\|^2_{\mathrm{L}^{4p}(\mathbb{R}^d)}, && (3.56)\\
T_5^\varepsilon &\lesssim_{\alpha,p} \varepsilon^2\mu_d(\tfrac{1}{\varepsilon})\Big(\|f\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^{\alpha\frac{p-1}{4p}}\,\mu_d(|\cdot|)^{\frac12}Dg\|^2_{\mathrm{L}^{4p}(\mathbb{R}^d)}\\
&\qquad\qquad +\|g\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^{\alpha\frac{p-1}{4p}}\,\mu_d(|\cdot|)^{\frac12}Df\|^2_{\mathrm{L}^{4p}(\mathbb{R}^d)}\Big). && (3.57)
\end{aligned}
$$

We split the proof into three steps, proving the above five estimates in the first two steps, and deducing the conclusion in the third one.

Step 1. Equation for the two-scale expansion error $w_{f,\varepsilon}$ on $\mathbb{R}^d$:

$$
-\nabla^*\cdot A\nabla w_{f,\varepsilon}=\nabla_l^*\Big(\sigma_{jkl}(\cdot-e_k)\nabla_k^*\nabla_j\bar u_\varepsilon+\phi_j(\cdot+e_k)A_{lk}\nabla_k\nabla_j\bar u_\varepsilon\Big). \qquad (3.58)
$$

This is the discrete counterpart of similar identities in [204, 203].

Using equations (3.18) and (3.34) in the form $-\nabla^*\cdot A\nabla(u_\varepsilon(\varepsilon\cdot))=-\nabla^*\cdot A_{\mathrm{hom}}\nabla\bar u_\varepsilon$, and using the discrete Leibniz rule (3.49), we obtain

$$
\begin{aligned}
-\nabla^*\cdot A\nabla w_{f,\varepsilon}&=-\nabla^*\cdot A\nabla(u_\varepsilon(\varepsilon\cdot)-\bar u_\varepsilon-\phi_j\nabla_j\bar u_\varepsilon)\\
&=-\nabla^*\cdot A_{\mathrm{hom}}\nabla\bar u_\varepsilon+\nabla^*\cdot A\nabla\bar u_\varepsilon+\nabla^*\cdot(A\nabla\phi_j\nabla_j\bar u_\varepsilon)+\nabla^*\cdot(Ae_k\phi_j(\cdot+e_k)\nabla_k\nabla_j\bar u_\varepsilon).
\end{aligned}
$$

Rearranging the terms and using the definition (3.33) of σ_j, this turns into

$$
\begin{aligned}
-\nabla^*\cdot A\nabla w_{f,\varepsilon} &= \nabla^*\cdot\Big((A(\nabla\phi_j+e_j)-A_{\mathrm{hom}}e_j)\nabla_j\bar u_\varepsilon\Big)+\nabla^*\cdot(Ae_k\phi_j(\cdot+e_k)\nabla_k\nabla_j\bar u_\varepsilon)\\
&= \nabla^*\cdot\Big((\nabla^*\cdot\sigma_j)\nabla_j\bar u_\varepsilon\Big)+\nabla^*\cdot(Ae_k\phi_j(\cdot+e_k)\nabla_k\nabla_j\bar u_\varepsilon).
\end{aligned}
$$

Using again the discrete Leibniz rule (3.49) and the skew-symmetry (3.32) of σ_j, we find

$$
\begin{aligned}
\nabla^*\cdot\Big((\nabla^*\cdot\sigma_j)\nabla_j\bar u_\varepsilon\Big) &= \nabla_k^*(\nabla_l^*\sigma_{jkl}\nabla_j\bar u_\varepsilon)=\underbrace{\nabla_k^*\nabla_l^*\sigma_{jkl}\nabla_j\bar u_\varepsilon}_{=0}+\nabla_l^*\sigma_{jkl}(\cdot-e_k)\nabla_k^*\nabla_j\bar u_\varepsilon\\
&= \nabla_l^*(\sigma_{jkl}(\cdot-e_k)\nabla_k^*\nabla_j\bar u_\varepsilon)-\underbrace{\sigma_{jkl}(\cdot-e_k-e_l)\nabla_k^*\nabla_l^*\nabla_j\bar u_\varepsilon}_{=0},
\end{aligned}
$$

and the conclusion (3.58) follows.

Step 2. Proof of estimates (3.53)–(3.57).

We start with the first term T_1^ε. For $b \in \mathcal{B}$ we use the notation $b = (z_b, z_b + \xi_b)$. Since $|\Delta_b A(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} T_1^\varepsilon &\lesssim \varepsilon^{d-2} \sum_{b\in\mathcal{B}} \mathbb{E}\left[|\nabla\phi(z_b) + \mathrm{Id}\,|^2 \Big(\int_{Q(z_b)} |g_\varepsilon||\nabla w_{f,\varepsilon}^b + \phi_i^b \nabla\nabla_i \bar u_\varepsilon|\Big)^2\right] \\ &\lesssim \varepsilon^{d-2}\, \mathbb{E}\left[\sum_{b\in\mathcal{B}} |\nabla\phi(z_b) + \mathrm{Id}\,|^4 \Big(\int_{Q(z_b)} |g_\varepsilon|^2\Big)^2\right]^{\frac12} \mathbb{E}\left[\sum_{b\in\mathcal{B}} \Big(\int_{Q(z_b)} |\nabla w_{f,\varepsilon}^b + \phi_i^b \nabla\nabla_i \bar u_\varepsilon|^2\Big)^2\right]^{\frac12}, \end{aligned}$$

and hence, using the moment bounds of Lemma 3.3.3 and the exchangeability of (A, A^b),

$$T_1^\varepsilon \lesssim \varepsilon^{d-2}\, \|g_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\, \mathbb{E}\left[\sum_{b\in\mathcal{B}} \Big(\int_{Q(z_b)} |\nabla w_{f,\varepsilon} + \phi_i \nabla\nabla_i \bar u_\varepsilon|^2\Big)^2\right]^{\frac12}.$$

We argue as in (3.44), rewriting the second right-hand side factor as a norm of averages at the scale r_*, smuggling in a suitable power of the weight w_ε, and applying Hölder's inequality, for all $p > 1$ and all $\alpha > d$,

$$T_1^\varepsilon \lesssim_{\alpha,p} \varepsilon^{\frac d2(1+\frac1p)-2}\, \|g_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\, \mathbb{E}\left[\int_{\mathbb{R}^d} w_\varepsilon(z)^{\alpha(p-1)} \Big(\fint_{B_*(z)} |\nabla w_{f,\varepsilon}|^2\Big)^{2p} dz + \int_{\mathbb{R}^d} w_\varepsilon^{\alpha(p-1)} |\phi|^{4p} |\nabla^2 \bar u_\varepsilon|^{4p}\right]^{\frac1{2p}}. \tag{3.59}$$

By large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.3.4) applied to equation (3.58) for $w_{f,\varepsilon}$, we deduce for all $0 < p-1 \ll 1$ and all $0 < \alpha - d \ll 1$,

$$T_1^\varepsilon \lesssim_{\alpha,p} \varepsilon^{\frac d2(1+\frac1p)-2}\, \|g_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\, \mathbb{E}\left[r_*(0)^{\alpha(p-1)} \int_{\mathbb{R}^d} w_\varepsilon^{\alpha(p-1)} \Big(|\sigma|^{4p} + |\phi|^{4p} + \sum_{k=1}^d |\phi(\cdot + e_k)|^{4p}\Big) |\nabla^2 \bar u_\varepsilon|^{4p}\right]^{\frac1{2p}}.$$

By the moment bounds of Lemmas 3.3.3 and 3.3.4, this yields

$$T_1^\varepsilon \lesssim_{\alpha,p} \varepsilon^{\frac d2(1+\frac1p)-2}\, \|g_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)} \|w_\varepsilon^{\alpha\frac{p-1}{4p}}\, \mu_d(|\cdot|)^{\frac12} \nabla^2 \bar u_\varepsilon\|^2_{\mathrm{L}^{4p}(\mathbb{R}^d)}.$$

Using the standard weighted Calderón-Zygmund theory applied to the discrete constant-coefficient equation (3.34) for $\bar u_\varepsilon$ (cf. Lemma 3.3.4 with $r_* = 1$), noting that for all $\chi, \zeta \in C_c^\infty(\mathbb{R}^d)$ and all $q < \infty$ the inequality $|\nabla(\zeta(\varepsilon x))| \le \varepsilon \int_0^1 |D\zeta(\varepsilon x + \varepsilon t e_k)| dt$ leads to

$$\int_{\mathbb{R}^d} \chi |\nabla(\zeta(\varepsilon\cdot))|^q \le \varepsilon^q \int_{\mathbb{R}^d} \Big(\sup_{B(x)} |\chi|\Big) |D\zeta(\varepsilon x)|^q dx \le \varepsilon^{q-d} \int_{\mathbb{R}^d} \Big(\sup_{B(\frac x\varepsilon)} |\chi|\Big) |D\zeta(x)|^q dx, \tag{3.60}$$

rescaling the integrals, and estimating $\mu_d(|\frac{\cdot}{\varepsilon}|) \le \mu_d(\frac1\varepsilon)\mu_d(|\cdot|)$, the conclusion (3.53) follows.

We turn to the second term T_2^ε. Since $|\Delta_b A(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$, the Cauchy-Schwarz inequality yields

$$T_2^\varepsilon \lesssim \varepsilon^{d-2}\, \mathbb{E}\left[\sum_{b\in\mathcal{B}} |\phi(z_b + e_k)|^2 \Big(\int_{Q(z_b)} |\nabla_k g_\varepsilon|^2\Big) \Big(\int_{Q(z_b)} |\nabla(u_\varepsilon^b(\varepsilon\cdot))|^2\Big)\right].$$

Bounding the second local integral by an integral at the scale r_*^b, using the notation $B_*^b(z) := B_{r_*^b(z)}(z)$, applying the Cauchy-Schwarz inequality, and using the moment bounds of Lemmas 3.3.3 and 3.3.4, we find

$$T_2^\varepsilon \lesssim \varepsilon^{d-2} \, \|\mu_d(|\cdot|)^{\frac12} \nabla g_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)} \, \mathbb{E}\left[\sum_{b\in\mathcal{B}} \Big(\fint_{B_*^b(z_b)\cup Q(z_b)} |\nabla(u_\varepsilon^b(\varepsilon\cdot))|^2\Big)^2\right]^{\frac12},$$

and hence, by exchangeability of (A, A^b),

$$T_2^\varepsilon \lesssim \varepsilon^{d-2} \, \|\mu_d(|\cdot|)^{\frac12} \nabla g_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)} \, \mathbb{E}\left[\int_{\mathbb{R}^d} \Big(\fint_{B_*(z)} |\nabla(u_\varepsilon(\varepsilon\cdot))|^2\Big)^2 dz\right]^{\frac12}.$$

By large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.3.4) applied to equation (3.18) for u_ε in the form $-\nabla^*\cdot A\nabla(u_\varepsilon(\varepsilon\cdot)) = \nabla^*\cdot(\varepsilon f_\varepsilon)$, we deduce

$$T_2^\varepsilon \lesssim \varepsilon^{d} \, \|\mu_d(|\cdot|)^{\frac12} \nabla g_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)} \, \|f_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)},$$

and the conclusion (3.54) follows similarly as above.

The proof of (3.55) for T_3^ε is more direct. Indeed, using the moment bounds of Lemma 3.3.3, decomposing $\nabla(g_{\varepsilon,i}\nabla u_\varepsilon) = \nabla g_{\varepsilon,i}\otimes\nabla u_\varepsilon + g_{\varepsilon,i}(\cdot+e_k)e_k\otimes\nabla_k\nabla u_\varepsilon$, and applying the Cauchy-Schwarz inequality, we find

$$\begin{aligned} T_3^\varepsilon &\lesssim \varepsilon^{d-2}\sum_{k=1}^d \mathbb{E}\left[\sum_{b\in\mathcal{B}} |\phi(z_b+e_k)|^2 |\nabla\phi^b(z_b)+\mathrm{Id}\,|^2 \Big(\int_{Q(z_b)} |\nabla(g_\varepsilon\nabla\bar u_\varepsilon)|\Big)^2\right] \\ &\lesssim \varepsilon^{d-2}\int_{\mathbb{R}^d} \mu_d(|\cdot|)|\nabla(g_\varepsilon\nabla\bar u_\varepsilon)|^2 \\ &\lesssim \varepsilon^{d-2}\Big(\|\nabla\bar u_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|\mu_d(|\cdot|)\nabla g_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)} + \|g_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|\mu_d(|\cdot|)\nabla^2\bar u_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\Big), \end{aligned}$$

and the conclusion (3.55) follows as above.

We turn to the fourth term T_4^ε. Smuggling a power $\alpha\frac{p-1}{2p}$ of the weight w_ε, applying Hölder's inequality with exponents $(\frac{2p}{p-1}, 2p, 2)$, using the moment bounds of Lemma 3.3.4, and the exchangeability of (A, A^b), we obtain for all $p>1$ and all $\alpha>d$,

$$\begin{aligned} T_4^\varepsilon &\lesssim \varepsilon^{d-2}\,\mathbb{E}\left[\sum_{b\in\mathcal{B}} r_*(z_b)^d r_*^b(z_b)^d \Big(\fint_{B_*(z_b)\cup Q(z_b)} |\nabla r_\varepsilon|^2\Big)\Big(\fint_{B_*^b(z_b)\cup Q(z_b)} |\nabla(u_\varepsilon^b(\varepsilon\cdot))|^2\Big) dz\right] \\ &\lesssim \varepsilon^{\frac d2(1+\frac1p)-2}\,\mathbb{E}\left[\int_{\mathbb{R}^d} w_\varepsilon(z)^{\alpha(p-1)}\Big(\fint_{B_*(z)}|\nabla r_\varepsilon|^2\Big)^{2p} dz\right]^{\frac{1}{2p}} \mathbb{E}\left[\int_{\mathbb{R}^d}\Big(\fint_{B_*(z)}|\nabla(u_\varepsilon(\varepsilon\cdot))|^2\Big)^2 dz\right]^{\frac12}. \end{aligned}$$

Using the large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.3.4) applied to equation (3.37) for r_ε and to equation (3.18) for u_ε, and using the moment bounds of Lemma 3.3.3, we deduce for all $0<p-1\ll 1$ and all $0<\alpha-d\ll 1$,

$$T_4^\varepsilon \lesssim \varepsilon^{\frac d2(1+\frac1p)} \, \|w_\varepsilon^{\alpha\frac{p-1}{4p}} \mu_d(|\cdot|)^{\frac12}\nabla g_\varepsilon\|^2_{\mathrm{L}^{4p}(\mathbb{R}^d)} \|f_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)},$$

and the conclusion (3.56) follows as before.

Finally, we turn to the last term T_5^ε. Using (3.43) in the form $|\nabla\phi^b(z_b)+\mathrm{Id}\,|\lesssim|\nabla\phi(z_b)+\mathrm{Id}\,|$, smuggling in a power $\alpha\frac{p-1}{2p}$ of the weight w_ε, applying Hölder's inequality with exponents $(\frac{2p}{p-1},\frac{2p}{p+1})$, using the moment bounds of Lemmas 3.3.3 and 3.3.4, we obtain for all $p>1$ and all $\alpha>d$,

$$\begin{aligned} T_5^\varepsilon &\lesssim \varepsilon^{d-2}\,\mathbb{E}\left[\int_{\mathbb{R}^d}|\nabla\phi(z)+\mathrm{Id}\,|^2\Big(\int_{Q_2(z)}|\nabla R_\varepsilon|^2\Big)dz\right] \\ &\lesssim \varepsilon^{\frac{d}{2}(1+\frac1p)-2}\,\mathbb{E}\left[\int_{\mathbb{R}^d}w_\varepsilon(z)^{\alpha\frac{p-1}{p+1}}\Big(\fint_{B_*(z)}|\nabla R_\varepsilon|^2\Big)^{\frac{2p}{p+1}}dz\right]^{\frac{p+1}{2p}}. \end{aligned}$$

Using the large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.3.4) applied to equation (3.38) for R_ε, and using the moment bounds of Lemma 3.3.3, we deduce for all $0<p-1\ll1$ and all $0<\alpha-d\ll1$,

$$T_5^\varepsilon\lesssim\varepsilon^{\frac{d}{2}(1+\frac1p)-2}\,\|w_\varepsilon^{\alpha\frac{p-1}{4p}}\mu_d(|\cdot|)^{\frac12}\nabla(g_\varepsilon\nabla\bar u_\varepsilon)\|^2_{\mathrm{L}^{\frac{4p}{p+1}}(\mathbb{R}^d)}.$$

Decomposing $\nabla(g_{\varepsilon,i}\nabla u_\varepsilon)=\nabla g_{\varepsilon,i}\otimes\nabla u_\varepsilon+g_{\varepsilon,i}(\cdot+e_k)e_k\otimes\nabla_k\nabla u_\varepsilon$ and suitably applying Hölder's inequality with exponents $(\frac{p+1}{p},p+1)$, the conclusion (3.57) follows as before.

Step 3. Conclusion.

Inserting estimates (3.53)–(3.57) into (3.52) yields for all $0<p-1\ll1$ and all $\alpha>d\frac{p-1}{4p}$,

$$\|E_0^\varepsilon(f,g)\|_{\mathrm{L}^2(\Omega)}\lesssim_{\alpha,p}\varepsilon\mu_d(\tfrac1\varepsilon)^{\frac12}\Big(\|f\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Dg\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}+\|g\|_{\mathrm{L}^4(\mathbb{R}^d)}\|w_1^\alpha Df\|_{\mathrm{L}^{4p}(\mathbb{R}^d)}\Big),\tag{3.61}$$

and it remains to deduce the corresponding result for $E^\varepsilon(f,g)$. In terms of $\tilde u_\varepsilon:=\bar u_\varepsilon(\frac{\cdot}{\varepsilon})$ and $\tilde v_\varepsilon:=\bar v_\varepsilon(\frac{\cdot}{\varepsilon})$, equations (3.34) take the form

$$-\nabla_\varepsilon^*\cdot A_{\mathrm{hom}}\nabla_\varepsilon\tilde u_\varepsilon=\nabla_\varepsilon^*\cdot f,\qquad -\nabla_\varepsilon^*\cdot A_{\mathrm{hom}}^*\nabla_\varepsilon\tilde v_\varepsilon=\nabla_\varepsilon^*\cdot g.\tag{3.62}$$

The definitions of E^ε and E_0^ε then lead to the relation

$$E^\varepsilon(f,g)\;=\;E_0^\varepsilon(f,g)+I_0^\varepsilon\big((\nabla_\varepsilon\tilde u_\varepsilon-D\bar u)\otimes g\big),\tag{3.63}$$

and it remains to treat the discretization error $I_0^\varepsilon((\nabla_\varepsilon\tilde u_\varepsilon-D\bar u)\otimes g)$. By Proposition 3.2.1, it is enough to establish for all $1<p<\infty$ and all $0\le\alpha<d\frac{p-1}{p}$,

$$\|w_1^\alpha(\nabla_\varepsilon\tilde u_\varepsilon-D\bar u)\|_{\mathrm{L}^p(\mathbb{R}^d)}\lesssim_{\alpha,p}\varepsilon\|w_1^\alpha Df\|_{\mathrm{L}^p(\mathbb{R}^d)}.\tag{3.64}$$

For that purpose, we note that $\bar u$ is an approximate solution of the discrete equation (3.62). Indeed, integrating equation (3.19) for $\bar u$ on a unit cube yields

$$0=\int_{[-1,0]^d}D\cdot(A_{\mathrm{hom}}D\bar u+f)(\cdot+\varepsilon y)dy=\nabla_\varepsilon^*\cdot(A_{\mathrm{hom}}D\bar u+f)+\nabla_\varepsilon^*\cdot T_\varepsilon,\tag{3.65}$$

where error term T_ε satisfies, for all $1\le p<\infty$ and all $0\le\alpha<\infty$,

$$\|w_1^\alpha T_\varepsilon\|_{\mathrm{L}^p(\mathbb{R}^d)}\lesssim_\alpha\varepsilon\|w_1^\alpha D(A_{\mathrm{hom}}D\bar u+f)\|_{\mathrm{L}^p(\mathbb{R}^d)}\lesssim\varepsilon\|w_1^\alpha Df\|_{\mathrm{L}^p(\mathbb{R}^d)}+\varepsilon\|w_1^\alpha D^2\bar u\|_{\mathrm{L}^p(\mathbb{R}^d)}.\tag{3.66}$$

Comparing equations (3.62) and (3.65), the difference $\bar u-\tilde u_\varepsilon$ satisfies

$$-\nabla_\varepsilon^*\cdot A_{\mathrm{hom}}\nabla_\varepsilon(\bar u-\tilde u_\varepsilon)=\nabla_\varepsilon^*\cdot T_\varepsilon-\nabla_\varepsilon^*\cdot A_{\mathrm{hom}}(\nabla_\varepsilon\bar u-D\bar u).$$

Hence, using the standard weighted Calderón-Zygmund theory applied to this discrete constant-coefficient equation, we obtain, for all $1 < p < \infty$ and all $0 \le \alpha < d\frac{p-1}{p}$,

$$\|w_1^\alpha \nabla_\varepsilon(\bar u - \tilde u_\varepsilon)\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_{\alpha,p} \|w_1^\alpha T_\varepsilon\|_{\mathrm{L}^p(\mathbb{R}^d)} + \|w_1^\alpha(\nabla_\varepsilon \bar u - D\bar u)\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

Noting that the second right-hand side term is bounded by $\varepsilon\|w_1^\alpha D^2\bar u\|_{\mathrm{L}^p(\mathbb{R}^d)}$, and applying (3.66), we deduce

$$\|w_1^\alpha(\nabla_\varepsilon \tilde u_\varepsilon - D\bar u)\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_{\alpha,p} \varepsilon\|w_1^\alpha Df\|_{\mathrm{L}^p(\mathbb{R}^d)} + \varepsilon\|w_1^\alpha D^2\bar u\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

Using the standard weighted Calderón-Zygmund theory applied to the constant-coefficient equation (3.19) for $\bar u$, the claim (3.64) follows.

3.3.5 Proof of Corollary 3.2.4

We start with the proof of (3.23) for I_1^ε. By integration by parts, equations (3.62) and (3.18) for $\tilde v_\varepsilon$, $\tilde u_\varepsilon$, and u_ε lead to

$$\begin{aligned}\int g\cdot\nabla_\varepsilon(u_\varepsilon - \tilde u_\varepsilon) &\overset{(3.62)}{=} -\int \nabla_\varepsilon \tilde v_\varepsilon \cdot A_{\mathrm{hom}}\nabla_\varepsilon(u_\varepsilon-\tilde u_\varepsilon) \overset{(3.62)}{=} -\int \nabla_\varepsilon\tilde v_\varepsilon\cdot f - \int\nabla_\varepsilon\tilde v_\varepsilon\cdot A_{\mathrm{hom}}\nabla_\varepsilon u_\varepsilon\\ &\overset{(3.18)}{=} \int \nabla_\varepsilon\tilde v_\varepsilon\cdot(A_\varepsilon\nabla_\varepsilon u_\varepsilon - A_{\mathrm{hom}}\nabla_\varepsilon u_\varepsilon).\end{aligned}$$

Subtracting the expectation of both sides yields a discrete version of identity (3.7). In terms of I_0^ε, I_1^ε, and E_0^ε (cf. Section 3.2 and (3.35)), this takes on the following guise,

$$I_1^\varepsilon(f,g) - I_0^\varepsilon(D\bar u\otimes D\bar v) = I_0^\varepsilon(\nabla_\varepsilon\tilde u_\varepsilon\otimes\nabla_\varepsilon\tilde v_\varepsilon - D\bar u\otimes D\bar v) + E_0^\varepsilon(f,\nabla_\varepsilon\tilde v_\varepsilon). \tag{3.67}$$

Using (3.64) and the standard weighted Calderón-Zygmund theory applied to the constant-coefficient equations (3.19) and (3.62), the conclusion (3.23) for I_1^ε follows from (3.61) together with Proposition 3.2.1.

We turn to the proof of (3.23) for I_2^ε. By definition of I_0^ε, I_1^ε, I_2^ε, and E^ε (cf. Section 3.2), we find

$$I_2^\varepsilon(f,g) = E^\varepsilon(f,g) + I_1^\varepsilon(f,A_{\mathrm{hom}}^*g) + I_0^\varepsilon(D\bar u\otimes g).$$

Inserting identities (3.63) and (3.67) (with g replaced by A_{hom}^*g and thus $\bar v$ replaced by the solution $\bar v^\circ$ of $-D\cdot A_{\mathrm{hom}}^*D\bar v^\circ = D\cdot A_{\mathrm{hom}}^*g$, so that $\bar{\mathcal{P}}_L^*g = D\bar v^\circ + g$), the conclusion (3.23) follows similarly as for I_1^ε.

We now turn to the proof of (3.24). Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decaying functions, and consider the subspace $\mathcal{K}_\varepsilon := \{g\in\mathcal{S}(\mathbb{R}^d)^d : \bar v_\varepsilon\in\mathcal{S}(\mathbb{R}^d)\}$, cf. (3.34). Given some fixed $\chi\in C_c^\infty(\mathbb{R}^d)$, set $\chi_L := \chi(L\cdot)$ for $L\ge 1$. For $g\in\mathcal{K}_\varepsilon$, we compute by integration by parts, using equation (3.21) for ϕ_j and equation (3.34) for $\bar v_\varepsilon$, together with the discrete Leibniz rule (3.49),

$$\begin{aligned}\int_{\mathbb{R}^d}\chi_L\nabla\bar v_\varepsilon\cdot e_i &= \int_{\mathbb{R}^d}\chi_L\nabla\bar v_\varepsilon\cdot\big(A(\nabla\phi_i+e_i) - A_{\mathrm{hom}}(\nabla\phi_i+e_i)\big)\\ &\overset{(3.21)}{=} -\int_{\mathbb{R}^d}\nabla(\bar v_\varepsilon\chi_L)\cdot A_{\mathrm{hom}}\nabla\phi_i - \int_{\mathbb{R}^d}\bar v_\varepsilon(\cdot+e_j)\nabla_j\chi_L\,\Xi_{ij}\\ &\overset{(3.34)}{=} \varepsilon\int_{\mathbb{R}^d}\chi_L g_\varepsilon\cdot\nabla\phi_i + \varepsilon\int_{\mathbb{R}^d}\phi_i(\cdot+e_j)g_{\varepsilon,j}\nabla_j\chi_L + \int_{\mathbb{R}^d}\phi_i(\cdot+e_j)\nabla_j\chi_L e_j\cdot A_{\mathrm{hom}}\nabla\bar v_\varepsilon\\ &\qquad - \int_{\mathbb{R}^d}\bar v_\varepsilon(\cdot+e_j)\nabla_j\chi_L e_j\cdot A_{\mathrm{hom}}\nabla\phi_i - \int_{\mathbb{R}^d}\bar v_\varepsilon(\cdot+e_j)\nabla_j\chi_L\,\Xi_{ij}.\end{aligned}$$

For fixed ε and $g \in \mathcal{K}_\varepsilon$, using the moment bounds of Lemma 3.3.3 and the rapid decay at infinity of g and $\bar v_\varepsilon$, we may pass to the limit $L \uparrow \infty$ in both sides in $\mathrm{L}^2(\Omega)$, and we deduce almost surely

$$\int_{\mathbb{R}^d} \nabla \bar v_\varepsilon : e_j \;=\; \varepsilon \int_{\mathbb{R}^d} g_\varepsilon \cdot \nabla \phi_j,$$

that is, after rescaling,

$$J_1^\varepsilon(e_j \otimes g) = I_0^\varepsilon(e_j \otimes \nabla_\varepsilon \tilde v_\varepsilon). \tag{3.68}$$

We now argue that for all $\varepsilon > 0$ this almost-sure identity can be extended to hold in $\mathrm{L}^2(\Omega)$ for all $g \in C_c^\infty(\mathbb{R}^d)^d$. First note that Proposition 3.2.1 combined with the standard weighted Calderón-Zygmund theory for the constant-coefficient equation (3.62) yields for all $0 < p-1 \ll 1$ and all $d\frac{p-1}{4p} < \alpha < d\frac{2p-1}{4p}$,

$$\mathbb{E}\left[|I_0^\varepsilon(e_j \otimes \nabla_\varepsilon \tilde v_\varepsilon)|^2\right]^{\frac12} \;\lesssim_{\alpha,p}\; \|w_1^{2\alpha} \nabla_\varepsilon \tilde v_\varepsilon\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \;\lesssim_{\alpha,p}\; \|w_1^{2\alpha} g\|_{\mathrm{L}^{2p}(\mathbb{R}^d)},$$

and similarly, arguing as in the proof of Proposition 3.2.1,

$$\mathbb{E}\left[|J_1^\varepsilon(e_j \otimes g)|^2\right]^{\frac12} \;\lesssim\; \|w_1^{2\alpha} g\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}.$$

Hence, it suffices to check the following density result: for all test functions $g \in C_c^\infty(\mathbb{R}^d)^d$ there exist a sequence $(g_n)_n$ of elements of $\mathcal{K}_\varepsilon$ such that $\|w_1^{2\alpha}(g_n - g)\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \to 0$ holds for some $0 < p-1 \ll 1$ and some $\alpha > d\frac{p-1}{4p}$. Let $g \in C_c^\infty(\mathbb{R}^d)^d$ be fixed. Up to a convolution argument on large scales, we may already assume that the Fourier transform $\hat g$ has compact support, say contained in B_R. Since the (continuum) Fourier symbol of the discrete Helmholtz projection $\nabla_\epsilon(\nabla_\epsilon^* \cdot \bar a \nabla_\epsilon)^{-1}\nabla_\epsilon^* \cdot$ is bounded and smooth outside of the dual lattice $(\frac{2\pi}{\epsilon}\mathbb{Z})^d$, a function $g_n \in \mathcal{S}(\mathbb{R}^d)^d$ actually belongs to $\mathcal{K}_\epsilon$ whenever its Fourier transform $\hat g_n$ vanishes in a neighborhood of $(\frac{2\pi}{\epsilon}\mathbb{Z})^d$. Choosing $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi = 1$ in a neighborhood of 0, and defining

$$\chi_n \;:=\; 1 - \sum_{z \in (\frac{2\pi}{\varepsilon}\mathbb{Z})^d} \chi(n(\cdot - z)),$$

the function $g_n \in \mathcal{S}(\mathbb{R}^d)^d$ defined by $\hat g_n := \chi_n \hat g$ thus belongs to $\mathcal{K}_\varepsilon$. For $p \ge 1$, setting $q := \frac{2p-1}{2p}$, since $\hat g$ is compactly supported in B_R, the Hausdorff-Young inequality leads to

$$\begin{aligned}\|w_1^{2\alpha}(g_n - g)\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \;&\le\; \|(\chi_n - 1)\hat g\|_{W^{2\alpha,q}(\mathbb{R}^d)} \;\lesssim_\alpha\; \|\chi_n - 1\|_{W^{2\alpha,q}(B_R)} \|\hat g\|_{W^{2\alpha,\infty}(\mathbb{R}^d)} \\ &\lesssim\; \|\chi_n - 1\|_{W^{2\alpha,q}(B_R)} \|w_1^{2\alpha} g\|_{\mathrm{L}^1(\mathbb{R}^d)}.\end{aligned}$$

For $2\alpha < \frac{d}{q} = d\frac{2p-1}{2p}$, reflecting the fact that the Sobolev space $W^{2\alpha,q}(\mathbb{R}^d)$ fails to embed into the space of continuous functions, there holds $\chi_n \to 1$ in $W^{2\alpha,q}_{\mathrm{loc}}(\mathbb{R}^d)$ as $n \uparrow \infty$, and hence $\|w_1^{2\alpha}(g_n - g)\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \to 0$. This establishes the claimed density result, and we conclude that identity (3.68) can be extended in $\mathrm{L}^2(\Omega)$ to all $g \in C_c^\infty(\mathbb{R}^d)^d$. The estimate (3.24) for J_1^ε then follows from the discretization error estimate (3.64) together with Proposition 3.2.1. The estimate (3.24) for J_2^ε is obtained in a similar way.

3.4 Asymptotic normality

We turn to the normal approximation result for the homogenization commutator Ξ as stated in Proposition 3.2.7.

3.4.1 Structure of the proof and auxiliary results

The main tool to prove Proposition 3.2.7 is the following suitable form of a second-order Poincaré inequality à la Chatterjee [112]. Based on Stein's method, it can be shown to hold for any product measure $\mathbb{P}$ on Ω. (The proof follows from [112, Theorem 2.2] and from [282, Theorem 4.2] in the case of the Wasserstein and of the Kolmogorov metric, respectively, combined with the spectral gap estimate of Lemma 3.3.1; see also Theorem 4.6.2 in Chapter 4.) Let us first fix some more notation. Let $X = X(A)$ be a random variable on Ω, that is, a measurable function of $(a(b))_{b\in\mathcal{B}}$. For all $E \subset \mathcal{B}$ we denote by A^E the random field that coincides with A on all edges $b \notin E$ and with the i.i.d. copy A' on all edges $b \in E$. In particular, A and A^E always have the same law. We use the abbreviation $X^E := X(A^E)$ and define $\Delta_b X^E := X^E - X^{E\cup\{b\}}$. As before, we write for simplicity $X^b := X^{\{b\}}$, and similarly $X^{b,b'} := X^{\{b,b'\}}$. In particular, $\Delta_b\Delta_{b'}X = X - X^b - X^{b'} + X^{b,b'}$.

Lemma 3.4.1 ([112, 282])**.** *Let $\mathbb{P}$ be a product measure. For all $X = X(A) \in \mathrm{L}^2(\Omega)$, we have*

$$
\begin{aligned}
(\mathrm{d_W}+\mathrm{d_K})\left(\frac{X-\mathbb{E}[X]}{\mathrm{Var}[X]^{\frac12}},\mathcal{N}\right) \lesssim\ & \frac{1}{\mathrm{Var}[X]^{\frac32}}\sum_{b\in\mathcal{B}_L}\mathbb{E}\left[|\Delta_b X|^6\right]^{\frac12}\\
&+\frac{1}{\mathrm{Var}[X]}\bigg(\sum_{b\in\mathcal{B}}\Big(\sum_{e'\in\mathcal{B}}\mathbb{E}\left[|\Delta_{b'}X|^4\right]^{\frac14}\mathbb{E}\left[|\Delta_b\Delta_{b'}X|^4\right]^{\frac14}\Big)^2\bigg)^{\frac12}. \qquad \Diamond
\end{aligned}
$$

In addition, we make crucial use of the following optimal annealed estimate on the mixed gradient of the Green's function, first proved by Marahrens and Otto [313].

Lemma 3.4.2 ([313])**.** *Let $d \ge 2$ and let $\mathbb{P}$ be a product measure. For all $y \in \mathbb{Z}^d$ there exists a function $\nabla G(\cdot, y)$ that is the unique decaying solution in $\mathbb{Z}^d$ of*

$$-\nabla^*\cdot A\nabla G(\cdot,y) = \delta(\cdot - y).$$

It satisfies the following moment bound: for all $q < \infty$ and all $x, y \in \mathbb{Z}^d$,

$$\mathbb{E}\left[|\nabla\nabla G(x,y)|^q\right]^{\frac1q} \lesssim_q (1+|x-y|)^{-d},$$

where $\nabla\nabla$ denotes the mixed second gradient. $\Diamond$

3.4.2 Proof of Proposition 3.2.7

Let $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$, and set $F_\varepsilon := F(\varepsilon\cdot)$. By Lemma 3.4.1, it is enough to estimate the following two contributions,

$$K_1^\varepsilon := \sum_{b\in\mathcal{B}}\mathbb{E}\left[|\Delta_b I_0^\varepsilon(F)|^6\right]^{\frac12}, \qquad K_2^\varepsilon := \sum_{b\in\mathcal{B}}\bigg(\sum_{b'\in\mathcal{B}}\mathbb{E}\left[|\Delta_{b'}I_0^\varepsilon(F)|^4\right]^{\frac14}\mathbb{E}\left[|\Delta_b\Delta_{b'}I_0^\varepsilon(F)|^4\right]^{\frac14}\bigg)^2.$$

We split the proof into three steps: we start with an auxiliary estimate, and then estimate K_1^ε and K_2^ε separately.

Step 1. Auxiliary result: for all $\zeta \in C_c^\infty(\mathbb{R}^d)$, all $1 \le p < \infty$, and all $r \ge 0$,

$$\int_{\mathbb{R}^d}\log^r(2+|z|)\bigg(\int_{\mathbb{R}^d}\frac{|\zeta(x)|}{(1+|x-z|)^d}dx\bigg)^p dz \ \lesssim_{p,r} \int_{\mathbb{R}^d}\log^{p+r}(2+|x|)\,|\zeta(x)|^p\,dx. \qquad (3.69)$$

Let $\alpha > 0$ be fixed. Smuggling in a power $\alpha\frac{p-1}{p}$ of the weight $1+|x|$, and applying Hölder's inequality with exponent p, we find

$$\int_{\mathbb{R}^d} \log^r(2+|z|)\Big(\int_{\mathbb{R}^d} \frac{|\zeta(x)|}{(1+|x-z|)^d}dx\Big)^p dz$$
$$\le \int_{\mathbb{R}^d} \log^r(2+|z|)\Big(\int_{\mathbb{R}^d} \frac{(1+|x|)^{\alpha(p-1)}|\zeta(x)|^p}{(1+|x-z|)^d}dx\Big)\Big(\int_{\mathbb{R}^d} \frac{dx}{(1+|x-z|)^d(1+|x|)^\alpha}\Big)^{p-1}dz.$$

Estimating the last integral in brackets leads to

$$\begin{aligned}
&\int_{\mathbb{R}^d} \log^r(2+|z|)\Big(\int_{\mathbb{R}^d} \frac{|\zeta(x)|}{(1+|x-z|)^d}dx\Big)^p dz\\
&\lesssim_{\alpha,p} \int_{\mathbb{R}^d} \frac{\log^{p+r-1}(2+|z|)}{(1+|z|)^{\alpha(p-1)}}\Big(\int_{\mathbb{R}^d} \frac{(1+|x|)^{\alpha(p-1)}|\zeta(x)|^p}{(1+|x-z|)^d}dx\Big)dz\\
&= \int_{\mathbb{R}^d} (1+|x|)^{\alpha(p-1)}|\zeta(x)|^p\Big(\int_{\mathbb{R}^d} \frac{\log^{p+r-1}(2+|z|)}{(1+|x-z|)^d(1+|z|)^{\alpha(p-1)}}dz\Big)dx.
\end{aligned}$$

Estimating the last integral in brackets yields the conclusion (3.69).

Step 2. Proof of
$$K_1^\varepsilon \lesssim \varepsilon^{\frac{d}{2}}\big(\|F\|^3_{\mathrm{L}^3(\mathbb{R}^d)} + \|\log(2+|\cdot|)\,\mu_d(|\cdot|)^{\frac12}DF\|^3_{\mathrm{L}^3(\mathbb{R}^d)}\big).$$

After integration by parts, the representation formula for the vertical derivative $\Delta_b\Xi$ in (3.47) leads to

$$\begin{aligned}
\Delta_b I_0^\varepsilon(F) = \varepsilon^{\frac{d}{2}}\int_{\mathbb{R}^d} F_{\varepsilon,ij}(\nabla\phi_j^*+e_j)\cdot\Delta_b A(\nabla\phi_i^b+e_i) + \varepsilon^{\frac{d}{2}}\int_{\mathbb{R}^d}\phi_j^*(\cdot+e_k)\nabla_k F_{\varepsilon,ij}e_k\cdot\Delta_b A(\nabla\phi_i^b+e_i)\\
+\varepsilon^{\frac{d}{2}}\int_{\mathbb{R}^d}\big(\phi_j^*(\cdot+e_k)A_{kl}\nabla_k F_{\varepsilon,ij}+\sigma^*_{jkl}(\cdot-e_k)\nabla_k^*F_{\varepsilon,ij}\big)\nabla_l\Delta_b\phi_i.
\end{aligned}$$

Given $b=(z_b,z_b+\xi_b)$, noting that the Green representation formula applied to equation (3.41) yields for all $x\in\mathbb{Z}^d$,

$$\nabla\Delta_b\phi_i(x) \;=\; -\nabla\nabla G(x,z_b)\Delta_b A(z_b)(\nabla\phi_i^b(z_b)+e_i), \tag{3.70}$$

inserting this into the above representation formula for $\Delta_b I_0^\varepsilon(F)$, noting that $|\Delta_b A(x)|\lesssim \mathbb{1}_{Q(z_b)}(x)$, and using the moment bounds of Lemmas 3.3.3 and 3.4.2, we obtain for all $q<\infty$,

$$\mathbb{E}\left[|\Delta_b I_0^\varepsilon(F)|^q\right]^{\frac1q} \;\lesssim_q\; \varepsilon^{\frac{d}{2}}\int_{Q(z_b)}|F_\varepsilon| + \varepsilon^{\frac{d}{2}}\int_{\mathbb{R}^d}\frac{\mu_d(|x|)^{\frac12}|\nabla F_\varepsilon(x)|}{(1+|x-z_b|)^d}dx. \tag{3.71}$$

Inserting this estimate into the definition of K_1^ε, and appealing to (3.69) with $p=3$, $r=0$, and $\zeta=\mu_d(|\cdot|)^{\frac12}\nabla F_\varepsilon$, we deduce

$$\begin{aligned}
K_1^\varepsilon \;&\lesssim\; \varepsilon^{\frac{3d}{2}}\int_{\mathbb{R}^d}|F_\varepsilon|^3 + \varepsilon^{\frac{3d}{2}}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\frac{\mu_d(|x|)^{\frac12}|\nabla F_\varepsilon(x)|}{(1+|x-z|^d)}dx\Big)^3dz\\
&\lesssim\; \varepsilon^{\frac{3d}{2}}\int_{\mathbb{R}^d}|F_\varepsilon|^3 + \varepsilon^{\frac{3d}{2}}\int_{\mathbb{R}^d}\log^3(2+|\cdot|)\,\mu_d(|\cdot|)^{\frac32}|\nabla F_\varepsilon|^3.
\end{aligned}$$

Rescaling the integrals and using (3.60), the conclusion follows.

Step 3. Proof of

$$K_2^\varepsilon \lesssim \varepsilon^d \log^2(2+\tfrac{1}{\varepsilon})\Big(\|F\|^4_{\mathrm{L}^4(\mathbb{R}^d)} + \|\log(2+|\cdot|)F\|^4_{\mathrm{L}^4(\mathbb{R}^d)}\Big)$$
$$+ \varepsilon^{d+2}\log^4(2+\tfrac{1}{\varepsilon})\mu_d(\tfrac{1}{\varepsilon})\Big(\|\log(2+|\cdot|)F\|^4_{\mathrm{L}^4(\mathbb{R}^d)} + \|\log^2(2+|\cdot|)\mu_d(|\cdot|)^{\frac12}DF\|^4_{\mathrm{L}^4(\mathbb{R}^d)}\Big)$$
$$+ \varepsilon^{d+4}\log^6(2+\tfrac{1}{\varepsilon})\mu_d(\tfrac{1}{\varepsilon})^2\|\log^2(2+|\cdot|)\mu_d(|\cdot|)^{\frac12}DF\|^4_{\mathrm{L}^4(\mathbb{R}^d)}.$$

We need to iterate the vertical derivative and estimate $\Delta_b\Delta_{b'}I_0^\varepsilon(F)$. By definition of the homogenization commutator, we find

$$\begin{aligned}\Delta_b\Delta_{b'}I_0^\varepsilon(F) &= \varepsilon^{\frac d2}\Delta_{b'}\int_{\mathbb{R}^d}F_{\varepsilon,ij}e_j\cdot\Delta_bA(\nabla\phi_i^b+e_i)+\varepsilon^{\frac d2}\Delta_{b'}\int_{\mathbb{R}^d}F_{\varepsilon,ij}e_j\cdot(A-A_{\mathrm{hom}})\nabla\Delta_b\phi_i\\ &= \varepsilon^{\frac d2}\int_{\mathbb{R}^d}F_{\varepsilon,ij}e_j\cdot\Delta_b\Delta_{b'}A(\nabla\phi_i^{b,b'}+e_i)+\varepsilon^{\frac d2}\int_{\mathbb{R}^d}F_{\varepsilon,ij}e_j\cdot\big(\Delta_bA\nabla\Delta_{b'}\phi_i^b+\Delta_{b'}A\nabla\Delta_b\phi_i^{b'}\big)\\ &\qquad+\varepsilon^{\frac d2}\int_{\mathbb{R}^d}F_{\varepsilon,ij}e_j\cdot(A-A_{\mathrm{hom}})\nabla\Delta_b\Delta_{b'}\phi_i. \end{aligned}\tag{3.72}$$

In order to avoid additional logarithmic factors, we need to suitably rewrite the last right-hand side term, and we argue similarly as in the proof of (3.47). Using the definition (3.33) of σ_j^* in the form $(A^*-A^*_{\mathrm{hom}})e_j=-A^*\nabla\phi_j^*+\nabla^*\cdot\sigma_j^*$, applying the discrete Leibniz rule (3.49), and using the skew-symmetry (3.32) of σ_i, we find

$$\begin{aligned}e_j\cdot(A-A_{\mathrm{hom}})\nabla\Delta_b\Delta_{b'}\phi_i &= (\nabla^*\cdot\sigma_j^*)\cdot\nabla\Delta_b\Delta_{b'}\phi_i-\nabla\phi_j^*\cdot A\nabla\Delta_b\Delta_{b'}\phi_i\\ &= \nabla_k\big(\sigma_{jlk}^*(\cdot-e_k)\nabla_l\Delta_b\Delta_{b'}\phi_i\big)-\nabla_k^*\big(\phi_j^*(\cdot+e_k)e_k\cdot A\nabla\Delta_b\Delta_{b'}\phi_i\big)+\phi_j^*\nabla^*\cdot A\nabla\Delta_b\Delta_{b'}\phi_i.\end{aligned}$$

Noting that the vertical derivative of equation (3.41) takes the form

$$-\nabla^*\cdot A\nabla\Delta_b\Delta_{b'}\phi_i=\nabla^*\cdot\Delta_{b'}A\nabla\Delta_b\phi_i^{b'}+\nabla^*\cdot\Delta_b\Delta_{b'}A(\nabla\phi_i^{b,b'}+e_i)+\nabla^*\cdot\Delta_bA\nabla\Delta_{b'}\phi_i^b,\tag{3.73}$$

and combining this with the above yields

$$\begin{aligned}e_j\cdot(A-A_{\mathrm{hom}})\nabla\Delta_b\Delta_{b'}\phi_i &= \nabla_k\big(\sigma_{jlk}^*(\cdot-e_k)\nabla_l\Delta_b\Delta_{b'}\phi_i\big)-\nabla_k^*\big(\phi_j^*(\cdot+e_k)e_k\cdot A\nabla\Delta_b\Delta_{b'}\phi_i\big)\\ &\quad-\phi_j^*\nabla^*\cdot\Delta_{b'}A\nabla\Delta_b\phi_i^{b'}-\phi_j^*\nabla^*\cdot\Delta_b\Delta_{b'}A(\nabla\phi_i^{b,b'}+e_i)-\phi_j^*\nabla^*\cdot\Delta_bA\nabla\Delta_{b'}\phi_i^b.\end{aligned}$$

Inserting this into (3.72), integrating by parts, and applying the discrete Leibniz rule (3.49), we are led to the following representation formula,

$$\begin{aligned}\Delta_b\Delta_{b'}I_0^\varepsilon(F) &= \varepsilon^{\frac d2}\int_{\mathbb{R}^d}F_{\varepsilon,ij}(\nabla\phi_j^*+e_j)\cdot\Delta_b\Delta_{b'}A(\nabla\phi_i^{b,b'}+e_i)\\ &\quad+\varepsilon^{\frac d2}\int_{\mathbb{R}^d}F_{\varepsilon,ij}(\nabla\phi_j^*+e_j)\cdot\big(\Delta_bA\nabla\Delta_{b'}\phi_i^b+\Delta_{b'}A\nabla\Delta_b\phi_i^{b'}\big)\\ &\quad+\varepsilon^{\frac d2}\int_{\mathbb{R}^d}\big(\sigma_{jkl}^*(\cdot-e_k)\nabla_k^*F_{\varepsilon,ij}+\phi_j^*(\cdot+e_k)A_{kl}\nabla_kF_{\varepsilon,ij}\big)\nabla_l\Delta_b\Delta_{b'}\phi_i\\ &\quad+\varepsilon^{\frac d2}\int_{\mathbb{R}^d}\phi_j^*(\cdot+e_k)\nabla_kF_{\varepsilon,ij}e_k\cdot\big(\Delta_bA\nabla\Delta_{b'}\phi_i^b+\Delta_{b'}A\nabla\Delta_b\phi_i^{b'}\big)\\ &\quad+\varepsilon^{\frac d2}\int_{\mathbb{R}^d}\phi_j^*(\cdot+e_k)\nabla_kF_{\varepsilon,ij}e_k\cdot\Delta_b\Delta_{b'}A(\nabla\phi_i^{b,b'}+e_i).\end{aligned}\tag{3.74}$$

We need to estimate the moment of each right-hand side term. Fix momentarily $b=(z_b,z_b+\xi_b)$ and $b'=(z_{b'},z_{b'}+\xi_{b'})$. Applying Lemmas 3.3.3 and 3.4.2 to the Green representation formula (3.70) for $\nabla\Delta_b\phi$, we find for all $q<\infty$,

$$\mathbb{E}\left[|\nabla\Delta_b\phi(x)|^q\right]^{\frac1q}\lesssim_q(1+|x-z_b|)^{-d}.$$

We then turn to the second vertical derivatives. We obviously have $\Delta_b\Delta_{b'}A = \mathbb{1}_{b=b'}\Delta_b A$. Next, the Green representation formula applied to equation (3.73) yields

$$\begin{aligned}\nabla\Delta_b\Delta_{b'}\phi_j(x) = -\nabla\nabla G(x,z_{b'})\cdot\Delta_{b'}A(z_{b'})\nabla\Delta_b\phi_j^{b'}(z_{b'}) - \nabla\nabla G(x,z_b)\cdot\Delta_b A(z_b)\nabla\Delta_{b'}\phi_j^b(z_b)\\ - \mathbb{1}_{b=b'}\nabla\nabla G(x,z_b)\cdot\Delta_b A(z_b)(\nabla\phi_j^b + e_j),\end{aligned}$$

so that, for all $q<\infty$, Lemmas 3.3.3 and 3.4.2 lead to

$$\mathbb{E}\left[|\nabla\Delta_b\Delta_{b'}\phi(x)|^q\right]^{\frac1q} \lesssim_q (1+|z_b-z_{b'}|)^{-d}\big((1+|x-z_{b'}|)^{-d} + (1+|x-z_b|)^{-d}\big).$$

Inserting these estimates into (3.74), we obtain

$$\begin{aligned}\mathbb{E}\left[|\Delta_b\Delta_{b'}I_0^\varepsilon(F)|^4\right]^{\frac14} \lesssim \frac{\varepsilon^{\frac d2}}{(1+|z_b-z_{b'}|)^d}\bigg(\int_{Q(z_b)}|F_\varepsilon| + \int_{Q(z_{b'})}|F_\varepsilon|\\ + \int_{\mathbb{R}^d}\frac{\mu_d(|x|)^{\frac12}|\nabla F_\varepsilon(x)|}{(1+|x-z_{b'}|)^d}dx + \int_{\mathbb{R}^d}\frac{\mu_d(|x|)^{\frac12}|\nabla F_\varepsilon(x)|}{(1+|x-z_b|)^d}dx\bigg).\end{aligned}$$

Combining this with (3.71) and with the definition of K_2^ε, using the short-hand notation

$$I(\zeta)(z) := \int_{\mathbb{R}^d}\frac{|\zeta(x)|}{(1+|x-z|)^d}dx, \qquad G_\varepsilon := \mu_d(|\cdot|)^{\frac12}\nabla F_\varepsilon,$$

we deduce

$$\begin{aligned}K_2^\varepsilon \lesssim \varepsilon^{2d}\int_{\mathbb{R}^d}\Big(|F_\varepsilon|^2|I(F_\varepsilon)|^2 + |I(|F_\varepsilon|^2)|^2 + |I(F_\varepsilon)|^2|I(G_\varepsilon)|^2 + |I(F_\varepsilon I(G_\varepsilon))|^2\\ + |F_\varepsilon|^2|I(I(G_\varepsilon))|^2 + |I(|I(G_\varepsilon)|^2)|^2 + |I(G_\varepsilon)|^2|I(I(G_\varepsilon))|^2\Big).\end{aligned}$$

Using the Cauchy-Schwarz inequality, and making a multiple use of (3.69) in the form

$$\|\log^r(2+|\cdot|)\,I(\zeta)\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_{p,r} \|\log^{r+1}(2+|\cdot|)\,\zeta\|_{\mathrm{L}^p(\mathbb{R}^d)},$$

we are led to

$$\begin{aligned}K_2^\varepsilon \lesssim \varepsilon^{2d}\Big(&\|F_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|\log(2+|\cdot|)F_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)} + \|\log^{\frac12}(2+|\cdot|)F_\varepsilon\|^4_{\mathrm{L}^4(\mathbb{R}^d)}\\ &+ \|F_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|\log^2(2+|\cdot|)G_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)} + \|\log(2+|\cdot|)F_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|\log(2+|\cdot|)G_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\\ &+ \|\log^{\frac32}(2+|\cdot|)G_\varepsilon\|^4_{\mathrm{L}^4(\mathbb{R}^d)} + \|\log(2+|\cdot|)G_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\|\log^2(2+|\cdot|)G_\varepsilon\|^2_{\mathrm{L}^4(\mathbb{R}^d)}\Big).\end{aligned}$$

Inserting the definition of G_ε, rescaling the integrals, and using (3.60), the conclusion follows.

3.5 Covariance structure

In this section, we turn to the limiting covariance structure of the homogenization commutator, as stated in Proposition 3.2.9.

3.5.1 Structure of the proof and auxiliary results

The main tool to prove Proposition 3.2.9 is the following stronger version of the spectral gap estimate of Lemma 3.3.1, which gives an identity (rather than a bound) for the variance of a random variable in terms of its variations. This is an i.i.d. version of the so-called Helffer-Sjöstrand representation formula [238, 399] (see also [335, 329]) and it holds for any product measure $\mathbb{P}$ on Ω. A proof is included for completeness in Subsection 3.5 below. It is more conveniently formulated in terms of $\tilde\Delta_b X := X - \mathbb{E}_{a(b)}[X]$, where $\mathbb{E}_{a(b)}[\cdot] := \mathbb{E}\left[\,\cdot\,\|\,(a(b'))_{b'\neq b}\right]$ denotes the expectation with respect to the random variable $a(b)$ only. This is a natural variant of the vertical derivative Δ_b and satisfies $\mathbb{E}\big[|\tilde\Delta_b X|^2\big] = \frac12\mathbb{E}\big[|\Delta_b X|^2\big]$. Note that by definition $\tilde\Delta_b X = \mathbb{E}_{a^b(b)}[\Delta_b X]$, where $\mathbb{E}_{a^b(b)}[\cdot]$ denotes the expectation with respect to the random variable $a^b(b)$ only.

Lemma 3.5.1. *Let $\mathbb{P}$ be a product measure. For all $X = X(A) \in \mathrm{L}^2(\Omega)$ we have*

$$\mathrm{Var}\,[X] = \sum_{b\in\mathcal{B}} \mathbb{E}\left[(\tilde\Delta_b X)\,\mathcal{T}\,(\tilde\Delta_b X)\right],$$

where $\mathcal{T} := (\sum_{b\in\mathcal{B}}\tilde\Delta_b\tilde\Delta_b)^{-1}$ is a self-adjoint positive operator on $\mathrm{L}^2(\Omega)/\mathbb{R} := \{X\in\mathrm{L}^2(\Omega) : \mathbb{E}[X]=0\}$ with operator norm bounded by 1. In particular, it implies the following covariance inequality: for all $X,Y\in\mathrm{L}^2(\Omega)$ we have

$$\mathrm{Cov}\,[X;Y] \le \frac12\sum_{b\in\mathcal{B}}\mathbb{E}\left[|\Delta_b X|^2\right]^{\frac12}\mathbb{E}\left[|\Delta_b Y|^2\right]^{\frac12}. \qquad \Diamond$$

The proof of Proposition 3.2.9(i) below further implies that the effective fluctuation tensor $\mathcal{Q}$ is given by the following formula, with the notation $b_n := (0, e_n)$,

$$\begin{aligned}\mathcal{Q}_{ijkl} &:= \sum_{n=1}^d \mathbb{E}\left[\big(M^n_{ij} - \mathbb{E}\big[M^n_{ij}\big]\big)\,\mathcal{T}\,\big(M^n_{kl} - \mathbb{E}\big[M^n_{kl}\big]\big)\right], &(3.75)\\ M^n_{ij} &:= \mathbb{E}_{a^{b_n}(b_n)}\Big[\big(a(b_n) - a^{b_n}(b_n)\big)\big(e_n\cdot(\nabla\phi^*_j(0)+e_j)\big)\big(e_n\cdot(\nabla\phi^{b_n}_i(0)+e_i)\big)\Big],\end{aligned}$$

in terms of the abstract operator $\mathcal{T}$ defined above. Although not convenient for numerical approximation of $\mathcal{Q}$, this formula allows to easily deduce the non-degeneracy result contained in Proposition 3.2.9(ii). In addition, this is key to the proof of Theorem 3.1.3 on the RVE method.

3.5.2 Proof of Lemma 3.5.1

We start with some observations on the difference operator $\tilde\Delta_b$ on $\mathrm{L}^2(\Omega)$. For all $X, Y \in \mathrm{L}^2(\Omega)$, by exchangeability of (A, A^b), we find

$$\begin{aligned}\mathbb{E}\big[X\tilde\Delta_b Y\big] = \mathbb{E}\big[XY\big] - \mathbb{E}\big[X\mathbb{E}_{a(b)}[Y]\big] &= \mathbb{E}\big[XY\big] - \mathbb{E}\big[\mathbb{E}_{a(b)}[X]\mathbb{E}_{a(b)}[Y]\big]\\ &= \mathbb{E}\big[XY\big] - \mathbb{E}\big[Y\mathbb{E}_{a(b)}[X]\big] = \mathbb{E}\big[Y\tilde\Delta_b X\big],\end{aligned}$$

so that $\tilde\Delta_b$ is symmetric on $\mathrm{L}^2(\Omega)$. In addition, we easily compute, for all $b, b' \in \mathcal{B}$,

$$\tilde\Delta_b\tilde\Delta_b = \tilde\Delta_b, \qquad \tilde\Delta_b\tilde\Delta_{b'} = \tilde\Delta_{b'}\tilde\Delta_b. \tag{3.76}$$

With these observations at hand, we now turn to the study of the (densely defined) operator $\mathcal{S} := \sum_{b\in\mathcal{B}} \tilde\Delta_b\tilde\Delta_b$ on $\mathrm{L}^2(\Omega)$. More precisely, we consider the space $\mathrm{L}^2(\Omega)/\mathbb{R} := \{X \in \mathrm{L}^2(\Omega) : \mathbb{E}[X] = 0\}$ of mean-zero square-integrable random variables, and we show that $\mathcal{S}$ is an essentially self-adjoint, non-negative operator on $\mathrm{L}^2(\Omega)/\mathbb{R}$ with dense image. First, since $\mathbb{E}\big[\tilde\Delta_b X\big] = 0$ for all $b \in \mathcal{B}$ and $X \in \mathrm{L}^2(\Omega)$, the image $\operatorname{Im}\mathcal{S}$ is clearly contained in $\mathrm{L}^2(\Omega)/\mathbb{R}$. Second, for all $X \in \mathrm{L}^2(\Omega)$ in the domain of $\mathcal{S}$, we compute

$$\mathbb{E}[X\mathcal{S}X] = \sum_{b\in\mathcal{B}} \mathbb{E}\big[|\tilde\Delta_b X|^2\big] \ge 0,$$

which shows that $\mathcal{S}$ is non-negative. Third, if $X \in \mathrm{L}^2(\Omega)/\mathbb{R}$ in the domain of $\mathcal{S}$ is orthogonal to the image $\operatorname{Im}\mathcal{S}$, then we deduce

$$0 = \mathbb{E}[X\mathcal{S}X] = \sum_{b\in\mathcal{B}} \mathbb{E}\big[|\tilde\Delta_b X|^2\big],$$

so that $\tilde\Delta_b X = 0$ almost surely for all $b \in \mathcal{B}$, which implies that X is constant.

These properties of $\mathcal{S}$ allow us to define (densely) the inverse $\mathcal{T} := \mathcal{S}^{-1}$ as an essentially self-adjoint, non-negative operator on $\mathrm{L}^2(\Omega)/\mathbb{R}$. Finally, the spectral gap of Lemma 3.3.1 implies, for all $X \in \mathrm{L}^2(\Omega)/\mathbb{R}$ in the domain of $\mathcal{S}$,

$$\|X\|^2_{\mathrm{L}^2(\Omega)} = \operatorname{Var}[X] \le \sum_{b\in\mathcal{B}} \mathbb{E}\big[|\tilde\Delta_b X|^2\big] = \mathbb{E}[X\mathcal{S}X] \le \|X\|_{\mathrm{L}^2(\Omega)}\|\mathcal{S}X\|_{\mathrm{L}^2(\Omega)},$$

and hence $\|X\|_{\mathrm{L}^2(\Omega)} \le \|\mathcal{S}X\|_{\mathrm{L}^2(\Omega)}$, which implies that $\mathcal{T} = \mathcal{S}^{-1}$ on $\mathrm{L}^2(\Omega)/\mathbb{R}$ has operator norm bounded by 1.

It remains to establish the representation formula for the variance. By density, it suffices to prove it for all $X \in \operatorname{Im}\mathcal{S}$. Writing $X = \mathcal{S}Y$ for some $Y \in \mathrm{L}^2(\Omega)/\mathbb{R}$, we decompose

$$\operatorname{Var}[X] = \mathbb{E}[X\mathcal{S}Y] = \sum_{b\in\mathcal{B}} \mathbb{E}\big[\tilde\Delta_b X\tilde\Delta_b Y\big] = \sum_{b\in\mathcal{B}} \mathbb{E}\big[(\tilde\Delta_b X)(\tilde\Delta_b \mathcal{T}X)\big].$$

Since the commutation relations (3.76) ensure that $\tilde\Delta_b\mathcal{S} = \mathcal{S}\tilde\Delta_b$ holds on the domain of $\mathcal{S}$ in $\mathrm{L}^2(\Omega)$, we deduce $\tilde\Delta_b\mathcal{T} = \mathcal{T}\tilde\Delta_b$ on $\mathrm{L}^2(\Omega)/\mathbb{R}$, and the above then leads to the desired representation

$$\operatorname{Var}[X] = \sum_{b\in\mathcal{B}} \mathbb{E}\big[(\tilde\Delta_b X)\mathcal{T}(\tilde\Delta_b X)\big].$$

3.5.3 Proof of Proposition 3.2.9(i)

By polarization and linearity, it is enough to prove (3.25) with $\mathcal{F} = G \in C_c^\infty(\mathbb{R}^d)^{d\times d}$. We thus need to establish the convergence of the variance

$$\nu_\varepsilon := \operatorname{Var}\left[\varepsilon^{-\frac d2}\int_{\mathbb{R}^d} F : \Xi(\tfrac{\cdot}{\varepsilon})\right] = \operatorname{Var}\left[\varepsilon^{\frac d2}\int_{\mathbb{R}^d} F_\varepsilon : \Xi\right],$$

where we have set $F_\varepsilon := F(\varepsilon\cdot)$. We split the proof into two steps.

Step 1. Proof of (3.25).

Applying the Helffer-Sjöstrand representation of Lemma 3.5.1 to the variance ν_ε, we are led to

$$\nu_\varepsilon = \varepsilon^d \sum_{b\in\mathcal{B}} \mathbb{E}\left[\Big(\tilde\Delta_b \int_{\mathbb{R}^d} F_\varepsilon : \Xi\Big)\mathcal{T}\Big(\tilde\Delta_b \int_{\mathbb{R}^d} F_\varepsilon : \Xi\Big)\right]. \tag{3.77}$$

We now appeal to (3.47) in the form

$$\Delta_b \int_{\mathbb{R}^d} F_\varepsilon : \Xi = \int_{\mathbb{R}^d} F_{\varepsilon,ij}(\nabla\phi_j^* + e_j)\cdot \Delta_b A(\nabla\phi_i^b + e_i) \\ + \int_{\mathbb{R}^d} \phi_j^*(\cdot + e_k)\nabla_k F_{\varepsilon,ij} e_k \cdot \Delta_b A(\nabla\phi_i^b + e_i) + \int_{\mathbb{R}^d} \nabla h_{\varepsilon,i}\cdot \Delta_b A(\nabla\phi_i^b + e_i),$$

where the auxiliary field $h_{\varepsilon,i}$ is the unique Lax-Milgram solution in $\mathbb{R}^d$ of

$$-\nabla^*\cdot A^*\nabla h_{\varepsilon,i} = \nabla_l^*\big(\phi_j^*(\cdot + e_k)A_{kl}\nabla_k F_{\varepsilon,ij} + \sigma_{jkl}^*(\cdot - e_k)\nabla_k^* F_{\varepsilon,ij}\big). \tag{3.78}$$

Recalling that $\tilde\Delta_b X = \mathbb{E}_{a^b(b)}[\Delta_b X]$, inserting this representation formula into (3.77), extracting the first term U_ε defined below, and using that $\mathcal{T}$ on $\mathrm{L}^2(\Omega)/\mathbb{R}$ has operator norm bounded by 1, we find

$$|\nu_\varepsilon - \varepsilon^d U_\varepsilon| \le \varepsilon^d \sum_{b\in\mathcal{B}} \Big(S_\varepsilon^b T_\varepsilon^b + \frac{1}{2}(T_\varepsilon^b)^2\Big), \tag{3.79}$$

where for convenience we define

$$\begin{aligned} U_\varepsilon &:= \sum_{b\in\mathcal{B}} \mathbb{E}\Big[(V_\varepsilon^b - \mathbb{E}[V_\varepsilon^b])\,\mathcal{T}\,(V_\varepsilon^b - \mathbb{E}[V_\varepsilon^b])\Big], \\ V_\varepsilon^b &:= \mathbb{E}_{a^b(b)}\Big[\int_{\mathbb{R}^d} F_{\varepsilon,ij}(\nabla\phi_j^* + e_j)\cdot\Delta_b A(\nabla\phi_i^b + e_i)\Big], \end{aligned}$$

while for all $b\in\mathcal{B}$ the error terms are given by

$$S_\varepsilon^b := \mathbb{E}\bigg[\Big(\int_{\mathbb{R}^d} |\Delta_b A||\nabla\phi^* + \mathrm{Id}\,||\nabla\phi^b + \mathrm{Id}\,||F_\varepsilon|\Big)^2\bigg]^{\frac12},$$

and by $T_\varepsilon^b := T_{\varepsilon,1}^b + T_{\varepsilon,2}^b$ with

$$\begin{aligned} T_{\varepsilon,1}^b &:= \sum_{k=1}^d \mathbb{E}\bigg[\Big(\int_{\mathbb{R}^d} |\Delta_b A||\phi^*(\cdot + e_k)||\nabla\phi^b + \mathrm{Id}\,||\nabla F_\varepsilon|\Big)^2\bigg]^{\frac12}, \\ T_{\varepsilon,2}^b &:= \mathbb{E}\bigg[\Big(\int_{\mathbb{R}^d} |\Delta_b A||\nabla\phi^b + \mathrm{Id}\,||\nabla h_\varepsilon|\Big)^2\bigg]^{\frac12}. \end{aligned}$$

We start with the analysis of U_ε. Writing $\Delta_b A(x) = (a(b) - a^b(b))\mathbb{1}_{Q(z_b)}(x)\xi_b\otimes\xi_b$ for $b = (z_b, z_b + \xi_b)$, we may compute

$$V_\varepsilon^b = \Big(\int_{Q(z_b)} F_{\varepsilon,ij}\Big)\,\mathbb{E}_{a^b(b)}\Big[(a(b) - a^b(b))\big(\xi_b\cdot(\nabla\phi_j^*(z_b) + e_j)\big)\big(\xi_b\cdot(\nabla\phi_i^b(z_b) + e_i)\big)\Big],$$

so that, by stationarity,

$$\varepsilon^d U_\varepsilon = \mathcal{Q}_{ijkl}\,\varepsilon^d \sum_{z\in\mathbb{Z}^d} \Big(\int_{Q(z)} F_{\varepsilon,ij}\Big)\Big(\int_{Q(z)} F_{\varepsilon,kl}\Big),$$

where the coefficient $\mathcal{Q}_{ijkl}$ is defined in (3.75) above. Since $\mathcal{T}$ on $\mathrm{L}^2(\Omega)/\mathbb{R}$ has operator norm bounded by 1, the moment bounds of Lemma 3.3.3 yield

$$|\mathcal{Q}_{ijkl}| \lesssim \sum_{n=1}^d \mathbb{E}\Big[|\nabla\phi^* + \mathrm{Id}\,|^2|\nabla\phi^{b_n} + \mathrm{Id}\,|^2\Big] \lesssim 1.$$

We may then estimate the discretization error

$$\Big|\varepsilon^d U_\varepsilon - \mathcal{Q}_{ijkl} \int_{\mathbb{R}^d} F_{ij}F_{kl}\Big| \;=\; \Big|\varepsilon^d U_\varepsilon - \mathcal{Q}_{ijkl}\,\varepsilon^d \int_{\mathbb{R}^d} F_{\varepsilon,ij}F_{\varepsilon,kl}\Big| \;\lesssim\; \varepsilon^d \sum_{z\in\mathbb{Z}^d} \int_{Q(z)} \Big|F_\varepsilon(x) - \int_{Q(z)} F_\varepsilon\Big|^2 dx$$
$$\lesssim\; \varepsilon^d \int_{\mathbb{R}^d} |DF_\varepsilon|^2 \;=\; \varepsilon^2 \int_{\mathbb{R}^d} |DF|^2. \tag{3.80}$$

We now turn to the estimate of the right-hand side of (3.79). Using $|\Delta_b A(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$ and the moment bounds of Lemma 3.3.3, we obtain

$$S_\varepsilon^b \lesssim \mathbb{E}\Big[|\nabla\phi^* + \mathrm{Id}\,|^2|\nabla\phi^b + \mathrm{Id}\,|^2\Big]^{\frac12} \int_{Q(z_b)} |F_\varepsilon| \lesssim \int_{Q(z_b)} |F_\varepsilon|.$$

Hence, by the Cauchy-Schwarz inequality,

$$\sum_{b\in\mathcal{B}} S_\varepsilon^b T_\varepsilon^b \lesssim \sum_{b\in\mathcal{B}} T_\varepsilon^b \int_{Q(z_b)} |F_\varepsilon| \lesssim \|F_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}^d)}\Big(\sum_{b\in\mathcal{B}} (T_\varepsilon^b)^2\Big)^{\frac12} \lesssim \varepsilon^{-\frac{d}{2}}\|F\|_{\mathrm{L}^2(\mathbb{R}^d)}\Big(\sum_{b\in\mathcal{B}} (T_\varepsilon^b)^2\Big)^{\frac12}, \tag{3.81}$$

and it remains to estimate

$$\sum_{b\in\mathcal{B}} (T_\varepsilon^b)^2 \le 2\sum_{b\in\mathcal{B}} (T_{\varepsilon,1}^b)^2 + 2\sum_{b\in\mathcal{B}} (T_{\varepsilon,2}^b)^2.$$

First, using $|\Delta_b A(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$ and the moment bounds of Lemma 3.3.3, we find

$$\varepsilon^d \sum_{b\in\mathcal{B}} (T_{\varepsilon,1}^b)^2 \;\lesssim_{\alpha,p}\; \varepsilon^d \|\mu_d(|\cdot|)^{\frac12}\nabla F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^d)}. \tag{3.82}$$

Second, arguing as in the proof of Proposition 3.2.1 (cf. (3.45)), using the large-scale weighted Calderón-Zygmund theory (cf. Lemma 3.3.4) applied to equation (3.78) for h_ε, we obtain for all $0 < p-1 \ll 1$ and all $\alpha > d$,

$$\varepsilon^d \sum_{b\in\mathcal{B}} (T_{\varepsilon,2}^b)^2 \;\lesssim_{\alpha,p}\; \varepsilon^{\frac{d}{p}} \|w_\varepsilon^{\alpha\frac{p-1}{2p}}\, \mu_d(|\cdot|)^{\frac12}\nabla F_\varepsilon\|^2_{\mathrm{L}^{2p}(\mathbb{R}^d)}. \tag{3.83}$$

Rescaling the integrals and using (3.60) and Hölder's inequality, we find

$$\varepsilon^d \sum_{b\in\mathcal{B}} (T_\varepsilon^b)^2 \;\lesssim_{\alpha,p}\; \varepsilon^2 \mu_d(\tfrac{1}{\varepsilon}) \|w_1^{\alpha\frac{p-1}{2p}}\, \mu_d(|\cdot|)^{\frac12} DF\|^2_{\mathrm{L}^{2p}(\mathbb{R}^d)}.$$

and the conclusion (3.25) follows.

Step 2. Proof of the Green-Kubo formula (3.26).

In order to establish (3.26), it suffices to repeat the argument of Step 1 with the test function $\mathcal{F} = \mathbb{1}_Q\, e_i \otimes e_j$ (hence $F_\varepsilon = \mathbb{1}_{\frac{1}{\varepsilon}Q}\, e_i \otimes e_j$), for some fixed $1 \le i,j \le d$. Lemma 3.5.1 again leads to (3.79), and we briefly indicate how to analyze the different terms in the present setting. First, the estimate (3.80) is replaced by the following (no summation over repeated indices),

$$|\varepsilon^d U_\varepsilon - \mathcal{Q}_{ijij}| \;\lesssim\; \varepsilon^d \sum_{z\in\mathbb{Z}^d} \int_Q \Big(\mathbb{1}_{\frac{1}{\varepsilon}Q}(z+x) - \int_Q \mathbb{1}_{\frac{1}{\varepsilon}Q}(z+y)dy\Big)^2 dx$$
$$\le\; \varepsilon^d \sum_{z\in\mathbb{Z}^d} \mathbb{1}_{(z+Q)\cap\partial(\frac{1}{\varepsilon}Q)\neq\varnothing} \;\lesssim\; \varepsilon.$$

Second, the estimate (3.81) remains unchanged. Third, using estimates (3.82) and (3.83), and noting that $|\nabla F_\varepsilon| \lesssim \mathbb{1}_{A_\varepsilon}$ with $A_\varepsilon := B + \partial Q_{\frac{1}{\varepsilon}}$ and that $w_\varepsilon \lesssim 1$ and $\mu_d(|\cdot|) \lesssim \mu_d(\frac{1}{\varepsilon})$ on A_ε, we deduce

$$\begin{aligned}\sum_{b\in\mathcal{B}} (T^b_{\varepsilon,1})^2 &\lesssim \mu_d(\tfrac{1}{\varepsilon})|A_\varepsilon| \lesssim \varepsilon^{1-d}\mu_d(\tfrac{1}{\varepsilon}),\\ \sum_{b\in\mathcal{B}} (T^b_{\varepsilon,2})^2 &\lesssim \varepsilon^{-d\frac{p-1}{p}}\mu_d(\tfrac{1}{\varepsilon})|A_\varepsilon|^{\frac{1}{p}} \lesssim \varepsilon^{\frac{1}{p}-d}\mu_d(\tfrac{1}{\varepsilon}),\end{aligned}$$

and the conclusion (3.26) follows.

3.5.4 Proof of Proposition 3.2.9(ii)

We turn to the proof of the non-degeneracy of $\mathcal{Q}$. Given a fixed direction $\xi \in \mathbb{R}^d \setminus \{0\}$, and letting ϕ_ξ denote the corrector in this direction, we may write, in view of formula (3.75) (with $\phi^*_\xi = \phi_\xi$ by symmetry of the coefficients),

$$(\xi\otimes\xi) : \mathcal{Q}\,(\xi\otimes\xi) = \sum_{n=1}^d \mathbb{E}\big[(\xi\cdot M^n\xi)\mathcal{T}(\xi\cdot M^n\xi)\big], \tag{3.84}$$

$$\xi\cdot M^n\xi := \mathbb{E}_{a^{b_n}(b_n)}\Big[\big(a(b_n)-a^{b_n}(b_n)\big)\big(e_n\cdot(\nabla\phi_\xi(0)+\xi)\big)\big(e_n\cdot(\nabla\phi_\xi^{b_n}(0)+\xi)\big)\Big],$$

since the exchangeability of (A, A^{b_n}) indeed yields $\mathbb{E}\big[\xi\cdot M^n\xi\big] = 0$ for all n. We start with a suitable reformulation of $\xi\cdot M^n\xi$. Considering the difference of the corrector equation (3.21) for ϕ_ξ and $\phi_\xi^{b_n}$ in the form $-\nabla^*\cdot A^{b_n}\nabla(\phi_\xi^{b_n}-\phi_\xi) = \nabla^*\cdot(A^{b_n}-A)(\nabla\phi_\xi+\xi)$, an integration by parts yields

$$\begin{aligned}\int_{\mathbb{R}^d}\nabla(\phi_\xi^{b_n}-\phi_\xi)\cdot A^{b_n}\nabla(\phi_\xi^{b_n}-\phi_\xi) &= -\int_{\mathbb{R}^d}\nabla(\phi_\xi^{b_n}-\phi_\xi)\cdot(A^{b_n}-A)(\nabla\phi_\xi+\xi)\\ &= (a(b_n)-a^{b_n}(b_n))(e_n\cdot\nabla(\phi_\xi^{b_n}-\phi_\xi)(0))(e_n\cdot(\nabla\phi_\xi(0)+\xi)).\end{aligned}$$

Hence, by definition of $\xi\cdot M^n\xi$,

$$\xi\cdot M^n\xi = \mathbb{E}_{a^{b_n}(b_n)}\bigg[\int_{\mathbb{R}^d}\nabla(\phi_\xi^{b_n}-\phi_\xi)\cdot A^{b_n}\nabla(\phi_\xi^{b_n}-\phi_\xi)\bigg] + (a(b_n)-\mathbb{E}\left[a(b_n)\right])(e_n\cdot(\nabla\phi_\xi(0)+\xi))^2. \tag{3.85}$$

We now argue by contradiction. If $(\xi\otimes\xi) : \mathcal{Q}\,(\xi\otimes\xi) = 0$, then by formula (3.84) and by the non-negativity of $\mathcal{T}$ we would have $\mathbb{E}\big[(\xi\cdot M^n\xi)\mathcal{T}(\xi\cdot M^n\xi)\big] = 0$ for all n. Let $1\le n\le d$ be momentarily fixed. Recalling that $\mathcal{T} = \mathcal{S}^{-1}$ with $\mathcal{S} = \sum_{b\in\mathcal{B}}\tilde\Delta_b\tilde\Delta_b$, this would imply

$$0 = \mathbb{E}\left[(\mathcal{T}(\xi\cdot M^n\xi))\mathcal{S}(\mathcal{T}(\xi\cdot M^n\xi))\right] = \sum_{b\in\mathcal{B}}\mathbb{E}\left[\big|\tilde\Delta_b\mathcal{T}(\xi\cdot M^n\xi)\big|^2\right],$$

hence $\mathcal{T}(\xi\cdot M^n\xi) = 0$, and thus $\xi\cdot M^n\xi = 0$ almost surely. Formula (3.85) would then imply

$$(a(b_n)-\mathbb{E}\left[a(b_n)\right])(e_n\cdot(\nabla\phi_\xi(0)+\xi))^2 = -\mathbb{E}_{a^{b_n}(b_n)}\bigg[\int_{\mathbb{R}^d}\nabla(\phi_\xi^{b_n}-\phi_\xi)\cdot A^{b_n}\nabla(\phi_\xi^{b_n}-\phi_\xi)\bigg], \tag{3.86}$$

almost surely. Since the law of $a(b_n)$ is non-degenerate, the event $a(b_n) > \mathbb{E}\left[a(b_n)\right]$ occurs with a positive probability. Conditioning on this event, the left-hand side in (3.86) is non-negative, and the non-positivity of the right-hand side would then imply that both sides vanish, that is,

$$e_n\cdot(\nabla\phi_\xi(0)+\xi) = 0 \qquad\text{and}\qquad \mathbb{E}_{a^{b_n}(b_n)}\bigg[\int_{\mathbb{R}^d}\nabla(\phi_\xi^{b_n}-\phi_\xi)\cdot A^{b_n}\nabla(\phi_\xi^{b_n}-\phi_\xi)\bigg] = 0,$$

almost surely. Since the integrand in this last expectation is non-negative, we would deduce that the event $a(b_n) > \mathbb{E}[a(b_n)]$ entails $e_n \cdot (\nabla\phi_\xi(0)+\xi) = 0$ and $\nabla\phi_\xi(0) = \nabla\phi_\xi^{b_n}(0)$, and thus also $e_n \cdot (\nabla\phi_\xi^{b_n}(0)+\xi) = 0$ almost surely. Since this last event is independent of $a(b_n)$, hence of the conditioning event, we would deduce unconditionally $e_n \cdot (\nabla\phi_\xi^{b_n}(0)+\xi) = 0$ almost surely. By exchangeability of (A, A^{b_n}), this means $e_n \cdot (\nabla\phi_\xi(0)+\xi) = 0$ almost surely. As this holds for any n, we would conclude $\nabla\phi_\xi(0)+\xi = 0$ almost surely, and taking the expectation would lead to a contradiction.

3.6 Approximation of the fluctuation tensor

In this section, we analyze the RVE method for the approximation of the fluctuation tensor $\mathcal{Q}$ as stated in Theorem 3.1.3.

3.6.1 Structure of the proof and auxiliary results

The estimate on the standard deviation is obtained similarly as the CLT scaling in Proposition 3.2.1, noting that the large-scale Calderón-Zygmund result of Lemma 3.3.4 also holds for the periodized operator $-\nabla^* \cdot A_L \nabla$ on Q_L.[4] The characterization (3.16) of $\mathcal{Q}$ and the estimate on the systematic error are deduced as corollaries of formula (3.75) for the fluctuation tensor $\mathcal{Q}$, together with the following crucial estimates on the periodized corrector ϕ_L. (The first estimate on $\nabla\phi_L$ is stated as such in [206, Proposition 1], and the second estimate follows from a decomposition of the difference $\nabla\phi_L - \nabla\phi$ via massive versions of the corrector, applying [207, Lemma 2.3 and equation (2.68)], and optimizing the mass.)

Lemma 3.6.1 ([206, 207])**.** *Let $d \geq 2$ and let $\mathbb{P}$ be a product measure. For all $L \geq 2$ and all $q < \infty$ we have*

$$\mathbb{E}\left[|\nabla\phi_L|^q\right]^{\frac{1}{q}} \lesssim_q 1, \qquad \textit{and} \qquad \mathbb{E}\left[|\nabla(\phi_L-\phi)(0)|^q\right]^{\frac{1}{q}} \lesssim_q L^{-\frac{d}{2}} \log^{\frac{d}{2}} L. \qquad \lozenge$$

3.6.2 Proof of Theorem 3.1.3

We split the proof into two steps: we first estimate the variance of the RVE approximation, and then we turn to the characterization (3.16) of $\mathcal{Q}$ and to the systematic error of the RVE approximation.

Step 1. Proof of $|\mathrm{Var}[\mathcal{Q}_{L,N}]|^{\frac{1}{2}} \lesssim N^{-\frac{1}{2}}$.

Since the realizations $A_{\mathrm{hom},L}^{(n)}$ are i.i.d. copies of $A_{\mathrm{hom},L}$, the definition (3.17) of $\mathcal{Q}_{L,N}$ leads after straightforward computations to

$$\mathrm{Var}[\mathcal{Q}_{L,N}] = N^{-1}\mathrm{Var}\left[\left(L^{\frac{d}{2}}A_{\mathrm{hom},L}^* - \mathbb{E}\big[L^{\frac{d}{2}}A_{\mathrm{hom},L}^*\big]\right)^{\otimes 2}\right],$$

and hence,

$$|\mathrm{Var}[\mathcal{Q}_{L,N}]| \lesssim N^{-1}\,\mathbb{E}\left[\left|L^{\frac{d}{2}}(A_{\mathrm{hom},L} - \mathbb{E}[A_{\mathrm{hom},L}])\right|^4\right].$$

Arguing as in [206, Lemma 2] (see also Proposition 4.3.1 in Chapter 4), the spectral gap estimate of Lemma 3.3.1 is seen to imply the following inequality: for all $X = X(A) \in \mathrm{L}^4(\Omega)$,

$$\mathbb{E}\left[(X-\mathbb{E}[X])^4\right] \,\leq\, 4\,\mathbb{E}\left[\left(\sum_{b\in\mathcal{B}}|\Delta_b X|^2\right)^2\right].$$

4. The only issue concerns the corresponding moment bound $\mathbb{E}\big[r_{*,L}^q\big] \lesssim_q 1$ for all $q<\infty$, which by definition of $r_{*,L}$ (cf. [204]) is a consequence of a sup-bound based on the version of Lemma 3.3.3 for the periodized correctors (ϕ_L, σ_L) (cf. [206]).

Applying this inequality to (each component of) $X = A_{\mathrm{hom},L}$, we deduce

$$|\mathrm{Var}\,[\mathcal{Q}_{L,N}]| \lesssim N^{-1}\,\mathbb{E}\left[\bigg(\sum_{b\in\mathcal{B}_L}\Big(L^{-\frac{d}{2}}\int_{Q_L}\Delta_b\big(A_L(\nabla\phi_L+\mathrm{Id})\big)\Big)^2\bigg)^2\right].$$

Arguing as in the proof of Proposition 3.2.1 (with ε replaced by $\frac{1}{L}$ and F_ε replaced by Id), using the periodized version of Lemma 3.3.4 and the moment bounds of Lemma 3.6.1, the conclusion follows.

Step 2. Proof of (3.16) and of $|\mathbb{E}\,[\mathcal{Q}_{L,N}] - \mathcal{Q}| \lesssim L^{-\frac{d}{2}}\log^{\frac{d}{2}} L$.

Since the realizations $A^{(n)}_{\mathrm{hom},L}$ are i.i.d. copies of $A_{\mathrm{hom},L}$, the definition (3.17) of $\mathcal{Q}_{L,N}$ yields after straightforward computations $\mathbb{E}\,[\mathcal{Q}_{L,N}] = \mathrm{Var}\big[L^{\frac{d}{2}}A^*_{\mathrm{hom},L}\big]$, that is,

$$\begin{aligned}\mathbb{E}\,[(\mathcal{Q}_{L,N})_{ijkl}] \;&=\; \mathrm{Cov}\left[L^{\frac{d}{2}}A_{\mathrm{hom},L,ji}; L^{\frac{d}{2}}A_{\mathrm{hom},L,lk}\right]\\ &=\; L^{-d}\;\mathrm{Cov}\left[\int_{Q_L} e_j\cdot A_L(\nabla\phi_{L,i}+e_i); \int_{Q_L} e_l\cdot A_L(\nabla\phi_{L,k}+e_k)\right].\end{aligned} \tag{3.87}$$

For $b\in\mathcal{B}$, we write $b=(z_b, z_b+\xi_b)$. Using the periodized corrector equation (3.27) and its vertical derivative, and recalling that $\Delta_b A_L(x) = (a(b)-a^b(b))\mathbb{1}_{Q(z_b)}(x)\xi_b\otimes\xi_b$ for $b\in\mathcal{B}_L$ and $x\in Q_L$, we find

$$\begin{aligned}\Delta_b\int_{Q_L} e_j\cdot A_L(\nabla\phi_{L,i}+e_i) &= \int_{Q_L} e_j\cdot\Delta_b A_L(\nabla\phi^b_{L,i}+e_i) + \int_{Q_L} e_j\cdot A_L\nabla\Delta_b\phi_{L,i}\\ &=\; \int_{Q_L} e_j\cdot\Delta_b A_L(\nabla\phi^b_{L,i}+e_i) - \int_{Q_L}\nabla\phi^*_{L,j}\cdot A_L\nabla\Delta_b\phi_{L,i}\\ &=\; \int_{Q_L}(\nabla\phi^*_{L,j}+e_j)\cdot\Delta_b A_L(\nabla\phi^b_{L,i}+e_i)\\ &=\; (a(b)-a^b(b))(\xi_b\cdot(\nabla\phi^*_{L,j}(z_b)+e_j))(\xi_b\cdot(\nabla\phi^b_{L,i}(z_b)+e_i)).\end{aligned} \tag{3.88}$$

Applying the Helffer-Sjöstrand representation formula of Lemma 3.5.1 to the covariance in (3.87), we thus find by stationarity, as in the proof of Proposition 3.2.9(i),

$$\mathbb{E}\,[(\mathcal{Q}_{L,N})_{ijkl}] \;=\; \sum_{n=1}^d\mathbb{E}\left[M^n_{ij,L}\,\mathcal{T}\,M^n_{kl,L}\right],$$

where we have set

$$M^n_{ij,L} \;:=\; \mathbb{E}_{a^{b_n}(b_n)}\Big[(a(b_n)-a^{b_n}(b_n))(e_n\cdot(\nabla\phi^*_{L,j}(0)+e_j))(e_n\cdot(\nabla\phi^{b_n}_{L,i}(0)+e_i))\Big].$$

Noting that (3.88) implies $\mathbb{E}\big[M^n_{ij,L}\big] = 0$, comparing the above identity for $\mathbb{E}\,[(\mathcal{Q}_{L,N})_{ijkl}]$ with formula (3.75) for $\mathcal{Q}$, and using that the operator $\mathcal{T}$ on $\mathrm{L}^2(\Omega)/\mathbb{R}$ has operator norm bounded by 1, we deduce

$$\big|\mathbb{E}\,[(\mathcal{Q}_{L,N})_{ijkl}] - \mathcal{Q}_{ijkl}\big| \;\lesssim\; \mathbb{E}\left[|\nabla(\phi_L-\phi)(0)|^4\right]^{\frac14}\left(\mathbb{E}\left[|\nabla\phi_L|^4\right]+\mathbb{E}\left[|\nabla\phi|^4\right]\right)^{\frac34},$$

and the conclusion follows from Lemmas 3.3.3 and 3.6.1.

3.A Appendix: Massive-term approximation

This appendix is devoted to the analysis of the approximation of the fluctuation tensor $\mathcal{Q}$ by replacing the corrector ϕ by its massive approximations. More precisely, for all $T > 0$, we denote by $\phi_{T,i}$ the so-called massive corrector in the direction e_i, defined as the unique stationary solution in $\mathbb{Z}^d$ of

$$\frac{1}{T}\phi_{T,i} - \nabla^* \cdot A(\nabla\phi_{T,i} + e_i) = 0, \tag{3.89}$$

and we set $\phi_T := (\phi_{T,i})_{i=1}^d$. The massive approximation of the homogenization commutator Ξ is then naturally defined by

$$\Xi_{T,i} := A(\nabla\phi_{T,i} + e_i) - A_{\mathrm{hom}}(\nabla\phi_{T,i} + e_i), \qquad \Xi_{T,ij} := (\Xi_{T,i})_j, \tag{3.90}$$

and we consider the random functional $I^\varepsilon_{0,T} : F \mapsto I^\varepsilon_{0,T}(F)$ given for all $F \in C^\infty_c(\mathbb{R}^d)^{d\times d}$ by

$$I^\varepsilon_{0,T}(F) := \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} F(x) : \Xi_T(\tfrac{x}{\varepsilon})dx. \tag{3.91}$$

We establish the following result for the massive approximation of the homogenization commutator and of the fluctuation tensor. (Note that the use of Richardson extrapolations for the massive corrector [202, 199, 206] could be shown to improve the suboptimal convergence rate in (3.95) below into $T^{-\frac{d}{4}}$ in all dimensions $d \geq 2$.)

Proposition 3.A.1. *Let $d \geq 2$, let $\mathbb{P}$ be a product measure, and let μ_d be defined in* (3.13)*. For all $T \geq 1$,*

(i) We define the symmetric 4-tensor $\mathcal{Q}_T$ by

$$\mathcal{Q}_T := \int_{\mathbb{R}^d} \mathrm{Cov}\left[\Xi_T(x); \Xi_T(0)\right] dx,$$

where the integral is absolutely convergent.

(ii) For all $\varepsilon > 0$ and all $F, G \in C^\infty_c(\mathbb{R}^d)^{d\times d}$ we have

$$\left| \mathrm{Cov}\left[I^\varepsilon_{0,T}(F); I^\varepsilon_{0,T}(G)\right] - \int_{\mathbb{R}^d} F(x) : \mathcal{Q}_T\, G(x)dx \right| \lesssim (\varepsilon\sqrt{T})^2 \|(DF, DG)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}, \tag{3.92}$$

and for all $L \geq 1$ we have

$$\left| \mathcal{Q}_{T,ijkl} - \int_{Q_{2L}} \frac{|Q_L \cap (x + Q_L)|}{|Q_L|} \mathrm{Cov}\left[\Xi_{T,ij}(x); \Xi_{T,kl}(0)\right] dx \right| \lesssim \frac{\sqrt{T}}{L}. \tag{3.93}$$

(iii) For all $\varepsilon > 0$ and all $F \in C^\infty_c(\mathbb{R}^d)^{d\times d}$, we have for all $0 < p - 1 \ll 1$ and all $\alpha > d\frac{p-1}{4p}$, in the regime $\sqrt{T} \leq C^{-1}_{\alpha,p}\varepsilon^{-1}$ for some (large enough) constant $C_{\alpha,p} \simeq_{\alpha,p} 1$,

$$\mathrm{Var}\left[I^\varepsilon_{0,T}(F) - I^\varepsilon_0(F)\right] \lesssim_p \frac{\mu_d(\frac{1}{\varepsilon})}{T} \Big(\|w_1^{2\alpha} F\|^2_{\mathrm{L}^{2p}(\mathbb{R}^d)} + \|w_1^{2\alpha} DF\|^2_{\mathrm{L}^{2p}(\mathbb{R}^d)} \Big). \tag{3.94}$$

In particular, combined with Proposition 3.2.9, this leads to

$$|\mathcal{Q}_T - \mathcal{Q}| \lesssim \sqrt{\frac{\mu_d(T)}{T}}. \tag{3.95}$$

◊

3.A.1 Structure of the proof and auxiliary lemmas

The proof of Proposition 3.A.1 makes crucial use of the spectral gap estimate of Lemma 3.3.1, as well as of the following estimates on the massive corrector ϕ_T (cf. [210, 206]).

Lemma 3.A.2 ([210, 206])**.** *Let $d \geq 2$, let $\mathbb{P}$ ba a product measure, and let μ_d be defined in* (3.13)*. For all $T \geq 1$ and all $q < \infty$ we have*

$$\mathbb{E}\left[|\nabla\phi_T|^q\right]^{\frac{1}{q}} \lesssim_q 1, \qquad \mathbb{E}\left[|\phi_T|^q\right]^{\frac{1}{q}} \lesssim_q \mu_d(T)^{\frac{1}{2}},$$

and

$$\mathbb{E}\left[|\nabla(\phi_T - \phi)|^q\right]^{\frac{1}{q}} \lesssim_q \begin{cases} T^{-\frac{d}{4}} & : \quad 2 \leq d < 4, \\ T^{-1}\log^{\frac{1}{2}} T & : \quad d = 4, \\ T^{-1} & : \quad d > 4. \end{cases} \qquad \lozenge$$

In addition, the proof requires an L^p-regularity result for the massive operator $\frac{1}{T} - \nabla^* \cdot A\nabla$, at least in a perturbative regime. This is achieved in the form of the following weighted Meyers' estimate. In the case $T = \infty$, such an estimate was first used in the context of homogenization by Conlon and Spencer [125]. For the massive case $T < \infty$ a proof is included in Section 3.A for the reader's convenience.

Lemma 3.A.3. *Let $d \geq 1$ and let $w_\varepsilon(x) := 1 + \varepsilon|x|$. For all $|p-2| \ll 1$ and $|\alpha| \ll 1$, in the regime $\sqrt{T} \leq C_{\alpha,p}^{-1}\varepsilon^{-1}$ for some (large enough) constant $C_{\alpha,p} \simeq_{\alpha,p} 1$, for all (sufficiently fast) decaying scalar fields u_T, h and vector field f related in $\mathbb{R}^d$ by*

$$\frac{1}{T}u_T - \nabla^* \cdot A\nabla u_T = \frac{1}{T}h + \nabla^* \cdot f,$$

we have

$$\frac{1}{\sqrt{T}}\|w_\varepsilon^\alpha u_T\|_{\mathrm{L}^p(\mathbb{R}^d)} + \|w_\varepsilon^\alpha \nabla u_T\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_{\alpha,p} \frac{1}{\sqrt{T}}\|w_\varepsilon^\alpha h\|_{\mathrm{L}^p(\mathbb{R}^d)} + \|w_\varepsilon^\alpha f\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

The same result holds in the limiting case $T = \infty$, for all $\varepsilon > 0$. $\lozenge$

On top of the spectral gap estimate, we need for the proof of items (i) and (ii) to make use of the covariance inequality of Lemma 3.5.1. Finally, in order to reach the optimal convergence rate in item (ii), we make use of the following annealed massive Green's function estimates by Marahrens and Otto [313] (see also [201] for the adaptation to the massive Green's function).

Lemma 3.A.4. *Let $d \geq 2$ and let $\mathbb{P}$ be a product measure. For all $T \geq 1$ and all $y \in \mathbb{Z}^d$, there exists a function $G_T(\cdot, y)$ that is the unique decaying solution in $\mathbb{Z}^d$ of*

$$T^{-1}G_T(\cdot,y) - \nabla^* \cdot A\nabla G_T(\cdot,y) = \delta(\cdot - y),$$

and for some $C > 0$ it satisfies the following moment bound, for all $q < \infty$ and all $x, y \in \mathbb{Z}^d$,

$$\mathbb{E}\left[|\nabla\nabla G_T(x,y)|^q\right]^{\frac{1}{q}} \lesssim_q (1 + |x-y|)^{-d} e^{-\frac{1}{C\sqrt{T}}|x-y|},$$

where $\nabla\nabla$ denotes the mixed second gradient. $\lozenge$

3.A.2 Proof of Proposition 3.A.1

We split the proof into three steps, proving each item separately.

Step 1. Proof of (i).

Define $K_T : \mathbb{R}^d \to \mathbb{R}^{d\times d\times d\times d}$ as $K_T(x) := \mathrm{Cov}\,[\Xi_T(x); \Xi_T(0)]$. We claim that $K_T \in \mathrm{L}^1(\mathbb{R}^d)$ and shall define $\mathcal{Q}_T := \int_{\mathbb{R}^d} K_T$. In order to prove the desired integrability, we appeal to the covariance inequality of Lemma 3.5.1,

$$|K_T(x)| \le \sum_{b\in\mathcal{B}} \mathbb{E}\left[|\Delta_b \Xi_T(x)|^2\right]^{\frac12} \mathbb{E}\left[|\Delta_b \Xi_T(0)|^2\right]^{\frac12}. \tag{3.96}$$

We compute the vertical derivative

$$\Delta_b \Xi_{T,i} = \Delta_b A(\nabla \phi^b_{T,i} + e_i) + (A - A_{\mathrm{hom}})\nabla \Delta_b \phi_{T,i}.$$

For $b \in \mathcal{B}$, taking the vertical derivative of the massive corrector equation (3.89) in the form

$$T^{-1}\Delta_b \phi_{T,i} - \nabla^* \cdot A\nabla \Delta_b \phi_{T,i} \;=\; \nabla^* \cdot \Delta_b A(\nabla \phi^b_{T,i} + e_i), \tag{3.97}$$

writing $b = (z_b, z_b + \xi_b)$, and recalling that $|\Delta_b A(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$, the Green representation formula yields for all $x \in \mathbb{Z}^d$,

$$\nabla \Delta_b \phi_{T,i}(x) \;=\; -\nabla\nabla G_T(x, z_b)\Delta_b A(x)(\nabla \phi^b_{T,i}(z_b) + e_i).$$

Using the moment bounds of Lemmas 3.A.2 and 3.A.4, we deduce in particular

$$\mathbb{E}\left[|\Delta_b \Xi_T(x)|^2\right]^{\frac12} \;\lesssim\; (1 + |x - z_b|)^{-d}\, e^{-\frac{1}{C\sqrt{T}}|x-z_b|},$$

and (3.96) then leads to the rough bound

$$|K_T(x)| \lesssim \int_{\mathbb{R}^d} (1 + |x - z|)^{-d} e^{-\frac{1}{C\sqrt{T}}|x-z|} (1 + |z|)^{-d} e^{-\frac{1}{C\sqrt{T}}|z|} dz \;\lesssim (1 + |x|)^{-d} e^{-\frac{1}{C\sqrt{T}}|x|}, \tag{3.98}$$

hence $\int_{\mathbb{R}^d} |K_T| \lesssim \log T$, and the conclusion follows.

Step 2. Proof of (ii).

We start with the proof of (3.92). By polarization and linearity, it is enough to prove the result for $F = G$. In this case, by stationarity, the definition of $\mathcal{Q}_T$ leads to

$$\begin{aligned}
&\mathrm{Cov}\left[I^\varepsilon_{0,T}(F); I^\varepsilon_{0,T}(F)\right] - \int_{\mathbb{R}^d} F(x) : \mathcal{Q}_T\, F(x)dx \\
&= \;\varepsilon^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x) : K_T\Big(\frac{y-x}{\varepsilon}\Big)\, F(y)dxdy - \varepsilon^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x) : K_T\Big(\frac{y}{\varepsilon}\Big)\, F(x)dxdy \\
&= \;-\frac{\varepsilon^{-d}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (F(x+y) - F(x)) : K_T\Big(\frac{y}{\varepsilon}\Big)\, (F(x+y) - F(x))dxdy,
\end{aligned}$$

and hence,

$$\Big|\mathrm{Cov}\left[I^\varepsilon_{0,T}(F); I^\varepsilon_{0,T}(F)\right] - \int_{\mathbb{R}^d} F(x) : \mathcal{Q}_T\, F(x)dx\Big| \le \frac12 \|DF\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\varepsilon y|^2 |K_T(y)|dy.$$

The conclusion (3.92) then follows from (3.98) and a direct computation. We turn to the proof of (3.93). For any Lipschitz domain $U \subset \mathbb{R}^d$, computing

$$\begin{aligned}
&\mathrm{Cov}\left[I^\varepsilon_{0,T}(\mathbb{1}_U e_i \otimes e_j); I^\varepsilon_{0,T}(\mathbb{1}_U e_k \otimes e_l)\right] - |U|\mathcal{Q}_{T,ijkl} \\
&\quad= \varepsilon^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}_U(x)(\mathbb{1}_U(x+y) - \mathbb{1}_U(x))K_{T,ijkl}\Big(\frac{y}{\varepsilon}\Big)dxdy = -\int_U \int_{z:x+\varepsilon z\notin U} K_{T,ijkl}(z)dzdx,
\end{aligned}$$

the estimate (3.98) leads to

$$\Big|\operatorname{Cov}\left[I^{\varepsilon}_{0,T}(\mathbb{1}_U e_i\otimes e_j);I^{\varepsilon}_{0,T}(\mathbb{1}_U e_k\otimes e_l)\right]-|U|\mathcal{Q}_{T,ijkl}\Big| \lesssim \int_U\int_{z:x+\varepsilon z\notin U}(1+|z|)^{-d}e^{-\frac{1}{C\sqrt{T}}|z|}dzdx$$
$$\lesssim \int_U\int_{|z|>\frac{1}{\varepsilon}d(x,\partial U)}(1+|z|)^{-d}e^{-\frac{1}{C\sqrt{T}}|z|}dzdx$$
$$\lesssim |\partial U|\int_0^\infty\int_{|z|>\frac{t}{\varepsilon}}(1+|z|)^{-d}e^{-\frac{1}{C\sqrt{T}}|z|}dzdt \lesssim |\partial U|\varepsilon\sqrt{T},$$

and (3.93) follows for the choice $U=Q$ and $\varepsilon=\frac{1}{L}$.

Step 3. Proof of (iii).

We split this step into two further substeps: first we carefully decompose the vertical derivative of the difference $\Xi_T-\Xi$, and then we apply the spectral gap estimate. Let $F\in C^\infty_c(\mathbb{R}^d)^{d\times d}$ and set $F_\varepsilon:=F(\varepsilon\cdot)$.

Substep 3.1. Representation formula for the vertical derivative of the difference $\Xi_T-\Xi$: for all $e\in\mathcal{B}$,

$$\Delta_e\int_{\mathbb{R}^d}F_\varepsilon:(\Xi_T-\Xi)=\int_{\mathbb{R}^d}\big(F_{\varepsilon,ij}(\nabla\phi^*_j+e_j)+\phi^*_j(\cdot+e_k)\nabla_k F_{\varepsilon,ij}e_k\big)\cdot\Delta_b A\nabla(\phi^b_{T,i}-\phi^b_i)$$
$$-\int_{\mathbb{R}^d}\nabla s_{\varepsilon,T,i}\cdot\Delta_b A(\nabla\phi^b_{T,i}+e_i)+\int_{\mathbb{R}^d}\nabla s_{\varepsilon,i}\cdot\Delta_b A(\nabla\phi^b_i+e_i), \quad (3.99)$$

where $s_{\varepsilon,T,i}$ and $s_{\varepsilon,i}$ denote the unique decaying solutions in $\mathbb{R}^d$ of

$$\begin{aligned}\frac{1}{T}s_{\varepsilon,T,i}-\nabla^*\cdot A^*\nabla s_{\varepsilon,T,i} &= \frac{1}{T}F_{\varepsilon,ij}\phi^*_j-\nabla^*_l\big(\phi^*_j(\cdot+e_k)\nabla_k F_{\varepsilon,ij}A_{kl}+\sigma^*_{jkl}(\cdot-e_k)\nabla^*_k F_{\varepsilon,ij}\big), \quad (3.100)\\ -\nabla^*\cdot A^*\nabla s_{\varepsilon,i} &= -\nabla^*_l\big(\phi^*_j(\cdot+e_k)\nabla_k F_{\varepsilon,ij}A_{kl}+\sigma^*_{jkl}(\cdot-e_k)\nabla^*_k F_{\varepsilon,ij}\big).\end{aligned}$$

By the definitions (3.22) and (3.90) of the homogenization commutators Ξ and Ξ_T, using the definition (3.33) of the flux corrector σ_j in the form $(A^*-A^*_{\mathrm{hom}})e_j=-A^*\nabla\phi^*_j+\nabla^*\cdot\sigma^*_j$, we deduce

$$\begin{aligned}\Delta_b(\Xi_{T,ij}-\Xi_{ij}) &= e_j\cdot\Delta_b A\nabla(\phi^b_{T,i}-\phi^b_i)+e_j\cdot(A-A_{\mathrm{hom}})\nabla(\Delta_b\phi_{T,i}-\Delta_e\phi_i)\\ &= e.\Delta_b A\nabla(\phi^b_{T,i}-\phi^b_i)-\nabla\phi^*_j\cdot A\nabla(\Delta_b\phi_{T,i}-\Delta_b\phi_i)+(\nabla^*\cdot\sigma^*_j)\cdot\nabla(\Delta_b\phi_{T,i}-\Delta_b\phi_i). \quad (3.101)\end{aligned}$$

Integrating by parts, using the discrete Leibniz rule (3.49), taking the vertical derivative of the corrector equations (3.21) and (3.89) in the form

$$\frac{1}{T}\Delta_b\phi_{T,i}-\nabla^*\cdot A\nabla(\Delta_b\phi_{T,i}-\Delta_b\phi_i)=\nabla^*\cdot\Delta_b A\nabla(\phi^b_{T,i}-\phi^b_i),$$

and using the skew-symmetry (3.32) of σ^*_j, we are led to

$$\Delta_e\int_{\mathbb{R}^d}F_\varepsilon:(\Xi_T-\Xi)=\int_{\mathbb{R}^d}F_{\varepsilon,ij}(\nabla\phi^*_j+e_j)\cdot\Delta_b A\nabla(\phi^b_{T,i}-\phi^b_i)+\int_{\mathbb{R}^d}\phi^*_j(\cdot+e_k)\nabla_k F_{\varepsilon,ij}e_k\cdot\Delta_b A\nabla(\phi^b_{T,i}-\phi^b_i)$$
$$+\frac{1}{T}\int_{\mathbb{R}^d}F_{\varepsilon,ij}\phi^*_j\Delta_b\phi_{T,i}+\int_{\mathbb{R}^d}\phi^*_j(\cdot+e_k)\nabla_k F_{\varepsilon,ij}A_{kl}\nabla_l(\Delta_b\phi_{T,i}-\Delta_b\phi_i)$$
$$+\int_{\mathbb{R}^d}\sigma^*_{jkl}(\cdot-e_k)\nabla^*_k F_{\varepsilon,ij}\nabla_l(\Delta_b\phi_{T,i}-\Delta_b\phi_i).$$

Injecting the definitions (3.100) of $s_{\varepsilon,T,i}$ and $s_{\varepsilon,i}$, and using equations (3.41) and (3.97), the conclusion follows.

Substep 3.2. Application of the spectral gap and conclusion.

For $b \in \mathcal{B}$ write $b = (z_b, z_b + \xi_b)$. Applying the spectral gap estimate of Lemma 3.3.1, injecting formula (3.99), recalling that $|\Delta_b A(x)| \lesssim \mathbb{1}_{Q(z_b)}(x)$, applying the moment bounds of Lemma 3.3.3, and noting that the argument in (3.43) implies $|\nabla\phi^b(z_b) + \mathrm{Id}\,| \lesssim |\nabla\phi(z_b) + \mathrm{Id}\,|$ and $|\nabla\phi_T^b(z_b) + \mathrm{Id}\,| \lesssim |\nabla\phi_T(z_b) + \mathrm{Id}\,|$, we obtain

$$\mathrm{Var}\left[\varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} F_\varepsilon : (\Xi_T - \Xi)\right] \lesssim \varepsilon^d\, \mathbb{E}\big[|\nabla(\phi_T - \phi)|^4\big]^{\frac12} \int_{\mathbb{R}^d} \big(|F_\varepsilon|^2 + \mu_d(|\cdot|)|\nabla F_\varepsilon|^2\big) \\ + \varepsilon^d\, \mathbb{E}\left[\int_{\mathbb{R}^d} |\nabla\phi_T + \mathrm{Id}\,|^2 |\nabla s_{\varepsilon,T}|^2\right] + \varepsilon^d\, \mathbb{E}\left[\int_{\mathbb{R}^d} |\nabla\phi + \mathrm{Id}\,|^2 |\nabla s_\varepsilon|^2\right]. \quad (3.102)$$

Smuggling in a power $\alpha\frac{p-1}{p}$ of the weight $w_\varepsilon(z) := 1+\varepsilon|z|$, applying Hölder's inequality with exponent p, and using the moment bounds of Lemma 3.A.2, we obtain for all $p > 1$ and $\alpha > d$,

$$\begin{aligned} \varepsilon^d\, \mathbb{E}\left[\int_{\mathbb{R}^d} |\nabla\phi_T + \mathrm{Id}\,|^2 |\nabla s_{\varepsilon,T}|^2\right] \quad &\le \quad \varepsilon^d\, \mathbb{E}\left[\int_{\mathbb{R}^d} w_\varepsilon^{\alpha(p-1)} |\nabla s_{\varepsilon,T}|^{2p}\right]^{\frac1p} \mathbb{E}\left[\int_{\mathbb{R}^d} w_\varepsilon^{-\alpha} |\nabla\phi_T + \mathrm{Id}\,|^{\frac{2p}{p-1}}\right]^{\frac{p-1}{p}} \\ &\lesssim_{\alpha,p} \quad \varepsilon^{\frac{d}{p}}\, \mathbb{E}\left[\int_{\mathbb{R}^d} w_\varepsilon^{\alpha(p-1)} |\nabla s_{\varepsilon,T}|^{2p}\right]^{\frac1p}. \end{aligned}$$

By the weighted Meyers' estimate of Lemma 3.A.3 applied to equation (3.100) and by the moment bounds of Lemma 3.3.3, this yields for all $0 < p-1 \ll 1$ and all $0 < \alpha - d \ll 1$,

$$\begin{aligned} &\varepsilon^d\, \mathbb{E}\left[\int_{\mathbb{R}^d} |\nabla\phi_T + \mathrm{Id}\,|^2 |\nabla s_{\varepsilon,T}|^2\right] \\ &\lesssim_p \quad T^{-1}\varepsilon^{\frac{d}{p}} \Big(\int_{\mathbb{R}^d} w_\varepsilon^{\alpha(p-1)} \mu_d(|\cdot|)^p |F_\varepsilon|^{2p}\Big)^{\frac1p} + \varepsilon^{\frac{d}{p}} \Big(\int_{\mathbb{R}^d} w_\varepsilon^{2d(p-1)} \mu_d(|\cdot|)^p |\nabla F_\varepsilon|^{2p}\Big)^{\frac1p} \\ &\lesssim_p \quad T^{-1}\mu_d(\tfrac{1}{\varepsilon}) \|w_1^{d\frac{p-1}{p}} \mu_d(|\cdot|)^{\frac12} F\|^2_{\mathrm{L}^{2p}(\mathbb{R}^d)} + \varepsilon^2 \mu_d(\tfrac{1}{\varepsilon}) \|w_1^{\alpha\frac{p-1}{2p}} \mu_d(|\cdot|)^{\frac12} DF\|^2_{\mathrm{L}^{2p}(\mathbb{R}^d)}. \end{aligned}$$

The last right-hand side term in (3.102) can be estimated similarly, applying Lemma 3.A.3 in the limiting case $T = \infty$ (or equivalently, repeating the argument in Step 2 of the proof of Proposition 3.2.1 using the large-scale weighted Calderón-Zygmund theory for $-\nabla^* \cdot A\nabla$). Combining this with (3.102) and with Lemma 3.A.2, the conclusion (3.94) follows.

3.A.3 Proof of Lemma 3.A.3

For simplicity we only treat the continuum case. (The adaptation to the present discrete setting requires to argue that the discrete Fourier multipliers are controlled by their continuum counterparts introduced below, see e.g. [206, Section 7.4] for similar arguments.) For that purpose, let us consider the solution u_T to

$$\frac{1}{T}u_T - \triangle u_T = \frac{1}{T}h + D\cdot f,$$

where $\triangle = D\cdot D$ denotes the continuum Laplacian. First, the energy estimate takes the form

$$\Big\|\frac{1}{\sqrt{T}}u_T\Big\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|Du_T\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \Big\|\frac{1}{\sqrt{T}}h\Big\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}. \quad (3.103)$$

Second, we show that the following extended weighted Calderón-Zygmund estimate holds: for all $\alpha \in \mathbb{R}$ and $1 < r < \infty$, in the regime $\varepsilon\sqrt{T} \le C_{\alpha,r}^{-1}$ for some (large enough) constant $C_{\alpha,r} \simeq_{\alpha,r} 1$, we have

$$\frac{1}{\sqrt{T}}\|w_\varepsilon^{-\alpha}u_T\|_{\mathrm{L}^r(\mathbb{R}^d)} + \|w_\varepsilon^{-\alpha}Du_T\|_{\mathrm{L}^r(\mathbb{R}^d)} \lesssim_{\alpha,r} \frac{1}{\sqrt{T}}\|w_\varepsilon^{-\alpha}h\|_{\mathrm{L}^r(\mathbb{R}^d)} + \|w_\varepsilon^{-\alpha}f\|_{\mathrm{L}^r(\mathbb{R}^d)}. \quad (3.104)$$

Indeed, the Fourier symbol relating $(\frac{1}{\sqrt{T}}u_T, Du_T)$ to $(\frac{1}{\sqrt{T}}h, f)$ is given by

$$M_T(k) := \left(\begin{array}{cc} \frac{1}{T}\frac{1}{\frac{1}{T}+|k|^2} & \frac{1}{\sqrt{T}}\frac{k'}{\frac{1}{T}+|k|^2} \\ \frac{1}{\sqrt{T}}\frac{k}{\frac{1}{T}+|k|^2} & \frac{k\otimes k}{\frac{1}{T}+|k|^2} \end{array} \right),$$

or equivalently, $M_T(k) = \tilde{M}(\sqrt{T}k)$, in terms of

$$\tilde{M}(k) = \left(\begin{array}{cc} \frac{1}{1+|k|^2} & \frac{k'}{1+|k|^2} \\ \frac{k}{1+|k|^2} & \frac{k\otimes k}{1+|k|^2} \end{array} \right).$$

We note that M_T is well-behaved: we have $|D^m\tilde{M}(k)| \lesssim_m (1+|k|)^{-m}$, hence $|D^m M_T(k)| \lesssim_m |k|^{-m}$ for all $m \geq 0$. An extended Calderón-Zygmund estimate then follows from Mikhlin's multiplier theorem: for all $1 < r < \infty$,

$$\Big\|\frac{1}{\sqrt{T}}u_T\Big\|_{\mathrm{L}^r(\mathbb{R}^d)} + \|Du_T\|_{\mathrm{L}^r(\mathbb{R}^d)} \lesssim_r \Big\|\frac{1}{\sqrt{T}}h\Big\|_{\mathrm{L}^r(\mathbb{R}^d)} + \|f\|_{\mathrm{L}^r(\mathbb{R}^d)}. \tag{3.105}$$

It remains to note that this estimate allows to deal directly with the Leibniz terms when smuggling in the weight w_ε^α. More precisely, we define $\tilde{w}_\varepsilon(x) := (1+(\varepsilon|x|)^2)^{\frac{1}{2}}$, and we write the equation for u_T as

$$\frac{1}{T}\tilde{w}_\varepsilon^\alpha u_T - \triangle(\tilde{w}_\varepsilon^\alpha u_T) = \frac{1}{T}\tilde{w}_\varepsilon^\alpha h + D\cdot(\tilde{w}_\varepsilon^\alpha f) - (f+2Du_T)\cdot D\tilde{w}_\varepsilon^\alpha - u_T\triangle\tilde{w}_\varepsilon^\alpha.$$

Applying (3.105) to this equation, we obtain

$$\begin{aligned}\frac{1}{\sqrt{T}}\|\tilde{w}_\varepsilon^\alpha u_T\|_{\mathrm{L}^r(\mathbb{R}^d)} + \|\tilde{w}_\varepsilon^\alpha Du_T\|_{\mathrm{L}^r(\mathbb{R}^d)} \lesssim_r \frac{1}{\sqrt{T}}\|\tilde{w}_\varepsilon^\alpha h\|_{\mathrm{L}^r(\mathbb{R}^d)} + \|\tilde{w}_\varepsilon^\alpha f\|_{\mathrm{L}^r(\mathbb{R}^d)} \\ + \|u_T D\tilde{w}_\varepsilon^\alpha\|_{\mathrm{L}^r(\mathbb{R}^d)} + \sqrt{T}\||D\tilde{w}_\varepsilon^\alpha|(f+2Du_T)\|_{\mathrm{L}^r(\mathbb{R}^d)} + \sqrt{T}\|u_T\triangle\tilde{w}_\varepsilon^\alpha\|_{\mathrm{L}^r(\mathbb{R}^d)},\end{aligned}$$

and, for all $\alpha \in \mathbb{R}$, the bounds $|D\tilde{w}_\varepsilon^\alpha| \lesssim_\alpha \varepsilon\tilde{w}_\varepsilon^\alpha$ and $|\triangle\tilde{w}_\varepsilon^\alpha| \lesssim_\alpha \varepsilon^2\tilde{w}_\varepsilon^\alpha$ then lead to

$$\begin{aligned}\frac{1}{\sqrt{T}}\|\tilde{w}_\varepsilon^\alpha u_T\|_{\mathrm{L}^r(\mathbb{R}^d)} + \|\tilde{w}_\varepsilon^\alpha Du_T\|_{\mathrm{L}^r(\mathbb{R}^d)} \lesssim_{\alpha,r} \frac{1}{\sqrt{T}}\|\tilde{w}_\varepsilon^\alpha h\|_{\mathrm{L}^r(\mathbb{R}^d)} + (1+\varepsilon\sqrt{T})\|\tilde{w}_\varepsilon^\alpha f\|_{\mathrm{L}^r(\mathbb{R}^d)} \\ + \varepsilon\sqrt{T}\|\tilde{w}_\varepsilon^\alpha Du_T\|_{\mathrm{L}^r(\mathbb{R}^d)} + \frac{\varepsilon\sqrt{T}+(\varepsilon\sqrt{T})^2}{\sqrt{T}}\|\tilde{w}_\varepsilon^\alpha u_T\|_{\mathrm{L}^r(\mathbb{R}^d)}.\end{aligned}$$

Noting that $\tilde{w}_\varepsilon \simeq w_\varepsilon$, the claim (3.104) follows.

Meyers' perturbative argument [322] requires that the constant in the extended weighted Calderón-Zygmund estimate (3.104) converges to 1 as $r \to 2$ and $\alpha \to 0$. The energy estimate (3.103) ensures that this constant equals 1 for $r = 2$ and $\alpha = 0$. The (upper semi-)continuity in r for fixed α follows from complex interpolation (see e.g. [52, Theorem 4.4.1]) of Lebesgue spaces (see e.g. [52, Theorem 5.1.1]). The continuity in α for fixed r follows from the real interpolation theorem of Stein and Weiss (see e.g. [52, Theorem 5.4.1]). Since both moduli of continuity are (locally) uniform (in fact, uniformly Lipschitz in the logarithmic scale), one so obtains joint continuity in r and α. This allows to then carry out Meyers' perturbative argument [322] to pass from the constant to the variable-coefficient case, and the conclusion follows.

Chapter 4

Weighted functional inequalities for correlated random fields

Consider an ergodic stationary random field A on the ambient space $\mathbb{R}^d$. As experienced in the previous chapter, functional inequalities in the probability space (like spectral gaps, covariance inequalities, logarithmic Sobolev inequalities, or second-order Poincaré inequalities à la Chatterjee) provide a sensitivity calculus with respect to A that is a very convenient tool to establish quantitative error estimates e.g. in the field of stochastic homogenization. In addition, these inequalities (in particular spectral gaps and logarithmic Sobolev inequalities) are also well-known in mathematical physics as powerful tools to prove nonlinear concentration of measure properties for nonlinear functions $X(A)$ in terms of assumptions on A.

These inequalities are however very stringent: they require A to have an integrable covariance function and they are only known to hold for a restricted class of laws (like product measures, Gaussian measures, or more general Gibbs measures with nicely behaved Hamiltonians). In the present chapter, we introduce new weighted versions of these inequalities that relax the integrability condition for the covariance function and broaden the class of admissible laws, while still ensuring strong concentration properties.

We then develop a constructive approach to produce random fields that satisfy such weighted functional inequalities. The construction is based on product structures in higher-dimensional spaces and relies on devising approximate chain rules for nonlinear and random changes of variables for random fields. This approach allows us to treat all the examples of heterogeneous materials encountered in the applied sciences [413], covering in particular Gaussian fields with non-necessarily integrable covariance function, Poisson random inclusions with (unbounded) random radii, random parking and Matérn-type processes, as well as Poisson random tessellations (Voronoi or Delaunay). These weighted functional inequalities, which we primarily develop here in view of their application to quantitative stochastic homogenization, are of independent interest.

As an application, we prove specific concentration results for averages of approximately local functions of the field A, which constitutes the main stochastic ingredient to the quenched large-scale regularity theory for random elliptic operators by Armstrong, Mourrat, and Smart [36, 34] and by Gloria, Neukamm, and Otto [204]. In addition, applied to random sequential adsorption models in stochastic geometry, weighted second-order Poincaré inequalities allow us to complete and improve previous results by Penrose and Yukich [360, 391] on the jamming limit, and to propose and fully analyze a more efficient algorithm to approximate the latter.

This chapter corresponds to the three articles [162, 163, 164] jointly written with Antoine Gloria.

Contents

4.1 Introduction

4.1.1 General overview

We consider a stationary random field A on the ambient space $\mathbb{R}^d$. Functional inequalities like spectral gaps, covariance, or logarithmic Sobolev inequalities are powerful tools to prove nonlinear concentration of measure properties and CLT scalings for nonlinear functions $X(A)$ of the random field. Besides their well-known applications in mathematical physics (e.g. for the study of interacting particle systems like the Ising model or for interface models), these inequalities provide a sensitivity calculus that is a very convenient tool and was recently used to establish quantitative stochastic homogenization results, starting with the inspiring unpublished work by Naddaf and Spencer [334], successfully followed by [209, 210, 206, 212, 205, 204, 203] and by our contribution in Chapter 3.

These functional inequalities have nevertheless two main limitations. On the one hand, whereas only few examples are known to satisfy them (besides product measures, Gaussian measures with integrable covariance function, and more general Gibbs measures with nicely behaved Hamiltonians), these inequalities are not robust with respect to various simple constructions: for instance, a Poisson point process satisfies a spectral gap, but the random field corresponding to the Voronoi tessellation of a Poisson point process does not. On the other hand, these functional inequalities require random fields to have an integrable covariance, which prevents one from considering fields with heavier tails. The aim of this chapter is to introduce a weaker notion of *weighted functional inequalities*, which is at the same time more robust to perturbations and less restrictive in terms of integrability of the covariance function, while still ensuring strong concentration properties. The main motivation is that most arguments making use of standard functional inequalities could be adapted to the very general setting of weighted functional inequalities that we develop here, under reasonable decay assumptions on the weight (see e.g. [203] in the context of stochastic homogenization).

In many fields of mathematics, complex objects in a low-dimensional space can be described as the projection of a simpler object that lives in a higher-dimensional space. A prototypical example is given by quasi-periodic structures. Conversely, suitable projections can be a powerful way to generate many (possibly complex) lower-dimensional objects from simpler higher-dimensional objects while preserving some essential properties, which is a useful point of view for modeling. For quasi-periodic functions, the simple high-dimensional objects are periodic functions (on a high-dimensional torus), the projection corresponds to the composition with a winding matrix, and the preserved essential property is some quantitative averaging property. In the present chapter, we apply this approach to functional inequalities.

Consider a random field $A = \Phi(A_0)$ on $\mathbb{R}^d$ obtained as the image by some "projection" Φ of some higher-dimensional random field A_0 on $\mathbb{R}^d \times \mathbb{R}^l$. A first natural question is to understand in what sense a (standard) functional inequality satisfied by A_0 can be transferred to A. A possible answer is given by the class of *weighted functional inequalities* that we introduce in Section 4.1.2 below. These functional inequalities are to be seen as a fine quantification of ergodicity, and indeed we quickly establish in Section 4.2 the relation between the weight and the decay of correlations as well as the relation to standard notions of mixing.

A second natural question is to understand to what extent such weighted functional inequalities ensure concentration properties. This is addressed in Section 4.3, and our analysis shows in particular that the concentration properties implied by weighted functional inequalities are in general stronger than those implied by the corresponding α-mixing (cf. applications in Section 4.7.2).

A third natural question is whether such inequalities are indeed more robust and less restrictive than standard functional inequalities. We answer this question in Section 4.4 by developing an abstract yet constructive approach to weighted inequalities, which amounts to making suitable assumptions on the "projection operator" Φ. In Section 4.5, we make use of this constructive approach to prove the validity of weighted functional inequalities for various examples of random fields considered in the literature (and cover in particular the above mentioned Voronoi tessellation of a Poisson point process). More precisely, our results allow to establish weighted functional inequalities for the following three classes of random fields,

(I) *Gaussian-like fields*: A is (possibly the image by a Lipschitz function of) the convolution of some white noise with some kernel, which leads to Gaussian fields with arbitrary covariance function.

(II) *Independent coloring of random geometric patterns*: A is characterized by a random geometric pattern completed by an independent product structure. The random geometric pattern is typically constructed starting from a point process (e.g. Poisson, random parking, or Matérn-type processes) by considering inclusions centered at the points, or (Voronoi or Delaunay) tessellations. The associated product structure then determines the values of A on the cells of the random pattern, or even completes the description of the random pattern (e.g. conferring random sizes and shapes to the inclusions). This leads to possibly long-range correlations of the geometric pattern.

(III) *Dependent coloring of random geometric patterns*: This corresponds to (II) for a coloring that does not come from a product structure but from a field that is itself correlated (e.g. of the class (I)). This leads to possibly long-range correlations of the colors of the inclusions (in the sense of e.g. value of A, size, or orientation of the inclusions), on top of the correlations of the geometric pattern.

The above three classes of random fields encompass all the examples considered in [413], a reference textbook on random heterogeneous structures for materials science, which brings the use of functional inequalities (in their weighted versions) in stochastic homogenization to the state-of-the-art of materials science.

In Section 4.6 we turn to Chatterjee's version of Stein's method in the form of second-order Poincaré inequalities [112, 113]: while first-order functional inequalities (like spectral gap or logarithmic Sobolev inequality) quantify the distance to constants for nonlinear functions $X(A)$ in terms of their local dependence on the random field A, the so-called second-order inequalities quantify their distance to normality. In the study of fluctuations in stochastic homogenization, the first use of second-order Poincaré inequalities is due to Nolen [346, 347], successfully followed by [207, 225, 328] and by our contribution in Chapter 3. Like first-order inequalities, second-order Poincaré inequalities are very restrictive and are only known to hold for product measures [112] and for Gaussian measures with integrable covariance function [113, 349]. Adapting our constructive approach of Section 4.4, we go beyond these examples and similarly establish the validity of suitable weighted versions of second-order Poincaré inequalities for various prototypical random fields with strong correlations.

In Section 4.7 we appeal to these weighted functional inequalities in order to study the simplest random variables possible, that is, (linear) spatial averages of (a possibly nonlinear yet approximately local transformation of) the random field A itself. Although the point of first- and second-order functional inequalities is to address concentration and approximate normality properties for general nonlinear functions of correlated random fields, this application to linear random variables is non-trivial, and is motivated by two different applications: quantitative stochastic homogenization, and fluctuations in stochastic geometry.

First, in the field of quantitative stochastic homogenization of random elliptic operators in divergence form, various quantities of interest are known to behave essentially like spatial averages of (an approximately local function of) the random field. In particular, the concentration properties of such spatial averages happen to be precisely the stochastic ingredient needed to establish sharp integrability estimates on the validity of the quenched large-scale regularity theory for random elliptic systems in divergence form as developed by Armstrong, Mourrat, and Smart [36, 34] and by Gloria, Neukamm, and Otto [204]. The added value of our results in this chapter is to emphasize that concentration properties implied by weighted functional inequalities are in general stronger than those implied by the corresponding α-mixing.

Second, regarding fluctuations in stochastic geometry, we are more precisely interested in random sequential adsorption (RSA) models and in fluctuations of the jamming limit. In that context, Stein's method was first used in combination with stabilization properties by Penrose and Yukich [360], and followed by [391, 286]. In order to analyze RSA processes, Penrose and Yukich [359] introduced a crucial notion of stabilization radius having its origins in the works of Lee [294, 295] (which is also our main inspiration for the constructive approach to weighted functional inequalities that we develop in Section 4.4), and this paved the way to a series of strong results on the jamming limit [357, 359, 358, 360, 391, 286]. Based on weighted first- and second-order functional inequalities, we revisit and complete this series of articles.

4.1.2 Weighted first-order functional inequalities

Let $A : \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a jointly measurable random field on $\mathbb{R}^d$, constructed on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A spectral gap in probability for A is a functional inequality which allows one to control the variance of any function $X(A)$ in terms of its local dependence on A, that is, in terms of some "derivative" of $X(A)$ with respect to local restrictions of A. In this section, we briefly recall the precise definitions of standard first-order functional inequalities and introduce their weighted versions. (The formulation of second-order functional inequalities is less canonical and is postponed to Section 4.6.)

A map $\tilde{\partial} : \mathcal{B}(\mathbb{R}^d) \times \mathrm{Mes}(\Omega; \mathbb{R}) \to \mathrm{Mes}(\Omega; [0, \infty])$ is called a *(wide-sense) derivative with respect to A* if, for all $\sigma(A)$-measurable random variables $X(A), Y(A)$, all $\lambda, \mu \in \mathbb{R}$, and all Borel subsets $S \subset \mathbb{R}^d$,

(i) the random variable $\tilde{\partial}_{A,S} X(A)$ is $\sigma(A)$-measurable, and it vanishes a.s. whenever $X(A)$ is $\sigma(A|_{\mathbb{R}^d \setminus S})$-measurable;

(ii) we have
$$\big|\tilde{\partial}_{A,S}(\lambda X(A) + \mu Y(A))\big| \le |\lambda|\, \tilde{\partial}_{A,S} X(A) + |\mu|\, \tilde{\partial}_{A,S} Y(A);$$

(iii) for all $R > 0$ the maps $\mathbb{R}^d \times \Omega \to [0, \infty] : (x, \omega) \mapsto \big(\tilde{\partial}_{A,B_R(x)} X(A)\big)(\omega)$ and $\mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to [0, \infty] : (r, x, \omega) \mapsto \big(\tilde{\partial}_{A,B_r(x)} X(A)\big)(\omega)$ are measurable.

We then call $\tilde{\partial}_{A,S} X(A)$ a *(wide-sense) derivative of $X(A)$ with respect to A on S*, which we think of as a quantification of the functional dependence of $X(A)$ with respect to the restriction $A|_S$ of A on S. Given such a (wide-sense) derivative $\tilde{\partial}$ (see below for typical choices), we recall the definition of the following standard functional inequalities (cf. Lemma 3.3.1 in the i.i.d. discrete setting).

Definition 4.1.1. We say that A satisfies the *(standard) spectral gap* ($\tilde{\partial}$-SG) *with radius $R > 0$ and constant $C < \infty$* if for all $\sigma(A)$-measurable random variable $X(A)$ we have

$$\mathrm{Var}\,[X(A)] \le C \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\tilde{\partial}_{A,B_R(x)} X(A)\Big)^2\right] dx; \tag{4.1}$$

it satisfies the *(standard) covariance inequality* ($\tilde{\partial}$-CI) *with radius $R > 0$ and constant $C < \infty$* if for all $\sigma(A)$-measurable random variables $X(A)$ and $Y(A)$ we have

$$\mathrm{Cov}\,[X(A); Y(A)] \le C \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\tilde{\partial}_{A,B_R(x)} X(A)\Big)^2\right]^{\frac12} \mathbb{E}\left[\Big(\tilde{\partial}_{A,B_R(x)} Y(A)\Big)^2\right]^{\frac12} dx; \tag{4.2}$$

it satisfies the *(standard) logarithmic Sobolev inequality* ($\tilde\partial$-LSI) *with radius* $R>0$ *and constant* $C<\infty$ if for all $\sigma(A)$-measurable random variable $Z(A)$ we have

$$\begin{aligned}\mathrm{Ent}\big[Z(A)^2\big] := \mathbb{E}\left[Z(A)^2\log Z(A)^2\right] - \mathbb{E}\left[Z(A)^2\right]\log\mathbb{E}\left[Z(A)^2\right]\\ \le C\int_{\mathbb{R}^d}\mathbb{E}\left[\Big(\tilde\partial_{A,B_R(x)}Z(A)\Big)^2\right]dx.\end{aligned}\quad (4.3)$$

◊

Recall that ($\tilde\partial$-CI) and ($\tilde\partial$-LSI) both imply ($\tilde\partial$-SG). The spectral gap (4.1) indeed follows from the covariance inequality (4.2) for the choice $Y=X$, while it follows from the logarithmic Sobolev inequality (4.3) for the choice $Z=1+\varepsilon X$ in the limit $\varepsilon\downarrow 0$.

In the continuum setting that we consider in this chapter, there is no canonical choice of a derivative with respect to the field A, and we describe below three such possible notions. We start with the derivative most commonly used in the literature (see e.g. [293]).

— As in the discrete setting, the so-called *Glauber derivative* ∂^{G} is defined as follows, letting A' denote an i.i.d. copy of A, and denoting by $\mathbb{E}'[\cdot]$ the expectation with respect to A' only,

$$\partial^{\mathrm{G}}_{A,S}X(A) := \mathbb{E}'\big[\big(X(A)-X(A')\big)^2 \,\big\|\, A'|_{\mathbb{R}^d\setminus S} = A|_{\mathbb{R}^d\setminus S}\big]^{\frac12}, \quad (4.4)$$

or equivalently, expanding the square,

$$\partial^{\mathrm{G}}_{A,S}X(A) = \Big(X(A)^2 - 2X(A)\,\mathbb{E}\big[X(A)\,\|\,A|_{\mathbb{R}^d\setminus S}\big] + \mathbb{E}\big[X(A)^2\,\|\,A|_{\mathbb{R}^d\setminus S}\big]\Big)^{\frac12}.$$

— The *oscillation* ∂^{osc}, as used for instance in [211, 212], is formally defined by

$$\begin{aligned}\partial^{\mathrm{osc}}_{A,S}\,X(A) \quad &:= \quad \sup_{A,S}\mathrm{ess}\,X(A) - \inf_{A,S}\mathrm{ess}\,X(A)\\ \text{“=”} \quad &\sup\mathrm{ess}\Big\{X(\tilde A):\tilde A\in\mathrm{Mes}(\mathbb{R}^d;\mathbb{R}),\ \tilde A|_{\mathbb{R}^d\setminus S}=A|_{\mathbb{R}^d\setminus S}\Big\}\\ &-\inf\mathrm{ess}\Big\{X(\tilde A):\tilde A\in\mathrm{Mes}(\mathbb{R}^d;\mathbb{R}),\ \tilde A|_{\mathbb{R}^d\setminus S}=A|_{\mathbb{R}^d\setminus S}\Big\},\end{aligned}\quad (4.5)$$

where the essential supremum and infimum are taken with respect to the measure induced by the field A on the space $\mathrm{Mes}(\mathbb{R}^d;\mathbb{R})$ (endowed with the cylindrical σ-algebra). This definition (4.5) of $\partial^{\mathrm{osc}}_{A,S}X(A)$ is not measurable in general, and we rather define

$$\partial^{\mathrm{osc}}_{A,S}\,X(A) := \mathcal{M}[X\|A|_{\mathbb{R}^d\setminus S}] + \mathcal{M}[-X\|A|_{\mathbb{R}^d\setminus S}]$$

in terms of the conditional essential supremum $\mathcal{M}[\cdot\|A_{\mathbb{R}^d\setminus S}]$ given $\sigma(A|_{\mathbb{R}^d\setminus S})$, as introduced in [47] (using a Radon-Nikodym theorem in L^∞ due to [46]). Alternatively, we may simply define $\partial^{\mathrm{osc}}_{A,S}X(A)$ as the measurable envelope of (4.5) (as e.g. in [211, 212]).

— The (integrated) *functional* (or Malliavin-type) *derivative* ∂^{fct}, as used in the first version of [204] and in [181], is the closest generalization of the usual partial derivatives often used in the discrete setting. Let us denote by $M\subset\mathrm{L}^\infty(\mathbb{R}^d)$ some open set such that the random field A takes its values in M. Given a $\sigma(A)$-measurable random variable $X(A)$, and given an extension $\tilde X: M\to\mathbb{R}$, its Fréchet derivative $\partial\tilde X(A)/\partial A\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is defined for any compactly supported perturbation $\delta A\in\mathrm{L}^\infty(\mathbb{R}^d)$ by

$$\lim_{t\to0}\frac{\tilde X(A+t\delta A)-\tilde X(A)}{t} = \int_{\mathbb{R}^d}\delta A(x)\frac{\partial\tilde X(A)}{\partial A}(x)\,dx,$$

if the limit exists. Since we are interested in the local averages of this derivative, we rather define for all bounded Borel subset $S \subset \mathbb{R}^d$,

$$\partial^{\mathrm{fct}}_{A,S} X(A) = \int_S \Big| \frac{\partial \tilde{X}(A)}{\partial A}(x) \Big| dx.$$

This derivative is additive with respect to the set S: for all disjoint Borel subsets $S_1, S_2 \subset \mathbb{R}^d$ we have $\partial^{\mathrm{fct}}_{A,S_1 \cup S_2} X(A) = \partial^{\mathrm{fct}}_{A,S_1} X(A) + \partial^{\mathrm{fct}}_{A,S_2} X(A)$.

It is clear by definition that the oscillation ∂^{osc} dominates the Glauber derivative ∂^{G}. Henceforth we use the notation $\tilde{\partial}$ for any of the above-defined (wide-sense) derivatives with respect to the random field A.

Let us now briefly discuss the applicability of these standard functional inequalities. On the one hand, classical arguments yield the following sufficient criterion. A standard proof is included for completeness in Appendix 4.A and will be referred to at several places in the sequel of this chapter. (Note that the logarithmic Sobolev inequality (LSI) is only established with the oscillation ∂^{osc}, while the version with the Glauber derivative ∂^{G} is well-known to be much more restrictive, crucially depending on the law of the underlying product structure.)

Proposition 4.1.2. *Let A_0 be a random field on $\mathbb{R}^d$ with values in some measurable space such that restrictions $A_0|_S$ and $A_0|_T$ are independent for all disjoint Borel subsets $S, T \subset \mathbb{R}^d$. Let A be a random field on $\mathbb{R}^d$ that is an R-local transformation of A_0, in the sense that for all $S \subset \mathbb{R}^d$ the restriction $A|_S$ is $\sigma(A_0|_{S+B_R})$-measurable. Then the field A satisfies* (∂^{G}-SG), (∂^{G}-CI), *and* (∂^{osc}-LSI) *with radius $R + \varepsilon$ for all $\varepsilon > 0$.* ♢

Note that any field satisfying the assumption in this criterion has finite range of dependence. Conversely any field that satisfies (CI) has necessarily finite range of dependence (cf. Proposition 4.2.1(iii) below). One does not expect, however, finite range of dependence to be a sufficient condition for the validity of (SG) in general (compare indeed with the constructions in [91, 76]).

On the other hand, in the Gaussian setting, a complete characterization of standard functional inequalities is available: if A is a jointly measurable stationary Gaussian random field on $\mathbb{R}^d$ with covariance function $\mathcal{C}(x) := \mathrm{Cov}\left[A(x); A(0)\right]$, then ($\partial^{\mathrm{fct}}$-SG) and ($\partial^{\mathrm{fct}}$-LSI) are essentially equivalent to the integrability of the covariance function, while (∂^{fct}-CI) is equivalent to the finiteness of the range of dependence (cf. Proposition 4.2.1 and Corollary 4.5.1(i) below).

As these examples show (see also the necessary conditions in Proposition 4.2.1 below), the standard functional inequalities (SG), (LSI), and (CI) are very restrictive in the sense that they can only hold for fields with sufficiently fast decaying correlations, which excludes many examples of practical interest (typically to stochastic homogenization, cf. [413]). One possible explanation why these standard functional inequalities are particularly restrictive is that the right-hand sides in (4.1), (4.2), and (4.3) only take into account functional dependences at distance at most R. The definition below relaxes these standard functional inequalities by explicitly taking into account dependences upon derivatives with respect to A restricted on arbitrarily large sets, according to some given weight.

Definition 4.1.3. Given an integrable function $\pi : \mathbb{R}_+ \to \mathbb{R}_+$, we say that A satisfies the *weighted spectral gap* ($\tilde{\partial}$-WSG) *with weight* π if for all $\sigma(A)$-measurable random variable $X(A)$ we have

$$\mathrm{Var}\left[X(A)\right] \le \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde{\partial}_{A,B_{\ell+1}(x)} X(A)\Big)^2 dx\, (\ell+1)^{-d} \pi(\ell)\, d\ell\right]; \tag{4.6}$$

it satisfies the *weighted covariance inequality* ($\tilde{\partial}$-WCI) *with weight* π if for all $\sigma(A)$-measurable random variables $X(A)$ and $Y(A)$ we have

$$\mathrm{Cov}\left[X(A); Y(A)\right] \le \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\tilde{\partial}_{A,B_{\ell+1}(x)} X(A)\Big)^2\right]^{\frac12} \mathbb{E}\left[\Big(\tilde{\partial}_{A,B_{\ell+1}(x)} Y(A)\Big)^2\right]^{\frac12} dx\, (\ell+1)^{-d} \pi(\ell)\, d\ell;$$

it satisfies the *weighted logarithmic Sobolev inequality* ($\tilde\partial$-WLSI) *with weight* π if for all $\sigma(A)$-measurable random variable $Z(A)$ we have

$$\mathrm{Ent}\big[Z(A)^2\big] \le \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde\partial_{A,B_{\ell+1}(x)} Z(A)\Big)^2 dx\, (\ell+1)^{-d} \pi(\ell)\, d\ell\right]. \qquad \Diamond$$

Note that, as for standard functional inequalities, ($\tilde\partial$-WCI) and ($\tilde\partial$-WLSI) both imply ($\tilde\partial$-WSG). The standard functional inequalities of Definition 4.1.1 are recovered by taking a compactly supported weight π.

Although the Glauber derivative ∂^{G} and the functional derivative ∂^{fct} are particularly convenient measures of sensitivity of a random variable $X(A)$ with respect to local restrictions of A, they are essentially only adapted to product structures and to Gaussian-like random fields, respectively. On the other hand, the oscillation ∂^{osc} is adapted to a much larger variety of fields (cf. Section 4.4.2), but it involves taking (essential) suprema, which might be difficult to control in some applications.

In the course of this chapter, we consider various classes of random fields on $\mathbb{R}^d$ that can be constructed as (possibly random) projections of random fields having a product structure in a higher-dimensional space $\mathbb{R}^d \times \mathbb{R}^l$. Such projections naturally allow one to "deform" the underlying Glauber derivative in a way that cannot be strictly speaking written as a Glauber derivative, but which shares important properties (and in particular avoids taking suprema). The following definition (which can be skipped at the first reading) gives such a proxy for the Glauber derivative, which can typically be used in functional inequalities with a loss of integrability.

Definition 4.1.4. Given $l \ge 0$, let $\mathcal{X}$ be some random field on $\mathbb{R}^d \times \mathbb{R}^l$ with values in some measure space, and assume that the random field A under consideration is $\sigma(\mathcal{X})$-measurable, $A = A(\mathcal{X})$. Choose $\mathcal{X}'$ an i.i.d. copy of the field $\mathcal{X}$, and for all x, t let the perturbed field $\mathcal{X}^{x,t}$ be defined by $\mathcal{X}^{x,t}|_{(\mathbb{R}^d\times\mathbb{R}^l)\setminus(Q^d(x)\times Q^l(t))} = \mathcal{X}|_{(\mathbb{R}^d\times\mathbb{R}^l)\setminus(Q^d(x)\times Q^l(t))}$ and $\mathcal{X}^{x,t}|_{Q^d(x)\times Q^l(t)} = \mathcal{X}'|_{Q^d(x)\times Q^l(t)}$. We use the short-hand notation

$$\partial^{\mathrm{dis}}_{\ell,x,t} X(A) := \big(X(A) - X(A(\mathcal{X}^{x,t}))\big)\mathbb{1}_{A|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)} = A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}}, \tag{4.7}$$

which we abusively call a *discrete derivative*. Given a family $(\pi_\lambda)_\lambda$ of integrable functions $\pi_\lambda : \mathbb{R}^l \times \mathbb{R}_+ \to \mathbb{R}_+$, we say that A satisfies the spectral gap with loss (∂^{dis}-WSG') with weights $(\pi_\lambda)_\lambda$ if for all $\sigma(A)$-measurable random variables $X(A)$ and all $\lambda \in (0,1)$ we have

$$\mathrm{Var}\left[X(A)\right] \le \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^l} \mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t} X(A)\Big)^{\frac{2}{1-\lambda}}\right]^{1-\lambda} \pi_\lambda(t,\ell)\, dt\, dx\, d\ell.$$

Likewise, we define the corresponding weighted covariance inequality (∂^{dis}-WCI') and weighted logarithmic Sobolev inequality (∂^{dis}-WLSI'). $\Diamond$

4.2 Link to mixing properties

4.2.1 Decay of correlations

In this subsection we quantify the relation between the decay of correlations of the random field and the decay of the weight π in the corresponding weighted inequalities, extending the well-known result that the standard spectral gap and covariance inequality imply the integrability of the covariance and the finiteness of the range of dependence, respectively. Note in particular that ($\tilde\partial$-WCI) gives much more information than ($\tilde\partial$-WSG) on the covariance function. As shown in Corollary 4.5.1, this result is sharp: in the Gaussian case each of the necessary conditions below is (essentially) sufficient.

Proposition 4.2.1. *Let A be a jointly measurable stationary random field on $\mathbb{R}^d$ with $\mathbb{E}\left[|A|^2\right] < \infty$, and let $\mathcal{C}(x) := \mathrm{Cov}\left[A(0); A(x)\right]$ denote its covariance function. When using the derivative $\tilde\partial = \partial^{\mathrm{osc}}$, further assume that A is bounded (except in item (iii)).*

(i) If A satisfies ($\tilde\partial$-SG), and if the covariance function $\mathcal{C}$ is nonnegative, then $\mathcal{C}$ is integrable.

(ii) If A satisfies ($\tilde\partial$-WSG) with weight π, and if the covariance function $\mathcal{C}$ is nonnegative, then $\mathcal{C}$ is integrable whenever $\int_0^\infty \ell^d \pi(\ell) d\ell < \infty$. More generally, $\mathcal{C}$ satisfies

$$\int_{\mathbb{R}^d} (1+|x|)^{-\alpha}\mathcal{C}(x)dx \le C_\alpha \begin{cases} \int_0^\infty (\ell+1)^{d-\alpha}\pi(\ell)d\ell, & \text{if } 0 \le \alpha < d; \\ \int_0^\infty \log^2(2+\ell)\pi(\ell)d\ell, & \text{if } \alpha = d; \\ \int_0^\infty \pi(\ell)d\ell, & \text{if } \alpha > d. \end{cases}$$

(iii) If A satisfies ($\tilde\partial$-CI) with radius $R+\varepsilon$ for all $\varepsilon > 0$, then the range of dependence of A is bounded by $2R$ (that is, for all Borel subsets $S, T \subset \mathbb{R}^d$ the restrictions $A|_S$ and $A|_T$ are independent whenever $d(S,T) > 2R$).

(iv) If A satisfies ($\tilde\partial$-WCI) with weight π, then the covariance function satisfies for all $x \in \mathbb{R}^d$,

$$|\mathcal{C}(x)| \le C \int_{\frac12(|x|-2)\vee 0}^\infty \pi(\ell)d\ell. \qquad \Diamond$$

Proof. We split the proof into four steps.

Step 1. Proof of (i).

Let the field A satisfy ($\tilde\partial$-SG) with radius R. For any $L \ge 1$, the standard spectral gap applied to the $\sigma(A)$-measurable random variable $X(A) = \int_{B_L} A$ (which is well-defined by measurability and moment bounds on A) yields

$$\mathrm{Var}\left[\int_{B_L} A\right] \le C\mathbb{E}\left[\int_{\mathbb{R}^d}\Big(\tilde\partial_{A,B_R(x)}\int_{B_L} A\Big)^2 dx\right].$$

For each choice of the derivative $\tilde\partial$ (further assuming that A is bounded in the case $\tilde\partial = \partial^{\mathrm{osc}}$), we have

$$\mathbb{E}\left[\Big(\tilde\partial_{A,B_R(x)}\int_{B_L} A\Big)^2\right] \le C|B_R(x)\cap B_L|^2 \le C_R \mathbb{1}_{|x|\le R+L}.$$

Hence, for $L \ge 1$,

$$\int_{B_L}\int_{B_L} \mathrm{Cov}\left[A(x); A(y)\right] dxdy = \mathrm{Var}\left[\int_{B_L} A\right] \le C_R|B_{R+L}| \le C_R|B_L|.$$

Therefore, if $\mathcal{C}$ is nonnegative, we deduce

$$\int_{B_L}\mathcal{C} \lesssim \int_{B_L}\fint_{B_L}\mathcal{C}(x-y)dydx = \int_{B_L}\fint_{B_L}\mathrm{Cov}\left[A(x); A(y)\right] dydx \le C_R.$$

Letting $L \uparrow \infty$, we conclude that $\mathcal{C}$ is integrable.

Step 2. Proof of (ii).

Let the field A satisfy ($\tilde\partial$-WSG) with weight π, and assume that $\mathcal{C}$ is nonnegative. Repeating the argument of Step 1, we deduce for all $L \ge 1$,

$$\begin{aligned} L^d\int_{B_L}\mathcal{C}(x)dx \;&\lesssim\; \mathbb{E}\left[\Big(\int_{B_L}(A(x)-\mathbb{E}\left[A\right])dx\Big)^2\right] \;\le\; \int_0^\infty\int_{\mathbb{R}^d}|B_{\ell+1}(x)\cap B_L|^2dx\,(\ell+1)^{-d}\pi(\ell)d\ell \\ &\lesssim\; \int_0^L L^d(\ell+1)^d\pi(\ell)d\ell + \int_L^\infty L^{2d}\pi(\ell)d\ell \;\lesssim\; L^d\int_0^\infty(\ell+1)^d\pi(\ell)d\ell, \end{aligned}$$

which shows that $\mathcal{C}$ is integrable if $\int_0^\infty (\ell+1)^d \pi(\ell) d\ell < \infty$.

Let now $\alpha > 0$ be fixed, and let $\gamma := \frac{1}{2}(d+\alpha)$. Assume that $\alpha \neq d$ (the case $\alpha = d$ can be treated similarly and yields the logarithmic correction). For all $L \geq 1$, the weighted spectral gap applied to the $\sigma(A)$-measurable random variable $X(A) = \int_{B_L} (1+|y|)^{-\gamma} A(y) dy$ yields

$$\begin{aligned}\operatorname{Var}\left[\int_{B_L} (1+|y|)^{-\gamma} A(y) dy\right] &\leq \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde\partial_{A,B_{\ell+1}(x)} \int_{B_L} (1+|y|)^{-\gamma} A(y) dy\Big)^2 dx\, (\ell+1)^{-d} \pi(\ell) d\ell\right] \\ &\leq C \int_0^\infty \int_{\mathbb{R}^d} \Big(\int_{B_L \cap B_{\ell+1}(x)} (1+|y|)^{-\gamma} dy\Big)^2 dx\, (\ell+1)^{-d} \pi(\ell) d\ell.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{B_{2L}} \Big(\int_{B_L(-x)} (1+|x+y|)^{-\gamma} (1+|y|)^{-\gamma} dy\Big) \mathcal{C}(x) dx \\ &\qquad\qquad = \operatorname{Var}\left[\int_{B_L} (1+|y|)^{-\gamma} A(y) dy\right] \leq C_\alpha \int_0^\infty (\ell+1)^{(d-\alpha)\vee 0} \pi(\ell) d\ell,\end{aligned}$$

which yields the claim by passing to the limit $L \uparrow \infty$.

Step 3. Proof of (iii).

Let the field A satisfy ($\tilde\partial$-CI) with radius $R+\varepsilon$ for any $\varepsilon > 0$. Given two Borel subsets $S, T \subset \mathbb{R}^d$ with $d(S,T) > 2R$, choosing $\varepsilon := \frac{1}{3}(d(S,T) - 2R)$, and noting that the sets $S + B_{R+\varepsilon}$ and $T + B_{R+\varepsilon}$ are disjoint, the covariance inequality ($\tilde\partial$-CI) with radius $R+\varepsilon$ implies for any $G \in \sigma(A|_S)$ and $H \in \sigma(A|_T)$,

$$|\operatorname{Cov}[\mathbb{1}_G; \mathbb{1}_H]| \leq C_\varepsilon \int_{(S+B_{R+\varepsilon}) \cap (T+B_{R+\varepsilon})} \mathbb{E}\left[\Big(\tilde\partial_{A,B_{R+\varepsilon}(x)} \mathbb{1}_G\Big)^2\right]^{\frac12} \mathbb{E}\left[\Big(\tilde\partial_{A,B_{R+\varepsilon}(x)} \mathbb{1}_H\Big)^2\right]^{\frac12} dx \;= 0.$$

This shows that the σ-algebras $\sigma(A|_S)$ and $\sigma(A|_T)$ are independent.

Step 4. Proof of (iv).

Let the field A satisfy ($\tilde\partial$-WCI) with weight π. For all $x \in \mathbb{R}^d$ and all $\varepsilon > 0$, the covariance inequality applied to the $\sigma(A)$-measurable random variables $\fint_{B_\varepsilon(x)} A$ and $\fint_{B_\varepsilon} A$ yields

$$\begin{aligned}&\Big|\fint_{B_\varepsilon(x)} \fint_{B_\varepsilon} \mathcal{C}(y-z) dy dz\Big| = \Big|\operatorname{Cov}\Big[\fint_{B_\varepsilon(x)} A; \fint_{B_\varepsilon} A\Big]\Big| \\ &\leq \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\tilde\partial_{A,B_{\ell+1}(y)} \fint_{B_\varepsilon(x)} A\Big)^2\right]^{\frac12} \mathbb{E}\left[\Big(\tilde\partial_{A,B_{\ell+1}(y)} \fint_{B_\varepsilon} A\Big)^2\right]^{\frac12} dy\, (\ell+1)^{-d} \pi(\ell) d\ell \\ &\leq \int_0^\infty \int_{\mathbb{R}^d} \varepsilon^{-d} |B_\varepsilon(x) \cap B_{\ell+1}(y)| \varepsilon^{-d} |B_\varepsilon \cap B_{\ell+1}(y)| dy\, (\ell+1)^{-d} \pi(\ell) d\ell.\end{aligned}$$

Letting $\varepsilon \downarrow 0$ and using the continuity of the function $\mathcal{C}$ (as a consequence of the stochastic continuity of the field A, which follows from its joint measurability), we deduce the claim: for all $x \in \mathbb{R}^d$,

$$|\mathcal{C}(x)| \leq C \int_0^\infty |B_{\ell+1}(x) \cap B_{\ell+1}|\, (\ell+1)^{-d} \pi(\ell) d\ell \;\leq\; C \int_{\frac12(|x|-2)\vee 0}^\infty \pi(\ell) d\ell. \qquad \square$$

As the above proposition shows, if the weight π satisfies $\int_0^\infty (\ell+1)^d \pi(\ell) d\ell < \infty$, both ($\tilde\partial$-SG) and ($\tilde\partial$-WSG) with weight π imply that $\mathcal{C}$ is integrable. The following proposition establishes that (∂^{fct}-SG) and (∂^{fct}-WSG) are actually equivalent for such weights π. This result does not hold if ∂^{fct} is replaced by another derivative or if SG is replaced by CI.

Proposition 4.2.2. *Let A satisfy* (∂^{fct}-WSG) *(resp.* (∂^{fct}-WLSI)*) with some weight π. If $\int_0^\infty(\ell+1)^d\pi(\ell)d\ell<\infty$, then A satisfies* (∂^{fct}-SG) *(resp.* (∂^{fct}-LSI)*) with any radius $R>0$.* $\Diamond$

Proof. Let $\varepsilon\in(0,1)$ be fixed. Let $X(A)$ be some $\sigma(A)$-measurable random variable. Cover the cube $Q_\ell(x)$ with the cubes $Q_\varepsilon(z_i^{x,\ell})$, $i=1,\ldots,\lceil r/\varepsilon\rceil^d$, where $z_i^{x,\ell}\in\varepsilon\mathbb{Z}^d$ is an enumeration of $Q_{\varepsilon\lceil\ell/\varepsilon\rceil}(x)\cap\varepsilon\mathbb{Z}^d$. We then estimate

$$\Big(\int_{Q_\ell(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2\le\Big(\sum_{i=1}^{\lceil\ell/\varepsilon\rceil^d}\int_{Q_\varepsilon(z_i^{x,\ell})}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2\le(1+\ell/\varepsilon)^d\sum_{i=1}^{\lceil\ell/\varepsilon\rceil^d}\Big(\int_{Q_\varepsilon(z_i^{x,\ell})}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2.$$

For all $\ell>0$, and $y\in\varepsilon\mathbb{Z}^d$, there are at most $\lceil\ell\rceil^d$ possible values of $x\in\mathbb{Z}^d$ such that $y\in\{z_i^{x,\ell}:i=1,\ldots,\ell^d\}$, so that we obtain

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\Big(\int_{Q_\ell(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2&\le\frac{(\ell+1)^d}{\varepsilon^d}\sum_{x\in\mathbb{Z}^d}\sum_{i=1}^{\lceil\ell/\varepsilon\rceil^d}\Big(\int_{Q_\varepsilon(z_i^{x,\ell})}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2\\&\le\frac{(\ell+1)^{2d}}{\varepsilon^d}\sum_{y\in\varepsilon\mathbb{Z}^d}\Big(\int_{Q_\varepsilon(y)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2,\end{aligned}$$

which directly yields, bounding integrals on cubes by integral on balls, and sums by integrals,

$$\begin{aligned}\int_{\mathbb{R}^d}\Big(\fint_{B_{\ell+1}(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2dx&\le(\ell+1)^{-2d}\sum_{x\in\mathbb{Z}^d}\Big(\int_{Q_{1+\sqrt{d}+\ell}(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2\\&\lesssim\varepsilon^{-d}\sum_{y\in\varepsilon\mathbb{Z}^d/\sqrt{d}}\Big(\int_{Q_{\varepsilon/\sqrt{d}}(y)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2\lesssim\int_{\mathbb{R}^d}\Big(\fint_{B_\varepsilon(y)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2dy.\end{aligned}\tag{4.8}$$

If A satisfies (∂^{fct}-WSG) with weight π, we deduce from the above inequality that for all $\varepsilon\in(0,1)$,

$$\begin{aligned}\mathrm{Var}[X(A)]&\le\int_0^\infty\int_{\mathbb{R}^d}\mathbb{E}\Big[\Big(\int_{B_{\ell+1}(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2\Big]dx\,(\ell+1)^{-d}\pi(\ell)d\ell\\&\lesssim\varepsilon^{-2d}\Big(\int_0^\infty(\ell+1)^d\pi(\ell)d\ell\Big)\int_{\mathbb{R}^d}\mathbb{E}\Big(\int_{B_\varepsilon(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2dx,\end{aligned}$$

which shows that the field A also satisfies (∂^{fct}-SG) if $\int_0^\infty(\ell+1)^d\pi(\ell)d\ell<\infty$. $\square$

4.2.2 Ergodicity and mixing

In the previous subsection we established the link between weighted functional inequality and the decay of the covariance function. We now turn to ergodicity properties, and further investigate the relation between weighted spectral gaps and standard mixing conditions.

Let us first recall some terminology. The random field A is said to be *strongly mixing* if for all $\sigma(A)$-measurable random variable $X(A)$ and all Borel subsets $E,E'\subset\mathbb{R}$ we have

$$\mathbb{P}\big[X(A)\in E,\,X(A(\cdot+x))\in E'\big]\xrightarrow{|x|\uparrow\infty}\mathbb{P}[X(A)\in E]\,\mathbb{P}\big[X(A)\in E'\big].$$

This qualitative property can be quantified into *strong mixing conditions.* A classical way to measure the dependence between two sub-σ-algebras $\mathcal{G}_1,\mathcal{G}_2\subset\mathcal{A}$ is the following *α-mixing coefficient*, first introduced by Rosenblatt [373],

$$\alpha(\mathcal{G}_1,\mathcal{G}_2):=\sup\big\{|\mathbb{P}[G_1\cap G_2]-\mathbb{P}[G_1]\mathbb{P}[G_2]|\,:\,G_1\in\mathcal{G}_1,G_2\in\mathcal{G}_2\big\}.$$

Applied to the random field A, this leads to the following measure of mixing: For all diameters $D \in (0,\infty]$ and distances $R > 0$, we set

$$\tilde{\alpha}(R,D;A) \\ := \sup\big\{\alpha(\sigma(A|_{S_1}),\sigma(A|_{S_2}))\,:\,S_1,S_2\in\mathcal{B}(\mathbb{R}^d),\,d(S_1,S_2)\geq R,\,\mathrm{diam}(S_1),\mathrm{diam}(S_2)\leq D\big\}. \quad (4.9)$$

We say that the field A is *α-mixing* if for all diameter $D \in (0,\infty)$ we have $\tilde{\alpha}(R,D;A) \xrightarrow{R\uparrow\infty} 0$. Note that α-mixing is the weakest of the usual strong mixing conditions (see e.g. [156]), although it is in general strictly stronger than qualitative strong mixing.

The following result makes explicit the connection between weighted spectral gaps and α-mixing properties. Note that this result is essentially sharp: on the one hand, in the Gaussian case, as shown in Corollary 4.5.1, each of the necessary conditions in (i), (ii), and (iv) below is (essentially) sufficient, and on the other hand the R-scaling in the estimate in (iii) can be checked to be sharp at least in some specific examples.

Proposition 4.2.3. *Let A be a jointly measurable stationary random field on $\mathbb{R}^d$.*

(i) If A satisfies ($\tilde\partial$-WSG) with integrable weight π, then A is ergodic.

(ii) If A satisfies ($\tilde\partial$-WCI) with integrable weight π, then A is strongly mixing.

(iii) If A satisfies ($\tilde\partial$-WCI) with weight π and with derivative $\tilde\partial = \partial^{\mathrm{G}}$ or ∂^{osc}, then A is α-mixing with coefficient $\tilde{\alpha}(R,D;A) \lesssim (1+\frac{D}{R})^d \int_{R-1}^\infty \pi(\ell)d\ell$.

(iv) If A satisfies ($\tilde\partial$-CI) with radius $R+\varepsilon > 0$ for all $\varepsilon > 0$, then $\tilde{\alpha}(r,\infty;A) = 0$ for all $r > 2R$. ♢

Remark 4.2.4. Item (iii) is expected to fail in general for the derivative $\tilde\partial = \partial^{\mathrm{fct}}$. Indeed, as shown in Corollary 4.5.1, if A is a stationary Gaussian random field with covariance function $\mathcal{C}$ satisfying $|\mathcal{C}(x)| \simeq (1+|x|)^{-\alpha}$ for all x, for some $\alpha > 0$, then the field A satisfies (∂^{fct}-WCI) with weight $\pi(r) \simeq (1+r)^{-\alpha-1}$. Therefore, if item (iii) above was true with $\tilde\partial = \partial^{\mathrm{fct}}$, we would deduce in this Gaussian example $\tilde{\alpha}(R,D;A) \lesssim (1+(D/R)^d)R^{-\alpha}$, which is however expected to fail (the correct scaling is rather expected to be $R^{d-\alpha}$ for $\alpha > d$, cf. [156, Corollary 2 of Section 2.1.1] or [249, Corollary p.195]). ♢

Proof of Proposition 4.2.3. Item (iv) follows from Proposition 4.2.1. We split the rest of the proof into three steps.

Step 1. Proof of (i).

Let the field A satisfy ($\tilde\partial$-WSG) with weight π. To prove ergodicity, it suffices to show that for all integrable $\sigma(A)$-measurable random variables $X(A)$ we have

$$\lim_{L\uparrow\infty}\mathbb{E}\left[\Big|\fint_{B_L} X(A(x+\cdot))dx - \mathbb{E}\left[X(A)\right]\Big|\right] = 0.$$

By an approximation argument in $\mathrm{L}^2(\Omega)$, we may assume that $X(A)$ is bounded and is $\sigma(A|_{B_R})$-measurable for some $R > 0$. The spectral gap ($\tilde\partial$-WSG) applied to the $\sigma(A)$-measurable random variable $\fint_{B_L} X(A(\cdot+x))dx$ yields

$$S_L := \mathbb{E}\left[\Big|\fint_{B_L} X(A(x+\cdot))dx - \mathbb{E}\left[X(A)\right]\Big|\right]^2 \\ \leq \mathrm{Var}\left[\fint_{B_L} X(A(x+\cdot))dx\right] \leq \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d}\Big(\fint_{B_L}\tilde\partial_{A,B_{\ell+1}(y)}X(A(x+\cdot))dx\Big)^2 dy\,(\ell+1)^{-d}\pi(\ell)d\ell\right],$$

and therefore

$$S_L \le \mathbb{E}\Big[\int_0^\infty \int_{\mathbb{R}^d} \fint_{B_L} \fint_{B_L} \tilde\partial_{A,B_{\ell+1}(y)} X(A(x+\cdot))\, \tilde\partial_{A,B_{\ell+1}(y)} X(A(x'+\cdot)) dx dx' dy (\ell+1)^{-d}\pi(\ell) d\ell\Big].$$

By assumption, $\tilde\partial_{A,B_{\ell+1}(y)} X(A(x+\cdot)) = 0$ whenever $B_R(x)\cap B_{\ell+1}(y)=\varnothing$, i.e. whenever $|x-y| > R+\ell+1$. For the choices $\tilde\partial = \partial^{\mathrm{osc}}$ and ∂^{G}, we also have $\tilde\partial_{A,B_{\ell+1}(y)} X(A(x+\cdot)) \le 2\|X\|_{\mathrm{L}^\infty}$, so that the above yields

$$\begin{aligned} S_L &\le 4\|X\|_{\mathrm{L}^\infty}^2 \int_0^\infty \int_{\mathbb{R}^d} \fint_{B_L} \fint_{B_L} \mathbb{1}_{|x-y|\le R+\ell+1} \mathbb{1}_{|x'-y|\le R+\ell+1} dx dx' dy\, (\ell+1)^{-d}\pi(\ell) d\ell \\ &= 4\|X\|_{\mathrm{L}^\infty}^2 L^{-2d} \int_0^\infty \Big(\int_{B_L} \int_{B_{R+\ell+1}(x)} |B_L \cap B_{R+\ell+1}(y)| dy dx\Big) (\ell+1)^{-d}\pi(\ell) d\ell \\ &\le 4\|X\|_{\mathrm{L}^\infty}^2 \int_0^\infty (R+\ell+1)^d \Big(\frac{R+\ell}{L}\wedge 1\Big)^d (\ell+1)^{-d}\pi(\ell) d\ell, \end{aligned}$$

where the right-hand side obviously goes to 0 as $L\uparrow\infty$ whenever $\int_0^\infty \pi(\ell)d\ell<\infty$. This proves ergodicity for the choices $\tilde\partial = \partial^{\mathrm{osc}}$ and ∂^{G}.

It remains to treat the case $\tilde\partial = \partial^{\mathrm{fct}}$. An additional approximation argument is then needed in order to restrict attention to those random variables $X(A)$ such that the derivative $\tilde\partial_{A,B_{\ell+1}(x)} X(A)$ is pointwise bounded. The stochastic continuity of the field A (which follows from its joint measurability) ensures that the $\sigma(A|_{B_R})$-measurable random variable $X(A)$ is actually $\sigma(A|_{\mathbb{Q}^d\cap B_R})$-measurable. A standard approximation argument then allows to construct a sequence $(x_n)_n \subset B_R$ and a sequence $(X_n(A))_n$ of random variables such that $X_n(A)$ is $\sigma((A(x_k))_{k=1}^n)$-measurable and converges to $X(A)$ in $\mathrm{L}^2(\Omega)$. By definition, we may write $X_n(A) = f_n(A(x_1),\ldots,A(x_n))$ for some Borel function $f_n : (\mathbb{R}^k)^n \to \mathbb{R}$. Another standard approximation argument now allows to replace the Borel maps f_n's by smooth functions. We end up with a sequence that approximates $X(A)$ in $\mathrm{L}^2(\Omega)$, and such that the elements have pointwise bounded $\tilde\partial$-derivative. For these approximations, the conclusion follows as before.

Step 2. Proof of (ii).

Let the field A satisfy ($\tilde\partial$-WCI) with weight π. To prove strong mixing, it suffices to show that for all bounded $\sigma(A)$-measurable random variables $X(A)$ and $Y(A)$ we have $\mathrm{Cov}\,[X(A);Y(A(x+\cdot))]\to 0$ as $|x|\to\infty$ (since the desired property then follows by choosing the random variables $X(A), Y(A)$ to be any pair of indicator functions). Again, a standard approximation argument allows one to consider bounded $\sigma(A|_{B_R})$-measurable random variables $X(A), Y(A)$ for some $R>0$. Given $x\in\mathbb{R}^d$, apply the covariance inequality ($\tilde\partial$-WCI) to $X(A)$ and $Y(A(\cdot+x))$ to obtain

$$\begin{aligned} &\big|\,\mathrm{Cov}\,[X(A);Y(A(x+\cdot))]\,\big| \\ &\qquad \le \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\tilde\partial_{A,B_{\ell+1}(y)} X(A)\Big)^2\Big]^{\frac12}\, \mathbb{E}\Big[\Big(\tilde\partial_{A,B_{\ell+1}(y)} Y(A(x+\cdot))\Big)^2\Big]^{\frac12} dy\, (\ell+1)^{-d}\pi(\ell) d\ell. \end{aligned}$$

By assumption, $\tilde\partial_{A,B_{\ell+1}(y)} X(A) = 0$ whenever $B_R\cap B_{\ell+1}(y)=\varnothing$, i.e. whenever $|y|>R+\ell+1$. For the choices $\tilde\partial = \partial^{\mathrm{osc}}$ and ∂^{G}, we have in addition $\tilde\partial_{A,B_{\ell+1}(y)} X(A) \le 2\|X\|_{\mathrm{L}^\infty}$, so that the above directly yields

$$\begin{aligned} \big|\,\mathrm{Cov}\,[X(A);Y(A(x+\cdot))]\,\big| &\le 4\|X\|_{\mathrm{L}^\infty}\|Y\|_{\mathrm{L}^\infty} \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{|y|\le R+\ell+1}\mathbb{1}_{|x-y|\le R+\ell+1}\, dy\, (\ell+1)^{-d}\pi(\ell) d\ell \\ &\lesssim \|X\|_{\mathrm{L}^\infty}\|Y\|_{\mathrm{L}^\infty} \int_0^\infty (R+\ell+1)^d (\ell+1)^{-d}\pi(\ell) d\ell \end{aligned}$$

where the right-hand side goes to 0 as $|x| \to \infty$ whenever $\int_0^\infty \pi(\ell)d\ell < \infty$. This proves strong mixing for the choices $\tilde{\partial} = \partial^{\mathrm{osc}}$ and ∂^{G}. In the case $\tilde{\partial} = \partial^{\mathrm{fct}}$, an additional approximation argument is needed as in Step 1 in order to restrict to random variables $X(A)$ such that $\tilde{\partial}_{A,B_{\ell+1}(y)}X(A)$ is pointwise bounded.

Step 3. Proof of (iii).

Let the field A satisfy ($\tilde{\partial}$-WCI) with weight π, and with derivative $\tilde{\partial} = \partial^{\mathrm{osc}}$ or ∂^{G}. Given Borel subsets $S, T \subset \mathbb{R}^d$ with diameter $\le D$ and with $d(S,T) \ge 2R$, the covariance inequality ($\tilde{\partial}$-WCI) for this choice of derivatives yields for all bounded random variables $X(A)$ and $Y(A)$, respectively $\sigma(A|_S)$-measurable and $\sigma(A|_T)$-measurable,

$$\begin{aligned}
&\big|\operatorname{Cov}[X(A);Y(A)]\big| \\
&\le \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\tilde{\partial}_{A,B_{\ell+1}(x)}X(A)\Big)^2\Big]^{\frac12} \mathbb{E}\Big[\Big(\tilde{\partial}_{A,B_{\ell+1}(x)}Y(A)\Big)^2\Big]^{\frac12} dx\,(\ell+1)^{-d}\pi(\ell)d\ell \\
&\le 4\|X(A)\|_{\mathrm{L}^\infty}\|Y(A)\|_{\mathrm{L}^\infty} \int_0^\infty \big|(S+B_{\ell+1}) \cap (T+B_{\ell+1})\big|\,(\ell+1)^{-d}\pi(\ell)d\ell \\
&\lesssim \|X(A)\|_{\mathrm{L}^\infty}\|Y(A)\|_{\mathrm{L}^\infty} \int_{R-1}^\infty (\ell+D+1)^d(\ell+1)^{-d}\pi(\ell)d\ell \\
&\le \|X(A)\|_{\mathrm{L}^\infty}\|Y(A)\|_{\mathrm{L}^\infty}\Big(1+\frac{D}{R}\Big)^d \int_{R-1}^\infty \pi(\ell)d\ell,
\end{aligned}$$

from which the claim follows by choosing for $X(A), Y(A)$ any pair of indicator functions. □

4.3 Moment bounds and concentration properties

In this section, we investigate the concentration properties that are implied by weighted spectral gaps, according to both the choice of the derivative and the decay of the weight. Although the results are new, the proofs rely mainly on standard Herbst-type arguments.

4.3.1 Control of higher moments

As for standard functional inequalities, weighted functional inequalities allow one to control higher moments of random variables. Note that these properties depend crucially on the choice of the derivative.

Proposition 4.3.1. *Assume that the random field A satisfies ($\tilde{\partial}$-WSG) with integrable weight $\pi : \mathbb{R}_+ \to \mathbb{R}_+$. Then there exists $C < \infty$ (depending only on π and d) such that for all $1 \le p < \infty$ and all $\sigma(A)$-measurable random variables $X(A)$ we have*

(i) if $\tilde{\partial} = \partial^{\mathrm{G}}$ or ∂^{fct},

$$\mathbb{E}\Big[\big(X(A)-\mathbb{E}[X(A)]\big)^{2p}\Big] \le (Cp^2)^p\,\mathbb{E}\bigg[\bigg(\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde{\partial}_{A,B_{\ell+1}(x)}X(A)\Big)^2 dx\,(\ell+1)^{-d}\pi(\ell)d\ell\bigg)^p\bigg],$$

where the multiplicative factor $(Cp^2)^p$ can be upgraded to $(Cp)^p$ if the field A further satisfies ($\tilde{\partial}$-WLSI);

(ii) if $\tilde{\partial} = \partial^{\mathrm{osc}}$,

$$\begin{aligned}
&\mathbb{E}\Big[\big(X(A)-\mathbb{E}[X(A)]\big)^{2p}\Big] \\
&\qquad \le (Cp^2)^p\,\mathbb{E}\bigg[\int_0^\infty \bigg(\int_{\mathbb{R}^d} \Big(\tilde{\partial}_{A,B_{2(\ell+1)}(x)}X(A)\Big)^2 dx\bigg)^p (\ell+1)^{-dp}\pi(\ell)d\ell\bigg]. \quad \Diamond
\end{aligned}$$

Proof. Let $X(A)$ be $\sigma(A)$-measurable. We may assume without loss of generality that $\mathbb{E}[X(A)]=0$. We split the proof into two steps.

Step 1. Proof of (i) and (ii) for ($\tilde\partial$-WSG).

Applying the spectral gap ($\tilde\partial$-WSG) to the $\sigma(A)$-measurable random variable $|X(A)|^p$ yields

$$\mathbb{E}\left[X(A)^{2p}\right] \le \mathbb{E}\left[|X(A)|^p\right]^2 + \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde\partial_{A,B_{\ell+1}(x)}\big(|X(A)|^p\big)\Big)^2 dx\, (\ell+1)^{-d}\pi(\ell)d\ell\right]. \tag{4.10}$$

For $p>2$, Hölder's and Young's inequalities with exponents $(\frac{2(p-1)}{p-2},\frac{2(p-1)}{p})$ and $(\frac{p-1}{p-2},p-1)$, respectively, imply for all $\delta>0$,

$$\begin{aligned}\mathbb{E}\left[|X(A)|^p\right]^2 = \mathbb{E}\left[|X(A)|^{p\frac{p-2}{p-1}}|X(A)|^{\frac{p}{p-1}}\right]^2 &\le \mathbb{E}\left[X(A)^{2p}\right]^{\frac{p-2}{p-1}}\mathbb{E}\left[X(A)^2\right]^{\frac{p}{p-1}}\\ &\le \frac{p-2}{p-1}\delta\,\mathbb{E}\left[X(A)^{2p}\right] + \frac{1}{p-1}\delta^{2-p}\,\mathbb{E}\left[X(A)^2\right]^p.\end{aligned}$$

while for $p\le 2$ Jensen's inequality simply yields $\mathbb{E}\left[|X(A)|^p\right]^2 \le \mathbb{E}\left[X(A)^2\right]^p$. Injecting these estimates into (4.10) for some $\delta \gtrsim 1$ small enough, we conclude for all $1\le p<\infty$,

$$\mathbb{E}\left[X(A)^{2p}\right] \le p^{-1}C^p\,\mathbb{E}\left[X(A)^2\right]^p + C\,\mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde\partial_{A,B_{\ell+1}(x)}\big(|X(A)|^p\big)\Big)^2 dx\, (\ell+1)^{-d}\pi(\ell)d\ell\right].$$

Since $\mathbb{E}\left[X(A)^2\right] = \mathrm{Var}\left[X(A)\right]$ follows from the centering assumption, the first right-hand side term is estimated by the spectral gap ($\tilde\partial$-WSG). Further using Jensen's inequality, this leads to

$$\begin{aligned}\mathbb{E}\left[X(A)^{2p}\right] \le p^{-1}C^p\,\mathbb{E}&\left[\left(\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde\partial_{A,B_{\ell+1}(x)}X(A)\Big)^2 dx\, (\ell+1)^{-d}\pi(\ell)d\ell\right)^p\right]\\ &+ C\,\mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde\partial_{A,B_{\ell+1}(x)}\big(|X(A)|^p\big)\Big)^2 dx\, (\ell+1)^{-d}\pi(\ell)d\ell\right].\end{aligned} \tag{4.11}$$

We split the rest of this step into three further substeps, and treat separately ∂^{fct}, ∂^{G}, and ∂^{osc}.

Substep 1.1. Proof of (i) for $\tilde\partial=\partial^{\mathrm{fct}}$.

By the Leibniz rule, $\partial^{\mathrm{fct}}_{A,S}(|X(A)|^p) = p|X(A)|^{p-1}\partial^{\mathrm{fct}}_{A,S}X(A)$, so that Hölder's inequality with exponents $(\frac{p}{p-1},p)$ yields

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)}\big(|X(A)|^p\big)\Big)^2 dx\, (\ell+1)^{-d}\pi(\ell)d\ell\right]\\ &\le\ p^2\,\mathbb{E}\left[X(A)^{2(p-1)}\int_0^\infty \int_{\mathbb{R}^d} \Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)}X(A)\Big)^2 dx\, (\ell+1)^{-d}\pi(\ell)d\ell\right]\\ &\le\ p^2\,\mathbb{E}\left[X(A)^{2p}\right]^{1-\frac1p}\,\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{R}^d} \Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)}X(A)\Big)^2 dx\, (\ell+1)^{-d}\pi(\ell)d\ell\right)^p\right]^{\frac1p}.\end{aligned} \tag{4.12}$$

Combined with (4.11) and Young's inequality with exponents $(\frac{p}{p-1},p)$ to absorb the factor $\mathbb{E}\left[X(A)^{2p}\right]$ into the left-hand side, the conclusion of item (i) follows with the prefactor $(Cp^2)^p$.

Substep 1.2. Proof of (i) for $\tilde\partial=\partial^{\mathrm{G}}$.

The inequality $||a|^p-|b|^p| \le p|a-b|(|a|^{p-1}+|b|^{p-1})$ for all $a,b\in\mathbb{R}$ easily implies, by definition of the Glauber derivative (4.4),

$$\begin{aligned}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,S}\big(|X(A)|^p\big)\Big)^2\right] &= \mathbb{E}\left[\mathbb{E}'\left[\big(|X(A')|^p-|X(A)|^p\big)^2\,\big\|\,A'|_{\mathbb{R}^d\setminus S}=A|_{\mathbb{R}^d\setminus S}\right]\right]\\ &\le\ 2p^2\mathbb{E}\left[\mathbb{E}'\left[\big(X(A)^{2(p-1)}+X(A')^{2(p-1)}\big)\big(X(A')-X(A)\big)^2\,\big\|\,A'|_{\mathbb{R}^d\setminus S}=A|_{\mathbb{R}^d\setminus S}\right]\right]\\ &=\ 4p^2\mathbb{E}\left[X(A)^{2(p-1)}\Big(\partial^{\mathrm{G}}_{A,S}X(A)\Big)^2\right],\end{aligned}$$

and we are now back to the situation of Substep 1.1.

Substep 1.3. Proof of (ii).

Again, the inequality $||a|^p-|b|^p|\le p|a-b|(|a|^{p-1}+|b|^{p-1})$ for all $a,b\in\mathbb{R}$ implies

$$\partial^{\mathrm{osc}}_{A,S}\,|X(A)|^p\;\le\;2p\Big(\sup_{A,S}|X(A)|^{p-1}\Big)\,\partial^{\mathrm{osc}}_{A,S}\,X(A)\;\le\;2p\bigg(|X(A)|+\partial^{\mathrm{osc}}_{A,S}\,X(A)\bigg)^{p-1}\,\partial^{\mathrm{osc}}_{A,S}\,X(A).\quad(4.13)$$

We then make use of the following inequality that holds for some constant $C\simeq 1$ large enough (independent of p): for all $a,b\ge 0$, $(a+b)^{p-1}\le 2a^{p-1}+(Cp)^pb^{p-1}$. This allows one to rewrite (4.13) in the form

$$\partial^{\mathrm{osc}}_{A,S}\,|X(A)|^p\le 4p|X(A)|^{p-1}\,\partial^{\mathrm{osc}}_{A,S}\,X(A)+(Cp)^p(\partial^{\mathrm{osc}}_{A,S}\,X(A))^p.\quad(4.14)$$

Arguing as in Substep 1.1, we obtain by Hölder's inequality,

$$\begin{aligned}\mathbb{E}\bigg[\int_0^\infty\int_{\mathbb{R}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\,|X(A)|^p\Big)^2dx\,(\ell+1)^{-d}\pi(\ell)d\ell\bigg]&\\
\le Cp^2\,\mathbb{E}\left[X(A)^{2p}\right]^{1-\frac1p}\,\mathbb{E}\bigg[\bigg(\int_0^\infty\int_{\mathbb{R}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\,X(A)\Big)^2dx\,(\ell+1)^{-d}\pi(\ell)d\ell\bigg)^p\bigg]^{\frac1p}&\\
+(Cp^2)^p\,\mathbb{E}\bigg[\int_0^\infty\int_{\mathbb{R}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\,X(A)\Big)^{2p}dx\,(\ell+1)^{-d}\pi(\ell)d\ell\bigg]&.\end{aligned}$$

Combined with (4.11) and Young's inequality to absorb the factor $\mathbb{E}\big[X(A)^{2p}\big]$ into the left-hand side, this yields

$$\begin{aligned}\mathbb{E}\left[X(A)^{2p}\right]\le(Cp^2)^p\,\mathbb{E}\bigg[\bigg(\int_0^\infty\int_{\mathbb{R}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\,X(A)\Big)^2dx\,(\ell+1)^{-d}\pi(\ell)d\ell\bigg)^p\bigg]&\\
+(Cp^2)^p\,\mathbb{E}\bigg[\int_0^\infty\int_{\mathbb{R}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\,X(A)\Big)^{2p}dx\,(\ell+1)^{-d}\pi(\ell)d\ell\bigg]&.\end{aligned}$$

It remains to reformulate the second right-hand side term. By the discrete $\ell^1-\ell^p$ inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\,X(A)\Big)^{2p}dx&\le\sum_{z\in\frac{\ell+1}{\sqrt d}\mathbb{Z}^d}\int_{z+\frac{\ell+1}{\sqrt d}Q}\Big(\,\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\,X(A)\Big)^{2p}dx\\
&\le\Big(\frac{\ell+1}{\sqrt d}\Big)^d\sum_{z\in\frac{\ell+1}{\sqrt d}\mathbb{Z}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{\frac32(\ell+1)}(z)}\,X(A)\Big)^{2p}\\
&\le\Big(\frac{\ell+1}{\sqrt d}\Big)^d\bigg(\sum_{z\in\frac{\ell+1}{\sqrt d}\mathbb{Z}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{\frac32(\ell+1)}(z)}\,X(A)\Big)^2\bigg)^p\\
&\le\Big(\frac{\ell+1}{\sqrt d}\Big)^d\bigg(\sum_{z\in\frac{\ell+1}{\sqrt d}\mathbb{Z}^d}\fint_{z+\frac{\ell+1}{\sqrt d}Q}\Big(\,\partial^{\mathrm{osc}}_{A,B_{2(\ell+1)}(x)}\,X(A)\Big)^2dx\bigg)^p\\
&\le\Big(\frac{\sqrt d}{\ell+1}\Big)^{d(p-1)}\bigg(\int_{\mathbb{R}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{2(\ell+1)}(x)}\,X(A)\Big)^2dx\bigg)^p.\end{aligned}\quad(4.15)$$

Combined with the above, this yields

$$\begin{aligned}\mathbb{E}\left[X(A)^{2p}\right]\le(Cp^2)^p\,\mathbb{E}\bigg[\bigg(\int_0^\infty\int_{\mathbb{R}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\,X(A)\Big)^2dx\,(\ell+1)^{-d}\pi(\ell)d\ell\bigg)^p\bigg]&\\
+(Cp^2)^p\,\mathbb{E}\bigg[\int_0^\infty\bigg(\int_{\mathbb{R}^d}\Big(\,\partial^{\mathrm{osc}}_{A,B_{2(\ell+1)}(x)}\,X(A)\Big)^2dx\bigg)^p(\ell+1)^{-dp}\pi(\ell)d\ell\bigg]&.\end{aligned}$$

Since $\int_0^\infty \pi(\ell)d\ell < \infty$, the first right-hand side term can be absorbed into the second right-hand side term. Indeed, the triangle inequality and the Hölder inequality with exponents $(p, \frac{p}{p-1})$ combine to

$$\begin{aligned}
&\mathbb{E}\bigg[\Big(\int_0^\infty \int_{\mathbb{R}^d} \Big(\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\, X(A)\Big)^2 dx\,(\ell+1)^{-d}\pi(\ell)d\ell\Big)^p\bigg] \\
&\le \bigg(\int_0^\infty \mathbb{E}\bigg[\Big(\int_{\mathbb{R}^d} \Big(\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\, X(A)\Big)^2 dx\Big)^p\bigg]^{\frac{1}{p}} (\ell+1)^{-d}\pi(\ell)d\ell\bigg)^p \\
&= \bigg(\int_0^\infty \mathbb{E}\bigg[\Big(\int_{\mathbb{R}^d} \Big(\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\, X(A)\Big)^2 dx\Big)^p (\ell+1)^{-dp}\pi(\ell)\bigg]^{\frac{1}{p}} \pi(\ell)^{1-\frac{1}{p}}d\ell\bigg)^p \\
&\le \bigg(\int_0^\infty \pi(\ell)d\ell\bigg)^{p-1} \mathbb{E}\bigg[\int_0^\infty \Big(\int_{\mathbb{R}^d} \Big(\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\, X(A)\Big)^2 dx\Big)^p (\ell+1)^{-dp}\pi(\ell)d\ell\bigg],
\end{aligned}$$

and the conclusion of item (ii) follows.

Step 2. Improvement of (i) for ($\tilde\partial$-WLSI).

In this step, we argue that the prefactor $(Cp^2)^p$ in item (i) can be upgraded to $(Cp)^p$ if the field A satisfies the corresponding logarithmic Sobolev inequality ($\tilde\partial$-WLSI). Starting point is the following observation (see [7, Theorem 3.4] and [41, Proposition 5.4.2]): if the random variable $X(A)$ satisfies $\mathrm{Ent}\big[X(A)^{2p}\big] < \infty$, then we have

$$\mathbb{E}\big[X(A)^{2p}\big]^{\frac{1}{p}} - \mathbb{E}\big[X(A)^2\big] = \int_1^p \frac{1}{q^2}\mathbb{E}\big[X(A)^{2q}\big]^{\frac{1}{q}-1}\,\mathrm{Ent}\big[X(A)^{2q}\big]\,dq. \tag{4.16}$$

It remains to estimate the entropy $\mathrm{Ent}\big[X(A)^{2q}\big]$ for all $1 \le q \le p$. Applied to the $\sigma(A)$-measurable random variable $|X(A)|^q$, ($\tilde\partial$-WLSI) yields

$$\mathrm{Ent}\big[X(A)^{2q}\big] \le \mathbb{E}\bigg[\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde\partial_{A,B_{\ell+1}(x)}|X(A)|^q\Big)^2 dx\,(\ell+1)^{-d}\pi(\ell)d\ell\bigg].$$

For the choice $\tilde\partial = \partial^{\mathrm{G}}$ or ∂^{fct}, the argument of Substeps 1.1–1.2, cf. (4.12), applied to the above right-hand side yields

$$\mathrm{Ent}\big[X(A)^{2q}\big] \le Cq^2\,\mathbb{E}\big[X(A)^{2q}\big]^{1-\frac{1}{q}}\,\mathbb{E}\bigg[\Big(\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde\partial_{A,B_{\ell+1}(x)}X(A)\Big)^2 dx\,(\ell+1)^{-d}\pi(\ell)d\ell\Big)^q\bigg]^{\frac{1}{q}}.$$

Inserting this into (4.16), we obtain

$$\mathbb{E}\big[X(A)^{2p}\big]^{\frac{1}{p}} \le \mathbb{E}\big[X(A)^2\big] + C\int_1^p \mathbb{E}\bigg[\Big(\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde\partial_{A,B_{\ell+1}(x)}X(A)\Big)^2 dx\,(\ell+1)^{-d}\pi(\ell)d\ell\Big)^q\bigg]^{\frac{1}{q}} dq.$$

We then appeal to the spectral gap ($\tilde\partial$-WSG) (which follows from ($\tilde\partial$-WLSI)) to estimate the first right-hand side term, and use Jensen's inequality on the second right-hand side to obtain

$$\mathbb{E}\big[X(A)^{2p}\big]^{\frac{1}{p}} \le Cp\,\mathbb{E}\bigg[\Big(\int_0^\infty \int_{\mathbb{R}^d} \Big(\tilde\partial_{A,B_{\ell+1}(x)}X(A)\Big)^2 dx\,(\ell+1)^{-d}\pi(\ell)d\ell\Big)^p\bigg]^{\frac{1}{p}}.$$

This upgrades the prefactor in item (i) to $(Cp)^p$, as claimed. $\square$

4.3.2 Concentration properties

The following results establish concentration properties implied by weighted functional inequalities, and extend the known results for standard functional inequalities. Again, these properties depend crucially on the choice of the derivative. On the one hand, spectral gaps for the Glauber and functional derivatives imply exponential tail concentration, and the corresponding logarithmic Sobolev inequalities imply stronger Gaussian tail concentration. On the other hand, for other choices of the derivative, the failure of the Leibniz rule in general only yields weaker results (except when the weight has compact support or when additional properties are assumed on the random variable, cf. Propositions 4.3.3(i) and 4.7.3(iii) below). Most of the following results are direct consequences of the p-versions of Proposition 4.3.1. We start with the concentration properties for the Glauber and functional derivatives.

Proposition 4.3.2. *Assume that the random field A satisfies ($\tilde\partial$-WSG) with integrable weight $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ and derivative $\tilde\partial = \partial^{\mathrm{G}}$ or ∂^{fct}. We define the Lipschitz norm of a $\sigma(A)$-measurable random variable $X(A)$ with respect to the derivative $\tilde\partial$ and the weight π as*

$$\|\!| X |\!\|_{\tilde\partial,\pi} := \sup_A \mathrm{ess} \left(\int_0^\infty \int_{\mathbb{R}^d} \left(\tilde\partial_{A,B_{\ell+1}(x)} X(A) \right)^2 dx \, (\ell+1)^{-d} \pi(\ell) d\ell \right)^{\frac12}.$$

Then there exists a constant $C>0$ depending only on d and π such for all $\sigma(A)$-measurable random variables $X(A)$ with $\|\!| X |\!\|_{\tilde\partial,\pi} \le 1$ we have exponential tail concentration in the form

$$\mathbb{E}\left[\exp\left(\frac{1}{C}|X(A) - \mathbb{E}[X(A)]\,|\right)\right] \le 2,$$
$$\mathbb{P}[X(A) - \mathbb{E}[X(A)] \ge r] \le e^{-\frac{r}{C}}, \qquad \textit{for all } r \ge 0.$$

If in addition A satisfies ($\tilde\partial$-WLSI) with weight π, then for all $\sigma(A)$-measurable random variables $X(A)$ with $\|\!| X |\!\|_{\tilde\partial,\pi} \le 1$ we have Gaussian tail concentration in the form

$$\mathbb{E}\left[\exp\left(\frac{1}{C}\big(X(A) - \mathbb{E}[X(A)]\,\big)^2\right)\right] \le 2,$$
$$\mathbb{P}[X(A) - \mathbb{E}[X(A)] \ge r] \le e^{-\frac{r^2 \vee r}{C}}, \qquad \textit{for all } r \ge 0. \qquad \lozenge$$

We now turn to the case of the oscillation, which yields in general weaker concentration results due to the failure of the Leibniz rule.

Proposition 4.3.3.

(i) Assume that the random field A satisfies (∂^{osc}-SG) with radius $R>0$. Then for all $\sigma(A)$-measurable random variables $X(A)$ that satisfy

$$\|\!| X |\!\|_{\partial^{\mathrm{osc}},R} := \sup_A \mathrm{ess} \int_{\mathbb{R}^d} \left(\partial^{\mathrm{osc}}_{A,B_R(x)} X(A) \right)^2 dx \le 1,$$

we have exponential tail concentration in the form

$$\mathbb{E}\left[\exp\left(\frac{1}{C}|X(A) - \mathbb{E}[X(A)]\,|\right)\right] \le 2,$$
$$\mathbb{P}[X(A) - \mathbb{E}[X(A)] \ge r] \le e^{-\frac{r}{C}}, \qquad \textit{for all } r \ge 0.$$

If in addition A satisfies (∂^{osc}-LSI) with radius $R>0$ and if the random variable $X(A)$ further satisfies

$$L := \sup_x \sup_A \mathrm{ess}\, \partial^{\mathrm{osc}}_{A,B_R(x)} X(A) < \infty,$$

we have Poisson tail concentration in the form

$$\mathbb{E}\left[\exp\left(\frac{1}{C}\psi_L\big(|X(A)-\mathbb{E}\left[X(A)\right]|\big)\right)\right]\leq 2,\qquad \psi_L(u):=\frac{u}{L}\log\Big(1+\frac{Lu}{C}\Big),$$
$$\mathbb{P}\left[X(A)-\mathbb{E}\left[X(A)\right]\geq r\right]\leq e^{-\frac{1}{C}\psi_L(r)},\qquad \textit{for all } r\geq 0.$$

(ii) Assume that the random field A satisfies (∂^{osc}-WSG) with integrable weight $\pi:\mathbb{R}_+\to\mathbb{R}_+$. Let $X(A)$ be a $\sigma(A)$-measurable random variable, and assume that, for some $\kappa>0$, $p_0,\alpha\geq 0$, we have for all $p\geq p_0$,

$$\mathbb{E}\left[\int_0^\infty\left(\int_{\mathbb{R}^d}\Big(\partial^{\mathrm{osc}}_{A,B_{\ell+1}(x)}\,X(A)\Big)^2dx\right)^p(\ell+1)^{-dp}\pi(\ell)d\ell\right]\;\leq\;p^{\alpha p}\kappa. \tag{4.17}$$

Then there exists a constant $C>0$ depending only on d, π, p_0, and α (but not on κ) such that we have concentration in the form

$$\mathbb{E}\left[\psi_{p_0,\alpha}\Big(\frac{1}{C}|X(A)-\mathbb{E}\left[X(A)\right]|\Big)\right]\;\leq\;C\kappa,\qquad \psi_{p_0,\alpha}(u):=(1\wedge r^{2p_0})\exp(r^{\frac{2}{2+\alpha}}),$$
$$\mathbb{P}\left[\,|X(A)-\mathbb{E}\left[X(A)\right]|\geq r\right]\leq C\kappa\Big(\psi_{p_0,\alpha}\big(\tfrac{r}{C}\big)\Big)^{-1},\qquad \textit{for all } r\geq 0. \qquad \Diamond$$

Remark 4.3.4. Comments are in order.

- For spatial averages of (possibly nonlinear approximately local transformations of) the random field A, one can prove much stronger concentration results using the specific structure of averages, cf. Proposition 4.7.3(iii) below.
- Proposition 4.3.3(ii) above is used in two contexts. When the weight π is algebraic, the decay in (4.17) is typically independent of p (that is, $\alpha=0$, and κ is not to the power p so that it cannot be absorbed by rescaling of X), in which case κ is the driving quantity (see e.g. the application in Proposition 4.7.3(ii)). When the weight is super-algebraic, there can be an interplay between the decay of the weight and the power p, and an optimization may allow to put part of the decay to the power p at the price of losing some power of p itself — which leads to (4.17) for some $\alpha>0$ (after rescaling of X). $\Diamond$

We start with the proof of Proposition 4.3.2.

Proof of Proposition 4.3.2. If A satisfies ($\tilde\partial$-WSG) for $\tilde\partial=\partial^{\mathrm{G}}$ or ∂^{fct}, the assumption $|\!|\!|\,X\,|\!|\!|_{\tilde\partial,\pi}\leq 1$ allows to apply Proposition 4.3.1(i) in the form

$$\mathbb{E}\left[\big(X(A)-\mathbb{E}\left[X(A)\right]\big)^{2p}\right]\leq(Cp^2)^p, \tag{4.18}$$

for all $p\geq 1$. Summing this estimate over p, and recalling that $n^n\leq e^n n!$, the exponential concentration result (i) follows in the form

$$\mathbb{E}\left[\exp\Big(\frac{1}{C}|X(A)-\mathbb{E}\left[X(A)\right]|\Big)\right]\leq 2,$$

and hence by Markov's inequality, for all $r\geq 0$,

$$\mathbb{P}\left[\big|X(A)-\mathbb{E}\left[X(A)\right]\big|\geq r\right]\leq 2e^{-\frac{r}{C}}.$$

The stronger unilateral estimate without the factor 2 is obtained by a standard application of Herbst-type techniques as in [66, Section 4] (see also [292, Section 2.5]).

If A further satisfies ($\tilde{\partial}$-WLSI) for $\tilde{\partial} = \partial^{\mathrm{G}}$ or ∂^{fct}, Proposition 4.3.1(i) asserts that the right-hand side in (4.18) is replaced by $(Cp)^p$, which yields after summation the corresponding Gaussian concentration result (ii) in the form

$$\mathbb{E}\left[\exp\left(\frac{1}{C}\big(X(A)-\mathbb{E}\left[X(A)\right]\big)^2\right)\right] \le 2,$$

and hence by Markov's inequality, for all $r \ge 0$,

$$\mathbb{P}\left[\big|X(A)-\mathbb{E}\left[X(A)\right]\big| \ge r\right] \le 2e^{-\frac{r^2}{C}}.$$

The stronger unilateral estimate without the factor 2 is obtained by a standard application of Herbst's argument as e.g. in [293, Section 5.1]. $\square$

We now turn to the proof of Proposition 4.3.3.

Proof of Proposition 4.3.3. We split the proof into two steps, and prove (i) and (ii) separately.

Step 1. Proof of (i).

The exponential concentration result in (i) follows from Proposition 4.3.1(ii) (with compactly supported weight π) as in the proof of Proposition 4.3.2 above. Let us now turn to the Poisson concentration result; although it could similarly be proven by first deriving suitable moment bounds, the proof is more transparent using a variation of Herbst's argument. Let A satisfy (∂^{osc}-LSI) and let $X(A)$ satisfy $L := \sup_x \operatorname{sup\,ess}_A \partial^{\mathrm{osc}}_{A,B_R(x)} X(A) < \infty$ and $|\!|\!|\, X \,|\!|\!|_{\partial^{\mathrm{osc}},R} \le 1$. For all $t \in \mathbb{R}$, we apply (∂^{osc}-LSI) to the $\sigma(A)$-measurable random variable $e^{tX(A)/2}$,

$$\operatorname{Ent}\big[e^{tX(A)}\big] \le C\,\mathbb{E}\left[\int_{\mathbb{R}^d}\Big(\partial^{\mathrm{osc}}_{A,B_R(x)}\, e^{tX(A)/2}\Big)^2 dx\right]. \tag{4.19}$$

By the inequality $|e^a - e^b| \le (e^a + e^b)|a-b|$ for all $a, b \in \mathbb{R}$, the integrand turns into

$$\Big(\partial^{\mathrm{osc}}_{A,S}\, e^{tX(A)/2}\Big)^2 \le 2t^2 \sup_{A,S} e^{tX(A)}\Big(\partial^{\mathrm{osc}}_{A,S}\, X(A)\Big)^2 \le 2t^2 e^{tX(A)} \exp\Big(t\,\partial^{\mathrm{osc}}_{A,S}\, X(A)\Big)\Big(\partial^{\mathrm{osc}}_{A,S}\, X(A)\Big)^2. \tag{4.20}$$

Inserting this inequality into (4.19) and using the assumptions on $X(A)$, we obtain

$$\operatorname{Ent}\big[e^{tX(A)}\big] \le Ct^2 e^{tL}\,\mathbb{E}\big[e^{tX(A)}\big].$$

Compared to the standard Herbst argument, we have to deal here with the additional exponential factor e^{tL}. We may then appeal to [292, Corollary 2.12] which indeed yields the desired Poisson concentration. We include a proof for the reader's convenience. In terms of the Laplace transform $H(t) = \mathbb{E}\big[e^{tX(A)}\big]$, the above takes the form

$$tH'(t) - H(t)\log H(t) \le Ct^2 e^{tL} H(t),$$

or equivalently,

$$\frac{d}{dt}\Big(\frac{1}{t}\log H(t)\Big) \le Ce^{tL},$$

and hence by integration

$$H(t) \le \exp\Big(\frac{Ct}{L}(e^{tL}-1) + t\frac{H'(0)}{H(0)}\Big) = e^{\frac{Ct}{L}(e^{tL}-1)+t\mathbb{E}[X(A)]}.$$

The Markov inequality then implies for all $r,t\geq 0$,

$$\mathbb{P}\left[X(A)\geq\mathbb{E}\left[X(A)\right]+r\right]\;=\;\mathbb{P}\Big[e^{tX(A)}\geq e^{t\mathbb{E}[X(A)]+tr}\Big]\;\leq\;e^{-t\mathbb{E}[X(A)]-tr}\mathbb{E}\big[e^{tX(A)}\big]\;\leq\;e^{\frac{Ct}{L}(e^{tL}-1)-tr}. \tag{4.21}$$

Let $r\geq 0$ be momentarily fixed, and denote by $t_*\geq 0$ the value of $t\geq 0$ that minimizes $f_r(t):=\frac{Ct}{L}(e^{tL}-1)-tr$, that is the (unique) solution $t_*\geq 0$ of the equation

$$Ce^{t_*L}=(Lr+C)/(1+t_*L) \tag{4.22}$$

(note that f_r is strictly convex, $f_r(0)=0$, and $f_r'(0)\leq 0$). We now give two estimates on $f_r(t_*)$ depending on the value of r. Assume first that $r\geq\frac{2eC}{L}$. We may then compute

$$f_r(t_*):=\frac{Ct_*}{L}(e^{t_*L}-1)-t_*r\overset{(4.22)}{=}-\frac{t_*^2(Lr+C)}{1+t_*L}.$$

Using the bound $2t_*L\geq t_*L+\log(1+t_*L)\overset{(4.22)}{=}\log(1+Lr/C)$, and the fact that $t\mapsto-\frac{t^2(Lr+C)}{1+tL}$ is decreasing on $\mathbb{R}_+$, we obtain

$$f_r(t_*)\leq-\frac{Lr+C}{2L^2}\frac{\log(1+Lr/C)^2}{2+\log(1+Lr/C)}.$$

Hence, for $r\geq\frac{2eC}{L}$, we obtain using in addition $\log(1+Lr/C)\geq\log(1+2e)>9/5$,

$$f_r(t_*)\leq-\frac{r}{5L}\log\Big(1+\frac{Lr}{C}\Big). \tag{4.23}$$

We now turn to the case $0\leq r\leq\frac{2eC}{L}$. Comparing the minimal value $f_r(t_*)$ to the choice $t=\frac{r}{2eC}$, and using the bound $e^a-1\leq ea$ for $a\in[0,1]$, we obtain for all $r\leq\frac{2eC}{L}$,

$$f_r(t_*)\leq f_r\Big(\frac{r}{2eC}\Big)=\frac{r}{2eL}\big(e^{\frac{rL}{2eC}}-1\big)-\frac{r^2}{2eC}\leq-\frac{r^2}{4eC},$$

which yields, using that $\log(1+a)\leq a$ for all $a\geq 0$,

$$f_r(t_*)\leq-\frac{r}{4eL}\log\Big(1+\frac{Lr}{C}\Big)\leq-\frac{r}{11L}\log\Big(1+\frac{Lr}{C}\Big).$$

Combining this with (4.21) and (4.23), we conclude

$$\mathbb{P}\left[X(A)\geq\mathbb{E}\left[X(A)\right]+r\right]\leq e^{-\frac{r}{11L}\log(1+\frac{Lr}{C})},$$

and the corresponding integrability result follows by integration.

Step 2. Proof of (ii).

Let A satisfy (∂^{osc}-WSG) with weight π, and let the random variable $X(A)$ satisfy (4.17) for some $\kappa>0$, $p_0,\alpha\geq 0$. Proposition 4.3.1(ii) then yields for all $p\geq p_0$,

$$\mathbb{E}\left[\big(X(A)-\mathbb{E}\left[X(A)\right]\big)^{2p}\right]\leq C^pp^{(2+\alpha)p}\kappa,$$

or alternatively, for all $p\geq(2+\alpha)p_0$,

$$\mathbb{E}\left[\Big(\big|X(A)-\mathbb{E}\left[X(A)\right]\big|^{\frac{2}{2+\alpha}}\Big)^p\right]\leq C^pp!\,\kappa.$$

Summing this estimate over p, we obtain

$$\mathbb{E}\left[\tilde\psi_{p_0,\alpha}\Big(\frac{1}{C}|X(A)-\mathbb{E}\left[X(A)\right]|^{\frac{2}{2+\alpha}}\Big)\right]\;\leq\;\kappa,$$

where we have set $\tilde\psi_{p_0,\alpha}(u):=\sum_{n=0}^\infty\frac{u^{n+(2+\alpha)p_0}}{(n+(2+\alpha)p_0)!}$. Noting that $\tilde\psi_{p_0,\alpha}(u)\simeq_{p_0,\alpha}(1\wedge u)^{(2+\alpha)p_0}e^u$ holds for all $u\geq 0$, the conclusion follows. $\square$

4.4 Constructive approach to weighted functional inequalities

In this section we consider random fields that can be constructed as transformations of product structures. Under suitable assumptions we describe how the standard spectral gaps, covariance inequalities, and logarithmic Sobolev inequalities satisfied by "hidden product structures" are deformed into weighted functional inequalities for the random fields of interest. The analysis of the examples mentioned in the introduction is postponed to Section 4.5.

4.4.1 Transformation of product structures

Let the random field A on $\mathbb{R}^d$ be $\sigma(\mathcal{X})$-measurable for some random field $\mathcal{X}$ defined on some measure space X and with values in some measurable space M. Assume that we have a partition $X = \biguplus_{x\in\mathbb{Z}^d,t\in\mathbb{Z}^l} X_{x,t}$, on which $\mathcal{X}$ is *completely independent*, that is, the family of restrictions $(\mathcal{X}|_{X_{x,t}})_{x\in\mathbb{Z}^d,t\in\mathbb{Z}^l}$ are all independent.

In the sequel, the case $l = 0$ (that is, the case when there is no variable t) is referred to as the non-projective case, while the case $l \geq 1$ is referred to as the projective case. Note however that the non-projective case is a particular case of the projective one, simply defining $X_{x,0} = X_x$ and $X_{x,t} = \varnothing$ for all $t \neq 0$. The random field $\mathcal{X}$ can be e.g. a random field on $\mathbb{R}^d \times \mathbb{R}^l$ with values in some measure space (choosing $X = \mathbb{R}^d \times \mathbb{R}^l$, $X_{x,t} = Q^d(x) \times Q^l(t)$, and M the space of values), or a random point process (or more generally a random measure) on $\mathbb{R}^d \times \mathbb{R}^l \times X'$ for some measure space X' (choosing $X = \mathbb{Z}^d \times \mathbb{Z}^l \times X'$, $X_{x,t} = \{x\} \times \{t\} \times X'$, and M the space of measures on $Q^d \times Q^l \times X'$).

Let $\mathcal{X}'$ be some given i.i.d. copy of $\mathcal{X}$. For all x, t, we define a perturbed random field $\mathcal{X}^{x,t}$ by setting $\mathcal{X}^{x,t}|_{X\setminus X_{x,t}} = \mathcal{X}|_{X\setminus X_{x,t}}$ and $\mathcal{X}^{x,t}|_{X_{x,t}} = \mathcal{X}'|_{X_{x,t}}$. By complete independence, the random fields $\mathcal{X}$ and $\mathcal{X}^{x,t}$ (resp. $A = A(\mathcal{X})$ and $A(\mathcal{X}^{x,t})$) have the same law. Arguing as in the proof of Proposition 4.1.2 (cf. (4.106) and (4.107) in Appendix 4.A), the complete independence assumption ensures that $\mathcal{X}$ satisfies the following standard functional inequalities.

Proposition 4.4.1. *For all $\sigma(\mathcal{X})$-measurable random variables $Y(\mathcal{X})$ and $Z(\mathcal{X})$, we have*

$$\mathrm{Var}\left[Y(\mathcal{X})\right] \leq \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{Z}^l} \mathbb{E}\left[\big(Y(\mathcal{X}) - Y(\mathcal{X}^{x,t})\big)^2\right], \tag{4.24}$$

$$\mathrm{Ent}[Y(\mathcal{X})] \leq 2 \sum_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{Z}^l} \mathbb{E}\left[\sup_{\mathcal{X}'} \mathrm{ess}\, \big(Y(\mathcal{X}) - Y(\mathcal{X}^{x,t})\big)^2\right], \tag{4.25}$$

$$\mathrm{Cov}\left[Y(\mathcal{X}); Z(\mathcal{X})\right] \leq \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{Z}^l} \mathbb{E}\left[\big(Y(\mathcal{X}) - Y(\mathcal{X}^{x,t})\big)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\big(Z(\mathcal{X}) - Z(\mathcal{X}^{x,t})\big)^2\right]^{\frac{1}{2}}. \tag{4.26}$$

$\Diamond$

4.4.2 Abstract criteria and action radius

We now describe general situations for which the functional inequalities for the hidden product structure $\mathcal{X}$ are deformed into weighted inequalities for the random field A. We distinguish the following two cases:

- *deterministic localization*, that is, when the random field A is a deterministic convolution of some product structure, so that the dependence pattern is prescribed deterministically a priori; it leads to weighted functional inequalities with the functional derivative ∂^{fct};
- *random localization*, that is, when the dependence pattern is encoded by the underlying product structure $\mathcal{X}$ itself (and therefore may depend on the realization, whence the terminology "random"); the localization of the dependence pattern is then measured in terms of what we call the *action radius*; it leads to weighted inequalities with the derivatives ∂^{osc} and ∂^{dis}, and generalizes the idea of local transformations of Proposition 4.1.2.

The case of deterministic localization essentially concerns Gaussian fields, which have been thoroughly studied in the literature. Weighted functional inequalities for such random fields then follow from standard functional inequalities (typically formulated in terms of Malliavin calculus on Wiener space, see e.g. [248, 258, 348]) combined with a *deterministic* radial change of variables to reformulate the right-hand side (extracting a 1D weight from Hilbert norms encoding the covariance structure, cf. the proof of Theorem 4.B.2 below). The right-hand side of weighted functional inequalities is indeed more explicit (and flexible when it turns to estimates — see e.g. bounds by duality in [203]). A self-contained approach to deterministic localization is included in Appendix 4.B.

In the rest of this section we focus on the more original setting of *random* localization (which involves a *random* change of variable, due to the randomness of the dependence pattern). More precisely, we introduce the notion of *action radius* as a probabilistic measure of the localization of the dependence pattern. General criteria for weighted spectral gaps are then obtained in terms of the properties of this action radius. Various examples that are included in this framework are described in Section 4.5 below.

We use the same notation as above: A is a $\sigma(\mathcal{X})$-measurable random field on $\mathbb{R}^d$, where $\mathcal{X}$ is a completely independent random field on some measure space $X = \biguplus_{x\in\mathbb{Z}^d, t\in\mathbb{Z}^l} X_{x,t}$ with values in some measurable space M. The following definition is inspired by the notion of stabilization radius first introduced by Lee [294, 295] and crucially used in the works by Penrose, Schreiber, and Yukich on random sequential adsorption processes [359, 358, 360, 391] (see also [286]).

Definition 4.4.2. Given an i.i.d. copy $\mathcal{X}'$ of the field $\mathcal{X}$, an *action radius for A with respect to $\mathcal{X}$ on $X_{x,t}$* (with reference perturbation $\mathcal{X}'$), if it exists, is defined as a nonnegative $\sigma(\mathcal{X},\mathcal{X}')$-measurable random variable ρ such that we have a.s.,

$$A(\mathcal{X}^{x,t})\big|_{\mathbb{R}^d\setminus(Q(x)+B_\rho)} = A(\mathcal{X})|_{\mathbb{R}^d\setminus(Q(x)+B_\rho)}\,,$$

where the perturbed random field $\mathcal{X}^{x,t}$ is defined by $\mathcal{X}^{x,t}|_{X\setminus X_{x,t}} := \mathcal{X}|_{X\setminus X_{x,t}}$ and $\mathcal{X}^{x,t}|_{X_{x,t}} := \mathcal{X}'|_{X_{x,t}}$. $\diamondsuit$

Note that if $\mathcal{X} = A_0$ is a random field on $\mathbb{R}^d$, and if for some $R > 0$ the random field A is an R-local transformation of A_0 in the sense of Proposition 4.1.2, then the constant $\rho = R$ is an action radius for A with respect to A_0 on any set. Reinterpreted in the case when $\mathcal{X} = \mathcal{P}$ is a random point process on $\mathbb{R}^d\times\mathbb{R}^l\times X'$ for some measure space X', the above definition takes on the following guise: given a subset $E\times F\subset\mathbb{R}^d\times\mathbb{R}^l$ and given an i.i.d. copy $\mathcal{P}'$ of $\mathcal{P}$, an *action radius for A with respect to $\mathcal{P}$ on $E\times F$*, if it exists, is a nonnegative random variable ρ such that we have a.s.,

$$A\Big(\big(\mathcal{P}\setminus(E\times F\times X')\big)\bigcup\big(\mathcal{P}'\cap(E\times F\times X')\big)\Big)\Big|_{\mathbb{R}^d\setminus(E+B_\rho)} = A(\mathcal{P})|_{\mathbb{R}^d\setminus(E+B_\rho)}\,.$$

We display two general results, Theorems 4.4.3 and 4.4.5 below. The first result is a general criterion for the validity of weighted spectral gaps in terms of the properties of an action radius, whereas the second result is based on more elaborate properties of action radii and is useful to avoid loss of integrability in some situations. Note that the condition for the validity of the weighted logarithmic Sobolev inequality below is rather stringent (see Section 4.5 for examples).

Theorem 4.4.3. *Let the fields $A,\mathcal{X}$ be as above. Given an i.i.d. copy $\mathcal{X}'$ of the field $\mathcal{X}$, assume that:*

(a) For all x,t, there exists an action radius $\rho_{x,t}$ for A with respect to $\mathcal{X}$ in $X_{x,t}$.

(b) The transformation A of $\mathcal{X}$ is stationary, that is, the random fields $A(\mathcal{X}(\cdot+z,\cdot))$ and $A(\mathcal{X})(\cdot+z)$ have the same law for all $z\in\mathbb{Z}^d$. Moreover, the law of the action radius $\rho_{x,t}$ is independent of x.

Then the following holds.

(i) Setting

$$\pi(t,\ell) := \mathbb{P}\left[\ell - 1 \le \rho_{0,t} < \ell,\ A(\mathcal{X}^{0,t}) \neq A(\mathcal{X})\right],$$

we have for all $\sigma(A)$-measurable random variable $Z(A)$ and all $\lambda \in (0,1)$,

$$\mathrm{Var}\left[Z(A)\right] \le \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{\ell=1}^{\infty} \sum_{t\in\mathbb{Z}^l} \pi(t,\ell)^{\lambda}\, \mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell,x,t} Z(A)\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda} \tag{4.27}$$

and

$$\mathrm{Cov}\left[Y(A); Z(A)\right] \le \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{Z}^l} \left(\sum_{\ell=1}^{\infty} \pi(t,\ell)^{\lambda}\, \mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell,x,t} Y(A)\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda} \right)^{\frac{1}{2}} \times \left(\sum_{\ell'=1}^{\infty} \pi(t,\ell')^{\lambda}\, \mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell',x,t} Z(A)\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda} \right)^{\frac{1}{2}}, \tag{4.28}$$

where $\partial^{\mathrm{dis}}_{\ell,x,t} Z(A)$ is the notation defined in (4.7)*, that is,*

$$\partial^{\mathrm{dis}}_{\ell,x,t} Z(A) := \big(Z(A) - Z(A(\mathcal{X}^{x,t}))\big) \mathbb{1}_{A|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)} = A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}}.$$

In particular, for all $\lambda \in (0,1)$, if we set

$$\pi_\lambda(\ell) := (\ell+1)^d \sum_{t\in\mathbb{Z}^l} \mathbb{P}\left[\ell - 1 \le \rho_{0,t} < \ell,\ A(\mathcal{X}^{0,t}) \neq A(\mathcal{X})\right]^{\lambda},$$

we obtain for all $\sigma(A)$-measurable random variables $Z(A)$,

$$\mathrm{Var}\left[Z(A)\right] \le \frac{1}{2} \sum_{\ell=1}^{\infty} (\ell+1)^{-d} \pi_\lambda(\ell) \sum_{x\in\mathbb{Z}^d} \mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A,Q_{2\ell+1}(x)}\, Z(A)\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda}. \tag{4.29}$$

If in addition the random variable $\rho_{x,t}$ is $\sigma(\mathcal{X})$-measurable for all x,t, then we have

$$\mathrm{Ent}[Z(A)] \le 2 \sum_{\ell=1}^{\infty} (\ell+1)^{-d} \pi_\lambda(\ell) \sum_{x\in\mathbb{Z}^d} \mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A,Q_{2\ell+1}(x)}\, Z(A)\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda}. \tag{4.30}$$

(ii) Assume that for all x,t the action radius $\rho_{x,t}$ is independent of $A|_{\mathbb{R}^d\setminus(Q(x)+B_{f(\rho_{x,t})})}$ for some influence function *$f:\mathbb{R}_+\to\mathbb{R}_+$ with $f(u)\ge u$ for all u. Then, with the convention $0/0=0$, if we set*

$$\tilde{\pi}(t,\ell) := \mathbb{P}\left[\mathcal{X}^{0,t} \neq \mathcal{X}\right] \frac{\mathbb{P}\left[\ell - 1 \le \rho_{0,t} < \ell \,\|\, \mathcal{X}^{0,t} \neq \mathcal{X}\right]}{\mathbb{P}\left[\rho_{0,t} < \ell\right]}, \qquad \pi(\ell) := (\ell+1)^d \sum_{t\in\mathbb{Z}^l} \tilde{\pi}(t,\ell),$$

we have for all $\sigma(A)$-measurable random variables $Z(A)$,

$$\mathrm{Var}\left[Z(A)\right] \le \frac{1}{2} \sum_{\ell=1}^{\infty} (\ell+1)^{-d} \pi(\ell) \sum_{x\in\mathbb{Z}^d} \mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\, Z(A)\right)^2\right] \tag{4.31}$$

and

$$\mathrm{Cov}\left[Y(A); Z(A)\right] \le \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{t\in\mathbb{Z}^l} \left(\sum_{\ell=1}^{\infty} \tilde{\pi}(t,\ell)\, \mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\, Y(A)\right)^2\right] \right)^{\frac{1}{2}} \times \left(\sum_{\ell'=1}^{\infty} \tilde{\pi}(t,\ell')\, \mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell')+1}(x)}\, Z(A)\right)^2\right] \right)^{\frac{1}{2}}. \tag{4.32}$$

If in addition the random variable $\rho_{x,t}$ is $\sigma(\mathcal{X})$-measurable for all x, t, then we have

$$\mathrm{Ent}[Z(A)] \le 2\sum_{\ell=1}^{\infty}(\ell+1)^{-d}\pi(\ell)\sum_{x\in\mathbb{Z}^d}\mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\,Z(A)\Big)^2\right]. \tag{4.33}$$

$\Diamond$

Remark 4.4.4. The covariance inequalities (4.28) and (4.32) are not in the canonical form of Definition 4.1.3. However note that if $\tilde\pi(t,\ell)$ is non-increasing with respect to ℓ then the inequality (4.32) (and likewise for (4.28)) easily leads to

$$\begin{aligned}&\mathrm{Cov}\left[Y(A); Z(A)\right]\\ &\quad\le \sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}(\ell+1)^{-d}\Big(\sum_{\ell'=1}^{\ell}\pi(\ell')\Big)\,\mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\,Y(A)\Big)^2\right]^{\frac12}\mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)}\,Z(A)\Big)^2\right]^{\frac12},\end{aligned}$$

which is now in the correct form, although the weight $\sum_{\ell'=1}^{\ell}\pi(\ell')$ seems to be suboptimal whenever π has algebraic decay. $\Diamond$

We now turn to a more complex situation when the dependence pattern is intricate but sufficiently well controlled in terms of a family of action radii. The aim of the following is to avoid the loss of integrability which would follow from Theorem 4.4.3(i) in the case of the random parking process and of Poisson tessellations.

Theorem 4.4.5. *Let $A = A(\mathcal{X})$ be a $\sigma(\mathcal{X})$-measurable random field on $\mathbb{R}^d$, where $\mathcal{X}$ is a completely independent random field on some measure space $X = \biguplus_{x\in\mathbb{Z}^d} X_x$ with values in some measurable space M. For all $x \in \mathbb{Z}^d$, $\ell \in \mathbb{N}$, set $X_x^\ell := \bigcup_{y\in\mathbb{Z}^d:|x-y|_\infty\le\ell} X_y$. Given an i.i.d. copy $\mathcal{X}'$ of the field $\mathcal{X}$, let the perturbed field $\mathcal{X}^{x,\ell}$ be defined by*

$$\mathcal{X}^{x,\ell}|_{X\setminus X_x^\ell} = \mathcal{X}|_{X\setminus X_x^\ell}, \qquad \text{and} \qquad \mathcal{X}^{x,\ell}|_{X_x^\ell} = \mathcal{X}'|_{X_x^\ell},$$

and assume that:

(a) For all x, ℓ, there exists an action radius ρ_x^ℓ for A with respect to $\mathcal{X}$ in X_x^ℓ, that is, a nonnegative random variable ρ_x^ℓ such that we have a.s.,

$$A(\mathcal{X}^{x,\ell})|_{\mathbb{R}^d\setminus(Q_{2\ell+1}(x)+B_{\rho_x^\ell})} = A(\mathcal{X})|_{\mathbb{R}^d\setminus(Q_{2\ell+1}(x)+B_{\rho_x^\ell})}.$$

(b) The transformation A of $\mathcal{X}$ is stationary, that is, the random fields $A(\mathcal{X}(\cdot+z,\cdot))$ and $A(\mathcal{X})(\cdot+z)$ have the same law for all $z \in \mathbb{Z}^d$. Moreover, the law of the action radius ρ_x^ℓ is independent of x.

Further assume that

(c) For all x, ℓ, the random variable ρ_x^ℓ is $\sigma\Big(\mathcal{X}|_{X_x^{\ell+\rho_x^\ell}\setminus X_x^\ell}\Big)$-measurable.

(In particular, for all x, ℓ, R, given the event $\rho_x^\ell \le R$, the random variables ρ_x^ℓ and $\rho_x^{\ell+R}$ are independent.)

Let $R \ge 1$ be chosen large enough so that

$$\sup_{\ell\ge R}\mathbb{P}\big[\rho_x^\ell \ge \ell\big] \le \frac14, \tag{4.34}$$

let $\pi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function such that $\mathbb{P}\left[\ell/4 \le \rho_x^{\ell_0} < \ell\right] \le \pi_0(\ell)$ holds for all $0 \le \ell_0 \le \ell/4$, and define the weight

$$\pi(\ell) := (\ell+1)^d\begin{cases}1, & \text{if } \ell \le 4R;\\ 8\ell^{-1}\pi_0(\ell/4), & \text{if } \ell > 4R.\end{cases}$$

Then for all $\sigma(A)$-measurable random variables $Y(A), Z(A)$, we have

$$\operatorname{Var}[Z(A)] \;\le\; \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,B_{\sqrt{d}(2\ell+3)}(x)}\, Z(A)\Big)^2\right] dx\,(\ell+1)^{-d}\pi(\ell)d\ell, \tag{4.35}$$

$$\operatorname{Ent}[Z(A)] \;\le\; 2\int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,B_{\sqrt{d}(2\ell+3)}(x)}\, Z(A)\Big)^2\right] dx\,(\ell+1)^{-d}\pi(\ell)d\ell, \tag{4.36}$$

$$\begin{aligned}\operatorname{Cov}[Y(A);Z(A)] \;\le\; \frac{1}{2}\int_{\mathbb{R}^d} &\bigg(\int_0^\infty \mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,B_{\sqrt{d}(2\ell+3)}(x)}\, Y(A)\Big)^2\right](\ell+1)^{-d}\pi(\ell)d\ell\bigg)^{\frac12}\\ &\times\bigg(\int_0^\infty \mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,B_{\sqrt{d}(2\ell+3)}(x)}\, Z(A)\Big)^2\right](\ell+1)^{-d}\pi(\ell)d\ell\bigg)^{\frac12} dx.\end{aligned} \tag{4.37}$$

$\diamondsuit$

We start with the proof of Theorem 4.4.3, and then turn to the proof of Theorem 4.4.5.

Proof of Theorem 4.4.3. Recall that for all x,t the perturbed field $\mathcal{X}^{x,t}$ is defined by $\mathcal{X}^{x,t}|_{X\setminus X_{x,t}} = \mathcal{X}|_{X\setminus X_{x,t}}$ and $\mathcal{X}^{x,t}|_{X_{x,t}} = \mathcal{X}'|_{X_{x,t}}$. By complete independence of $\mathcal{X}$, the fields $\mathcal{X}$ and $\mathcal{X}^{x,t}$ (hence $A = A(\mathcal{X})$ and $A(\mathcal{X}^{x,t})$) have the same law. The strategy of the proof consists in deforming the functional inequalities of Proposition 4.4.1 with respect to the transformation $A(\mathcal{X})$ in terms of the action radius. We split the proof into four steps.

Step 1. Proof of the spectral gap (4.27).

Conditioning the right-hand side of (4.24) with respect to the values of the action radius $\rho_{x,t}$, applying the Hölder inequality, and using the stationarity assumption (b) to recognize the weight $\pi(t,\ell)$, we obtain for all $0<\lambda<1$,

$$\begin{aligned}\operatorname{Var}[Z(A)] &\le \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^l}\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\right]\\ &= \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}\sum_{t\in\mathbb{Z}^l}\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\mathbb{1}_{\ell-1\le\rho_{x,t}<\ell}\right]\\ &\le \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}\sum_{t\in\mathbb{Z}^l}\pi(t,\ell)^{\lambda}\,\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^{\frac{2}{1-\lambda}}\mathbb{1}_{\rho_{x,t}<\ell}\right]^{1-\lambda}.\end{aligned}$$

Noting that the event $\rho_{x,t}<\ell$ entails that $A|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)} = A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}$, the above can be rewritten as follows,

$$\operatorname{Var}[Z(A)] \le \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{\ell=1}^{\infty}\sum_{t\in\mathbb{Z}^l}\pi(t,\ell)^{\lambda}\,\mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t}Z\Big)^{\frac{2}{1-\lambda}}\right]^{1-\lambda},$$

that is, (4.27).

Step 2. Proof of the spectral gap (4.31).

For all x,t, conditioning with respect to the values of $\rho_{x,t}$, we may decompose

$$\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\right] = g_x^1(t)+g_x^2(t), \tag{4.38}$$

$$g_x^1(t) := \sum_{\ell=2}^{\infty}\mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\mathbb{1}_{\ell-1\le\rho_{x,t}<\ell}\right],$$

$$g_x^2(t) := \mathbb{E}\left[\big(Z(A)-Z(A(\mathcal{X}^{x,t}))\big)^2\mathbb{1}_{\rho_{x,t}<1}\right].$$

We first estimate the term $g_x^1(t)$. Recalling that the influence function f satisfies $f(u) \geq u$ for all u, we obtain

$$\begin{aligned}
g_x^1(t) &= \sum_{\ell=2}^{\infty} \mathbb{E}\left[\big(Z(A) - Z(A(\mathcal{X}^{x,t}))\big)^2 \mathbb{1}_{\mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}}} \mathbb{1}_{\ell-1 \leq \rho_{x,t} < \ell}\right] \\
&\leq \sum_{\ell=2}^{\infty} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)} Z(A)\Big)^2 \mathbb{1}_{\mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}}} \mathbb{1}_{\ell-1 \leq \rho_{x,t} < \ell}\right] \\
&= \sum_{\ell=2}^{\infty} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)} Z(A)\Big)^2 \mathbb{1}_{\mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}}} \,\Big\|\, \ell-1 \leq \rho_{x,t} < \ell\right] \mathbb{P}\left[\ell-1 \leq \rho_{x,t} < \ell\right].
\end{aligned}$$

By definition, given $\rho_{x,t} < \ell$, the restriction $A|_{\mathbb{R}^d \setminus Q_{2f(\ell)+1}(x)}$ is independent of $\mathcal{X}|_{X_{x,t}}$ and $\mathcal{X}'|_{X_{x,t}}$. The above thus yields

$$g_x^1(t) \leq \sum_{\ell=2}^{\infty} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)} Z(A)\Big)^2 \,\Big\|\, \ell-1 \leq \rho_{x,t} < \ell\right] \mathbb{P}\left[\ell-1 \leq \rho_{x,t} < \ell,\, \mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}}\right].$$

By assumption in item (ii), the restriction $A|_{\mathbb{R}^d \setminus Q_{2f(\rho_{x,t})+1}(x)}$ is independent of $\rho_{x,t}$, so that we may deduce

$$g_x^1(t) \leq \sum_{\ell=2}^{\infty} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)} Z(A)\Big)^2 \,\Big\|\, \rho_{x,t} < \ell\right] \mathbb{P}\left[\ell-1 \leq \rho_{x,t} < \ell,\, \mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}}\right].$$

To simplify notation, we set for all $\ell \geq 1$,

$$Y_\ell := \Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)} Z(A)\Big)^2.$$

Estimating

$$\mathbb{E}\left[Y_\ell \,\|\, \rho_{x,t} < \ell\right] \leq \frac{\mathbb{E}\left[Y_\ell\right]}{\mathbb{P}\left[\rho_{x,t} < \ell\right]},$$

and using the stationarity assumption (b) for the action radius, we may conclude

$$\begin{aligned}
g_x^1(t) &\leq \sum_{\ell=2}^{\infty} \frac{\mathbb{E}\left[Y_\ell\right]}{\mathbb{P}\left[\rho_{x,t} < \ell\right]} \mathbb{P}\left[\ell-1 \leq \rho_{x,t} < \ell,\, \mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}}\right] \\
&= \sum_{\ell=2}^{\infty} \frac{\mathbb{E}\left[Y_\ell\right]}{\mathbb{P}\left[\rho_{0,t} < \ell\right]} \mathbb{P}\left[\ell-1 \leq \rho_{0,t} < \ell,\, \mathcal{X}|_{X_{0,t}} \neq \mathcal{X}'|_{X_{0,t}}\right] \\
&= \sum_{\ell=2}^{\infty} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(\ell)+1}(x)} Z(A)\Big)^2\right] \mathbb{P}\left[\mathcal{X}|_{X_{0,t}} \neq \mathcal{X}'|_{X_{0,t}}\right] \frac{\mathbb{P}\left[\ell-1 \leq \rho_{0,t} < \ell \,\|\, \mathcal{X}|_{X_{0,t}} \neq \mathcal{X}'|_{X_{0,t}}\right]}{\mathbb{P}\left[\rho_{0,t} < \ell\right]}.
\end{aligned} \tag{4.39}$$

We now turn to the estimate of the term $g_x^2(t)$. Since the influence function f satisfies $f(u) \geq u$ for all u, we find

$$\begin{aligned}
g_x^2(t) &= \mathbb{E}\left[\Big(Z(A) - Z(A(\mathcal{X}^{x,t}))\Big)^2 \mathbb{1}_{\mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}}} \mathbb{1}_{\rho_{x,t} < 1}\right] \\
&\leq \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,Q_{2f(1)+1}(x)} Z(A)\Big)^2 \mathbb{1}_{\mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}}} \,\Big\|\, \rho_{x,t} < 1\right] \mathbb{P}\left[\rho_{x,t} < 1\right].
\end{aligned}$$

By definition, given $\rho_{x,t} < 1$, the restriction $A|_{\mathbb{R}^d \setminus Q_{2f(1)+1}(x)}$ is independent of $\mathcal{X}|_{X_{x,t}}$ and $\mathcal{X}'|_{X_{x,t}}$. The above thus yields

$$
\begin{aligned}
g_x^2(t) &\le \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A, Q_{2f(1)+1}(x)} \, Z(A)\Big)^2\right] \mathbb{P}\left[\mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}} \,\|\, \rho_{x,t} < 1\right] \\
&= \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A, Q_{2f(1)+1}(x)} \, Z(A)\Big)^2\right] \mathbb{P}\left[\mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}}\right] \frac{\mathbb{P}\left[\rho_{x,t} < 1 \,\|\, \mathcal{X}|_{X_{x,t}} \neq \mathcal{X}'|_{X_{x,t}}\right]}{\mathbb{P}\left[\rho_{x,t} < 1\right]}.
\end{aligned}
$$

Using the stationarity assumption (b) again, and combining this with (4.38) and (4.39), the conclusion (4.31) follows.

Step 3. Proof of the logarithmic Sobolev inequalities (4.30) and (4.33).

Conditioning the right-hand side of (4.25) with respect to the values of the action radius $\rho_{x,t}$, we obtain

$$
\begin{aligned}
\mathrm{Ent}[Z(A)] &\le 2 \sum_{x \in \mathbb{Z}^d} \sum_{\ell=1}^{\infty} \sum_{t \in \mathbb{Z}^l} \mathbb{E}\left[\sup_{\mathcal{X}'} \operatorname{ess} \Big(\big(Z(A(\mathcal{X})) - Z(A(\mathcal{X}^{x,t}))\big)^2 \mathbb{1}_{\ell-1 \le \rho_{x,t} < \ell}\Big)\right] \\
&\le 2 \sum_{x \in \mathbb{Z}^d} \sum_{\ell=1}^{\infty} \sum_{t \in \mathbb{Z}^l} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A, Q_{2\ell+1}(x)} \, Z(A)\Big)^2 \sup_{\mathcal{X}'} \operatorname{ess} \big(\mathbb{1}_{\ell-1 \le \rho_{x,t} < \ell}\big)\right].
\end{aligned}
$$

Hence, if for all x, t the random variable $\rho_{x,t}$ is $\sigma(\mathcal{X})$-measurable, we may deduce

$$
\mathrm{Ent}[Z(A)] \le 2 \sum_{x \in \mathbb{Z}^d} \sum_{\ell=1}^{\infty} \sum_{t \in \mathbb{Z}^l} \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A, Q_{2\ell+1}(x)} \, Z(A)\Big)^2 \mathbb{1}_{\ell-1 \le \rho_{x,t} < \ell}\right].
$$

The result (4.30) follows from the Hölder inequality, while the result (4.33) follows as in Step 2.

Step 4. Proof of the covariance inequalities (4.28) and (4.32).

Conditioning the right-hand side of (4.26) with respect to the values of the action radius $\rho_{x,t}$, we obtain

$$
\begin{aligned}
\mathrm{Cov}\left[Y(A); Z(A)\right] \le \frac{1}{2} \sum_{x \in \mathbb{Z}^d} \sum_{t \in \mathbb{Z}^l} &\left(\sum_{\ell=1}^{\infty} \mathbb{E}\left[\big(Y(A) - Y(A(\mathcal{X}^{x,t}))\big)^2 \mathbb{1}_{\ell-1 \le \rho_{x,t} < \ell}\right]\right)^{\frac{1}{2}} \\
&\times \left(\sum_{\ell'=1}^{\infty} \mathbb{E}\left[\big(Z(A) - Z(A(\mathcal{X}^{x,t}))\big)^2 \mathbb{1}_{\ell'-1 \le \rho_{x,t} < \ell'}\right]\right)^{\frac{1}{2}}.
\end{aligned}
$$

Now the sums over ℓ, ℓ' are estimated exactly as in Steps 1 and 2, and the results (4.28) and (4.32) follow. □

We now prove Theorem 4.4.5.

Proof of Theorem 4.4.5. We only prove the spectral gap (4.35). The proof of the logarithmic Sobolev inequality (4.36) and of the covariance inequality (4.37) is similar, based on (4.25) and (4.26), respectively. For all x, let the field $\mathcal{X}^x$ be defined by $\mathcal{X}^x|_{X \setminus X_x} = \mathcal{X}|_{X \setminus X_x}$ and $\mathcal{X}^x|_{X_x} = \mathcal{X}'|_{X_x}$, and recall that the spectral gap (4.24) for $\mathcal{X}$ takes the form

$$
\mathrm{Var}\left[Z(A)\right] \le \frac{1}{2} \sum_{x \in \mathbb{Z}^d} \mathbb{E}\left[\big(Z(A) - Z(A(\mathcal{X}^x))\big)^2\right].
$$

The conclusion (4.35) then follows provided we prove that for all $x \in \mathbb{Z}^d$,

$$\mathbb{E}\Big[\big(Z(A) - Z(A(\mathcal{X}^x))\big)^2\Big] \le \int_0^\infty \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{A,Q_{2\ell+1}(x)}\, Z(A)\Big)^2\Big]\,(\ell+1)^{-d}\pi(\ell)d\ell. \tag{4.40}$$

Without loss of generality, it suffices to consider the case $x = 0$. Moreover, by an approximation argument, we may assume that the random variable $Z(A)$ is bounded. For simplicity, we set $\rho(r) := r + \rho_0^r$ and $\partial^{\mathrm{osc}}_r := \partial^{\mathrm{osc}}_{A,Q_{2r+1}}$. Note that the choice (4.34) of R then takes the form

$$\sup_{\ell \ge R} \mathbb{P}\big[\rho(\ell) \ge 2\ell\big] \le \frac{1}{4}. \tag{4.41}$$

We split the proof into two steps.

Step 1. Conditioning argument.

In this step, we prove for all $r_2 \ge 2r_1 \ge 2R$,

$$\begin{aligned}\mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{r_2}\, Z(A)\Big)^2 \mathbb{1}_{\frac12 r_2 \le \rho(r_1) < r_2}\Big] &\le 2\,\mathbb{P}\left[\tfrac12 r_2 \le \rho(r_1) < r_2\right] \\ &\quad \times \bigg(\mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{2r_2}\, Z(A)\Big)^2\Big] + \sum_{\ell=2}^{\infty} \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{2^\ell r_2}\, Z(A)\Big)^2 \mathbb{1}_{2^{\ell-1} r_2 \le \rho(r_2) < 2^\ell r_2}\Big]\bigg).\end{aligned} \tag{4.42}$$

Conditioning the left-hand side with respect to the value of $\rho(r_2)$, we decompose

$$\begin{aligned}\mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{r_2}\, Z(A)\Big)^2 \mathbb{1}_{\frac12 r_2 \le \rho(r_1) < r_2}\Big] &\le \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{r_2}\, Z(A)\Big)^2 \mathbb{1}_{\frac12 r_2 \le \rho(r_1) < r_2} \mathbb{1}_{\rho(r_2) < 2r_2}\Big] \\ &\quad + \sum_{\ell=2}^{\infty} \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{r_2}\, Z(A)\Big)^2 \mathbb{1}_{\frac12 r_2 \le \rho(r_1) < r_2} \mathbb{1}_{2^{\ell-1} r_2 \le \rho(r_2) < 2^\ell r_2}\Big].\end{aligned} \tag{4.43}$$

We estimate each of the right-hand side terms separately. For that purpose, note that the definition of ρ and assumption (c) ensure that, given $\rho(\ell_1) \le \ell_2$ and $\rho(\ell_2) \le \ell_3$, the random variable $\rho(\ell_1)$ is independent of $\partial^{\mathrm{osc}}_{\ell_3}\, Z(A)$. This observation directly yields

$$\begin{aligned}&\mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{r_2}\, Z(A)\Big)^2 \mathbb{1}_{\frac12 r_2 \le \rho(r_1) < r_2} \mathbb{1}_{\rho(r_2) < 2r_2}\Big] \\ &\le\ \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{2r_2}\, Z(A)\Big)^2 \mathbb{1}_{\rho(r_1) \ge \frac12 r_2} \,\Big\|\, \rho(r_1) < r_2,\, \rho(r_2) < 2r_2\Big]\, \mathbb{P}\big[\rho(r_1) < r_2,\, \rho(r_2) < 2r_2\big] \\ &\le\ \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{2r_2}\, Z(A)\Big)^2\Big] \frac{\mathbb{P}\big[\frac12 r_2 \le \rho(r_1) < r_2\big]}{\mathbb{P}\left[\rho(r_1) < r_2,\, \rho(r_2) < 2r_2\right]} \\ &\le\ \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{2r_2}\, Z(A)\Big)^2\Big] \frac{\mathbb{P}\big[\frac12 r_2 \le \rho(r_1) < r_2\big]}{1 - \mathbb{P}\left[\rho(r_1) \ge r_2\right] - \mathbb{P}\left[\rho(r_2) \ge 2r_2\right]}.\end{aligned}$$

For $r_2 \ge 2r_1 \ge 2R$, the choice (4.41) of R yields

$$\mathbb{P}\left[\rho(r_1) \ge r_2\right] + \mathbb{P}\left[\rho(r_2) \ge 2r_2\right] \ \le\ \mathbb{P}\left[\rho(r_1) \ge 2r_1\right] + \mathbb{P}\left[\rho(r_2) \ge 2r_2\right] \le \frac12,$$

so that the above takes the simpler form

$$\mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{r_2}\, Z(A)\Big)^2 \mathbb{1}_{\frac12 r_2 \le \rho(r_1) < r_2} \mathbb{1}_{\rho(r_2) < 2r_2}\Big] \le 2\,\mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{2r_2}\, Z(A)\Big)^2\Big]\, \mathbb{P}\big[\tfrac12 r_2 \le \rho(r_1) < r_2\big]. \tag{4.44}$$

On the other hand, further recalling that assumption (c) ensures that given $\rho(\ell_1) \leq \ell_2$ the random variables $\rho(\ell_1)$ and $\rho(\ell_2)$ are independent, we similarly obtain

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\partial^{\rm osc}_{r_2} Z(A)\Big)^2 \mathbb{1}_{\frac12 r_2\le\rho(r_1)<r_2}\mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\Big]\\
&\quad\le\ \mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^\ell r_2} Z(A)\Big)^2 \mathbb{1}_{\rho(r_2)\ge 2^{\ell-1}r_2}\,\Big\|\,\rho(r_1)<r_2,\,\rho(r_2)<2^\ell r_2\Big]\\
&\qquad\qquad\qquad\times\mathbb{P}\big[\rho(r_1)\ge\tfrac12 r_2\,\|\,\rho(r_1)<r_2,\,\rho(r_2)<2^\ell r_2\big]\,\mathbb{P}\big[\rho(r_1)<r_2,\,\rho(r_2)<2^\ell r_2\big]\\
&\quad\le\ \mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^\ell r_2} Z(A)\Big)^2 \mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\Big]\,\frac{\mathbb{P}\big[\frac12 r_2\le\rho(r_1)<r_2\big]}{\mathbb{P}\big[\rho(r_1)<r_2,\,\rho(r_2)<2^\ell r_2\big]}\\
&\quad\le\ \mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^\ell r_2} Z(A)\Big)^2 \mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\Big]\,\frac{\mathbb{P}\big[\frac12 r_2\le\rho(r_1)<r_2\big]}{1-\mathbb{P}\big[\rho(r_1)\ge r_2\big]-\mathbb{P}\big[\rho(r_2)\ge 2^\ell r_2\big]}.
\end{aligned}$$

With the choice (4.41) of R, for $r_2 \geq 2r_1 \geq 2R$ and $\ell \geq 1$, this turns into

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\partial^{\rm osc}_{r_2} Z(A)\Big)^2 \mathbb{1}_{\frac12 r_2\le\rho(r_1)<r_2}\mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\Big]\\
&\qquad\qquad\le 2\,\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^\ell r_2} Z(A)\Big)^2 \mathbb{1}_{2^{\ell-1}r_2\le\rho(r_2)<2^\ell r_2}\Big]\,\mathbb{P}\big[\tfrac12 r_2\le\rho(r_1)<r_2\big].
\end{aligned}$$

Combining this with (4.43) and (4.44), the conclusion (4.42) follows.

Step 2. Proof of (4.40).

Conditioning the left-hand side of (4.40) with respect to the value of the action radius $\rho(0)$, we obtain

$$\mathbb{E}\Big[\big(Z(A)-Z(A(\mathcal{X}^x))\big)^2\Big]\le\mathbb{E}\Big[\Big(\partial^{\rm osc}_{R} Z(A)\Big)^2\Big]+\sum_{\ell=1}^\infty\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^\ell R} Z(A)\Big)^2\mathbb{1}_{2^{\ell-1}R\le\rho(0)<2^\ell R}\Big].$$

We now iteratively apply (4.42) to estimate the last right-hand side terms: with the short-hand notation $\pi(\ell_2;\ell_1) := \mathbb{P}\big[\frac12\ell_2 \leq \rho(\ell_1) < \ell_2\big]$, we obtain for all $n \geq 1$,

$$\begin{aligned}
\mathbb{E}\Big[\big(Z(A)-Z(A(\mathcal{X}^x))\big)^2\Big]&\le\mathbb{E}\Big[\Big(\partial^{\rm osc}_{R} Z(A)\Big)^2\Big]+2\sum_{\ell_1=1}^\infty\pi(2^{\ell_1}R;0)\,\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^{\ell_1+1}R} Z(A)\Big)^2\Big]\\
&\quad+2^2\sum_{\ell_1=1}^\infty\pi(2^{\ell_1}R;0)\sum_{\ell_2=\ell_1+2}^\infty\pi(2^{\ell_2}R;2^{\ell_1}R)\,\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^{\ell_2+1}R} Z(A)\Big)^2\Big]+\ldots\\
&\quad+2^n\sum_{\ell_1=1}^\infty\pi(2^{\ell_1}R;0)\sum_{\ell_2=\ell_1+2}^\infty\pi(2^{\ell_2}R;2^{\ell_1}R)\ldots\sum_{\ell_n=\ell_{n-1}+2}^\infty\pi(2^{\ell_n}R;2^{\ell_{n-1}}R)\,\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^{\ell_n+1}R} Z(A)\Big)^2\Big]\\
&\quad+2^n\sum_{\ell_1=1}^\infty\pi(2^{\ell_1}R;0)\sum_{\ell_2=\ell_1+2}^\infty\pi(2^{\ell_2}R;2^{\ell_1}R)\ldots\sum_{\ell_n=\ell_{n-1}+2}^\infty\pi(2^{\ell_n}R;2^{\ell_{n-1}}R)\\
&\qquad\times\sum_{\ell_{n+1}=\ell_n+2}^\infty\mathbb{E}\Big[\Big(\partial^{\rm osc}_{2^{\ell_{n+1}}R} Z(A)\Big)^2\mathbb{1}_{2^{\ell_{n+1}-1}R\le\rho(2^{\ell_n}R)<2^{\ell_{n+1}}R}\Big].
\end{aligned}$$

With the choice (4.41) of R in the form

$$\sup_{\ell_0\ge0}\sum_{\ell=\ell_0+2}^\infty\pi(2^\ell R;2^{\ell_0}R)=\sup_{\ell_0\ge0}\mathbb{P}\big[\rho(2^{\ell_0}R)\ge2^{\ell_0+1}R\big]\le\frac14,$$

the definition $\tilde{\pi}(\ell) := \sup_{\ell_0 : 0 \le \ell_0 \le \ell/4} \pi(\ell;\ell_0)$ of the weight, and recalling that the random variable $Z(A)$ is bounded, we deduce

$$\mathbb{E}\Big[\big(Z(A) - Z(A(\mathcal{X}^x))\big)^2\Big] \le \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{R}\, Z(A)\Big)^2\Big] \\ + 2\Big(\sum_{m=0}^{n-1} 2^{-m}\Big)\sum_{\ell=1}^{\infty} \tilde{\pi}(2^\ell R)\,\mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{2^{\ell+1}R}\, Z(A)\Big)^2\Big] + 2^{-n-2}\|Z\|_{\mathrm{L}^\infty}.$$

Letting $n \uparrow \infty$, we thus obtain

$$\mathbb{E}\Big[\big(Z(A) - Z(A(\mathcal{X}^x))\big)^2\Big] \le \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{R}\, Z(A)\Big)^2\Big] + 4\sum_{\ell=1}^{\infty} \tilde{\pi}(2^\ell R)\,\mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{2^{\ell+1}R}\, Z(A)\Big)^2\Big].$$

Comparing sums to integrals and using the definition of π, the conclusion (4.40) follows. □

4.4.3 Local operations

In this subsection, we describe two typical operations on random fields that do preserve functional inequalities: local transformations and gluing of independent random fields with respect to an independent pattern. These operations allow one to generate many variations around the examples of Section 4.5 below.

Local transformations

Given a random field A_0 on $\mathbb{R}^d$, we say that a random field A on $\mathbb{R}^d$ is a R-local transformation of A_0 (as in Proposition 4.1.2) if $A|_S$ is $\sigma(A|_{S+B_R})$-measurable for all Borel subsets $S \subset \mathbb{R}^d$. Important particular cases are local smoothing (e.g. by convolution with a smooth kernel with bounded support) and truncation (e.g. by applying a Lipschitz function).

Lemma 4.4.6. *If A_0, A are two random fields on $\mathbb{R}^d$, and if A is a R-local transformation of A_0, then we have for all Borel subsets $S \subset \mathbb{R}^d$ and all $\sigma(A)$-measurable random variables $X(A)$,*

$$\partial^{\mathrm{osc}}_{A_0,S}\, X(A(A_0)) \;\le\; \partial^{\mathrm{osc}}_{A,S+B_R}\, X(A)$$

and

$$\partial^{\mathrm{fct}}_{A_0,S} X(A(A_0)) \;\le\; R^d \left\|\frac{\partial A}{\partial A_0}\right\|_{\mathrm{L}^\infty} \partial^{\mathrm{fct}}_{A,S+B_R} X(A),$$

so that functional inequalities for A_0 with the oscillation or the functional derivative imply the corresponding functional inequalities for A with the oscillation or the functional derivative (provided $\partial A/\partial A_0$ is bounded if the functional derivative is used). ♢

Proof. By assumption, $A|_{\mathbb{R}^d\setminus(S+B_R)}$ is $\sigma(A_0|_{\mathbb{R}^d\setminus S})$-measurable, so that the σ-algebra $\sigma(A|_{\mathbb{R}^d\setminus(S+B_R)})$ is contained in $\sigma(A_0|_{\mathbb{R}^d\setminus B})$, and the inequality follows. □

Independent gluing

The following result shows how independent localized fields can be glued together. Since it is a direct consequence of the standard tensorization arguments used e.g. in the proof of Proposition 4.1.2, details are omitted.

Lemma 4.4.7. *Let A_1, A_2, and A_3 be three independent random fields on $\mathbb{R}^d$. Assume that $|A_1 - A_3| \le C$ a.s. for some deterministic constant $C > 0$, that A_2 has values in $[0,1]$, and consider the "glued" random field $A := A_2 A_1 + (1 - A_2)A_3$. If A_1, A_2, and A_3 satisfy different forms of weighted spectral gaps (resp. covariance inequality, resp. logarithmic Sobolev inequality), then the random field A satisfies the worst of these spectral gaps (resp. covariance inequality, resp. logarithmic Sobolev inequality), that is, with the right-hand side replaced by the sum of the corresponding right-hand sides.* ♢

4.5 Examples

In this section we consider four representative examples: Gaussian fields, tessellations associated with a Poisson point process, random parking bounded inclusions, and Poisson or random parking inclusions with unbounded radii. The main results are summarized in the table below.

Example of field	**Key property**	**Functional inequalities**
Gaussian random field	covariance function $\mathcal{C}$ $\sup_{B(x)} \|\mathcal{C}\| \le c(\|x\|)$	(∂^{fct}-WSG), (∂^{fct}-WLSI) weight $\pi(\ell) \simeq (-c'(\ell))_+$
Poisson tessellations (Voronoi/Delaunay)	$\sigma(\mathcal{X})$-measurable action radius	(∂^{osc}-WSG), (∂^{osc}-WLSI) weight $\pi(\ell) \simeq e^{-\frac{1}{C}\ell^d}$
Random parking bounded inclusions	$\sigma(\mathcal{X})$-measurable action radius & exponential stabilization	(∂^{osc}-WSG), (∂^{osc}-WLSI) weight $\pi(\ell) \simeq e^{-\frac{1}{C}\ell}$
Poisson random inclusions with random radii	radius law V $\gamma(\ell) := \mathbb{P}[\ell - 4 \le V < \ell + 2]$	(∂^{osc}-WSG) weight $\pi(\ell) \simeq (\ell+1)^d \gamma(\ell)$ (and (∂^{osc}-LSI) if V bounded)

4.5.1 Gaussian random fields

Gaussian random fields are the main examples of deterministically localized fields as introduced in Section 4.4.2. The following criterion is established in Appendix 4.B as a direct application of our general results on deterministically localized fields. Note that in items (ii)–(iii), the weights obtained for (∂^{fct}-WSG) and (∂^{fct}-WCI) typically have the same scaling. As shown in Proposition 4.2.1, this result is sharp: each sufficient condition is (essentially) necessary.

Corollary 4.5.1. *Let A be a jointly measurable stationary Gaussian random field on $\mathbb{R}^d$ with covariance function $\mathcal{C}(x) := \mathrm{Cov}[A(x); A(0)]$.*

(i) If $x \mapsto \sup_{B(x)} |\mathcal{C}|$ is integrable, then A satisfies (∂^{fct}-SG) and (∂^{fct}-LSI) with any radius $R > 0$.

(ii) If $\sup_{B(x)} |\mathcal{C}| \le c(|x|)$ holds for some Lipschitz function $c : \mathbb{R}_+ \to \mathbb{R}_+$, then A satisfies (∂^{fct}-WSG) and (∂^{fct}-WLSI) with weight $\pi(\ell) \simeq (-c'(\ell))_+$.

(iii) If $\mathcal{F}\mathcal{C} \in \mathrm{L}^1(\mathbb{R}^d)$ and if $\sup_{B(x)} |\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})| \le r(|x|)$ holds for some non-increasing Lipschitz function $r : \mathbb{R}_+ \to \mathbb{R}_+$, then A satisfies (∂^{fct}-WCI) with weight $\pi(\ell) \simeq (\ell+1)^d r(\ell)(-r'(\ell))$. ♢

4.5.2 Poisson random tessellations

In this section, we consider random fields that take i.i.d. values on the cells of a tessellation associated with a stationary random point process $\mathcal{P}$ on $\mathbb{R}^d$. Such random fields can be formalized as projections of decorated random point processes. Given a point process $\mathcal{P}$ on $\mathbb{R}^d$ and given a random element G with values in some measurable space X, we call *decorated random point process associated with $\mathcal{P}$ and G* a point process $\hat{\mathcal{P}}$ on $\mathbb{R}^d \times X$ defined as follows: choose a measurable enumeration $\mathcal{P} = \{X_j\}_j$, pick independently a sequence $(G_j)_j$ of i.i.d. copies of the random element G, and set $\hat{\mathcal{P}} := \{X_j, G_j\}_j$ (that is, in measure notation, $\hat{\mathcal{P}} := \sum_j \delta_{(X_j,G_j)}$). Note that by definition $\hat{\mathcal{P}}$ is completely independent whenever $\mathcal{P}$ is completely independent.

We focus here on the case when the underlying point process $\mathcal{P}$ is some Poisson point process $\mathcal{P} = \mathcal{P}_0$ on $\mathbb{R}^d$ with intensity $\mu = 1$. Choose a measurable random field V on $\mathbb{R}^d$, corresponding to the values on the cells. We study both Voronoi and Delaunay tessellations.

(1) *Voronoi tessellation:* Let $\hat{\mathcal{P}}_1 := \{X_j, V_j\}_j$ denote a decorated point process associated with the random point process $\mathcal{P}_0 := \{X_j\}_j$ and the random element V (hence $(V_j)_j$ is a sequence of i.i.d. copies of the random field V). We define a $\sigma(\hat{\mathcal{P}}_1)$-measurable random field A_1 as follows,

$$A_1(x) = \sum_j V_j(x)\mathbb{1}_{C_j}(x),$$

where $\{C_j\}_j$ denotes the partition of $\mathbb{R}^d$ into the Voronoi cells associated with the Poisson points $\{X_j\}_j$, that is,

$$C_j := \{x \in \mathbb{R}^d : |x - X_j| < |x - X_k|,\ \forall k \neq j\}.$$

(2) *Delaunay tessellation:* Let $\tilde{V} := (\tilde{V}_\zeta)_\zeta$ denote a family of i.i.d. copies of the random element V, indexed by sets ζ of $d+1$ distinct integers. We define a random field A_2 as follows,

$$A_2(x) = \sum_j \tilde{V}_{\zeta(D_j)}(x)\mathbb{1}_{D_j}(x),$$

where $\{D_j\}_j$ denotes the partition of $\mathbb{R}^d$ into the Delaunay d-simplices associated with the Poisson points $\{X_j\}_j$ (the Delaunay triangulation is indeed almost surely uniquely defined), and where $\zeta(D_j)$ denotes the set of the $d+1$ indices $i_1, \ldots, i_{d+1}$ of the vertices $X_{i_1}, \ldots, X_{i_{d+1}}$ of D_j.

Since large holes in the Poisson process have exponentially small probability, large cells in the corresponding Voronoi or Delaunay tessellations also have exponentially small probability. This allows one to prove the following weighted functional inequalities with stretched exponential weights.

Proposition 4.5.2. *For $s = 1, 2$, the above-defined random field A_s satisfies* (∂^{osc}-WSG), (∂^{osc}-WLSI), *and* (∂^{osc}-WCI) *with weight $\pi(\ell) = e^{-\frac{1}{C}\ell^d}$ for some constant $C > 0$. Moreover for all $\sigma(A_s)$-measurable random variables $Z(A_s)$ and all $\lambda \in (0,1)$ we have*

$$\mathrm{Var}\,[Z(A_s)] \le C \sum_{x\in\mathbb{Z}^d} \sum_{\ell=1}^{\infty} e^{-\frac{\lambda}{C}\ell^d}\, \mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell,x} Z(A_s)\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda},$$

with the notation $\partial^{\mathrm{dis}}_{\ell,x} Z(A_s)$ defined in (4.7) *(with $l = 0$).* ◊

Proof. We focus on the case of the Voronoi tessellation (the argument for the Delaunay tessellation is similar). We shall appeal to Theorem 4.4.5, and need to construct and control action radii, which we do in two separate steps. (The weighted spectral gap with loss and discrete derivative follows from Theorem 4.4.3(i).)

Step 1. Definition and properties of the action radius.

Let $x \in \mathbb{R}^d$, $\ell \in \mathbb{N}$ be fixed. Changing the point configuration of $\hat{\mathcal{P}}_1 = \{X_j, V_j\}_j$ inside $Q_{2\ell+1}(x) \times \mathbb{R}^{\mathbb{R}^d}$ only modifies the Voronoi tessellation (hence the field A_1) inside the set

$$V_{\mathcal{P}_0,\ell}(x) := \big\{y \in \mathbb{R}^d : \exists z \in Q_{2\ell+1}(x) \text{ such that } |y - z| \le |y - X| \text{ for all } X \in \mathcal{P}_0 \setminus Q_{2\ell+1}(x)\big\}.$$

An action radius for A_1 with respect to $\hat{\mathcal{P}}_1$ on $Q_{2\ell+1}(x) \times \mathbb{R}^{\mathbb{R}^d}$ is thus given by

$$\rho^\ell_x := 2\,\mathrm{diam}\, V_{\mathcal{P}_0,\ell}(x) + 1 - \ell,$$

and property (a) of Theorem 4.4.5 is proved. The stationarity property (b) follows by construction, and it remains to prove the measurability property (c). In particular, we need to prove that ρ_x^ℓ is $\sigma(\mathcal{P}_0|_{Q_{2(\ell+\rho_x^\ell)+1}(x)\setminus Q_{2\ell+1}(x)})$-measurable. Since ρ_x^ℓ is $\sigma(\mathcal{P}_0|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)})$-measurable by construction, it remains to prove it is $\sigma(\mathcal{P}_0|_{Q_{2(\ell+\rho_x^\ell)+1}(x)})$-measurable. To this aim, let $\tilde{\mathcal{P}}$ be an arbitrary locally finite point set and consider the compound point set $\tilde{\mathcal{P}}_{0,\ell}(x) = \mathcal{P}_0|_{Q_{2(\ell+\rho_x^\ell)+1}(x)} \cup \tilde{\mathcal{P}}|_{\mathbb{R}^d\setminus Q_{2(\ell+\rho_x^\ell)+1}(x)}$. The claimed measurability then follows from the identity $V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x) = V_{\mathcal{P}_0,\ell}(x)$. We start with the proof that $V_{\mathcal{P}_0,\ell}(x) \subset V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$. Let $y \in V_{\mathcal{P}_0,\ell}(x)$. Then for all $X \in \tilde{\mathcal{P}}_{0,\ell}(x)|_{\mathbb{R}^d\setminus Q_{2(\ell+\rho_x^\ell)+1}(x)}$ we have by the triangle inequality

$$|X-y| \ge |X-x| - |x-y| \ge \ell + \rho_x^\ell - \operatorname{diam} V_{\mathcal{P}_0,\ell}(x) \;=\; \operatorname{diam} V_{\mathcal{P}_0,\ell}(x) + 1 \ge |x-y|,$$

so that $y \in V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$. Let us turn to the converse inclusion. By definition, $V_{\mathcal{P}_0,\ell}(x)$ and $V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$ are convex, and thus simply connected. Set $\eta = \frac{1}{2}$ and consider $y \in (B_\eta + V_{\mathcal{P}_0,\ell}(x)) \setminus V_{\mathcal{P}_0,\ell}(x)$ (the η-fattened boundary of $V_{\mathcal{P}_0,\ell}(x)$). By definition we have $y \notin V_{\mathcal{P}_0,\ell}(x)$, so that for all $z \in Q_{2\ell+1}(x)$ there exists $X \in \mathcal{P}_0 \setminus Q_{2\ell+1}(x)$ such that $|y-z| > |y-X|$. Let us argue that $X \in Q_{2(\ell+\rho_x^\ell)+1}(x)$. Indeed, by the triangle inequality,

$$|X-x| \le |X-y| + |y-x| < |y-z| + |y-x| \le \operatorname{diam} V_{\mathcal{P}_0,\ell}(x) + \eta + \operatorname{diam} V_{\mathcal{P}_0,\ell}(x) + \eta \;=\; \rho_x^\ell + \ell.$$

Hence, we deduce $X \in \tilde{\mathcal{P}}_{0,\ell}(x)$, which in turn implies $y \notin V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$. This proves the inclusion $V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x) \subset V_{\mathcal{P}_0,\ell}(x) \cup (\mathbb{R}^d \setminus (B_\eta + V_{\mathcal{P}_0,\ell}(x)))$. Combined with the inclusion $V_{\mathcal{P}_0,\ell}(x) \subset V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$ and the fact that both sets are simply connected, this yields the desired identity $V_{\mathcal{P}_0,\ell}(x) = V_{\tilde{\mathcal{P}}_{0,\ell}(x),\ell}(x)$ and therefore proves the claimed measurability property (c). We then appeal to Theorem 4.4.5, and it remains to estimate the weights.

Step 2. Control of the weight.

By scaling and change of intensity, it is enough to consider $\ell = 0$ (we omit the sub- and superscripts ℓ in the notation) and a Poisson point process $\mathcal{P}_0$ of general intensity $\mu > 0$. Denote by $\mathcal{C}_i = \{x \in \mathbb{R}^d : x_i \ge \frac{5}{6}|x|\}$ the d cones in the canonical directions e_i of $\mathbb{R}^d$, and consider the $2d$ cones $\mathcal{C}_i^\pm := \pm(2e_i + C_i)$. By an elementary geometric argument, for some constant $C \simeq 1$ the following implication holds: for all $L > C$,

$$\sharp\big(\mathcal{P}_0 \cap \mathcal{C}_i^\pm \cap \{x : C \le |x_i| \le L\}\big) > 0 \text{ for all } i \text{ and } \pm \qquad\Longrightarrow\qquad \operatorname{diam} V_{\mathcal{P}_0}(0) \le CL.$$

A union bound then yields for all $L > C$,

$$\mathbb{P}\left[\operatorname{diam} V_{\mathcal{P}_0}(0) \ge L\right] \;\le\; \mathbb{P}\left[\exists 1 \le i \le d, \exists \pm : \sharp\big(\mathcal{P}_0 \cap \mathcal{C}_i^\pm \cap \{x : |x_i| \le \frac{1}{C}L\}\big) = 0\right] \;\le\; 2d\, e^{-\frac{\mu}{C}L^d}.$$

Combined with the definition of the action radius in Step 1, this implies the desired estimate. □

4.5.3 Random parking process

In this section we let $\mathcal{P}$ be the random parking point process on $\mathbb{R}^d$ with given radius $R > 0$. As shown by Penrose [357] (see also [213, Section 2.1]), the random parking point process $\mathcal{P}$ can be constructed as a transformation $\mathcal{P} = \Phi(\mathcal{P}_0)$ of a Poisson point process $\mathcal{P}_0$ on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity 1. Let us recall the graphical construction of this transformation Φ. We first construct an oriented graph on the points of $\mathcal{P}_0$ in $\mathbb{R}^d \times \mathbb{R}_+$, by putting an oriented edge from (x,t) to (x',t') whenever $B(x,R) \cap B(x',R) \ne \varnothing$ and $t < t'$ (or $t = t'$ and x precedes x' in the lexicographic order, say). We say that (x',t') is an offspring (resp. a descendant) of (x,t), if (x,t) is a direct ancestor (resp. an ancestor) of (x',t'), that is, if there is an edge (resp. a directed path) from (x,t) to (x',t'). The set $\mathcal{P} := \Phi(\mathcal{P}_0)$ is then constructed as follows. Let F_1 be the set of all roots in the oriented graph (that is, the points of $\mathcal{P}_0$ without ancestor), let G_1 be the set of points of $\mathcal{P}_0$ that are offsprings of points of F_1, and let $H_1 := F_1 \cup G_1$. Now consider the oriented graph induced on $\mathcal{P}_0 \setminus H_1$, and define F_2, G_2, H_2 in the same way, and so on. By construction, the sets $(F_j)_j$ and $(G_j)_j$ are all disjoint and constitute a partition of $\mathcal{P}_0$. We finally define $\mathcal{P} := \Phi(\mathcal{P}_0) := \bigcup_{j=1}^{\infty} F_j$.

In this setting we show that there exists an action radius with exponential moments for $\mathcal{P}$ with respect to $\mathcal{P}_0$. The proof follows from the exponential stabilization results of [391].

Proposition 4.5.3. *For all $x \in \mathbb{Z}^d$ and $\ell \geq 0$, the random parking point process $\mathcal{P}$ with radius $R > 0$ as constructed above admits an action radius ρ_x^ℓ with respect to $\mathcal{P}_0$ on $Q_{2\ell+1}(x) \times \mathbb{R}_+$, which satisfies for all $L \geq 0$,*

$$\mathbb{P}[\rho_x^\ell \geq L] \leq C_R(\ell+1)^d e^{-L/C_R},$$

and which is $\sigma\big(\mathcal{P}_0|_{((Q_{2(\ell+\rho_x^\ell)+1}(x)\setminus Q_{2\ell+1}(x))\times\mathbb{R}_+}\big)$-measurable. In particular, the point process $\mathcal{P}$ satisfies (∂^{osc}-WSG), (∂^{osc}-WLSI), *and* (∂^{osc}-WCI) *with weight $\pi(\ell) =: e^{-\ell/C_R}$.* $\Diamond$

Proof. The proof relies on the notion of *causal chains* defined in the proof of [391, Lemma 3.5] to which we refer the reader. Note that for all consecutive points (x,t) and (y,s) in a causal chain we necessarily have $|x-y| < 2R$. By definition, it follows that an action radius for $\mathcal{P}$ given $\mathcal{P}_0$ on $Q_{2\ell+1}(x) \times \mathbb{R}_+$ can be defined by the maximum of the distances $2R + d(y, Q_{2\ell+1}(x))$ on the set of points $(y,s) \in \mathcal{P}_0$ such that there exists a causal chain between a point of $\mathcal{P}_0$ in $((Q_{2\ell+1}(x)+B_{2R}) \setminus Q_{2\ell+1}(x)) \times \mathbb{R}_+$ and (y,s). We denote by ρ_x^ℓ this maximum. By construction, we note that this random variable ρ_x^ℓ is $\sigma\big(\mathcal{P}_0|_{((Q_{2\ell+1}(x)+B_{\rho_x^\ell})\setminus Q_{2\ell+1}(x))\times\mathbb{R}_+}\big)$-measurable.

It remains to estimate the decay of its probability law. First, note that by definition the event $\rho_x^\ell > L$ entails the existence of some $(y,s) \in \mathcal{P}_0$ with $y \in (Q_{2\ell+1}(x)+B_{L+2R}) \setminus (Q_{2\ell+1}(x)+B_L)$ and of a causal chain between a point of $((Q_{2\ell+1}(x)+B_{2R}) \setminus Q_{2\ell+1}(x)) \times \mathbb{R}_+$ and (y,s). Second, the exponential stabilization result of [391, Lemma 3.5] states that for all $z \in \mathbb{R}^d$ and all $L > 0$ the probability that there exists $(y,s) \in Q(z) \times \mathbb{R}_+$ and a causal chain from a point outside $(Q(z)+B_L) \times \mathbb{R}_+$ towards (y,s) is bounded by $C_R e^{-L/C_R}$. For $L \geq R$, covering $(Q_{2\ell+1}(x)+B_{L+2R}) \setminus (Q_{2\ell+1}(x)+B_L)$ with $C(L+\ell)^{d-1}R$ unit cubes and covering $Q_{2\ell+1}(x)+B_{2R}$ with $C(R+\ell)^d$ unit cubes, a union bound then yields

$$\mathbb{P}\big[\rho_x^\ell > L\big] \leq C_R\big(L^{d-1}+\ell^d\big)e^{-L/C_R} \leq C_R(\ell+1)^d e^{-L/C_R}.$$

All the assumptions of Theorem 4.4.5 are then satisfied with $\pi(\ell) = C_R e^{-\ell/C_R}$, and the conclusion follows. $\square$

4.5.4 Random inclusions with random radii

We consider typical examples of random fields on $\mathbb{R}^d$ taking random values on random inclusions centered at the points of some random point process $\mathcal{P}$. The inclusions are allowed to have i.i.d. random shapes (hence in particular i.i.d. random radii). For the random point process $\mathcal{P}$, we consider projections $\Phi(\mathcal{P}_0)$ of some Poisson point process $\mathcal{P}_0$ on $\mathbb{R}^d \times \mathbb{R}^l$ with intensity $\mu > 0$, and shall assume that for all $x \in \mathbb{Z}^d$ the process $\mathcal{P}$ admits an action radius ρ_x *with respect to* $\mathcal{P}_0$ *on* $Q(x) \times \mathbb{R}^l$.

We turn to the construction of the random inclusions. Let V be a nonnegative random variable (corresponding to the random radius of the inclusions). In order to define the random shapes, we consider the set Y of all nonempty Borel subsets $E \subset \mathbb{R}^d$ with $\sup_{x\in E}|x| = 1$, and endow it with the σ-algebra $\mathcal{Y}$ generated by all subsets of the form $\{E \in Y : x_0 \in E\}$ with $x_0 \in \mathbb{R}^d$. Let S be a random nonempty Borel subset of $\mathbb{R}^d$ with $\sup_{x\in S}|x| = 1$ a.s., that is, a random element in the measurable space Y. (Note that V and S need not be independent.) Let $\hat{\mathcal{P}}_0 := \{X_j, V_j, S_j\}_j$ be a decorated point process associated with the random point process $\mathcal{P}_0 = \{X_j\}_j$ and the random element (V,S). The collection of random inclusions is then given by $\{I_j\}_j$ with $I_j := X_j + V_j S_j$.

It remains to associate random values to the random inclusions. Since inclusions may intersect each other, several constructions can be considered; we focus on the following three typical choices.

(1) Given $\alpha, \beta \in \mathbb{R}$, we set $\hat{\mathcal{P}}_1 := \hat{\mathcal{P}}_0$, and we consider the $\sigma(\hat{\mathcal{P}}_1)$-measurable random field A_1 that is equal to α inside the inclusions, and to β outside. More precisely,

$$A_1 := \beta + (\alpha - \beta)\mathbb{1}_{\bigcup_j I_j}.$$

The simplest example is the random field A_1 obtained for $\mathcal{P}$ a Poisson point process on $\mathbb{R}^d$ with intensity $\mu = 1$, and for S the unit ball centered at the origin in $\mathbb{R}^d$; this is referred to as the *Poisson unbounded spherical inclusion model.*

(2) Let $\beta \in \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function, and let W be a measurable random field on $\mathbb{R}^d$. Let $\hat{\mathcal{P}}_2 := \{X_j, V_j, S_j, W_j\}$ be a decorated point process associated with $\hat{\mathcal{P}}_0$ and W. We then consider the $\sigma(\hat{\mathcal{P}}_2)$-measurable random field A_2 that is equal to $f(\sum_{j:x\in I_j} W_j)$ at any point x of the inclusions, and to β outside. More precisely,

$$A_2(x) := \beta + \bigg(f\Big(\sum_j W_j(x)\mathbb{1}_{I_j}(x)\Big) - \beta\bigg)\mathbb{1}_{\bigcup_j I_j}(x).$$

(Of course, this example can be generalized by considering more general functions than simple sums of the values W_j; the corresponding concentration properties will then remain the same.)

(3) Let $\beta \in \mathbb{R}$, let W be a measurable random field on $\mathbb{R}^d$, and let U denote a uniform random variable on $[0, 1]$. Let $\hat{\mathcal{P}}_3 := \{X_j, V_j, S_j, W_j, U_j\}$ be a decorated point process associated with $\hat{\mathcal{P}}_0$ and (W, U). Given a $\sigma(VS, W)$-measurable random variable $P(VS, W)$, we say that inclusion I_j has the priority on inclusion I_i if $P(V_jS_j, W_j) < P(V_iS_i, W_i)$ or if $P(V_jS_j, W_j) = P(V_iS_i, W_i)$ and $U_j < U_i$. Since the random variables $\{U_j\}_j$ are a.s. all distinct, this defines a priority order on the inclusions on a set of maximal probability. Let us then relabel the inclusions and values $\{(I_j, V_j)\}_j$ into a sequence $(I'_j, V'_j)_j$ in such a way that for all j the inclusion I'_j has the j-th highest priority. We then consider the $\sigma(\hat{\mathcal{P}}_3)$-measurable random field A_3 defined as follows,

$$A_3 := \beta + \sum_j (W'_j - \beta)\mathbb{1}_{I'_j \setminus \bigcup_{i:i<j} I'_i}.$$

(Note that this example includes in particular the case when the priority order is purely random (choosing $P \equiv 0$), as well as the case when the priority is given to inclusions with e.g. larger or smaller radius (choosing $P(VS, W) = V$ or $-V$, respectively).)

In each of these three examples, $s = 1, 2, 3$, the random field A_s is $\sigma(\hat{\mathcal{P}}_s)$-measurable, for some completely independent random point process $\hat{\mathcal{P}}_s$ on $\mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}_+ \times Y_s$ and some measurable space Y_s (the set $\mathbb{R}^d \times \mathbb{R}^l$ stands for the domain of the point process $\mathcal{P}_0 = \{X_j\}_j$, and the set $\mathbb{R}_+$ stands for the domain of the radius variables $\{V_j\}_j$). In order to recast this into the framework of Section 4.4.1, we may define $\mathcal{X}_s(x, t, v) := \mathcal{P}_s|_{Q(x)\times Q(t)\times Q(v)\times Y_s}$, so that $\mathcal{X}_s$ is a completely independent measurable random field on the space $X = \mathbb{Z}^d \times \mathbb{Z}^l \times \mathbb{Z}$ with values in the space of (locally finite) measures on $Q^d \times Q^l \times Q^1 \times Y_s$.

Rather than stating a general result, we focus on the representative examples of the Poisson and of the random parking point processes. For the latter, a refined analysis is needed to avoid a loss of integrability. Note that the proof below yields slightly more general results than contained in the statement (and can easily be adapted to various other situations).

Proposition 4.5.4. *Set* $\gamma(v) := \mathbb{P}[v - 1/2 \le V < v + 1/2]$.

(i) Assume that $\mathcal{P} = \mathcal{P}_0$ *is a Poisson point process on* $\mathbb{R}^d$ *with constant intensity* μ *(hence* $l = 0$*). Then, for each* $s = 1, 2, 3$*, the above-defined random field* A_s *satisfies* (∂^{osc}-WSG) *and* (∂^{osc}-WCI) *with weight* $\ell \mapsto \mu\,(\ell+1)^d \sup_{0\le u\le 2}\gamma(\frac{1}{\sqrt{d}}\ell - u)$*. In addition, for all* $\lambda \in (0,1)$,

$$\operatorname{Cov}[Y(A_s); Z(A_s)] \le \frac{(2\mu)^\lambda}{2}\sum_{x\in\mathbb{Z}^d}\sum_{v=0}^{\infty}\gamma(v)^\lambda \mathbb{E}\left[\left(\partial^{\text{dis}}_{v+1,x,v}Y\right)^{\frac{2}{1-\lambda}}\right]^{\frac{1-\lambda}{2}}\mathbb{E}\left[\left(\partial^{\text{dis}}_{v+1,x,v}Z\right)^{\frac{2}{1-\lambda}}\right]^{\frac{1-\lambda}{2}}, \tag{4.45}$$

where $\partial^{\text{dis}}_{v+1,x,v}Z$ *is the notation defined in* (4.7)*, that is,*

$$\partial^{\text{dis}}_{v+1,x,v}Z := Z(A_s) - Z(A_s(\mathcal{X}^{x,v})) = \big(Z(A_s) - Z(A_s(\mathcal{X}^{x,v}))\big)\mathbb{1}_{A_s|_{\mathbb{R}^d\setminus Q_{2v+3}(x)} = A_s(\mathcal{X}^{x,v})|_{\mathbb{R}^d\setminus Q_{2v+3}(x)}}.$$

In the case when the radius law V *is almost surely bounded by a deterministic constant, the standard logarithmic Sobolev inequality* (∂^{osc}-LSI) *holds.*

(ii) Assume that $\mathcal{P}$ *is a random parking point process on* $\mathbb{R}^d$ *with radius* $R > 0$ *as constructed in Section 4.5.3. Then, for each* $s = 1, 2, 3$*, the above-defined random field* A_s *satisfies* (∂^{osc}-WSG) *with weight* $\pi_R(\ell) := C_R\big(e^{-\ell/C_R} + (\ell+1)^d\gamma(\ell)\big)$*. More generally it satisfies the following covariance inequality: for all* $\sigma(A_s)$*-measurable random variables* $Y(A_s), Z(A_s)$ *we have*

$$\begin{aligned}\operatorname{Cov}[Y(A_s); Z(A_s)] \le \int_{\mathbb{R}^d}&\left(\int_0^\infty \mathbb{E}\left[\left(\partial^{\text{osc}}_{A_s,B_{2\ell+1}(x)}Y(A_s)\right)^2\right](\ell+1)^{-d}\pi_R(\ell)d\ell\right)^{\frac12}\\ &\times\left(\int_0^\infty \mathbb{E}\left[\left(\partial^{\text{osc}}_{A_s,B_{2\ell+1}(x)}Z(A_s)\right)^2\right](\ell+1)^{-d}\pi_R(\ell)d\ell\right)^{\frac12}dx.\end{aligned} \tag{4.46}$$

In addition, for all $\lambda \in (0,1)$,

$$\begin{aligned}\operatorname{Cov}[Y(A_s); Z(A_s)] \le \sum_{x\in\mathbb{Z}^d}\sum_{v=0}^{\infty}&\left(\sum_{\ell=1}^{\infty}\pi_R(v,\ell)^\lambda\,\mathbb{E}\left[\left(\partial^{\text{dis}}_{\ell,x,v}Y\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda}\right)^{\frac12}\\ &\times\left(\sum_{\ell'=1}^{\infty}\pi_R(v,\ell')^\lambda\,\mathbb{E}\left[\left(\partial^{\text{dis}}_{\ell',x,v}Z\right)^{\frac{2}{1-\lambda}}\right]^{1-\lambda}\right)^{\frac12},\end{aligned} \tag{4.47}$$

where we have set

$$\pi_R(v,\ell) := C_R\Big(\gamma(v)\mathbb{1}_{\ell-1\le v<\ell} + \gamma(v)\wedge\Big(e^{-\ell/C_R} + \sup_{r\ge\ell/2}\gamma(r)\Big)\Big),$$

and where $\partial^{\text{dis}}_{\ell,x,v}Z$ *is the notation defined in* (4.7)*, that is,*

$$\partial^{\text{dis}}_{\ell,x,v}Z := \big(Z(A_s) - Z(A_s(\mathcal{X}^{x,v}))\big)\mathbb{1}_{A_s|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)} = A_s(\mathcal{X}^{x,v})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}}.$$

In the case when the radius law V *is almost surely bounded by a deterministic constant, the logarithmic Sobolev inequality* (∂^{osc}-WLSI) *holds with weight* $\ell \mapsto C_R e^{-\ell/C_R}$. ◇

Proof. We split the proof into three steps. We first apply the general results of Theorem 4.4.3, and then treat more carefully the case of the random parking point process in order to avoid the loss of integrability.

Step 1. Proof of the covariance estimates with loss.

Assume for simplicity that the transformation Φ of $\mathcal{P}_0$ into $\mathcal{P} = \Phi(\mathcal{P}_0)$ does not add points and does not move points in the direction of $\mathbb{R}^d$: more precisely, this means that for any locally finite sequence $(x_j)_j \subset \mathbb{R}^d \times \mathbb{R}^l$ we have $\Phi((x_j)_j) = (p(x_j))_{j\in I}$ for some subset I of indices (depending on $(x_j)_j$), where $p : \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}^d$ denotes the projection onto the first factor. Further assume that for all locally finite $(x_j)_j \subset \mathbb{R}^d \times \mathbb{R}^l$, denoting $\Phi((x_j)_j) = (p(x_j))_{j\in I}$, we have $\Phi((x_j)_{j\in J}) = (p(x_j))_{j\in I}$ for all subset $J \supset I$. In this step, we show that, for each $s = 1, 2, 3$, the random field A_s satisfies for all $\sigma(A^s)$-measurable random variables $Y(A_s), Z(A_s)$ and all $\lambda \in (0,1)$,

$$\begin{aligned}\operatorname{Cov}[Y(A_s); Z(A_s)] \le \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{v=0}^{\infty}\bigg(\sum_{\ell=1}^{\infty}\pi(v,\ell)^{\lambda}\,\mathbb{E}\Big[\Big(\partial^{\mathrm{dis}}_{\ell,x,v}Y\Big)^{\frac{2}{1-\lambda}}\Big]^{1-\lambda}\bigg)^{\frac12}&\\ \times\bigg(\sum_{\ell'=1}^{\infty}\pi(v,\ell')^{\lambda}\,\mathbb{E}\Big[\Big(\partial^{\mathrm{dis}}_{\ell',x,v}Z\Big)^{\frac{2}{1-\lambda}}\Big]^{1-\lambda}\bigg)^{\frac12},& \qquad (4.48)\end{aligned}$$

where we have set

$$\begin{aligned}\pi(v,\ell) \,:=&\; 2\big(\mathbb{E}\left[\sharp(\mathcal{P}\cap Q)\right]+1\big)\\ &\times\Big(\gamma(v)\mathbb{1}_{\ell-1\le v<\ell}+\gamma(v)\wedge\mathbb{E}\left[\sharp(\mathcal{P}\cap Q_{2\rho_x+1}(x))\,\mathbb{P}\left[\ell-1\le\rho_x+V<\ell\,\|\,\rho_x\right]\right]\Big),\end{aligned}$$

and where $\partial^{\mathrm{dis}}_{\ell,x,v}Z$ is the notation defined in (4.7). Applying this in the case of the random parking process together with Proposition 4.5.3, the weight becomes

$$\pi(v,\ell) \le C_R\Big(\gamma(v)\mathbb{1}_{\ell-1\le v<\ell}+\gamma(v)\wedge\int_0^{\ell}\gamma(\ell-r)\,e^{-r/C_R}\,dr\Big),$$

and estimating the last integral leads to the desired result (4.47).

Let $\mathcal{X}'_s$ denote an i.i.d. copy of the field $\mathcal{X}_s$, and let $\hat{\mathcal{P}}'_s := \{X'_j, V'_j, Y'_{j,s}\}_j$ denote the corresponding i.i.d. copy of $\hat{\mathcal{P}}_s := \{X_j, V_j, Y_{j,s}\}_j$. For all x, v, let the perturbations $\mathcal{X}^{x,v}_s$ and $\hat{\mathcal{P}}^{x,v}_s$ be then defined as usual, and let $\mathcal{P}^{x,v}_0$ be the corresponding projected point process on $\mathbb{R}^d \times \mathbb{R}^l$. Let us consider $J_v(x,r)$ the set of all indices j such that the projection $p(X_j)$ belongs to $(\Phi(\mathcal{P}_0)\cup\Phi(\mathcal{P}^{x,v}_0))\cap(Q(x)+B_r)\setminus Q(x)$. Given the assumptions on the transformation Φ, an action radius for A_s with respect to $\mathcal{X}_s$ on $\{x\}\times\{v\}$ (or equivalently, with respect to $\hat{\mathcal{P}}_s$ on $Q(x)\times Q(v)\times Y_s$) is then given by

$$\rho^s_{x,v} := \big(v\vee(\rho_x+\max\{V_j : j\in J_v(x,\rho_x)\})\big)\,\mathbb{1}_{\mathcal{X}_s\neq\mathcal{X}^{x,v}_s}.$$

In order to prove (4.48), by Theorem 4.4.3(i), it remains to estimate the corresponding weights. First, for all $\ell \ge 0$, a union bound yields

$$\begin{aligned}&\mathbb{P}\left[\ell-1\le\rho_x+\max\{V_j : j\in J_v(x,\rho_x)\}<\ell\right]\\ &\quad\le\;\mathbb{E}\left[\sharp J_v(x,\rho_x)\,\mathbb{P}\left[\ell-1\le\rho_x+V<\ell\,\|\,\rho_x\right]\right]\\ &\quad\le\;\mathbb{E}\left[\sharp((\Phi(\mathcal{P}_0)\cup\Phi(\mathcal{P}^{x,v}_0))\cap Q_{2\rho_x+1}(x))\,\mathbb{P}\left[\ell-1\le\rho_x+V<\ell\,\|\,\rho_x\right]\right]\\ &\quad\le\;2\,\mathbb{E}\left[\sharp(\mathcal{P}\cap Q_{2\rho_x+1}(x))\,\mathbb{P}\left[\ell-1\le\rho_x+V<\ell\,\|\,\rho_x\right]\right].\end{aligned}$$

Let us now define $I_v(x)$ as the set of all indices j such that either $p(X_j)$ or $p(X'_j)$ belongs to $(\Phi(\mathcal{P}_0)\cup\Phi(\mathcal{P}^{x,v}_0))\cap Q(x)$. Given the assumptions on the transformation Φ, we may then compute, in terms of the probability law $\gamma(v) = \mathbb{P}\left[V\in Q(v)\right]$,

$$\mathbb{P}\left[A_s(\mathcal{X}^{x,v}_s)\neq A_s(\mathcal{X})\right]\;\le\;\mathbb{P}\left[\exists j\in I_v(x):\;V_j\in Q(v)\right]\;\le\;\gamma(v)\,\mathbb{E}\left[\sharp I_v(x)\right]\le 2\gamma(v)\,\mathbb{E}\left[\sharp(\mathcal{P}\cap Q)\right].$$

Combining the above estimates, we conclude

$$\begin{aligned}
&\mathbb{P}\left[\ell-1 \leq \rho_{x,v}^{s} < \ell,\ A(\mathcal{X}_s^{x,v}) \neq A(\mathcal{X}_s)\right] \\
&\quad \leq \ \big(2\gamma(v)\,\mathbb{E}\left[\sharp(\mathcal{P}\cap Q)\right]\big) \wedge \Big(\mathbb{1}_{\ell-1\leq v<\ell} + \mathbb{P}\left[\ell-1 \leq \rho_x + \max\{V_j : j \in J_v(x,\rho_x)\} < \ell\right]\Big) \\
&\quad \leq \ 2\big(\mathbb{E}\left[\sharp(\mathcal{P}\cap Q)\right]+1\big)\Big(\gamma(v)\mathbb{1}_{\ell-1\leq v<\ell} + \gamma(v) \wedge \mathbb{E}\left[\sharp(\mathcal{P}\cap Q_{2\rho_x+1}(x))\,\mathbb{P}\left[\ell-1 \leq \rho_x + V < \ell \,\|\, \rho_x\right]\right]\Big).
\end{aligned}$$

The result (4.48) then follows from Theorem 4.4.3(i).

Step 2. Proof of (i).

We repeat the analysis of Step 1 in the particular case of a Poisson point process $\mathcal{P} = \mathcal{P}_0$ on $\mathbb{R}^d$ with constant intensity $\mu > 0$. In this case, we have $\rho_x = 0$, hence $J_v(x,r) = \varnothing$, so that the action radius $\rho_{x,v}^{s}$ takes the simpler form

$$\rho_{x,v}^{s} = v\,\mathbb{1}_{\mathcal{X}_s \neq \mathcal{X}_s^{x,v}}.$$

Estimating

$$\begin{aligned}
\mathbb{P}\left[\ell-1 \leq \rho_{x,v}^{s} < \ell,\ A_s(\mathcal{X}_s^{x,v}) \neq A_s(\mathcal{X}_s)\right] &\leq \mathbb{P}\left[\ell-1 \leq \rho_{x,v}^{s} < \ell,\ \mathcal{X}_s^{x,v} \neq \mathcal{X}_s\right] \\
&\leq \mathbb{P}\left[\mathcal{X}_s^{x,v} \neq \mathcal{X}_s\right]\mathbb{1}_{\ell-1\leq v<\ell} \ \leq 2\mu\gamma(v)\,\mathbb{1}_{\ell-1\leq v<\ell},
\end{aligned}$$

the conclusion (4.45) follows from Theorem 4.4.3(i). It remains to prove (∂^{osc}-WCI). Since we obviously have $\mathbb{P}\big[\rho_{x,v}^{s} < \ell\big] = 1$ if $v < \ell$, we compute for all $x \in \mathbb{Z}^d$, $v \geq 0$, $\ell \geq 1$,

$$\frac{\mathbb{P}\left[\ell-1 \leq \rho_{x,v}^{s} < \ell,\ \mathcal{X}_s^{x,v} \neq \mathcal{X}\right]}{\mathbb{P}\left[\rho_{x,v}^{s} < \ell\right]} \leq \frac{2\mu\gamma(v)\,\mathbb{1}_{\ell-1\leq v<\ell}}{\mathbb{P}\left[\rho_{x,v}^{s} < \ell\right]} = 2\mu\gamma(v)\,\mathbb{1}_{\ell-1\leq v<\ell},$$

and Theorem 4.4.3(ii) with influence function $f(u) = u$ then yields

$$\operatorname{Cov}\left[Y(A_s); Z(A_s)\right] \leq \mu \sum_{x\in\mathbb{Z}^d}\sum_{v=0}^{\infty}\gamma(v)\,\mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A_s,Q_{2v+3}(x)}\,Y(A_s)\Big)^2\right]^{\frac12}\mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A_s,Q_{2v+3}(x)}\,Z(A_s)\Big)^2\right]^{\frac12}.$$

The desired covariance estimate (∂^{osc}-WCI) follows by taking local averages.

Step 3. Proof of (ii).

In this step, we consider the case when the stationary point process $\mathcal{P}$ satisfies a hard-core condition $\sharp(\mathcal{P}\cap Q) \leq C$ a.s. for some deterministic constant $C > 0$, and also satisfies the following covariance inequality (resp. the corresponding (∂^{osc}-WSG)) with some integrable weight π_0: for all $\sigma(\mathcal{P})$-measurable random variables $Y(\mathcal{P}), Z(\mathcal{P})$,

$$\begin{aligned}
\operatorname{Cov}\left[Y(\mathcal{P}); Z(\mathcal{P})\right] \leq \int_{\mathbb{R}^d}&\left(\int_0^\infty \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{\mathcal{P},B_{\ell+1}(x)}\,Y(\mathcal{P})\Big)^2\right](\ell+1)^{-d}\pi_0(\ell)d\ell\right)^{\frac12} \\
&\times\left(\int_0^\infty \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{\mathcal{P},B_{\ell+1}(x)}\,Z(\mathcal{P})\Big)^2\right](\ell+1)^{-d}\pi_0(\ell)d\ell\right)^{\frac12} dx.
\end{aligned}$$

We then show that, for each $s = 1,2,3$, the random field A_s satisfies the following covariance inequality (resp. the corresponding (∂^{osc}-WSG)): for all $\sigma(A_s)$-measurable random variables $Y(A_s), Z(A_s)$ we have

$$\begin{aligned}
\operatorname{Cov}\left[Y(A_s); Z(A_s)\right] \leq C\int_{\mathbb{R}^d}&\left(\int_0^\infty \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A_s,B_{2\ell+1}(x)}\,Y(A_s)\Big)^2\right](\ell+1)^{-d}\pi(\ell)d\ell\right)^{\frac12} \\
&\times\left(\int_0^\infty \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A_s,B_{2\ell+1}(x)}\,Z(A_s)\Big)^2\right](\ell+1)^{-d}\pi(\ell)d\ell\right)^{\frac12} dx, \quad (4.49)
\end{aligned}$$

where we have set $\pi(\ell) := \pi_0(\ell) + (\ell+1)^d\,\mathbb{P}[\ell-1 \le V < \ell]$. In particular, combined with Proposition 4.5.3, this implies the covariance inequality (4.46) in the case of the random parking point process.

To simplify notation, we only treat the case of the spectral gap inequality. Consider a measurable enumeration of the point process $\mathcal{P} = \{Z_j\}_j$, let $\{Z_j, V_j, Y_{s,j}\}$ be a decorated point process associated with $\mathcal{P}$ and the decoration law (V, Y_s), and let $\mathcal{D} := \{V_j, Y_{s,j}\}_j$ denote the decoration sequence. Since $\mathcal{P}$ and $\mathcal{D}$ are independent, the expectation $\mathbb{E}$ splits into $\mathbb{E} = \mathbb{E}_{\mathcal{P}}\mathbb{E}_{\mathcal{D}}$, where $\mathbb{E}_{\mathcal{P}}[\cdot] = \mathbb{E}[\cdot\|\mathcal{D}]$ denotes the expectation with respect to $\mathcal{P}$, and where $\mathbb{E}_{\mathcal{D}}[\cdot] = \mathbb{E}[\cdot\|\mathcal{P}]$ denotes the expectation with respect to $\mathcal{D}$. By tensorization of the variance as in (4.56), the spectral gap assumption for $\mathcal{P}$ and the standard spectral gap (4.24) for the i.i.d. sequence $\mathcal{D}$ then yields for all random variables $Z = Z(A_s)$,

$$\begin{aligned}
&\operatorname{Var}[Z(A_s)] = \mathbb{E}_{\mathcal{P}}\big[\operatorname{Var}_{\mathcal{D}}[Z(A_s)]\big] + \operatorname{Var}_{\mathcal{P}}\big[\mathbb{E}_{\mathcal{D}}[Z(A_s)]\big] \\
&\le \frac{1}{2}\sum_k \mathbb{E}\Big[\big(Z(A_s) - Z(A_s^k)\big)^2\Big] + \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{\mathcal{P},B_{\ell+1}(x)}\,\mathbb{E}_{\mathcal{D}}[Z(A_s)]\Big)^2\Big]\,dx\,(\ell+1)^{-d}\pi_0(\ell)d\ell, \quad (4.50)
\end{aligned}$$

where A_s^k corresponds to the field A_s with the decoration $(V_k, Y_{s,k})$ replaced by an i.i.d. copy $(V_k', Y_{s,k}')$. We separately estimate the two right-hand side terms in (4.50), and we begin with the first. For all $x \in \mathbb{R}^d$, we define the following two random variables,

$$N(x) := \sharp(\mathcal{P} \cap B(x)), \qquad R(x) := \max\{V_j : Z_j \in B(x)\}.$$

Let $R_0 \ge 1$ denote the smallest value such that $\mathbb{P}[V < R_0] \ge \frac{1}{2}$, which implies in particular by a union bound and by the hard-core assumption

$$\mathbb{P}[R(x) < R_0] = \mathbb{E}\big[\mathbb{P}[V < R_0]^{N(x)}\big] \ge \mathbb{E}\big[2^{-N(x)}\big] \ge 2^{-C}. \quad (4.51)$$

Conditioning with respect to the value of $R(x)$, we obtain

$$\begin{aligned}
&\sum_k \mathbb{E}\Big[\big(Z(A_s) - Z(A_s^k)\big)^2\Big] \\
&\quad\lesssim \int_{R_0}^\infty \int_{\mathbb{R}^d} \sum_k \mathbb{E}\Big[\big(Z(A_s) - Z(A_s^k)\big)^2 \mathbb{1}_{Z_k \in B(x)} \mathbb{1}_{\ell-1 \le R(x) < \ell}\Big]\,dx\,d\ell \\
&\qquad\qquad + \int_{\mathbb{R}^d} \sum_k \mathbb{E}\Big[\big(Z(A_s) - Z(A_s^k)\big)^2 \mathbb{1}_{Z_k \in B(x)} \mathbb{1}_{R(x) < R_0}\Big]\,dx \\
&\quad\le \int_{R_0}^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{A_s,B_{\ell+1}(x)}\,Z(A_s)\Big)^2 N(x)\,\mathbb{1}_{\ell-1 \le R(x) < \ell}\Big]\,dx\,d\ell \\
&\qquad\qquad + \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{A_s,B_{R_0+1}(x)}\,Z(A_s)\Big)^2 N(x)\Big]\,dx \\
&\quad= \int_{R_0}^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{A_s,B_{\ell+1}(x)}\,Z(A_s)\Big)^2 N(x)\,\mathbb{1}_{R(x) \ge \ell-1}\,\Big\|\,R(x) < \ell\Big]\,\mathbb{P}[R(x) < \ell]\,dx\,d\ell \\
&\qquad\qquad + \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{A_s,B_{R_0+1}(x)}\,Z(A_s)\Big)^2 N(x)\Big]\,dx.
\end{aligned}$$

Using the hard-core assumption in the form $N(x) \le C$ a.s., and noting that given $R(x) < \ell$ the random variable $R(x)$ is independent of $A_s|_{\mathbb{R}^d \setminus B_{\ell+1}(x)}$, we deduce

$$\begin{aligned}
\sum_k \mathbb{E}\Big[\big(Z(A_s) - Z(A_s^k)\big)^2\Big] &\lesssim \int_{R_0}^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{A_s,B_{\ell+1}(x)}\,Z(A_s)\Big)^2\Big]\,\frac{\mathbb{P}[\ell-1 \le R(x) < \ell]}{\mathbb{P}[R(x) < \ell]}\,dx\,d\ell \\
&\qquad + \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\partial^{\mathrm{osc}}_{A_s,B_{R_0+1}(x)}\,Z(A_s)\Big)^2\Big]\,dx.
\end{aligned}$$

Estimating by a union bound $\mathbb{P}[\ell-1 \le R(x) < \ell] \le C\,\mathbb{P}[\ell-1\le V<\ell]$, and making use of the property (4.51) of the choice of $R_0 \ge 1$, we conclude

$$\sum_k \mathbb{E}\Big[\big(Z(A_s)-Z(A_s^k)\big)^2\Big] \lesssim \int_{R_0}^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{A_s,B_{\ell+1}(x)}\,Z(A_s)\Big)^2\Big]\,\mathbb{P}[\ell-1\le V<\ell]\,dx\,d\ell + \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{A_s,B_{R_0+1}(x)}\,Z(A_s)\Big)^2\Big]\,dx. \quad (4.52)$$

It remains to estimate the second right-hand side term in (4.50). The hard-core assumption for $\mathcal{P}$ yields by stationarity $\sharp(\mathcal{P}\cap B_\ell(x)) \le C\ell^d$ a.s. Further noting that a union bound gives

$$\mathbb{P}\Big[r-1\le \max_{1\le j\le C\ell^d} V_j < r\Big] \le \sum_{j=1}^{C\ell^d} \mathbb{P}\Big[V_j \ge r-1,\ \text{and}\ V_k<r\ \forall 1\le k\le C\ell^d\Big] = C\ell^d\,\mathbb{P}[V<r]^{C\ell^d-1}\,\mathbb{P}[r-1\le V<r]\,,$$

and hence for all $r\ge R_0$,

$$\frac{\mathbb{P}\big[r-1\le \max_{1\le j\le C\ell^d} V_j<r\big]}{\mathbb{P}\big[\max_{1\le j\le C\ell^d} V_j<r\big]} \le C\ell^d\,\frac{\mathbb{P}[r-1\le V<r]}{\mathbb{P}[V<r]} \le 2C\ell^d\mathbb{P}[r-1\le V<r]\,,$$

we find, arguing similarly as above,

$$\int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{\mathcal{P},B_\ell(x)}\,\mathbb{E}_{\mathcal{D}}[Z(A_s)]\Big)^2\Big]\,dx\,(\ell+1)^{-d}\pi_0(\ell)d\ell$$
$$\lesssim \int_0^\infty\int_{R_0}^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{A_s,B_{\ell+r}(x)}\,Z(A_s)\Big)^2\Big]\,dx\,\mathbb{P}[r-1\le V<r]\,dr\,\pi_0(\ell)d\ell + \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(\,\partial^{\mathrm{osc}}_{A_s,B_{\ell+R_0}(x)}\,Z(A_s)\Big)^2\Big]\,dx\,\pi_0(\ell)d\ell. \quad (4.53)$$

Combining this with (4.50) and (4.52), the conclusion (4.49) follows in variance form. □

4.5.5 Dependent coloring of random geometric patterns

Up to here, besides Gaussian random fields, all examples of random fields that we have been considering correspond to random geometric patterns (various random point processes constructed from a higher-dimensional Poisson process or random tessellations) endowed with an independent coloring determining e.g. the size and shape of the cells and the value of the field in the cells. In the present subsection, we turn to the examples of type (III) mentioned in the introduction in Section 4.1.1, and consider *dependent* colorings of the random geometric patterns. The random field A is now a function of both a product structure (typically some decorated Poisson point process $\hat{\mathcal{P}}$), and of a random field G (e.g. a Gaussian random field) which typically has long-range correlations but is assumed to satisfy some weighted functional inequality. In other words, this amounts to mixing up all the previous examples. Rather than stating general results in this direction, we only treat a number of typical concrete examples in order to illustrate the robustness of the approach.

(1) The first example A_1 is a random field on $\mathbb{R}^d$ corresponding to random spherical inclusions centered at the points of a Poisson point process $\mathcal{P}$ of intensity $\mu=1$, with i.i.d. random radii of law V, but such that the values on the inclusions are determined by some random field G_1 with long-range correlations.
More precisely, we let $\hat{\mathcal{P}}_1 := \{\tilde X_j, \tilde V_j, \tilde U_j\}_j$ denote a decorated point process associated with $\mathcal{P}$ and

(V, U), where U denotes an independent uniform random variable on $[0,1]$. Independently of $\hat{\mathcal{P}}_1$ we choose a jointly measurable stationary bounded random field G_1 on $\mathbb{R}^d$, with typically long-range correlations. The collection of random inclusions is given by $\{\tilde{I}_1^j\}_j$ with $\tilde{I}_1^j := \tilde{X}_j + \tilde{V}_j B$. As in the third example of Section 4.5.4, we choose a $\sigma(V,U)$-measurable random variable $P(V,U)$, and we say that the inclusion $\tilde{I}_1^j$ has the priority on inclusion $\tilde{I}_1^i$ if $P(\tilde{V}_j, \tilde{U}_j) < P(\tilde{V}_i, \tilde{U}_i)$ or if $P(\tilde{V}_j, \tilde{U}_j) = P(\tilde{V}_i, \tilde{U}_i)$ and $\tilde{U}_j < \tilde{U}_i$. This defines a priority order on the inclusions on a set of maximal probability, and we then relabel the inclusions and the points of $\hat{\mathcal{P}}_1$ into a sequence $(I_1^j, X_j, V_j, U_j)_j$ such that for all j the inclusion I_1^j has the j-th highest priority. Given $\beta \in \mathbb{R}$, we then consider the $\sigma(\hat{\mathcal{P}}_1, G_1)$-measurable random field A_1 defined as follows,

$$A_1 := \beta + \sum_j \big(G_1(X_j) - \beta\big) \mathbb{1}_{I_1^j \setminus \bigcup_{i:i<j} I_1^i}.$$

(2) The second example A_2 is a random field on $\mathbb{R}^d$ corresponding to random inclusions centered at the points of a Poisson point process $\mathcal{P}$ of intensity $\mu = 1$, with i.i.d. random radii of law V, but with orientations determined by some random field G_2 with long-range correlations.
More precisely, we let $\hat{\mathcal{P}}_2 := \{X_j, V_j\}_j$ denote a decorated point process associated with $\mathcal{P}$ and V, we choose a reference shape $S \in \mathcal{B}(\mathbb{R}^d)$ with $0 \in S$, and independently of $\hat{\mathcal{P}}_2$ we choose a jointly measurable stationary bounded random field G_2 on $\mathbb{R}^d$ with values in the orthogonal group $O(d)$ in dimension d, and with typically long-range correlations. The collection of random inclusions is then given by $\{I_2^j\}_j$ with $I_2^j := X_j + G_2(X_j)S$. Given $\alpha, \beta \in \mathbb{R}$, and given a function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(t) = 1$ for $t \le 1$ and $\phi(t) = 0$ for $t \ge 2$, and with $\|\phi'\|_{\mathrm{L}^\infty} \lesssim 1$, we then consider the $\sigma(\hat{\mathcal{P}}_2, G_2)$-measurable random field A_2 defined as follows,

$$A_2(x) := \beta + (\alpha - \beta)\, \phi\Big(d\big(x\,,\, \cup_j I_2^j\big)\Big).$$

(Note that the smoothness of this interpolation ϕ between the values α and β is crucial for the arguments below.)

(3) The third example A_3 is a random field on $\mathbb{R}^d$ corresponding to the Voronoi tessellation associated with the points of a Poisson point process $\mathcal{P}$ of unit intensity, such that the values on the cells are determined by some random field G_3 with long-range correlations.
More precisely, we let $\hat{\mathcal{P}}_3 := \mathcal{P} = \{X_j\}_j$, and we let $\{C_j\}_j$ denote the partition of $\mathbb{R}^d$ into the Voronoi cells associated with the Poisson points $\{X_j\}_j$. Independently of $\hat{\mathcal{P}}_3$ we choose a jointly measurable stationary bounded random field G_3 on $\mathbb{R}^d$. We then consider the $\sigma(\hat{\mathcal{P}}_3, G_3)$-measurable random field A_3 defined as follows,

$$A_3(x) := \sum_j G_3(X_j) \mathbb{1}_{C_j}.$$

For each of these examples, we show functional inequalities with as derivative the supremum of the functional derivative ∂^{fct}, which we define by

$$\partial^{\sup}_{A,S} X(A) \;:= \sup_{A,S} \mathrm{ess} \int_S \Big| \frac{\partial \tilde{X}(A)}{\partial A} \Big|.$$

Note that provided A is bounded we have $\partial^{\mathrm{osc}}, \partial^{\mathrm{fct}} \lesssim \partial^{\sup}$. From the proofs in Sections 4.2 and 4.3, it is clear that weighted functional inequalities with $\partial^{\sup}$ imply the same concentration properties as the corresponding functional inequalities with ∂^{osc}. Note that the proof of the following result is quite robust and many variants could be considered.

Proposition 4.5.5. *For $s = 1, 2, 3$, assume that the random field G_s satisfies (∂^{fct}-WSG) for some integrable weight π_s. For $s = 1, 2$, set $\gamma(v) := \mathbb{P}[v - 4 \le V < v + 4]$. Then the following holds.*

(i) For $s=1,2$, the above-defined random field A_s satisfies the following weighted spectral gap: for all $\sigma(A_s)$-measurable random variable $Z(A_s)$ we have

$$\mathrm{Var}\left[Z(A_s)\right] \lesssim \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{\mathrm{sup}}_{A,B_{\ell+C(v+1)}(x)} Z(A_s)\right)^2\right] dx \big((\ell+1)^{-d} \wedge \gamma(v)\big)\pi_s(\ell) dv d\ell. \tag{4.54}$$

In the case when the random variable V is almost surely bounded by a deterministic constant, we rather obtain

$$\begin{aligned}\mathrm{Var}\left[Z(A_s)\right] \lesssim & \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\ \partial^{\mathrm{osc}}_{A_s,B_C(x)}\ Z(A_s)\right)^2\right] dx \\ & + \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{\mathrm{fct}}_{A_s,B_{\ell+C}(x)} Z(A_s)\right)^2\right] dx\,(\ell+1)^{-d}\pi_s(\ell) d\ell,\end{aligned} \tag{4.55}$$

and if the random field G_s further satisfies (∂^{fct}-WLSI) with weight π_s, then the corresponding logarithmic Sobolev inequality also holds, that is,

$$\begin{aligned}\mathrm{Ent}[Z(A_s)] \lesssim & \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\ \partial^{\mathrm{osc}}_{A_s,B_C(x)}\ Z(A_s)\right)^2\right] dx \\ & + \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{\mathrm{fct}}_{A_s,B_{\ell+C}(x)} Z(A_s)\right)^2\right] dx\,(\ell+1)^{-d}\pi_s(\ell) d\ell.\end{aligned}$$

(ii) The above-defined random field A_3 satisfies (∂^{sup}-WSG) with weight $\pi(\ell) := C(\pi_3(\ell) + e^{-\frac{1}{C}\ell^d})$. If the random field G_3 further satisfies (∂^{fct}-WLSI) with weight π_3, then A_3 also satisfies (∂^{sup}-WLSI) with weight π. ♢

Proof. For $s=1,2,3$, since $\hat{\mathcal{P}}_s$ and G_s are independent, the expectation $\mathbb{E}$ splits into $\mathbb{E} = \mathbb{E}_{\hat{\mathcal{P}}_s}\mathbb{E}_{G_s}$, where $\mathbb{E}_{\hat{\mathcal{P}}_s}[\cdot] = \mathbb{E}[\cdot \| G_s]$ denotes the expectation with respect to $\hat{\mathcal{P}}_s$, and where $\mathbb{E}_{G_s}[\cdot] = \mathbb{E}[\cdot\|\hat{\mathcal{P}}_s]$ denotes the expectation with respect to G_s. The variance and the entropy also tensorize: for all $\sigma(A_s)$-measurable random variables $Z(A_s)$,

$$\begin{aligned}\mathrm{Var}\left[Z(A_s)\right] &= \mathrm{Var}_{G_s}[\mathbb{E}_{\hat{\mathcal{P}}_s}[Z(A_s)]] + \mathbb{E}_{G_s}[\mathrm{Var}_{\hat{\mathcal{P}}_s}[Z(A_s)]], \\ \mathrm{Ent}[Z(A_s)] &= \mathrm{Ent}_{G_s}[\mathbb{E}_{\hat{\mathcal{P}}_s}[Z(A_s)]] + \mathbb{E}_{G_s}[\mathrm{Ent}_{\hat{\mathcal{P}}_s}[Z(A_s)]].\end{aligned} \tag{4.56}$$

In each of the examples under consideration, the estimate on the terms $\mathrm{Var}_{\hat{\mathcal{P}}_s}[Z(A_s)]$ and $\mathrm{Ent}_{\hat{\mathcal{P}}_s}[Z(A_s)]$ (with G_s "frozen") follows from the same arguments as in the proof of Propositions 4.5.2 and 4.5.4(i). We therefore focus on the estimates of $\mathrm{Var}_{G_s}[\mathbb{E}_{\hat{\mathcal{P}}_s}[Z(A_s)]]$ and $\mathrm{Ent}_{G_s}[\mathbb{E}_{\hat{\mathcal{P}}_s}[Z(A_s)]]$, and only treat the case of the variance. We split the proof into three steps.

Step 1. Cases $s=1,2$ with bounded radius law.

Since the random field G_s is assumed to satisfy (∂^{fct}-WSG) with weight π_s, we obtain

$$\begin{aligned}\mathrm{Var}_{G_s}[\mathbb{E}_{\hat{\mathcal{P}}_s}[Z(A_s)]] \,&\le\, \mathbb{E}_{\hat{\mathcal{P}}_s}[\mathrm{Var}_{G_s}[Z(A_s)]] \\ &\le\, \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \left(\partial^{\mathrm{fct}}_{G_s,B_{\ell+1}(x)} Z(A_s)\right)^2 dx\,(\ell+1)^{-d}\pi_s(\ell) d\ell\right].\end{aligned} \tag{4.57}$$

The chain rule yields

$$\begin{aligned}\partial^{\mathrm{fct}}_{G_s,B_{\ell+1}(x)} Z(A_s) \,&=\, \int_{B_{\ell+1}(x)} \Big|\frac{\partial Z(A_s(\hat{\mathcal{P}}_s, G_s))}{\partial G_s}(y)\Big| dy \\ &\le \int_{B_{\ell+1}(x)} \int_{\mathbb{R}^d} \Big|\frac{\partial Z(A_s)}{\partial A_s}(z)\Big|\Big|\frac{\partial A_s(\hat{\mathcal{P}}_s, G_s)(z)}{\partial G_s}(y)\Big| dz dy.\end{aligned}$$

Since A_s is $\sigma(\hat{\mathcal{P}}_s, \{G_s(X_j)\}_j)$-measurable, we obtain

$$\partial^{\mathrm{fct}}_{G_s,B_{\ell+1}(x)} Z(A_s) \le \sum_j \mathbb{1}_{X_j\in B_{\ell+1}(x)} \int_{\mathbb{R}^d} \Big|\frac{\partial Z(A_s)}{\partial A_s}(z)\Big|\Big|\frac{\partial A_s(\hat{\mathcal{P}}_s, G_s)(z)}{\partial G_s(X_j)}\Big| dz \tag{4.58}$$

in terms of the usual partial derivative of $A_s(\hat{\mathcal{P}}_s, G_s)(z)$ with respect to $G_s(X_j)$. We now need to compute this derivative in each of the considered examples. We claim that

$$\Big|\frac{\partial A_s(\hat{\mathcal{P}}_s, G_s)(z)}{\partial G_s(X_j)}\Big| \le C\mathbb{1}_{R_s^j}(z), \tag{4.59}$$

where

$$R_s^j := \begin{cases} I_1^j \setminus \bigcup_{i:i<j} I_1^i, & \text{if } s=1;\\ \big\{x \,:\, 0<d(x,I_2^j)<2\wedge d(x,I_2^k),\ \forall k\neq j\big\}, & \text{if } s=2;\\ C_j, & \text{if } s=3. \end{cases}$$

This claim (4.59) is obvious for $s=1$ and $s=3$. For $s=2$, the properties of ϕ and the definition of R_2^j yield

$$\Big|\frac{\partial A_2(\hat{\mathcal{P}}_2, G_2)(z)}{\partial G_2(X_j)}\Big| \le |\alpha-\beta|\Big|\phi'\Big(d\big(z\,,\,\cup_k I_2^k\big)\Big)\Big|\mathbb{1}_{R_2^j}(z) = |\alpha-\beta|\big|\phi'\big(d(z,I_2^j)\big)\big|\mathbb{1}_{R_2^j}(z),$$

which indeed implies (4.59). Now injecting (4.59) into (4.58), and noting that in each case the sets $\{R_s^j\}_j$ are disjoint, we obtain

$$\begin{aligned}\partial^{\mathrm{fct}}_{G_s,B_{\ell+1}(x)} Z(A_s) \le C\sum_j \mathbb{1}_{X_j\in B_{\ell+1}(x)} \int_{R_s^j}\Big|\frac{\partial Z(A_s)}{\partial A_s}\Big| &= C\int_{\bigcup_{j:X_j\in B_{\ell+1}(x)} R_s^j}\Big|\frac{\partial Z(A_s)}{\partial A_s}\Big|\\ &\le C\int_{B_{D_s(\ell,x)}(x)}\Big|\frac{\partial Z(A_s)}{\partial A_s}\Big|, \end{aligned}\tag{4.60}$$

with

$$D_s(\ell,x) := \sup\Big\{d(y,x)\,:\, y\in \bigcup_{j:X_j\in B_{\ell+1}(x)} R_s^j\Big\}.$$

For $s=1,2$ with radius law V almost surely bounded by a deterministic constant $R>0$, we obtain $D_1(\ell,x)\le \ell+R+1$ and $D_2(\ell,x)\le \ell+R+3$, and injecting (4.60) into (4.57) directly yields the result (4.55).

Step 2. Cases $s=1,2$ with unbounded radius law.

We now consider the cases $s=1,2$ with general unbounded radii. Without loss of generality we only treat $s=1$, in which case

$$D_1(\ell,x)\le \ell+1+\bar{D}_1(\ell,x), \qquad \bar{D}_1(\ell,x) := \max\big\{V_j \,:\, X_j\in B_{\ell+1}(x)\big\}.$$

Noting that the restriction $A_1|_{\mathbb{R}^d\setminus B_{\ell+1+\bar{D}_1(\ell,x)}(x)}$ is by construction independent of $\bar{D}_1(\ell,x)$, we obtain, conditioning on the values of $\bar D_1(\ell,x)$ and arguing as in Step 2 of the proof of Theorem 4.4.3,

$$\begin{aligned}
&\mathbb{E}\left[\Big(\int_{B_{\ell+1+\bar D_1(\ell,x)}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2\right]\\
&\le \sum_{i=1}^{\infty}\mathbb{P}\big[i-1\le \bar D_1(\ell,x)<i\big]\mathbb{E}\left[\Big(\int_{B_{\ell+i+1}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2\,\Big\|\, i-1\le \bar D_1(\ell,x)<i\right]\\
&\le \sum_{i=1}^{\infty}\mathbb{P}\big[i-1\le \bar D_1(\ell,x)<i\big]\mathbb{E}\left[\sup_{A_1,B_{\ell+i+1}(x)}\operatorname{ess}\Big(\int_{B_{\ell+i+1}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2\,\Big\|\, \bar D_1(\ell,x)<i\right]\\
&\le \sum_{i=1}^{\infty}\frac{\mathbb{P}\big[i-1\le \bar D_1(\ell,x)<i\big]}{\mathbb{P}\big[\bar D_1(\ell,x)<i\big]}\mathbb{E}\left[\sup_{A_1,B_{\ell+i+1}(x)}\operatorname{ess}\Big(\int_{B_{\ell+i+1}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2\right]. \qquad (4.61)
\end{aligned}$$

Now by definition of the decorated Poisson point process $\hat{\mathcal{P}}_1$, we may compute for all $i\ge 1$,

$$\begin{aligned}
\mathbb{P}\big[\bar D_1(\ell,x)\ge i-1\big]&=\mathbb{P}\big[\exists j\,:\,V_j\ge i-1 \text{ and } X_j\in B_{\ell+1}(x)\big]\\
&=e^{-|B_{\ell+1}|}\sum_{n=0}^{\infty}\frac{|B_{\ell+1}|^n}{n!}\big(1-(1-\mathbb{P}\left[V\ge i-1\right])^n\big)=1-e^{-|B_{\ell+1}|\,\mathbb{P}[V\ge i-1]},
\end{aligned}$$

and hence

$$\frac{\mathbb{P}\big[i-1\le \bar D_1(\ell,x)<i\big]}{\mathbb{P}\big[\bar D_1(\ell,x)<i\big]}=1-e^{-|B_{\ell+1}|\mathbb{P}[i-1\le V<i]}\le 1\wedge\big(C(\ell+1)^d\,\mathbb{P}\left[i-1\le V<i\right]\big).$$

Combining this computation with (4.57), (4.60) and (4.61), and setting $\gamma(v):=\mathbb{P}\left[v-2\le V<v+1\right]$, we obtain

$$\begin{aligned}
&\operatorname{Var}_{G_1}[\mathbb{E}_{\hat{\mathcal{P}}_1}[Z(A_1)]]\\
&\lesssim \mathbb{E}\left[\int_0^\infty\sum_{i=1}^\infty\int_{\mathbb{R}^d}\sup_{A_1,B_{\ell+i+1}(x)}\operatorname{ess}\Big(\int_{B_{\ell+i+1}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2dx\big((\ell+1)^{-d}\wedge\mathbb{P}\left[i-1\le V<i\right]\big)\pi_s(\ell)d\ell\right]\\
&\le \mathbb{E}\left[\int_0^\infty\int_0^\infty\int_{\mathbb{R}^d}\sup_{A_1,B_{\ell+v+2}(x)}\operatorname{ess}\Big(\int_{B_{\ell+v+2}(x)}\Big|\frac{\partial Z(A_1)}{\partial A_1}\Big|\Big)^2dx\,\big((\ell+1)^{-d}\wedge\gamma(v)\big)\pi_s(\ell)dvd\ell\right],
\end{aligned}$$

and the conclusion (4.54) follows.

Step 3. Case $s=3$.

We now turn to the case $s=3$, for which

$$D_3(\ell,x)\le \ell+1+\bar D_3(\ell,x),\qquad \bar D_3(\ell,x):=\max\big\{\operatorname{diam}(C_j)\,:\,X_j\in B_{\ell+1}(x)\big\}.$$

Noting that the restriction $A_3|_{\mathbb{R}^d\setminus B_{\ell+1+2\bar D_3(\ell,x)}(x)}$ is by construction independent of $\bar D_3(\ell,x)$ we obtain, after conditioning on the values of $\bar D_3(\ell,x)$ and arguing as in (4.61),

$$\begin{aligned}
\mathbb{E}\left[\Big(\int_{B_{\ell+1+\bar D_3(\ell,x)}(x)}\Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2\right]&\le \mathbb{E}\left[\sup_{A_3,B_{3\ell+1}(x)}\operatorname{ess}\Big(\int_{B_{3\ell+1}(x)}\Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2\right]\\
&\quad+\sum_{i=2\ell}^{\infty}\frac{\mathbb{P}\big[i-1\le \bar D_3(\ell,x)<i\big]}{\mathbb{P}\big[\bar D_3(\ell,x)<i\big]}\mathbb{E}\left[\sup_{A_3,B_{\ell+i+1}(x)}\operatorname{ess}\Big(\int_{B_{\ell+i+1}(x)}\Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2\right]. \qquad (4.62)
\end{aligned}$$

Similar computations as in Step 2 of the proof of Proposition 4.5.2 yield

$$\mathbb{P}\big[\bar{D}_3(\ell,x) \geq i\big] \leq Ce^{-\frac{1}{C}(i-\ell)_+^d}.$$

Combining this with (4.57), (4.60) and (4.62), we obtain

$$\begin{aligned}
&\mathrm{Var}_{G_3}[\mathbb{E}_{\hat{\mathcal{P}}_3}[Z(A_3)]] \\
&\lesssim \; \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \operatorname*{sup\,ess}_{A_3, B_{3\ell+1}(x)} \Big(\int_{B_{3\ell+1}(x)} \Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2 dx\, (\ell+1)^{-d}\pi_3(\ell) d\ell\right] \\
&\qquad + \mathbb{E}\left[\int_0^\infty \sum_{i=2\ell}^\infty e^{-\frac{1}{C}i^d} \int_{\mathbb{R}^d} \operatorname*{sup\,ess}_{A_3, B_{2i+1}(x)} \Big(\int_{B_{2i+1}(x)} \Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2 dx\, (\ell+1)^{-d}\pi_3(\ell) d\ell\right] \\
&\lesssim \; \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \operatorname*{sup\,ess}_{A_3, B_{3\ell+1}(x)} \Big(\int_{B_{3\ell+1}(x)} \Big|\frac{\partial Z(A_3)}{\partial A_3}\Big|\Big)^2 dx\, \big((\ell+1)^{-d}\pi_3(\ell) + e^{-\frac{1}{C}\ell^d}\big) d\ell\right],
\end{aligned}$$

and the result follows. □

4.6 Weighted second-order Poincaré inequalities

Chatterjee's second-order Poincaré inequalities are known to hold in total variation distance for Gaussian fields with integrable covariance function [113, 349], as well as in Wasserstein and Kolmogorov distance for general discrete product structures [112, 282]. Based on these results, similarly as for first-order functional inequalities in Sections 4.4 and 4.5, we prove the validity of *weighted* second-order Poincaré inequalities for correlated random fields that display a hidden product structure. Again, we distinguish between two prototypical classes of examples (cf. Section 4.4.2): deterministically localized fields (which essentially concern Gaussian fields), and randomly localized fields (in which case localization is quantified in terms of the action radius). These two situations are separately addressed in Sections 4.6.1 and 4.6.2 below.

Before we state the main results, let us comment on the existing literature. On the one hand, for Gaussian random fields, our results can be compared with [349, Theorem 1.1] (see also [348]), which establishes a similar (infinite-dimensional) second-order Poincaré inequality in terms of Malliavin calculus in abstract Wiener space (where the covariance structure is encoded in some Hilbert norm). The interest of our formulation is the explicit structure of the right-hand side in the form of a weighted inequality, in line with our approach to generalized first order functional inequalities.

On the other hand, for randomly localized fields, our approach to control distance to normality can be compared to [286], which develops a general strategy to prove approximate normality results for functionals of Poisson processes based on stabilization properties. In particular, this approach requires stabilization properties to be checked explicitly each time a normal approximation result is to be proved. In contrast, given a random field A which is a transformation of a Poisson process, our approach consists in exploiting stabilization properties of the transformation (in the form of a control on the action radius) to derive a "generalized" second-order functional inequality. This weighted second-order Poincaré inequality has the advantage to be intrinsic for the field A, and as such it can be subsequently applied to any random variable $X(A)$ without having to make further use of the stabilization properties of the transformation.

4.6.1 Deterministically localized fields

In this subsection we treat the main example of deterministically localized fields, that is, correlated Gaussian random fields. The main result of this section is a continuum version with nontrivial covariance structure of the second-order Poincaré inequality for i.i.d. Gaussian random variables due to Chatterjee [113], and based on Stein's method. As already discussed, this is to be compared with [349].

Theorem 4.6.1. *Let G be a jointly measurable stationary Gaussian random field on $\mathbb{R}^d$, characterized by its covariance $\mathcal{C}(x) := \mathrm{Cov}\,[G(x); G(0)]$, and assume that $|\mathcal{C}(x)| \le c(|x|)$ for some Lipschitz non-increasing map $c : \mathbb{R}_+ \to \mathbb{R}_+$. Let $h \in C^2(\mathbb{R})$ with $h', h'' \in \mathrm{L}^\infty(\mathbb{R})$, and let A be the random field on $\mathbb{R}^d$ defined by $A(x) := h(G(x))$ for all x. Then for all $\sigma(A)$-measurable random variable $X(A)$ and all $R > 0$ we have*

$$
\begin{aligned}
&\mathrm{d_{TV}}\left(\frac{X(A)-\mathbb{E}\,[X(A)]}{\sqrt{\mathrm{Var}\,[X(A)]}}, \mathcal{N}\right)^2 \qquad (4.63)\\
&\lesssim \frac{\|h'\|_{\mathrm{L}^\infty}^6}{(\mathrm{Var}\,[X(A)])^2}\mathbb{E}\left[\Big(\int_0^\infty\int_{\mathbb{R}^d}\Big(\int_{B_{2(\ell+1)}(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2 dx\,(\ell+1)^{-d}(-c'(\ell))d\ell\Big)^2\right]^{\frac12}\\
&\qquad\times\mathbb{E}\left[\Big(\int_0^\infty\int_0^\infty\iint_{\mathbb{R}^d\times\mathbb{R}^d}\Big(\iint_{B_{2(\ell_1+1)}(x_1)\times B_{2(\ell_2+1)}(x_2)}\Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2 dx_1dx_2\right.\\
&\qquad\qquad\left.\times(\ell_2+1)^{-d}(-c'(\ell_2))d\ell_2\,(\ell_1+1)^{-d}(-c'(\ell_1))d\ell_1\Big)^2\right]^{\frac12}\\
&\quad+\frac{\|h'\|_{\mathrm{L}^\infty}^2\|h''\|_{\mathrm{L}^\infty}^2}{(\mathrm{Var}\,[X(A)])^2}\int_{\mathbb{R}^d}\!\!\ldots\!\int_{\mathbb{R}^d} c(|x_1-x_2|-R)\,c(|x_2-x_3|-R)\,c(|x_3-x_4|-R)\\
&\qquad\times\prod_{i=1}^4\mathbb{E}\left[\Big(\fint_{B_R(x_i)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^4\right]^{\frac14}dx_1\ldots dx_4.
\end{aligned}
$$

If the covariance is integrable in the sense of $\|\bar{\mathcal{C}}\|_{\mathrm{L}^1} := \int(\sup_{B(x)}|\mathcal{C}|)dx < \infty$, then the above reduces to

$$
\begin{aligned}
&\mathrm{d_{TV}}\left(\frac{X(A)-\mathbb{E}\,[X(A)]}{\sqrt{\mathrm{Var}\,[X(A)]}}, \mathcal{N}\right)^2 \qquad (4.64)\\
&\lesssim \frac{\|h'\|_{\mathrm{L}^\infty}^6}{(\mathrm{Var}\,[X(A)])^2}\|\bar{\mathcal{C}}\|_{\mathrm{L}^1}^3\,\mathbb{E}\left[\Big(\int_{\mathbb{R}^d}\Big(\int_{B(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2dx\Big)^2\right]^{\frac12}\\
&\qquad\times\mathbb{E}\left[\Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d}\Big(\iint_{B(x)\times B(y)}\Big|\frac{\partial^2X(A)}{\partial A^2}\Big|\Big)^2dxdy\Big)^2\right]^{\frac12}\\
&\quad+\frac{\|h'\|_{\mathrm{L}^\infty}^2\|h''\|_{\mathrm{L}^\infty}^2}{(\mathrm{Var}\,[X(A)])^2}\|\bar{\mathcal{C}}\|_{\mathrm{L}^1}^3\,\mathbb{E}\left[\int_{\mathbb{R}^d}\Big(\int_{B(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^4dx\right]. \qquad \lozenge
\end{aligned}
$$

Proof. By scaling it does not restrict generality to assume $\mathbb{E}\,[X(A)] = 0$ and $\mathrm{Var}\,[X(A)] = 1$. We split the proof into three steps.

Step 1. Discrete setting.

In this step, we establish the discrete counterpart of the desired result, that is, a second-order Poincaré inequality à la Chatterjee for correlated Gaussian vectors. Let $V = (V_1, \ldots, V_N)$ denote a Gaussian random vector with covariance $\Sigma := \mathrm{Var}\,[V] \in \mathbb{R}^{N\times N}$. Let $h \in C^2(\mathbb{R})$, and for all i let $W_i := h(V_i)$. Given a smooth transformation $g : \mathbb{R}^N \to \mathbb{R}$, we consider the random variable $Z := g(W)$, which can also be represented as $Z := f(V)$ for some map $f : \mathbb{R}^N \to \mathbb{R}$. Assume that $\mathbb{E}[Z] = 0$ and $\mathrm{Var}[Z] = 1$. Let V' denote an i.i.d. copy of V, and for all $t \in [0,1]$ define $U_t := \sqrt{t}V + \sqrt{1-t}V'$ and $(Y_t)_i := h((U_t)_i)$. In this step, we establish the following variant of [113, Theorem 2.2],

$$\begin{aligned}\frac{1}{2}\,\mathrm{d_{TV}}(Z,\mathcal{N})^2 \le 2\|h'\|_{\mathrm{L}^\infty}^6 \int_0^1 \Big(1+\frac{2}{\sqrt{t}}\Big) \sum_{i,j,k,l,m,n} |\Sigma_{ij}||\Sigma_{kl}||\Sigma_{mn}| \\ \times \mathbb{E}\left[|\nabla_i g(Y_t)||\nabla^2_{jk} g(W)||\nabla^2_{lm} g(W)||\nabla_n g(Y_t)|\right] dt \\ + 2\|h'\|_{\mathrm{L}^\infty}^2 \|h''\|_{\mathrm{L}^\infty}^2 \int_0^1 \Big(1+\frac{2}{\sqrt{t}}\Big) \sum_{i,j,k,l} |\Sigma_{ij}||\Sigma_{jk}||\Sigma_{kl}| \\ \times \mathbb{E}\left[|\nabla_i g(Y_t)||\nabla_j g(W)||\nabla_k g(W)||\nabla_l g(Y_t)|\right] dt. \quad (4.65)\end{aligned}$$

For that purpose, we simply adapt the strategy of [113] to the case with a nontrivial covariance. Using the i.i.d. copy V' of V, we may decompose, for any smooth $\psi : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[Z\psi(Z)] &= \mathbb{E}\left[f(V)\psi(f(V)) - f(V')\psi(f(V))\right] = \mathbb{E}\left[\psi(f(V)) \int_0^1 \frac{d}{dt}\big(f(\sqrt{t}V + \sqrt{1-t}V')\big)dt\right] \\ &= \frac{1}{2}\mathbb{E}\left[\psi(f(V)) \int_0^1 \Big(\frac{V}{\sqrt{t}} - \frac{V'}{\sqrt{1-t}}\Big) \cdot \nabla f(\sqrt{t}V + \sqrt{1-t}V')dt\right],\end{aligned}$$

or alternatively, in terms of $U_t := \sqrt{t}V + \sqrt{1-t}V'$ and $V_t := \sqrt{1-t}V - \sqrt{t}V'$,

$$\mathbb{E}[Z\psi(Z)] = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{t}\sqrt{1-t}}\mathbb{E}\left[\psi(f(\sqrt{t}U_t + \sqrt{1-t}V_t))V_t \cdot \nabla f(U_t)\right] dt.$$

Noting that the Gaussian vectors U_t and V_t are independent of each other and have the same law as V, and that Gaussian integration by parts takes the form

$$\mathbb{E}[V\zeta(V)] = \Sigma\,\mathbb{E}[\nabla\zeta(V)], \qquad \zeta \in C^1_b(\mathbb{R}^N),$$

we deduce from the above,

$$\mathbb{E}[Z\psi(Z)] = \int_0^1 \frac{1}{2\sqrt{t}}\mathbb{E}\left[\psi'(f(\sqrt{t}U_t + \sqrt{1-t}V_t))\nabla f(\sqrt{t}U_t + \sqrt{1-t}V_t) \cdot \Sigma\nabla f(U_t)\right] dt.$$

Defining

$$T(V,V') := \int_0^1 \frac{1}{2\sqrt{t}}\nabla f(V) \cdot \Sigma\,\nabla f(U_t)dt, \quad (4.66)$$

we have thus proven the identity

$$\mathbb{E}[Z\psi(Z)] = \mathbb{E}\left[\psi'(Z)T(V,V')\right] = \mathbb{E}\left[\psi'(Z)\mathbb{E}\left[T(V,V') \,\|\, Z\right]\right].$$

In other words, we have constructed the so-called Stein factor $\mathbb{E}[T(V,V') \,\|\, Z]$ for Z. A standard use of Stein's method (see e.g. [113, Lemma 5.1]) then yields

$$\mathrm{d_{TV}}(Z,\mathcal{N}) \le 2\mathbb{E}\left[\left|\mathbb{E}\left[T(V,V') \,\|\, Z\right] - 1\right|\right] \le 2\mathrm{Var}\left[\mathbb{E}\left[T(V,V') \,\|\, V\right]\right]^{\frac{1}{2}}.$$

In order to estimate this last variance, we use the Gaussian Brascamp-Lieb inequality (see e.g. Proposition 4.B.1),

$$\begin{aligned}\frac{1}{2}\,\mathrm{d_{TV}}(Z,\mathcal{N})^2 &\le 2\mathbb{E}\left[\nabla_V\mathbb{E}\left[T(V,V') \,\|\, V\right] \cdot \Sigma\nabla_V\mathbb{E}\left[T(V,V') \,\|\, V\right]\right] \\ &= 2\mathbb{E}\left[\left|\Sigma^{1/2}\mathbb{E}\left[\nabla_V T(V,V') \,\|\, V\right]\right|^2\right] \le 2\mathbb{E}\left[\left|\Sigma^{1/2}\nabla_V T(V,V')\right|^2\right].\end{aligned}$$

An explicit computation of the gradient $\nabla_V T(V,V')$ based on definition (4.66) yields

$$\nabla_V T(V,V') = \int_0^1 \frac{1}{2\sqrt{t}} \nabla^2 f(V)\cdot \Sigma \nabla f(U_t)dt + \frac{1}{2}\int_0^1 \nabla f(V)\cdot \Sigma\, \nabla^2 f(U_t)dt.$$

Combined with the inequality $(a+b)^2 \le 2(a^2+b^2)$ for all $a,b\in\mathbb{R}$, we obtain

$$\begin{aligned}\frac{1}{2}\,\mathrm{d_{TV}}\left(Z,\mathcal{N}\right)^2 \le \int_0^1 \frac{1}{\sqrt{t}}\int_0^1 \frac{1}{\sqrt{s}}\,\mathbb{E}\left[\nabla f(U_t)\cdot\Sigma\nabla^2 f(V)\Sigma\nabla^2 f(V)\Sigma\nabla f(U_s)\right] dsdt \\ +\int_0^1\int_0^1 \mathbb{E}\left[\nabla f(V)\cdot\Sigma\nabla^2 f(U_t)\Sigma\nabla^2 f(U_s)\Sigma\nabla f(V)\right] dsdt.\end{aligned}$$

Using successively the inequality $x\cdot\Sigma y\le\frac{1}{2}(x\cdot\Sigma x+y\cdot\Sigma y)$, the identity $\int_0^1 t^{-1/2}dt=2$, and noting that (V,U_t) has the same distribution as (U_t,V), we are left with

$$\frac{1}{2}\,\mathrm{d_{TV}}\left(Z,\mathcal{N}\right)^2 \le \int_0^1\Big(1+\frac{2}{\sqrt{t}}\Big)\mathbb{E}\left[\nabla f(U_t)\cdot\Sigma\nabla^2 f(V)\Sigma\nabla^2 f(V)\Sigma\nabla f(U_t)\right] dt.$$

By definition $Z=f(V)=g(W)$ with $W_i=h(V_i)$, so that

$$\nabla_i f(V) = h'(V_i)\nabla_i g(W),\qquad\text{and}\qquad \nabla^2_{ij} f(V) = h'(V_i)h'(V_j)\nabla^2_{ij}g(W)+\delta_{ij}h''(V_i)\nabla_i g(W),$$

and the result (4.65) follows.

Step 2. Continuum counterparts.

By an approximation argument, the result (4.65) of Step 1 yields for all $\sigma(A)$-measurable random variables $X(A)$ with $\mathbb{E}\left[X(A)\right]=0$ and $\mathrm{Var}\left[X(A)\right]=1$,

$$\begin{aligned}\frac{1}{2}\,\mathrm{d_{TV}}\left(X(A),\mathcal{N}\right)^2 \le\ & 2\|h'\|^6_{\mathrm{L}^\infty}\int_0^1\Big(1+\frac{2}{\sqrt{t}}\Big)\int\ldots\int|\mathcal{C}(x_1-x_2)||\mathcal{C}(x_3-x_4)||\mathcal{C}(x_5-x_6)| \\ &\times\mathbb{E}\bigg[\Big|\frac{\partial X(A_t)}{\partial A_t}(x_1)\Big|\Big|\frac{\partial^2 X(A)}{\partial A^2}(x_2,x_3)\Big|\Big|\frac{\partial^2 X(A)}{\partial A^2}(x_4,x_5)\Big|\Big|\frac{\partial X(A_t)}{\partial A_t}(x_6)\Big|\bigg]dx_1\ldots dx_6 dt \\ &+2\|h'\|^2_{\mathrm{L}^\infty}\|h''\|^2_{\mathrm{L}^\infty}\int_0^1\Big(1+\frac{2}{\sqrt{t}}\Big)\int\ldots\int|\mathcal{C}(x_1-x_2)||\mathcal{C}(x_2-x_3)||\mathcal{C}(x_3-x_4)| \\ &\times\mathbb{E}\bigg[\Big|\frac{\partial X(A_t)}{\partial A_t}(x_1)\Big|\Big|\frac{\partial X(A)}{\partial A}(x_2)\Big|\Big|\frac{\partial X(A)}{\partial A}(x_3)\Big|\Big|\frac{\partial X(A_t)}{\partial A_t}(x_4)\Big|\bigg]dx_1\ldots dx_4 dt,\end{aligned}\tag{4.67}$$

where we have set $A_t(x):=h(\sqrt{t}G(x)+\sqrt{1-t}G'(x))$ for an i.i.d. copy G' of the Gaussian random field G (in particular note that A and A_t have the same law). This result is to be compared with [349].

Step 3. Conclusion.

In this step, we argue that (4.67) yields the desired second-order weighted Poincaré inequality. For all smooth $\zeta:\mathbb{R}^d\to\mathbb{R}$ and $\xi:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, we claim that the following estimate holds,

$$\begin{aligned}T:=\int\ldots\int|\zeta(x_1)||\xi(x_2,x_3)||\xi(x_4,x_5)||\zeta(x_6)||\mathcal{C}(x_1-x_2)||\mathcal{C}(x_3-x_4)||\mathcal{C}(x_5-x_6)|dx_1\ldots dx_6 \\ \le\bigg(\int_0^\infty d\ell_1(\ell_1{+}1)^{-d}(-c'(\ell_1))\int_0^\infty d\ell_2(\ell_2{+}1)^{-d}(-c'(\ell_2))\iint dx_1dx_2\Big(\iint_{B_{2(\ell_1+1)}(x_1)\times B_{2(\ell_2+1)}(x_2)}|\xi|\Big)^2\bigg) \\ \times\bigg(\int_0^\infty d\ell(\ell+1)^{-d}(-c'(\ell))\int dx\Big(\int_{B_{2(\ell+1)}(x)}|\zeta|\Big)^2\bigg).\end{aligned}\tag{4.68}$$

We postpone the proof of this estimate to the end of this step, and first show how it implies the desired result. We denote the two right-hand side terms of (4.67) by S_1 and S_2, respectively, and we start with the estimation of S_1. We apply inequality (4.68) to $\zeta(x) := (\partial X(A_t)/\partial A_t)(x)$ and $\xi(x,y) := (\partial^2 X(A)/\partial A^2)(x,y)$, use Cauchy-Schwarz' inequality in probability, and note that A_t has the same law as A for all t, so that

$$
\begin{aligned}
S_1 \le 2\|h'\|_{\mathrm{L}^\infty}^6 \int_0^1 \Big(1+\frac{2}{\sqrt{t}}\Big)\mathbb{E}\bigg[&\bigg(\int_0^\infty d\ell(\ell+1)^{-d}(-c'(\ell))\int dx\Big(\int_{B_{2(\ell+1)}(x)}\Big|\frac{\partial X(A_t)}{\partial A_t}\Big|\Big)^2\bigg)\\
&\times\bigg(\int_0^\infty d\ell_1(\ell_1+1)^{-d}(-c'(\ell_1))\int_0^\infty d\ell_2(\ell_2+1)^{-d}(-c'(\ell_2))\\
&\qquad\times\iint dxdy\Big(\iint_{B_{2(\ell_1+1)}(x)\times B_{2(\ell_2+1)}(y)}\Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2\bigg)\bigg]dt\\
\le 10\|h'\|_{\mathrm{L}^\infty}^6\mathbb{E}\bigg[&\bigg(\int_0^\infty d\ell(\ell+1)^{-d}(-c'(\ell))\int dx\Big(\int_{B_{2(\ell+1)}(x)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2\bigg)^2\bigg]^{\frac12}\\
\times\mathbb{E}\bigg[&\bigg(\int_0^\infty d\ell_1(\ell_1+1)^{-d}(-c'(\ell_1))\int_0^\infty d\ell_2(\ell_2+1)^{-d}(-c'(\ell_2))\\
&\qquad\times\iint dxdy\Big(\iint_{B_{2(\ell_1+1)}(x)\times B_{2(\ell_2+1)}(y)}\Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2\bigg)^2\bigg]^{\frac12}.
\end{aligned}
$$

We now turn to the second term S_2. Taking local spatial averages, using Hölder's inequality in probability, and recalling that A_t has the same law as A for all t, we obtain

$$
\begin{aligned}
S_2 \le 10\|h'\|_{\mathrm{L}^\infty}^2\|h''\|_{\mathrm{L}^\infty}^2\int\ldots\int &\bar c_R(|x_1-x_2|)\bar c_R(|x_2-x_3|)\bar c_R(|x_3-x_4|)\\
&\times\prod_{i=1}^4\mathbb{E}\bigg[\Big(\fint_{B_{R/2}(x_i)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^4\bigg]^{\frac14}dx_1\ldots dx_4,
\end{aligned}
$$

where we have set $\bar c_R(t) := \sup_{|u|\le R} c(t+u)$. Hence, since c is non-increasing,

$$
\begin{aligned}
S_2 \le 2^{4d}10\|h'\|_{\mathrm{L}^\infty}^2\|h''\|_{\mathrm{L}^\infty}^2\int\ldots\int &c(|x_1-x_2|-R)\,c(|x_2-x_3|-R)\,c(|x_3-x_4|-R)\\
&\times\prod_{i=1}^4\mathbb{E}\bigg[\Big(\fint_{B_R(x_i)}\Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^4\bigg]^{\frac14}dx_1\ldots dx_4.
\end{aligned}
$$

The result (4.63) follows by inserting the above estimates for S_1 and S_2 into (4.67).

We now prove the result (4.64) in the case when $\int\bar{\mathcal{C}}<\infty$, where we have set $\bar{\mathcal{C}}(x) := \sup_{B_2(x)}|\mathcal{C}|$. Using the inequality $2ab\le a^2+b^2$ for all $a,b\in\mathbb{R}$, we obtain

$$
\begin{aligned}
S_1 \lesssim \|h'\|_{\mathrm{L}^\infty}^6\int_0^1\Big(1+\frac{2}{\sqrt{t}}\Big)\int\ldots\int\mathbb{E}\bigg[&\Big(\int_{B(x_1)}\Big|\frac{\partial X(A_t)}{\partial A_t}\Big|\Big)^2\Big(\iint_{B(x_4)\times B(x_5)}\Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2\bigg]\\
&\times\bar{\mathcal{C}}(x_1-x_2)\bar{\mathcal{C}}(x_3-x_4)\bar{\mathcal{C}}(x_5-x_6)dx_1\ldots dx_6dt\\
\le \|h'\|_{\mathrm{L}^\infty}^6\|\bar{\mathcal{C}}\|_{\mathrm{L}^1}^3\int_0^1\Big(1+\frac{2}{\sqrt{t}}\Big)&\\
\times\iiint\mathbb{E}\bigg[&\Big(\int_{B(x_1)}\Big|\frac{\partial X(A_t)}{\partial A_t}\Big|\Big)^2\Big(\iint_{B(x_2)\times B(x_3)}\Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2\bigg]dx_1dx_2dx_3dt
\end{aligned}
$$

$$\lesssim \|h'\|_{\mathrm{L}^\infty}^6 \|\bar{\mathcal{C}}\|_{\mathrm{L}^1}^3 \, \mathbb{E}\bigg[\Big(\int \Big(\int_{B(x)} \Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^2 dx\Big)^2\bigg]^{\frac12} \mathbb{E}\bigg[\Big(\iint \Big(\iint_{B(x)\times B(y)} \Big|\frac{\partial^2 X(A)}{\partial A^2}\Big|\Big)^2 dxdy\Big)^2\bigg]^{\frac12}.$$

Likewise, using the inequality $a_1a_2a_3a_4 \le \frac14 \sum_{i=1}^4 a_i^4$, we obtain

$$S_2 \lesssim \|h'\|_{\mathrm{L}^\infty}^2 \|h''\|_{\mathrm{L}^\infty}^2 \|\bar{\mathcal{C}}\|_{\mathrm{L}^1}^3 \mathbb{E}\bigg[\int \Big(\int_{B(x)} \Big|\frac{\partial X(A)}{\partial A}\Big|\Big)^4 dx\bigg].$$

Combined with (4.67), these estimates yield the desired result (4.64).

It remains to prove the general estimate (4.68). Using radial coordinates, the left-hand side T takes the form

$$\begin{aligned} T \le \iiint dx_1dx_2dx_3 \int_0^\infty d\ell_1 c(\ell_1) \int_{\partial B_{\ell_1}} d\sigma(u_1) \ldots \int_0^\infty d\ell_3 c(\ell_3) \int_{\partial B_{\ell_3}} d\sigma(u_3) \\ \times |\zeta(x_1)||\xi(x_1+u_1,x_2)||\xi(x_2+u_2,x_3+u_3)||\zeta(x_3)|, \end{aligned}$$

which, by integration by parts, turns into

$$\begin{aligned} T \le \iiint dx_1dx_2dx_3 \int_0^\infty d\ell_1(-c'(\ell_1)) \int_0^\infty d\ell_2(-c'(\ell_2)) \int_0^\infty d\ell_3(-c'(\ell_3)) \\ \times |\zeta(x_1)||\zeta(x_3)|\Big(\int_{B_{\ell_1}(x_1)} |\xi(\cdot,x_2)|\Big)\Big(\int_{B_{\ell_2}(x_2)\times B_{\ell_3}(x_3)} |\xi|\Big). \end{aligned}$$

Taking local averages, and bounding $\int_{B_{\ell_1}(y_1)}$ by $\int_{B_{2(\ell_1+1)}(x_1)}$ for all $y_1 \in B_{\ell_1+1}(x_1)$, we directly deduce

$$\begin{aligned} T \le \iiint dx_1dx_2dx_3 \int_0^\infty d\ell_1(\ell_1+1)^{-d}(-c'(\ell_1)) \ldots \int_0^\infty d\ell_3(\ell_3+1)^{-d}(-c'(\ell_3)) \\ \times \Big(\int_{B_{2(\ell_1+1)}(x_1)\times B_{2(\ell_2+1)}(x_2)} |\xi|\Big)\Big(\int_{B_{2(\ell_2+1)}(x_2)\times B_{2(\ell_3+1)}(x_3)} |\xi|\Big)\Big(\int_{B_{2(\ell_1+1)}(x_1)} |\zeta|\Big)\Big(\int_{B_{2(\ell_3+1)}(x_3)} |\zeta|\Big), \end{aligned}$$

which, by the inequality $ab \le \frac12(a^2+b^2)$ for all $a,b \in \mathbb{R}$, yields

$$\begin{aligned} T \le \iiint dx_1dx_2dx_3 \int_0^\infty d\ell_1(\ell_1+1)^{-d}(-c'(\ell_1)) \ldots \int_0^\infty d\ell_3(\ell_3+1)^{-d}(-c'(\ell_3)) \\ \times \Big(\int_{B_{2(\ell_1+1)}(x_1)} |\zeta|\Big)^2 \Big(\int_{B_{2(\ell_2+1)}(x_2)\times B_{2(\ell_3+1)}(x_3)} |\xi|\Big)^2, \end{aligned}$$

that is, (4.68). □

4.6.2 Randomly localized fields

We use the same notation as in Section 4.4.1: A is a $\sigma(\mathcal{X})$-measurable random field on $\mathbb{R}^d$, where $\mathcal{X}$ is a completely independent random field on some measure space $X = \biguplus_{x\in\mathbb{Z}^d, t\in\mathbb{Z}^l} X_{x,t}$ with values in some measurable space M. In this subsection, we address the situation when the dependence pattern of A with respect to $\mathcal{X}$ is random in the sense that it is determined by the underlying product structure $\mathcal{X}$ itself. The following theorem establishes weighted second-order Poincaré inequalities for A, based on assumptions on the action radius (actually in a slightly stronger version than that introduced in Section 4.4.2, cf. assumption (a) below). The strategy consists in applying Chatterjee's second-order Poincaré inequality for $\mathcal{X}$ (cf. [112]), and then exploiting the localization properties of the action radius to devise an approximate chain rule and deduce a functional inequality for $A = A(\mathcal{X})$ itself. As already discussed, this is to be compared with [286].

Theorem 4.6.2. *Let A be a $\sigma(\mathcal{X})$-measurable random field on $\mathbb{R}^d$, where $\mathcal{X}$ is a completely independent random field on some measure space $X = \biguplus_{x\in\mathbb{Z}^d, t\in\mathbb{Z}^l} X_{x,t}$ with values in some measurable space M. Let $\mathcal{X}'$ be an i.i.d. copy of $\mathcal{X}$. For all $B \subset \mathbb{Z}^d \times \mathbb{Z}^l$, let the perturbed random field $\mathcal{X}^B$ be defined by*

$$\mathcal{X}^B|_{\cup_{(x,t)\in B} X_{x,t}} = \mathcal{X}'|_{\cup_{(x,t)\in B} X_{x,t}}, \qquad \mathcal{X}^B|_{\cup_{(x,t)\notin B} X_{x,t}} = \mathcal{X}|_{\cup_{(x,t)\notin B} X_{x,t}},$$

and for all $x, x' \in \mathbb{Z}^d$ and $t, t' \in \mathbb{Z}^l$ we set for simplicity $\mathcal{X}^{x,t} := \mathcal{X}^{\{(x,t)\}}$ and $\mathcal{X}^{x,t;x',t'} := \mathcal{X}^{\{(x,t),(x',t')\}}$. Assume that

(a) For all x, t and all $B \subset \mathbb{Z}^d \times \mathbb{Z}^l$, there exists an action radius $\rho_{x,t}(\mathcal{X}^B)$ for $A(\mathcal{X}^B)$ with respect to $\mathcal{X}^B$ in $X_{x,t}$ with reference perturbation $\mathcal{X}'$ (in the sense of Definition 4.4.2), and set

$$\tilde{\rho}_{x,t} := \sup\big\{\rho_{x,t}(\mathcal{X}^B) \,:\, B \subset \mathbb{Z}^d \times \mathbb{Z}^l\big\}.$$

(b) The transformation A of $\mathcal{X}$ is stationary, that is, the random fields $A(\mathcal{X}(\cdot + z, \cdot))$ and $A(\mathcal{X})(\cdot + z)$ have the same law for all $z \in \mathbb{Z}^d$. Moreover, for all t, B, the law of the action radius $\rho_{x,t}(\mathcal{X}^B)$ is independent of x. In particular, for all t, the law of $\tilde{\rho}_{x,t}$ is independent of x.

For all $t \in \mathbb{Z}^l$ and $\ell \geq 1$, define the weight

$$\pi(t,\ell) := \mathbb{P}\left[\ell - 1 \leq \tilde{\rho}_{0,t} < \ell\,,\ \mathcal{X} \neq \mathcal{X}^{0,t}\right].$$

Then the following results hold.

(i) For all $\sigma(A)$-measurable random variables $X = X(A)$, we have

$$\begin{aligned}
&\mathrm{d_W}\left(\frac{X - \mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}}, \mathcal{N}\right) \\
&\lesssim \frac{1}{\mathrm{Var}[X]} \inf_{0<\lambda<1} \left(\sum_{x,x',x''} \sum_{t,t',t''} \sum_{\ell,\ell',\ell''=1}^{\infty} \big(\pi(t,\ell)^{\frac13} \pi(t',\ell')^{\frac13} \pi(t'',\ell'')^{\frac13}\big)^{\lambda}\, \mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t} \partial^{\mathrm{dis}}_{\ell',x',t'} X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}} \right. \\
&\qquad \left. \times\, \mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t} \partial^{\mathrm{dis}}_{\ell'',x'',t''} X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}} \mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell',x',t'} X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}} \mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell'',x'',t''} X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}} \right)^{\frac12} \\
&\quad + \frac{1}{\mathrm{Var}[X]} \inf_{0<\lambda<1} \left(\sum_{x,x'} \sum_{t,t'} \sum_{\ell,\ell'=1}^{\infty} \big(\pi(t,\ell)^{\frac12} \pi(t',\ell')^{\frac12}\big)^{\lambda} \right. \\
&\qquad\qquad \left. \times\, \mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t} \partial^{\mathrm{dis}}_{\ell',x',t'} X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{2}} \mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell',x',t'} X\Big)^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{2}} \right)^{\frac12} \\
&\quad + \frac{1}{\mathrm{Var}[X]} \inf_{0<\lambda<1} \left(\sum_{x} \sum_{t} \sum_{\ell=1}^{\infty} \pi(t,\ell)^{\lambda}\, \mathbb{E}\left[(\partial^{\mathrm{dis}}_{\ell,x,t} X)^{\frac{4}{1-\lambda}}\right]^{1-\lambda} \right)^{\frac12} \\
&\quad + \frac{1}{\mathrm{Var}[X]^{3/2}} \inf_{0<\lambda<1} \sum_{x} \sum_{t} \sum_{\ell=1}^{\infty} \pi(t,\ell)^{\lambda}\, \mathbb{E}\left[\Big(\partial^{\mathrm{dis}}_{\ell,x,t} X\Big)^{\frac{3}{1-\lambda}}\right]^{1-\lambda},
\end{aligned} \tag{4.69}$$

where the sums in x, x', x'' (resp. in t, t', t'') implicitly run over $\mathbb{Z}^d$ (resp. over $\mathbb{Z}^l$), and where for all $x \in \mathbb{Z}^d$ and $t \in \mathbb{Z}^l$ we have defined the discrete derivative

$$\partial^{\mathrm{dis}}_{\ell,x,t} X := \big(X(A) - X(A(\mathcal{X}^{x,t}))\big) \mathbb{1}_{A(\mathcal{X}^{x,t})|_{\mathbb{R}^d \setminus Q_{2\ell+1}(x)} = A|_{\mathbb{R}^d \setminus Q_{2\ell+1}(x)}}$$

and the discrete second derivative

$$\begin{aligned}\partial^{\mathrm{dis}}_{\ell,x,t}\partial^{\mathrm{dis}}_{\ell',x',t'}X &:= \big(X(A)-X(A(\mathcal{X}^{x,t}))-X(A(\mathcal{X}^{x',t'}))+X(A(\mathcal{X}^{x,t;x',t'}))\big)\\ &\quad\times \mathbb{1}_{A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}=A|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}}\mathbb{1}_{A(\mathcal{X}^{x,t;x',t'})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}=A(\mathcal{X}^{x',t'})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}}\\ &\qquad\times \mathbb{1}_{A(\mathcal{X}^{x',t'})|_{\mathbb{R}^d\setminus Q_{2\ell'+1}(x')}=A|_{\mathbb{R}^d\setminus Q_{2\ell'+1}(x')}}\mathbb{1}_{A(\mathcal{X}^{x,t;x',t'})|_{\mathbb{R}^d\setminus Q_{2\ell'+1}(x')}=A(\mathcal{X}^{x,t})|_{\mathbb{R}^d\setminus Q_{2\ell'+1}(x')}}.\end{aligned}$$

(ii) For all $\sigma(A)$-measurable random variables $X=X(A)$, we have

$$\mathrm{d_K}\left(\frac{X-\mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}},\mathcal{N}\right)\lesssim \mathrm{RHS}_{(4.69)}(X)+G_1(X),\tag{4.70}$$

where $\mathrm{RHS}_{(4.69)}(X)$ *denote the right-hand side of* (4.69)*, and where we have set*

$$G_1(X) := \frac{1}{\mathrm{Var}[X]^{3/2}}\inf_{0<\lambda<1}\sum_x\sum_t\left(\sum_{\ell=1}^{\infty}\pi(t,\ell)^{\lambda}\,\mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell,x,t}X\right)^{\frac{6}{1-\lambda}}\right]^{1-\lambda}\right)^{\frac12}.$$

If in addition for all x,t there exists a $\sigma(\mathcal{X}|_{X_{x,t}},\mathcal{X}'|_{X_{x,t}})$-measurable action radius $\rho_{x,t}$ for $A(\mathcal{X})$ with respect to $\mathcal{X}$ on $X_{x,t}$, then we simply have $\tilde\rho_{x,t}=\rho_{x,t}$ for all x,t, the weights $\pi^{\frac13}$ and $\pi^{\frac12}$ can both be replaced by π in the first two right-hand side terms of (4.69) *and in the corresponding terms in* $\mathrm{RHS}_{(4.69)}(X)$ *in* (4.70)*, and the term $G_1(X)$ in* (4.70) *can be replaced by*

$$G_2(X) := \frac{1}{\mathrm{Var}[X]^{3/2}}\inf_{0<\lambda<1}\sum_x\sum_t\sum_{\ell=1}^{\infty}\pi(t,\ell)^{\lambda}\,\mathbb{E}\left[\left(\partial^{\mathrm{dis}}_{\ell,x,t}X\right)^{\frac{6}{1-\lambda}}\right]^{\frac{1-\lambda}{2}}. \qquad\Diamond$$

Remark 4.6.3. The additional term $G_1(X)$ in (4.70) typically dominates the right-hand side terms of (4.69). However they become of the same order if the weight π is super-algebraically decaying, or if the improved form of the above result holds (that is, with $G_1(x)$ replaced by $G_2(X)$). In each of the examples below, we are in one of these two situations, hence the above bounds on the Kolmogorov and on the Wasserstein distances essentially coincide. Otherwise, it might be advantageous to rather bound the Kolmogorov distance by the square-root of the Wasserstein distance and then use the above estimate for the latter. $\Diamond$

Before we turn to the proof of Theorem 4.6.2, we recall representative examples analyzed in [163, Section 3], and to which it applies. In each case, we quickly discuss the existence and properties of the action radius $\tilde\rho$ (which is a slightly stronger notion of action radius than the one ρ given in Definition 4.4.2 and needed for first-order weighted functional inequalities). For technical details we refer to Section 4.5, where the action radii ρ are constructed.

(A) *Poisson unbounded spherical inclusion model.* Consider a Poisson point process $\mathcal{P}$ of unit intensity on $\mathbb{R}^d$. For each Poisson point $x\in\mathcal{P}$ consider a random radius $r(x)$ (independent of the radii of other points and identically distributed according to some given law ν on $\mathbb{R}^+$), and define the inclusion $C_x:=B_{r(x)}(x)$. Consider the inclusion set $\mathcal{I}:=\cup_{x\in\mathcal{P}}C_x$, let $A_0,A_1\in\mathbb{R}$ be given values, and define a random field A on $\mathbb{R}^d$ by

$$A(x) := A_0\mathbb{1}_{x\notin\mathcal{I}}+A_1\mathbb{1}_{x\in\mathcal{I}},$$

that is, A takes value A_1 in the inclusions and A_0 outside. As argued in Section 4.5.4, A can be reformulated in the form addressed in Theorem 4.6.2 above with $l=1$, and for all x,t there exists a $\sigma(\mathcal{X}|_{X_{x,t}},\mathcal{X}'|_{X_{x,t}})$-measurable action radius $\rho_{x,t}:=t\,\mathbb{1}_{\mathcal{X}\neq\mathcal{X}^{x,t}}$ (cf. the proof of Proposition 4.5.4(i)). The improved form of the above result therefore holds with

$$\pi(t,\ell):=\mathbb{1}_{\ell-1\le t<\ell}\,\mathbb{P}\left[\mathcal{X}\neq\mathcal{X}^{0,t}\right]\le 2\,\nu([t-\tfrac12,t+\tfrac12))\,\mathbb{1}_{\ell-1\le t<\ell}.$$

(B) *Random parking process.* Consider the random parking point process $\mathcal{R}$ with unit radius on $\mathbb{R}^d$ (cf. Section 4.5.3 for a precise construction based on an underlying Poisson point process $\mathcal{P}_0$ of unit intensity on $\mathbb{R}^d \times \mathbb{R}_+$). As above, for all $x \in \mathcal{R}$ we denote by $C_x := B(x)$ the unit spherical inclusion centered at x (so that by definition of $\mathcal{R}$ all the inclusions are disjoint), we consider the inclusion set $\mathcal{I} := \cup_{x\in\mathcal{R}} C_x$, and we define a random field A on $\mathbb{R}^d$ by

$$A(x) := A_0 \mathbb{1}_{x\notin\mathcal{I}} + A_1 \mathbb{1}_{x\in\mathcal{I}}.$$

In the proof of Proposition 4.5.3 we have constructed for all x an action radius ρ_x with respect to the underlying Poisson point process $\mathcal{P}_0$ on $Q(x) \times \mathbb{R}_+$. By definition, this action radius satisfies $\rho_x(\mathcal{P}_0^B) \le \rho_x(\mathcal{P}_0 \cup \mathcal{P}_0')$ for all $B \subset \mathbb{Z}^d$: indeed, adding points in the Poisson point process $\mathcal{P}_0$ adds possible causal chains, hence increases the defined action radius. Therefore, we deduce $\tilde\rho_x \le \rho_x(\mathcal{P}_0 \cup \mathcal{P}_0')$. As $\mathcal{P}_0 \cup \mathcal{P}_0'$ is itself a Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+$ with doubled intensity, we conclude $\mathbb{P}[\tilde\rho_x \ge \ell] \le C\exp(-\frac{1}{C}\ell)$ as in Proposition 4.5.3, and we may apply Theorem 4.6.2 with $l = 0$ and exponential weight $\pi(\ell) \le C\exp(-\frac{1}{C}\ell)$.

(C) *Poisson random tessellations.* Consider a Poisson point process $\mathcal{P}$ on $\mathbb{R}^d$, and let $\mathcal{V}$ denote the associated Voronoi tessellation of $\mathbb{R}^d$, that is, a partition of $\mathbb{R}^d$ into convex polyhedra $V_x \in \mathcal{V}$ centered at the Poisson points $x \in \mathcal{P}$. For each point $x \in \mathcal{P}$ consider a random value $\alpha(x)$ (independent of the values at other points and identically distributed), and we define a random field A on $\mathbb{R}^d$ by

$$A(x) := \sum_{y\in\mathcal{P}} \alpha(y) \mathbb{1}_{x\in V_y}.$$

As argued in the proof of Proposition 4.5.2, A can be reformulated in the form addressed in Theorem 4.6.2 above with $l = 0$ and with weight

$$\pi(\ell) \le \mathbb{P}[\tilde\rho_x \ge \ell - 1] \le C\exp\Big(-\frac{1}{C}\ell^d\Big). \tag{4.71}$$

(More precisely, we argue as follows: Denote by $C_i := \{x \in \mathbb{R}^d : x_i \ge \frac56|x|\}$, $1 \le i \le d$, the d cones in the canonical directions e_i of $\mathbb{R}^d$, and consider the $2d$ cones $C_i^\pm := \pm(2e_i + C_i)$. For all x, let $\rho_x := \rho_x^0$ denote the action radius for A defined in the proof of Proposition 4.5.2, and let $\tilde\rho_x$ be defined as in the statement of Theorem 4.6.2 above. By construction, the inequality $\tilde\rho_x \le CL$ holds if for each cone $C_i^\pm$ there exists a cube $Q \subset C_i^\pm \cap \{x : |x_i| \le L\}$ such that $\mathcal{P}_0 \cap Q \ne \varnothing \ne \mathcal{P}_0' \cap Q$. By independence of $\mathcal{P}_0$ and $\mathcal{P}_0'$, and by a union bound, the claim (4.71) follows.)

Proof of Theorem 4.6.2. We split the proof into two steps. First note that by approximation it is enough to prove the result for $\sigma(\mathcal{X}|_{\cup_{(x,t)\in E}(Q(x)\times Q(t))})$-measurable random variables $X = X(\mathcal{X})$ for a finite set $E \subset \mathbb{Z}^d \times \mathbb{Z}^l$. Let such a finite set E and such a random variable X be fixed.

Step 1. Application of a result by Chatterjee.

By [112, Theorem 2.2] (together with the standard spectral gap (4.4.1)), we have

$$\begin{aligned}
\mathrm{d_W}\bigg(\frac{X-\mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}},\mathcal{N}\bigg) &\lesssim \frac{1}{\mathrm{Var}[X]^{3/2}} \sum_{x,t} \mathbb{E}\big[|\Delta_{x,t}X|^3\big] \\
&\quad + \frac{1}{\mathrm{Var}[X]}\bigg(\sum_{x,t}\mathbb{E}\Big[\Big|\sum_{x',t'}(\Delta_{x,t}\Delta_{x',t'}X)\overline{\Delta_{x',t'}X}\Big|^2\Big]\bigg)^{\frac12} \\
&\quad + \frac{1}{\mathrm{Var}[X]}\bigg(\sum_{x,t}\mathbb{E}\Big[\Big|\sum_{x',t'}(\Delta_{x,t}\overline{\Delta_{x',t'}X})\Delta_{x',t'}X\Big|^2\Big]\bigg)^{\frac12},
\end{aligned} \tag{4.72}$$

where the sums in (x,t) and (x',t') implicitly run over E, and where we have set

$$\Delta_{x,t}X(\mathcal{X}^B) := X(\mathcal{X}^B) - X(\mathcal{X}^{B\cup\{(x,t)\}}),$$
$$\overline{\Delta_{x,t}X} := \sum_{\substack{B\subset E\\(x,t)\notin B}} K_B \Delta_{x,t}X(\mathcal{X}^B), \qquad K_B := \frac{|B|!(|E|-|B|-1)!}{|E|!}.$$

Note that by definition $\sum_{B\subset E:(x,t)\notin B} K_B = 1$. By [282, Theorem 4.2] (together with the standard spectral gap (4.4.1)), the following estimate on the Kolmogorov distance also holds

$$\mathrm{d_K}\left(\frac{X-\mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}},\mathcal{N}\right) \lesssim \mathrm{RHS}(X) + \frac{1}{\mathrm{Var}[X]^{3/2}}\mathbb{E}\Big[\Big(\sum_{x,t}|\Delta_{x,t}X|^2\,\overline{\overline{\Delta_{x,t}X}}\Big)^2\Big]^{\frac12}$$
$$+\frac{1}{\mathrm{Var}[X]}\bigg(\sum_{x,t}\mathbb{E}\Big[\Big|\sum_{x',t'}(\Delta_{x,t}\Delta_{x',t'}X)\overline{\overline{\Delta_{x',t'}X}}\Big|^2\Big]\bigg)^{\frac12}$$
$$+\frac{1}{\mathrm{Var}[X]}\bigg(\sum_{x,t}\mathbb{E}\Big[\Big|\sum_{x',t'}(\Delta_{x,t}\overline{\overline{\Delta_{x',t'}X}})\Delta_{x',t'}X\Big|^2\Big]\bigg)^{\frac12}, \quad (4.73)$$

where $\mathrm{RHS}(X)$ stands for the right-hand side of (4.72) above, and

$$\overline{\overline{\Delta_{x,t}X}} := \sum_{\substack{B\subset E\\(x,t)\notin B}} K_B|\Delta_{x,t}X(\mathcal{X}^B)|.$$

Only the first right-hand side term of (4.73) (after $\mathrm{RHS}(X)$) will lead the correction $G_1(X)$ in (4.70) with respect to (4.69).

Step 2. Conditioning with respect to the action radius.

In this step we reformulate the right-hand sides of (4.72) and (4.73) by introducing the action radius $\rho_{x,t}$ for A with respect to $\mathcal{X}$. We only address the second right-hand side term in (4.72) since all the other terms can be treated similarly. To simplify notation, we write $z := (x,t)$ and $Q(z) := Q(x)\times Q(t)$. We start by expanding the square and by distinguishing cases when the differences Δ_z are taken at the same points,

$$\sum_z \mathbb{E}\left[\Big|\sum_{z'}(\Delta_z\Delta_{z'}X)\overline{\Delta_{z'}X}\Big|^2\right] \le \sum_{z,z',z''}\mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|\right]$$
$$=\sum_z\mathbb{E}\left[|\Delta_zX|^2|\overline{\Delta_zX}|^2\right] + 2\sum_{z\neq z'}\mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_zX||\overline{\Delta_zX}||\overline{\Delta_{z'}X}|\right]$$
$$+\sum_{z\neq z'}\mathbb{E}\left[|\Delta_z\Delta_{z'}X|^2|\overline{\Delta_{z'}X}|^2\right] + \sum_{\substack{z,z',z''\\ \text{distinct}}}\mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|\right], \quad (4.74)$$

where we used the fact that $\Delta_z\Delta_zX = \Delta_zX$. We then reformulate the four right-hand side terms by introducing the action radius. We only treat the last term in detail (the other terms are similar). Since the product $|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|$ vanishes whenever $\mathcal{X}|_{Q(z)} = \mathcal{X}'|_{Q(z)}$ or $\mathcal{X}|_{Q(z')} = \mathcal{X}'|_{Q(z')}$ or $\mathcal{X}|_{Q(z'')} = \mathcal{X}'|_{Q(z'')}$, we obtain after conditioning with respect to the values of $\tilde\rho_z$, $\tilde\rho_{z'}$ and

$\tilde\rho_{z''}$ (that is, the stronger notion of action radii defined in the statement),

$$\sum_{\substack{z,z',z''\\ \text{distinct}}} \mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|\right]$$
$$\le \sum_{\ell,\ell',\ell''=1}^{\infty}\sum_{\substack{z,z',z''\\ \text{distinct}}} \mathbb{E}\Big[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|$$
$$\times \mathbb{1}_{\ell-1\le\tilde\rho_z<\ell}\,\mathbb{1}_{\mathcal{X}|_{Q(z)}\ne\mathcal{X}'|_{Q(z)}}\,\mathbb{1}_{\ell'-1\le\tilde\rho_{z'}<\ell'}\,\mathbb{1}_{\mathcal{X}|_{Q(z')}\ne\mathcal{X}'|_{Q(z')}}\,\mathbb{1}_{\ell''-1\le\tilde\rho_{z''}<\ell''}\,\mathbb{1}_{\mathcal{X}|_{Q(z'')}\ne\mathcal{X}'|_{Q(z'')}}\Big].$$

Note that the event $\tilde\rho_z<\ell$ entails by definition $A(\mathcal{X}^B)|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)} = A(\mathcal{X}^{B\cup\{z\}})|_{\mathbb{R}^d\setminus Q_{2\ell+1}(x)}$ for all $B\subset E$. By Hölder's inequality and by definition of ∂^{dis} and $\partial^{\mathrm{dis}}\partial^{\mathrm{dis}}$, we then obtain for all $0<\lambda<1$,

$$\sum_{\substack{z,z',z''\\ \text{distinct}}} \mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|\right] \le \sum_{\ell,\ell',\ell''=1}^{\infty}\sum_{\substack{z,z',z''\\ \text{distinct}}}\sum_{\substack{B'\subset E\\ z'\notin B'}}K_{B'}\sum_{\substack{B''\subset E\\ z''\notin B''}}K_{B''}$$
$$\times\mathbb{E}\left[\mathbb{1}_{\ell-1\le\tilde\rho_z<\ell}\,\mathbb{1}_{\mathcal{X}|_{Q(z)}\ne\mathcal{X}'|_{Q(z)}}\,\mathbb{1}_{\ell'-1\le\tilde\rho_{z'}<\ell'}\,\mathbb{1}_{\mathcal{X}|_{Q(z')}\ne\mathcal{X}'|_{Q(z')}}\,\mathbb{1}_{\ell''-1\le\tilde\rho_{z''}<\ell''}\,\mathbb{1}_{\mathcal{X}|_{Q(z'')}\ne\mathcal{X}'|_{Q(z'')}}\right]^{\lambda}$$
$$\times\mathbb{E}\left[\Big(\big|\partial^{\mathrm{dis}}_{\ell,z}\partial^{\mathrm{dis}}_{\ell',z'}X(\mathcal{X})\big|\big|\partial^{\mathrm{dis}}_{\ell,z}\partial^{\mathrm{dis}}_{\ell'',z''}X(\mathcal{X})\big|\big|\partial^{\mathrm{dis}}_{\ell',z'}X(\mathcal{X}^{B'})\big|\big|\partial^{\mathrm{dis}}_{\ell'',z''}X(\mathcal{X}^{B''})\big|\Big)^{\frac{1}{1-\lambda}}\right]^{1-\lambda}.$$

Again applying Hölder's inequality, noting that $\sum_{B\subset E:z\notin B}K_B=1$, and recalling that $\mathcal{X}$ and $\mathcal{X}^B$ have the same law for all $B\subset E$, we conclude

$$\sum_{\substack{z,z',z''\\ \text{distinct}}} \mathbb{E}\left[|\Delta_z\Delta_{z'}X||\Delta_z\Delta_{z''}X||\overline{\Delta_{z'}X}||\overline{\Delta_{z''}X}|\right]$$
$$\le \sum_{\ell,\ell',\ell''=1}^{\infty}\sum_{z,z',z''}\big(\pi(t,\ell)\pi(t',\ell')\pi(t'',\ell'')\big)^{\frac{\lambda}{3}}\,\mathbb{E}\left[\big|\partial^{\mathrm{dis}}_{\ell,z}\partial^{\mathrm{dis}}_{\ell',z'}X\big|^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}\mathbb{E}\left[\big|\partial^{\mathrm{dis}}_{\ell,z}\partial^{\mathrm{dis}}_{\ell'',z''}X\big|^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}$$
$$\times\mathbb{E}\left[\big|\partial^{\mathrm{dis}}_{\ell',z'}X\big|^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}\mathbb{E}\left[\big|\partial^{\mathrm{dis}}_{\ell'',z''}X\big|^{\frac{4}{1-\lambda}}\right]^{\frac{1-\lambda}{4}}. \quad (4.75)$$

The other terms in (4.74) can be treated similarly, and the results (i)–(ii) follow. Finally note that if for all z there is an action radius ρ_z for A with respect to $\mathcal{X}$ on $Q(z)$ which is $\sigma(\mathcal{X}|_{Q(z)},\mathcal{X}'|_{Q(z)})$-measurable, then the complete independence of $\mathcal{X}$ ensures that $\tilde\rho_z$, $\tilde\rho_{z'}$ and $\tilde\rho_{z''}$ are independent for z,z',z'' distinct, so that we simply obtain

$$\mathbb{E}\left[\mathbb{1}_{\ell-1\le\tilde\rho_z<\ell}\,\mathbb{1}_{\mathcal{X}|_{Q(z)}\ne\mathcal{X}'|_{Q(z)}}\,\mathbb{1}_{\ell'-1\le\tilde\rho_{z'}<\ell'}\,\mathbb{1}_{\mathcal{X}|_{Q(z')}\ne\mathcal{X}'|_{Q(z')}}\,\mathbb{1}_{\ell''-1\le\tilde\rho_{z''}<\ell''}\,\mathbb{1}_{\mathcal{X}|_{Q(z'')}\ne\mathcal{X}'|_{Q(z'')}}\right]$$
$$=\pi(t,\ell)\pi(t',\ell')\pi(t'',\ell'').$$

The exponent $\frac13$ can then be removed from the weights in (4.75), and the corresponding improved result follows. □

4.7 Application to spatial averages of the random field

Although the primary aim of this chapter is to address concentration and approximate normality properties for general *nonlinear* functions of correlated random fields, we illustrate the use of weighted functional inequalities on the simplest functions possible, that is, (linear) spatial averages of (a possibly nonlinear yet approximately local transformation of) the random field itself.

Given a jointly measurable stationary random field A, we typically consider a $\sigma(A)$-measurable random variable $f(A)$ that is approximately 1-local with respect to the field A, in the following sense: for all $r > 0$ we assume

$$\sup_A \operatorname{ess} \big| f(A) - \mathbb{E}\left[f(A) \,\|\, A|_{B_r} \right] \big| \le C e^{-\frac{r}{C}}. \tag{4.76}$$

More precisely, given $\tilde\partial = \partial^{\mathrm{G}}$, ∂^{fct}, or ∂^{osc}, we will use the following finer notion of approximate 1-locality: for all $x \in \mathbb{R}^d$ and $\ell \ge 0$,

$$\sup_A \operatorname{ess} \tilde\partial_{A,B_{\ell+1}(x)} f(A) \le C e^{-\frac{1}{C}(|x|-\ell)_+}. \tag{4.77}$$

(An important particular case is when the random variable $f(A)$ is exactly 1-local, that is, when $f(A)$ is exactly $\sigma(A|_{B_1})$-measurable.) We then set $F(x) := f(A(\cdot + x))$ for all $x \in \mathbb{R}^d$, and for all $L \ge 0$ we consider the random variable

$$X_L := X_L(A) := \fint_{Q_L} (F - \mathbb{E}[F]),$$

that is, the spatial average of (the nonlinear approximately local transformation F of) the random field A on the cube of side-length L. Note that the results below hold in the same form if the random variable X_L is replaced by $L^{-d} \int_{Q_L} e^{-\frac{1}{L}|y|} (F(y) - \mathbb{E}[F]) dy$.

4.7.1 Scaling of spatial averages

We start with the scaling of the variance of the spatial average X_L. Note that a similar result holds in stochastic homogenization, where X_L is replaced by the spatial average of the square of the gradient of the extended corrector (cf. [203]).

Proposition 4.7.1. *If A satisfies ($\tilde\partial$-WSG) with integrable weight π and derivative $\tilde\partial = \partial^{\mathrm{G}}$, ∂^{fct}, or ∂^{osc}, and if the random variable $f(A)$ satisfies* (4.77)*, then we have for all $L > 0$,*

$$\operatorname{Var}[X_L] \lesssim \pi_*(L)^{-1},$$

where we define

$$\pi_*(\ell) := \Big(\fint_{B_\ell} \int_{|x|}^\infty \pi(s) ds dx \Big)^{-1}. \qquad \Diamond$$

Remark 4.7.2. If $\pi(\ell) \simeq (\ell+1)^{-1-\beta}$ for some $\beta > 0$, then we compute

$$\pi_*(\ell) \simeq \begin{cases} (\ell+1)^d \log^{-1}(2+\ell), & \text{if } \beta = d; \\ (\ell+1)^d, & \text{it } \beta > d. \end{cases}$$

In particular if correlations are integrable (corresponding to the case $\beta > d$), we recover the CLT scaling: $\operatorname{Var}[X_L] \lesssim \pi_*(L)^{-1} \simeq L^{-d}$ for all $L \ge 1$. $\Diamond$

Proof of Proposition 4.7.1. Let $L > 0$. Given $\tilde\partial = \partial^{\mathrm{G}}$, ∂^{fct}, and ∂^{osc}, assumption (4.77) yields

$$|\tilde\partial_{A,B_{\ell+1}(x)} X_L| \lesssim \fint_{Q_L} e^{-\frac{1}{C}(|x-y|-\ell)_+} dy \lesssim L^{-d} (L \wedge (\ell+1))^d e^{-\frac{1}{C}(|x|-L-\ell)_+},$$

so that the weighted spectral gap yields

$$\begin{aligned} \operatorname{Var}[X_L] &\lesssim \int_0^\infty \int_{\mathbb{R}^d} L^{-2d} (L \wedge (\ell+1))^{2d} e^{-\frac{1}{C}(|x|-L-\ell)_+} dx \, (\ell+1)^{-d} \pi(\ell) d\ell \\ &\lesssim \int_0^\infty L^{-2d} (L \wedge (\ell+1))^{2d} (L+\ell)^d (\ell+1)^{-d} \pi(\ell) d\ell \lesssim L^{-d} \int_0^L (\ell+1)^d \pi(\ell) d\ell + \int_L^\infty \pi(\ell) d\ell. \end{aligned}$$

An integration by parts yields $\pi_*(L)^{-1} \simeq L^{-d} \int_0^L \pi(\ell) \ell^d d\ell + \int_L^\infty \pi(\ell) d\ell$, and the conclusion follows. $\square$

4.7.2 Concentration properties of spatial averages

We turn to the concentration properties of the spatial average X_L. The following result shows that the scaling crucially depends on three properties: the type of weighted functional inequality, the type of derivative, and the decay of the weight. More importantly, we show that concentration properties implied by weighted functional inequalities are in general stronger than those implied by the corresponding α-mixing.

As emphasized in the introduction in Section 4.1.1, establishing such concentration properties turns out to be particularly relevant for quantitative stochastic homogenization, and more precisely to obtain sharp integrability estimates on the validity of the quenched large-scale regularity theory for random elliptic systems in divergence form (that is, operators of the form $-\nabla \cdot A\nabla$ with A a matrix-valued random coefficient field as considered throughout this chapter). More precisely, Gloria, Neukamm, and Otto [204] reduce the validity of this large-scale regularity to concentration properties of spatial averages X_L of the square of an approximately local version of the extended corrector (cf. [204, Proposition 3]), and then make direct use of weighted functional inequalities in the form of the concentration results below: large-scale regularity is indeed characterized in [204] by the so-called minimal radius r_*, an almost surely finite stationary random field whose stochastic integrability essentially coincides with the scaling in L of the probability $\mathbb{P}\left[X_L \geq \delta\right]$ for some fixed (small) $\delta > 0$, in the sense that a property of the form $\mathbb{P}\left[X_L \geq \delta\right] \leq g_\delta(L)^{-1}$ for all $L \geq 1$ essentially implies $\mathbb{E}\big[g_\delta(\frac{1}{C_\delta} r_*)\big] < \infty$. We believe that these concentration results can also be used within the approach to large-scale regularity by Armstrong and Smart [36] and Armstrong and Mourrat [34] (which is rather formulated in terms of α-mixing assumptions).

Proposition 4.7.3. *Assume that the random variable $f(A)$ satisfies (4.77).*

(i) Let A satisfy ($\tilde\partial$-WSG) with integrable weight π and derivative $\tilde\partial = \partial^{\mathrm{G}}$ or ∂^{fct}, and let π_* be defined as in Proposition 4.7.1. Then for all $\delta, L > 0$ we have

$$\mathbb{P}\left[X_L \geq \delta\right] \leq \exp\Big(-\frac{\delta}{C}\pi_*(L)^{\frac{1}{2}}\Big). \tag{4.78}$$

If in addition A satisfies ($\tilde\partial$-WLSI) with weight π, then for all $\delta, L > 0$ we have

$$\mathbb{P}\left[X_L \geq \delta\right] \leq \exp\Big(-\frac{\delta^2}{C}\pi_*(L)\Big). \tag{4.79}$$

(ii) Let A satisfy (∂^{osc}-WSG) with weight $\pi(\ell) \lesssim (\ell+1)^{-\beta-1}$ for some $\beta > 0$. Then for all $\delta, L > 0$ we have

$$\mathbb{P}\left[X_L \geq \delta\right] \leq Ce^{-\frac{1}{C}\delta}\big(1+\delta^{-\frac{2\beta}{d}}|\log\delta|\big)\,L^{-\beta}. \tag{4.80}$$

(iii) Let A satisfy (∂^{osc}-WSG) with weight $\pi(\ell) \lesssim \exp(-\frac{1}{C}\ell^\beta)$ for some $\beta > 0$. Then for all $\delta > 0$ and all $L \geq 1$ we have

$$\mathbb{P}\left[X_L \geq \delta\right] \leq \exp\Big(-\frac{\delta \wedge \delta^2}{C}L^{\beta\wedge\frac{d}{2}}\Big). \tag{4.81}$$

If in addition A satisfies (∂^{osc}-WLSI) with weight $\pi(\ell) \lesssim \exp(-\frac{1}{C}\ell^\beta)$ for some $\beta > 0$, then for all $\delta > 0$ and all $L \geq 1$ we have

$$\mathbb{P}\left[X_L \geq \delta\right] \leq \exp\Big(-\frac{\delta \wedge \delta^2}{C}L^{\beta\wedge d}\Big). \tag{4.82}$$

$\Diamond$

Remark 4.7.4. If we further assume that the random variable $f(A)$ is a.s. bounded by a deterministic constant $C_0 \geq 1$, then there holds $\mathbb{P}\left[|X_L| > C_0\right] = 0$, and hence in (4.81) and (4.82) we may replace $\delta \wedge \delta^2$ by $\frac{1}{C_0}\delta^2$. $\Diamond$

In the case of a super-algebraic weight, it is instructive to compare these (nonlinear) concentration results to the corresponding (linear) concentration result implied by the α-mixing properties of the field A (see also [34, Appendix A]). Note that the same result holds under the corresponding weighted covariance inequality (which is natural in view of Proposition 4.2.3(iii)). (As we are basically interested in the scaling in L, we do not try to optimize the $|\log\delta|$-dependence below.)

Proposition 4.7.5. *Given $\beta>0$, assume that the random field A either is α-mixing with*

$$\tilde\alpha(\ell,D;A)\lesssim(1+D)^C\exp\Big(-\frac{1}{C}\ell^\beta\Big),\qquad \textit{for all } D,\ell\ge0,$$

or satisfies ($\tilde\partial$-WCI) with weight $\pi(\ell)\lesssim\exp(-\frac1C\ell^\beta)$ and derivative $\tilde\partial=\partial^{\mathrm G}$ or ∂^{osc}. Further assume that the random variable $f(A)$ is a.s. bounded by a deterministic constant, that is, $\sup\mathrm{ess}_A|f(A)|\lesssim1$, *and that it satisfies* (4.76). *Then for all $\delta>0$ and all $L\ge1$ we have*

$$\mathbb P\left[X_L>\delta\right]\le C\exp\bigg(-\frac{\delta^2(|\log\delta|+1)^{-\frac{d\beta}{d+\beta}}}{C}\,L^{\frac{d\beta}{d+\beta}}\bigg).\qquad\Diamond$$

Remark 4.7.6. Let us briefly compare the concentration results of Propositions 4.7.3(iii) and 4.7.5. Assume that the random field A satisfies a weighted functional inequality with super-algebraic weight $\pi(\ell)\lesssim\exp(-\frac1C\ell^\beta)$ and derivative ∂^{osc}, and that A is α-mixing with $\tilde\alpha(\ell,D;A)\lesssim(1+D)^d\exp(-\frac1C\ell^\beta)$ (these assumptions are indeed compatible in view of Proposition 4.2.3(iii)). Then the decay in L of the probability $\mathbb P\left[X_L\ge\delta\right]$ obtained from the α-mixing is better than the one obtained from (∂^{osc}-WSG) only for $\beta>d$, and is always worse than the one obtained from (∂^{osc}-WLSI). Similarly, in the case of an algebraic weight $\pi(\ell)\lesssim(\ell+1)^{-\beta-1}$, the functional inequality (∂^{osc}-WSG) yields the optimal decay $L^{-\beta}$ (cf. Proposition 4.7.3(ii)), while one can check that the corresponding α-mixing only leads to this decay up to a small (sub-algebraic) loss. $\Diamond$

We start with the proof of Proposition 4.7.3.

Proof of Proposition 4.7.3. We split the proof into three steps. We start with the proofs of (4.78), (4.79), and (4.80), which directly follow from Propositions 4.3.2 and 4.3.3(ii). The proof of estimates (4.81) and (4.82) is more subtle and is based on a fine tuning of Herbst's argument using specific features of the random variable Z_L.

Step 1. Proof of (4.78), (4.79), and (4.80).

For $\tilde\partial=\partial^{\mathrm G}$ or ∂^{fct}, let the Lipschitz norm $|||\cdot|||_{\tilde\partial,\pi}$ be defined as in the statement of Proposition 4.3.2. The same computation as in the proof of Proposition 4.7.1 ensures that the random variable $Z_L:=\pi_*(L)^{1/2}X_L=\pi_*(L)^{1/2}\fint_{Q_L}(F-\mathbb E\left[F\right])$ satisfies

$$|||\,Z_L\,|||_{\tilde\partial,\pi}\lesssim1.$$

Hence, estimates (4.78) and (4.79) follow from Proposition 4.3.2. We now turn to the proof of (4.80). If A satisfies (∂^{osc}-WSG) with weight $\pi(\ell)\lesssim(\ell+1)^{-\beta-1}$, $\beta>0$, we compute for all $p\ge p_0>\frac\beta d$, using assumption (4.77),

$$\mathbb E\bigg[\int_0^\infty\Big(\int_{\mathbb R^d}\Big(\partial^{\mathrm{osc}}_{A,B_\ell(x)}X_L\Big)^2dx\Big)^p(\ell+1)^{-dp-\beta-1}d\ell\bigg]$$
$$\lesssim L^{-2dp}\int_1^\infty(L+\ell)^{dp}(L\wedge\ell)^{2dp}\ell^{-dp-\beta-1}d\ell\;\lesssim\;\big(1+(dp_0-\beta)^{-1}\big)\,L^{-\beta}.$$

Then applying Proposition 4.3.3(ii) and optimizing the choice of $p_0>\frac\beta d$, the result (4.80) follows.

Step 2. Proof of (4.81).

Let $L \geq 1$, and define $Z_L := L^{d/2} X_L$. As in the proof of Proposition 4.7.1, assumption (4.77) yields

$$\partial^{\mathrm{osc}}_{A,B_\ell(x)} Z_L \;\lesssim\; L^{-\frac{d}{2}} (L \wedge (\ell+1))^d e^{-\frac{1}{C}(|x|-L-\ell)_+}. \tag{4.83}$$

We make use of a variant of Herbst's argument as in [66, Section 4] (see also [292, Section 2.5]). For all $t \geq 0$ we apply (∂^{osc}-WSG) to the random variable $\exp(\frac{1}{2} t Z_L)$: using the inequality $|e^a - e^b| \leq (e^a + e^b)|a-b|$ for all $a, b \in \mathbb{R}$, we obtain

$$\begin{aligned}
\mathrm{Var}\big[e^{\frac{1}{2}tZ_L}\big] \;&\leq\; \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\bigg[\Big(\, \partial^{\mathrm{osc}}_{A,B_\ell(x)}\, e^{\frac{1}{2}tZ_L}\Big)^2\bigg]\, dx\, (\ell+1)^{-d} \pi(\ell) d\ell \\
&\lesssim\; t^2\, \mathbb{E}\left[e^{tZ_L}\right] \sup_A \operatorname{ess} \int_0^\infty \int_{\mathbb{R}^d} e^{t\partial^{\mathrm{osc}}_{A,B_\ell(x)} Z_L} \Big(\, \partial^{\mathrm{osc}}_{A,B_\ell(x)}\, Z_L\Big)^2 dx\, e^{-\frac{1}{C}\ell^\beta} d\ell,
\end{aligned}$$

and hence, in terms of the Laplace transform $H_L(t) := \mathbb{E}\left[e^{tZ_L}\right]$,

$$H_L(t) - H_L(t/2)^2 \leq\; t^2 H_L(t) \sup_A \operatorname{ess} \int_0^\infty \int_{\mathbb{R}^d} e^{t\partial^{\mathrm{osc}}_{A,B_\ell(x)} Z_L} \Big(\partial^{\mathrm{osc}}_{A,B_\ell(x)} Z_L\Big)^2 dx\, e^{-\frac{1}{C}\ell^\beta} d\ell.$$

Using the property (4.83) of the random variable Z_L, we find

$$\begin{aligned}
&H_L(t) - H_L(t/2)^2 \\
&\quad\lesssim\; t^2 H_L(t) \int_1^\infty \Big(\frac{L\wedge \ell}{\sqrt{L}}\Big)^{2d} \exp\bigg(Ct\Big(\frac{L\wedge\ell}{\sqrt{L}}\Big)^d - \frac{\ell^\beta}{C}\bigg) \int_{\mathbb{R}^d} e^{-\frac{1}{C}(|x|-L-\ell)_+} dx\, d\ell \\
&\quad\lesssim\; t^2 H_L(t) \int_1^\infty (L+\ell)^d \Big(\frac{L\wedge\ell}{\sqrt{L}}\Big)^{2d} \exp\bigg(Ct\Big(\frac{L\wedge\ell}{\sqrt{L}}\Big)^d - \frac{\ell^\beta}{C}\bigg) d\ell \\
&\quad\lesssim\; t^2 H_L(t) \bigg(\int_0^L \exp\bigg(Ct\Big(\frac{\ell}{\sqrt{L}}\Big)^d - \frac{\ell^\beta}{C}\bigg) d\ell + L^d e^{CtL^{\frac{d}{2}}} \int_L^\infty e^{-\frac{1}{C}\ell^\beta} d\ell\bigg).
\end{aligned} \tag{4.84}$$

Without loss of generality we may assume that $\beta \leq \frac{d}{2}$ (the statement (4.81) is indeed not improved for $\beta > \frac{d}{2}$). We then restrict to

$$0 \leq t \leq T := \frac{1}{K} L^{\beta - \frac{d}{2}}, \tag{4.85}$$

for some $K \gg 1$ to be chosen later (with in particular $K \geq 2C^2$). As a consequence of $\beta \leq \frac{d}{2}$, this choice yields $T \leq K^{-1}$. On the one hand, for all $0 \leq \ell \leq L$ and all $0 \leq t \leq T$, the choice of T with $K \geq 2C^2$ yields

$$Ct\Big(\frac{\ell}{\sqrt{L}}\Big)^d - \frac{\ell^\beta}{C} \;=\; -\frac{L^\beta}{C}\bigg(\Big(\frac{\ell}{L}\Big)^\beta - \frac{C^2 t}{L^{\beta-\frac{d}{2}}}\Big(\frac{\ell}{L}\Big)^d\bigg) \leq -\frac{L^\beta}{C}\bigg(\Big(\frac{\ell}{L}\Big)^\beta - \frac{1}{2}\Big(\frac{\ell}{L}\Big)^d\bigg) \leq -\frac{\ell^\beta}{2C},$$

and hence

$$\int_0^L \exp\bigg(Ct\Big(\frac{\ell}{\sqrt{L}}\Big)^d - \frac{\ell^\beta}{C}\bigg) d\ell \lesssim \int_0^\infty e^{-\frac{\ell^\beta}{2C}} d\ell \lesssim 1.$$

On the other hand, for all $0 \leq t \leq T$, the choice of T with $K \geq 2C^2$ yields

$$L^d e^{CtL^{\frac{d}{2}}} \int_L^\infty e^{-\frac{1}{C}\ell^\beta} d\ell \lesssim \exp\Big(CtL^{\frac{d}{2}} - \frac{L^\beta}{2C}\Big) \leq \exp\Big(\frac{CL^\beta}{K} - \frac{L^\beta}{2C}\Big) \leq 1.$$

Injecting these estimates into (4.84), we obtain for all $0 \leq t \leq T$,

$$H_L(t) - H_L(t/2)^2 \leq Ct^2 H_L(t),$$

and hence

$$H_L(t) \leq \frac{H_L(t/2)^2}{1-Ct^2}.$$

Applying the same inequality for $t/2$, iterating, and noting that $H_L(2^{-n}t)^{2^n} \to e^{t\mathbb{E}[Z_L]} = 1$ as $n \uparrow \infty$, we obtain for all $0 \leq t \leq T$,

$$H_L(t) \leq \prod_{n=0}^{\infty} \big(1 - C(2^{-n}t)^2\big)^{-2^n}.$$

For K large enough such that $CT^2 \leq CK^{-2} \leq \frac{1}{2}$, the inequality $\log(1-x) \geq -2x$ for all $0 \leq x \leq \frac{1}{2}$ then yields for all $0 \leq t \leq T$,

$$\log H_L(t) \leq -\sum_{n=0}^{\infty} 2^n \log\big(1 - C(2^{-n}t)^2\big) \leq 2Ct^2 \sum_{n=0}^{\infty} 2^{-n} \lesssim t^2,$$

and thus $H_L(T) \leq e^{CT^2}$. Using Markov's inequality and the choice (4.85) of T, we deduce for all $r \geq 0$,

$$\mathbb{P}\left[Z_L > r\right] \leq e^{-Tr + CT^2} = \exp\Big(- \frac{L^{\beta-\frac{d}{2}} r}{K} + \frac{C}{K^2} L^{2\beta-d}\Big).$$

With the choice $r = \delta L^{\frac{d}{2}}$ for $\delta > 0$, this turns into

$$\mathbb{P}\left[X_L > \delta\right] \leq \exp\Big(- \frac{\delta}{K} L^{\beta} + \frac{C}{K^2} L^{2\beta-d}\Big) \leq \exp\Big(- \frac{1}{K}\Big(\delta - \frac{C}{K}\Big) L^{\beta}\Big).$$

Choosing $K \simeq 1 \vee \delta^{-1}$ large enough, the desired estimate (4.81) follows.

Step 2. Proof of (4.82).

Let $L \geq 1$, and define $Z_L := L^{d/2} X_L$. We make use of Herbst's classical argument as presented e.g. in [293, Section 5.1]. For all $t \geq 0$ we apply (∂^{osc}-WLSI) to the random variable $\exp(\frac{1}{2}tZ_L)$,

$$\mathrm{Ent}\big[e^{tZ_L}\big] \;\leq\; \int_0^\infty \int \mathbb{E}\left[\Big(\partial^{\mathrm{osc}}_{A,B_\ell(x)} e^{\frac{1}{2}tZ_L}\Big)^2\right] dx\, (\ell+1)^{-d} \pi(\ell) d\ell.$$

Estimating the right-hand side as in (4.84), we obtain in terms of $H_L(t) := \mathbb{E}\big[e^{tZ_L}\big]$,

$$\frac{d}{dt}\Big(\frac{1}{t}\log H_L(t)\Big) \;\lesssim\; \int_0^L \exp\Big(Ct\Big(\frac{\ell}{\sqrt{L}}\Big)^d - \frac{\ell^\beta}{C}\Big) d\ell + L^d e^{CtL^{\frac{d}{2}}} \int_L^\infty e^{-\frac{1}{C}\ell^\beta} d\ell.$$

Without loss of generality we may assume that $\beta \leq d$ (the statement (4.82) is indeed not improved for $\beta > d$). We then restrict to

$$0 \leq t \leq T := \frac{1}{K} L^{\beta - \frac{d}{2}}, \tag{4.86}$$

for some $K \gg 1$ to be chosen later (with in particular $K \geq 2C$). Arguing as in Step 1, we obtain for all $0 \leq t \leq T$,

$$\frac{d}{dt}\Big(\frac{1}{t}\log H_L(t)\Big) \;\lesssim\; 1,$$

which yields by integration with respect to t on $[0, T]$,

$$\frac{1}{T}\log H_L(T) = \frac{1}{T}\log H_L(T) - \mathbb{E}\left[Z_L\right] \lesssim T,$$

that is, $H_L(T) \;\leq\; e^{CT^2}$. The desired estimate (4.82) then follows as in Step 1, using Markov's inequality and choosing K large enough. $\square$

We now turn to the proof of Proposition 4.7.5.

Proof of Proposition 4.7.5. Without loss of generality we may assume that $\sup\mathrm{ess}_A |f(A)| \le 1$, which implies $\mathbb{P}[|X_L| > 1] = 0$. It is then sufficient to establish the result for $0 < \delta \le 1$. We split the proof into two steps. In the first step we prove the result in the case when the random variable $f(A)$ is exactly 1-local. We then extend the result in Step 2 when $f(A)$ is only approximately local in the sense (4.76). Since (WCI) implies α-mixing by Proposition 4.2.3(iii), it is enough to prove the result under the sole assumption of α-mixing.

Step 1. Exactly 1-local random variable $f(A)$.

In this step we assume in addition that $f(A)$ is $\sigma(A|_{B_1})$-measurable, and we prove that for all $\delta, L > 0$,

$$\mathbb{P}[X_L > \delta] \le C \exp\Big(-\frac{\delta^2}{C} L^{\frac{d\beta}{d+\beta}}\Big). \tag{4.87}$$

Let $p \ge 1$ be an integer and let $R > 0$. Setting

$$E_{R,p} := \{(x_1, \dots, x_{2p}) \in (Q_L)^{2p} : |x_1 - x_j| > R,\ \forall j \ne 1\},$$

and noting that for all x the random variable $F(x)$ is $\sigma(A|_{B(x)})$-measurable, α-mixing leads to

$$\Big| \int \dots \int_{E_{R,p}} \mathbb{E}\left[(F(x_1) - \mathbb{E}[F]) \dots (F(x_{2p}) - \mathbb{E}[F])\right] dx_1 \dots dx_{2p} \Big| \\ \le\ C^p L^{2dp} \tilde\alpha(R-2, \sqrt{d}L + 2; A)\ \lesssim\ C^p L^{2dp+C} e^{-\frac{1}{C}R^\beta}. \tag{4.88}$$

Using this estimate, we compute

$$\mathbb{E}\big[X_L^{2p}\big] = \fint_{Q_L} \dots \fint_{Q_L} \mathbb{E}\left[(F(x_1) - \mathbb{E}[F]) \dots (F(x_{2p}) - \mathbb{E}[F])\right] dx_1 \dots dx_{2p} \\ \le C^p L^C e^{-\frac{1}{C}R^\beta} + C^p L^{-2dp} \int_{Q_L} \dots \int_{Q_L} \mathbb{1}_{\forall i,\, \exists j \ne i:\, |x_i - x_j| \le R}\, dx_1 \dots dx_{2p}. \tag{4.89}$$

We consider the partitions $P := \{P_1, \dots, P_{N_P}\}$ of the index set $[2p] := \{1, \dots, 2p\}$ into nonempty subsets of cardinality ≥ 2 (that is, $\cup_j P_j = [2p]$, $\sharp P_j \ge 2$ for all j, and $P_j \cap P_l = \varnothing$ for all $j \ne l$), and we use the notation $P \vdash_2 [2p]$ for such partitions. The above then takes the form

$$\mathbb{E}\big[X_L^{2p}\big] \le C^p L^C e^{-\frac{1}{C}R^\beta} + C^p L^{-2dp} \sum_{P \vdash_2 [2p]} L^{dN_P} R^{d(2p-N_P)}.$$

Since for all $1 \le k \le p$ the number of partitions $P \vdash_2 [2p]$ with $N_P = k$ is bounded by the Stirling number of the second kind $\left\{ {2p \atop k} \right\} \le \frac{1}{2}\binom{2p}{k} k^{2p-k} \le C^p p^{2p} k^{2(p-k)} (2p-k)^{-(2p-k)}$, we deduce

$$\mathbb{E}\big[X_L^{2p}\big] \le C^p L^C e^{-\frac{1}{C}R^\beta} + C^p \Big(\frac{R}{L}\Big)^{dp} \sum_{k=1}^{p} \frac{p^{2p} k^{2(p-k)}}{(2p-k)^{2p-k}} \Big(\frac{R}{L}\Big)^{d(p-k)},$$

and hence by Markov's inequality, for all $\delta > 0$,

$$\mathbb{P}[X_L > \delta] \le \delta^{-2p} C^p L^C e^{-\frac{1}{C}R^\beta} + \delta^{-2p} C^p \Big(\frac{R}{L}\Big)^{dp} \sum_{k=1}^{p} \frac{p^{2p} k^{2(p-k)}}{(2p-k)^{2p-k}} \Big(\frac{R}{L}\Big)^{d(p-k)}. \tag{4.90}$$

Recall that we may restrict to $0 < \delta \le 1$. Choosing $R = L^\alpha$, $p = \delta^2 C_0^{-1} L^{\alpha\beta}$, and $\alpha = \frac{d}{d+\beta}$, for some $C_0 \simeq 1$ large enough, the estimate (4.90) above leads to

$$\mathbb{P}[X_L > \delta] \le C e^{-\frac{1}{C}L^{\alpha\beta}} + \delta^{-2p} C^p L^{-\alpha\beta p} \sum_{k=1}^{p} \frac{p^{p+k} k^{2(p-k)}}{(2p-k)^{2p-k}}.$$

Noting that the summand is increasing in k, and using the choice of p with C_0 large enough, we deduce

$$\mathbb{P}\left[X_L > \delta\right] \le Ce^{-\frac{1}{C}L^{\alpha\beta}} + \delta^{-2p}C^pL^{-\alpha\beta p}p^p \le Ce^{-\frac{1}{C}\delta^2L^{\alpha\beta}}, \tag{4.91}$$

from which the desired result (4.87) follows.

Step 2. Approximately 1-local random variable $f(A)$.

For all $r > 1$, we define the (r-local) random variable $f_r(A) := \mathbb{E}\left[f(A) \,\|\, A|_{B_r}\right]$, and we set $F_r(x) := f_r(A(\cdot + x))$ and $X_{r,L} := \fint_{Q_L}(F_r - \mathbb{E}\left[F_r\right])$. The approximate locality assumption (4.76) implies a.s. for all $r, L > 0$,

$$|X_{r,L} - X_L| \le Ce^{-\frac{r}{C}}. \tag{4.92}$$

Setting $\tilde F_r(x) := F(rx)$ and $A_r(x) := A(rx)$, we note that for all $x \in \mathbb{R}^d$ the random variable $\tilde F_r(x)$ is $\sigma(A|_{B_r(rx)})$-measurable, that is, $\sigma(A_r|_{B(x)})$-measurable. For all $r \ge 1$, the α-mixing assumption on A implies that the contracted random field A_r satisfies α-mixing with coefficient

$$\tilde\alpha_r(\ell, D; A_r) := \Big((1 + rD)^C \exp(-\frac{1}{C}(r\ell)^\beta)\Big) \wedge 1 \;\le\; C(1 + rD)^C \exp(-\frac{1}{C}(r\ell)^\beta),$$

so that the α-mixing coefficient is basically unchanged for $r \ge 1$. We may therefore apply Step 1 in the following form for all $\delta, L > 0$ and all $r \ge 1$,

$$\mathbb{P}\left[X_{r,L} > \delta\right] = \mathbb{P}\left[\fint_{Q_{L/r}}(\tilde F_r - \mathbb{E}\big[\tilde F_r\big]) > \delta\right] \le C\exp\Big(-\frac{\delta^2}{C}\Big(\frac{L}{r}\Big)^{\frac{d\beta}{d+\beta}}\Big),$$

where the constant $C \ge 1$ is independent of r. Combining this with (4.92) and choosing $r := C|\log(\frac{\delta}{eC})| \ge 1$, we obtain for all $0 < \delta \le 1$ and $L > 0$,

$$\mathbb{P}\left[X_L > \delta\right] \le \mathbb{P}\left[X_{r,L} > \delta - Ce^{-\frac{r}{C}}\right] \le \mathbb{P}\left[X_{r,L} > \tfrac{\delta}{2}\right] \le C\exp\Big(-\frac{\delta^2}{C}\Big(\frac{L}{|\log(\frac{\delta}{eC})|}\Big)^{\frac{d\beta}{d+\beta}}\Big),$$

and the conclusion follows. $\square$

4.7.3 Approximate normality of spatial averages

We now turn to the approximate normality of the spatial average X_L. We focus for simplicity on the case $f(A) = A(0)$, so that $X_L = \fint_{Q_L}(A - \mathbb{E}\left[A\right])$ is the spatial average of the random field itself (more general cases can be considered as well, at the price of further assumptions on second vertical derivatives of f). We study two prototypical examples: Gaussian random fields, and Poisson random inclusions with (unbounded) random radii. A similar result is expected to hold in stochastic homogenization, where Z_L is replaced by the spatial average of the homogenization commutator (cf. Chapter 3).

Proposition 4.7.7. *Let $X_L := \fint_{Q_L}(A - \mathbb{E}\left[A\right])$, and consider the two examples separately.*

(i) Let G be a jointly measurable stationary Gaussian random field on $\mathbb{R}^d$, characterized by its covariance $\mathcal{C}(x) := \mathrm{Cov}\left[G(x); G(0)\right]$, and assume that $\sup_{B(x)}|\mathcal{C}| \le c(|x|)$ for some Lipschitz non-increasing map $c : \mathbb{R}_+ \to \mathbb{R}_+$. Let $h \in C^2(\mathbb{R})$ with $h', h'' \in \mathrm{L}^\infty(\mathbb{R})$, and let A be the random field on $\mathbb{R}^d$ defined by $A(x) := h(G(x))$ for all x. Set $\pi(\ell) = -c'(\ell)$, and define

$$\pi_*(\ell) \;:=\; \Big(\fint_{B_\ell}\int_{|x|}^\infty \pi(s)dsdx\Big)^{-1}.$$

Then Proposition 4.7.1 ensures that the rescaled random variable $Z_L := \pi_*(L)^{1/2} X_L$ satisfies $\sigma_L^2 := \mathrm{Var}[Z_L] \lesssim 1$. Moreover we have for all $L \geq 1$,

$$\mathrm{d_{TV}}\left(\frac{Z_L}{\sigma_L}, \mathcal{N}\right) \lesssim \sigma_L^{-2} \pi_*(L)^{-\frac{1}{2}}. \tag{4.93}$$

(ii) Let the random field A be given by the Poisson unbounded spherical inclusion model with radius law ν (cf. example (A) in Section 4.6.2), and assume that the law ν satisfies for some $\beta > 0$,

$$\gamma(\ell) := \nu([\ell, \ell+1)) \lesssim \ell^{-3d-\beta-1}.$$

Then Proposition 4.7.1 holds with weight $\pi(\ell) = (\ell+1)^{-2d-\beta-1}$ and $\pi_*(L) = L^d$, and the rescaled random variable $Z_L := L^{d/2} X_L$ satisfies $\sigma_L^2 := \mathrm{Var}[Z_L] \lesssim 1$. Moreover we have for all $L \geq 1$,

$$\mathrm{d_W}\left(\frac{Z_L}{\sigma_L}, \mathcal{N}\right) + \mathrm{d_K}\left(\frac{Z_L}{\sigma_L}, \mathcal{N}\right) \lesssim \left(1 + \sigma_L^{-3}\right) L^{-\frac{d}{2}} \left(1 + L^{d-\beta}\right)^{\frac{1}{2}}. \qquad \diamondsuit$$

Remarks 4.7.8. Comments are in order.

— As e.g. in [112], we consider that estimating $\sigma_L \lesssim 1$ from below is a separate issue. In the Gaussian case with integrable covariance function, we do not believe this is essential: in that case, if h is for instance an increasing function, then one can indeed prove $\sigma_L \gtrsim 1$ (see e.g. [207] for a similar argument in stochastic homogenization, starting from a lower bound for variances proved in [422]). In the Gaussian case with non-integrable covariance, the question of bounding σ_L from below is more subtle. It is typically related to the Hermite rank of the function h and may lead to different scalings than π_*, in which case approximate normality may fail. We refer the reader to the recent works [224, 291] in the context of 1D stochastic homogenization, and more generally to [408].

— Before we turn to the proof of the above result, let us discuss its optimality. We believe that for Gaussian random fields item (i) is generically optimal, but optimality is much less clear for item (ii) as the comparison to results based on α-mixing suggests. For this discussion, we restrict to the more documented case of dimension $d = 1$. Two results are available on approximate normality for spatial averages of α-mixing random fields. The first result is classical and due to Ibragimov (see e.g. [75]): it ensures that a *qualitative* CLT holds for $Z_L := L^{1/2} X_L$ whenever for some $\kappa > 1$ the field A satisfies $\tilde\alpha(R, \infty; A) \lesssim R^{-\kappa}$ for all $R \geq 1$. The second result is due to Pène [356, Theorem 1.1] and essentially shows that Z_L satisfies a *quantitative* CLT in 1-Wasserstein distance with optimal rate $L^{-1/2}$ whenever for some $\kappa > 2$ there holds $\tilde\alpha(R, \infty; A) \lesssim R^{-\kappa}$ for all $R \geq 1$. Let us compare these results with the statement of item (ii) above. For the Poisson unbounded spherical inclusion model with radius law ν (cf. example (A) in Subsection 4.6.2), assuming that $\gamma(\ell) := \nu([\ell, \ell+1)) \simeq (\ell+1)^{-\kappa-d-1}$ with $\kappa > 0$, we proved in Propositions 4.2.3(iii) and 4.5.4(i) that for any fixed diameter $D > 0$ the α-mixing coefficient satisfies $\tilde\alpha(R, D; A) \lesssim_D R^{-\kappa}$ for all $R \geq 1$, while item (ii) above for $d = 1$ yields a qualitative CLT whenever $\kappa > 2$, and a CLT in 1-Wasserstein distance with optimal rate $L^{-1/2}$ whenever $\kappa > 3$. Comparing this with the results by Ibragimov and by Pène, there is thus a discrepancy in the critical values of κ, suggesting that item (ii) might not be optimal. Nevertheless, in the Poisson unbounded spherical model under consideration one can prove that $\inf_{R \geq 1} \tilde\alpha(R, \infty; A) > 0$, so that strictly speaking the results by Ibragimov and Pène do not apply — no general CLT result seems to be known based on the decay of α-mixing coefficients on bounded sets only. $\diamondsuit$

Proof of Proposition 4.7.7. We split the proof into two steps.

Step 1. Proof of item (i).

By [163, Corollary 3.1], we may apply [162, Proposition 4.1] with the weight $\pi(\ell) = -c'(\ell)$, which then yields $\sigma_L \lesssim 1$. We now apply Theorem 4.6.1 to Z_L, which greatly simplifies in this precise linear situation since second derivatives of Z_L with respect to A vanish identically. More precisely, for all $L \geq 1$ with the choice $R := 1$, it leads to

$$\mathrm{d}_{\mathrm{TV}}\left(\frac{Z_L}{\sigma_L}, \mathcal{N}\right)^2 \lesssim \frac{1}{\sigma_L^4} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} c(|x_1 - x_2| - 1)\, c(|x_2 - x_3| - 1)\, c(|x_3 - x_4| - 1) \\ \times \prod_{i=1}^{4} \mathbb{E}\left[\Big(\int_{B(x_i)} \Big|\frac{\partial Z_L}{\partial A}\Big|\Big)^4\right]^{\frac{1}{4}} dx_1 \ldots dx_4.$$

For all $x \in \mathbb{R}^d$ a direct calculation yields

$$\int_{B(x)} \Big|\frac{\partial Z_L}{\partial A}\Big| \lesssim \pi_*(L)^{\frac{1}{2}} L^{-d} \mathbb{1}_{|x| \lesssim L},$$

so that the above turns into

$$\mathrm{d}_{\mathrm{TV}}\left(\frac{Z_L}{\sigma_L}, \mathcal{N}\right)^2 \lesssim \frac{1}{\sigma_L^4} L^{-4d} \pi_*(L)^2 \int_{B_{CL}} \cdots \int_{B_{CL}} c(|x_1 - x_2|) c(|x_2 - x_3|) c(|x_3 - x_4|) dx_1 \ldots dx_4 \\ \lesssim \frac{1}{\sigma_L^4} \pi_*(L)^2 \Big(L^{-d} \int_{B_{CL}} c(|x|) dx\Big)^3.$$

Recalling that c is non-increasing and that $\pi(\ell) = -c'(\ell)$, we compute

$$L^{-d} \int_{B_{CL}} c(|x|) dx \simeq L^{-d} \int_{B_L} c(|x|) dx \simeq \fint_{B_L} \int_{|x|}^{\infty} \pi(s) ds dx = \pi_*(L)^{-1}.$$

The claim (4.93) then follows from the combination of these last two estimates.

Step 2. Proof of item (ii).

By [163, Proposition 3.4], we may apply [162, Proposition 4.1] with the weight

$$\pi(\ell) \simeq (\ell + 1)^d \sup_{|u| \leq 2} \gamma(\ell + u - 1) \lesssim \ell^{-2d-\beta-1},$$

which implies $\pi_*(L) \simeq L^d$ and hence $\sigma_L \lesssim 1$. We then apply Theorem 4.6.2 to Z_L. For all $x, x' \in \mathbb{Z}^d$ and $\ell, \ell' \in \mathbb{N}$, we have

$$|\partial^{\mathrm{dis}}_{\ell,x} Z_L| \lesssim L^{-\frac{d}{2}} |B_{\ell+1}(x) \cap Q_L| \lesssim L^{-\frac{d}{2}} \big(L \wedge (\ell + 1)\big)^d \mathbb{1}_{|x| \lesssim L + \ell},$$

and also

$$|\partial^{\mathrm{dis}}_{\ell,x} \partial^{\mathrm{dis}}_{\ell',x'} Z_L| \lesssim L^{-\frac{d}{2}} |B_{\ell'+1}(x') \cap B_{\ell+1}(x) \cap Q_L| \\ \lesssim L^{-\frac{d}{2}} \big(L \wedge (\ell + 1) \wedge (\ell' + 1)\big)^d \mathbb{1}_{|x'| \lesssim L + \ell'} \mathbb{1}_{|x| \lesssim L + \ell} \mathbb{1}_{|x - x'| \lesssim \ell + \ell'}.$$

As these right-hand sides are deterministic, we may actually apply Theorem 4.6.2 with the borderline exponent $\lambda = 1$, which yields

$$\begin{aligned}
&\mathrm{d_W}\left(\frac{Z_L}{\sigma_L},\mathcal{N}\right) + \mathrm{d_K}\left(\frac{Z_L}{\sigma_L},\mathcal{N}\right) \\
&\lesssim_\mu \quad \frac{1}{\sigma_L^2}\Bigg(\sum_{x,x',x''}\sum_{\ell,\ell',\ell''=0}^{\infty}\gamma(\ell)\gamma(\ell')\gamma(\ell'')\,\sup_A \mathrm{ess}\,|\partial^{\mathrm{dis}}_{\ell,x}\partial^{\mathrm{dis}}_{\ell',x'}Z_L| \\
&\qquad\qquad\qquad \times \sup_A \mathrm{ess}\,|\partial^{\mathrm{dis}}_{\ell,x}\partial^{\mathrm{dis}}_{\ell'',x''}Z_L|\,\sup_A \mathrm{ess}\,|\partial^{\mathrm{dis}}_{\ell',x'}Z_L|\,\sup_A \mathrm{ess}\,|\partial^{\mathrm{dis}}_{\ell'',x''}Z_L|\Bigg)^{\frac12} \\
&\qquad + \frac{1}{\sigma_L^2}\Bigg(\sum_{x,x'}\sum_{\ell,\ell'=0}^{\infty}\gamma(\ell)\gamma(\ell')\,\sup_A \mathrm{ess}\,|\partial^{\mathrm{dis}}_{\ell,x}\partial^{\mathrm{dis}}_{\ell',x'}Z_L|^2\,\sup_A \mathrm{ess}\,|\partial^{\mathrm{dis}}_{\ell',x'}Z_L|^2\Bigg)^{\frac12} \\
&\qquad + \frac{1}{\sigma_L^2}\Bigg(\sum_{x}\sum_{\ell=0}^{\infty}\gamma(\ell)\,\sup_A \mathrm{ess}\,|\partial^{\mathrm{dis}}_{\ell,x}Z_L|^4\Bigg)^{\frac12} + \frac{1}{\sigma_L^3}\sum_{x}\sum_{\ell=0}^{\infty}\gamma(\ell)\,\sup_A \mathrm{ess}\,|\partial^{\mathrm{dis}}_{\ell,x}Z_L|^3.
\end{aligned}$$

We denote by $I_1,\ldots,I_4$ the four right-hand side terms. Given the bound $\gamma(\ell) \lesssim \ell^{-\beta'-1}$ for some $\beta' > 0$, straightforward calculations left to the reader yield for all $L \geq 1$,

$$I_1 \lesssim \frac{1}{\sigma_L^2}L^{-\frac{d}{2}}(1\vee L^{2d-\beta'})^{\frac32}, \qquad I_2 \lesssim \frac{1}{\sigma_L^2}L^{-\frac{d}{2}}(1\vee L^{3d-\beta'})^{\frac12}(1\vee L^{2d-\beta'})^{\frac12},$$
$$I_3 \lesssim \frac{1}{\sigma_L^2}L^{-\frac{d}{2}}(1\vee L^{4d-\beta'})^{\frac12}, \qquad I_4 \lesssim \frac{1}{\sigma_L^3}L^{-\frac{d}{2}}(1\vee L^{3d-\beta'}).$$

The dominating term with respect to scaling in L is the third one I_3, and the claim then follows by taking $\beta' := 3d + \beta$ for $\beta > 0$. □

4.7.4 Random sequential adsorption and the jamming limit

We consider the problem of sequential packing at saturation, following the presentation in [391]. Let $R > 0$, and let $(U_{i,R})_{i\geq 1}$ be a sequence of i.i.d. random points uniformly distributed on the cube Q_R. Let $\mathcal{S}$ be a fixed bounded closed convex set in $\mathbb{R}^d$ with non-empty interior and centered at the origin 0 of $\mathbb{R}^d$ (that is, a reference "solid"), and for $i \geq 1$ let $\mathcal{S}_{i,R}$ be the translate of $\mathcal{S}$ with center at $U_{i,R}$. Then $\mathcal{S}_R := (\mathcal{S}_{i,R})_{i\geq1}$ is an infinite sequence of solids centered at uniform random positions in Q_R (the centers lie in Q_R but the solids themselves need not lie wholly inside Q_R). Let the first solid $S_{1,R}$ be packed, and recursively for $i \geq 2$ let the i-th solid $\mathcal{S}_{i,R}$ be packed if it does not overlap any solid in $\{\mathcal{S}_{1,R},\ldots,\mathcal{S}_{i-1,R}\}$ which has already been packed. If not packed, the i-th solid is discarded. This process, known as random sequential adsorption (RSA) with infinite input on the domain Q_R, is irreversible and terminates when it is not possible to accept additional solids. The jamming number $\mathcal{N}_R := \mathcal{N}_R(\mathcal{S}_R)$ denotes the number of solids packed in Q_R at termination. We are then interested in the asymptotic behavior of $R^{-d}\mathcal{N}_R$ in the infinite volume regime $R \uparrow \infty$, the limit of which (if it exists) is called the *jamming limit*.

In any dimension $d \geq 1$ and for any choice of the reference solid $\mathcal{S}$, Penrose [357] established the existence of the jamming limit, as well as the existence of the infinite volume limit for the distribution of the centers of packed solids, which defines a point process ξ on the whole of $\mathbb{R}^d$. (In the model case $\mathcal{S} := B_1$, this locally finite random measure ξ is referred to as the random parking point process with unit radius.) As already recalled in Section 4.5.3 (with unit balls replaced by translates of $\mathcal{S}$), the key argument in [357] relies on a graphical construction for ξ as a transformation $\xi = \Phi(\mathcal{P}_0)$ of a unit intensity Poisson point process $\mathcal{P}_0$ on the extended space $\mathbb{R}^d \times \mathbb{R}_+$.

In [391], Schreiber, Penrose, and Yukich further showed in any dimension $d \geq 1$ that the rescaled variance $R^{-d}\mathrm{Var}\,[\mathcal{N}_R]$ converges to a positive limit (without rate) and that $\mathcal{N}_R$ satisfies a CLT, that is, the fluctuations of the random variable $\mathcal{N}_R$ are asymptotically normal. They also quantified the rate of convergence to the normal, as well as the rate of convergence of $R^{-d}\mathbb{E}\big[\mathcal{N}_R\big]$ to the jamming limit. The numerical approximation of the value of the jamming limit has been the object of several works, including [413, Chapter 11.4] and [415]. As is clear from the analysis, the speed of convergence of $R^{-d}\mathbb{E}\big[\mathcal{N}_R\big]$ towards its limit is dominated by a boundary effect (the error scales like R^{-1}).

In order to avoid this boundary effect and to obtain better rates of convergence, we may replace $\mathcal{N}_R$ by the number $\tilde{\mathcal{N}}_R$ of packed solids with periodic boundary conditions on Q_R: we say that the i-th solid $\mathcal{S}_{i,R}$ is packed with periodic boundary conditions if its *periodic extension* $\mathcal{S}_{i,R} + R\mathbb{Z}^d$ does not overlap with any solid in $\{\mathcal{S}_{1,R}, \ldots, \mathcal{S}_{i-1,R}\}$ which has already been packed. The following shows that this allows one to get rid of the boundary effect, yields optimal estimates, and therefore suggests a more efficient way to approximate the jamming limit numerically.

Theorem 4.7.9. *For all $R \geq 0$, let $\tilde{\mathcal{N}}_R := \tilde{\mathcal{N}}_R(\mathcal{S}_R)$ be the number of packed solids of $\mathcal{S}_R$ with periodic boundary conditions as defined above. There are constants $\mu := \mu(\mathcal{S}, d) \in (0, \infty)$ (the jamming limit) and $\sigma^2 := \sigma^2(\mathcal{S}, d) \in (0, \infty)$ such that as $R \uparrow \infty$ we have*

$$|R^{-d}\mathbb{E}\big[\tilde{\mathcal{N}}_R\big] - \mu| \lesssim e^{-\frac{1}{C}R}, \tag{4.94}$$

$$|R^{-d}\mathrm{Var}\big[\tilde{\mathcal{N}}_R\big] - \sigma^2| \lesssim e^{-\frac{1}{C}R}, \tag{4.95}$$

and

$$\mathrm{d_W}\left(R^{\frac{d}{2}}(R^{-d}\tilde{\mathcal{N}}_R - \mu), \mathcal{N}(\sigma^2)\right) + \mathrm{d_K}\left(R^{\frac{d}{2}}(R^{-d}\tilde{\mathcal{N}}_R - \mu), \mathcal{N}(\sigma^2)\right) \lesssim R^{-\frac{d}{2}}, \tag{4.96}$$

where $\mathcal{N}(\sigma^2)$ denotes a centered normal random variable with variance σ^2. $\diamondsuit$

Estimates (4.94) and (4.95) are a consequence of the stabilization properties established in [391]. Note that (4.96) is the best one can hope for: If we considered a Poisson point process instead of the random parking process, then $\tilde{\mathcal{N}}_R$ would be the number of Poisson points in Q_R, the constant μ would be the intensity of the process, we would have $\sigma^2 = \mu$, and (4.96) would be sharp. The proof of (4.96) combines (4.94) and (4.95) to a normal approximation result, which is itself a slight improvement of [391, Theorem 1.1] in the sense that it avoids the spurious logarithmic correction $\log^{3d}(R)$. This improvement is a direct consequence of Theorem 4.6.2 (it also follows from [286, Theorem 6.1], but the proof we display here is more direct).

Proof of Theorem 4.7.9. Denote by ξ_R the (R-periodic extension of the) random parking measure on Q_R with periodic boundary conditions (that is, the measure obtained as the sum of Dirac masses at the centers of the periodically packed solids in Q_R). Also denote by $\xi = \xi_\infty$ the corresponding random parking measure on the whole space $\mathbb{R}^d$. Note that by definition both measures ξ_R and ξ are stationary, and we have $\xi_R(Q_R) = \tilde{\mathcal{N}}_R$.

Let us first introduce a natural pairing between ξ_R and ξ based on the graphical construction recalled above. Replacing the original Poisson point process $\mathcal{P}_0$ by $\mathcal{P}_0 \cap (Q_R \times \mathbb{R}_+) + R\mathbb{Z}^d$ (that is, the R-periodization of the restriction of $\mathcal{P}_0$ to $Q_R \times \mathbb{R}^+$), and then running the same graphical construction as above, we obtain a version of the R-periodic random parking measure ξ_R. Using this version, we view both ξ_R and ξ as $\sigma(\mathcal{P}_0)$-measurable random measures for the same underlying Poisson point process $\mathcal{P}_0$. Note however that with this coupling the pair (ξ_R, ξ) is no longer stationary.

We split the proof into three steps. In the first step we recall the construction of action radii for ξ_R and ξ. We then prove (4.94) and (4.95) using the exponentially decaying tail of the constructed action radii (or alternatively, the weighted covariance inequality of Proposition 4.5.3, and finally we prove (4.96) by appealing to Theorem 4.6.2.

Step 1. Construction and properties of action radii.

In this step we claim for all y that ξ admits an action radius ρ_y with respect to $\mathcal{P}_0$ on $Q(y) \times \mathbb{R}^+$, that the restriction $\xi_R|_{Q_R}$ admits an action radius $\rho_{R,y}$ with respect to $\mathcal{P}_0$ on $Q(y) \times \mathbb{R}^+$, and that we have

$$\mathbb{P}\left[\rho_y > \ell\right] + \mathbb{P}\left[\rho_{R,y} > \ell\right] \lesssim e^{-\frac{1}{C}\ell}.$$

In particular, we show that this implies

$$\sup_{y \in Q_{R/2}} \mathbb{P}\left[\xi(Q(y)) \neq \xi_R(Q(y))\right] \lesssim e^{-\frac{1}{C}R}. \tag{4.97}$$

The construction and tail behavior of the action radius ρ_y follows from Proposition 4.5.3 (with $\ell = 0$). Let the action radius $\rho_{R,y}$ be constructed similarly (simply replacing $\mathcal{P}_0$ by the point set $\mathcal{P}_0 \cap (Q_R \times \mathbb{R}_+) + R\mathbb{Z}^d$). A careful inspection of the proof of [391, Lemma 3.5] reveals that the same exponential tail behavior holds for $\rho_{R,y}$ uniformly in $R > 0$. It remains to argue in favor of (4.97), which simply follows from the exponential tail behavior of the action radii in the form

$$\sup_{y \in Q_{R/2}} \mathbb{P}\left[\xi(Q(y)) \neq \xi_R(Q(y))\right] \leq \sup_{y \in Q_{R/2}} \mathbb{P}\left[Q(y) + B_{\rho_y} \not\subset Q_R\right] \lesssim e^{-\frac{1}{C}R}.$$

Step 2. Proof of (4.94) and (4.95).

By stationarity of ξ_R and ξ we find $\mathbb{E}\left[\xi_R(Q_R)\right] = R^d \mathbb{E}\left[\xi_R(Q)\right]$ and $\mathbb{E}\left[\xi(Q_R)\right] = \mu R^d$ with $\mu := \mathbb{E}\left[\xi(Q)\right]$. We define

$$\sigma^2 := \int_{\mathbb{R}^d} \mathrm{Cov}\left[\xi(Q(x)); \xi(Q)\right] dx \tag{4.98}$$

and shall prove (4.94) and (4.95) in the form

$$|R^{-d}\mathbb{E}\big[\tilde{\mathcal{N}}_R\big] - \mu| \lesssim e^{-\frac{1}{C}R} \qquad \text{and} \qquad |R^{-d}\mathrm{Var}\big[\tilde{\mathcal{N}}_R\big] - \sigma^2| \lesssim e^{-\frac{1}{C}R}. \tag{4.99}$$

The estimate for the convergence of the mean follows from (4.97) in the form

$$|R^{-d}\mathbb{E}\big[\tilde{\mathcal{N}}_R\big] - \mu| = |\mathbb{E}\big[\xi_R(Q) - \xi(Q)\big]| \leq \operatorname{sup\,ess}\big(\xi_R(Q) + \xi(Q)\big)\, \mathbb{P}\left[\xi_R(Q) \neq \xi(Q)\right] \lesssim e^{-\frac{1}{C}R}.$$

We now appeal to the covariance inequality of [163, Proposition 3.3] to prove both the existence of σ^2 (by showing that the integral (4.98) is absolutely convergent) and the estimate for the convergence of the variance in (4.99). Rather than using the complete covariance inequality, it is actually sufficient here to make direct use of the constructed action radii ρ_0 and $\rho_{R,0}$ of Step 1. For $|y| \geq \sqrt{d} + 1$, noting that given $\rho_0 \vee \rho_y \leq \frac{1}{2}(|y| - \sqrt{d})$ the random variables $\xi(Q(y))$ and $\xi(Q)$ are by definition independent, we obtain

$$\begin{aligned}
&\mathrm{Cov}\left[\xi(Q(y)); \xi(Q)\right] \\
&\quad = \mathbb{E}\big[(\xi(Q(y)) - \mu)(\xi(Q) - \mu)\mathbb{1}_{\rho_0 \vee \rho_y > \frac{1}{2}(|y| - \sqrt{d})}\big] \\
&\qquad + \mathbb{E}\big[(\xi(Q(y)) - \mu)(\xi(Q) - \mu) \,\big\|\, \rho_0 \vee \rho_y \leq \tfrac{1}{2}(|y| - \sqrt{d})\big]\, \mathbb{P}\big[\rho_0 \vee \rho_y \leq \tfrac{1}{2}(|y| - \sqrt{d})\big] \\
&\quad = \mathbb{E}\big[(\xi(Q(y)) - \mu)(\xi(Q) - \mu)\mathbb{1}_{\rho_0 \vee \rho_y > \frac{1}{2}(|y| - \sqrt{d})}\big] \\
&\qquad + \mathbb{P}\big[\rho_0 \vee \rho_y \leq \tfrac{1}{2}(|y| - \sqrt{d})\big]^{-1} \mathbb{E}\big[(\xi(Q(y)) - \mu)\mathbb{1}_{\rho_0 \vee \rho_y \leq \frac{1}{2}(|y| - \sqrt{d})}\big]\, \mathbb{E}\big[(\xi(Q) - \mu)\mathbb{1}_{\rho_0 \vee \rho_y \leq \frac{1}{2}(|y| - \sqrt{d})}\big] \\
&\quad = \mathbb{E}\big[(\xi(Q(y)) - \mu)(\xi(Q) - \mu)\mathbb{1}_{\rho_0 \vee \rho_y > \frac{1}{2}(|y| - \sqrt{d})}\big] \\
&\qquad + \big(1 - \mathbb{P}\big[\rho_0 \vee \rho_y > \tfrac{1}{2}(|y| - \sqrt{d})\big]\big)^{-1} \\
&\qquad\qquad \times \mathbb{E}\big[(\xi(Q(y)) - \mu)\mathbb{1}_{\rho_0 \vee \rho_y > \frac{1}{2}(|y| - \sqrt{d})}\big]\, \mathbb{E}\big[(\xi(Q) - \mu)\mathbb{1}_{\rho_0 \vee \rho_y > \frac{1}{2}(|y| - \sqrt{d})}\big],
\end{aligned}$$

and hence, for all $|y| \geq C$ with $C \simeq 1$ large enough such that

$$\mathbb{P}\big[\rho_0 \vee \rho_y > \tfrac{1}{2}(|y| - \sqrt{d})\big] \leq 2\, \mathbb{P}\big[\rho_0 > \tfrac{1}{2}(|y| - \sqrt{d})\big] \leq \frac{1}{2},$$

we conclude

$$|\operatorname{Cov}[\xi(Q(y));\xi(Q)]| \lesssim \sup\operatorname{ess}\big(\xi(Q)^2\big)\,\mathbb{P}\big[\rho_0 > \tfrac{1}{2}(|y|-\sqrt{d})\big] \lesssim e^{-\frac{1}{C}|y|}. \tag{4.100}$$

Arguing similarly for ξ_R with ρ_0 replaced by $\rho_{R,0}$, we deduce for all $y \in Q_R$,

$$|\operatorname{Cov}[\xi_R(Q(y));\xi_R(Q)]| \lesssim e^{-\frac{1}{C}|y|}. \tag{4.101}$$

The estimate (4.100) implies in particular that the integral for σ^2 in (4.98) is well-defined. It remains to prove the estimate for the convergence of the variance in (4.99). By R-periodicity and stationarity of ξ_R, we find

$$\begin{aligned} R^{-d}\operatorname{Var}[\tilde{\mathcal{N}}_R] \;=\; R^{-d}\operatorname{Var}\Big[\int_{Q_R}\xi_R(Q(y))dy\Big] \;&=\; \fint_{Q_R}\int_{Q_R}\operatorname{Cov}[\xi_R(Q(x-y));\xi_R(Q)]\,dxdy \\ &=\; \int_{Q_R}\operatorname{Cov}[\xi_R(Q(y));\xi_R(Q)]\,dy, \end{aligned}$$

so that we may decompose

$$\begin{aligned} \sigma^2 - R^{-d}\operatorname{Var}[\tilde{\mathcal{N}}_R] \;=\; &\int_{\mathbb{R}^d\setminus Q_{R/2}}\operatorname{Cov}[\xi(Q(y));\xi(Q)]\,dy - \int_{Q_R\setminus Q_{R/2}}\operatorname{Cov}[\xi_R(Q(y));\xi_R(Q)]\,dy \\ &+\int_{Q_{R/2}}\big(\operatorname{Cov}[\xi(Q(y));\xi(Q)] - \operatorname{Cov}[\xi_R(Q(y));\xi_R(Q)]\big)dy. \end{aligned} \tag{4.102}$$

We estimate each of the three right-hand side terms separately. On the one hand, the estimates (4.100) and (4.101) yield

$$\Big|\int_{\mathbb{R}^d\setminus Q_{R/2}}\operatorname{Cov}[\xi(Q(y));\xi(Q)]\,dy\Big| \lesssim \int_{\mathbb{R}^d\setminus Q_{R/2}} e^{-\frac{1}{C}|y|}dy \lesssim e^{-\frac{1}{C}R}.$$

and

$$\Big|\int_{Q_R\setminus Q_{R/2}}\operatorname{Cov}[\xi_R(Q(y));\xi_R(Q)]\,dy\Big| \lesssim \int_{Q_R\setminus Q_{R/2}} e^{-\frac{1}{C}|y|}dy \lesssim e^{-\frac{1}{C}R}.$$

On the other hand, using (4.97), we obtain

$$\begin{aligned} &\Big|\int_{Q_{R/2}}\big(\operatorname{Cov}[\xi(Q(y));\xi(Q)] - \operatorname{Cov}[\xi_R(Q(y));\xi_R(Q)]\big)dy\Big| \\ &\le \int_{Q_{R/2}}\mathbb{E}\big[\,|\xi(Q)-\mathbb{E}[\xi(Q)]|\,|\xi(Q(y))-\xi_R(Q(y))|\,\big] \\ &\qquad\qquad + \int_{Q_{R/2}}\mathbb{E}\big[\,|\xi_R(Q(y))-\mathbb{E}[\xi_R(Q(y))]|\,|\xi(Q)-\xi_R(Q)|\,\big]\,dy \\ &\lesssim\; R^d \sup\operatorname{ess}\big(\xi(Q)+\xi_R(Q)\big)\sup_{y\in Q_{R/2}}\mathbb{P}[\xi(Q(y))\neq\xi_R(Q(y))] \;\lesssim\; e^{-\frac{1}{C}R}. \end{aligned}$$

Injecting these estimates into (4.102), the conclusion (4.99) for the convergence of the variance follows.

Step 3. Proof of (4.96).

We claim that it is enough to prove the normal approximation estimate

$$\mathrm{d_W}\left(\frac{\mathcal{N}_R-\mathbb{E}[\mathcal{N}_R]}{\sqrt{\operatorname{Var}[\mathcal{N}_R]}},\mathcal{N}\right) + \mathrm{d_K}\left(\frac{\mathcal{N}_R-\mathbb{E}[\mathcal{N}_R]}{\sqrt{\operatorname{Var}[\mathcal{N}_R]}},\mathcal{N}\right) \;\lesssim\; R^{-\frac{d}{2}}. \tag{4.103}$$

Indeed, the result (4.96) then follows from (4.103), (4.94), and (4.95) by the triangle inequality. We omit the proof of (4.103), which is identical to the proof of Proposition 4.7.7(ii) (the correction $L^{d-\beta}$ disappears here since the weight is exponential). □

4.A Appendix: Criterion for standard functional inequalities

In this appendix, we give a proof of Proposition 4.1.2.

Proof of Proposition 4.1.2. Let $\varepsilon > 0$ be fixed, and consider the partition $(Q_z)_{z\in\mathbb{Z}^d}$ of $\mathbb{R}^d$ defined by $Q_z = \varepsilon z + \varepsilon Q$. Choose an i.i.d. copy A_0' of the field A_0, and for all z define the random field A_0^z by $A_0^z|_{\mathbb{R}^d\setminus Q_z} := A_0|_{\mathbb{R}^d\setminus Q_z}$ and $A_0^z|_{Q_z} := A_0'|_{Q_z}$. We split the proof into three steps.

Step 1. Tensorization argument.

Choose an enumeration $(z_n)_n$ of $\mathbb{Z}^d$, and for all n let Π_n and $\mathbb{E}_n$ denote the linear maps on $\mathrm{L}^2(\Omega)$ defined by

$$\Pi_n[X] := \mathbb{E}\big[X \,\big\|\, A_0|_{\bigcup_{k=1}^n Q_{z_k}}\big], \qquad\qquad \mathbb{E}_n[X] := \mathbb{E}\big[X \,\big\|\, A_0|_{\mathbb{R}^d\setminus Q_{z_n}}\big].$$

Also define

$$\mathrm{Cov}_n[X;Y] := \mathbb{E}_n[XY] - \mathbb{E}_n[X]\mathbb{E}_n[Y], \qquad \mathrm{Var}_n[X] := \mathrm{Cov}_n[X;X],$$
$$\mathrm{Ent}_n[X^2] := \mathbb{E}_n\big[X^2 \log(X^2/\mathbb{E}_n[X^2])\big].$$

In this step, we make use of a martingale argument à la Lu-Yau [310] to show the following tensorization identities for the covariance and for the entropy: for all $\sigma(A_0)$-measurable random variables $X(A_0)$ and $Y(A_0)$, we have

$$\big|\,\mathrm{Cov}\left[X(A_0);Y(A_0)\right]\big| \;\le\; \sum_{k=1}^\infty \mathbb{E}\left[\big|\,\mathrm{Cov}_k\left[\Pi_k[X(A_0)];\Pi_k[Y(A_0)]\right]\big|\right], \tag{4.104}$$

$$\mathrm{Ent}\big[X(A_0)^2\big] \;\le\; \sum_{k=1}^\infty \mathbb{E}\left[\mathrm{Ent}_k\big[\Pi_k[X(A_0)^2]\big]\right]. \tag{4.105}$$

First note that for all $\sigma(A_0)$-measurable random variables $X(A_0)\in\mathrm{L}^2(\Omega)$, the properties of conditional expectations ensure that $\Pi_n[X(A_0)]\to X(A_0)$ in $\mathrm{L}^2(\Omega)$ as $n\uparrow\infty$. We then decompose the covariance into the following telescopic sum

$$\begin{aligned}\mathrm{Cov}\left[\Pi_n[X(A_0)];\Pi_n[Y(A_0)]\right] &= \sum_{k=1}^n \Big(\mathbb{E}\left[\Pi_k[X(A_0)]\Pi_k[Y(A_0)]\right] - \mathbb{E}\left[\Pi_{k-1}[X(A_0)]\Pi_{k-1}[Y(A_0)]\right]\Big)\\ &= \sum_{k=1}^n \mathbb{E}\left[\mathrm{Cov}_k\left[\Pi_k[X(A_0)];\Pi_k[Y(A_0)]\right]\right],\end{aligned}$$

so that the result (4.104) follows by taking the limit $n\uparrow\infty$. Likewise, we decompose the entropy into the following telescopic sum

$$\begin{aligned}\mathrm{Ent}\big[\Pi_n[X(A_0)^2]\big] &= \sum_{k=1}^n \Big(\mathbb{E}\left[\Pi_k[X(A_0)^2]\log(\Pi_k[X(A_0)^2])\right] - \mathbb{E}\left[\Pi_{k-1}[X(A_0)^2]\log(\Pi_{k-1}[X(A_0)^2])\right]\Big)\\ &= \sum_{k=1}^n \mathbb{E}\left[\mathrm{Ent}_k\big[\Pi_k[X(A_0)^2]\big]\right],\end{aligned}$$

and the result (4.105) follows in the limit $n\uparrow\infty$.

Step 2. Preliminary versions of (CI) and (LSI).

In this step, we prove that for all $\sigma(A_0)$-measurable random variables $X(A_0)$ and $Y(A_0)$ we have

$$\begin{aligned} |\operatorname{Cov}[X(A_0);Y(A_0)]| &\le \frac{1}{2}\sum_{k=1}^{\infty}\mathbb{E}\left[\left|\Pi_k\left[X(A_0)-X(A_0^{z_k})\right]\right|\left|\Pi_k\left[Y(A_0)-Y(A_0^{z_k})\right]\right|\right] \\ &\le \frac{1}{2}\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\left(X(A_0)-X(A_0^z)\right)^2\right]^{\frac12}\mathbb{E}\left[\left(Y(A_0)-Y(A_0^z)\right)^2\right]^{\frac12}, \end{aligned} \tag{4.106}$$

and

$$\operatorname{Ent}[X(A_0)] \le 2\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\sup_{A_0'}\operatorname{ess}\left(X(A_0)-X(A_0^z)\right)^2\right]. \tag{4.107}$$

We first prove (4.106): we appeal to (4.104) in the form

$$\begin{aligned} |\operatorname{Cov}[X(A_0);Y(A_0)]| &\le \frac{1}{2}\sum_{k=1}^{\infty}\mathbb{E}\left[\left|\mathbb{E}_k\left[\Pi_k[X(A_0)-X(A_0^{z_k})]\,\Pi_k[Y(A_0)-Y(A_0^{z_k})]\right]\right|\right] \\ &\le \frac{1}{2}\sum_{k=1}^{\infty}\mathbb{E}\left[\left|\Pi_k[X(A_0)-X(A_0^{z_k})]\right|\left|\Pi_k[Y(A_0)-Y(A_0^{z_k})]\right|\right], \end{aligned}$$

which directly yields (4.106) by Cauchy-Schwarz' inequality. Likewise, we argue that (4.107) follows from (4.105). To this aim, we have to reformulate the right-hand side of (4.105): using the inequality $a\log a - a + 1 \le (a-1)^2$ for all $a \ge 0$, we obtain for all $k \ge 0$,

$$\begin{aligned} \operatorname{Ent}_k\left[\Pi_k[X(A_0)^2]\right] &\le \mathbb{E}_k[\Pi_k[X(A_0)^2]]\,\mathbb{E}_k\left[\Big(\frac{\Pi_k[X(A_0)^2]}{\mathbb{E}_k[\Pi_k[X(A_0)^2]]}-1\Big)^2\right] \\ &= \frac{\operatorname{Var}_k\left[\Pi_k[X(A_0)^2]\right]}{\mathbb{E}_k[\Pi_k[X(A_0)^2]]} \\ &= \frac{\mathbb{E}_k\left[(\Pi_k[X(A_0)^2]-\Pi_k[X(A_0^{z_k})^2])^2\right]}{2\,\mathbb{E}_k[\Pi_k[X(A_0)^2]]} \\ &= \frac{\mathbb{E}_k\left[(\Pi_k[(X(A_0)-X(A_0^{z_k}))(X(A_0)+X(A_0^{z_k}))])^2\right]}{2\,\mathbb{E}_k[\Pi_k[X(A_0)^2]]} \\ &\le \frac{\mathbb{E}_k\left[\Pi_k[(X(A_0)-X(A_0^{z_k}))^2]\,\Pi_k[(X(A_0)+X(A_0^{z_k}))^2]\right]}{2\,\mathbb{E}_k[\Pi_k[X(A_0)^2]]}. \end{aligned}$$

Since $(A_0, A_0^{z_k})$ and $(A_0^{z_k}, A_0)$ have the same law by complete independence, the above implies, using the inequality $(a+b)^2 \le 2(a^2+b^2)$ for all $a,b\in\mathbb{R}$,

$$\begin{aligned} \operatorname{Ent}_k\left[\Pi_k[X(A_0)^2]\right] &\le \frac{2\,\mathbb{E}_k\left[\Pi_k[(X(A_0)-X(A_0^{z_k}))^2]\,\Pi_k[X(A_0^{z_k})^2]\right]}{\mathbb{E}_k[\Pi_k[X(A_0^{z_k})^2]]} \\ &\le 2\sup_{A_0'|_{Q_{z_k}}}\operatorname{ess}\Pi_k[(X(A_0)-X(A_0^{z_k}))^2] \le 2\,\Pi_k\Big[\sup_{A_0'|_{Q_{z_k}}}\operatorname{ess}(X(A_0)-X(A_0^{z_k}))^2\Big]. \end{aligned}$$

Estimate (4.107) now follows from (4.105).

Step 3. Proof of (CI) and (LSI).

We start with the proof of (CI). Since $A = A(A_0)$ is $\sigma(A_0)$-measurable, (4.106) yields for all $\sigma(A)$-measurable random variables $X(A)$ and $Y(A)$,

$$\big|\operatorname{Cov}[X(A);Y(A)]\big| \le \frac{1}{2}\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\left(X(A)-X(A(A_0^z))\right)^2\right]^{\frac12}\mathbb{E}\left[\left(Y(A)-Y(A(A_0^z))\right)^2\right]^{\frac12}.$$

Using that $\mathbb{E}\big[X(A)\,\|\,A_0|_{\mathbb{R}^d\setminus Q_z}\big]=\mathbb{E}\big[X(A(A_0^z))\,\|\,A_0|_{\mathbb{R}^d\setminus Q_z}\big]$ by complete independence of the field A_0,

$$\mathbb{E}\left[\big(X(A)-X(A(A_0^z))\big)^2\right]=\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A_0,Q_z}X(A(A_0))\Big)^2\right].$$

Since the conditional expectation $\mathbb{E}\big[\cdot\,\|\,A_0|_{\mathbb{R}^d\setminus Q_z}\big]$ coincides with the L^2-projection onto the $\sigma(A_0|_{\mathbb{R}^d\setminus Q_z})$-measurable functions, and since $\mathbb{E}\big[X(A)\,\|\,A|_{\mathbb{R}^d\setminus(Q_z+B_R)}\big]$ is $\sigma(A|_{\mathbb{R}^d\setminus(Q_z+B_R)})$-measurable and therefore $\sigma(A_0|_{\mathbb{R}^d\setminus Q_z})$-measurable by assumption, we have

$$\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A_0,Q_z}X(A(A_0))\Big)^2\right]\le\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,Q_z+B_R}X(A)\Big)^2\right].$$

Combining these two observations, we deduce that for all $\sigma(A)$-measurable random variables $X(A)$ and $Y(A)$,

$$\big|\operatorname{Cov}[X(A);Y(A)]\big|\le\frac{1}{2}\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,Q_z+B_R}X(A)\Big)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,Q_z+B_R}Y(A)\Big)^2\right]^{\frac{1}{2}}.$$

By taking local averages, this turns into

$$\begin{aligned}\big|\operatorname{Cov}[X(A);Y(A)]\big| &\le \frac{\varepsilon^{-d}}{2}\sum_{z\in\mathbb{Z}^d}\int_{\varepsilon Q}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,y+\varepsilon z+\varepsilon Q+B_R}X(A)\Big)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,y+\varepsilon z+\varepsilon Q+B_R}Y(A)\Big)^2\right]^{\frac{1}{2}}dy\\ &= \frac{\varepsilon^{-d}}{2}\int_{\mathbb{R}^d}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,y+\varepsilon Q+B_R}X(A)\Big)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,y+\varepsilon z+\varepsilon Q+B_R}Y(A)\Big)^2\right]^{\frac{1}{2}}dy\\ &\le \frac{\varepsilon^{-d}}{2}\int_{\mathbb{R}^d}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,B_{R+\varepsilon\sqrt{d}/2}(y)}X(A)\Big)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\Big(\partial^{\mathrm{G}}_{A,B_{R+\varepsilon\sqrt{d}/2}(y)}Y(A)\Big)^2\right]^{\frac{1}{2}}dy,\end{aligned}$$

that is, (CI) for any radius larger than R.

We then turn to the proof of (LSI). For all $\sigma(A)$-measurable random variables $X(A)$, the estimate (4.107) yields

$$\operatorname{Ent}[X(A)]\;\le\;2\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\operatorname*{sup\,ess}_{A_0'}\big(X(A(A_0))-X(A(A_0^z))\big)^2\right]\;\le\;2\sum_{z\in\mathbb{Z}^d}\mathbb{E}\left[\Big(\,\partial^{\mathrm{osc}}_{A,Q_z+B_R}\,X(A)\Big)^2\right].$$

The desired result (LSI) then follows from taking local averages. □

4.B Appendix: Abstract criteria for deterministically localized fields

In this appendix, we discuss general criteria for weighted functional inequalities in the case when the random field A is deterministically localized in the sense of Section 4.4.2. To be precise we focus on the typical example of a convolution of a random noise. In this case we prove the validity of a Brascamp-Lieb inequality from which the desired weighted functional inequalities follow. Although Gaussian random fields are the most prominent examples of this framework, we develop the general argument in a slightly more abstract setting. (Note that we choose to argue by approximation and reduce to discrete fields, rather than appeal to Malliavin calculus and associated functional analysis.)

Let W be a random noise on $\mathbb{R}^d$, that is, a mean-zero stationary completely independent second-order random Borel measure on $\mathbb{R}^d$ (see e.g. [367, Section 2]). More precisely, W associates a random variable $W(E)$ to any bounded Borel subset $E\subset\mathbb{R}^d$, in such a way that

(i) $\mathbb{E}[W(E)] = 0$ and $\mathbb{E}\left[|W(E)|^2\right] < \infty$ for all bounded Borel subset $E \subset \mathbb{R}^d$;

(ii) if $(E_n)_n$ is a family of disjoint Borel subsets of $\mathbb{R}^d$, then $W\left(\bigcup_{n=1}^\infty E_n\right) = \sum_{n=1}^\infty W(E_n)$ in the L^2-sense;

(iii) $(W(x+E), W(x+E'))$ has the same law as $(W(E), W(E'))$ for any two bounded Borel subsets $E, E' \subset \mathbb{R}^d$ and any $x \in \mathbb{R}^d$;

(iv) $W(E_1), \ldots, W(E_n)$ are independent for any disjoint Borel sets $E_1, \ldots, E_n \subset \mathbb{R}^d$ and any $n \in \mathbb{N}$.

Stationarity implies in particular that the Borel measure $\mathbb{E}\left[|dW|^2\right]$ is proportional to the Lebesgue measure: $\mathbb{E}\left[|dW|^2\right] = \lambda dx$, for some constant $\lambda \geq 0$ that is called the intensity of the random noise W.

Given a (deterministic) nonnegative Borel function $F \in \mathrm{L}^2(\mathbb{R}^d)$ and a constant $m \in \mathbb{R}^d$, we now define a measurable random field A on $\mathbb{R}^d$ by the following convolution,

$$A(y) = m + \int_{\mathbb{R}^d} F(y-z)dW(z), \tag{4.108}$$

the covariance function of which is then given by

$$\mathcal{C}(x) := \mathrm{Cov}\left[A(x); A(0)\right] = \lambda \int_{\mathbb{R}^d} F(x-z)F(z)dz. \tag{4.109}$$

The following result (which is rather standard) shows that a Brascamp-Lieb inequality holds for such random fields whenever the random noise W satisfies a standard spectral gap, thus mimicking the well-known situation of Gaussian fields. (For Gaussian fields, a discrete version of the Brascamp-Lieb inequality (4.111) below was first due to [81], while a discrete version of the inequality in covariance form (4.112) and in entropy form (4.115) is due to [320] and to [67, Proposition 3.4], respectively.)

Proposition 4.B.1 (Brascamp-Lieb type inequalities). *Let W be a random noise on $\mathbb{R}^d$ with intensity λ, let the stationary random field A on $\mathbb{R}^d$ be given by (4.108), and let $\mathcal{C}$ denote its covariance function.*

(i) Assume that for all $\eta > 0$ the random variable $W(\eta Q)$ satisfies the following spectral gap: for any smooth function ϕ,

$$\mathrm{Var}\left[\phi(W(\eta Q))\right] \leq C\lambda\eta^d \mathbb{E}\left[\phi'(W(\eta Q))^2\right]. \tag{4.110}$$

Then the random field A satisfies the following Brascamp-Lieb inequality: for all $\sigma(A)$-measurable random variables $X(A)$,

$$\mathrm{Var}\left[X(A)\right] \leq C\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\frac{\partial X(A)}{\partial A}(z)\right|\left|\frac{\partial X(A)}{\partial A}(z')\right| |\mathcal{C}(z-z')| dz dz'\right]. \tag{4.111}$$

Moreover, the following Brascamp-Lieb inequality in covariance form holds: for all $\sigma(A)$-measurable random variables $X(A), Y(A)$ we have

$$\begin{aligned}\mathrm{Cov}\left[X(A); Y(A)\right] \leq C \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\int_{\mathbb{R}^d}\left|\frac{\partial X(A)}{\partial A}(z)\right| |\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})(x-z)|\, dz\right)^2\right]^{\frac12} \\ \times \mathbb{E}\left[\left(\int_{\mathbb{R}^d}\left|\frac{\partial Y(A)}{\partial A}(z')\right| |\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})(x-z')|\, dz'\right)^2\right]^{\frac12} dx,\end{aligned} \tag{4.112}$$

and in particular

$$\mathrm{Cov}\left[X(A); Y(A)\right] \leq C \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{E}\left[\left|\frac{\partial X(A)}{\partial A}(z)\right|^2\right]^{\frac12} \mathbb{E}\left[\left|\frac{\partial Y(A)}{\partial A}(z')\right|^2\right]^{\frac12} \tilde{\mathcal{C}}(z-z') dz dz', \tag{4.113}$$

in terms of

$$\tilde{\mathcal{C}}(x) := \int |\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})(x-y)||\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})(y)| dy.$$

(ii) *Assume that for all $\eta > 0$ the random variable $W(\eta Q)$ satisfies the corresponding logarithmic Sobolev inequality: for any smooth function ϕ,*

$$\operatorname{Ent}\big[\phi(W(\eta Q))^2\big] \le C\lambda\eta^d \mathbb{E}\left[\phi'(W(\eta Q))^2\right]. \tag{4.114}$$

Then the random field A satisfies the corresponding Brascamp-Lieb inequality in logarithmic Sobolev form: for all $\sigma(A)$-measurable random variables $X(A)$,

$$\operatorname{Ent}[X(A)] \le C\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big|\frac{\partial X(A)}{\partial A}(z)\Big|\Big|\frac{\partial X(A)}{\partial A}(z')\Big||\mathcal{C}(z-z')|dzdz'\right]. \tag{4.115}$$

$\Diamond$

In the following theorem, we show that Brascamp-Lieb inequalities imply weighted functional inequalities, using a suitable radial change of variables.

Theorem 4.B.2. *Let A be a jointly measurable stationary random field on $\mathbb{R}^d$, let $\mathcal{C}$ denote its covariance function. Assume that A satisfies the Brascamp-Lieb inequality* (4.111) *(resp. in logarithmic Sobolev form* (4.115)*).*

(i) *If the map $x \mapsto \sup_{B(x)}|\mathcal{C}|$ is integrable, then the field A satisfies* (∂^{fct}-SG) *(resp.* (∂^{fct}-LSI)*) for any radius $R > 0$.*

(ii) *If $\sup_{B(x)}|\mathcal{C}| \le c(|x|)$ holds for some Lipschitz function $c : \mathbb{R}_+ \to \mathbb{R}_+$, then the field A satisfies* (∂^{fct}-WSG) *(resp.* (∂^{fct}-WLSI)*) with weight $\pi(\ell) \simeq (-c'(\ell))_+$.*

(iii) *If A further satisfies the Brascamp-Lieb inequality in covariance form* (4.112)*, and if there holds $\sup_{B(x)}|\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})| \le r(|x|)$ for some non-increasing Lipschitz function $r : \mathbb{R}_+ \to \mathbb{R}_+$, then A satisfies* (∂^{fct}-WCI) *with weight $\pi(\ell) \simeq (\ell+1)^d\, r(\ell)(-r'(\ell))$.* $\Diamond$

Note that in items (ii)–(iii), the weights obtained for (∂^{fct}-WSG) and (∂^{fct}-WCI) typically have the same scaling. We start with the proof of Proposition 4.B.1, and then turn to the proof of Theorem 4.B.2. Finally, we focus on the example of Gaussian random fields and deduce Corollary 4.5.1.

Proof of Proposition 4.B.1. For all $\varepsilon > 0$, consider the following approximations of the random field A,

$$A_\varepsilon(x) := \sum_{y,z\in\varepsilon\mathbb{Z}^d} \mathbb{1}_{Q_\varepsilon(z)}(x)\, W(Q_\varepsilon(y)) \fint_{Q_\varepsilon(z)}\fint_{Q_\varepsilon(y)} F(z'-y')dz'dy'.$$

By an approximation argument, we may reduce the proof of the proposition to the proof of the following discrete counterpart: given a random vector $W := (W_1,\ldots,W_N)$ with N independent components, and given a linear transformation $F \in \mathbb{R}^{N\times N}$, the transformed random vector $A := (A_1,\ldots,A_N) := FW$ satisfies:

(i') If for all $1 \le j \le N$ the random variable W_j satisfies the standard spectral gap

$$\operatorname{Var}[\phi(W_j)] \le C\mathbb{E}\left[\phi'(W_j)^2\right]$$

for all smooth functions $\phi : \mathbb{R} \to \mathbb{R}$, then the random vector A satisfies for all smooth functions $X, Y : \mathbb{R}^N \to \mathbb{R}$

$$\operatorname{Var}[X(A)] \le C\sum_{i,j=1}^N |(FF^t)_{ij}|\;\mathbb{E}\left[\Big|\frac{\partial X(A)}{\partial A_i}\Big|\Big|\frac{\partial X(A)}{\partial A_j}\Big|\right], \tag{4.116}$$

and also

$$\operatorname{Cov}[X(A);Y(A)] \le \sum_{i=1}^N \mathbb{E}\left[\Big(\sum_{j=1}^N \frac{\partial X(A)}{\partial A_j}F_{ji}\Big)^2\right]^{\frac12}\mathbb{E}\left[\Big(\sum_{k=1}^N \frac{\partial Y(A)}{\partial A_k}F_{ki}\Big)^2\right]^{\frac12}. \tag{4.117}$$

(ii') If for all $1 \leq j \leq N$ the random variable W_j satisfies the standard logarithmic Sobolev inequality

$$\mathrm{Ent}\big[\phi(W_j)^2\big] \leq C\mathbb{E}\left[\phi'(W_j)^2\right]$$

for all smooth functions $\phi : \mathbb{R} \to \mathbb{R}$, then the random vector A satisfies for all smooth functions $X : \mathbb{R}^N \to \mathbb{R}$,

$$\mathrm{Ent}\big[X(A)^2\big] \leq C \sum_{i,j=1}^{N} |(FF^t)_{ij}|\, \mathbb{E}\left[\Big|\frac{\partial X(A)}{\partial A_i}\Big|\Big|\frac{\partial X(A)}{\partial A_j}\Big|\right]. \tag{4.118}$$

We start with the proof of item (i'). Using the tensorization identity (4.104), the spectral gap assumption yields

$$\mathrm{Var}\left[X(A)\right] \leq \sum_{i=1}^{N} \mathbb{E}\left[\mathrm{Var}\left[X(A) \,\|\, (W_j)_{j:j\neq i}\right]\right] \leq \sum_{i=1}^{N} \mathbb{E}\left[\Big(\frac{\partial X(A)}{\partial W_i}\Big)^2\right],$$

and hence, by the chain rule,

$$\begin{aligned}\mathrm{Var}\left[X(A)\right] \leq \sum_{i=1}^{N} \mathbb{E}\left[\Big(\sum_{j=1}^{N} \frac{\partial X(A)}{\partial A_j} F_{ji}\Big)^2\right] &= \mathbb{E}\left[\nabla X(A) \cdot (FF^t)\nabla X(A)\right] \\ &\leq \sum_{i,j=1}^{N} |(FF^t)_{ij}|\, \mathbb{E}\left[\Big|\frac{\partial X(A)}{\partial A_i}\Big|\Big|\frac{\partial X(A)}{\partial A_j}\Big|\right]. \end{aligned} \tag{4.119}$$

In covariance form, using again the tensorization identity (4.104), the spectral gap assumption yields

$$\begin{aligned}\mathrm{Cov}\left[X(A); Y(A)\right] &\leq \sum_{i=1}^{N} \mathbb{E}\left[\mathrm{Var}\left[X(A) \,\|\, (W_j)_{j:j\neq i}\right]\right]^{\frac{1}{2}} \mathbb{E}\left[\mathrm{Var}\left[Y(A) \,\|\, (W_j)_{j:j\neq i}\right]\right]^{\frac{1}{2}} \\ &\leq \sum_{i=1}^{N} \mathbb{E}\left[\Big(\frac{\partial X(A)}{\partial W_i}\Big)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\Big(\frac{\partial Y(A)}{\partial W_i}\Big)^2\right]^{\frac{1}{2}},\end{aligned}$$

and the result (4.117) follows from the chain rule. We now turn to the proof of item (ii'). Using the tensorization identity (4.105), the logarithmic Sobolev inequality assumption yields

$$\begin{aligned}\mathrm{Ent}\big[X(A)^2\big] &\leq \sum_{i=1}^{N} \mathbb{E}\left[\mathrm{Ent}\left[\mathbb{E}\left[X(A)^2 \,\|\, (W_j)_{j:j\leq i}\right] \,\|\, (W_j)_{j:j\neq i}\right]\right] \\ &\leq C\sum_{i=1}^{N} \mathbb{E}\left[\Big|\frac{\partial}{\partial W_i}\mathbb{E}\left[X(A)^2 \,\|\, (W_j)_{j:j\leq i}\right]^{\frac{1}{2}}\Big|^2\right] \\ &= C\sum_{i=1}^{N} \mathbb{E}\left[\mathbb{E}\left[X(A)^2 \,\|\, (W_j)_{j:j\leq i}\right]^{-1} \Big|\mathbb{E}\left[X(A)\frac{\partial X(A)}{\partial W_i} \,\Big\|\, (W_j)_{j:j\leq i}\right]\Big|^2\right] \\ &\leq C\sum_{i=1}^{N} \mathbb{E}\left[\Big|\frac{\partial X(A)}{\partial W_i}\Big|^2\right].\end{aligned}$$

Now arguing as in (4.119), the result of item (ii') follows. $\square$

We turn to Theorem 4.B.2.

Proof of Theorem 4.B.2. We focus on items (i) and (ii) for the variance and the covariance (the arguments for the entropy are similar). Assume that A satisfies the Brascamp-Lieb inequality (4.111). If $x \mapsto \sup_{B(x)} |\mathcal{C}|$ is integrable, the inequality $|ab| \le (a^2+b^2)/2$ for $a, b \in \mathbb{R}$ directly yields for all $\sigma(A)$-measurable random variables $X(A)$ and all $R > 0$ (after taking local averages),

$$\begin{aligned}\mathrm{Var}\left[X(A)\right] &\le C\,\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big|\frac{\partial X(A)}{\partial A}(z)\Big|\Big|\frac{\partial X(A)}{\partial A}(z')\Big||\mathcal{C}(z-z')|dzdz'\right]\\ &\le 2C\Big\|\sup_{B_{2R}(\cdot)}|\mathcal{C}|\Big\|_{\mathrm{L}^1}\mathbb{E}\left[\int_{\mathbb{R}^d}\bigg(\fint_{B_R(z)}\Big|\frac{\partial X(A)}{\partial A}\Big|\bigg)^2dz\right].\end{aligned}$$

Now assume that the covariance function $\mathcal{C}$ is not integrable, and that $\sup_{B(x)}|\mathcal{C}| \le c(|x|)$ for some Lipschitz function $c : \mathbb{R}_+ \to \mathbb{R}_+$. Given a $\sigma(A)$-measurable random variable $X(A)$, we consider the projection $X_R(A) := \mathbb{E}[X(A)\|A|_{B_R}]$, for $R > 0$. Taking local averages, using polar coordinates, and integrating by parts (note that there is no boundary term since the Fréchet derivative $\partial X_R(A)/\partial A$ is compactly supported in B_R), the Brascamp-Lieb inequality (4.111) yields

$$\begin{aligned}&\mathrm{Var}\left[X_R(A)\right]\\ &\quad\le\ C\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\int_0^\infty\Big|\frac{\partial X_R(A)}{\partial A}(z)\Big|\fint_{B(z+\ell u)}\Big|\frac{\partial X_R(A)}{\partial A}(u')\Big|du'\ell^{d-1}c(\ell)d\ell d\sigma(u)dz\right]\\ &\quad=\ C\mathbb{E}\left[\int_{\mathbb{R}^d}\Big|\frac{\partial X_R(A)}{\partial A}(z)\Big|\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_0^\ell\fint_{B(z+su)}\Big|\frac{\partial X_R(A)}{\partial A}(u')\Big|du's^{d-1}ds(-c'(\ell))d\ell d\sigma(u)dz\right]\\ &\quad\le\ C\mathbb{E}\left[\int_{\mathbb{R}^d}\Big|\frac{\partial X_R(A)}{\partial A}(z)\Big|\int_0^\infty\bigg(\int_{B_{\ell+1}(z)}\Big|\frac{\partial X_R(A)}{\partial A}\Big|\bigg)(-c'(\ell))d\ell dz\right].\end{aligned}$$

Reorganizing the integrals, and taking local spatial averages, we conclude

$$\begin{aligned}\mathrm{Var}\left[X_R(A)\right]\ &\lesssim\ \mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\Big|\frac{\partial X_R(A)}{\partial A}(z)\Big|\Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(z)}X_R(A)\Big)dz(-c'(\ell))_+d\ell\right]\\ &\lesssim\ \mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\int_{B_{\ell+1}}\Big|\frac{\partial X_R}{\partial A}(z+y)\Big|\Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(z+y)}X_R(A)\Big)dydz\,(\ell+1)^{-d}(-c'(\ell))_+d\ell\right]\\ &\lesssim\ \mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\Big(\partial^{\mathrm{fct}}_{A,B_{2(\ell+1)}(z)}X_R(A)\Big)^2dz\,(\ell+1)^{-d}(-c'(\ell))_+d\ell\right]\\ &\lesssim\ \mathbb{E}\left[\int_0^\infty\int_{\mathbb{R}^d}\Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(z)}X_R(A)\Big)^2dz\,(\ell+1)^{-d}(-c'(\ell))_+d\ell\right],\end{aligned}$$

where in the last line we used the (sub)additivity of $S \mapsto \partial^{\mathrm{fct}}_{A,S}$. By Jensen's inequality in the form

$$\mathbb{E}\left[\Big(\partial^{\mathrm{fct}}_{A,S}X_R(A)\Big)^2\right] \le \mathbb{E}\left[\Big(\mathbb{E}\left[\partial^{\mathrm{fct}}_{A,S}X(A)\,\Big\|\,A|_{B_R}\right]\Big)^2\right] \le \mathbb{E}\left[\Big(\partial^{\mathrm{fct}}_{A,S}X(A)\Big)^2\right],$$

and passing to the limit $R \uparrow \infty$, the conclusion (∂^{fct}-WSG) follows. Let us now turn to the case when the field A satisfies the Brascamp-Lieb inequality in covariance form (4.112). Assuming that $\sup_{B(x)}|\mathcal{F}^{-1}(\sqrt{\mathcal{F}\mathcal{C}})| \le r(|x|)$ for some Lipschitz function $r : \mathbb{R}_+ \to \mathbb{R}_+$, a radial integration by parts similar as above yields

$$\begin{aligned}\mathrm{Cov}\left[X_R(A);Y_R(A)\right] \lesssim \int_{\mathbb{R}^d}\mathbb{E}\left[\bigg(\int_0^\infty\Big(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)}X_R(A)\Big)(-r'(\ell))_+\,d\ell\bigg)^2\right]^{\frac12}&\\ \times\,\mathbb{E}\left[\bigg(\int_0^\infty\Big(\partial^{\mathrm{fct}}_{A,B_{\ell'+1}(x)}Y_R(A)\Big)(-r'(\ell'))_+\,d\ell'\bigg)^2\right]^{\frac12}dx.&\end{aligned}$$

By the triangle inequality, this turns into

$$\begin{aligned}
&\mathrm{Cov}\left[X_R(A);Y_R(A)\right]\\
&\lesssim \int_0^\infty\int_0^\infty\int_{\mathbb{R}^d}\mathbb{E}\left[\left(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)}X_R(A)\right)^2\right]^{\frac12}\mathbb{E}\left[\left(\partial^{\mathrm{fct}}_{A,B_{\ell'+1}(x)}Y_R(A)\right)^2\right]^{\frac12}dx(-r'(\ell))_+\,d\ell(-r'(\ell'))_+\,d\ell'\\
&\le 2\int_0^\infty\int_{\mathbb{R}^d}\mathbb{E}\left[\left(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)}X_R(A)\right)^2\right]^{\frac12}\mathbb{E}\left[\left(\partial^{\mathrm{fct}}_{A,B_{\ell+1}(x)}Y_R(A)\right)^2\right]^{\frac12}dx\Big(\int_0^\ell(-r'(\ell'))_+\,d\ell'\Big)(-r'(\ell))_+\,d\ell,
\end{aligned}$$

and the conclusion (∂^{fct}-WCI) follows after passing to the limit $R\uparrow\infty$. □

We may finally turn to the proof of Corollary 4.5.1.

Proof of Corollary 4.5.1. Let W denote a Gaussian white noise with intensity 1, that is, a random noise W on $\mathbb{R}^d$ such that for all bounded Borel subsets $E\subset\mathbb{R}^d$ the random variable $W(E)$ has a centered Gaussian law with variance $\mathbb{E}\big[W(E)^2\big]=|E|$. As shown in [152, Section XI.8], a stationary Gaussian random field A on $\mathbb{R}^d$ can be rewritten as a convolution (4.108) with a Gaussian white noise whenever the field A has an absolutely continuous spectral measure, or equivalently, whenever the Fourier transform $\mathcal{F}\mathcal{C}$ of the covariance function is in $\mathrm{L}^1(\mathbb{R}^d)$. Under such a restriction on $\mathcal{C}$, since Gaussian random variables satisfy the standard spectral gap (4.110) and logarithmic Sobolev inequality (4.114) (cf. [221]), we can directly apply Proposition 4.B.1 and Theorem 4.B.2 to establish the validity of weighted spectral gaps, covariance, and logarithmic Sobolev inequalities.

It remains to show that this regularity restriction on $\mathcal{C}$ can be relaxed in the case of spectral gaps and logarithmic Sobolev inequalities. To this end, it suffices to prove that the conclusion of Proposition 4.B.1 (that is, the validity of Brascamp-Lieb type inequalities) always holds for any jointly measurable Gaussian stationary random field A. This is achieved by an approximation argument. We focus on the Brascamp-Lieb inequality (4.111), while the argument is analogous for (4.115). As an approximation argument shows, it is enough to establish (4.111) for those random variables $X(A)$ that depend on A only via their spatial averages on the partition $\{Q_\varepsilon(z)\}_{z\in B_R\cap\varepsilon\mathbb{Z}^d}$ with $\varepsilon,R>0$. Let us introduce the following notation for these averages,

$$A_\varepsilon(z):=\fint_{Q_\varepsilon(z)}A,\qquad\text{for } z\in\varepsilon\mathbb{Z}^d. \tag{4.120}$$

In this case, the Fréchet derivative $\{\frac{\partial X}{\partial A}(x)\}_{x\in\mathbb{R}^d}$ and the partial derivatives $\{\frac{\partial X}{\partial A_\varepsilon(z)}\}_{z\in\varepsilon\mathbb{Z}^d}$ of $X(A)$ are related via

$$\varepsilon^d\frac{\partial X}{\partial A}(x)=\frac{\partial X}{\partial A_\varepsilon(z)},\qquad\text{for } x\in Q_\varepsilon(z),\ z\in\varepsilon\mathbb{Z}^d. \tag{4.121}$$

We infer from (4.120) that $\{A_\varepsilon(z)\}_{z\in\varepsilon\mathbb{Z}^d}$ is a discrete centered Gaussian random field (which is now stationary with respect to the action of $\varepsilon\mathbb{Z}^d$), characterized by its covariance

$$\mathcal{C}_\varepsilon(z-z'):=\fint_{Q_\varepsilon(z)}\fint_{Q_\varepsilon(z')}\mathcal{C}(x-x')dx'dx. \tag{4.122}$$

By the discrete result (4.116) obtained in the proof of Proposition 4.B.1 (based on the standard spectral gap for Gaussian random variables [221]), we deduce for all $\varepsilon,R>0$ and all random variables $X(A)$ that depend on A only via its spatial averages on the partition $\{Q_\varepsilon(z)\}_{z\in B_R\cap\varepsilon\mathbb{Z}^d}$,

$$\mathrm{Var}\left[X\right]\le C\sum_{z\in B_R\cap\varepsilon\mathbb{Z}^d}\ \sum_{z'\in B_R\cap\varepsilon\mathbb{Z}^d}|\mathcal{C}_\varepsilon(z-z')|\,\mathbb{E}\left[\Big|\frac{\partial X}{\partial A_\varepsilon(z)}\Big|\Big|\frac{\partial X}{\partial A_\varepsilon(z')}\Big|\right].$$

Injecting (4.121) and (4.122), the conclusion (4.111) follows. □

4.C Appendix: Functional inequalities for Poisson point process

For the Poisson point process, we have established the standard functional inequalities (∂^{G}-SG) and (∂^{osc}-LSI) (cf. Proposition 4.1.2). In this appendix we shortly comment on another possible natural form of a standard functional inequality for a point process $\mathcal{P}$ on $\mathbb{R}^d$, rather using the usual partial derivatives with respect to the point locations. More precisely, we say that the point process $\mathcal{P} = \{x_k\}_k$ satisfies the spectral gap (∂°-SG) if for all $\sigma(\mathcal{P})$-measurable random variable $X = X(\mathcal{P})$ we have

$$\mathrm{Var}\,[X] \le C\mathbb{E}\left[\|\partial^\circ X\|^2\right] := C\mathbb{E}\left[\sum_k |\nabla_{x_k} F|^2\right],$$

and that it satisfies the logarithmic Sobolev inequality (∂°-LSI) if for all $\sigma(\mathcal{P})$-measurable random variable $X = X(\mathcal{P})$ we have

$$\mathrm{Ent}\big[X^2\big] \le C\mathbb{E}\left[\|\partial^\circ X\|^2\right] := C\mathbb{E}\left[\sum_k |\nabla_{x_k} X|^2\right].$$

For the very same reason why the Poisson distribution on $\mathbb{N}$ does not satisfy a logarithmic Sobolev inequality (cf. e.g. [292, p.65]), we easily deduce the following negative result for the Poisson point process.

Lemma 4.C.1. *A Poisson point process $\mathcal{P}$ on $\mathbb{R}^d$ does not satisfy* (∂°-LSI). $\lozenge$

Proof. Let χ be a smooth cut-off function equal to 1 on the unit cube Q, and to 0 outside $2Q$. For all $n \ge 1$, define a random variable $X_n := X_n(\mathcal{P})$ as follows: denoting by $y_n \in \mathcal{P}$ the n-th closest point to 0 (more precisely, the point $x \in \mathcal{P}$ such that the maximum $\max_{1\le i\le d} |(x)_i|$ is the n-th smallest, which is a.s. well defined), we set

$$X_n(\mathcal{P}) := \mathbb{1}_{|\mathcal{P}\cap Q|\ge n} + \chi(y_n)\mathbb{1}_{|\mathcal{P}\cap Q|=n-1}.$$

We then find

$$\mathbb{P}\left[|\mathcal{P}\cap Q| \ge n\right] \le \mathbb{E}\left[X_n^2\right] \le \mathbb{P}\left[|\mathcal{P}\cap 2Q| \ge n\right],$$

and also

$$\mathbb{E}\left[X_n^2 \log X_n^2\right] = \mathbb{E}\left[\mathbb{1}_{|\mathcal{P}\cap Q|=n-1}\chi(y_n)^2 \log \chi(y_n)^2\right] \ge -e^{-1}\mathbb{P}\left[|\mathcal{P}\cap Q| = n-1\right],$$

which yields, for all n large enough,

$$\mathrm{Ent}\big[X_n^2\big] \ge -e^{-1}\mathbb{P}\left[|\mathcal{P}\cap Q| = n-1\right] - \mathbb{P}\left[|\mathcal{P}\cap Q| \ge n\right] \log \mathbb{P}\left[|\mathcal{P}\cap Q| \ge n\right].$$

On the other hand, we compute the carré-du-champ

$$\mathbb{E}\left[\sum_k |\nabla_{x_k} X_n|^2\right] \le \mathbb{E}\left[\mathbb{1}_{|\mathcal{P}\cap Q|=n-1}|\nabla\chi(y_n)|^2\right] \le C\mathbb{P}\left[|\mathcal{P}\cap Q| = n-1\right].$$

The inequality (∂°-LSI) applied to X_n would then imply for all n large enough,

$$-\mathbb{P}\left[|\mathcal{P}\cap Q| \ge n\right] \log \mathbb{P}\left[|\mathcal{P}\cap Q| \ge n\right] \le C\mathbb{P}\left[|\mathcal{P}\cap Q| = n-1\right],$$

hence

$$n\log n \simeq -\log \mathbb{P}\left[|\mathcal{P}\cap Q| \ge n\right] \le C\frac{\mathbb{P}\left[|\mathcal{P}\cap Q| = n-1\right]}{\mathbb{P}\left[|\mathcal{P}\cap Q| \ge n\right]} \simeq n,$$

a contradiction. $\square$

Nevertheless, as first mentioned to us by Felix Otto, the corresponding spectral gap (∂°-SG) does hold for the Poisson point process.

Proposition 4.C.2. *Any Poisson point process $\mathcal{P}$ on $\mathbb{R}^d$ satisfies (∂°-SG) (with constant $C = \frac{5}{4}$, independent of the intensity of the process).* $\Diamond$

Proof. As $\mathcal{P}$ is completely independent, it suffices to prove the result for the restriction of $\mathcal{P} = \{x_k\}_{k\in\mathbb{N}}$ on the unit cube Q. Denote $N := |\mathcal{P}\cap Q|$, let $X := X(\mathcal{P}|_Q)$ be a $\sigma(\mathcal{P}|_Q)$-measurable random variable, and assume that X is of class $C^1(Q^{\mathbb{N}})$. By tensorization of the variance we may decompose

$$\mathrm{Var}\left[X\right] = \mathbb{E}\left[\mathrm{Var}\left[X \parallel N\right]\right] + \mathrm{Var}\left[\mathbb{E}\left[X \parallel N\right]\right] = \sum_{n=0}^{\infty} \mathrm{Var}\left[X \parallel N = n\right] \mathbb{P}\left[N = n\right] + \mathrm{Var}\left[\mathbb{E}\left[X \parallel N\right]\right],$$

thus distinguishing between the variability of X for a fixed number of points, and the variability due to changes in the number of points. We begin with the first contribution. As $[\mathcal{P}|N = n]$ is distributed as the tensor product of n independent uniform distributions on Q, we directly find

$$\mathrm{Var}\left[X \parallel N = n\right] \le \mathbb{E}\left[\sum_k |\nabla_{x_k} X|^2 \,\middle\|\, N = n\right],$$

and hence

$$\mathrm{Var}\left[X\right] \le \mathbb{E}\left[\sum_k |\nabla_{x_k} X|^2\right] + \mathrm{Var}\left[\mathbb{E}\left[X \parallel N\right]\right],$$

so that it remains to consider the last contribution. By definition, N is distributed according to a Poisson law on $\mathbb{N}$. By the spectral gap satisfied by the Poisson law (cf. e.g. [292, (5.13)]), we have

$$\mathrm{Var}\left[\mathbb{E}\left[X \parallel N\right]\right] \le \mathbb{E}\left[N\right] \sum_{n=0}^{\infty} (\mathbb{E}\left[X \parallel N = n+1\right] - \mathbb{E}\left[X \parallel N = n\right])^2\, \mathbb{P}\left[N = n\right], \tag{4.123}$$

and we are thus reduced to understanding the variations of $\mathbb{E}\left[X \parallel N\right]$ as the value of N varies. As X only depends on the coordinates inside Q, we may write, with obvious notation,

$$\begin{aligned}
&\mathbb{E}\left[X \parallel N = n+1\right] - \mathbb{E}\left[X \parallel N = n\right] \\
&\qquad = \int_Q \!\!\cdots\! \int_Q dx_1 \ldots dx_n \int_Q dx_{n+1} \big(X(x_1,\ldots,x_{n+1}) - X(x_1,\ldots,x_n,p(x_{n+1}))\big),
\end{aligned}$$

where $p(x_{n+1})$ denotes the projection of x_{n+1} on ∂Q. We may then estimate

$$\begin{aligned}
&|\mathbb{E}\left[X \parallel N = n+1\right] - \mathbb{E}\left[X \parallel N = n\right]| \\
&\quad\le \int_Q \!\!\cdots\! \int_Q dx_1 \ldots dx_n \int_Q dx_{n+1} \int_0^1 dt\; |\nabla_{n+1} X(x_1,\ldots,x_n, t x_{n+1} + (1-t) p(x_{n+1}))| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times |x_{n+1} - p(x_{n+1})| \\
&\quad\le \frac{1}{2} \int_Q \!\!\cdots\! \int_Q |\nabla_{n+1} X(x_1,\ldots,x_{n+1})|\, dx_1 \ldots dx_{n+1} \\
&\quad\le \frac{1}{2} \Big(\int_Q \!\!\cdots\! \int_Q |\nabla_{n+1} X(x_1,\ldots,x_{n+1})|^2 dx_1 \ldots dx_{n+1} \Big)^{\frac{1}{2}} \\
&\quad= \frac{1}{2} \Big(\frac{1}{n+1} \sum_{k=1}^{n+1} \int_Q \!\!\cdots\! \int_Q |\nabla_k X(x_1,\ldots,x_{n+1})|^2 dx_1 \ldots dx_{n+1} \Big)^{\frac{1}{2}},
\end{aligned}$$

and hence,

$$\left|\mathbb{E}\left[X \,\|\, N=n+1\right]-\mathbb{E}\left[X \,\|\, N=n\right]\right| \leq \frac{1}{2(n+1)^{\frac{1}{2}}} \mathbb{E}\left[\sum_k |\nabla_{x_k} X|^2 \,\middle\|\, N=n+1\right]^{\frac{1}{2}}. \qquad (4.124)$$

Injecting this into (4.123), and noting that the Poisson law satisfies

$$\mathbb{E}\left[N\right] \mathbb{P}\left[N=n\right]=(n+1) \mathbb{P}\left[N=n+1\right],$$

we deduce

$$\begin{aligned}
\operatorname{Var}\left[\mathbb{E}\left[X \,\|\, N\right]\right] &\leq \frac{\mathbb{E}\left[N\right]}{4} \sum_{n=0}^{\infty} \mathbb{E}\left[\sum_k |\nabla_{x_k} X|^2 \,\middle\|\, N=n+1\right] \frac{\mathbb{P}\left[N=n\right]}{n+1} \\
&= \frac{1}{4} \sum_{n=0}^{\infty} \mathbb{E}\left[\sum_k |\nabla_{x_k} X|^2 \,\middle\|\, N=n+1\right] \mathbb{P}\left[N=n+1\right] \leq \frac{1}{4} \mathbb{E}\left[\sum_k |\nabla_{x_k} X|^2\right],
\end{aligned}$$

and the result follows. □

Chapter 5

Clausius-Mossotti formulas and beyond

This chapter is concerned with the behavior of the homogenized coefficients associated with some random stationary ergodic medium under a Bernoulli perturbation. Introducing a new family of energy estimates that combine probability and physical spaces, we prove the analyticity of the perturbed homogenized coefficients with respect to the Bernoulli parameter. Our approach holds under the minimal assumptions of stationarity and ergodicity (together with a crucial assumption of bounded penetrability of the random inclusions), both in the scalar and vector cases, and it leads to semi-explicit formulas for each derivative that essentially coincide with the so-called cluster expansions used by physicists. In particular, the first term in this expansion yields the celebrated (electric and elastic) Clausius-Mossotti formulas for isotropic spherical random inclusions in an isotropic reference medium. This work constitutes the first general proof of these formulas in the case of random inclusions and solves a 150-year-old problem. Under suitable strong quantitative ergodicity assumptions, similar expansions are further obtained for the perturbed effective fluctuation tensor of Chapter 3.

This chapter corresponds to the article [165] jointly written with Antoine Gloria, to the exception of the additional results of Appendix 5.A concerning the case with *unbounded* penetrability of the random inclusions, and of the results of Section 5.1.3 and Appendix 5.B on the effective fluctuation tensor.

Contents

5.1 Introduction

5.1.1 General overview

Let $\rho = (q_n)_n$ be a stationary and ergodic random point process in $\mathbb{R}^d$ of unit intensity. To each point q_n we associate a family of independent Bernoulli variables $[0,1] \ni p \mapsto b_n^{(p)}$ that take value 1 with probability p and value 0 with probability $1-p$. Given $\alpha, \beta > 0$, we define a family of random matrix fields $A^{(p)}$ on $\mathbb{R}^d$ as follows,

$$A^{(p)}(x) := \alpha \operatorname{Id} + \sum_{n=1}^{\infty} b_n^{(p)} (\beta - \alpha) \operatorname{Id} \mathbb{1}_{B(q_n)}(x),$$

where $B(q_n)$ denotes the unit ball centered at q_n (above we have assumed that the balls $B(q_n)$ are disjoint for simplicity of the discussion). The random matrix fields $A^{(p)}$ are stationary and ergodic, so that the standard stochastic homogenization theory [354] yields the existence of associated homogenized matrices $A_{\mathrm{hom}}^{(p)}$. For p small, we expect $A_{\mathrm{hom}}^{(p)}$ to be a perturbation of order p of the unperturbed medium $\alpha \operatorname{Id}$. Indeed, in the second half of the 19th century, Mossotti [324], Maxwell [317], Clausius [119], Lorentz [308], and Lorenz [309] proposed the following expansion in the scalar case,

$$A_{\mathrm{hom}}^{(p)} = \alpha \operatorname{Id} + v_p \frac{\alpha d(\beta - \alpha)}{\beta + \alpha(d-1)} \operatorname{Id} + o(p), \tag{5.1}$$

where at first order the correction is proportional to the volume fraction $v_p \simeq p$ of chosen inclusions (cf. (5.27) below). This formula is called the *Clausius-Mossotti relation* in the dielectric context, *Maxwell's formula* in the conductivity context, and the *Lorentz-Lorenz equation* in the refractivity context in optics (see e.g. [283, 315] for a detailed historical notice); we choose to adopt in the sequel the naming Clausius-Mossotti. The counterpart of this formula for linear elasticity (see Corollary 5.1.5 below) is attributed to Bruggeman [89], Skorohod [400], Hill [241], and Budniansky [90].

The rigorous proof of these electric and elastic Clausius-Mossotti formulas has remained a challenge since then. The first justification of the electric version is due to Almog in dimension $d > 2$, whose results in [14, 15, 13], combined with elementary homogenization theory, precisely yield (5.1) (the convergence rate obtained in [15, Theorem 1] is lost when combined with homogenization). The proof is based on (scalar) potential theory and crucially relies on the facts that $d > 2$, that $A^{(p)}$ is everywhere a multiple of the identity, and that the inclusions are disjoint, but it requires particularly weak assumptions on the random structure. Another contribution is due to Mourrat [327], who considered a discrete scalar elliptic equation instead of a continuum elliptic equation for all $d \geq 2$. In the case treated in [327], $A^{(p)}$ is a discrete set of i.i.d. conductivities β and α with probabilities p and $1-p$. The extension of Mourrat's results to the present continuum setting (which is made possible by the more recent contributions [201, 212]) would yield the improvement of (5.1) to

$$A_{\mathrm{hom}}^{(p)} = \alpha \operatorname{Id} + v_p \frac{\alpha d(\beta - \alpha)}{\beta + \alpha(d-1)} \operatorname{Id} + O_\gamma(p^{2-\gamma}), \tag{5.2}$$

for any $\gamma > 0$. The assumptions of [327] are however very stringent, as the proof crucially relies on estimates from the *quantitative* theory of stochastic homogenization, which typically hold under quantitative ergodicity assumptions such as spectral gap (see also Chapter 4) or strong mixing assumptions, whence the corresponding restriction on the random point set ρ. There is a small gap in the proof of (the discrete counterpart of) (5.2) (a quantitative estimate of $|a_1^\circ(\mu) - a_1^\circ(0)|$, see [327, (8.2)], is missing to complete the proof of [327, Theorem 11.3]), which can however be fixed using the quantitative results of [209, 210, 313] (cf. proof of Theorem 5.A.1 in Appendix 5.A). Mourrat also made the nice observation that the reference medium needs not be the unperturbed background medium (of conductivity α), and essentially proved that $p \mapsto A_{\text{hom}}^{(p)}$ is $C^{1,1-\gamma}$ on the whole interval $[0, 1]$ for all $\gamma > 0$ in the discrete setting.

The idea of perturbing a non-uniform reference medium first appeared in the work [20] by Anantharaman and Le Bris, who considered the perturbation of a periodic array of inclusions by i.i.d. Bernoulli variables (the inclusions are independently deleted with probability p). The corresponding matrix field $A^{(p)}$ is then a random ergodic field with discrete stationarity. As above, for p small, one expects $A_{\text{hom}}^{(p)}$ to be a perturbation of $A_{\text{hom}}^{(0)}$ (the homogenized coefficients of the unperturbed *periodic* medium) of order p. Anantharaman and Le Bris considered the approximation $A_L^{(p)}$ of $A_{\text{hom}}^{(p)}$ obtained by periodizing the random medium $A^{(p)}$ on a cube of size L. The qualitative homogenization theory ensures that almost surely the random approximation $A_L^{(p)}$ of $A_{\text{hom}}^{(p)}$ converges to $A_{\text{hom}}^{(p)}$ (note that $A_L^{(0)}$ is deterministic and coincides with $A_{\text{hom}}^{(0)}$ for all integer L). Although not formulated in this way, they essentially proved that the approximation $p \mapsto \mathbb{E}[A_L^{(p)}]$ is C^2 at $p = 0$, and obtained bounds on the first two derivatives $\partial_p \mathbb{E}[A_L^{(p)}]|_{p=0}$ and $\partial_p^2 \mathbb{E}[A_L^{(p)}]|_{p=0}$ that are uniform in L. This is however not quite enough to prove that $p \mapsto A_{\text{hom}}^{(p)}$ is itself C^2 (or C^1) at zero.

In the present contribution we shall prove in a very general setting (which includes both the examples studied by Mourrat and by Anantharaman and Le Bris) that the map $p \mapsto A_{\text{hom}}^{(p)}$ is analytic on $[0, 1]$ (see Theorem 5.1.1 below). Our result holds under the mildest statistical assumptions on the reference medium $x \mapsto A^{(0)}(x)$ and on the point process, that is, (discrete or continuum) stationarity and ergodicity. We also make the crucial assumption that the number of intersections between the inclusions at every point is uniformly bounded (see however the discussion in Section 5.1.5 and the weaker results of Appendix 5.A). We believe that a suitable adaptation of our arguments may allow to treat the case when the Bernoulli law is replaced by more general laws as considered in [21, Section 3]. Although our results are much stronger than those of Mourrat [327], our proof was mainly inspired by the ingenious computations of Anantharaman and Le Bris (see in particular [19, Proposition 3.4]), and only relies on soft arguments. In particular the crucial ingredient of our proof is a new family of energy estimates that combine both physical and probability spaces (see Proposition 5.2.6 below). Since the proof only uses ingredients that are available for systems, our results hold not only for scalar equations, but also for uniformly elliptic systems and for linear elasticity. In the case of an isotropic constant background medium perturbed by randomly distributed isotropic spherical inclusions, this proves the celebrated (electric) Clausius-Mossotti formula (5.1) with an optimal error estimate (see Corollary 5.1.4),

$$A_{\text{hom}}^{(p)} = \alpha \operatorname{Id} + v_p \frac{\alpha d(\beta - \alpha)}{\beta + \alpha(d-1)} \operatorname{Id} + O(p^2), \tag{5.3}$$

as well as its elastic counterpart (see Corollary 5.1.5), under the weakest statistical assumptions possible.

Our proof makes use of the standard modification of the corrector equation by a massive term of magnitude T^{-1}, and the derivatives of $A_{\text{hom}}^{(p)}$ with respect to p are given by the limits as $T \uparrow \infty$ of the derivatives of a deterministic approximation $A_T^{(p)}$ of $A_{\text{hom}}^{(p)}$ ($\sqrt{T}$ plays a similar role as the period L in [20]). Interestingly, in the case when the inclusions are disjoint, the fact that $p \mapsto A_{\text{hom}}^{(p)}$ is $C^{1,1}$ on $[0, 1]$ can be obtained as a corollary of the (classical) energy estimates of [20, 19] (applied to $A_T^{(p)}$ instead of $A_L^{(p)}$), further using that they hold for any $p \in [0, 1]$ by Mourrat's observation (see Section 5.1.4). The proof that $p \mapsto A_{\text{hom}}^{(p)}$ is analytic is however more subtle and is based on the new family of energy estimates.

There are two motivations to go beyond the $C^{1,1}$ result. First, this gives a definite answer to the maximal regularity of the map $p \mapsto A_{\text{hom}}^{(p)}$. Analyticity was indeed conjectured in applied mechanics and applied physics (see for instance [413, Chapters 18 and 19]). This analyticity result contrasts very much with the corresponding regularity of a similar periodic model originally studied by Maxwell

[317] and Rayleigh [368] (see also [265, Section 1.7] and [51]). The latter consists of a homogeneous medium periodically perturbed by inclusions of volume p located at integer points of $\mathbb{R}^d$ (note that the perturbed medium is still $\mathbb{Z}^d$-periodic), in which case the associated map $p \mapsto A_{\text{hom}}^{(p)}$ is $C^{3+\frac{4}{d}}$ in dimension d, but not more. The second motivation stems from numerical analysis. Indeed the main motivation for [20, 21] is to exploit the perturbative character of $A_{\text{hom}}^{(p)}$ for p small (seen as a model for defects) and use the first two terms of a Taylor-expansion at zero as a good approximation for $A_{\text{hom}}^{(p)}$, the accuracy of which can be optimally quantified by Theorem 5.1.1 below. Note that Legoll and Minvielle [296] also made an original use of this approximation of $A_{\text{hom}}^{(p)}$ in a control variate method to reduce the variance for the approximation of homogenized coefficients.

As emphasized in [20], a second natural question is to quantify the convergence speed of computable massive approximations of the derivatives of $A_{\text{hom}}^{(p)}$. As opposed to Theorem 5.1.1, which is a *qualitative* result, establishing such a convergence speed requires to quantify the speed of convergence in the ergodic theorem for stochastic homogenization, which is a *quantitative* result. We give such a result under the assumption that the speed of convergence of A_T to A_{hom} can be quantified (see Corollary 5.1.6), which is indeed known to hold under suitable strong quantitative mixing assumptions (as assumed by Mourrat [327]) using e.g. the results of [210, 212, 203].

Our approach to prove Theorem 5.1.1 is constructive and explicit bounds are obtained on the derivatives. We do not know whether there is an abstract alternative to establish analyticity. In [120], Cohen, Devore and Schwab obtained a result of the same flavor using a complexification method: they proved the analyticity of the solution of linear elliptic PDEs with respect to parameters in the coefficients in the framework of a chaos expansion. Their setting is however very much different from the setting of Theorem 5.1.1. Indeed, as emphasized in Remark 5.1.7 below, the solution of interest here (the corrector) is in general not even differentiable with respect to the Bernoulli parameter (the homogenized coefficients are analytic because of subtle cancellations).

For the clarity of the exposition, although our proof of Theorem 5.1.1 holds in the case of uniformly elliptic systems and of linear elasticity (provided the elasticity tensor is uniformly very strongly elliptic, as standard in homogenization), for non-symmetric coefficients, and for discrete elliptic equations, we use continuum scalar notation and assume that the coefficients are symmetric. For non-symmetric coefficients, it is indeed enough to consider, in addition to the primal corrector equation, the dual corrector equation (associated with the pointwise transpose coefficients), which would only make notation heavier. In addition we assume that the coefficients enjoy continuum stationarity (in the case of $\mathbb{Z}^d$-stationarity, the expectation would simply be replaced everywhere by the expectation of the integral over the unit cube). Note that our result also covers the case of laminates, or more generally the case when the heterogeneous coefficients are random in some direction(s) and invariant along the other(s) (cf. the example of cylindrical fibers considered in [20] and encountered in practice).

The rest of the chapter is organized as follows. In Section 5.1.2 we introduce the main notation and state the main results of the chapter: the analyticity of the homogenized coefficients with respect to the Bernoulli parameter and the validity of the Clausius-Mossotti formulas. In Section 5.1.3, we state our parallel partial expansion results for the effective fluctuation tensor. In Section 5.1.4, we present the general strategy of the proof of the main results under the additional simplifying assumption that the inclusions are disjoint. Section 5.2 is dedicated to the introduction and proof of auxiliary results, and in particular of the improved energy estimates. The main results are then proved in Section 5.3. In Appendix 5.A, we relax the boundedness assumption on the degree of intersections between the random inclusions (cf. (5.4) below), and we establish weaker regularity results based on the *quantitative* theory of stochastic homogenization. The proof of our results on the effective fluctuation tensor is of a similar spirit and is postponed to Appendix 5.B.

5.1.2 Main results

Assumptions

Let A be a random field. We choose a point process $\rho = (q_n)_n$, and random bounded inclusions $(J_n)_n$ centered at the points q_n. To the inclusions $(J_n)_n$ we attach i.i.d. Bernoulli variables $(b_n^{(p)})_n$ with parameter $p \in [0,1]$, and we perturb A on J_n if $b_n^{(p)} = 1$. The only assumptions we need here are stationarity and ergodicity, as well as some deterministic bound on the degree of intersections between the inclusions. More precise definitions are given below.

Point process. Let ρ be a (locally finite) ergodic stationary point process on $\mathbb{R}^d$, and choose for convenience a measurable enumeration $\rho = (q_n)_{n=1}^{\infty}$. For any open set D of $\mathbb{R}^d$, we denote by $\rho(D) := \#\{q_n \in D, n \in \mathbb{N}\}$ the number of points of ρ in D.

Inclusions centered at the point process. Let $R > 0$ be fixed. For all n, let J_n° be random Borel subsets $J_n^\circ \subset B_R(\subset \mathbb{R}^d)$ (maybe depending on $\rho = (q_n)_n$). This defines random bounded Borel inclusions $J_n := q_n + J_n^\circ$. We assume that this inclusion process is stationary, in the sense that the random set $\bigcup_n J_n$ is stationary. Moreover, we further assume that the intersections between the inclusions J_n's are of degree bounded by some deterministic constant $\Gamma \in \mathbb{N}$; by stationarity, this just means

$$\#\{n \in \mathbb{N} : 0 \in J_n\} \le \Gamma, \qquad \text{almost surely.} \tag{5.4}$$

In physics (see e.g. [413, Section 3.1]), the constant Γ in (5.4) is called the impenetrability parameter: different inclusions may penetrate each other but only with the fixed finite maximum degree Γ. As explained in Section 5.1.5, we believe that this assumption is actually necessary for our analyticity result.

As $J_n \subset B_R(q_n)$, assumption (5.4) is trivially satisfied if we assume $\rho(Q) \le \theta_0$ a.s. (thus forbidding arbitrary large clusters in the point process), but it is important to note that the only problem is the possibility of intersections of arbitrary large degree, which has a priori nothing to do with the point process ρ itself. In the case of inclusions with inner radius bounded from below, however, assumption (5.4) is equivalent to an assumption on ρ of the form $\rho(Q) \le \theta_0$ almost surely.

Reference random fields. Given $0 < \lambda \le 1$, denote by $\mathcal{M}_\lambda$ the space of uniformly elliptic symmetric $d \times d$-matrices M satisfying $\lambda|\xi|^2 \le \xi \cdot M\xi \le |\xi|^2$ for all $\xi \in \mathbb{R}^d$. Let A, A' be two $\mathcal{M}_\lambda$-valued ergodic stationary random fields on $\mathbb{R}^d$. Note that A and A' do not need to be independent of the point process ρ, and we simply assume that this dependence is local, in the sense that $A(0)$ and $A'(0)$ only depend on ρ via the restriction $\rho|_{B_r}$ for some given deterministic $r > 0$.

Bernoulli perturbation of A. For any fixed $p \in [0,1]$, we choose a sequence $(b_n^{(p)})_n$ of i.i.d. Bernoulli random variables with $\mathbb{P}[b_n^{(p)} = 1] = p$, independent of all previous random elements. We can now consider the following p-perturbed random field, which is a perturbation of the random field A on the inclusions for which $b_n^{(p)} = 1$:

$$A^{(p)} = A \mathbb{1}_{\mathbb{R}^d \setminus \bigcup_{n \in E^{(p)}} J_n} + A' \mathbb{1}_{\bigcup_{n \in E^{(p)}} J_n},$$

where we have set $E^{(p)} := \{n \in \mathbb{N} : b_n^{(p)} = 1\}$.

In the case when the inclusions are disjoint, the p-perturbed random field $A^{(p)}$ can be rewritten as follows:

$$A^{(p)} = \sum_n \Big(b_n^{(p)} A' + (1 - b_n^{(p)}) A\Big) \mathbb{1}_{J_n} + A \mathbb{1}_{\mathbb{R}^d \setminus \bigcup_n J_n}.$$

Moreover, in that case, as A' is allowed to depend (locally) on the inclusion process, the following interesting particular example can be considered: choose a sequence $(A'_n)_n$ of i.i.d. $\mathcal{M}_\lambda$-valued random fields, and define

$$A' := \mathrm{Id}\,\mathbb{1}_{\mathbb{R}^d \setminus \bigcup_n J_n} + \sum_n A'_n \mathbb{1}_{J_n}, \tag{5.5}$$

so that $A^{(p)}$ takes the form

$$A^{(p)} = \sum_n \Big(b_n^{(p)} A'_n + (1 - b_n^{(p)}) A \Big) \mathbb{1}_{J_n} + A \mathbb{1}_{\mathbb{R}^d \setminus \bigcup_n J_n}.$$

Probability space and product structure. Let us now briefly comment on the underlying probability space. Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ be a probability space on which the (stationary) random elements ρ, $(J_n)_n$, A, and A' are defined. For all $p \in [0,1]$ and $n \in \mathbb{N}$, let $\Omega_{2,n}^{(p)} := \{b_n^{(p)} \in \{0,1\}\}$, endowed with the trivial σ-algebra $\mathcal{F}_{2,n}^{(p)}$, and let $\mathbb{P}_{2,n}^{(p)}$ be the Bernoulli measure of parameter p on $\Omega_{2,n}^{(p)}$. The probability space we consider in this chapter is the product space $(\Omega, \mathcal{F}, \mathbb{P})$ of $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_{2,n}^{(p)}, \mathcal{F}_{2,n}^{(p)}, \mathbb{P}_{2,n}^{(p)})$ for all $p \in [0,1]$ and $n \in \mathbb{N}$ (with the cylindrical σ-algebra). With $\mathbb{E}$, $\mathbb{E}_1$, $\mathbb{E}_{2,n}^{(p)}$ the expectations with respect to the measures $\mathbb{P}$, $\mathbb{P}_1$, $\mathbb{P}_{2,n}^{(p)}$, respectively, we have by definition

$$\mathbb{E} = \mathbb{E}_1 \prod_{p \in [0,1]} \prod_{n \in \mathbb{N}} \mathbb{E}_{2,n}^{(p)}.$$

The independence of the Bernoulli variables, at the origin of this product structure, then takes the form: for any integrable random variables $\chi_n^{(p)}$ and $\eta_n^{(p)}$ defined on $\Omega_1 \times \prod_{m, m \neq n} \Omega_{2,m}^{(p)}$ and $\Omega_{2,n}^{(p)}$, respectively,

$$\mathbb{E}[\chi_n^{(p)} \eta_n^{(p)}] = \mathbb{E}[\chi_n^{(p)}] \mathbb{E}[\eta_n^{(p)}]. \tag{5.6}$$

Note that for all $p \in [0,1]$ the random field $A^{(p)}$ is defined on $\Omega_1 \times \prod_{n \in \mathbb{N}} \Omega_{2,n}^{(p)}$ and is stationary and ergodic for the measure

$$\mathbb{P}_1 \otimes \bigotimes_{n \in \mathbb{N}} \mathbb{P}_{2,n}^{(p)}.$$

As such, it can be viewed as a stationary and ergodic random field on $(\Omega, \mathcal{F}, \mathbb{P})$.

Typical examples. One typical example is that of spherical inclusions $J_n = B_R(q_n)$ centered at the points of any ergodic stationary random point process ρ with minimal distance bounded away from 0. In that case, ρ can be chosen as the hardcore Poisson point process (that is, a modified Poisson process for which points that are at a distance less than $2R_0$ are deleted; see also the hardcore construction in Step 1 of the proof of Theorem 5.1.1 in Section 5.3.2) or the random parking measure (see [357, 213]). Instead of spherical inclusions, we can consider more general (and random) shapes $J_n = q_n + J_n^\circ$, where the J_n°'s are i.i.d. copies of some random Borel set, with $J_n^\circ \subset B_R$ a.s.

Another interesting example is when ρ is a Poisson process (or *any* other ergodic stationary point process) and when J_n is the Voronoi cell at q_n, intersected with the ball at q_n of radius R, say. We could alternatively choose for J_n the largest ball of radius less than R centered at q_n and completely included in the Voronoi cell at q_n. In this case, the number of inclusions per unit volume is not necessarily uniformly bounded.

Notation

We start with the definition of homogenized coefficients, correctors, and approximate correctors. We then introduce the crucial notion of difference operators, and conclude with the introduction of a notational system for perturbed coefficients which will turn out to be very convenient when it comes to the (numerous) combinatorial arguments involved in the proofs.

Correctors, approximate correctors, and homogenized coefficients. For any (possibly infinite) subset $E \subset \mathbb{N}$, we define $A^E := A + C^E$, where $C^E := (A' - A)\mathbb{1}_{J^E}$ and $J^E := \bigcup_{n \in E} J_n$. In these terms, with $E^{(p)} := \{n \in \mathbb{N} : b_n^{(p)} = 1\}$, we have $A^{E^{(p)}} = A + C^{E^{(p)}}$, and we use the short-hand notation $C^{(p)} := C^{E^{(p)}}$ and $A^{(p)} := A^{E^{(p)}}$.

For all $T > 0$, we define the *approximate correctors* $\phi_{T,\xi}$ and $\phi_{T,\xi}^E$ in direction ξ, $|\xi| = 1$, associated with any field A and any A^E, respectively, as the unique solutions in the space $\{v \in H^1_{\mathrm{loc}}(\mathbb{R}^d) : \sup_z \int_{B(z)} (|v|^2 + |\nabla v|^2) < \infty\}$ of the equations

$$\frac{1}{T}\phi_{T,\xi} - \nabla \cdot A(\nabla \phi_{T,\xi} + \xi) = 0, \qquad \text{and} \qquad \frac{1}{T}\phi_{T,\xi}^E - \nabla \cdot A^E(\nabla \phi_{T,\xi}^E + \xi) = 0. \tag{5.7}$$

These solutions satisfy the following energy estimates (see [212, Lemma 2.7]):

$$\sup_z \fint_{B_{\sqrt{T}}(z)} (T^{-1}|\phi_{T,\xi}|^2 + |\nabla \phi_{T,\xi}|^2) \lesssim 1, \qquad \sup_z \fint_{B_{\sqrt{T}}(z)} (T^{-1}|\phi_{T,\xi}^E|^2 + |\nabla \phi_{T,\xi}^E|^2) \lesssim 1. \tag{5.8}$$

To shorten notation, we write $\phi_{T,\xi}^{(p)}$ for $\phi_{T,\xi}^{E^{(p)}}$. For all $p \in [0,1]$, as the random field $A^{(p)}$ is ergodic and stationary, we have (combine for instance the ergodic theorem with [354] in the symmetric case, and with [199, Theorem 1] in the non-symmetric case)

$$\lim_{T \uparrow \infty} \mathbb{E}[|\nabla \phi_{T,\xi}^{(p)} - \nabla \phi_\xi^{(p)}|^2] \;=\; 0, \tag{5.9}$$

where $\nabla \phi_\xi^{(p)}$ is the gradient of the corrector, i.e. the gradient of the unique measurable random map $\phi_\xi^{(p)} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ solution of the equation

$$-\nabla \cdot A^{(p)}(\nabla \phi_\xi^{(p)} + \xi) = 0$$

on $\mathbb{R}^d$ that satisfies $\phi_\xi^{(p)}(0) = 0$ almost surely and such that $\nabla \phi_\xi^{(p)}$ is stationary and has bounded second moment. (Note that $\phi_{T,\xi}$ exists for any matrix field A if $T > 0$, whereas ϕ_ξ only exists almost surely for a stationary random field A.) As usual, the *homogenized coefficients* are then given by

$$\xi \cdot A_{\mathrm{hom}}^{(p)} \xi = \mathbb{E}\big[(\nabla \phi_\xi^{(p)} + \xi) \cdot A(\nabla \phi_\xi^{(p)} + \xi)\big] = \mathbb{E}\big[\xi \cdot A^{(p)}(\nabla \phi_\xi^{(p)} + \xi)\big].$$

They can be approximated by symmetric or non-symmetric approximate homogenized coefficients:

$$\xi \cdot A_{\mathrm{hom}}^{(p)} \xi = \lim_{T \uparrow \infty} \mathbb{E}\big[(\nabla \phi_{T,\xi}^{(p)} + \xi) \cdot A^{(p)}(\nabla \phi_{T,\xi}^{(p)} + \xi)\big] = \lim_{T \uparrow \infty} \mathbb{E}\big[\xi \cdot A^{(p)}(\nabla \phi_{T,\xi}^{(p)} + \xi)\big]. \tag{5.10}$$

We denote by $\xi \cdot A_T^{(p)} \xi := \mathbb{E}\big[\xi \cdot A^{(p)}(\nabla \phi_{T,\xi}^{(p)} + \xi)\big]$ the non-symmetric approximate homogenized coefficients. When ξ is fixed, we simply write ϕ_T for $\phi_{T,\xi}$, ϕ for ϕ_ξ, etc.

Difference operators. The aim of this chapter is to understand how $A_{\mathrm{hom}}^{(p)}$ depends on p for p close to 0. We shall first study the easier map $p \mapsto A_T^{(p)}$, seen as a function of the approximate corrector $\phi_T^{(p)}$. Following physicists we introduce for all $n \in \mathbb{N}$ a difference operator $\delta^{\{n\}}$ acting generically on measurable functions of $(\Omega, \mathcal{F})$, and in particular on approximate correctors as follows: for all $H \subset \mathbb{N}$,

$$\delta^{\{n\}} \phi_T^H := \phi_T^{H \cup \{n\}} - \phi_T^H.$$

This operator yields a natural measure of the sensitivity of the corrector ϕ_T^H with respect to the perturbation of the medium at inclusion J_n. This is to be compared to the *vertical derivative* first used in [209] in the context of quantitative stochastic homogenization and to the *randomized derivatives* introduced by Chatterjee [112] in the context of Stein's method (see also the Hoeffding decompositions in [282], where these randomized derivatives are used up to any order). For all finite $F \subset \mathbb{N}$, we further introduce the higher-order difference operator $\delta^F = \prod_{n\in F} \delta^{\{n\}}$; more explicitly, this difference operator δ^F acting on approximate correctors ϕ_T^H (for any $H \subset \mathbb{N}$) is defined as follows:

$$\delta^F \phi_T^H := \sum_{l=0}^{|F|} (-1)^{|F|-l} \sum_{\substack{G\subset F\\ |G|=l}} \phi_T^{G\cup H} = \sum_{G\subset F} (-1)^{|F\setminus G|} \phi_T^{G\cup H}, \tag{5.11}$$

with the convention $\delta^\varnothing \phi_T^H = (\phi_T^H)^\varnothing := \phi_T^H$. Physicists have introduced such operators to derive *cluster expansions* (see [413]), which are used as formal proxies for Taylor expansions with respect to the Bernoulli perturbation: up to order k in the parameter p, the cluster expansion for the perturbed corrector reads, for small $p \geq 0$,

$$\phi_T^{(p)} \rightsquigarrow \phi_T + \sum_{n\in E^{(p)}} \delta^{\{n\}}\phi_T + \frac{1}{2!} \sum_{\substack{n_1,n_2\in E^{(p)}\\ \text{distinct}}} \delta^{\{n_1,n_2\}}\phi_T + \ldots + \frac{1}{k!} \sum_{\substack{n_1,\ldots,n_k\in E^{(p)}\\ \text{distinct}}} \delta^{\{n_1,\ldots,n_k\}}\phi_T,$$

which we rewrite in the more compact form

$$\phi_T^{(p)} \rightsquigarrow \sum_{j=0}^{k} \sum_{\substack{F\subset E^{(p)}\\ |F|=j}} \delta^F \phi_T = \sum_{j=0}^{k} \sum_{\substack{F\subset E^{(p)}\\ |F|=j}} \sum_{G\subset F} (-1)^{|F\setminus G|} \phi_T^G, \tag{5.12}$$

where $\sum_{|G|=j}$ denotes the sum over j-uplets of integers (when $j = 0$, this sum reduces to the single term $G = \varnothing$). Intuitively, $\phi_T^{(p)}$ is expected to be close to a series where terms of order j involve a correction due to the interaction of j inclusions (and therefore derivatives of order j). Whereas the cluster formula for $A_{\mathrm{hom}}^{(p)}$ in Corollary 5.1.2 below holds under the mildest statistical assumptions on the coefficients, the validity of the expansion (5.12) is expected to require strong mixing assumptions (see [327] for the first order). This illustrates again the fact that averaged quantities (e.g. homogenized coefficients) are better behaved than pointwise quantities (e.g. correctors).

For convenience, we also set

$$\delta_\xi^F \phi_T^H := \delta^F(\phi_T^H + \xi\cdot x), \tag{5.13}$$

that is, in terms of gradients,

$$\nabla\delta_\xi^F \phi_T^H = \sum_{l=0}^{|F|} (-1)^{|F|-l} \sum_{\substack{G\subset F\\ |G|=l}} (\nabla\phi_T^{G\cup H} + \xi) = \sum_{G\subset F} (-1)^{|F\setminus G|} (\nabla\phi_T^{G\cup H} + \xi). \tag{5.14}$$

By the binomial formula $\sum_{l=0}^{|F|} \binom{|F|}{l} (-1)^{|F|-l} = 0$, we have $\nabla\delta_\xi^F \phi_T^H = \nabla\delta^F\phi_T^H$ for all $F \neq \varnothing$ and $H \subset \mathbb{N}$, $\nabla\delta_\xi^\varnothing \phi_T^H = \nabla\phi_T^H + \xi$ for all $H \subset \mathbb{N}$, and, for all finite sets $F, G, H \subset \mathbb{N}$,

$$\nabla\delta_\xi^G \phi_T^{F\cup H} = \sum_{S\subset F} \nabla\delta_\xi^{S\cup G}\phi_T^H. \tag{5.15}$$

Inclusion-exclusion formula. When the inclusions are disjoint, we have

$$C^{(p)} = \sum_{n\in E^{(p)}} C^{\{n\}}. \tag{5.16}$$

However, when inclusions may overlap, this formula no longer holds since intersections may be accounted for several times. In the rest of this paragraph we define a suitable system of notation to deal with these intersections.

For any (possibly infinite) subset $E \subset \mathbb{N}$, we set $A_E := A + C_E$, where $C_E := (A' - A)\mathbb{1}_{J_E}$ and $J_E := \bigcap_{n \in E} J_n$. Note that $J_{\{n\}} = J^{\{n\}} = J_n$, and $C^{\{n\}} = C_{\{n\}}$. For non-necessarily disjoint inclusions, $C^{(p)}$ is then given by the following general inclusion-exclusion formula,

$$C^{(p)} = \sum_{n \in E^{(p)}} C_{\{n\}} - \sum_{n_1 < n_2 \in E^{(p)}} C_{\{n_1,n_2\}} + \sum_{n_1<n_2<n_3 \in E^{(p)}} C_{\{n_1,n_2,n_3\}} - \dots$$
$$= \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{\substack{F \subset E^{(p)} \\ |F|=k}} C_F. \quad (5.17)$$

Since the inclusions J_n's have a diameter bounded by $2R$ and $\rho(B_{2R})$ is almost surely finite, the sum (5.17) is locally finite almost surely. Recalling that by assumption (5.4) the degree of the intersections of the inclusions is bounded by Γ, we deduce that we must have $C_F \equiv 0$ for all $|F| > \Gamma$. Therefore, the inclusion-exclusion formula (5.17) actually reads

$$C^{(p)} = \sum_{k=1}^{\Gamma} (-1)^{k+1} \sum_{\substack{F \subset E^{(p)} \\ |F|=k}} C_F. \quad (5.18)$$

We shall need further notation in the proofs. For all $E, F \subset \mathbb{N}$, $E \neq \varnothing$, we set $J_{E\|F} := (\bigcap_{n\in E} J_n) \setminus (\bigcup_{n \in F} J_n)$ and $J^E_{\|F} := (\bigcup_{n \in E} J_n) \setminus (\bigcup_{n\in F} J_n)$, and then

$$C_{E\|F} := (A'-A)\mathbb{1}_{J_{E\|F}}, \qquad \text{and} \qquad C^E_{\|F} := (A'-A)\mathbb{1}_{J^E_{\|F}}.$$

In particular, we have $C_{E\|\varnothing} = C_E$, $C^E_{\|\varnothing} = C^E$, and $C^{\varnothing}_{\|F} = 0$. For simplicity of notation (except in the proof of Lemma 5.2.1), we also set $C_{\varnothing\|F} = 0 = C_\varnothing$. The inclusion-exclusion formula then yields for all $G, H \subset \mathbb{N}$, $G \neq \varnothing$,

$$C^H = \sum_{S \subset H} (-1)^{|S|+1} C_S, \quad (5.19)$$
$$C^H_{\|G} = \sum_{S \subset H} (-1)^{|S|+1} C_{S\|G}, \quad (5.20)$$
$$C_{G\|H} = \sum_{S \subset H} (-1)^{|S|} C_{S \cup G}. \quad (5.21)$$

We shall also use the symbols $\simeq$, $\gtrsim$ and $\lesssim$ for $=$, $\geq$, $\leq$ up to constants that only depend on R, Γ, d, and λ. Subscripts are used to indicate additional dependence of the constants, e. g. $\lesssim_\eta$ means that the multiplicative constant depends on η, next to R, Γ, d, and λ. Throughout, we will denote by C any positive constant with $C \simeq 1$, whose value may vary from line to line.

Statement of main results

Our main result asserts the analyticity of the map $p \mapsto A_{\rm hom}^{(p)}$ corresponding to the perturbed coefficients.

Theorem 5.1.1 (Analyticity of the homogenized coefficients)**.** *Under the above assumptions, the map $p \mapsto A_{\rm hom}^{(p)}$ is analytic on $[0,1]$ and there exists a constant $0 < c \le 1$ such that, for all $p_0 \in [0,1]$ and all $-p_0 \wedge c \le p \le (1-p_0) \wedge c$,*

$$A_{\rm hom}^{(p_0+p)} = A_{\rm hom}^{(p_0)} + \sum_{j=1}^{\infty} \frac{p^j}{j!} A_{\rm hom}^{(p_0),j}, \tag{5.22}$$

where the series converges, and where, for all $j \ge 1$, $A_{\rm hom}^{(p_0),j}$ denotes the (well-defined) j-th derivative of the map $p \mapsto A_{\rm hom}^{(p)}$ at p_0. $\lozenge$

Since our proof is constructive, we obtain formulas for the derivatives. These formulas involve two approximation arguments: the addition of a massive term T^{-1} in the corrector equation to deal with integrability issues at large distances, and a hardcore approximation of the point process to deal with integrability issues at short distances.

Corollary 5.1.2 (Formulas for derivatives)**.** *Let the above assumptions prevail. We can construct a sequence $(\rho_\theta)_\theta$ of hardcore approximations of the stationary point process ρ in the following sense: for any $\theta > 0$, ρ_θ is an ergodic stationary point process on $\mathbb{R}^d$ such that $\rho_\theta \subset \rho$, $\rho_\theta(Q) \le \theta$ a.s., and $\rho_\theta \uparrow \rho$ locally almost surely as $\theta \uparrow \infty$. For any $F, G \subset \mathbb{N}$, denote by $A_\theta^F, (C_\theta)_{F\|G}, (C_\theta)_F$ the coefficients $A^F, C_{F\|G}, C_F$ corresponding to ρ_θ in place of ρ, and further denote by $\phi_{T,\theta,\xi}^F$ the approximate corrector $\phi_{T,\xi}^F$ associated with the coefficients corresponding to ρ_θ in place of ρ.*

Then, for all $k \ge 1$ and all $p_0 \in [0,1]$, the k-th derivative $A_{\rm hom}^{(p_0),k}$ at p_0 satisfies the following three equivalent formulas, for all ξ,

$$\xi \cdot A_{\rm hom}^{(p_0),k} \xi = k! \lim_{T\uparrow\infty} \lim_{\theta\uparrow\infty} \sum_{|F|=k} \sum_{G \subsetneq F} (-1)^{|F\setminus G|+1} \mathbb{E}\left[\nabla \delta_\xi^G \phi_{T,\theta,\xi}^{E^{(p_0)}\setminus F} \cdot (C_\theta)_{F\setminus G\|G} (\nabla \phi_{T,\theta,\xi}^{E^{(p_0)}\cup F} + \xi)\right] \tag{5.23}$$

$$= k! \lim_{T\uparrow\infty} \lim_{\theta\uparrow\infty} \sum_{|F|=k} \sum_{\substack{G \subset F \\ G \ne \varnothing}} (-1)^{|G|+1} \mathbb{E}\left[(\nabla \phi_{T,\theta,\xi}^{E^{(p_0)}\setminus F} + \xi) \cdot (C_\theta)_G \nabla \delta_\xi^{F\setminus G} \phi_{T,\theta,\xi}^{G\cup(E^{(p_0)}\setminus F)}\right] \tag{5.24}$$

$$= k! \lim_{T\uparrow\infty} \lim_{\theta\uparrow\infty} \sum_{|F|=k} \mathbb{E}\bigg[\sum_{G\subset F} (-1)^{|F\setminus G|} \xi \cdot A_\theta^{G\cup(E^{(p_0)}\setminus F)} (\nabla \phi_{T,\theta,\xi}^{G\cup(E^{(p_0)}\setminus F)} + \xi)\bigg], \tag{5.25}$$

where the limits exist and where the sums are absolutely convergent for any fixed $T, \theta < \infty$ (recall that $\sum_{|F|=k}$ stands for the sum running over all the k-uplets of distinct positive integers).

Moreover, in the case when the point process ρ satisfies $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \ge 1$, then the limits in θ as well as all subscripts θ can be omitted in the above formulas (5.23)–(5.25). Finally, in the case $k = 1$, and under the additional assumption that $\rho(Q) \le \theta_0$ a.s. for some fixed $\theta_0 > 0$, we can pass to the limit in T inside the sum in (5.24): for all $p_0 \in [0,1]$,

$$\xi \cdot A_{\rm hom}^{(p_0),1} \xi = \sum_n \mathbb{E}\left[(\nabla \phi_\xi^{E^{(p_0)}\setminus\{n\}} + \xi) \cdot C^{\{n\}} (\nabla \phi_\xi^{E^{(p_0)}\cup\{n\}} + \xi)\right], \tag{5.26}$$

where the sum is still absolutely convergent. $\lozenge$

Formula (5.25) is the rigorous version of the so-called *cluster expansion formula* formally used by physicists (see [413]) as well as in [20]: it compares the homogenized coefficients corresponding to the coefficients obtained with a finite number of perturbed inclusions. In particular, the k-th derivative of $A_{\text{hom}}^{(p)}$ with respect to p is obtained by considering k perturbed inclusions. Note that these cluster expansion formulas can be rewritten as in (5.12) using the difference operators defined in (5.11): for all $k \geq 1$,

$$\xi \cdot A_{\text{hom}}^{(p_0),k} \xi = k! \lim_{T\uparrow\infty} \lim_{\theta\uparrow\infty} \sum_{|F|=k} \mathbb{E}\Big[\delta^F \Big(\xi \cdot A_\theta^{E^{(p_0)}\setminus F} (\nabla \phi_{T,\theta,\xi}^{E^{(p_0)}\setminus F} + \xi)\Big)\Big],$$

where δ^F now acts on the random variable $\xi \cdot A_\theta^{E^{(p_0)}\setminus F}(0)(\nabla \phi_{T,\theta,\xi}^{E^{(p_0)}\setminus F}(0) + \xi)$. For $k = 1$, it essentially coincides with the formula obtained by Mourrat in [327],

$$\xi \cdot A_{\text{hom}}^{(p_0),1} \xi = \lim_{T\uparrow\infty} \lim_{\theta\uparrow\infty} \sum_n \mathbb{E}\left[\xi \cdot A_\theta^{E^{(p_0)}\cup\{n\}} (\nabla \phi_{T,\theta,\xi}^{E^{(p_0)}\cup\{n\}} + \xi) - \xi \cdot A_\theta^{E^{(p_0)}\setminus\{n\}} (\nabla \phi_{T,\theta,\xi}^{E^{(p_0)}\setminus\{n\}} + \xi)\right].$$

Also note that, in the particular case when the inclusions J_n's are disjoint, formula (5.24) takes the following simpler form: for all $p_0 \in [0, 1]$ and $k \geq 1$,

$$\xi \cdot A_{\text{hom}}^{(p_0),k} \xi \;=\; k! \lim_{T\uparrow\infty} \lim_{\theta\uparrow\infty} \sum_{|F|=k} \sum_{n\in F} \mathbb{E}\left[(\nabla \phi_{T,\theta,\xi}^{E^{(p_0)}\setminus F} + \xi) \cdot (C_\theta)^{\{n\}} \nabla \delta_\xi^{F\setminus\{n\}} \phi_{T,\theta,\xi}^{\{n\}\cup(E^{(p_0)}\setminus F)}\right],$$

which further reduces, under the additional assumption that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \geq 1$, to

$$\xi \cdot A_{\text{hom}}^{(p_0),k} \xi \;=\; k! \lim_{T\uparrow\infty} \sum_{|F|=k} \sum_{n\in F} \mathbb{E}\left[(\nabla \phi_{T,\xi}^{E^{(p_0)}\setminus F} + \xi) \cdot C^{\{n\}} \nabla \delta_\xi^{F\setminus\{n\}} \phi_{T,\xi}^{\{n\}\cup(E^{(p_0)}\setminus F)}\right].$$

As a direct consequence of Theorem 5.1.1 we obtain the following universality principle, well-known by physicists: at first order in the volume fraction of the perturbation, the perturbed homogenized coefficient does not depend on the underlying point process ρ. More precisely,

Corollary 5.1.3 (First-order universality principle)**.** *On top of the above assumptions, assume that $\mathbb{E}[\rho(Q)^2] < \infty$. Then, we may define the volume fraction of the perturbation by the limit*

$$v_p := \lim_{L\uparrow\infty} \frac{\mathbb{E}\left[|LQ \cap \bigcup_{n\in E^{(p)}} J_n|\right]}{L^d}, \tag{5.27}$$

and there exists some matrix K such that for all $p \geq 0$,

$$A_{\text{hom}}^{(p)} = A_{\text{hom}}^{(0)} + K v_p + O(v_p^2).$$

If the point process ρ is independent of A, of A' (or else of $(A'_n)_n$ in the particular example (5.5)*) and of the random volumes $|J_n^\circ|$'s, then the constant K does not depend on the choice of the underlying point process ρ.* $\Diamond$

Since the formulas given by Corollary 5.1.2 for the k-th derivative $A_{\text{hom}}^{(0),k}$ of $A_{\text{hom}}^{(p)}$ at 0 involve terms of the form $\mathbb{E}[\sum_{n_1,\ldots,n_k} f(q_{n_1}, \ldots, q_{n_k})]$, they depend on moments of ρ up to order k, so that stronger dependence on the point process ρ is expected for higher-order terms (see indeed [413, p. 493–494]).

Formula (5.26) for the first derivative has the advantage of being exact (there is no limit left wrt T), and, at $p_0 = 0$, it is given by the solution of the corrector equation corresponding to a single inclusion. In particular, this makes explicit calculations possible for spherical inclusions, and allows us to prove the celebrated Clausius-Mossotti formula in a very general context.

Corollary 5.1.4 (Electric Clausius-Mossotti formula)**.** *On top of the above assumptions, assume that the inclusions are spherical, i.e.* $J_n = B_R(q_n)$*, and that both the unperturbed and perturbed coefficients are constant and isotropic:* $A = \alpha\,\mathrm{Id}$ *and* $A' = \beta\,\mathrm{Id}$*. Denoting by* v_p *the volume fraction* (5.27) *of the perturbation, we then have, for all* $p \geq 0$*,*

$$A_{\mathrm{hom}}^{(p)} \;=\; \alpha\,\mathrm{Id} + v_p \frac{\alpha d(\beta-\alpha)}{\beta+\alpha(d-1)}\,\mathrm{Id} + O(v_p^2). \qquad \Diamond$$

As pointed out in Section 5.1.1, all our results also hold for linear elasticity. This allows us to give the first rigorous proof of the elastic Clausius-Mossotti formula for random inclusions. Recall that an isotropic stiffness tensor A has the form $\frac{1}{2}\xi : A : \xi = G|\xi|^2 + \frac{\lambda}{2}(\mathrm{Tr}\,\xi)^2$, where G and λ are the Lamé coefficients, to which we associate the bulk modulus $K = \lambda + 2G/d$ and shear modulus G.

Corollary 5.1.5 (Elastic Clausius-Mossotti formula)**.** *On top of the above assumptions, assume that the inclusions are spherical, i.e.* $J_n = B_R(q_n)$*, and that both the unperturbed and perturbed stiffness tensors* A *and* A' *are constant and isotropic, and denote by* $K, G > 0$ *and* $K', G' > 0$ *their respective bulk and shear moduli. Let* A^1 *be the stiffness matrix of an isotropic medium of bulk modulus* $K_1 = K + (K'-K)\frac{K+\beta}{K'+\beta}$ *and shear modulus* $G_1 = G + (G'-G)\frac{G+\alpha}{G'+\alpha}$*, where we have set*

$$\alpha = G\frac{d^2K + 2(d+1)(d-2)G}{2d(K+2G)}, \qquad \beta = 2G\frac{d-1}{d}. \tag{5.28}$$

Denoting by v_p *the volume fraction* (5.27) *of the perturbation, we then have, for all* $p \geq 0$*,*

$$A_{\mathrm{hom}}^{(p)} \;=\; A + v_p A^1 + O(v_p^2). \qquad \Diamond$$

Corollaries 5.1.4 and 5.1.5 treat spherical inclusions, in which case the solution of the corrector equation with a single inclusion can be calculated explicitly, so that (5.26) can be turned into an explicit formula. In the case of ellipsoidal inclusions, explicit calculations can also be made in terms of the so-called depolarization coefficients in the electric case (see [402]), or in terms of the Eshelby tensor in the elastic case (see [178], and also [332] for more precise analytic computations), so that an explicit formula for the first derivative $A_{\mathrm{hom}}^{(0),1}$ can also be derived. The comparison of these results for spherical and ellipsoidal inclusions illustrates the fact that the first derivative already heavily depends on the geometry of the microstructure (see e.g. [413, Section 19.1.2]).

An explicit formula could in principle also be obtained for the second derivative at $p_0 = 0$ for spherical inclusions, since the corrector equation for two disjoint spheres can be solved analytically as well (see [374] and [259, Section 5]).

Formulas (5.23), (5.24) and (5.25) for the derivatives as given by Theorem 5.1.1 are expressed as limits in terms of the approximate corrector gradient. For practical purposes, it may be important to prove rates of convergence for these limits. This is a quantitative ergodic result and therefore requires quantitative ergodic assumptions. In what follows we assume that a quantitative convergence result is available for the convergence of $A_T^{(p)} := \mathbb{E}[\xi \cdot A^{(p)}(\nabla\phi_T^{(p)} + \xi)]$ to $A_{\mathrm{hom}}^{(p)}$ (through the convergence of $\nabla\phi_T^{(p)}$ to $\nabla\phi^{(p)}$) and show how this rate is inherited by their derivatives with respect to p.

Corollary 5.1.6. *On top of the above assumptions, assume that* $\mathbb{E}[\rho(Q)^s] < \infty$ *for all* $s \geq 1$*, and further assume that there exists a function* γ *such that, for all* $T > 0$ *and* $p \in [0,1]$*,*

$$\mathbb{E}[|\nabla(\phi_T^{(p)} - \phi_{2T}^{(p)})|^2] \lesssim \gamma(T)^2. \tag{5.29}$$

Let $p \in [0,1]$ *be fixed. Recall the formulas for the approximate derivatives of* $A_{\mathrm{hom}}^{(p)}$*: for all* $k \geq 0$*,*

$$\xi \cdot A_T^{(p),k}\xi := k! \sum_{|F|=k} \sum_{G \subset F} (-1)^{|F \setminus G|} \mathbb{E}\left[\xi \cdot A^{G \cup E^{(p_0)} \setminus F}(\nabla\phi_{T,\xi}^{G \cup E^{(p_0)} \setminus F} + \xi)\right]. \tag{5.30}$$

Then, there is a constant $C \simeq 1$ such that, for all $k \geq 0$, we have

$$\left|A_T^{(p),k} - A_{2T}^{(p),k}\right| \leq k! C^k \gamma(T)^{2^{-k}}.$$

In particular, if $\gamma(T) \lesssim T^{-\alpha}$ for some $\alpha > 0$, this yields for some constant $C \simeq_\alpha 1$,

$$\left|A_T^{(p),k} - A_{\mathrm{hom}}^{(p),k}\right| \leq k! C^k T^{-2^{-k}\alpha}. \qquad \Diamond$$

It is not clear to us whether Corollary 5.1.6 is optimal, and symmetric approximations could yield better rates. Such improvements, which would require nontrivial arguments based on the quantitative theory of stochastic homogenization, are not the goal of this chapter. Note that in the case of fast decaying correlations the optimal expected rate $\gamma(T)$ for the approximate corrector gradient in (5.29) is as follows [210, 212, 203],

$$\gamma(T)^2 = \begin{cases} T^{-1}, & \text{if } d = 2; \\ T^{-3/2}, & \text{if } d = 3; \\ T^{-2}\log T, & \text{if } d = 4; \\ T^{-2}, & \text{if } d > 4. \end{cases} \tag{5.31}$$

Note that in higher dimensions these convergence rates are improved when using suitable extrapolation techniques (cf. [202, 199, 206, 203]).

Let us emphasize an observation by Anantharaman and Le Bris in [20] on the regularity of the corrector with respect to p in the case of disjoint inclusions — which dramatically contrasts with the analyticity of the perturbed homogenized coefficients.

Remark 5.1.7. By testing the equation $-\nabla \cdot A^{(p)}\nabla(\phi^{(p)} - \phi) = \nabla \cdot C^{(p)}(\nabla\phi + \xi)$ in probability, we have

$$\mathbb{E}[|\nabla(\phi^{(p)} - \phi)|^2] \lesssim \mathbb{E}[|C^{(p)}|^2(1 + |\nabla\phi|^2)] \simeq p.$$

We believe that this scaling is in general optimal, so that the map $[0,1] \to \mathrm{L}^2(\Omega) : p \mapsto \nabla\phi^{(p)}(0)$ is expected to be in general nowhere differentiable. Since we prove that the homogenized coefficients are analytic, this illustrates that averaged quantities behave much better than pointwise quantities (like the corrector gradient). $\Diamond$

5.1.3 Corresponding results on the effective fluctuation tensor

As established in Chapter 3, under strong quantitative ergodicity assumptions, the random fluctuations of the solution $u_\varepsilon^{(p)}$ of $-\nabla \cdot A^{(p)}(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon^{(p)} = f$, those of the corrector $\nabla\phi^{(p)}$, and those of the flux of the corrector $A^{(p)}(\nabla\phi^{(p)} + \mathrm{Id})$ are driven by the random fluctuations of the corresponding homogenization commutator, which are asymptotically Gaussian with limiting variance given by the effective fluctuation tensor $\mathcal{Q}^{(p)}$. As shown in Appendix 3.A, this symmetric fourth-order tensor can be computed e.g. via massive approximations, which we take as a definition here,

$$\mathcal{Q}^{(p)} := \lim_{T\uparrow\infty} \mathcal{Q}_T^{(p)}, \qquad (\xi_1 \otimes \xi_2) : \mathcal{Q}_T^{(p)}\,(\xi_1 \otimes \xi_2) := \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\xi_1 \otimes \xi_2 : \Xi_T^{(p)}(x)\right)\left(\xi_1 \otimes \xi_2 : \Xi_T^{(p)}(0)\right)\right] dx, \tag{5.32}$$

where the limit exists and the integral is absolutely convergent, and where we have set

$$\xi_1 \otimes \xi_2 : \Xi_T^{(p)} := \xi_2 \cdot (A^{(p)} - A_{\mathrm{hom}}^{(p)})(\nabla\phi_{T,\xi_1}^{(p)} + \xi_1).$$

We may then prove the following, which parallels the expansion result for the homogenized coefficients.

Theorem 5.1.8. *Under the above assumptions together with suitable strong quantitative ergodicity hypotheses (see Appendix 5.B), the map $p \mapsto \mathcal{Q}^{(p)}$ is of class $C^{1,1-\gamma}$ for all $\gamma > 0$, and it satisfies for all $p_0 \in [0,1]$ and all $-p_0 \le p \le 1-p_0$,*

$$\big|\mathcal{Q}^{(p_0+p)} - \mathcal{Q}^{(p_0)} - p\mathcal{Q}^{(p_0),1}\big| \lesssim p^2|\log p|, \tag{5.33}$$

where the first derivative $\mathcal{Q}^{(0),1}$ at $p_0 = 0$ is given by

$$\begin{aligned}
&\xi_1 \otimes \xi_2 : \mathcal{Q}^{(0),1} : \xi_1 \otimes \xi_2 \\
&\qquad = \lim_{T\uparrow\infty} \bigg(\int_{\mathbb{R}^d} \mathrm{Cov}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot\sum_n C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(x)\ ;\ \xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(0)\Big]dx \\
&\qquad\qquad + \int_{\mathbb{R}^d} \mathrm{Cov}\Big[\xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(x)\ ;\ (\nabla\phi_{T,\xi_2}+\xi_2)\cdot\sum_n C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(0)\Big]dx \\
&\qquad + \sum_n \int_{\mathbb{R}^d} \mathbb{E}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(x)\,(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(0)\Big]\,dx\bigg),
\end{aligned} \tag{5.34}$$

where the limit exists and where the sums are absolutely convergent for any fixed $T < \infty$. If in addition the inclusions are spherical (i.e. $J_n = B_R(q_n)$) and if both reference coefficients are constant and isotropic (i.e. $A = \alpha\,\mathrm{Id}$ and $A' = \beta\,\mathrm{Id}$), then in terms of the volume fraction (5.27) *of the perturbation we have for small $p \ge 0$,*

$$\mathcal{Q}^{(p)} \;=\; v_p|B_R|\left(\frac{1}{v_p}\big(A_{\mathrm{hom}}^{(p)} - A_{\mathrm{hom}}^{(0)}\big)\right)^{\otimes 2} \;+\; O(p^2|\log p|), \tag{5.35}$$

and the corresponding result holds in the case of linear elasticity. $\Diamond$

We believe that this result can be straightforwardly extended into a C^∞ regularity result for the map $p \mapsto \mathcal{Q}^{(p)}$ (up to combinatorial technicalities in the proof), so that in particular the errors in (5.33) and (5.35) could be replaced by $O(p^2)$. Note that the exact link (5.35) between the effective fluctuation tensor and the homogenized coefficients at first order is only expected to hold in the case of spherical inclusions.

5.1.4 Strategy of the proof

In this section we present the strategy of the proof of Theorem 5.1.1. The key ingredient is a new family of energy estimates, the proof of which essentially combines combinatorial and induction arguments. When inclusions are disjoint, the combinatorics is significantly less involved than in the general case. In order to focus only on the core of the proof of Theorem 5.1.1 and to avoid additional combinatorial technicalities in this presentation, we shall momentarily assume that the inclusions are disjoint.

Fix some direction $\xi \in \mathbb{R}^d$, $|\xi| = 1$. The aim of the present chapter is to investigate the difference

$$\Delta^{(p)} := \xi\cdot(A_{\mathrm{hom}}^{(p)} - A_{\mathrm{hom}})\xi,$$

and express it as a convergent power series in the variable p around 0. Since the approximate correctors behave much better than the correctors themselves, we start with the analysis of the approximate difference

$$\Delta_T^{(p)} := \xi\cdot(A_T^{(p)} - A_T)\xi,$$

for fixed $T > 0$. Indeed, the approximate difference is a good proxy for the difference since $\lim_T \Delta_T^{(p)} = \Delta^{(p)}$ by (5.10). Next we rewrite the approximate difference in a form which is more suitable for the analysis. By definition,

$$\Delta_T^{(p)} = \mathbb{E}[\xi \cdot A^{(p)}(\nabla\phi_T^{(p)} + \xi)] - \mathbb{E}[\xi \cdot A(\nabla\phi_T + \xi)]$$
$$= \mathbb{E}[\xi \cdot C^{(p)}(\nabla\phi_T^{(p)} + \xi)] + \mathbb{E}[\xi \cdot A\nabla(\phi_T^{(p)} - \phi_T)]. \quad (5.36)$$

The first term is already in a nice form (since it is of order p), while the second term is not (recall Remark 5.1.7: an energy estimate would only imply that it is of order $\sqrt{p}$). In the following lemma, we make use of the corrector equation to unravel some cancellations.

Lemma 5.1.9. *The approximate difference $\Delta_T^{(p)}$ satisfies*

$$\Delta_T^{(p)} = \mathbb{E}[(\nabla\phi_T + \xi) \cdot C^{(p)}(\nabla\phi_T^{(p)} + \xi)]. \qquad \Diamond$$

Proof. Using that $A^{(p)} = A + C^{(p)}$, the second term of (5.36) turns into

$$\begin{aligned}
&\mathbb{E}[\xi \cdot A\nabla(\phi_T^{(p)} - \phi_T)] \\
&= \mathbb{E}[(\nabla\phi_T^{(p)} + \xi) \cdot A\nabla(\phi_T^{(p)} - \phi_T)] - \mathbb{E}[\nabla\phi_T^{(p)} \cdot A\nabla(\phi_T^{(p)} - \phi_T)] \\
&= \mathbb{E}[(\nabla\phi_T^{(p)} + \xi) \cdot A^{(p)}\nabla(\phi_T^{(p)} - \phi_T)] - \mathbb{E}[(\nabla\phi_T^{(p)} + \xi) \cdot C^{(p)}\nabla(\phi_T^{(p)} - \phi_T)] \\
&\quad - \mathbb{E}[\nabla\phi_T^{(p)} \cdot A^{(p)}(\nabla\phi_T^{(p)} + \xi)] + \mathbb{E}[\nabla\phi_T^{(p)} \cdot A(\nabla\phi_T + \xi)] + \mathbb{E}[\nabla\phi_T^{(p)} \cdot C^{(p)}(\nabla\phi_T^{(p)} + \xi)].
\end{aligned}$$

By symmetry of the coefficients A and $C^{(p)}$, reorganizing the terms yields

$$\mathbb{E}\left[\xi \cdot A\nabla(\phi_T^{(p)} - \phi_T)\right] = -\mathbb{E}\left[\nabla\phi_T \cdot A^{(p)}(\nabla\phi_T^{(p)} + \xi)\right]$$
$$+ \mathbb{E}\left[\nabla\phi_T^{(p)} \cdot A(\nabla\phi_T + \xi)\right] + \mathbb{E}\left[\nabla\phi_T \cdot C^{(p)}(\nabla\phi_T^{(p)} + \xi)\right]. \quad (5.37)$$

The sum of the first two terms of the right-hand side of (5.37) coincides with the sum of the weak formulations in probability of the equations

$$\frac{1}{T}\phi_T^{(p)} - \nabla \cdot A^{(p)}(\nabla\phi_T^{(p)} + \xi) = 0 \qquad \text{and} \qquad \frac{1}{T}\phi_T - \nabla \cdot A(\nabla\phi_T + \xi) = 0,$$

tested with ϕ_T and $\phi_T^{(p)}$ respectively, so that (5.37) reduces to

$$\mathbb{E}[\xi \cdot A\nabla(\phi_T^{(p)} - \phi_T)] = \mathbb{E}[\nabla\phi_T \cdot C^{(p)}(\nabla\phi_T^{(p)} + \xi)],$$

and the result then follows from (5.36). $\square$

Assuming that the inclusions are disjoint, we may use the inclusion-exclusion formula (5.18) in the elementary form of (5.16), so that the result of Lemma 5.1.9 above turns into

$$\Delta_T^{(p)} = \sum_n \mathbb{E}\left[(\nabla\phi_T + \xi) \cdot C^{\{n\}}(\nabla\phi_T^{(p)} + \xi)\mathbb{1}_{n \in E^{(p)}}\right],$$

or alternatively, using the constraint $n \in E^{(p)}$ to replace $\phi_T^{(p)}$ by $\phi_T^{E^{(p)} \cup \{n\}}$,

$$\Delta_T^{(p)} = \sum_n \mathbb{E}\left[(\nabla\phi_T + \xi) \cdot C^{\{n\}}(\nabla\phi_T^{E^{(p)} \cup \{n\}} + \xi)\mathbb{1}_{n \in E^{(p)}}\right].$$

Note that this sum is absolutely convergent since the $C^{\{n\}}$'s are assumed to have disjoint supports. As $\mathbb{1}_{n\in E^{(p)}}$ only depends on $b_n^{(p)}$ and as $(\nabla\phi_T+\xi)\cdot C^{\{n\}}(\nabla\phi_T^{E^{(p)}\cup\{n\}}+\xi)$ does not depend on $b_n^{(p)}$, we have by independence (5.6), using that $b_n^{(p)}$ is a Bernoulli random variable of parameter p,

$$\Delta_T^{(p)} = p\sum_n \mathbb{E}\left[(\nabla\phi_T+\xi)\cdot C^{\{n\}}(\nabla\phi_T^{E^{(p)}\cup\{n\}}+\xi)\right].$$

We further decompose the right-hand side

$$\Delta_T^{(p)} = p\sum_n \mathbb{E}\left[(\nabla\phi_T+\xi)\cdot C^{\{n\}}(\nabla\phi_T^{\{n\}}+\xi)\right] + p\sum_n \mathbb{E}\left[(\nabla\phi_T+\xi)\cdot C^{\{n\}}\nabla(\phi_T^{E^{(p)}\cup\{n\}}-\phi_T^{\{n\}})\right],$$

and observe that the second sum is a difference of the same nature as $\Delta_T^{(p)}$, which begs for an induction argument, and the following lemma is indeed proved by induction (see Lemma 5.3.1 for a more general statement).

Lemma 5.1.10. *Assume that the inclusions J_n's are disjoint and that $\mathbb{E}[\rho(Q)^s]<\infty$ for all $s\ge 1$. For all $k\ge 0$, all $T>0$, and all $p\in[0,1]$, we have*

$$\Delta_T^{(p)} = \sum_{j=1}^k p^j\Delta_T^j + p^{k+1}E_T^{(p),k+1} \tag{5.38}$$

where, for all $0\le j\le k$, the approximate derivatives Δ_T^j and the error $E_T^{(p),k+1}$ are given by

$$\Delta_T^j := \sum_{|F|=j}\sum_{n\in F}\mathbb{E}\left[\nabla\delta_\xi^{F\setminus\{n\}}\phi_T\cdot C^{\{n\}}(\nabla\phi_T^F+\xi)\right], \tag{5.39}$$

$$E_T^{(p),k+1} := \sum_{|F|=k+1}\sum_{n\in F}\mathbb{E}\left[\nabla\delta_\xi^{F\setminus\{n\}}\phi_T\cdot C^{\{n\}}(\nabla\phi_T^{E^{(p)}\cup F}+\xi)\right], \tag{5.40}$$

and the sums in (5.39) *and* (5.40) *are absolutely convergent.* $\Diamond$

Since the combinatorics in the proof of Lemma 5.3.1 is not more involved than for the proof of Lemma 5.1.10, we refer the reader to the proof of the former.

If we can prove that $|E_T^{(p),k}|\le C^k$ for all $k\ge 1$ and for some constant $C\simeq 1$ (independent of $T>0$ and of $p\in[0,1]$), then we can easily pass to the limit $T\uparrow\infty$ in the expansion (5.38) and obtain a convergent power-series expansion for the exact difference $\Delta^{(p)}$ itself around $p=0$.

The following lemma shows that a new family of energy estimates is needed to control the error terms. We display the proof of this lemma, which is significantly simpler than the corresponding proof in the general case of non-necessarily disjoint inclusions (see Proposition 5.3.2).

Lemma 5.1.11. *Assume that the inclusions J_n's are disjoint and that $\mathbb{E}[\rho(Q)^s]<\infty$ for all $s\ge 1$. Then, there is a constant $C\simeq 1$ (independent of T, of p, and of the moments of ρ) such that, for all $k\ge 0$, $T>0$, and $p\in[0,1]$, the error $E_T^{(p),k+1}$ defined in Lemma 5.3.1 satisfies*

$$|E_T^{(p),k+1}| \lesssim \sum_{j=0}^k \mathbb{E}\Big[\sum_{|G|=j}\Big|\sum_{\substack{|F|=k-j\\ F\cap G=\varnothing}}\nabla\delta_\xi^{F\cup G}\phi_T\Big|^2\Big] + \sum_{j=0}^{k+1}\mathbb{E}\Big[\sum_{|G|=j}|\nabla\delta_\xi^G\phi_T^{(p)}|^2\Big]. \tag{5.41}$$

$\Diamond$

Proof. Let $k \geq 0$. First rewrite the error as follows,

$$E_T^{(p),k+1} = \sum_{|F|=k} \sum_{n \notin F} \mathbb{E}\left[\nabla\delta_\xi^F \phi_T \cdot C^{\{n\}}(\nabla\phi_T^{E^{(p)}\cup F\cup\{n\}} + \xi)\right]. \tag{5.42}$$

Recalling identity $\sum_{G\subset H} \nabla\delta_\xi^G \phi_T = \nabla\phi_T^H + \xi$ for all $H \subset \mathbb{N}$, we deduce

$$\nabla\phi_T^{E^{(p)}\cup F\cup\{n\}} + \xi = \sum_{G\subset F} \nabla\delta_\xi^G \phi_T^{(p)} + \sum_{G\subset F} \nabla\delta_\xi^{G\cup\{n\}} \phi_T^{(p)},$$

so that (5.42) turns into

$$E_T^{(p),k+1} = \sum_{|F|=k} \sum_{G\subset F} \sum_{n\notin F} \mathbb{E}\left[\nabla\delta_\xi^F \phi_T \cdot C^{\{n\}}(\nabla\delta_\xi^G \phi_T^{(p)} + \nabla\delta_\xi^{G\cup\{n\}} \phi_T^{(p)})\right],$$

or equivalently

$$E_T^{(p),k+1} = \sum_{j=0}^{k} \sum_{|G|=j} \sum_{n\notin G} \sum_{\substack{|F|=k-j \\ F\cap(G\cup\{n\})=\varnothing}} \mathbb{E}\left[\nabla\delta_\xi^{F\cup G} \phi_T \cdot C^{\{n\}}(\nabla\delta_\xi^G \phi_T^{(p)} + \nabla\delta_\xi^{G\cup\{n\}} \phi_T^{(p)})\right]. \tag{5.43}$$

For all $n \notin G$ and all maps f, we obviously have (compare with the more general statement (5.80))

$$\sum_{\substack{|F|=k-j \\ F\cap(G\cup\{n\})=\varnothing}} f(F,G,n) = \sum_{\substack{|F|=k-j \\ F\cap G=\varnothing}} f(F,G,n) - \sum_{\substack{|F|=k-j-1 \\ F\cap(G\cup\{n\})=\varnothing}} f(F\cup\{n\},G,n),$$

so that we may rearrange the terms in (5.43) as follows,

$$\begin{aligned}|E_T^{(p),k+1}| \lesssim{}& \sum_{j=0}^{k} \sum_{|G|=j} \sum_{n\notin G} \mathbb{E}\left[\mathbb{1}_{J_n}\Big| \sum_{\substack{|F|=k-j \\ F\cap G=\varnothing}} \nabla\delta_\xi^{F\cup G} \phi_T\Big| \Big(|\nabla\delta_\xi^G \phi_T^{(p)}| + |\nabla\delta_\xi^{G\cup\{n\}} \phi_T^{(p)}|\Big)\right] \\ &+ \sum_{j=0}^{k} \sum_{|G|=j} \sum_{n\notin G} \mathbb{E}\left[\mathbb{1}_{J_n}\Big| \sum_{\substack{|F|=k-j-1 \\ F\cap(G\cup\{n\})=\varnothing}} \nabla\delta_\xi^{F\cup G\cup\{n\}} \phi_T\Big| \Big(|\nabla\delta_\xi^G \phi_T^{(p)}| + |\nabla\delta_\xi^{G\cup\{n\}} \phi_T^{(p)}|\Big)\right].\end{aligned}$$

By Young's inequality and the fact that the inclusions J_n's are disjoint, this yields

$$\begin{aligned}|E_T^{(p),k+1}| \lesssim{}& \sum_{j=0}^{k} \sum_{|G|=j} \left(\mathbb{E}\left[\Big| \sum_{\substack{|F|=k-j \\ F\cap G=\varnothing}} \nabla\delta_\xi^{F\cup G} \phi_T\Big|^2\right] + \mathbb{E}[|\nabla\delta_\xi^G \phi_T^{(p)}|^2]\right) \\ &+ \sum_{j=0}^{k} \sum_{|G|=j} \sum_{n\notin G} \left(\mathbb{E}\left[\mathbb{1}_{J_n}\Big| \sum_{\substack{|F|=k-j-1 \\ F\cap(G\cup\{n\})=\varnothing}} \nabla\delta_\xi^{F\cup G\cup\{n\}} \phi_T\Big|^2\right] + \mathbb{E}[\mathbb{1}_{J_n}|\nabla\delta_\xi^{G\cup\{n\}} \phi_T^{(p)}|^2]\right),\end{aligned}$$

and the announced result already follows. ☐

In view of (5.41) it is enough to prove the following family of energy estimates: there exists $C \simeq 1$ such that for all $k \geq j \geq 0$ we have

$$\mathbb{E}\left[\sum_{|G|=j} \Big| \sum_{\substack{|F|=k-j \\ F\cap G=\varnothing}} \nabla\delta_\xi^{F\cup G} \phi_T^{(p)}\Big|^2\right] \leq C^k. \tag{5.44}$$

On the one hand, a straightforward energy estimate directly yields (cf. Lemma 5.2.5)

$$\mathbb{E}\Big[\Big|\sum_n \nabla\delta_\xi^{\{n\}}\phi_T^{(p)}\Big|^2\Big] \lesssim 1. \tag{5.45}$$

On the other hand, a simple induction argument yields for some $C \simeq 1$ and all $j \geq 0$ (cf. Lemma 5.2.4),

$$\mathbb{E}\Big[\sum_{|F|=j} |\nabla\delta_\xi^F\phi_T^{(p)}|^2\Big] \leq C^j. \tag{5.46}$$

For $j \leq 2$, this estimate already appears in [20] (with however the massive term approximation replaced by the approximation by periodization). As mentioned in Section 5.1.1, in view of Lemmas 5.1.10 and 5.1.11, these uniform bounds (combined with the fact that the estimates are independent of p and combined with some invariance argument due to the structure of Bernoulli random variables, see Step 3 of the proof of Theorem 5.1.1 in Section 5.3.2) imply that $p \mapsto A_{\mathrm{hom}}^{(p)}$ is $C^{1,1}$ on $[0,1]$.

Before we describe the complete induction strategy used in Section 5.2.3 to prove (5.44), let us start by showing it in action, proving the result for $k=2$ based on the corresponding result for $k=1$ (that is, (5.45) and (5.46) for $j=1$). This proof is instructive in three respects: it implements the general induction strategy in the first nontrivial step, it shows that we need to use several forms of the equation satisfied by $\delta^{\{n,m\}}\phi_T$, and it suggests that the proof of these equivalent forms relies on combinatorial arguments.

Lemma 5.1.12. *Assume that the inclusions J_n's are disjoint and that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \geq 1$. Then, for all $T > 0$ and $p \in [0,1]$,*

$$\mathbb{E}\Big[\sum_{m\neq n} |\nabla\delta^{\{n,m\}}\phi_T^{(p)}|^2\Big] \;\lesssim\; 1, \tag{5.47}$$

$$\mathbb{E}\Big[\sum_n\Big|\sum_{m,m\neq n} \nabla\delta^{\{n,m\}}\phi_T^{(p)}\Big|^2\Big] \;\lesssim\; 1, \tag{5.48}$$

$$\mathbb{E}\Big[\Big|\sum_{m\neq n} \nabla\delta^{\{n,m\}}\phi_T^{(p)}\Big|^2\Big] \;\lesssim\; 1. \tag{5.49}$$

$\Diamond$

Proof. For notational convenience, we consider $p=0$ only. In what follows we take for granted that the series we consider are all absolutely converging, which is indeed ensured for fixed T by the (suboptimal) estimates of Lemma 5.2.2. We split the proof into four steps. In the first step we give three forms of the equation satisfied by $\delta^{\{n,m\}}\phi_T$. In the second step we prove (5.47) based on one equation and (5.45). In the third step we prove (5.48) based on another equation, (5.45), (5.46) for $j=1$, and (5.47). In the last step we prove (5.49) based on a third form of the equation, (5.45), (5.46) for $j=1$, (5.47), and (5.48).

Step 1. Equations satisfied by $\delta^{\{n,m\}}\phi_T$.

Let $m \neq n$. The equation satisfied by the difference $\delta^{\{n,m\}}\phi_T$ can be written in several forms, with perturbed or unperturbed operators. With the unperturbed operator, we have

$$\begin{aligned}&\frac{1}{T}\delta^{\{n,m\}}\phi_T - \nabla\cdot A\nabla\delta^{\{n,m\}}\phi_T\\ &= \nabla\cdot C^{\{n,m\}}(\nabla\phi_T^{\{n,m\}}+\xi) - \nabla\cdot C^{\{n\}}(\nabla\phi_T^{\{n\}}+\xi) - \nabla\cdot C^{\{m\}}(\nabla\phi_T^{\{m\}}+\xi).\end{aligned}$$

By the inclusion-exclusion formula in the simple form $C^{\{n,m\}} = C^{\{n\}} + C^{\{m\}}$ (due to disjointness of the inclusions), the equation takes the form

$$\frac{1}{T}\delta^{\{n,m\}}\phi_T - \nabla\cdot A\nabla\delta^{\{n,m\}}\phi_T = \nabla\cdot C^{\{n\}}\nabla\delta^{\{m\}}\phi_T^{\{n\}} + \nabla\cdot C^{\{m\}}\nabla\delta^{\{n\}}\phi_T^{\{m\}}. \tag{5.50}$$

This equation will be used to prove (5.49). A combinatorial argument (which is elementary here because the difference operators are of order 2 only and the inclusions are disjoint) allows one to turn the equation satisfied by $\delta^{\{n,m\}}\phi_T$ into

$$\frac{1}{T}\delta^{\{n,m\}}\phi_T - \nabla\cdot A^{\{n\}}\nabla\delta^{\{n,m\}}\phi_T \;=\; \nabla\cdot C^{\{m\}}\nabla\delta^{\{n\}}\phi_T^{\{m\}} + \nabla\cdot C^{\{n\}}\nabla\delta^{\{m\}}\phi_T, \tag{5.51}$$

which involves a perturbed operator (with the partially perturbed coefficients $A^{\{n\}}$), and will be used to prove (5.48). The third and last version of the equation takes the form

$$\frac{1}{T}\delta^{\{n,m\}}\phi_T - \nabla\cdot A^{\{n,m\}}\nabla\delta^{\{n,m\}}\phi_T \;=\; \nabla\cdot C^{\{m\}}\nabla\delta^{\{n\}}\phi_T + \nabla\cdot C^{\{n\}}\nabla\delta^{\{m\}}\phi_T, \tag{5.52}$$

which involves the completely perturbed operator, and will be used to prove (5.47).

Step 2. Proof of (5.47).

The starting point is (5.52), the right-hand side of which only involves first-order differences of the unperturbed corrector ϕ_T. Although the argument of the expectation in the left-hand side of (5.47) is stationary, the equation (5.52) is not stationary. We shall first obtain energy estimates associated with (5.52) which are localized in space. It is only after summing these estimates over n and m, taking the expectation, and passing to the limit in the localization parameter that the desired estimate (5.47) in expectation will come out in the form

$$\mathbb{E}\Big[\sum_{n\neq m}|\nabla\delta^{\{n,m\}}\phi_T|^2\Big] \;\le\; C\,\mathbb{E}\Big[\sum_{n}|\nabla\delta^{\{n\}}\phi_T|^2\Big], \tag{5.53}$$

to be combined with (5.46) for $j=1$.

For all $N\ge 0$, we then introduce a cut-off function χ_N for B_N in B_{2N} such that $|\nabla\chi_N|\lesssim 1/N$, and test equation (5.52) with test function $\chi_N\delta^{\{n,m\}}\phi_T\in H^1(\mathbb{R}^d)$. This yields for all $n\neq m$ after integration by parts, using the properties of χ_N, and rearranging the terms,

$$\begin{aligned}\int_{B_N}|\nabla\delta^{\{n,m\}}\phi_T|^2 \;\le\; C\int_{B_{2N}}&(\mathbb{1}_{J_n}|\nabla\delta^{\{m\}}\phi_T| + \mathbb{1}_{J_m}|\nabla\delta^{\{n\}}\phi_T|)|\nabla\delta^{\{n,m\}}\phi_T|\\ &+\frac{C}{N}\int_{B_{2N}}|\nabla\delta^{\{n,m\}}\phi_T||\delta^{\{n,m\}}\phi_T| + \frac{C}{N}\int_{B_{2N}}(\mathbb{1}_{J_n}|\nabla\delta^{\{m\}}\phi_T| + \mathbb{1}_{J_m}|\nabla\delta^{\{n\}}\phi_T|)|\delta^{\{n,m\}}\phi_T|.\end{aligned}$$

We use Young's inequality on each term (to ultimately absorb part of the right-hand side into the left-hand side), sum this inequality over $n,m\in\mathbb{N}$ with $n\neq m$, and take the expectation to obtain

$$\begin{aligned}\int_{B_N}\mathbb{E}\Big[\sum_{n\neq m}|\nabla\delta^{\{n,m\}}\phi_T|^2\Big] \;\le\; C\int_{B_{2N}}&\mathbb{E}\Big[\sum_{n\neq m}(\mathbb{1}_{J_n}|\nabla\delta^{\{m\}}\phi_T|^2 + \mathbb{1}_{J_m}|\nabla\delta^{\{n\}}\phi_T|^2)\Big]\\ &+\frac{1}{C}\int_{B_{2N}}\mathbb{E}\Big[\sum_{n\neq m}|\nabla\delta^{\{n,m\}}\phi_T|^2\Big] + \frac{C}{N}\int_{B_{2N}}\mathbb{E}\Big[\sum_{n\neq m}|\delta^{\{n,m\}}\phi_T|^2\Big],\end{aligned}$$

where all the terms make sense and are finite by Lemma 5.2.2. Since all the arguments of the expectations are now stationary, one may get rid of the integrals, which allows one to absorb the second right-hand side term into the left-hand side (choosing $C>0$ big enough) and obtain

$$\mathbb{E}\Big[\sum_{n\neq m}|\nabla\delta^{\{n,m\}}\phi_T|^2\Big] \;\le\; C\,\mathbb{E}\Big[\sum_{n\neq m}\mathbb{1}_{J_n}|\nabla\delta^{\{m\}}\phi_T|^2\Big] + \frac{C}{N}\,\mathbb{E}\Big[\sum_{n\neq m}|\delta^{\{n,m\}}\phi_T|^2\Big].$$

We are now in position to conclude: by taking $N \uparrow \infty$ we get rid of the second term of the right-hand side, so that (5.53) follows.

Step 3. Proof of (5.48).

The desired estimate is a consequence of (5.45), of (5.46) for $j=1$, of (5.47), and of

$$\mathbb{E}\Big[\sum_n \Big|\sum_{m,m\neq n} \nabla\delta^{\{n,m\}}\phi_T\Big|^2\Big] \lesssim \mathbb{E}\Big[\sum_{n\neq m} |\nabla\delta^{\{n,m\}}\phi_T|^2\Big] + \mathbb{E}\Big[\Big|\sum_n \nabla\delta^{\{n\}}\phi_T\Big|^2\Big] + \mathbb{E}\Big[\sum_n |\nabla\delta^{\{n\}}\phi_T|^2\Big]. \tag{5.54}$$

The starting point is equation (5.52) that we first sum over m for $m\neq n$,

$$\frac{1}{T}\sum_{m,m\neq n}\delta^{\{n,m\}}\phi_T - \nabla\cdot A^{\{n\}}\nabla\sum_{m,m\neq n}\delta^{\{n,m\}}\phi_T = \nabla\cdot\sum_{m,m\neq n} C^{\{m\}}\nabla\delta^{\{n\}}\phi_T^{\{m\}} + \nabla\cdot C^{\{n\}}\nabla\sum_{m,m\neq n}\delta^{\{m\}}\phi_T.$$

Following the approach of Step 2, we test this equation in space with $\chi_N\sum_{m,m\neq n}\delta^{\{n,m\}}\phi_T$ and the same cut-off χ_N. We obtain after summing the estimate over n, taking the expectation, and passing to the limit $N\uparrow\infty$,

$$\mathbb{E}\Big[\sum_n \Big|\nabla\sum_{m,m\neq n}\delta^{\{n,m\}}\phi_T\Big|^2\Big] \lesssim \mathbb{E}\Big[\sum_n \Big|\sum_{m,m\neq n} C^{\{m\}}\nabla\delta^{\{n\}}\phi_T^{\{m\}}\Big|^2\Big] + \mathbb{E}\Big[\sum_n \Big|C^{\{n\}}\sum_{m,m\neq n}\nabla\delta^{\{m\}}\phi_T\Big|^2\Big],$$

and hence, using that $\mathbb{1}_{J_n}\mathbb{1}_{J_m}=0$ for $n\neq m$ by the disjointness of the inclusions,

$$\mathbb{E}\Big[\sum_n \Big|\nabla\sum_{m,m\neq n}\delta^{\{n,m\}}\phi_T\Big|^2\Big] \lesssim \mathbb{E}\Big[\sum_n\sum_{m,m\neq n}\mathbb{1}_{J_m}|\nabla\delta^{\{n\}}\phi_T^{\{m\}}|^2\Big] + \mathbb{E}\Big[\sum_n \mathbb{1}_{J_n}\Big|\nabla\sum_{m,m\neq n}\delta^{\{m\}}\phi_T\Big|^2\Big].$$

By the decomposition $\delta^{\{n\}}\phi_T^{\{m\}} = \delta^{\{n\}}\phi_T + \delta^{\{n,m\}}\phi_T$ and the inequality $\sum_m \mathbb{1}_{J_m}\leq 1$, the first right-hand side term turns into

$$\mathbb{E}\Big[\sum_n\sum_{m,m\neq n}\mathbb{1}_{J_m}|\nabla\delta^{\{n\}}\phi_T^{\{m\}}|^2\Big] \lesssim \mathbb{E}\Big[\sum_n |\nabla\delta^{\{n\}}\phi_T|^2\Big] + \mathbb{E}\Big[\sum_{m\neq n}|\nabla\delta^{\{n,m\}}\phi_T|^2\Big].$$

The desired inequality (5.54) is then obtained by transforming the second right-hand side term as follows: we complete the sum over m, use the triangle inequality and the inequality $\sum_n \mathbb{1}_{J_n}\leq 1$, so that

$$\mathbb{E}\Big[\sum_n \mathbb{1}_{J_n}\Big|\nabla\sum_{m,m\neq n}\delta^{\{m\}}\phi_T\Big|^2\Big] \lesssim \mathbb{E}\Big[\Big|\sum_m \nabla\delta^{\{m\}}\phi_T\Big|^2\Big] + \mathbb{E}\Big[\sum_n |\nabla\delta^{\{n\}}\phi_T|^2\Big]. \tag{5.55}$$

Step 4. Proof of (5.49).

The desired estimate is a consequence of (5.45), of (5.46) for $j=1$, of (5.48), and of

$$\mathbb{E}\Big[\Big|\sum_{n\neq m}\nabla\delta^{\{n,m\}}\phi_T\Big|^2\Big] \lesssim \mathbb{E}\Big[\sum_n\Big|\sum_{m,m\neq n}\nabla\delta^{\{n,m\}}\phi_T\Big|^2\Big] + \mathbb{E}\Big[\Big|\sum_n \nabla\delta^{\{n\}}\phi_T\Big|^2\Big] + \mathbb{E}\Big[\sum_n |\nabla\delta^{\{n\}}\phi_T|^2\Big]. \tag{5.56}$$

The starting point is equation (5.50), that we sum over $n\neq m$:

$$\frac{1}{T}\sum_{n\neq m}\delta^{\{n,m\}}\phi_T - \nabla\cdot A\nabla\sum_{n\neq m}\delta^{\{n,m\}}\phi_T = 2\nabla\cdot\sum_{n\neq m} C^{\{n\}}\nabla\delta^{\{m\}}\phi_T^{\{n\}}.$$

Proceeding again as in Step 2, this yields

$$\mathbb{E}\Big[\Big|\sum_{n\neq m}\nabla\delta^{\{n,m\}}\phi_T\Big|^2\Big]\lesssim\mathbb{E}\Big[\Big|\sum_{n\neq m}C^{\{n\}}\nabla\delta^{\{m\}}\phi_T^{\{n\}}\Big|^2\Big].$$

(Note that since each term of the equation is stationary after summation over n and m, this coincides with the energy estimate in probability.) Since the inclusions are disjoint, we are left with

$$\mathbb{E}\Big[\Big|\sum_{n\neq m}\nabla\delta^{\{n,m\}}\phi_T\Big|^2\Big]\lesssim\mathbb{E}\Big[\sum_n\mathbb{1}_{J_n}\Big|\sum_{m,m\neq n}\nabla\delta^{\{m\}}\phi_T^{\{n\}}\Big|^2\Big].\tag{5.57}$$

By the decomposition $\delta^{\{m\}}\phi_T^{\{n\}}=\delta^{\{m\}}\phi_T+\delta^{\{n,m\}}\phi_T$, this turns into

$$\mathbb{E}\Big[\Big|\sum_{n\neq m}\nabla\delta^{\{n,m\}}\phi_T\Big|^2\Big]\lesssim\mathbb{E}\Big[\sum_n\mathbb{1}_{J_n}\Big|\sum_{m,m\neq n}\nabla\delta^{\{m\}}\phi_T\Big|^2\Big]+\mathbb{E}\Big[\sum_n\Big|\sum_{m,m\neq n}\nabla\delta^{\{n,m\}}\phi_T\Big|^2\Big],$$

and the desired inequality (5.56) follows from (5.55). □

This lemma easily implies that the map $p\mapsto A_{\mathrm{hom}}^{(p)}$ is $C^{1,1}$ on $[0,1]$. As already mentioned, the above proof illustrates the induction argument that we shall use in the proofs of Lemma 5.2.4 and of Proposition 5.2.6 below.

In the proof of Lemma 5.2.4, we shall always consider the equation for $\delta^F\phi_T$ with coefficients A^F, so that the right-hand side will only involve unperturbed correctors, and then sum over F the resulting energy estimate (first localized in space).

The proof of Proposition 5.2.6 is more involved. Call $P(j,k)$ the property (5.44). We make a first induction on k and then on j. Note that at step k there are k different forms of the equation satisfied by $\delta^F\phi_T$ (for $|F|=k$). By Lemma 5.2.4, $P(k,k)$ holds for all $k\in\mathbb{N}$. Then, given $P(k+1,k+1)$ and $P(i,l)$ for all $i\le l\le k$, we shall prove $P(k+1-j,k+1)$ iteratively starting with $j=1$. Indeed, $P(k+1-j,k+1)$ will follow from $P(k+1-j',k+1)$ for $j'<j$ and $P(j',l)$ for all $j'\le l\le k$, using the form of the equation where the coefficients are $k+1-j$ times perturbed. The last step $P(0,k+1)$ is similar to Step 4 in the proof above and relies on the equation with the unperturbed operator.

To be more precise, in the case of disjoint inclusions, the family of equations is as follows (see Lemma 5.2.1 for the general case): for all disjoint subsets $F,G,H\subset\mathbb{N}$, with F,G finite, $F\cup G\neq\varnothing$,

$$\frac{1}{T}\delta_\xi^{F\cup G}\phi_T^H-\nabla\cdot A^{F\cup H}\nabla\delta_\xi^{F\cup G}\phi_T^H=\sum_{n\in F}\nabla\cdot C^{\{n\}}\nabla\delta_\xi^{(F\setminus\{n\})\cup G}\phi_T^H+\sum_{n\in G}\nabla\cdot C^{\{n\}}\nabla\delta_\xi^{F\cup(G\setminus\{n\})}\phi_T^{H\cup\{n\}}.$$

5.1.5 Perspectives

A particularly interesting open question concerns the understanding of the maximal regularity of the perturbed homogenized coefficient $p\mapsto A_{\mathrm{hom}}^{(p)}$ in the case when the inclusions are allowed to intersect unboundedly. Assuming that the random variable $\Gamma:=\#\{n\in\mathbb{N}:0\in J_n\}$ only satisfies a superalgebraic moment bound $\mathbb{E}\left[\Gamma^k\right]\le L(k)<\infty$ for all $k\ge1$, the k-th derivative of the perturbed homogenized coefficient is expected to be bounded by $L(\frac{2}{\varepsilon}k)^\varepsilon k!C_\varepsilon^k$ for any $\varepsilon>0$ small enough. In particular this suggests a loss of analyticity whenever the inclusions intersect unboundedly. More generally, if for some $\alpha\ge0$ there holds $L(k)\le(Ck^\alpha)^k$ for all $k\ge1$ (the case of bounded penetrability corresponds to $\alpha=0$, and the example of Poisson unit inclusions corresponds to $\alpha=1$), then the perturbed homogenized coefficient should be of Gevrey class with index $1+2\alpha$.

As is clear from an inspection of the proof of Theorem 5.A.1 (cf. (5.131)), the missing ingredient is an improvement of integrability for the a priori estimates (5.44): we would need to show for some

$\varepsilon > 0$ that there holds for all $k \geq j \geq 0$,

$$\mathbb{E}\bigg[\bigg(\sum_{|G|=j}\bigg|\sum_{\substack{|F|=k-j\\F\cap G=\varnothing}}\nabla\delta_\xi^{F\cup G}\phi_T^{(p)}\bigg|^2\bigg)^{1+\varepsilon}\bigg] \leq C_\varepsilon^k. \tag{5.58}$$

This does however not seem to follow from soft Meyers-type arguments. In Appendix 5.A, we show that quantitative stochastic homogenization methods easily lead to bounds on the left-hand side of (5.58) with suboptimal k-dependence, and we establish in this way a C^∞ regularity result (cf. Theorem 5.A.1). We believe that suitable refinements (replacing the brutal estimate of Lemma 5.A.4 by the finer inductive argument of the proof of Proposition 5.2.6) should also lead to the optimal expected Gevrey regularity, but we have not pursued in that direction as we do not expect quantitative stochastic homogenization methods to be necessary here. The general expected Gevrey result without additional mixing assumptions is thus left as an open problem.

It is also not clear how much the Bernoulli law can be relaxed to allow for correlations. In particular it would be interesting to determine how the regularity of the homogenized coefficients depends on the decay of correlations of the generalization of the Bernoulli law.

As already mentioned, the convergence rates in Corollary 5.1.6 could be substantially improved using symmetric approximations and extrapolation methods [202, 199], which would be a useful result for numerical purposes.

Also, we believe that the $C^{1,1-}$ regularity of the perturbed effective fluctuation tensor obtained in Theorem 5.1.8 could easily be extended to a C^∞ regularity result, up to minor technicalities. In contrast, the question of analyticity is left as a completely open question even in the case of disjoint inclusions.

5.2 Auxiliary results and improved energy estimates

5.2.1 Perturbed corrector equations

We start by making precise the equations satisfied by the map $\delta_\xi^F\phi_T^G$ for disjoint subsets $F, G \subset \mathbb{N}$, which will be used abundantly in the sequel of this chapter. The proof of this lemma (like many other auxiliary results of this chapter) is purely combinatorial.

Lemma 5.2.1. *For all disjoint subsets $F, H \subset \mathbb{N}$, with F finite, $F \neq \varnothing$, and for all $T > 0$, the map $\delta_\xi^F\phi_T^H$ defined in* (5.11) *satisfies the following two equations (weakly) in $\mathbb{R}^d$,*

$$\frac{1}{T}\delta_\xi^F\phi_T^H - \nabla\cdot A^{F\cup H}\nabla\delta_\xi^F\phi_T^H = \sum_{S\subset F}(-1)^{|S|+1}\nabla\cdot C_{S\|H\cup F\setminus S}\nabla\delta_\xi^{F\setminus S}\phi_T^H, \tag{5.59}$$

and

$$\frac{1}{T}\delta_\xi^F\phi_T^H - \nabla\cdot A^{H}\nabla\delta_\xi^F\phi_T^H = \sum_{S\subset F}(-1)^{|S|+1}\nabla\cdot C_{S\|H}\nabla\delta_\xi^{F\setminus S}\phi_T^{S\cup H}. \tag{5.60}$$

More generally, for all disjoint subsets $F, G, H \subset \mathbb{N}$, with F, G finite, $F \cup G \neq \varnothing$, and for all $T > 0$, the map $\delta_\xi^{F\cup G}\phi_T^H$ defined in (5.11) *satisfies the following equation (weakly) in $\mathbb{R}^d$,*

$$\frac{1}{T}\delta_\xi^{F\cup G}\phi_T^H - \nabla\cdot A^{F\cup H}\nabla\delta_\xi^{F\cup G}\phi_T^H = \sum_{S\subset F}\sum_{U\subset G}(-1)^{|S|+|U|+1}\nabla\cdot C_{S\cup U\|H\cup(F\setminus S)}\nabla\delta_\xi^{(F\setminus S)\cup(G\setminus U)}\phi_T^{U\cup H}. \tag{5.61}$$

$\Diamond$

Proof. Let $T > 0$ be fixed. Without loss of generality we may assume that $H = \varnothing$. We first prove (5.60), from which we shall then deduce (5.61). Equation (5.59) is a particular case of (5.61) with $G = \varnothing$.

Step 1. Proof of (5.60).

Let $F \subset \mathbb{N}$ be a finite nonempty subset. By definition (5.14) of $\delta_\xi^F \phi_T$ and the inclusion-exclusion identity (5.19), we have

$$\begin{aligned}\frac{1}{T}\delta_\xi^F \phi_T - \nabla \cdot A \nabla \delta_\xi^F \phi_T &= \sum_{H \subset F} (-1)^{|F \setminus H|} \nabla \cdot C^H (\nabla \phi_T^H + \xi) \\ &= \sum_{H \subset F} \sum_{S \subset H} (-1)^{|F \setminus H|} (-1)^{|S|+1} \nabla \cdot C_S (\nabla \phi_T^H + \xi) \\ &= \sum_{S \subset F} (-1)^{|S|+1} \nabla \cdot C_S \sum_{H \subset F \setminus S} (-1)^{|(F \setminus S) \setminus H|} (\nabla \phi_T^{H \cup S} + \xi).\end{aligned}$$

Recognizing the definition of $\delta_\xi^{F \setminus S} \phi_T^S$, this yields

$$\frac{1}{T}\delta_\xi^F \phi_T - \nabla \cdot A \nabla \delta_\xi^F \phi_T = \sum_{S \subset F} (-1)^{|S|+1} \nabla \cdot C_S \nabla \delta_\xi^{F \setminus S} \phi_T^S,$$

and proves the validity of equation (5.60).

Step 2. A combinatorial identity.

For any finite subsets $K, L, M \subset \mathbb{N}$ (with K and L non-empty), we use the following notation:

$$C_{L\|M}^K := (A' - A)\mathbb{1}_{J_{L\|M}^K}, \qquad J_{L\|M}^K = \Big(\bigcup_{n \in K} J_n\Big) \cap \Big(\bigcap_{n \in L} J_n\Big) \setminus \Big(\bigcup_{n \in M} J_n\Big).$$

In this proof, and in this proof only, when K or L is empty, we further set $J_{\varnothing\|M}^K = J_{\|M}^K$ and $J_{L\|M}^\varnothing = 0$. We now check the following general, purely combinatorial identity: for any finite disjoint subsets $U, F \subset \mathbb{N}$ and for any $S \subsetneq F$,

$$(-1)^{|F \setminus S|} C_{(F \setminus S) \cup U \| S} = \sum_{H \subsetneq F \setminus S} (-1)^{|F \setminus (H \cup S)|} C_{U \| H \cup S}^{F \setminus (H \cup S)}. \tag{5.62}$$

It is obviously enough to prove this identity for $U = S = \varnothing$. Setting $G := F \setminus S$, we need to prove that, for any finite subset $G \subset \mathbb{N}$,

$$(-1)^{|G|} C_G = \sum_{H \subset G} (-1)^{|G \setminus H|} C_{\|H}^{G \setminus H}. \tag{5.63}$$

Using the inclusion-exclusion identity (5.20) in form of $C_{\|H}^{G \setminus H} = \sum_{S \subset G \setminus H} (-1)^{|S|+1} C_{S\|H}$, we have

$$\sum_{H \subset G} (-1)^{|G \setminus H|} C_{\|H}^{G \setminus H} = \sum_{H \subset G} (-1)^{|G \setminus H|} \sum_{S \subset G \setminus H} (-1)^{|S|+1} C_{S\|H} = \sum_{S \subset G} (-1)^{|S|+1} \sum_{H \subset G \setminus S} (-1)^{|G \setminus H|} C_{S\|H}.$$

Using then (5.21) in form of $C_{S\|H} = \mathbb{1}_{S \neq \varnothing} \sum_{U \subset H} (-1)^{|U|} C_{S \cup U}$, this turns into

$$\begin{aligned}\sum_{H \subset G} (-1)^{|G \setminus H|} C_{\|H}^{G \setminus H} &= \sum_{\substack{S \subset G \\ S \neq \varnothing}} (-1)^{|S|+1} \sum_{H \subset G \setminus S} (-1)^{|G \setminus H|} \sum_{U \subset H} (-1)^{|U|} C_{S \cup U} \\ &= \sum_{\substack{S \subset G \\ S \neq \varnothing}} (-1)^{|S|+1} \sum_{U \subset G \setminus S} (-1)^{|U|} C_{S \cup U} \sum_{H \subset G \setminus (S \cup U)} (-1)^{|G \setminus (H \cup U)|}.\end{aligned}$$

Using twice the binomial identity in the form $\sum_{J\subset K}(-1)^{|K\setminus J|}=\mathbb{1}_{K=\varnothing}$, this reduces to

$$\begin{aligned}\sum_{H\subset G}(-1)^{|G\setminus H|}C^{G\setminus H}_{\|H}&=\sum_{\substack{S\subset G\\ S\neq\varnothing}}(-1)^{|S|+1}\sum_{U\subset G\setminus S}(-1)^{|U|+|S|}C_{S\cup U}\mathbb{1}_{U=G\setminus S}\\&=(-1)^{|G|}C_G\sum_{\substack{S\subset G\\ S\neq\varnothing}}(-1)^{|S|+1}=(-1)^{|G|}C_G,\end{aligned}$$

and identity (5.63) is proven.

Step 3. Proof of (5.61).

Let $F,G\subset\mathbb{N}$ be two fixed disjoint finite subsets, with $F\cup G\neq\varnothing$. Equation (5.61) (with $H=\varnothing$) is obviously a direct corollary of (5.60) (with F replaced by $F\cup G$ and with $H=\varnothing$) provided we prove the identity

$$-C^F\nabla\delta_\xi^{F\cup G}\phi_T+A_{F,G}=B_{F,G},\tag{5.64}$$

where we have defined

$$\begin{aligned}A_{F,G}&:=\sum_{S\subset F\cup G}(-1)^{|S|+1}C_S\nabla\delta_\xi^{(F\cup G)\setminus S}\phi_T^S\\&=\sum_{S\subset F}\sum_{U\subset G}(-1)^{|S|+|U|+1}C_{S\cup U}\nabla\delta_\xi^{(F\setminus S)\cup(G\setminus U)}\phi_T^{S\cup U},\\B_{F,G}&:=\sum_{S\subset F}\sum_{U\subset G}(-1)^{|S|+|U|+1}C_{S\cup U\|F\setminus S}\nabla\delta_\xi^{(F\setminus S)\cup(G\setminus U)}\phi_T^U.\end{aligned}$$

Let us first rewrite $A_{F,G}$ in a more suitable way. We appeal to the definition (5.14) of $\nabla\delta_\xi^{(F\setminus S)\cup(G\setminus U)}\phi_T^{S\cup U}$, then make the change of variables $H\cup S\rightsquigarrow H$ and $U\cup W\rightsquigarrow W$, and conclude by using (5.19),

$$\begin{aligned}A_{F,G}&=\sum_{S\subset F}\sum_{U\subset G}(-1)^{|S|+|U|+1}C_{S\cup U}\sum_{H\subset F\setminus S}\sum_{W\subset G\setminus U}(-1)^{|G\setminus(U\cup W)|}(-1)^{|F\setminus(H\cup S)|}(\nabla\phi_T^{S\cup U\cup H\cup W}+\xi)\\&=\sum_{H\subset F}\sum_{W\subset G}(-1)^{|G\setminus W|}(-1)^{|F\setminus H|}\sum_{S\subset H}\sum_{U\subset W}(-1)^{|S|+|U|+1}C_{S\cup U}(\nabla\phi_T^{H\cup W}+\xi)\\&\overset{(5.19)}{=}\sum_{H\subset F}\sum_{W\subset G}(-1)^{|G\setminus W|+|F\setminus H|}C^{H\cup W}(\nabla\phi_T^{H\cup W}+\xi).\end{aligned}\tag{5.65}$$

We now treat $B_{F,G}$. The change of variables $F\setminus S\rightsquigarrow S$ yields

$$\begin{aligned}B_{F,G}&=\sum_{U\subset G}(-1)^{|U|+1}\sum_{S\subset F}(-1)^{|F\setminus S|}C_{(F\setminus S)\cup U\|S}\nabla\delta_\xi^{S\cup(G\setminus U)}\phi_T^U\\&=\underbrace{\sum_{U\subset G}(-1)^{|U|+1}\sum_{S\subsetneq F}(-1)^{|F\setminus S|}C_{(F\setminus S)\cup U\|S}\nabla\delta_\xi^{S\cup(G\setminus U)}\phi_T^U}_{=:B^1_{F,G}}+\underbrace{\sum_{U\subset G}(-1)^{|U|+1}C_{U\|F}\nabla\delta_\xi^{F\cup(G\setminus U)}\phi_T^U}_{=:B^2_{F,G}}.\end{aligned}\tag{5.66}$$

We treat both terms $B^1_{F,G}$ and $B^2_{F,G}$ separately. The combinatorial identity (5.62) and the change of variables $H\cup S\rightsquigarrow H$ yield

$$\begin{aligned}B^1_{F,G}&=\sum_{U\subset G}(-1)^{|U|+1}\sum_{S\subsetneq F}\sum_{H\subsetneq F\setminus S}(-1)^{|F\setminus(H\cup S)|}C^{F\setminus(H\cup S)}_{U\|H\cup S}\nabla\delta_\xi^{S\cup(G\setminus U)}\phi_T^U\\&=\sum_{U\subset G}(-1)^{|U|+1}\sum_{H\subsetneq F}(-1)^{|F\setminus H|}C^{F\setminus H}_{U\|H}\sum_{S\subset H}\nabla\delta_\xi^{S\cup(G\setminus U)}\phi_T^U.\end{aligned}$$

By the identity (5.15) in the form $\sum_{S\subset H}\nabla\delta_\xi^{S\cup(G\setminus U)}\phi_T^U=\nabla\delta_\xi^{G\setminus U}\phi_T^{H\cup U}$, this turns into

$$B^1_{F,G}=\sum_{U\subset G}(-1)^{|U|+1}\sum_{H\subsetneq F}(-1)^{|F\setminus H|}C^{F\setminus H}_{U\|H}\nabla\delta_\xi^{G\setminus U}\phi_T^{H\cup U},$$

and thus, by definition (5.11)–(5.13) of $\delta_\xi^{G\setminus U}$ and the change of variables $U\cup W\rightsquigarrow W$,

$$\begin{aligned}B^1_{F,G}&=\sum_{U\subset G}(-1)^{|U|+1}\sum_{H\subsetneq F}(-1)^{|F\setminus H|}\sum_{W\subset G\setminus U}(-1)^{|G\setminus(U\cup W)}C^{F\setminus H}_{U\|H}(\nabla\phi_T^{H\cup U\cup W}+\xi)\\&=\sum_{H\subsetneq F}(-1)^{|F\setminus H|}\sum_{W\subset G}(-1)^{|G\setminus W|}\sum_{U\subset W}(-1)^{|U|+1}C^{F\setminus H}_{U\|H}(\nabla\phi_T^{H\cup W}+\xi).\end{aligned}$$

Noting that, by the usual inclusion-exclusion formula,

$$\sum_{U\subset W}(-1)^{|U|+1}C^{F\setminus H}_{U\|H}=-C^{F\setminus H}_{\|H}+C^{F\setminus H}_{\|H}\sum_{\substack{U\subset W\\U\neq\varnothing}}(-1)^{|U|+1}\mathbb{1}_{J_U}=-C^{F\setminus H}_{\|H}+C^{F\setminus H}_{\|H}\mathbb{1}_{J^W}=-C^{F\setminus H}_{\|H\cup W},$$

we conclude that

$$B^1_{F,G}=-\sum_{H\subset F}(-1)^{|F\setminus H|}\sum_{W\subset G}(-1)^{|G\setminus W|}C^{F\setminus H}_{\|H\cup W}(\nabla\phi_T^{H\cup W}+\xi).\tag{5.67}$$

For the second term $B^2_{F,G}$ in (5.66), we argue as in Step 1, and obtain

$$\begin{aligned}B^2_{F,G}&=\sum_{H\subset F}(-1)^{|F\setminus H|}\sum_{\substack{U\subset G\\U\neq\varnothing}}(-1)^{|U|+1}C_{U\|F}\sum_{W\subset G\setminus U}(-1)^{|G\setminus(U\cup W)|}(\nabla\phi_T^{U\cup W\cup H}+\xi)\\&=\sum_{H\subset F}(-1)^{|F\setminus H|}\sum_{W\subset G}(-1)^{|G\setminus W|}\sum_{\substack{U\subset W\\U\neq\varnothing}}(-1)^{|U|+1}C_{U\|F}(\nabla\phi_T^{W\cup H}+\xi)\\&=\sum_{H\subset F}\sum_{W\subset G}(-1)^{|F\setminus H|+|G\setminus W|}C^W_{\|F}(\nabla\phi_T^{W\cup H}+\xi).\end{aligned}\tag{5.68}$$

Combining (5.65), (5.66), (5.67) and (5.68) then yields

$$A_{F,G}-B_{F,G}=\sum_{H\subset F}\sum_{W\subset G}(-1)^{|G\setminus W|+|F\setminus H|}\Big(C^{H\cup W}+C^{F\setminus H}_{\|H\cup W}-C^W_{\|F}\Big)(\nabla\phi_T^{H\cup W}+\xi),$$

which proves (5.64) by definition (5.11)–(5.13) of $\delta_\xi^{F\cup G}\phi_T$. ☐

5.2.2 Basic energy estimates

The advantage of the massive term approximations ϕ_T^F is to localize the dependence with respect to the coefficients to a ball of radius $\sqrt{T}$ (up to exponentially small corrections). While this regularization in T allows us to get rid of convergence issues at infinity, convergence problems may also occur at short distances because of high concentrations of the point process ρ. In order to avoid such issues, we further assume that ρ has all its moments finite. The assumption $T<\infty$ and the finite moments assumption are crucial to make rigorous all subsequent formal computations. Under these assumptions we shall prove some basic energy estimates that are uniform with respect to the regularization parameter T and to the moments bounds on ρ; these estimates will be substantially improved in next section.

Lemma 5.2.2. *Assume that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \geq 1$. Then, for all $L \simeq 1$, $k \geq 0$, $H \subset \mathbb{N}$, $s \geq 1$, and $T > 0$, the following estimate holds*

$$\mathbb{E}\bigg[\bigg(\sum_{|F|=k} T^{-\frac{1}{2}}(\delta_\xi^F \phi_T^H)_L(0) + (\nabla \delta_\xi^F \phi_T^H)_L(0)\bigg)^s\bigg] \leq C_T^{sk}\, \mathbb{E}[\rho(B_R)^{sk}] < \infty,$$

for some constant $C_T \simeq_T 1$, where $\delta_\xi^F \phi_T^H$ is as in (5.13)*, and where we write $(f)_L(x) := (\fint_{B_L(x)} |f|^2)^{\frac{1}{2}}$ for the local quadratic average of any map f.* $\Diamond$

Proof. Since our argument is deterministic (we take the expectation only at the very end), we can assume w.l.o.g. that $H = \varnothing$. By (5.59) in Lemma 5.2.1, $\delta_\xi^F \phi_T$ satisfies

$$\frac{1}{T}\delta_\xi^F \phi_T - \nabla \cdot A^F \nabla \delta_\xi^F \phi_T = \sum_{S \subset F} (-1)^{|S|+1} \nabla \cdot C_{S\|F\setminus S} \nabla \delta_\xi^{F\setminus S} \phi_T.$$

Let $z \in \mathbb{R}^d$ and set $\eta_T^z(x) := e^{-c|x-z|/\sqrt{T}}$ with $c > 0$ to be chosen later. Testing this equation with $\eta_T^z \delta_\xi^F \phi_T$ in the whole space, and noting that $|\nabla \eta_T^z| = c\eta_T^z/\sqrt{T}$, we obtain the starting point for Caccioppoli's inequality

$$\begin{aligned}\frac{1}{T}\int_{\mathbb{R}^d} \eta_T^z |\delta_\xi^F \phi_T|^2 + \int_{\mathbb{R}^d} \eta_T^z |\nabla \delta_\xi^F \phi_T|^2 \lesssim &\sum_{S\subset F} \int_{J_{S\|F\setminus S}} \eta_T^z |\nabla \delta_\xi^F \phi_T|\, |\nabla \delta_\xi^{F\setminus S} \phi_T| \\ &+ \frac{c}{\sqrt{T}} \sum_{S\subset F} \int_{J_{S\|F\setminus S}} \eta_T^z |\delta_\xi^F \phi_T|\, |\nabla \delta_\xi^{F\setminus S} \phi_T| + \frac{c}{\sqrt{T}} \int_{\mathbb{R}^d} \eta_T^z |\delta_\xi^F \phi_T|\, |\nabla \delta_\xi^F \phi_T|.\end{aligned}$$

It is crucial to note here that, for fixed F, the sets $J_{S\|F\setminus S}$, $S \subset F$, are all disjoint. By Young's inequality, and choosing $c > 0$ small enough so that one may absorb all the terms but two in the left-hand side, this turns into

$$\frac{1}{T}\int_{\mathbb{R}^d} \eta_T^z |\delta_\xi^F \phi_T|^2 + \int_{\mathbb{R}^d} \eta_T^z |\nabla \delta_\xi^F \phi_T|^2 \lesssim \sum_{S\subset F} \int_{J_{S\|F\setminus S}} \eta_T^z |\nabla \delta_\xi^{F\setminus S} \phi_T|^2.$$

For $L \simeq 1$, taking the square root of both sides yields

$$\begin{aligned}e^{-cL/\sqrt{T}} \Big(T^{-\frac{1}{2}}(\delta_\xi^F \phi_T)_L(z) + (\nabla \delta_\xi^F \phi_T)_L(z)\Big) &\lesssim \bigg(\sum_{S\subset F} \int_{J_{S\|F\setminus S}} \eta_T^z |\nabla \delta_\xi^{F\setminus S} \phi_T|^2\bigg)^{\frac{1}{2}} \\ &\leq \sum_{S\subset F} \bigg(\int_{J_{S\|F\setminus S}} \eta_T^z |\nabla \delta_\xi^{F\setminus S} \phi_T|^2\bigg)^{\frac{1}{2}}. \qquad (5.69)\end{aligned}$$

Now note that, relabeling the sum in terms of $F \setminus S$, and using that $J_{S\|F\setminus S} = \varnothing$ whenever $S = \varnothing$, we get

$$\sum_{|F|=k} \sum_{S\subset F} \bigg(\int_{J_{S\|F\setminus S}} \eta_T^z |\nabla \delta_\xi^{F\setminus S} \phi_T|^2\bigg)^{\frac{1}{2}} \leq \sum_{|F|\leq k-1} \sum_{|S|\leq k} \bigg(\int_{J_{S\|F}} \eta_T^z |\nabla \delta_\xi^F \phi_T|^2\bigg)^{\frac{1}{2}},$$

and hence, as $J_{S\|F} \subset J_n \subset B_R(q_n)$ for any $n \in S$,

$$\sum_{|F|=k} \sum_{S\subset F} \bigg(\int_{J_{S\|F\setminus S}} \eta_T^z |\nabla \delta_\xi^{F\setminus S} \phi_T|^2\bigg)^{\frac{1}{2}} \leq \sum_{|F|\leq k-1} \sum_{n} \bigg(\int_{B_R(q_n)} \eta_T^z |\nabla \delta_\xi^F \phi_T|^2\bigg)^{\frac{1}{2}} \sum_{\substack{|S|\leq k\\ n\in S}} \mathbb{1}_{J_{S\|F}\neq\varnothing}. \qquad (5.70)$$

We bound the last sum as follows: for any fixed n, recalling that by assumption (5.4) the intersections of the inclusions J_n's are of degree at most $\Gamma \simeq 1$,

$$\sum_{\substack{|S|\le k\\ n\in S}} \mathbb{1}_{J_{S\|F}\neq\varnothing} \le \sum_{\substack{|S|\le k\\ n\in S}} \mathbb{1}_{J_S\neq\varnothing} \le \sum_{j=1}^{k} \binom{\Gamma-1}{j-1} \le 2^{\Gamma-1} \lesssim 1. \tag{5.71}$$

As we have $\eta_T^z(x) \le e^{-c|q_n-z|/\sqrt{T}} e^{cR/\sqrt{T}}$ for all $x \in B_R(q_n)$, we can then deduce from (5.70) and (5.71),

$$\sum_{|F|=k} \sum_{S\subset F} \bigg(\int_{J_{S\|F\setminus S}} \eta_T^z |\nabla \delta_\xi^{F\setminus S}\phi_T|^2\bigg)^{\frac12} \lesssim e^{cR/\sqrt{T}} \sum_{|F|\le k-1} \sum_n e^{-c|q_n-z|/\sqrt{T}} (\nabla\delta_\xi^F \phi_T)_R(q_n).$$

We then sum (5.69) over $|F| = k$, $k \ge 1$, and use the above estimate to get for any $z \in \mathbb{R}^d$,

$$\begin{aligned} S_T^k(z) &:= T^{-\frac12} \sum_{|F|=k} (\delta_\xi^F \phi_T)_L(z) + \sum_{|F|=k} (\nabla\delta_\xi^F \phi_T)_L(z) \\ &\lesssim e^{c(L+R)/\sqrt{T}} \sum_n e^{-c|q_n-z|/\sqrt{T}} \sum_{|F|\le k-1} (\nabla\delta_\xi^F\phi_T)_R(q_n). \end{aligned}$$

Combining this with (5.8), we conclude by induction that, for some (deterministic) constant $C_T \simeq_T 1$,

$$S_T^k(z) \lesssim C_T^k \sum_{j=1}^{k} \underbrace{\sum_{n_1,\ldots,n_j} e^{-c|q_{n_1}-z|/\sqrt{T}} \prod_{i=2}^{j} e^{-c|q_{n_{i-1}}-q_{n_i}|/\sqrt{T}}}_{=:\, I_T^j(z)}.$$

It only remains to compute the sum $I_T^j(z)$. For that purpose, we compare sums to integrals

$$I_T^j(z) \le e^{cjR/\sqrt{T}} \sum_{n_1,\ldots,n_j} \int_{B_R(q_{n_1})} \cdots \int_{B_R(q_{n_j})} e^{-c|x_1-z|/\sqrt{T}} \prod_{i=2}^{j} e^{-c|x_{i-1}-x_i|/\sqrt{T}} dx_j \ldots dx_1,$$

and hence,

$$I_T^j(z) \le e^{cjR/\sqrt{T}} \int_{(\mathbb{R}^d)^j} e^{-c|x_1-z|/\sqrt{T}} \rho(B_R(x_1)) \prod_{i=2}^{j} \Big(e^{-c|x_{i-1}-x_i|/\sqrt{T}} \rho(B_R(x_i))\Big) dx_1 \ldots dx_j.$$

Taking expectation of $I_T^j(z)^s$, for some $s \ge 1$, and applying the triangle and the Hölder inequalities, we obtain

$$\mathbb{E}[I_T^j(z)^s]^{1/s} \le e^{cjR/\sqrt{T}} \mathbb{E}[\rho(B_R)^{sj}]^{1/s} \int_{(\mathbb{R}^d)^j} e^{-c|x_1-z|/\sqrt{T}} \prod_{i=2}^{j} e^{-c|x_{i-1}-x_i|/\sqrt{T}} dx_1 \ldots dx_j,$$

which finally gives, by an obvious change of variables,

$$\mathbb{E}[I_T^j(z)^s]^{\frac1s} \le e^{cjR/\sqrt{T}} \mathbb{E}[\rho(B_R)^{sj}]^{\frac1s} \bigg(\int_{\mathbb{R}^d} e^{-c|x|/\sqrt{T}} dx\bigg)^j = C^j T^{jd/2} e^{cjR/\sqrt{T}} \mathbb{E}[\rho(B_R)^{sj}]^{\frac1s},$$

and the announced result is then proved. $\square$

Based on this deterministic estimate, we prove a lemma which will be crucial to give sense to formal calculations, and implies the absolute convergence of all the series we will be considering for fixed T, at least under the additional assumption that moments of ρ are finite. Note that the result obviously also holds for $\nabla\phi_T^F$ replaced e.g. by $\nabla\phi_T^{E^{(p)}\cup F}$.

Lemma 5.2.3. *Assume that $\mathbb{E}[\rho(Q)^s]<\infty$ for all $s\ge 1$. For all $T>0$, and $k\ge 1$, we have*

$$S_T^k := \sum_{|F|=k}\sum_{G\subset F}\mathbb{E}\left[|C_{F\setminus G}|\,|\nabla\delta_\xi^G\phi_T|\,(1+|\nabla\phi_T^F|)\right] < \infty, \tag{5.72}$$

$$\sum_{|F|=k}\sum_{G\subset F}\mathbb{E}\Big[|C_G|\,|\nabla\delta_\xi^{F\setminus G}\phi_T^G|\,(1+|\nabla\phi_T^F|)\Big] < \infty, \tag{5.73}$$

$$\sum_{|F|=k}\mathbb{E}\left[|C_F|\,(1+|\nabla\phi_T|)(1+|\nabla\phi_T^F|)\right] < \infty. \tag{5.74}$$

$\Diamond$

Proof. We only prove (5.72); the proofs of the other statements are similar. Let $k\ge 0$ be fixed. By stationarity we add a local average over the ball B_L, say, with $L\simeq 1$, we apply the Cauchy-Schwarz inequality, and note that, for all $x\in B_L$,

$$|C_H(x)|\lesssim \mathbb{1}_{x\in J_n,\forall n\in H}\le \mathbb{1}_{x\in B_R(q_n),\forall n\in H}=\mathbb{1}_{q_n\in B_R(x),\forall n\in H}\le \mathbb{1}_{q_n\in B_{R+L},\forall n\in H}=:\chi_L(H),$$

so that we can write by the change of variables $F\rightsquigarrow F\cup G$,

$$\begin{aligned}S_T^k &\lesssim \sum_{|F|=k}\sum_{G\subset F}\mathbb{E}\Big[\chi_L(F\setminus G)(\nabla\delta_\xi^G\phi_T)_L(1+(\nabla\phi_T^F)_L)\Big]\\ &\le \sum_{|F|\le k}\sum_{|G|\le k}\mathbb{E}\Big[\chi_L(F)(\nabla\delta_\xi^G\phi_T)_L(1+(\nabla\phi_T^{F\cup G})_L)\Big].\end{aligned}$$

Using the deterministic estimate (5.8) in the form of $(\nabla\phi_T^{F\cup G})_L\lesssim T^{\frac d2}$, this yields

$$S_T^k\lesssim T^{\frac d2}\mathbb{E}\bigg[\Big(\sum_{|F|\le k}\chi_L(F)\Big)\Big(\sum_{|G|\le k}(\nabla\delta_\xi^G\phi_T)_L\Big)\bigg].$$

The first sum can be estimated as follows: since $\binom{n}{i}\le\frac{1}{i!}n^i\le(en/i)^i$ and $\sum_{i=1}^\infty(e/i)^i\lesssim 1$,

$$\sum_{|F|\le k}\chi_L(F)=\sum_{|F|\le k}\mathbb{1}_{q_n\in B_{R+L},\forall n\in F}\le\sum_{i=0}^k\binom{\rho(B_{R+L})}{i}\lesssim\rho(B_{R+L})^k,$$

so that we conclude by Lemma 5.2.2 and the Cauchy-Schwarz inequality that

$$S_T^k\lesssim T^{d/2}\mathbb{E}[\rho(B_{R+L})^{2k}]^{1/2}\mathbb{E}\bigg[\Big(\sum_{|G|\le k}(\nabla\delta_\xi^G\phi_T)_L\Big)^2\bigg]^{1/2}<\infty. \qquad\square$$

We now turn to energy estimates that hold uniformly with respect to T and the moments bounds on ρ. The following two estimates will be further improved in the next section.

Lemma 5.2.4. *Assume that $\mathbb{E}[\rho(Q)^s]<\infty$ for all $s\ge 1$. Then, there exists a constant $C\simeq 1$ (independent of T and of the moments of ρ) such that, for all $k\ge 0$ and $T>0$,*

$$\mathbb{E}\bigg[\sum_{|F|=k}|\nabla\delta_\xi^F\phi_T|^2\bigg]\le C^{k+1}. \qquad\Diamond$$

Proof. For $k=0$, the result $\mathbb{E}[|\nabla\phi_T+\xi|^2]\lesssim 1$ reduces to the basic energy estimate (5.8) on the modified corrector. We now argue by induction. Assume that the result holds for some fixed $k\geq 0$. From (5.59) in Lemma 5.2.1, we learn that $\delta_\xi^F\phi_T$ satisfies on $\mathbb{R}^d$,

$$\frac{1}{T}\delta_\xi^F\phi_T-\nabla\cdot A^F\nabla\delta_\xi^F\phi_T=\sum_{S\subset F}(-1)^{|S|+1}\nabla\cdot C_{S\|F\setminus S}\nabla\delta_\xi^{F\setminus S}\phi_T.$$

We test this equation with $\chi_N\delta_\xi^F\phi_T$, where χ_N is a cut-off function for B_N in B_{2N} such that $|\nabla\chi_N|\lesssim 1/N$. This yields

$$\int_{B_N}|\nabla\delta_\xi^F\phi_T|^2\lesssim\sum_{S\subset F}\int_{B_{2N}}\mathbb{1}_{J_{S\|F\setminus S}}|\nabla\delta_\xi^F\phi_T|\,|\nabla\delta_\xi^{F\setminus S}\phi_T| \\ +\frac{1}{N}\int_{B_{2N}}|\delta_\xi^F\phi_T|\,|\nabla\delta_\xi^F\phi_T|+\frac{1}{N}\sum_{S\subset F}\int_{B_{2N}}\mathbb{1}_{J_{S\|F\setminus S}}|\delta_\xi^F\phi_T|\,|\nabla\delta_\xi^{F\setminus S}\phi_T|. \tag{5.75}$$

We then take the expectation, sum over $|F|=k-1$ (all the sums are convergent by Lemmas 5.2.2 and 5.2.3), and divide by N^d:

$$\mathbb{E}\Big[\fint_{B_N}\sum_{|F|=k+1}|\nabla\delta_\xi^F\phi_T|^2\Big]\lesssim\mathbb{E}\Big[\fint_{B_{2N}}\sum_{|F|=k+1}\sum_{S\subset F}\mathbb{1}_{J_{S\|F\setminus S}}|\nabla\delta_\xi^F\phi_T|\,|\nabla\delta_\xi^{F\setminus S}\phi_T|\Big] \\ +\frac{1}{N}\mathbb{E}\Big[\fint_{B_{2N}}\sum_{|F|=k+1}|\delta_\xi^F\phi_T|\,|\nabla\delta_\xi^F\phi_T|\Big]+\frac{1}{N}\mathbb{E}\Big[\fint_{B_{2N}}\sum_{|F|=k+1}\sum_{S\subset F}\mathbb{1}_{J_{S\|F\setminus S}}|\delta_\xi^F\phi_T|\,|\nabla\delta_\xi^{F\setminus S}\phi_T|\Big].$$

Since each sum above is absolutely convergent and defines an integrable stationary random field (the expectation of which obviously does not depend on the point it is taken), this inequality also takes the form

$$\mathbb{E}\Big[\sum_{|F|=k+1}|\nabla\delta_\xi^F\phi_T|^2\Big]\lesssim\mathbb{E}\Big[\sum_{|F|=k+1}\sum_{S\subset F}\mathbb{1}_{J_{S\|F\setminus S}}|\nabla\delta_\xi^F\phi_T|\,|\nabla\delta_\xi^{F\setminus S}\phi_T|\Big] \\ +\frac{1}{N}\mathbb{E}\Big[\sum_{|F|=k+1}|\delta_\xi^F\phi_T|\,|\nabla\delta_\xi^F\phi_T|\Big]+\frac{1}{N}\mathbb{E}\Big[\sum_{|F|=k+1}\sum_{S\subset F}\mathbb{1}_{J_{S\|F\setminus S}}|\delta_\xi^F\phi_T|\,|\nabla\delta_\xi^{F\setminus S}\phi_T|\Big].$$

Taking the limit $N\uparrow\infty$ then yields

$$\mathbb{E}\Big[\sum_{|F|=k+1}|\nabla\delta_\xi^F\phi_T|^2\Big]\lesssim\mathbb{E}\Big[\sum_{|F|=k+1}\sum_{S\subset F}\mathbb{1}_{J_{S\|F\setminus S}}|\nabla\delta_\xi^F\phi_T|\,|\nabla\delta_\xi^{F\setminus S}\phi_T|\Big]. \tag{5.76}$$

By Young's inequality and the disjointness of the sets $J_{S\|F\setminus S}$, $S\subset F$ (for fixed F), in the form of $\sum_{S\subset F}\mathbb{1}_{J_{S\|F\setminus S}}\leq 1$, we may absorb part of the right-hand side into the left-hand side, and obtain

$$\mathbb{E}\Big[\sum_{|F|=k+1}|\nabla\delta_\xi^F\phi_T|^2\Big]\lesssim\mathbb{E}\Big[\sum_{|F|=k+1}\sum_{S\subset F}\mathbb{1}_{J_{S\|F\setminus S}}|\nabla\delta_\xi^{F\setminus S}\phi_T|^2\Big]\leq\mathbb{E}\Big[\sum_{|F|\leq k}|\nabla\delta_\xi^F\phi_T|^2\sum_{|S|\leq k+1}\mathbb{1}_{J_{S\|F}}\Big], \tag{5.77}$$

where we used that $J_{\varnothing\|F}=\varnothing$. By assumption (5.4), proceeding as for (5.71), we have

$$\sum_{|S|\leq k+1}\mathbb{1}_{J_{S\|F}}(0)\leq\sum_{|S|\leq k+1}\mathbb{1}_{J_S}(0)\leq\sum_{j=0}^{k+1}\binom{\Gamma}{j}\leq 2^\Gamma\lesssim 1, \tag{5.78}$$

so that (5.77) finally turns into

$$\mathbb{E}\Big[\sum_{|F|=k+1}|\nabla\delta_\xi^F\phi_T|^2\Big]\lesssim\mathbb{E}\Big[\sum_{|F|\le k}|\nabla\delta_\xi^F\phi_T|^2\Big],$$

from which the desired conclusion follows by the induction assumption. ☐

For sums $\sum_{|F|=k}$ of size $k=1$, the following result is easily proven as an energy estimate in the probability space; for general k it also holds but the proof relies on a subtle induction and combinatorial argument, which is presented in the next section.

Lemma 5.2.5. *Assume that $\mathbb{E}[\rho(Q)^s]<\infty$ for all $s\ge 1$. Then, for all $T>0$, we have (uniformly in T and in the moments of ρ)*

$$\mathbb{E}\Big[\Big|\sum_n\nabla\delta_\xi^{\{n\}}\phi_T\Big|^2\Big]\lesssim 1.$$

◊

Proof. By Lemma 5.2.2 for $k=1$, the sum $\sum_n\delta_\xi^{\{n\}}\phi_T$ is well-defined in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and satisfies the following equation on $\mathbb{R}^d$,

$$\frac{1}{T}\sum_n\delta_\xi^{\{n\}}\phi_T-\nabla\cdot A\nabla\sum_n\delta_\xi^{\{n\}}\phi_T=\nabla\cdot\sum_n C^{\{n\}}(\nabla\phi_T^{\{n\}}+\xi).$$

We then test this equation with $\chi_N(\sum_n\delta_\xi^{\{n\}}\phi_T)$ for some cut-off χ_N for B_N in B_{2N} such that $|\nabla\chi_N|\lesssim 1/N$. Since $\sum_n\nabla\delta_\xi^{\{n\}}\phi_T$ is stationary, we may proceed as for the proof of (5.76) in Lemma 5.2.4, and obtain after taking the expectation and the limit $N\uparrow\infty$ (or equivalently testing the equation in probability),

$$\mathbb{E}\Big[\Big|\sum_n\nabla\delta_\xi^{\{n\}}\phi_T\Big|^2\Big]\lesssim\mathbb{E}\Big[\Big|\sum_n C^{\{n\}}(\nabla\phi_T^{\{n\}}+\xi)\Big|^2\Big]\lesssim\mathbb{E}\Big[\Big(\sum_n\mathbb{1}_{J_n}\Big)\Big(\sum_n\mathbb{1}_{J_n}(1+|\nabla\phi_T^{\{n\}}|^2)\Big)\Big].$$

By assumption (5.4), proceeding as for (5.71), we have $\sum_n\mathbb{1}_{J_n}(0)\lesssim 1$, so that, using in addition the decomposition $1+|\nabla\phi_T^{\{n\}}|^2\lesssim(1+|\nabla\phi_T|^2)+|\nabla\delta_\xi^{\{n\}}\phi_T|^2$, we obtain

$$\mathbb{E}\Big[\Big|\sum_n\nabla\delta_\xi^{\{n\}}\phi_T\Big|^2\Big]\lesssim\mathbb{E}[1+|\nabla\phi_T|^2]+\mathbb{E}\Big[\sum_n|\nabla\delta_\xi^{\{n\}}\phi_T|^2\Big]\lesssim 1,$$

where the last inequality follows from Lemma 5.2.4 with $k=1$. ☐

5.2.3 Improved energy estimates

In this section, we prove the following generalization of Lemma 5.2.5 to any order k in the following form (choosing $j=k$ in (5.79) below): there is a constant $C\simeq 1$ such that, for all $k\ge 1$ and $T>0$,

$$\mathbb{E}\Big[\Big|\sum_{|F|=k}\nabla\delta^F\phi_T\Big|^2\Big]\le C^{k+1},$$

and we give an interpolation result between this inequality and the energy estimates of Lemma 5.2.4.

Proposition 5.2.6. *Assume that $\mathbb{E}[\rho(Q)^s]<\infty$ for all $s\ge 1$. Then, there exists a constant $C\simeq 1$ (independent of T and of the moments of ρ) such that, for all $T>0$, $k\ge 0$, and $0\le j\le k$,*

$$S_j^k:=\mathbb{E}\Big[\sum_{|G|=k-j}\Big|\sum_{\substack{|F|=j\\ F\cap G=\varnothing}}\nabla\delta^{F\cup G}\phi_T\Big|^2\Big]\le C^{k+1}.\qquad(5.79)$$

◊

Proof. We proceed by induction and split the proof into two steps.

Step 1. Preliminary.

For $k = j = 0$, the estimate $S_0^0 = \mathbb{E}[|\nabla\phi_T + \xi|^2] \lesssim 1$ reduces to the energy estimate for the modified corrector. (Note also that the estimate $S_1^1 \lesssim 1$ already follows from Lemma 5.2.5 — but this will not be used here.) We now argue by a double induction argument. Since the result is proven for $k = 0$, we may indeed argue by induction on k: we assume that $S_j^{k'} \le C^{k'+1}$ for all $0 \le j \le k'$ and for all $0 \le k' \le k$, and shall prove that $S_j^{k+1} \le C^{k+2}$ for all $0 \le j \le k+1$. Since Lemma 5.2.4 implies the desired result for $j = 0$, we may as well argue by induction on j: we further assume that $S_{j'}^{k+1} \le C^{k+2}$ for all $0 \le j' \le j$, for some $0 \le j < k+1$, and shall prove that $S_{j+1}^{k+1} \le C^{k+2}$.

Before we turn to Step 2, we state another combinatorial inequality we shall need in the proof. Let $G, S \subset \mathbb{N}$ be finite fixed disjoint subsets. We claim that

$$\Bigg|\sum_{\substack{|F|=k\\ F\cap(G\cup S)=\varnothing}} \nabla\delta_\xi^{F\cup G}\phi_T\Bigg| \le \sum_{l=0}^{|S|}\sum_{\substack{|L|=l\\ L\subset S}}\Bigg|\sum_{\substack{|F|=k-l\\ F\cap(G\cup L)=\varnothing}} \nabla\delta_\xi^{F\cup G\cup L}\phi_T\Bigg|. \tag{5.80}$$

We first rewrite

$$S_{G,S}^k := \sum_{\substack{|F|=k\\ F\cap(G\cup S)=\varnothing}} \nabla\delta_\xi^{F\cup G}\phi_T = \sum_{\substack{|F|=k\\ F\cap G=\varnothing}} \mathbb{1}_{F\cap S=\varnothing}\nabla\delta_\xi^{F\cup G}\phi_T,$$

where we can decompose, by the usual inclusion-exclusion argument,

$$\mathbb{1}_{F\cap S=\varnothing} = 1 - \mathbb{1}_{F\cap S\neq\varnothing} = 1 - \sum_{l=1}^{|S|}(-1)^{l+1}\sum_{\substack{|L|=l\\ L\subset S}}\mathbb{1}_{L\subset F} = \sum_{l=0}^{|S|}(-1)^l\sum_{\substack{|L|=l\\ L\subset S}}\mathbb{1}_{L\subset F},$$

so that $S_{G,S}^k$ becomes, by a change of variables,

$$S_{G,S}^k = \sum_{l=0}^{|S|}(-1)^l\sum_{\substack{|L|=l\\ L\subset S}}\sum_{\substack{|F|=k\\ F\cap G=\varnothing}}\mathbb{1}_{L\subset F}\nabla\delta_\xi^{F\cup G}\phi_T = \sum_{l=0}^{|S|}(-1)^l\sum_{\substack{|L|=l\\ L\subset S}}\sum_{\substack{|F|=k-l\\ F\cap(G\cup L)=\varnothing}}\nabla\delta_\xi^{F\cup G\cup L}\phi_T,$$

and the claim (5.80) then follows from the triangle inequality.

Step 2. Bound on S_{j+1}^{k+1}.

Let $G \subset \mathbb{N}$ be a finite subset. By Lemma 5.2.2 we may sum equation (5.61) of Lemma 5.2.1 (with $H = \varnothing$) for $\delta^{F\cup G}\phi_T$ over $|F| = j+1$, $F \cap G = \varnothing$, which yields the following equation for the sum $\sum_{\substack{|F|=j+1\\ F\cap G=\varnothing}}\delta^{F\cup G}\phi_T$ on $\mathbb{R}^d$:

$$\begin{aligned}
&\frac{1}{T}\sum_{\substack{|F|=j+1\\ F\cap G=\varnothing}}\delta_\xi^{F\cup G}\phi_T - \nabla\cdot A^G\nabla\sum_{\substack{|F|=j+1\\ F\cap G=\varnothing}}\delta_\xi^{F\cup G}\phi_T\\
&\quad= \nabla\cdot\sum_{\substack{|F|=j+1\\ F\cap G=\varnothing}}\sum_{S\subset F}\sum_{U\subset G}(-1)^{|S|+|U|+1}C_{S\cup U\|G\setminus U}\nabla\delta_\xi^{(F\setminus S)\cup(G\setminus U)}\phi_T^S\\
&\quad= \nabla\cdot\sum_{U\subset G}\sum_{\substack{|S|\le j+1\\ S\cap G=\varnothing}}(-1)^{|S|+|U|+1}C_{S\cup U\|G\setminus U}\sum_{\substack{|F|=j+1-|S|\\ F\cap(G\cup S)=\varnothing}}\nabla\delta_\xi^{F\cup(G\setminus U)}\phi_T^S.
\end{aligned}$$

We test this equation with $\chi_N\sum_{|F|=j+1,F\cap G=\varnothing}\delta_\xi^{F\cup G}\phi_T$, where χ_N is a cut-off function for B_N in B_{2N} such that $|\nabla\chi_N| \lesssim 1/N$, we take the sum over $|G| = (k+1)-(j+1) = k-j$ (which is again absolutely converging by Lemma 5.2.2), take the expectation, and then use stationarity to pass to the limit $N\uparrow\infty$, as in the proof of (5.76) in Lemma 5.2.4. This yields

$$S_{j+1}^{k+1}\lesssim\mathbb{E}\Bigg[\sum_{|G|=k-j}\bigg|\sum_{U\subset G}\sum_{\substack{|S|\le(j+1)\wedge\Gamma\\ S\cap G=\varnothing,S\cup U\neq\varnothing}}(-1)^{|S|+|U|+1}C_{S\cup U\|G\setminus U}\sum_{\substack{|F|=j+1-|S|\\ F\cap(G\cup S)=\varnothing}}\nabla\delta_\xi^{F\cup(G\setminus U)}\phi_T^S\bigg|^2\Bigg],$$

where the additional restriction $S\cup U\neq\varnothing$ follows from the fact that $C_{S\cup U\|G\setminus U}$ vanishes identically otherwise and where we have further restricted to $|S|\le\Gamma$ since by assumption (5.4) there is no intersection of degree larger than Γ. Since we have $|C_{S\cup U\|G\setminus U}|\lesssim\mathbb{1}_{J_S}\mathbb{1}_{J_{U\|G\setminus U}}$ (using here notation $\mathbb{1}_{J_\varnothing}=1$), and the $J_{U\|G\setminus U}$'s are disjoint for $U\subset G$ (for fixed G), we deduce

$$S_{j+1}^{k+1}\lesssim\mathbb{E}\Bigg[\sum_{|G|=k-j}\sum_{U\subset G}\mathbb{1}_{J_{U\|G\setminus U}}\bigg(\sum_{\substack{|S|\le(j+1)\wedge\Gamma\\ S\cap G=\varnothing,S\cup U\neq\varnothing}}\mathbb{1}_{J_S}\bigg|\sum_{\substack{|F|=j+1-|S|\\ F\cap(G\cup S)=\varnothing}}\nabla\delta_\xi^{F\cup(G\setminus U)}\phi_T^S\bigg|\bigg)^2\Bigg].$$

As for (5.78), we have $\sum_{|S|\le j+1}\mathbb{1}_{J_S}(0)\lesssim1$, so that by the Cauchy-Schwarz inequality, this estimate turns into

$$S_{j+1}^{k+1}\lesssim\mathbb{E}\Bigg[\sum_{|G|=k-j}\sum_{U\subset G}\mathbb{1}_{J_U}\sum_{\substack{|S|\le(j+1)\wedge\Gamma\\ S\cap G=\varnothing,S\cup U\neq\varnothing}}\mathbb{1}_{J_S}\bigg|\sum_{\substack{|F|=j+1-|S|\\ F\cap(G\cup S)=\varnothing}}\nabla\delta_\xi^{F\cup(G\setminus U)}\phi_T^S\bigg|^2\Bigg].$$

Now using the decomposition $\nabla\delta_\xi^{F\cup(G\setminus U)}\phi_T^S=\sum_{R\subset S}\nabla\delta_\xi^{F\cup R\cup(G\setminus U)}\phi_T$ (that is, (5.14) with $H=\varnothing$, $G\rightsquigarrow F\cup(G\setminus U)$ and $F\rightsquigarrow S$), together with the observation that

$$\mathbb{1}_{J_S}\bigg(\sum_{R\subset S}a_R\bigg)^2\le\mathbb{1}_{J_S}\bigg(\sum_{R\subset S}\mathbb{1}_{J_R}a_R\bigg)^2\lesssim\mathbb{1}_{J_S}\sum_{R\subset S}a_R^2,$$

which follows again from combining the Cauchy-Schwarz inequality with inequality $\sum_{|R|\le j+1}\mathbb{1}_{J_R}\lesssim1$, we obtain

$$\begin{aligned}S_{j+1}^{k+1}&\lesssim\mathbb{E}\Bigg[\sum_{|G|=k-j}\sum_{U\subset G}\mathbb{1}_{J_U}\sum_{\substack{|S|\le(j+1)\wedge\Gamma\\ S\cap G=\varnothing,S\cup U\neq\varnothing}}\mathbb{1}_{J_S}\sum_{R\subset S}\bigg|\sum_{\substack{|F|=j+1-|S|\\ F\cap(G\cup S)=\varnothing}}\nabla\delta_\xi^{F\cup R\cup(G\setminus U)}\phi_T\bigg|^2\Bigg]\\&\le\sum_{i=0}^{(j+1)\wedge\Gamma}\mathbb{E}\Bigg[\sum_{|U|\le k-j}\mathbb{1}_{J_U}\sum_{\substack{|G|\le k-j\\ G\cap U=\varnothing}}\delta_{ijk}^G\sum_{\substack{|S|=i\\ S\cap G=\varnothing}}\mathbb{1}_{J_S}\sum_{R\subset S}\bigg|\sum_{\substack{|F|=j+1-i\\ F\cap(G\cup U\cup S)=\varnothing}}\nabla\delta_\xi^{F\cup R\cup G}\phi_T\bigg|^2\Bigg],\end{aligned}$$

where we have set $\delta_{ijk}^G=0$ when simultaneously $|G|=k-j$ and $i=0$, and $\delta_{ijk}^G=1$ otherwise. By (5.80) and the inequality $\sum_{|L|\le j+1}\mathbb{1}_{J_L}\lesssim1$, for any $R\subset S$ and any $G\cap U=\varnothing=S\cap G$, we have

$$\begin{aligned}\mathbb{1}_{J_S\cap J_U}\bigg|\sum_{\substack{|F|=j+1-i\\ F\cap(G\cup U\cup S)=\varnothing}}\nabla\delta_\xi^{F\cup R\cup G}\phi_T\bigg|^2&\le\bigg(\sum_{l=0}^{j+1-i}\sum_{\substack{|L|=l\\ L\subset U\cup S\setminus R}}\mathbb{1}_{J_L}\bigg|\sum_{\substack{|F|=j+1-i-l\\ F\cap(L\cup R\cup G)=\varnothing}}\nabla\delta_\xi^{F\cup L\cup R\cup G}\phi_T\bigg|\bigg)^2\\&\lesssim\sum_{l=0}^{j+1-i}\sum_{\substack{|L|=l\\ L\subset U\cup S\setminus R}}\mathbb{1}_{J_L}\bigg|\sum_{\substack{|F|=j+1-i-l\\ F\cap(L\cup R\cup G)=\varnothing}}\nabla\delta_\xi^{F\cup L\cup R\cup G}\phi_T\bigg|^2,\end{aligned}\qquad(5.81)$$

and hence we obtain, using $\sum_{|U|\le k}\mathbb{1}_{J_U}\lesssim 1$ again, and using $U\cup(S\setminus R)\subset\mathbb{N}\setminus(G\cup R)$,

$$S_{j+1}^{k+1}\lesssim\sum_{i=0}^{(j+1)\wedge\Gamma}\sum_{l=0}^{j+1-i}\mathbb{E}\bigg[\sum_{|U|\le k-j}\mathbb{1}_{J_U}\sum_{\substack{|G|\le k-j\\ G\cap U=\varnothing}}\delta_{ijk}^{G}$$
$$\sum_{\substack{|S|=i\\ S\cap G=\varnothing}}\mathbb{1}_{J_S}\sum_{R\subset S}\sum_{\substack{|L|=l\\ L\subset U\cup S\setminus R}}\mathbb{1}_{J_L}\bigg|\sum_{\substack{|F|=j+1-i-l\\ F\cap(L\cup R\cup G)=\varnothing}}\nabla\delta_\xi^{F\cup L\cup R\cup G}\phi_T\bigg|^2\bigg]$$
$$\lesssim\sum_{i=0}^{(j+1)\wedge\Gamma}\sum_{l=0}^{j+1-i}\mathbb{E}\bigg[\sum_{|G|\le k-j}\delta_{ijk}^{G}\sum_{\substack{|R|\le i\\ R\cap G=\varnothing}}\mathbb{1}_{J_R}\sum_{\substack{|L|=l\\ L\cap(G\cup R)=\varnothing}}\mathbb{1}_{J_L}\bigg|\sum_{\substack{|F|=j+1-i-l\\ F\cap(L\cup R\cup G)=\varnothing}}\nabla\delta_\xi^{F\cup L\cup R\cup G}\phi_T\bigg|^2\bigg].$$

Successively using $\Gamma\lesssim 1$ in the form of $\sum_{i=0}^{(j+1)\wedge\Gamma}\sum_{|R|\le i}\lesssim\sum_{i=0}^{(j+1)\wedge\Gamma}\sum_{|R|=i}$ and $\sum_L\mathbb{1}_{J_L}\lesssim 1$, we obtain by the change of variables $L\cup R\rightsquigarrow R$,

$$S_{j+1}^{k+1}\lesssim\sum_{i=0}^{(j+1)\wedge\Gamma}\sum_{l=0}^{j+1-i}\mathbb{E}\bigg[\sum_{|R|=i}\mathbb{1}_{J_R}\sum_{\substack{|L|=l\\ L\cap R=\varnothing}}\mathbb{1}_{J_L}\sum_{\substack{|G|\le k-j\\ G\cap(L\cup R)=\varnothing}}\delta_{ijk}^{G}\bigg|\sum_{\substack{|F|=j+1-i-l\\ F\cap(L\cup R\cup G)=\varnothing}}\nabla\delta_\xi^{F\cup L\cup R\cup G}\phi_T\bigg|^2\bigg]$$
$$\lesssim\sum_{i=0}^{(j+1)\wedge\Gamma}\sum_{l=0}^{k-j}\mathbb{E}\bigg[\sum_{|R|=i}\mathbb{1}_{J_R}\sum_{\substack{|G|=l\\ R\cap G=\varnothing}}\delta_{ijk}^{G}\bigg|\sum_{\substack{|F|=j+1-i\\ F\cap(G\cup R)=\varnothing}}\nabla\delta_\xi^{F\cup R\cup G}\phi_T\bigg|^2\bigg],$$

or equivalently, recalling the definition of the δ_{ijk}^{G}'s and of the S_j^k's,

$$S_{j+1}^{k+1}\lesssim\sum_{l=0}^{k-j-1}S_{j+1}^{l+j+1}+\sum_{i=1}^{j+1}\sum_{l=0}^{k-j}\mathbb{E}\bigg[\sum_{|R|=i}\mathbb{1}_{J_R}\sum_{\substack{|G|=l\\ R\cap G=\varnothing}}\bigg|\sum_{\substack{|F|=j+1-i\\ F\cap(G\cup R)=\varnothing}}\nabla\delta_\xi^{F\cup R\cup G}\phi_T\bigg|^2\bigg].\tag{5.82}$$

Using again the fact that $\sum_{|R|=i}\mathbb{1}_{J_R}\lesssim 1$, we can bound

$$\mathbb{E}\bigg[\sum_{|R|=i}\mathbb{1}_{J_R}\sum_{\substack{|G|=l\\ R\cap G=\varnothing}}\bigg|\sum_{\substack{|F|=j+1-i\\ F\cap(G\cup R)=\varnothing}}\nabla\delta_\xi^{F\cup R\cup G}\phi_T\bigg|^2\bigg]=\mathbb{E}\bigg[\sum_{|G|=i+l}\sum_{\substack{R\subset G\\ |R|=i}}\mathbb{1}_{J_R}\bigg|\sum_{\substack{|F|=j+1-i\\ F\cap G=\varnothing}}\nabla\delta_\xi^{F\cup G}\phi_T\bigg|^2\bigg]$$
$$\lesssim\mathbb{E}\bigg[\sum_{|G|=i+l}\bigg|\sum_{\substack{|F|=j+1-i\\ F\cap G=\varnothing}}\nabla\delta_\xi^{F\cup G}\phi_T\bigg|^2\bigg]=S_{j+1-i}^{l+j+1},$$

so that (5.82) turns into

$$S_{j+1}^{k+1}\lesssim\sum_{l=0}^{k-j-1}S_{j+1}^{l+j+1}+\sum_{i=1}^{j+1}\sum_{l=0}^{k-j}S_{j+1-i}^{l+j+1}=\sum_{l=0}^{k}S_{j+1}^{l}+\sum_{i=0}^{j}\sum_{l=j+1}^{k+1}S_i^l.$$

As the right-hand side only involves the $S_{j'}^{k'}$'s with $k'\le k$ or with $k'=k+1$, $j'\le j$, we conclude that $S_{j+1}^{k+1}\le C^{k+2}$ by the induction assumption. □

5.3 Proofs of the main results

In this section, we prove the analyticity of the perturbed coefficients (Theorem 5.1.1) and the analytical formulas for the derivatives (Corollary 5.1.2), from which we further deduce the Clausius-Mossotti formulas (Corollaries 5.1.4 and 5.1.5).

5.3.1 Approximate derivatives at $p = 0$

In this subsection we devise analytical formulas for the derivatives of the map $p \mapsto A_T^{(p)}$ at $p = 0$ under the assumptions that $T > 0$ and $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \geq 1$. We shall show in particular that $A_T^{(p)}$ is C^∞ at $p = 0$. These results, which rely on the improved energy estimates of Proposition 5.3.2, constitute the core of the proof of Theorem 5.1.1.

Fix some direction $\xi \in \mathbb{R}^d$, $|\xi| = 1$. As in Section 5.1.4, we consider the exact and approximate differences

$$\Delta^{(p)} := \xi \cdot (A_{\mathrm{hom}}^{(p)} - A_{\mathrm{hom}})\xi, \qquad \Delta_T^{(p)} := \xi \cdot (A_T^{(p)} - A_T)\xi,$$

and we recall that $\lim_T \Delta_T^{(p)} = \Delta^{(p)}$ follows from (5.10). By Lemma 5.1.9 the approximate difference satisfies

$$\Delta_T^{(p)} = \mathbb{E}[(\nabla\phi_T + \xi) \cdot C^{(p)}(\nabla\phi_T^{(p)} + \xi)]. \tag{5.83}$$

By assumption (5.4), we may now appeal to the inclusion-exclusion formula in the form of (5.18), so that (5.83) turns into

$$\Delta_T^{(p)} = \sum_{j=1}^{\Gamma}(-1)^{j+1} \sum_{|F|=j} \mathbb{E}\left[(\nabla\phi_T + \xi) \cdot C_F(\nabla\phi_T^{E^{(p)}\cup F} + \xi)\mathbb{1}_{F\subset E^{(p)}}\right],$$

where the sum is absolutely convergent by (5.74) in Lemma 5.2.3. Using that the event $[F \subset E^{(p)}]$ is by definition independent of the rest of the summand, and that we have i.i.d. Bernoulli variables of parameter p, this identity takes the form

$$\Delta_T^{(p)} = \sum_{j=1}^{\Gamma}(-1)^{j+1}p^j\, \mathbb{E}\left[\sum_{|F|=j}(\nabla\phi_T + \xi) \cdot C_F(\nabla\phi_T^{E^{(p)}\cup F} + \xi)\right], \tag{5.84}$$

which can be further decomposed as

$$\begin{aligned}\Delta_T^{(p)} = \sum_{j=1}^{\Gamma}(-1)^{j+1}p^j\, \mathbb{E}\left[\sum_{|F|=j}(\nabla\phi_T + \xi) \cdot C_F(\nabla\phi_T^{F} + \xi)\right]& \\ + \sum_{j=1}^{\Gamma}(-1)^{j+1}p^j\, \mathbb{E}\left[\sum_{|F|=j}(\nabla\phi_T + \xi) \cdot C_F\nabla(\phi_T^{E^{(p)}\cup F} - \phi_T^F)\right]&,\end{aligned}$$

where the sums are still absolutely convergent by (5.74) in Lemma 5.2.3. The first term of the first sum (i.e. corresponding to the choice $j = 1$) is of order p and coincides with the argument of the limit in (5.23) for $k = 1$. The second sum can be rewritten as a sum of errors of order at least p^2, which can then be combined with the corresponding (higher-order) terms in the first sum, and an induction argument finally allows us to prove the following decomposition.

Lemma 5.3.1. *Assume that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \geq 1$. For any $k \geq 0$ and any $p \in [0, 1]$, we have*

$$\Delta_T^{(p)} = \sum_{j=1}^{k} p^j \Delta_T^j + \sum_{j=k+1}^{k+\Gamma} p^j E_T^{(p),j,k} \tag{5.85}$$

where, for all $j > k \geq 0$, the approximate derivatives Δ_T^j and the errors $E_T^{(p),j,k}$ are given by

$$\begin{aligned}\Delta_T^j &:= \sum_{|F|=j}\sum_{G\subset F}(-1)^{|F\setminus G|+1}\mathbb{E}\left[\nabla\delta_\xi^G\phi_T \cdot C_{F\setminus G\|G}(\nabla\phi_T^F + \xi)\right], & (5.86)\\ E_T^{(p),j,k} &:= \sum_{|F|=j}\sum_{\substack{G\subset F\\|G|\leq k}}(-1)^{|F\setminus G|+1}\mathbb{E}\left[\nabla\delta_\xi^G\phi_T \cdot C_{F\setminus G\|G}(\nabla\phi_T^{E^{(p)}\cup F} + \xi)\right], & (5.87)\end{aligned}$$

and the sums $\sum_{|F|=j}\sum_{G\subset F}$ *in* (5.86) *and* (5.87) *are absolutely convergent for fixed* T. ◊

Proof. We proceed by induction. For $k=0$, (5.85) reduces to (5.84). Assume now that (5.85) holds true for some $k\geq 0$. First of all, we decompose $E_T^{(p),k+1,k}$ as follows:

$$E_T^{(p),k+1,k}=\Delta_T^{k+1}+G_T^{(p),k}, \tag{5.88}$$

where the error reads

$$\begin{aligned} G_T^{(p),k}&:=\sum_{|F|=k+1}\sum_{\substack{G\subset F\\ |G|\leq k}}(-1)^{|F\setminus G|+1}\mathbb{E}\left[\nabla\delta_\xi^G\phi_T\cdot C_{F\setminus G\|G}\nabla(\phi_T^{E^{(p)}\cup F}-\phi_T^F)\right]\\ &=\sum_{|F|=k+1}\sum_{j=1}^{k+1}(-1)^{j+1}\sum_{\substack{G\subset F\\ |G|=j}}\mathbb{E}\left[\nabla\delta_\xi^{F\setminus G}\phi_T\cdot C_{G\|F\setminus G}\nabla(\phi_T^{E^{(p)}\cup F}-\phi_T^F)\right], \end{aligned}$$

since the summand for $G=F$ in (5.86) vanishes (cf. $C_{\varnothing\|F}\equiv 0$).

Given $|F|=k+1$, recall that (5.59) in Lemma 5.2.1 (for $H=\varnothing$) asserts that $\delta_\xi^F\phi_T$ solves

$$\frac{1}{T}\delta_\xi^F\phi_T-\nabla\cdot A^F\nabla\delta_\xi^F\phi_T=\sum_{j=1}^{k+1}(-1)^{j+1}\sum_{\substack{G\subset F\\ |G|=j}}\nabla\cdot C_{G\|F\setminus G}\nabla\delta_\xi^{F\setminus G}\phi_T,$$

and also recall that since $A^{E^{(p)}\cup F}=A^F+C_{\|F}^{(p)}$, $\phi_T^{E^{(p)}\cup F}-\phi_T^F$ solves

$$\frac{1}{T}(\phi_T^{E^{(p)}\cup F}-\phi_T^F)-\nabla\cdot A^F\nabla(\phi_T^{E^{(p)}\cup F}-\phi_T^F)=\nabla\cdot C_{\|F}^{(p)}(\nabla\phi_T^{E^{(p)}\cup F}+\xi).$$

Successively testing these equations with $\phi_T^{E^{(p)}\cup F}-\phi_T^F$ and $\delta^F\phi_T$ respectively (as for the proof of (5.76) in Lemma 5.2.4, still noting that all the sums converge absolutely by Lemmas 5.2.2 and 5.2.3), we get

$$\begin{aligned} G_T^{(p),k}&=-\frac{1}{T}\sum_{|F|=k+1}\mathbb{E}\left[\delta_\xi^F\phi_T(\phi_T^{E^{(p)}\cup F}-\phi_T^F)\right]-\sum_{|F|=k+1}\mathbb{E}\left[\nabla\delta_\xi^F\phi_T\cdot A^F\nabla(\phi_T^{E^{(p)}\cup F}-\phi_T^F)\right]\\ &=\sum_{|F|=k+1}\mathbb{E}\left[\nabla\delta_\xi^F\phi_T\cdot C_{\|F}^{(p)}(\nabla\phi_T^{E^{(p)}\cup F}+\xi)\right]. \end{aligned}$$

Hence, using the inclusion-exclusion formula (5.18) as before (cf. (5.84)) and the independence, this yields

$$G_T^{(p),k}=\sum_{j=1}^{\Gamma}(-1)^{j+1}p^j\sum_{|G|=j}\sum_{\substack{|F|=k+1\\ G\cap F=\varnothing}}\mathbb{E}\left[\nabla\delta_\xi^F\phi_T\cdot C_{G\|F}(\nabla\phi_T^{E^{(p)}\cup F\cup G}+\xi)\right],$$

and hence, relabeling the sums,

$$p^{k+1}G_T^{(p),k}=\sum_{j=k+2}^{k+\Gamma+1}(-1)^{j-k}p^j\sum_{|F|=j}\sum_{\substack{G\subset F\\ |G|=k+1}}\mathbb{E}\left[\nabla\delta_\xi^G\phi_T\cdot C_{F\setminus G\|G}(\nabla\phi_T^{E^{(p)}\cup F}+\xi)\right]. \tag{5.89}$$

By the induction assumption (5.85) at order k and the decomposition (5.88), we thus have

$$\Delta_T^{(p)}=\sum_{j=1}^{k+1}p^j\Delta_T^j+p^{k+1}G_T^{(p),k}+\sum_{j=k+2}^{k+\Gamma}p^j\sum_{|F|=j}\sum_{\substack{G\subset F\\ |G|\leq k}}(-1)^{|F\setminus G|+1}\mathbb{E}\left[\nabla\delta_\xi^G\phi_T\cdot C_{F\setminus G\|G}(\nabla\phi_T^{E^{(p)}\cup F}+\xi)\right].$$

Combined with (5.89), this yields

$$\Delta_T^{(p)} = \sum_{j=1}^{k+1} p^j \Delta_T^j + \sum_{j=k+2}^{k+\Gamma+1} p^j \sum_{|F|=j} \sum_{\substack{G\subset F\\|G|\le k+1}} (-1)^{|G|+1}\mathbb{E}\left[\nabla\delta_\xi^G\phi_T \cdot C_{F\setminus G\|G}(\nabla\phi_T^{E^{(p)}\cup F}+\xi)\right],$$

that is $\Delta_T^{(p)} = \sum_{j=1}^{k+1} p^j\Delta_T^j + \sum_{j=k+2}^{k+\Gamma+1} p^j E_T^{(p),j,k+1}$, and therefore (5.85) at step $k+1$. □

We now prove that the approximate derivatives are bounded uniformly in T and in the moments of ρ, as a consequence of the improved energy estimates of Proposition 5.2.6.

Proposition 5.3.2. *Assume that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \ge 1$. Then, there is a constant $C \simeq 1$ (independent of T and of the moments of ρ) such that, for any $k \ge 1$, the approximate k-th derivative Δ_T^k defined in* (5.86) *satisfies*

$$|\Delta_T^k| \le C^k.$$

Likewise, for any $j > k \ge 1$ and any $p \in [0,1]$, the error $E_T^{(p),j,k}$ defined in (5.87) *satisfies*

$$|E_T^{(p),j,k}| \le C^j. \qquad \Diamond$$

Proof. The estimates of the errors $E_T^{(p),j,k}$'s are obtained using the same arguments as for the estimates of the approximate derivatives Δ_T^j's, and we only display the proof of the latter. Since $\nabla\phi_T^F + \xi = \sum_{S\subset F}\nabla\delta_\xi^S\phi_T$ (cf. (5.15) with $G = H = \varnothing$) and $C_{F\setminus G\|G} = C_{F\setminus G} + \sum_{U\subset G, U\neq\varnothing}(-1)^{|U|}C_{U\cup(F\setminus G)}$ for any $G \subsetneq F$ (cf. (5.21)), and $C_{\varnothing\|G} \equiv 0$, we may rewrite formula (5.86) as follows:

$$\Delta_T^k = \underbrace{\sum_{|F|=k}\sum_{G\subsetneq F}\sum_{S\subset F}(-1)^{|F\setminus G|+1}\mathbb{E}[\nabla\delta_\xi^G\phi_T\cdot C_{F\setminus G}\nabla\delta_\xi^S\phi_T]}_{=:\Delta_{T,1}^k}$$
$$+ \underbrace{\sum_{|F|=k}\sum_{G\subsetneq F}\sum_{S\subset F}\sum_{\substack{U\subset G\\U\neq\varnothing}}(-1)^{|F\setminus G|+|U|+1}\mathbb{E}[\nabla\delta_\xi^G\phi_T\cdot C_{U\cup(F\setminus G)}\nabla\delta_\xi^S\phi_T]}_{=:\Delta_{T,2}^k}.$$

We treat each term separately. By the change of variables $G \rightsquigarrow G\cup U$, $S \rightsquigarrow S\cup U$, and $F \rightsquigarrow F\cup G\cup S\cup U$ (with F, G, S, U disjoint), we rewrite $\Delta_{T,1}^k$ as

$$\Delta_{T,1}^k = \sum_{l=0}^{k-1}\sum_{i=0}^{k-l}\sum_{j=0}^{l}\sum_{|F|=k-l-i}\sum_{\substack{|G|=l-j\\G\cap F=\varnothing}}\sum_{\substack{|S|=i\\S\cap(F\cup G)=\varnothing}}\sum_{\substack{|U|=j\\U\cap(F\cup G\cup S)=\varnothing}}(-1)^{|F|+|S|+1}\mathbb{E}[\nabla\delta_\xi^{G\cup U}\phi_T\cdot C_{F\cup S}\nabla\delta_\xi^{S\cup U}\phi_T],$$

so that, by the triangle inequality,

$$|\Delta_{T,1}^k| \lesssim \sum_{l=0}^{k-1}\sum_{i=0}^{k-l}\sum_{j=0}^{l}\sum_{|F|=k-l-i}\sum_{\substack{|S|=i\\S\cap F=\varnothing}}\sum_{\substack{|U|=j\\U\cap(F\cup S)=\varnothing}}\mathbb{E}\left[\mathbb{1}_{J_{F\cup S}}|\nabla\delta_\xi^{S\cup U}\phi_T|\left|\sum_{\substack{|G|=l-j\\G\cap(F\cup S\cup U)=\varnothing}}\nabla\delta_\xi^{G\cup U}\phi_T\right|\right].$$

Recall from (5.80) in the proof of Proposition 5.2.6 that

$$\left|\sum_{\substack{|G|=l-j\\G\cap(F\cup S\cup U)=\varnothing}}\nabla\delta_\xi^{G\cup U}\phi_T\right| \le \sum_{u=0}^{|F|}\sum_{s=0}^{|S|}\sum_{\substack{|W|=u\\W\subset F}}\sum_{\substack{|H|=s\\H\subset S}}\left|\sum_{\substack{|G|=l-j-u-s\\G\cap(W\cup H\cup U)=\varnothing}}\nabla\delta_\xi^{G\cup W\cup H\cup U}\phi_T\right|.$$

Hence, by the change of variables $F \rightsquigarrow F \setminus W$ and $S \rightsquigarrow S \setminus H$, and the notation $\delta_{F,S,U,W,H} = 1$ if F, S, U, W, H are disjoint, and $\delta_{F,S,U,W,H} = 0$ otherwise, this yields

$$|\Delta^k_{T,1}| \lesssim \sum_{l=0}^{k-1} \sum_{i=0}^{k-l} \sum_{j=0}^{l} \sum_{u=0}^{k-l-i} \sum_{s=0}^{i} \sum_{|F|=k-l-i-u} \sum_{|S|=i-s} \sum_{|U|=j} \sum_{|W|=u} \sum_{|H|=s} \delta_{F,S,U,W,H}$$
$$\times \mathbb{E}\bigg[\mathbb{1}_{J_{F \cup W \cup S \cup H}} |\nabla \delta^{S \cup H \cup U}_{\xi} \phi_T| \bigg| \sum_{\substack{|G|=l-j-u-s \\ G \cap (W \cup H \cup U) = \varnothing}} \nabla \delta^{G \cup W \cup H \cup U}_{\xi} \phi_T \bigg| \bigg].$$

We rearrange the sums suitably, and use the notation $\mathbb{1}_{J_\varnothing} = 1$ (so that we have $\mathbb{1}_{J_{L \cup K}} = \mathbb{1}_{J_L} \mathbb{1}_{J_K}$) to obtain

$$|\Delta^k_{T,1}| \lesssim \sum_{l=0}^{k-1} \sum_{i=0}^{k-l} \sum_{j=0}^{l} \sum_{u=0}^{k-l-i} \sum_{s=0}^{i} \sum_{|U|=j} \sum_{\substack{|H|=s \\ H \cap U = \varnothing}} \mathbb{E}\bigg[\mathbb{1}_{J_H} \Big(\sum_{|F|=k-l-i-u} \mathbb{1}_{J_F} \Big) \Big(\sum_{\substack{|S|=i-s \\ S \cap (H \cup U) = \varnothing}} \mathbb{1}_{J_S} |\nabla \delta^{S \cup H \cup U}_{\xi} \phi_T| \Big)$$
$$\times \Big(\sum_{\substack{|W|=u \\ W \cap (H \cup U) = \varnothing}} \mathbb{1}_{J_W} \bigg| \sum_{\substack{|G|=l-j-u-s \\ G \cap (W \cup H \cup U) = \varnothing}} \nabla \delta^{G \cup W \cup H \cup U}_{\xi} \phi_T \bigg| \Big) \bigg].$$

Recalling that $\sum_{L \subset \mathbb{N}} \mathbb{1}_{J_L}(0) \lesssim 1$ by (5.78) (as a consequence of assumption (5.4)), we deduce from a multiple use of the Cauchy-Schwarz inequality

$$|\Delta^k_{T,1}| \lesssim \sum_{l=0}^{k-1} \sum_{i=0}^{k-l} \sum_{j=0}^{l} \sum_{u=0}^{k-l-i} \sum_{s=0}^{i} \sum_{|U|=j} \sum_{\substack{|H|=s \\ H \cap U = \varnothing}} \mathbb{E}\bigg[\mathbb{1}_{J_H} \Big(\sum_{\substack{|S|=i-s \\ S \cap (H \cup U) = \varnothing}} \mathbb{1}_{J_S} |\nabla \delta^{S \cup H \cup U}_{\xi} \phi_T|^2 \Big)^{\frac{1}{2}}$$
$$\times \Big(\sum_{\substack{|W|=u \\ W \cap (H \cup U) = \varnothing}} \mathbb{1}_{J_W} \bigg| \sum_{\substack{|G|=l-j-u-s \\ G \cap (W \cup H \cup U) = \varnothing}} \nabla \delta^{G \cup W \cup H \cup U}_{\xi} \phi_T \bigg|^2 \Big)^{\frac{1}{2}} \bigg],$$

and hence, by the Jensen inequality,

$$|\Delta^k_{T,1}| \lesssim \sum_{l=0}^{k-1} \sum_{i=0}^{k-l} \sum_{j=0}^{l} \sum_{u=0}^{k-l-i} \sum_{s=0}^{i} \mathbb{E}\bigg[\sum_{|U|=j} \sum_{\substack{|H|=s \\ H \cap U = \varnothing}} \mathbb{1}_{J_H} \sum_{\substack{|S|=i-s \\ S \cap (H \cup U) = \varnothing}} \mathbb{1}_{J_S} |\nabla \delta^{S \cup H \cup U}_{\xi} \phi_T|^2 \bigg]$$
$$+ \sum_{l=0}^{k-1} \sum_{i=0}^{k-l} \sum_{j=0}^{l} \sum_{u=0}^{k-l-i} \sum_{s=0}^{i} \mathbb{E}\bigg[\sum_{|U|=j} \sum_{\substack{|H|=s \\ H \cap U = \varnothing}} \mathbb{1}_{J_H} \sum_{\substack{|W|=u \\ W \cap (H \cup U) = \varnothing}} \mathbb{1}_{J_W} \bigg| \sum_{\substack{|G|=l-j-u-s \\ G \cap (W \cup H \cup U) = \varnothing}} \nabla \delta^{G \cup W \cup H \cup U}_{\xi} \phi_T \bigg|^2 \bigg].$$

By the changes of variables $S \cup H \cup U \rightsquigarrow U$ in the first term and $W \cup H \cup U \rightsquigarrow U$ in the second term, and using that $\sum_{|H| \le k} \sum_{|W| \le k} \mathbb{1}_{J_H} \mathbb{1}_{J_W} \lesssim 1$, this finally yields

$$|\Delta^k_{T,1}| \lesssim \sum_{j=0}^{k} \mathbb{E}\bigg[\sum_{|U|=j} |\nabla \delta^U_{\xi} \phi_T|^2 \bigg] + \sum_{j=0}^{k-1} \sum_{i=0}^{j} \mathbb{E}\bigg[\sum_{|U|=j-i} \bigg| \sum_{\substack{|G|=i \\ G \cap U = \varnothing}} \nabla \delta^{G \cup U}_{\xi} \phi_T \bigg|^2 \bigg]. \tag{5.90}$$

The improved energy estimates of Proposition 5.2.6 then allow us to conclude that $|\Delta^k_{T,1}| \lesssim C^k$ for some $C \simeq 1$. As we can easily argue in a similar way for $\Delta^k_{T,2}$, the conclusion follows. $\square$

The combination of Lemma 5.3.1 and Proposition 5.3.2 immediately yields the following result.

Corollary 5.3.3. *Assume that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \geq 1$. Then, there exists a constant $C \simeq 1$ (independent of T and of the moments of ρ) such that, for all $k \geq 0$ and all $p \in [0,1]$, we have*

$$\left| \Delta_T^{(p)} - \sum_{j=1}^{k} p^j \Delta_T^j \right| \leq (Cp)^{k+1}. \qquad \Diamond$$

The following lemma provides useful alternative formulas for the approximate derivatives Δ_T^j's (which coincide with the argument of the limit in (5.23) for $p_0 = 0$), showing that they coincide with the arguments of the limits in (5.24) and (5.25) for $p_0 = 0$.

Lemma 5.3.4. *Assume that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \geq 1$. For all $T > 0$, the approximate derivatives Δ_T^j's, $j \geq 1$, given by* (5.86)*, satisfy the following two equivalent formulas:*

$$\Delta_T^j = \sum_{|F|=j} \sum_{G \subset F} (-1)^{|G|+1} \mathbb{E}\left[(\nabla \phi_T + \xi) \cdot C_G \nabla \delta_\xi^{F \setminus G} \phi_T^G \right] \tag{5.91}$$

$$= \sum_{|F|=j} \sum_{G \subset F} (-1)^{|F \setminus G|} \mathbb{E}[\xi \cdot A^{F \setminus G} (\nabla \phi_T^G + \xi)], \tag{5.92}$$

where both sums $\sum_{|F|=j}$ are absolutely convergent. $\Diamond$

Before we turn to the proof of this lemma, let us comment on the equivalent formulas (5.86), (5.91) and (5.92). Formula (5.86) is the natural formula that we obtain by expanding the difference quotient (see proof of (5.84) and of Lemma 5.3.1), formula (5.91) is the easiest to use in practice (see e.g. Corollaries 5.1.4 and 5.1.5), while formula (5.92) is the cluster-expansion formula used by physicists.

Proof. We split the proof into two steps. We first prove (5.91), from which (5.92) is an easy consequence.

Step 1. Proof of (5.91).

All absolute convergence issues that we need here (for fixed T) simply follow as before from Lemma 5.2.3 or similar statements (based on Lemma 5.2.2). For the clarity of the exposition, we discard this issue in the proof. Let $j \geq 1$ be fixed. Separating the cases $G = \varnothing$ and $G \neq \varnothing$, and noting that $C_{F \setminus G \| G}$ vanishes whenever $G = F$, the very definition (5.86) of Δ_T^j reads

$$\Delta_T^j = \sum_{|F|=j} \sum_{\substack{G \subsetneq F \\ G \neq \varnothing}} (-1)^{|F \setminus G|+1} \mathbb{E}\left[\nabla \delta_\xi^G \phi_T \cdot C_{F \setminus G \| G} (\nabla \phi_T^F + \xi) \right]$$
$$+ (-1)^{j+1} \sum_{|F|=j} \mathbb{E}\left[(\nabla \phi_T + \xi) \cdot C_F (\nabla \phi_T^F + \xi) \right].$$

For any $|F| = j$, $G \subsetneq F$, $G \neq \varnothing$, by (5.60) in Lemma 5.2.1, $\delta_\xi^{F \setminus G} \phi_T^G$ satisfies

$$\frac{1}{T} \delta_\xi^{F \setminus G} \phi_T^G - \nabla \cdot A^G \nabla \delta_\xi^{F \setminus G} \phi_T^G = \sum_{S \subset F \setminus G} (-1)^{|S|+1} \nabla \cdot C_{S \| G} \nabla \delta_\xi^{F \setminus (G \cup S)} \phi_T^{G \cup S}.$$

Testing this equation with $\delta_\xi^G \phi_T$ (as in the proof of (5.76) in Lemma 5.2.4) yields

$$\begin{aligned}\Delta_T^j = &-\frac{1}{T}\sum_{|F|=j}\sum_{\substack{G\subsetneq F\\ G\neq\varnothing}} \mathbb{E}\left[\delta_\xi^G\phi_T\delta_\xi^{F\setminus G}\phi_T^G\right] - \sum_{|F|=j}\sum_{\substack{G\subsetneq F\\ G\neq\varnothing}} \mathbb{E}\left[\nabla\delta_\xi^G\phi_T\cdot A^G\nabla\delta_\xi^{F\setminus G}\phi_T^G\right]\\ &+\sum_{|F|=j}\sum_{\substack{G\subsetneq F\\ G\neq\varnothing}}\sum_{S\subsetneq F\setminus G}(-1)^{|S|}\mathbb{E}\left[\nabla\delta_\xi^G\phi_T\cdot C_{S\|G}\nabla\delta_\xi^{F\setminus(G\cup S)}\phi_T^{G\cup S}\right]\\ &+(-1)^{j+1}\sum_{|F|=j}\mathbb{E}\left[(\nabla\phi_T+\xi)\cdot C_F(\nabla\phi_T^F+\xi)\right].\end{aligned}$$

Now, for $G\neq\varnothing$, by (5.59) in Lemma 5.2.1 (with $H=\varnothing$), $\delta_\xi^G\phi_T$ solves

$$\frac{1}{T}\delta_\xi^G\phi_T - \nabla\cdot A^G\nabla\delta_\xi^G\phi_T = \sum_{S\subset G}(-1)^{|S|+1}\nabla\cdot C_{S\|G\setminus S}\nabla\delta_\xi^{G\setminus S}\phi_T.$$

Testing this equation with $\delta_\xi^{F\setminus G}\phi_T^G$ yields

$$\begin{aligned}\Delta_T^j = &-\sum_{|F|=j}\sum_{\substack{G\subsetneq F\\ G\neq\varnothing}}\sum_{S\subset G}(-1)^{|S|}\mathbb{E}\left[\nabla\delta_\xi^{G\setminus S}\phi_T\cdot C_{S\|G\setminus S}\nabla\delta_\xi^{F\setminus G}\phi_T^G\right]\\ &+\sum_{|F|=j}\sum_{\substack{G\subsetneq F\\ G\neq\varnothing}}\sum_{S\subsetneq F\setminus G}(-1)^{|S|}\mathbb{E}\left[\nabla\delta_\xi^G\phi_T\cdot C_{S\|G}\nabla\delta_\xi^{F\setminus(G\cup S)}\phi_T^{G\cup S}\right]\\ &+(-1)^{j+1}\sum_{|F|=j}\mathbb{E}\left[(\nabla\phi_T+\xi)\cdot C_F(\nabla\phi_T^F+\xi)\right],\end{aligned}$$

and therefore

$$\begin{aligned}\Delta_T^j = &-\sum_{|F|=j}\sum_{G\subsetneq F}\sum_{S\subsetneq G}(-1)^{|S|}\mathbb{E}\left[\nabla\delta_\xi^{G\setminus S}\phi_T\cdot C_{S\|G\setminus S}\nabla\delta_\xi^{F\setminus G}\phi_T^G\right]\\ &+\sum_{|F|=j}\sum_{\substack{G\subsetneq F\\ G\neq\varnothing}}\sum_{S\subsetneq F\setminus G}(-1)^{|S|}\mathbb{E}\left[\nabla\delta_\xi^G\phi_T\cdot C_{S\|G}\nabla\delta_\xi^{F\setminus(G\cup S)}\phi_T^{G\cup S}\right]\\ &+\sum_{|F|=j}\sum_{G\subset F}(-1)^{|G|+1}\mathbb{E}\left[(\nabla\phi_T+\xi)\cdot C_G\nabla\delta_\xi^{F\setminus G}\phi_T^G\right].\end{aligned}$$

With the change of variables $G\rightsquigarrow G\setminus S$ in the first term, we observe that the first two groups of sums cancel, so that we are left with

$$\Delta_T^j = \sum_{|F|=j}\sum_{G\subset F}(-1)^{|G|+1}\mathbb{E}\left[(\nabla\phi_T+\xi)\cdot C_G\nabla\delta_\xi^{F\setminus G}\phi_T^G\right],$$

that is, (5.91).

Step 2. Proof of (5.92).

Absolute convergence issues for this part of the proof (which do not straightforwardly follow from Lemmas 5.2.3 and 5.2.2) will be addressed at the end of this step. Let $j\geq 1$ be fixed. Formula (5.91) gives

$$\Delta_T^j = \underbrace{\sum_{|F|=j}\sum_{G\subset F}(-1)^{|G|+1}\mathbb{E}\left[\xi\cdot C_G\nabla\delta_\xi^{F\setminus G}\phi_T^G\right]}_{=:S_T^{j,1}} + \underbrace{\sum_{|F|=j}\sum_{G\subset F}(-1)^{|G|+1}\mathbb{E}\left[\nabla\phi_T\cdot C_G\nabla\delta_\xi^{F\setminus G}\phi_T^G\right]}_{=:S_T^{j,2}}. \quad (5.93)$$

By (5.60) in Lemma 5.2.1 (with $H = \varnothing$), $\delta_\xi^F \phi_T$ solves

$$\frac{1}{T}\delta_\xi^F \phi_T - \nabla \cdot A\nabla\delta_\xi^F \phi_T = \sum_{G\subset F}(-1)^{|G|+1}\nabla \cdot C_G \nabla \delta_\xi^{F\setminus G}\phi_T^G,$$

whereas ϕ_T solves

$$\frac{1}{T}\phi_T - \nabla \cdot A(\nabla\phi_T + \xi) = 0.$$

On the one hand, testing these equations with ϕ_T and $\delta_\xi^F \phi_T$ respectively (as in the proof of (5.76) in Lemma 5.2.4), we obtain

$$\begin{aligned} S_T^{j,2} &= -\frac{1}{T}\sum_{|F|=j}\mathbb{E}\left[\phi_T\delta_\xi^F\phi_T\right] - \sum_{|F|=j}\mathbb{E}\left[\nabla\phi_T \cdot A\nabla\delta_\xi^F\phi_T\right] \\ &= \sum_{|F|=j}\mathbb{E}\left[(\nabla\phi_T+\xi)\cdot A\nabla\delta_\xi^F\phi_T\right] - \sum_{|F|=j}\mathbb{E}\left[\nabla\phi_T \cdot A\nabla\delta_\xi^F\phi_T\right] \\ &= \sum_{|F|=j}\mathbb{E}\left[\xi\cdot A\nabla\delta_\xi^F\phi_T\right], \end{aligned} \tag{5.94}$$

and therefore

$$S_T^{j,2} = \sum_{|F|=j}\sum_{G\subset F}(-1)^{|F\setminus G|}\mathbb{E}\left[\xi\cdot A(\nabla\phi_T^G+\xi)\right]. \tag{5.95}$$

On the other hand, $S_T^{j,1}$ can be rewritten as follows:

$$S_T^{j,1} = \sum_{|F|=j}\sum_{G\subset F}(-1)^{|G|+1}\sum_{S\subset F\setminus G}(-1)^{F\setminus(S\cup G)}\mathbb{E}\left[\xi\cdot C_G(\nabla\phi_T^{S\cup G}+\xi)\right],$$

which yields by the change of variables $S\cup G \rightsquigarrow U$

$$\begin{aligned} S_T^{j,1} &= \sum_{|F|=j}\sum_{U\subset F}(-1)^{|F\setminus U|}\mathbb{E}\left[\xi\cdot\left(\sum_{G\subset U}(-1)^{|G|+1}C_G\right)(\nabla\phi_T^U+\xi)\right] \\ &= \sum_{|F|=j}\sum_{U\subset F}(-1)^{|F\setminus U|}\mathbb{E}\left[\xi\cdot C^U(\nabla\phi_T^U+\xi)\right]. \end{aligned} \tag{5.96}$$

The desired result follows from the combination of (5.93), (5.95), and (5.96). Note that the sum defining $S_T^{j,1}$ in (5.93) is absolutely convergent by virtue of (5.73) in Lemma 5.2.3, and hence the sum $\sum_{|F|=j}$ in (5.96) is also absolutely convergent, since its terms have just been rewritten but are still the same. Likewise, the sum in the right-hand side of (5.94) is absolutely convergent by Lemma 5.2.2 (thus justifying the testing argument), so that the sum in (5.95) must also converge absolutely. This finally proves that the sum $\sum_{|F|=j}$ in (5.92) is absolutely convergent too (which would not be clear a priori without performing this decomposition). $\square$

5.3.2 Proof of Theorem 5.1.1 and Corollary 5.1.2: analyticity

Let $\xi \in \mathbb{R}^d$, $|\xi| = 1$ be fixed. It suffices to prove Theorem 5.1.1 and Corollary 5.1.2 for that fixed choice of ξ. What needs to be done is to pass to the limit $T \uparrow \infty$ in Corollary 5.3.3, and get rid of the additional assumption that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \geq 1$. For that second purpose, given a point process ρ, we introduce approximations for which all moments exist: more precisely, we shall construct hardcore approximations ρ_θ of ρ, apply Corollary 5.3.3 for these approximations, and then pass to the limit in both the parameters T and θ. We split the proof into five steps.

Step 1. Hardcore approximations of ρ.

Let $\theta > 0$ be fixed. In this first step, we construct hardcore approximations ρ_θ of the stationary point process ρ in the following sense: for any $\theta > 0$, ρ_θ is an ergodic stationary point process on $\mathbb{R}^d$ such that $\rho_\theta \subset \rho$ and $\rho_\theta(Q) \le \theta$ a.s., and moreover $\rho_\theta \uparrow \rho$ locally almost surely as $\theta \uparrow \infty$. For any $\theta > 0$, we choose a measurable enumeration $\rho_\theta = (q_n^\theta)_n$. We then define A_θ^F as the coefficients obtained when replacing ρ by ρ_θ in A^F. Similarly, we define $\phi_{T,\theta}^F$ the approximate corrector and $A_{T,\theta}^{(p)}$ the approximate homogenized coefficients associated with $A_\theta^F, A_\theta^{(p)}$ instead of $A^F, A^{(p)}$. We then also prove the following convergence properties, which will be crucial in the next step: for fixed $p \in [0,1]$ and $T > 0$, we have

$$\mathbb{E}[|\nabla(\phi_{T,\theta}^{(p)} - \phi_T^{(p)})|^2] \xrightarrow{\theta\uparrow\infty} 0, \tag{5.97}$$

and therefore

$$|A_{T,\theta}^{(p)} - A_T^{(p)}| \xrightarrow{\theta\uparrow\infty} 0. \tag{5.98}$$

We first give a possible construction of such an approximating sequence $(\rho_\theta)_\theta$. Consider the measurable enumeration $\rho = (q_n)_n$, choose independently a sequence $(U_n)_n$ of i.i.d. random variables that are uniformly distributed on $(0,1)$, and consider the decorated process $(q_n, U_n)_n$. We then build an oriented graph on the points $(q_n, U_n)_n$ in $\mathbb{R}^d \times [0,1]$ as follows: we put an oriented edge from (q,u) to (q',u') whenever $(q + \frac{1}{\theta}Q) \cap (q' + \frac{1}{\theta}Q) \neq \varnothing$ and $u < u'$ (or $u = u'$ and q precedes q' in the lexicographic order, say). We say that (q',u') is an offspring (resp. a descendant) of (q,u) if (q,u) is a direct ancestor (resp. an ancestor) of (q',u'), i.e. if there is an edge (resp. a directed path) from (q,u) to (q',u') in the oriented graph constructed above. We now construct ρ_θ as follows. Let F_1 be the set of all roots in the oriented graph (i.e. the points of $\mathcal{P}_0$ without ancestor), let G_1 be the set of points of $\mathcal{P}_0$ that are offsprings of points of F_1, and let $H_1 = F_1 \cup G_1$. Now consider the oriented graph induced on $(q_n, U_n)_n \setminus H_1$, and define F_2, G_2, H_2 in the same way, and so on. By construction, the sets F_i and G_i are all disjoint and constitute a partition of the collection $(q_n, U_n)_n$. Finally define $\rho_\theta := \pi_1(\bigcup_i F_i)$, where π_1 is the projection on the first factor, $\pi_1(q,u) = q$. We easily check that ρ_θ defines a stationary point process on $\mathbb{R}^d$ and satisfies the required properties. Ergodicity of ρ_θ easily follows from that of ρ exactly in the same way as for the random parking measure in [213, Step 4 of the proof of Proposition 2.1].

It only remains to prove the convergence property (5.97). For that purpose, we write the equation satisfied by the difference $\phi_{T,\theta}^{(p)} - \phi_T^{(p)}$,

$$\frac{1}{T}(\phi_{T,\theta}^{(p)} - \phi_T^{(p)}) - \nabla \cdot A_\theta^{(p)} \nabla(\phi_{T,\theta}^{(p)} - \phi_T^{(p)}) = \nabla \cdot (A_\theta^{(p)} - A^{(p)})(\nabla\phi_T^{(p)} + \xi).$$

Testing this equation in probability with $\phi_{T,\theta}^{(p)} - \phi_T^{(p)}$ itself yields

$$\mathbb{E}[|\nabla(\phi_{T,\theta}^{(p)} - \phi_T^{(p)})|^2] \lesssim \mathbb{E}[|A_\theta^{(p)} - A^{(p)}|^2\,(|\nabla\phi_T^{(p)}|^2 + 1)].$$

By assumption, $A(0)$ and $A'(0)$ only depend on ρ via the restriction $\rho|_{B_r}$ for some given $r > 0$, so that the same property holds by definition for $A^{(p)}(0)$. Hence for some $L > 0$,

$$\mathbb{E}[|\nabla(\phi_{T,\theta}^{(p)} - \phi_T^{(p)})|^2] \lesssim \mathbb{E}[\mathbb{1}_{\rho_\theta|_{B_L} \neq \rho|_{B_L}}\,(|\nabla\phi_T^{(p)}|^2 + 1)].$$

Now the desired result simply follows from dominated convergence and the basic energy estimate $\mathbb{E}[|\nabla\phi_T^{(p)}|^2 + 1] \lesssim 1$, recalling that by definition we have almost surely as $\theta \uparrow \infty$,

$$\mathbb{1}_{\rho_\theta|_{B_L} \neq \rho|_{B_L}} \to 0.$$

Step 2. Reduction by regularization.

In this step, we prove Theorem 5.1.1 and Corollary 5.1.2 provided we have that, for fixed T and under the additional assumption $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \geq 1$, the map $p \mapsto \xi \cdot A_T^{(p)}\xi$ satisfies, for any $p_0 \in [0,1]$ and any $k \geq 1$, for all $-p_0 \leq p \leq 1-p_0$, $|p| \leq 1/C_{p_0}$,

$$\left| \xi \cdot A_T^{(p_0+p)}\xi - \xi \cdot A_T^{(p_0)}\xi - \sum_{j=1}^{k} p^j \Delta_T^{(p_0),j} \right| \leq (pC_{p_0})^{k+1}, \tag{5.99}$$

for some constant $C_{p_0} \simeq_{p_0} 1$, where the $\Delta_T^{(p_0),j}$'s are equivalently given by the arguments of any of the limits (5.23), (5.24) and (5.25), and further satisfy the bounds $|\Delta_T^{(p_0),j}| \leq C^j$ for all $j \geq 1$ (uniformly in T, p_0 and the moments of ρ).

Let $p_0 \in [0,1]$ be fixed. Consider the approximations ρ_θ introduced in Step 1, and apply (5.99) with ρ replaced by ρ_θ (where obviously all moments of ρ_θ are finite). For any $k \geq 1$, it follows from (5.99) that the map $p \mapsto \xi \cdot A_{T,\theta}^{(p)}\xi$ is smooth (on the whole of $[0,1]$), and a Taylor expansion of the map around p_0 up to order k gives, by Lagrange's remainder theorem, for all $-p_0 \leq p \leq 1-p_0$,

$$\left| \xi \cdot A_{T,\theta}^{(p_0+p)}\xi - \xi \cdot A_{T,\theta}^{(p_0)}\xi - \sum_{j=1}^{k} p^j \Delta_{T,\theta}^{(p_0),j} \right| \leq p^{k+1} \sup_{u\in[0,1]} |\Delta_{T,\theta}^{(p_0+up),k+1}| \leq (Cp)^{k+1}. \tag{5.100}$$

From (5.10) and (5.98), we learn that

$$\lim_{T\uparrow\infty} \lim_{\theta\uparrow\infty} (\xi \cdot A_{T,\theta}^{(p_0+p)}\xi - \xi \cdot A_{T,\theta}^{(p_0)}\xi) = \lim_{T\uparrow\infty} (\xi \cdot A_T^{(p_0+p)}\xi - \xi \cdot A_T^{(p_0)}\xi) = \xi \cdot A_{\mathrm{hom}}^{(p_0+p)}\xi - \xi \cdot A_{\mathrm{hom}}^{(p_0)}\xi. \tag{5.101}$$

Hence, in order to pass to the limit $T, \theta \uparrow \infty$ in (5.100), it is enough to prove that the limits

$$\Delta^{(p_0),j} := \lim_{T\uparrow\infty} \lim_{\theta\uparrow\infty} \Delta_{T,\theta}^{(p_0),j} \tag{5.102}$$

all exist in $\mathbb{R}$, for all $j \geq 1$. The combination of (5.101) and (5.102) indeed yields that for any $k \geq 1$, for all $-p_0 \leq p \leq 1-p_0$, we have

$$\left| \xi \cdot A_{\mathrm{hom}}^{(p_0+p)}\xi - \xi \cdot A_{\mathrm{hom}}^{(p_0)}\xi - \sum_{j=1}^{k} p^j \Delta^{(p_0),j} \right| \leq (Cp)^{k+1},$$

which is equivalent to the analyticity statement of Theorem 5.1.1 (with convergence of the Taylor series at p_0 for all perturbations p of magnitude $|p| < 1/C$, $-p_0 \leq p \leq 1-p_0$), and the derivatives $j!\Delta^{(p_0),j}$'s are then given by the desired well-defined limits stated in Corollary 5.1.2. In the particular case when the process ρ has all its moments finite, the regularization in θ can be omitted (so that only the limit in T remains). The proof of formula (5.26) in Corollary 5.1.2 is postponed to Step 5.

We prove (5.102) by induction. The proof of the statement for $j = 1$ is similar to the proof of the induction step, and we only display the latter. Assume that the limits $\Delta^{(p_0),j} = \lim_T \lim_\theta \Delta_{T,\theta}^{(p_0),j}$ exist in $\mathbb{R}$ for all $1 \leq j \leq k$, for some $k \geq 1$. We shall then prove that the limit $\Delta^{(p_0),k+1} = \lim_T \lim_\theta \Delta_{T,\theta}^{(p_0),k+1}$ also exists in $\mathbb{R}$. As $\Delta_{T,\theta}^{(p_0),k+1}$ is bounded uniformly in T, θ, it converges to some limit $L_T^{(p_0)} \in \mathbb{R}$ as $\theta \uparrow \infty$ up to extraction. Passing to the limit $\theta \uparrow \infty$ along a subsequence in inequality (5.100) with k replaced by $k+1$, and using the induction assumptions and (5.101), we obtain for any $-p_0 \leq p \leq 1-p_0$,

$$\left| \xi \cdot A_T^{(p_0+p)}\xi - \xi \cdot A_T^{(p_0)}\xi - \sum_{j=1}^{k} p^j \lim_\theta \Delta_{T,\theta}^{(p_0),j} - p^{k+1} L_T^{(p_0)} \right| \leq (Cp)^{k+2}.$$

This proves that $L^{(p_0)}$ satisfies

$$L_T^{(p_0)} = \lim_{\substack{p\to 0\\ -p_0\le p\le 1-p_0}} \bigg(p^{-k-1}(\xi\cdot A_T^{(p_0+p)}\xi - \xi\cdot A_T^{(p_0)}\xi) - \sum_{j=1}^k p^{j-k-1}\lim_\theta \Delta_{T,\theta}^{(p_0),j}\bigg),$$

where in particular the limit must exist. Since the right-hand side does not depend on the extraction, $L_T^{(p_0)}$ is uniquely defined, and $L_T^{(p_0)} = \lim_\theta \Delta^{(p),k+1}$ does exists in $\mathbb{R}$. A similar argument for the limit in T shows that $\lim_T \lim_\theta \Delta_{T,\theta}^{(p_0),k+1}$ exists in $\mathbb{R}$, so that (5.102) is proved.

Step 3. Reduction by restriction to $p_0 = 0$.

Let $T > 0$ be fixed and assume that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \ge 1$. In the present step, we prove that it suffices to check the result (5.99) at $p_0 = 0$: more precisely, it suffices to show that the map $p \mapsto \xi\cdot A_T^{(p)}\xi$ satisfies, for all $k \ge 1$ and $p \in [0,1]$,

$$\bigg|\xi\cdot A_T^{(p)}\xi - \xi\cdot A_T\xi - \sum_{j=1}^k p^j \Delta_T^j\bigg| \le (Cp)^{k+1}, \tag{5.103}$$

for some constant $C \simeq 1$, where, for any $j \ge 1$, Δ_T^j is equivalently given by formulas (5.86), (5.91) and (5.92), and satisfies the bound $|\Delta_T^j| \le C^j$ for all $j \ge 1$, uniformly in T and the moments of ρ.

First consider $p_0 \in [0,1)$ and positive perturbations $p_0 + p$ with $p \ge 0$. For $0 \le p \le 1 - p_0$, choose a sequence $(d_n^{(p_0,p)})_n$ of i.i.d. Bernoulli random variables (independent of all the others) with parameter $\mathbb{P}[d_n^{(p_0,p)} = 1] = p/(1-p_0)$, and consider the twice perturbed coefficients

$$\begin{aligned} A^{(p_0,p)} &= A^{(p_0)}\mathbb{1}_{\mathbb{R}^d\setminus J_n} + \sum_n \Big(d_n^{(p_0,p)}A' + (1-d_n^{(p_0,p)})A^{(p_0)}\Big)\,\mathbb{1}_{J_n} \\ &= A\mathbb{1}_{\mathbb{R}^d\setminus\bigcup_n J_n} + \sum_n \Big((1-d_n^{(p_0,p)})(1-b_n^{(p_0)})A + (d_n^{(p_0,p)} + b_n^{(p_0)}(1-d_n^{(p_0,p)}))A'\Big)\,\mathbb{1}_{J_n}. \end{aligned}$$

The field $A^{(p_0,p)}$ has by definition the same distribution as $A^{(p_0+p)}$, and it is a perturbation of $A^{(p_0)}$ with perturbation parameter $p/(1-p_0)$ (and with perturbed medium A'). Applying to $A^{(p_0,p)}$ the result (5.103) around 0 (which is assumed to hold), we deduce that the map $p \mapsto \xi\cdot A_T^{(p_0,p)}\xi = \xi\cdot A_T^{(p_0+p)}\xi$ satisfies for all $k \ge 1$ and all $0 \le p \le 1-p_0$,

$$\bigg|\xi\cdot A_T^{(p_0+p)}\xi - \xi\cdot A_T^{(p_0)}\xi - \sum_{j=1}^k \frac{p^j}{(1-p_0)^j}\tilde\Delta_T^{(p_0),j}\bigg| \le \Big(\frac{Cp}{1-p_0}\Big)^{k+1}, \tag{5.104}$$

where, for any $j \ge 1$, $\tilde\Delta_T^{(p_0),j}$ is the j-th right-derivative at 0 of the map $p \mapsto \xi\cdot\tilde A_T^{(p)}\xi$, corresponding to the "reference" coefficients $\tilde A := A^{(p_0)}$ and the "perturbed" coefficients A'. The cluster formula (5.92) reads in that case

$$\tilde\Delta_T^{(p_0),j} := \sum_{|F|=j}\mathbb{E}\bigg[\sum_{G\subset F}(-1)^{|F\setminus G|}\xi\cdot A^{E^{(p_0)}\cup G}(\nabla\phi_T^{E^{(p_0)}\cup G}+\xi)\bigg], \tag{5.105}$$

where the sum is absolutely convergent. Now note that the argument of the expectation vanishes whenever $F\cap E^{(p_0)} \ne \varnothing$, while, otherwise, if $F\cap E^{(p_0)} = \varnothing$, the argument of the expectation equals

$$\sum_{G\subset F}(-1)^{|F\setminus G|}\xi\cdot A^{G\cup(E^{(p_0)}\setminus F)}(\nabla\phi_T^{G\cup(E^{(p_0)}\setminus F)}+\xi).$$

As this expression is obviously independent of the event $[F\cap E^{(p_0)}=\varnothing]$, we can then rewrite

$$\begin{aligned}\tilde\Delta_T^{(p_0),j}&=\sum_{|F|=j}\mathbb{E}\bigg[\mathbb{1}_{F\cap E^{(p_0)}=\varnothing}\sum_{G\subset F}(-1)^{|F\setminus G|}\xi\cdot A^{E^{(p_0)}\cup G}(\nabla\phi_T^{E^{(p_0)}\cup G}+\xi)\bigg]\\&=\sum_{|F|=j}\mathbb{P}[F\cap E^{(p_0)}=\varnothing]\mathbb{E}\bigg[\sum_{G\subset F}(-1)^{|F\setminus G|}\xi\cdot A^{E^{(p_0)}\cup G}(\nabla\phi_T^{E^{(p_0)}\cup G}+\xi)\bigg]\\&=(1-p_0)^j\Delta_T^{(p_0),k},\end{aligned}\tag{5.106}$$

where $\Delta_T^{(p_0),j}$ is defined as the argument of the limit (5.25). The expansion (5.104) then becomes, for any $k\ge 1$, for all $0\le p\le 1-p_0$,

$$\bigg|\xi\cdot A_T^{(p_0+p)}\xi-\xi\cdot A_T^{(p_0)}\xi-\sum_{j=1}^k p^j\Delta_T^{(p_0),j}\bigg|\le\Big(\frac{Cp}{1-p_0}\Big)^{k+1}.\tag{5.107}$$

Moreover, recalling the bound $|\Delta_T^j|\le C^j$ for the right-derivatives at 0, which is assumed to hold for any choice of the coefficient (the constant C only depends on R,Γ,d,λ), we conclude, for all $p_0\in[0,1)$,

$$|\Delta_T^{(p_0),j}|\le C^j(1-p_0)^{-j}.\tag{5.108}$$

Note that this estimate for the derivatives deteriorates when p_0 gets closer to 1. This difficulty is overcome by considering negative perturbations, that is, looking at left-derivatives, as we do now.

Let us now consider $p_0\in(0,1]$ and negative perturbations at that point. For $0\le p\le p_0$, choose a sequence $(d_n^{(p_0,-p)})_n$ of i.i.d. Bernoulli random variables (independent of all the others) with $\mathbb{P}[d_n^{(p_0,-p)}=1]=p/p_0$, and consider the twice perturbed coefficients

$$\begin{aligned}A^{(p_0,-p)}&=A^{(p_0)}\mathbb{1}_{\mathbb{R}^d\setminus J_n}+\sum_n\Big(d_n^{(p_0,-p)}A+(1-d_n^{(p_0,-p)})A^{(p_0)}\Big)\mathbb{1}_{J_n}\\&=A\mathbb{1}_{\mathbb{R}^d\setminus\bigcup_n J_n}+\sum_n\Big((d_n^{(p_0,-p)}+(1-d_n^{(p_0,-p)})(1-b_n^{(p_0)}))A+b_n^{(p_0)}(1-d_n^{(p_0,-p)})A'\Big)\mathbb{1}_{J_n}.\end{aligned}$$

The field $A^{(p_0,p)}$ has by definition the same distribution as $A^{(p_0-p)}$, and it is a perturbation of $A^{(p_0)}$ with perturbation parameter p/p_0 (and with "perturbed" medium A, instead of A'). Applying to $A^{(p_0,p)}$ the result (5.103) around 0 (which is assumed to hold), we deduce that the map $p\mapsto\xi\cdot A_T^{(p_0,p)}\xi=\xi\cdot A_T^{(p_0-p)}\xi$ satisfies, for any $k\ge1$, for all $0\le p\le p_0$,

$$\bigg|\xi\cdot A_T^{(p_0-p)}\xi-\xi\cdot A_T^{(p_0)}\xi-\sum_{j=1}^k\frac{p^j}{p_0^j}\hat\Delta_T^{(p_0),j}\bigg|\le\Big(\frac{Cp}{p_0}\Big)^{k+1},\tag{5.109}$$

where, for any $j\ge1$, $\hat\Delta_T^{(p_0),j}$ is the j-th right-derivative of the map $p\mapsto\hat A_T^{(p)}$, corresponding to the "reference" coefficients $\hat A:=A^{(p_0)}$ and the "perturbed" coefficients A. The cluster formula (5.92) gives in this case

$$\hat\Delta_T^{(p_0),j}:=\sum_{|F|=j}\mathbb{E}\bigg[\sum_{G\subset F}(-1)^{|F\setminus G|}\xi\cdot A^{E^{(p_0)}\setminus G}(\nabla\phi_T^{E^{(p_0)}\setminus G}+\xi)\bigg],$$

where the sum is absolutely convergent. Arguing as above, the argument of the expectation vanishes unless $F\subset E^{(p_0)}$; hence, by the independence assumption, we obtain

$$\hat\Delta_T^{(p_0),j}=p_0^j\sum_{|F|=j}\mathbb{E}\bigg[\sum_{G\subset F}(-1)^{|F\setminus G|}\xi\cdot A^{(E^{(p_0)}\setminus F)\cup(F\setminus G)}(\nabla\phi_T^{(E^{(p_0)}\setminus F)\cup(F\setminus G)}+\xi)\bigg],$$

or equivalently, by the change of variables $F \setminus G \rightsquigarrow H$,

$$\hat{\Delta}_T^{(p_0),j} = (-1)^j p_0^j \sum_{|F|=j} \mathbb{E}\Big[\sum_{H\subset F} (-1)^{|F\setminus H|} \xi \cdot A^{H\cup(E^{(p_0)}\setminus F)}(\nabla\phi_T^{H\cup(E^{(p_0)}\setminus F)} + \xi)\Big] = (-1)^j p_0^j \Delta_T^{(p_0),j},$$

where, as before, $\Delta_T^{(p_0),k}$ is defined as the argument of the same limit (5.25). The expansion (5.104) then becomes, for any $k \geq 1$, for all $0 \leq p \leq p_0$,

$$\left| \xi \cdot A_T^{(p_0-p)}\xi - \xi \cdot A_T^{(p_0)}\xi - \sum_{j=1}^k (-p)^j \Delta_T^{(p_0),j} \right| \leq \left(\frac{Cp}{p_0}\right)^{k+1}. \tag{5.110}$$

By the bounds $|\Delta_T^j| \leq C^j$ for the right-derivatives at 0, we conclude, for all $p_0 \in [0,1)$,

$$|\Delta_T^{(p_0),j}| \leq C^j p_0^{-j}. \tag{5.111}$$

Combining (5.107) and (5.110) then directly yields the desired result (5.99). Moreover, combining (5.108) and (5.111) gives, for any $j \geq 1$, the uniform bound

$$|\Delta_T^{(p_0),j}| \leq \min\{C^j p_0^{-j}, C^j(1-p_0)^{-j}\} \leq (2C)^j.$$

Finally, arguing as in Lemma 5.3.4 (where the argument is performed at $p_0 = 0$ and proves the equivalence between formulas (5.86), (5.91) and (5.92)), we see that, for fixed T, θ, the cluster formula for $\Delta_T^{(p_0),j}$, that is the argument of the limit (5.25), is equivalent to the formulas given by the argument of the limits (5.23) and (5.24).

Step 4. Conclusion.

Let $T > 0$ be fixed, and assume that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \geq 1$. For any $k \geq 1$, Corollary 5.3.3 exactly asserts (5.103), Lemma 5.3.4 ensures that the Δ_T^j's are equivalently given by formulas (5.86), (5.91) and (5.92), and Proposition 5.3.2 gives the uniform bounds $|\Delta_T^j| \leq C^j$, for all $j \geq 1$. By the previous steps, this proves Theorem 5.1.1 and Corollary 5.1.2.

Step 5. Exact formula for the first derivative.

In this last step, we further assume that $\rho(Q) \leq \theta_0$ a.s. (so that in particular all the moments are bounded, and we can thus everywhere omit the regularization in θ), and we prove under that assumption the validity of formula (5.26) in Corollary 5.1.2. More precisely, we need to prove that we can pass to the limit in T inside the formula for the first approximate derivative

$$\Delta_T^{(p_0),1} = \sum_n \mathbb{E}\Big[(\nabla\phi_T^{E^{(p_0)}\setminus\{n\}} + \xi) \cdot C^{\{n\}}(\nabla\phi_T^{\{n\}\cup E^{(p_0)}} + \xi)\Big],$$

i.e. we prove that the well-defined limit $\Delta^{(p_0),1} = \lim_T \Delta_T^{(p_0),1}$ is given by the following formula:

$$\Delta^{(p_0),1} = \sum_n \mathbb{E}\Big[(\nabla\phi^{E^{(p_0)}\setminus\{n\}} + \xi) \cdot C^{\{n\}}(\nabla\phi^{\{n\}\cup E^{(p_0)}} + \xi)\Big], \tag{5.112}$$

with an abolutely converging sum. As before, we can restrict to $p_0 = 0$, and shall prove that the limit $\Delta^1 := \lim_T \Delta_T^1$ exists and is given by

$$\Delta^1 = \sum_n \mathbb{E}\Big[(\nabla\phi + \xi) \cdot C^{\{n\}}(\nabla\phi^{\{n\}} + \xi)\Big].$$

We start by showing that the sum is absolutely convergent. Decomposing $\nabla\phi^{\{n\}} = \nabla\delta^{\{n\}}\phi + \nabla\phi$, using assumption (5.4) in the form $\sum_n \mathbb{1}_{J_n} \lesssim 1$ (see also (5.78)), and recalling the elementary energy estimate $\mathbb{E}[1+|\nabla\phi|^2] \lesssim 1$ (see (5.8)), we have

$$|\Delta^1| \lesssim \sum_n \mathbb{E}[\mathbb{1}_{J_n}(1+|\nabla\phi|^2+|\nabla\phi^{\{n\}}|^2)] \lesssim 1+\sum_n \mathbb{E}[|\nabla\delta^{\{n\}}\phi|^2],$$

where the last sum is finite by Lemma 5.2.4 (for $k=1$) and the fact that $\nabla\delta^{\{n\}}\phi_T \rightharpoonup \nabla\delta^{\{n\}}\phi$ weakly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d;\mathrm{L}^2(\Omega))$.

We now prove that $\lim_T \Delta^1_T = \Delta^1$. Given $L \simeq 1$, the additional assumption $\rho(Q) \le \theta_0$ a.s. implies by stationarity $\rho(B_{R+L}) \le C\theta_0 =: Z$, with $C \simeq 1$. Hence, we can choose the measurable enumeration $(q_n)_n$ of the point process ρ in such a way that $B_{R+L} \cap (q_n)_n \subset (q_n)_{n=1}^Z$. Defining

$$a^n_L := \mathbb{E}\bigg[\fint_{B_L} (\nabla\phi+\xi)\cdot C^{\{n\}}(\nabla\phi^{\{n\}}+\xi)\bigg], \quad \text{and} \quad a^n_{T,L} := \mathbb{E}\bigg[\fint_{B_L} (\nabla\phi_T+\xi)\cdot C^{\{n\}}(\nabla\phi^{\{n\}}_T+\xi)\bigg],$$

we observe $\Delta^1 = \sum_{n=1}^Z a^n_L$ and $\Delta^1_T = \sum_{n=1}^Z a^n_{T,L}$. Indeed, by stationarity (together with absolute convergence), $\Delta^1 = \sum_{n=1}^\infty a^n_L$, so that $\Delta^1 = \sum_{n=1}^Z a^n_L$ by the choice of the measurable enumeration, and likewise for Δ^1_T. Therefore, it is enough to prove $\lim_T a^n_{T,L} = a^n_L$ for any $1 \le n \le \Gamma$. Since $\nabla\phi^{\{n\}}_T \rightharpoonup \nabla\phi^{\{n\}}$ weakly and $\nabla\phi_T \to \nabla\phi$ strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d,\mathrm{L}^2(\Omega))$ (see [199, Theorem 1]), we directly get $a^n_{T,L} \to a^n_L$ as $T \uparrow \infty$, for any n, as desired. $\square$

5.3.3 Proof of Corollaries 5.1.3, 5.1.4 and 5.1.5: Clausius-Mossotti formulas

In this section we further assume that $\mathbb{E}[\rho(Q)^2] < \infty$. (Note that, in the case of Corollaries 5.1.4 and 5.1.5, this directly follows from assumption (5.4) together with the fact that we are then dealing with ball inclusions of fixed radius.)

First-order universality principle

Set $J^{(p)} := \bigcup_{n\in E^{(p)}} J_n$. The volume fraction v_p of the perturbation is defined as follows:

$$v_p := \lim_{L\uparrow\infty} \frac{\mathbb{E}[|LQ\cap J^{(p)}|]}{L^d},$$

or equivalently, by stationarity of the inclusion process,

$$v_p = \lim_{L\uparrow\infty} L^{-d} \sum_{z\in LQ\cap\mathbb{Z}^d} \mathbb{E}[|(z+Q)\cap J^{(p)}|] = \mathbb{E}[|Q\cap J^{(p)}|].$$

An inclusion-exclusion argument gives

$$\sum_n \mathbb{E}[\mathbb{1}_{n\in E^{(p)}}|Q\cap J_n|] - \sum_{n\ne m} \mathbb{E}[\mathbb{1}_{n,m\in E^{(p)}}|Q\cap J_n\cap J_m|] \le v_p \le \sum_n \mathbb{E}[\mathbb{1}_{n\in E^{(p)}}|Q\cap J_n|].$$

By the independence between the Bernoulli process $E^{(p)}$ and all the other random variables, this turns into

$$p\sum_n \mathbb{E}[|Q\cap J_n|] - p^2\sum_{n\ne m} \mathbb{E}[|Q\cap J_n\cap J_m|] \le v_p \le p\sum_n \mathbb{E}[|Q\cap J_n|].$$

As $J_n \subset B_R(q_n)$ for all n, we note that, by the assumption $\mathbb{E}[\rho(Q)^2] < \infty$,

$$\sum_{n\ne m} \mathbb{E}[|Q\cap J_n\cap J_m|] \le \mathbb{E}\bigg[\sum_{n\ne m} \mathbb{1}_{q_n,q_m\in Q+B_R}\bigg] = \mathbb{E}[\rho(Q+B_R)^2] < \infty,$$

so that we have indeed proven

$$v_p = p\sum_n \mathbb{E}[|Q\cap J_n|] + O(p^2) =: p\gamma + O(p^2).$$

If $\gamma = 0$, then $\cup_n J_n = \varnothing$ so that $A_{\mathrm{hom}}^{(p)} = A_{\mathrm{hom}}$ and $v_p = 0$, and the conclusion is trivial. If $\gamma \neq 0$, since

$$\gamma \le \mathbb{E}\bigg[\sum_n \mathbb{1}_{q_n\in Q+B_R}\bigg] = \mathbb{E}[\rho(Q+B_R)] < \infty,$$

we have $v_p \simeq_\gamma p + O(p^2)$. In particular, the expansion in p in Theorem 5.1.1 at first order can as well be rewritten as an expansion in v_p: at first order at $p_0 = 0$, we have, for any $p \ge 0$,

$$A_{\mathrm{hom}}^{(p)} = A_{\mathrm{hom}} + Kv_p + O(v_p^2),$$

where K is given by

$$\xi\cdot K\xi = \frac{1}{\gamma}\xi\cdot A_{\mathrm{hom}}^{(0),1}\xi = \frac{1}{\gamma}\mathbb{E}\left[\sum_n (\nabla\phi+\xi)\cdot C^{\{n\}}(\nabla\phi^{\{n\}}+\xi)\right].$$

If in addition the random volumes $|J_n^\circ|$'s are i.i.d. and independent of the point process ρ (and of its enumeration), then γ can be computed more explicitly. By stationarity of the inclusion process, for any $L > 0$,

$$\gamma = \lim_{L\uparrow\infty} L^{-d}\mathbb{E}\bigg[\sum_n |LQ\cap J_n|\bigg],$$

where we can estimate

$$\mathbb{E}\bigg[\sum_n |LQ\cap J_n|\bigg] \le \sum_n \mathbb{E}[\mathbb{1}_{q_n\in LQ}|J_n|] = \mathbb{E}[|J_0^\circ|]\mathbb{E}[\rho(LQ)], \tag{5.113}$$

and also

$$\mathbb{E}\bigg[\sum_n |LQ\cap J_n|\bigg] \ge \sum_n \mathbb{E}[\mathbb{1}_{q_n\in (L-R)Q}|J_n|] = \mathbb{E}[|J_0^\circ|]\mathbb{E}[\rho((L-R)Q)]. \tag{5.114}$$

Now, for all continuous and integrable functions $f:\mathbb{R}^d\to\mathbb{R}$, we have $\mathbb{E}[\sum_n f(q_n)] = \mathbb{E}[\int f d\rho] = \int f d\mathbb{E}[\rho]$. Since ρ is stationary, the Borel measure $\mathbb{E}[\rho]$ is translation-invariant, and hence, since it is locally finite by definition, it is a multiple of the Lebesgue measure: $\mathbb{E}[\rho] = \sigma dx$ for some constant $\sigma\in\mathbb{R}^+$, which is characterized e.g. by $\sigma = \mathbb{E}[\rho(Q)]$. In these terms, (5.113) and (5.114) give

$$\sigma\mathbb{E}[|J_0^\circ|] = \mathbb{E}[|J_0^\circ|]\lim_L L^{-d}\mathbb{E}[\rho((L-R)Q)] \le \gamma \le \mathbb{E}[|J_0^\circ|]\lim_L L^{-d}\mathbb{E}[\rho(LQ)] = \sigma\mathbb{E}[|J_0^\circ|],$$

which means $\gamma = \sigma\mathbb{E}[|J_0^\circ|]$, and thus

$$v_p = p\sigma\mathbb{E}[|J_0^\circ|]. \tag{5.115}$$

The matrix K then takes the form

$$\xi\cdot K\xi = \frac{1}{\sigma\mathbb{E}[|J_0^\circ|]}\mathbb{E}\bigg[\sum_n (\nabla\phi+\xi)\cdot C^{\{n\}}(\nabla\phi^{\{n\}}+\xi)\bigg].$$

Further assuming that ρ is independent of A, of $(A'_n)_n$ (as well as of the random volumes $|J_n^\circ|$'s), we note that the random variable $\mathbb{E}[(\nabla\phi(0)+\xi)\cdot C^{\{n\}}(0)(\nabla\phi^{\{n\}}(0)+\xi)\|\rho]$ only depends on the point process ρ through the point q_n, so that it can be written as $f(q_n)$ for some measurable function f. In these terms, we get

$$\xi\cdot K\xi = \frac{1}{\sigma\mathbb{E}[|J_0^\circ|]}\mathbb{E}\bigg[\sum_n f(q_n)\bigg] = \frac{1}{\mathbb{E}[J_0^\circ|]}\int_{\mathbb{R}^d} f(x)dx,$$

which does clearly no longer depend on the choice of the point process ρ. This proves Corollary 5.1.3. $\square$

Electric Clausius-Mossotti formula

We consider the case when the inclusions are spherical $J_n = B_R(q_n)$, and the unperturbed and perturbed coefficients have the form $A = \alpha\,\mathrm{Id}$ and $A' = \beta\,\mathrm{Id}$ respectively. We shall compute explicitly the first derivative $\xi \cdot A_{\mathrm{hom}}^{(0),1}\xi$ of the perturbed homogenized coefficient at 0, as given by formula (5.26). As inclusions are balls of fixed radius R, assumption (5.4) implies $\rho(Q) \le \theta_0$ a.s. for some constant $\theta_0 > 0$, so that we can indeed apply formula (5.26).

Since A is constant, the unique gradient solution of $-\nabla \cdot A(\nabla\phi_\xi + \xi) = 0$ is clearly $\nabla\phi_\xi = 0$. Let now n be fixed. The solution $\phi_\xi^{\{n\}} \in H^1(\mathbb{R}^d)$ of $-\nabla \cdot A^{\{n\}}(\nabla\phi_\xi^{\{n\}} + \xi) = 0$ is easily checked to be unique if it exists, and its existence follows from a direct computation. Since $A^{\{n\}}$ is constant both inside and outside $B_R(q_n)$, the solution $\phi_\xi^{\{n\}}$ is radial and of the form

$$\phi_\xi^{\{n\}}(x) = \psi_\xi(x - q_n) = \begin{cases} C(x - q_n) \cdot \xi, & \text{for } |x - q_n| < R; \\ C' \frac{(x-q_n)\cdot\xi}{|x-q_n|^d}, & \text{for } |x - q_n| > R; \end{cases}$$

so that its gradient satisfies

$$\nabla\phi_\xi^{\{n\}}(x) = \nabla\psi_\xi(x - q_n) = \begin{cases} C\xi, & \text{for } |x - q_n| < R; \\ \frac{C'}{|x-q_n|^d}\left(\xi - d\frac{(x-q_n)\cdot\xi}{|x-q_n|}\frac{x-q_n}{|x-q_n|}\right), & \text{for } |x - q_n| > R. \end{cases} \tag{5.116}$$

Since $\phi_\xi^{\{n\}}$ is radial and in $H^1(\mathbb{R}^d)$, it is continuous, which implies that $C' = CR^d$. The normal component of $A^{\{n\}}(\nabla\phi_\xi^{\{n\}} + \xi)$ must also be continuous through the sphere, so that we conclude

$$C = \frac{\alpha - \beta}{\beta + \alpha(d-1)}. \tag{5.117}$$

This allows us to turn (5.26) into an explicit formula for the first derivative $A_{\mathrm{hom}}^{(0),1}$:

$$\begin{aligned} \xi \cdot A_{\mathrm{hom}}^{(0),1}\xi &= \sum_n \mathbb{E}[(\nabla\phi + \xi) \cdot C^{\{n\}}(\nabla\phi^{\{n\}} + \xi)] \\ &= \sum_n \mathbb{E}[\xi \cdot (\beta - \alpha)\mathbb{1}_{B_R(q_n)}(C\xi + \xi)] = (1 + C)(\beta - \alpha)\,\mathbb{E}\Big[\sum_n \mathbb{1}_{q_n \in B_R}\Big]. \end{aligned}$$

From the above paragraph, we learn that $\mathbb{E}[\sum_n f(q_n)] = \sigma \int f(x)dx$ for any continous and integrable function f. Hence,

$$\xi \cdot A_{\mathrm{hom}}^{(0),1}\xi = (1 + C)(\beta - \alpha)\,\mathbb{E}\Big[\sum_n \mathbb{1}_{B_R}(q_n)\Big] = \sigma|B_R|(1 + C)(\beta - \alpha),$$

so that expression (5.117) for C yields

$$\xi \cdot A_{\mathrm{hom}}^{(0),1}\xi = \sigma|B_R|\frac{\alpha d(\beta - \alpha)}{\beta + \alpha(d-1)}.$$

In the present case, formula (5.115) holds true and gives $v_p = p\sigma|B_R|$. The conclusion of Corollary 5.1.4 now follows from Theorem 5.1.1. □

Elastic Clausius-Mossotti formula

We consider the case of spherical inclusions $J_n = B_R(q_n)$ and assume that both the unperturbed stiffness tensor A and the perturbed stiffness tensor A' are constant and isotropic — we denote by K, G and K', G' their respective bulk and shear moduli. We shall compute explicitly in that case the first derivative $\xi \cdot A_{\mathrm{hom}}^{(0),1}\xi$ of the perturbed homogenized stiffness tensor $A_{\mathrm{hom}}^{(p)}$ at 0, as given by formula (5.26). Indeed, as inclusions are balls of fixed radius R, assumption (5.4) implies $\rho(Q) \le \theta_0$ a.s. for some constant $\theta_0 > 0$, so that we can apply formula (5.26).

Let $\xi \in \mathbb{R}^{d\times d}$ be symmetric. Since A is constant, the unique gradient solution of $-\nabla \cdot A : (\nabla\phi_\xi + \xi) = 0$ is clearly $\nabla\phi_\xi = 0$. Let now n be fixed. As shown e.g. in Section 17.2.1 of [413], equation $-\nabla\cdot A^{\{n\}} : (\nabla\phi_\xi^{\{n\}} + \xi)$ admits a (necessarily unique) solution in $H^1(\mathbb{R}^d)$. Inside the inclusion $B_R(q_n)$, that is, for all $|x - q_n| < R$ (see [413, equation (17.84)]),

$$\nabla\phi_\xi^{\{n\}}(x) + \xi = \mathrm{Id}\,\frac{\mathrm{Tr}\,\xi}{d}\,\frac{K+\beta}{K'+\beta} + \left(\xi - \mathrm{Id}\,\frac{\mathrm{Tr}\,\xi}{d}\right)\frac{G+\alpha}{G'+\alpha}, \tag{5.118}$$

where α, β are defined by (5.28). Recalling that $\xi : A : \chi = 2G\xi : \chi + \lambda\,\mathrm{Tr}\,\xi\,\mathrm{Tr}\,\chi$ for any symmetric $\chi \in \mathbb{R}^{d\times d}$, we can now explicitly compute formula (5.26) for the first derivative $A_{\mathrm{hom}}^{(0),1}$,

$$\begin{aligned}\frac12 \xi : A_{\mathrm{hom}}^{(0),1} : \xi &= \frac12 \sum_n \mathbb{E}\left[\mathbb{1}_{B_R(q_n)}\xi : (A'-A) : (\nabla\phi_\xi^{\{n\}} + \xi)\right]\\ &= \sum_n \mathbb{E}\left[\mathbb{1}_{B_R(q_n)}\Big((G'-G)\xi : (\nabla\phi_\xi^{\{n\}}+\xi) + \frac12(\lambda'-\lambda)\,\mathrm{Tr}\,\xi\,\mathrm{Tr}(\nabla\phi_\xi^{\{n\}}+\xi)\Big)\right],\end{aligned}$$

and hence, using (5.118), and recalling that $\mathbb{E}[\sum_n \mathbb{1}_{B_R(q_n)}] = \sigma|B_R|$,

$$\frac{1}{2\sigma|B_R|}\xi : A_{\mathrm{hom}}^{(0),1} : \xi = \frac1d(\mathrm{Tr}\,\xi)^2\frac{K+\beta}{K'+\beta}\Big((G'-G) + \frac d2(\lambda'-\lambda)\Big) + \Big(|\xi|^2 - \frac1d(\mathrm{Tr}\,\xi)^2\Big)(G'-G)\frac{G+\alpha}{G'+\alpha}.$$

In terms of bulk moduli $K = \lambda + 2G/d$ and $K' = \lambda' + 2G'/d$, this takes the form

$$\frac{1}{2\sigma|B_R|}\xi : A_{\mathrm{hom}}^{(0),1} : \xi = \frac12(\mathrm{Tr}\,\xi)^2(K'-K)\frac{K+\beta}{K'+\beta} + \Big(|\xi|^2 - \frac1d(\mathrm{Tr}\,\xi)^2\Big)(G'-G)\frac{G+\alpha}{G'+\alpha}. \tag{5.119}$$

As formula (5.115) again holds true in the present case and gives $v_p = p\sigma|B_R|$, Corollary 5.1.5 now follows by Theorem 5.1.1. ☐

5.3.4 Proof of Corollary 5.1.6: convergence rates

Let $p_0 \in [0,1]$ be fixed, and assume that $\mathbb{E}[\rho(Q)^s] < \infty$ for all $s \ge 1$. Estimate (5.100) in the proof of Theorem 5.1.1 (see Section 5.3.2) then yields, for all $k \ge 1$,

$$\begin{aligned}\left|\sum_{j=1}^k p^j(\Delta_T^{(p_0),j} - \Delta_{2T}^{(p_0),j})\right| &\le (Cp)^{k+1} + |A_T^{(p_0+p)} - A_{2T}^{(p_0+p)}| + |A_T^{(p_0)} - A_{2T}^{(p_0)}|\\ &\lesssim (Cp)^{k+1} + \mathbb{E}[|\nabla(\phi_T^{(p_0+p)} - \phi_{2T}^{(p_0+p)})|] + \mathbb{E}[|\nabla(\phi_T^{(p_0)} - \phi_{2T}^{(p_0)})|],\end{aligned}$$

and hence, combining this with assumption (5.29),

$$\left|\sum_{j=1}^k p^j(\Delta_T^{(p_0),j} - \Delta_{2T}^{(p_0),j})\right| \le (Cp)^{k+1} + C\gamma(T). \tag{5.120}$$

By induction, we easily see that this implies, for all $j \geq 1$,

$$|\Delta_T^{(p_0),j} - \Delta_{2T}^{(p_0),j}| \leq (2C)^{j+1}\gamma(T)^{2^{-j}}. \tag{5.121}$$

Estimate (5.120) with $k = 1$ gives $|\Delta_T^{(p_0),j} - \Delta_{2T}^{(p_0),j}| \leq C^2 p + C\gamma(T)/p$, which turns into (5.121) for $j = 1$ with the choice $p = \gamma(T)^{\frac{1}{2}}$. Assume now that the result (5.121) is proven for all $0 \leq j \leq J$. Then, equation (5.120) for $k = J+1$ gives

$$|\Delta_T^{(p_0),J+1} - \Delta_{2T}^{(p_0),J+1}| \leq C^{J+2}p + Cp^{-J-1}\gamma(T) + \sum_{j=1}^{J}(2C)^j p^{j-J-1}\gamma(T)^{2^{-j}}.$$

With the choice $p = \gamma(T)^{2^{-J-1}}$, and noting that $(l+1)2^{-l} \leq 1$ for any $l \in \mathbb{N}$, this turns into

$$\begin{aligned}
|\Delta_T^{(p_0),J+1} - \Delta_{2T}^{(p_0),J+1}| &\leq C^{J+2}\gamma(T)^{2^{-J-1}} + C\gamma(T)^{1-(J+1)2^{-J-1}} \\
&\qquad + \sum_{j=1}^{J}(2C)^{j+1}\gamma(T)^{2^{-j}(1-(J+1-j)2^{-(J+1-j)})} \\
&\leq C^{J+2}\gamma(T)^{2^{-J-1}} + C\gamma(T)^{2^{-1}} + \sum_{j=1}^{J}(2C)^{j+1}\gamma(T)^{2^{-j-1}} \\
&\leq C^{J+2}\gamma(T)^{2^{-J-1}}\Big(2 + \sum_{j=1}^{J} 2^{j+1}\Big) \leq (2C)^{J+2}\gamma(T)^{2^{-J-1}},
\end{aligned}$$

which proves (5.121) by induction, and concludes the proof of Corollary 5.1.6. $\square$

5.A Appendix: Relaxing the finite penetrability assumption

As discussed in Section 5.1.5, the finite penetrability assumption (5.4) is crucially used in the proof of the analyticity result, and it is unclear to us how it can be relaxed in general. In particular, our approach cannot treat the natural example of Poisson spherical inclusions (that is, $\rho = (q_n)_n$ is a Poisson point process and we set $J_n := B(q_n)$ with e.g. constant reference coefficients A and A'). In this specific example, the perturbed random fields $A^{(p)}$ all satisfy a standard spectral gap in the probability space and the point process satisfies $\mathbb{E}\left[\rho(Q)^s\right] < \infty$ for all $s \geq 1$ (and even $\mathbb{E}[e^{c\rho(Q)}] < \infty$ for all $c > 0$). In such a situation, the quantitative theory of stochastic homogenization provides additional analytical tools (as used in the discrete setting in Mourrat's contribution [327]), which allow to prove that the map $p \mapsto A_{\mathrm{hom}}^{(p)}$ is at least C^∞ on $[0, 1]$ with derivatives given by the same explicit formulas as before, in particular justifying the Clausius-Mossotti formulas in the form of Corollaries 5.1.4 and 5.1.5. More precisely, the strategy consists in establishing the version (5.58) with improved integrability of the energy estimates of Theorem 5.2.6, based on Green's representation formulas and optimal annealed estimates on Green's functions [313, 201]. Let us emphasize that this approach requires quantitative ergodicity assumptions on the random fields themselves (and not only on the point process), which contrasts dramatically with the other results in this chapter.

5.A.1 Main result

Assumptions and notation

Let the same assumptions and notation hold as in Section 5.1.2, except that the finite penetrability assumption (5.4) is replaced by the following assumption on the point process ρ: for all $s \geq 1$ we have

$$M_s^s := \mathbb{E}[\rho(Q)^s] < \infty. \tag{5.122}$$

For all subset $E \subset \mathbb{N}$ and all $T > 0$, we let $G_T^E : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ denote the Green's function associated with the (massive) random elliptic operator $\frac{1}{T} - \nabla \cdot A^E \nabla$, which is defined for all $y \in \mathbb{R}^d$ and all $\omega \in \Omega$ as the unique distributional solution in $W^{1,1}(\mathbb{R}^d)$, continuous in $\mathbb{R}^d \setminus \{y\}$, of the equation

$$\frac{1}{T} G_T^E(x, y; \omega) - \nabla_x \cdot A(x, \omega) \nabla_x G_T^E(x, y; \omega) = \delta(x - y).$$

We let $\nabla_1 \nabla_2 G_T^E$ denote the mixed gradient of this Green's function, and we use the short-hand notation $G_T := G_T^\varnothing$ and $G_T^{(p)} := G_T^{E^{(p)}}$. Denoting as follows the local quadratic averages of maps $f : \mathbb{R}^d \to \mathbb{R}^m$ and $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^m$,

$$(f)_L(x) := \left(\fint_{B_L(x)} |f|^2 \right)^{1/2}, \qquad (g)_{L_1, L_2}(x, y) = \left(\fint_{B_L(x)} \fint_{B_L(y)} |g(v, w)|^2 dw dv \right)^{1/2},$$

we assume that the following quantitative properties also hold for all $p \in [0, 1]$ and all $\theta \in (0, \infty]$ (where we set $\rho_\infty := \rho$ and where the ρ_θ's are the hardcore approximations of ρ as constructed in Step 1 of Section 5.3.2),

(H$_1$) *Corrector gradient bounds*: for all $r \geq 1$ and $L \simeq 1$, we have (uniformly in $\theta > 0$)

$$\mathbb{E}\left[(\nabla \phi_{T,\theta}^{(p)})_L^{2r} \right]^{\frac{1}{2r}} \lesssim_r 1;$$

(H$_2$) *Optimal annealed bounds on the Green's function*: for all $r \geq 1$, $L_1, L_2 \simeq 1$ and $|y| \geq L_1 + L_2 + 1$, we have (uniformly in $\theta > 0$)

$$\mathbb{E}\left[(\nabla_2 G_{T,\theta}^{(p)})_{L_1, L_2}^r (0, y) \right]^{\frac{1}{r}} \lesssim_r |y|^{1-d} e^{-\frac{1}{C\sqrt{T}}|y|},$$

$$\mathbb{E}\left[(\nabla_1 \nabla_2 G_{T,\theta}^{(p)})_{L_1, L_2}^r (0, y) \right]^{\frac{1}{r}} \lesssim_r |y|^{-d} e^{-\frac{1}{C\sqrt{T}}|y|} =: \gamma_T(y);$$

(H$_3$) *Convergence of correctors*: there exists a function $\gamma(T) \le T^{-\varepsilon}$, $\varepsilon > 0$, such that

$$\mathbb{E}\big[|\nabla(\phi_T^{(p)} - \phi_{2T}^{(p)})|^2\big] \lesssim \gamma(T)^2,$$

which implies in particular

$$|A_{\mathrm{hom}}^{(p)} - A_T^{(p)}| \lesssim \gamma(T).$$

Note that these assumptions are relevant to the quantitative theory of stochastic homogenization, as they are known to hold under the assumption that the coefficients satisfy suitable functional inequalities in the probability space [209, 210, 212, 313, 201, 203, 49]. In the case of fast decaying correlations the optimal expected rate $\gamma(T)$ in (H$_3$) is again given by (5.31). In particular, these assumptions hold e.g. for the example of Poisson spherical inclusions with constant reference fields A and A'.

Statement of main result

We establish the following C^∞ regularity result for the perturbed homogenized coefficients with respect to the Bernoulli parameter.

Theorem 5.A.1. *Under the above assumptions, the map $p \mapsto A_{\mathrm{hom}}^{(p)}$ is of class C^∞ on $[0,1]$, and the derivatives are equivalently given by each of the three formulas (5.23), (5.24), and (5.25) of Corollary 5.1.2, where the limits exist and where the sums are absolutely convergent for any fixed $T, \theta < \infty$. Moreover, at $p_0 = 0$, the θ-regularizations can be omitted in each of these formulas. In particular, the first-order invariance principle of Corollary 5.1.3 is still valid, as well as the electric and elastic Clausius-Mossotti formulas of Corollaries 5.1.4 and 5.1.5.* $\lozenge$

Remark 5.A.2. With similar quantitative methods as those developed in this appendix, we may actually prove the following cluster expansion formula for the approximate corrector $\phi_T^{(p)}$: for all $k \ge 1$, $\varepsilon, p \in (0,1)$, we have

$$\mathbb{E}\bigg[\Big(\nabla\phi_T^{(p)} - \sum_{j=0}^{k} \sum_{\substack{F \subset E^{(p)} \\ |F|=j}} \nabla\delta^F \phi_T\Big)_L^{1+\varepsilon}\bigg]^{\frac{1}{1+\varepsilon}} \lesssim_{\varepsilon,k} p^{\frac{1-\varepsilon}{1+\varepsilon}(k+1)}(1+\log T)^{k+1}.$$

In the spirit of Mourrat's work [327], we could deduce Theorem 5.A.1 from such an expansion. Nevertheless, we prefer to keep here the same proof strategy as in the rest of the chapter. $\lozenge$

5.A.2 Proof of Theorem 5.A.1

Green's function estimates

We start with estimating perturbed Green's functions in terms of unperturbed ones. Such estimates are crucial since only the Green's functions G_T and $G_T^{(p)}$ are associated with stationary coefficients and satisfy the optimal annealed estimates of assumption (H$_2$).

Lemma 5.A.3. *For all $F \subset \mathbb{N}$ and $n \in F$, for all $L_1, L_2 \simeq 1$ and all x, y with $|x-y| \ge L_1 + L_2 + 1$, we have*

$$\begin{aligned}(\nabla_1\nabla_2 G_T^F)_{L_1,L_2}(x,y) &\lesssim (\nabla_1\nabla_2 G_T^{F\setminus\{n\}})_{L_1,L_2}(x,y)\\ &\qquad + \Big((\nabla_1\nabla_2 G_T^{F\setminus\{n\}})_{R,L_2}(q_n,y) \wedge (\nabla_1\nabla_2 G_T^{F\setminus\{n\}})_{L_1,R}(x,q_n)\Big).\end{aligned}$$

In particular, for all $x_0 \in \mathbb{R}^d$, $L \simeq 1$, and all distinct $n_1, \ldots, n_k \in \mathbb{N}$, we have

$$(\nabla_1 \nabla_2 G_T^{\{n_1,\ldots,n_k\}})_{L,R}(x_0, q_{n_1}) \lesssim C^k \sum_{j=1}^{k} (\nabla_1 \nabla_2 G_T)_{L,R}(x_0, q_{n_j}). \qquad \Diamond$$

Proof. We can assume smoothness of all considered coefficients, since the general result is then deduced by a standard approximation argument. The difference $\nabla_y (G_T^F - G_T^{F\setminus\{n\}})(\cdot, y)$ by definition satisfies the following equation, for $|q_n - y| > R$,

$$\frac{1}{T}\nabla_y (G_T^F - G_T^{F\setminus\{n\}})(x,y) - \nabla_x \cdot A^F(x) \nabla_x \nabla_y (G_T^F - G_T^{F\setminus\{n\}})(x,y) \\ = \nabla_x \cdot C_{\{n\}\|F\setminus\{n\}}(x) \nabla_x \nabla_y G_T^{F\setminus\{n\}}(x,y).$$

As a function of x, the right-hand side is the divergence of a bounded compactly supported function. This implies that the solution $\nabla_y (G_T^F - G_T^{F\setminus\{n\}})(\cdot, y)$ belongs to $H^1(\mathbb{R}^d)$. Testing the equation against $\nabla_y (G_T^F - G_T^{F\setminus\{n\}})(\cdot, y)$ then yields, for all $|q_n - y| > R$,

$$\int_{\mathbb{R}^d} |\nabla_x \nabla_y (G_T^F - G_T^{F\setminus\{n\}})(x,y)|^2 dx \lesssim \int_{B_R(q_n)} |\nabla_x \nabla_y G_T^{F\setminus\{n\}}(x,y)|^2 dx,$$

and hence

$$(\nabla_1 \nabla_2 G_T^F)_{L_1,L_2}(x,y) \lesssim (\nabla_1 \nabla_2 G_T^{F\setminus\{n\}})_{L_1,L_2}(x,y) + (\nabla_1 \nabla_2 G_T^{F\setminus\{n\}})_{R,L_2}(q_n,y).$$

The statement follows by symmetry. $\square$

With this result at hand, we now establish the following useful deterministic refinement of the a priori estimate of Lemma 5.2.2. Note that the prefactor obtained in the proof below is a priori of order C^{k^2}, thus forbidding any hope of establishing analyticity nor any Gevrey regularity. (See however the discussion in Section 5.1.5.)

Lemma 5.A.4. *For all $x \in \mathbb{R}^d$ and all $F, H \subset \mathbb{N}$ with $|F| = k \geq 1$, setting $q_{n_0} := x$, we have*

$$\sum_{|F|=k} (\nabla \delta^F \phi_T^H)_R(x) \\ \lesssim_k \sum_{j=0}^{k-1} \sum_{\substack{n_1,\ldots,n_{j+1} \\ distinct}} (\nabla \phi_T^H + \xi)_R(q_{n_{j+1}}) \bigg(\prod_{l=0}^{j} \sum_{i=l+1}^{j+1} 1 \wedge (\nabla_1 \nabla_2 G_T^H)_{3R,3R}(q_{n_l}, q_{n_i}) \bigg) \bigg(\prod_{i=1}^{j+1} \rho(B_{2R}(q_{n_i}))^k \bigg). \ \Diamond$$

Proof. As the estimate is deterministic, it is enough to consider the case $H = \varnothing$. Let $|F| = k \geq 1$. By Lemma 5.2.1, $\delta^F \phi_T$ satisfies

$$\frac{1}{T}\delta^F \phi_T - \nabla \cdot A^F \nabla \delta^F \phi_T = \sum_{S \subset F} (-1)^{|S|+1} \nabla \cdot C_{S\|F\setminus S} \nabla \delta_\xi^{F\setminus S} \phi_T.$$

Let $\mathcal{F}_1 \biguplus \mathcal{F}_2 = \{S : S \subset F\}$ be a given partition of the set of all subsets of F. For all $S \in \mathcal{F}_2$, define ψ_1 and ψ_2^S as the (unique) solutions in $H^1(\mathbb{R}^d)$ of equations

$$\frac{1}{T}\psi_1 - \nabla \cdot A^F \nabla \psi_1 = \sum_{S \in \mathcal{F}_1} (-1)^{|S|+1} \nabla \cdot C_{S\|F\setminus S} \nabla \delta_\xi^{F\setminus S} \phi_T,$$

$$\frac{1}{T}\psi_2^S - \nabla \cdot A^F \nabla \psi_2^S = (-1)^{|S|+1} \nabla \cdot C_{S\|F\setminus S} \nabla \delta_\xi^{F\setminus S} \phi_T. \qquad (5.123)$$

By linearity, we have $\delta^F\phi_T = \psi_1 + \sum_{S\in\mathcal{F}_2}\psi_2^S$. On the one hand, assuming smoothness of the coefficients, the Green's formula yields

$$|\nabla\psi_1(x)| \lesssim \sum_{S\in\mathcal{F}_1}\int_{J_{S\|F\setminus S}} |\nabla_1\nabla_2 G_T^F(x,y)|\,|\nabla\delta_\xi^{F\setminus S}\phi_T(y)|dy,$$

and hence, taking local averages,

$$(\nabla\psi_1)_R(x) \lesssim \sum_{S\in\mathcal{F}_1}\bigg(\int_{J_{S\|F\setminus S}}(\nabla_1\nabla_2 G_T^F(\cdot,y))_R^2(x)dy\bigg)^{1/2}\bigg(\int_{J_{S\|F\setminus S}}|\nabla\delta_\xi^{F\setminus S}\phi_T|^2\bigg)^{1/2}.$$

On the other hand, for all $S\in\mathcal{F}_2$, an a priori estimate for equation (5.123) yields

$$\int_{\mathbb{R}^d}|\nabla\psi_2^S|^2 \lesssim \int_{J_{S\|F\setminus S}}|\nabla\delta_\xi^{F\setminus S}\phi_T|^2.$$

Combining these estimates with the decomposition $\delta^F\phi_T = \psi_1 + \sum_{S\in\mathcal{F}_2}\psi_2^S$, and suitably choosing the partition $\mathcal{F}_1\biguplus\mathcal{F}_2$, we conclude

$$(\nabla\delta^F\phi_T)_R(x) \lesssim \sum_{S\subset F}\bigg(1\wedge\int_{J_{S\|F\setminus S}}(\nabla_1\nabla_2 G_T^F(\cdot,y))_R^2(x)dy\bigg)^{1/2}\bigg(\int_{J_{S\|F\setminus S}}|\nabla\delta_\xi^{F\setminus S}\phi_T|^2\bigg)^{1/2},$$

or equivalently, noting that for $n\in S$ we have $J_{S\|F\setminus S}\subset J_n\subset B_R(q_n)$,

$$(\nabla\delta^F\phi_T)_R(x) \lesssim \sum_{n\in F}\big(1\wedge(\nabla_1\nabla_2 G_T^F)_{R,R}(x,q_n)\big)\sum_{\substack{S\subset F\\ n\in S}}(\nabla\delta_\xi^{F\setminus S}\phi_T)_R(q_n)\,\mathbb{1}_{J_S\neq\varnothing}.$$

Setting

$$S_T^F(x) := \sum_{n\in F} 1\wedge(\nabla_1\nabla_2 G_T)_{R,R}(x,q_n),$$

and noting that for $n\in F$ Lemma 5.A.3 gives $1\wedge(\nabla_1\nabla_2 G_T^F)_{R,R}(x,q_n)\le C^k S_T^F(x)$, the above takes the form

$$(\nabla\delta^F\phi_T)_R(x) \le C^k S_T^F(x)\sum_{S\subset F}\mathbb{1}_{J_{F\setminus S}\neq\varnothing}\sum_{n\in F\setminus S}(\nabla\delta^S\phi_T)_R(q_n).$$

By induction, a repeated use of this estimate leads to

$$\begin{aligned}(\nabla\delta^F\phi_T)_R(x) \lesssim_k S_T^F(x)\sum_{j=0}^{k-1}\sum_{S_1\subset F}\mathbb{1}_{J_{F\setminus S_1}\neq\varnothing}\sum_{n_1\in F\setminus S_1}S_T^{S_1}(q_{n_1})\sum_{S_2\subset S_1}\mathbb{1}_{J_{S_1\setminus S_2}\neq\varnothing}\sum_{n_2\in S_1\setminus S_2}S_T^{S_2}(q_{n_2})\ldots\\ \ldots\sum_{S_j\subset S_{j-1}}\mathbb{1}_{J_{S_{j-1}\setminus S_j}\neq\varnothing}\sum_{n_j\in S_{j-1}\setminus S_j}S_T^{S_j}(q_{n_j})\mathbb{1}_{J_{S_j}\neq\varnothing}\sum_{m\in S_j}(\nabla\phi_T+\xi)_R(q_m),\end{aligned}$$

where we have set $S_{-1}:=F$. Further setting $q_{n_0}:=x$, and disjointifying the subsets S_j's, this becomes

$$\begin{aligned}&(\nabla\delta^F\phi_T)_R(x)\\ &\lesssim_k \sum_{j=0}^{k-1}\sum_{\substack{S_1,\ldots,S_{j+1}\\ F=\biguplus_i S_i}}\mathbb{1}_{J_{S_1},\ldots,J_{S_{j+1}}\neq\varnothing}\sum_{n_1\in S_1}\cdots\sum_{n_{j+1}\in S_{j+1}}\bigg(\prod_{l=0}^{j}S_T^{\biguplus_{i=l+1}^{j+1}S_i}(q_{n_l})\bigg)(\nabla\phi_T+\xi)_R(q_{n_{j+1}}).\end{aligned}\tag{5.124}$$

Setting $\gamma_T(x,y) := 1 \wedge (\nabla_1\nabla_2 G_T)_{R,R}(x,y)$ and $\tilde\gamma_T(x,y) := 1 \wedge (\nabla_1\nabla_2 G_T)_{3R,3R}(x,y)$, we find

$$S_T^{\biguplus_{i=l+1}^{j+1} S_i}(q_{n_l}) = \sum_{i=l+1}^{j+1} \sum_{r\in S_i} \gamma_T(q_{n_l}, q_r),$$

and hence, given $J_{S_1},\ldots,J_{S_{j+1}} \neq \varnothing$ and $n_1 \in S_1,\ldots,n_{j+1}\in S_{j+1}$, we may estimate

$$S_T^{\biguplus_{i=l+1}^{j+1} S_i}(q_{n_l}) \lesssim \sum_{i=l+1}^{j+1} |S_i| \tilde\gamma_T(q_{n_l}, q_{n_i}) \lesssim_k \sum_{i=l+1}^{j+1} \tilde\gamma_T(q_{n_l}, q_{n_i}).$$

Combining this estimate with (5.124), summing over all subsets $F \subset \mathbb{N}$ with $|F| = k$, and relabeling the sums, we obtain

$$\begin{aligned}\sum_{|F|=k} (\nabla\delta^F\phi_T)_R(x) \lesssim_k \sum_{j=0}^{k-1} \sum_{\substack{n_1,\ldots,n_{j+1}\\ \text{distinct}}} \Big(\prod_{l=0}^{j} \sum_{i=l+1}^{j+1} \tilde\gamma_T(q_{n_l}, q_{n_i})\Big) (\nabla\phi_T + \xi)_R(q_{n_{j+1}}) \\ \times \sum_{|F|=k-j} \sum_{\substack{S_1,\ldots,S_{j+1}\\ F=\biguplus_i S_i}} \mathbb{1}_{J_{S_1}\cap J_{n_1},\ldots,J_{S_{j+1}}\cap J_{n_{j+1}} \neq \varnothing}. \qquad (5.125)\end{aligned}$$

The last sum is estimated as follows,

$$\sum_{|F|=k-j} \sum_{\substack{S_1,\ldots,S_{j+1}\\ F=\biguplus_i S_i}} \mathbb{1}_{J_{S_1}\cap J_{n_1},\ldots,J_{S_{j+1}}\cap J_{n_{j+1}} \neq \varnothing} \le \prod_{i=1}^{j+1} \sum_{|S|\le k} \mathbb{1}_{J_S\cap J_{n_i}\neq\varnothing} \le \prod_{i=1}^{j+1} \sum_{l=0}^{k} \binom{\rho(B_{2R}(q_{n_i}))}{l},$$

and hence, using that $\binom{n}{k} \le (en/k)^k$ and $\sum_{l=0}^\infty (e/l)^l \lesssim 1$,

$$\sum_{|F|=k-j} \sum_{\substack{S_1,\ldots,S_{j+1}\\ F=\biguplus_i S_i}} \mathbb{1}_{J_{S_1}\cap J_{n_1},\ldots,J_{S_{j+1}}\cap J_{n_{j+1}} \neq \varnothing} \lesssim \prod_{i=1}^{j+1} \rho(B_{2R}(q_{n_i}))^k.$$

Injecting this into (5.125), the conclusion follows. □

Random integration lemma

The following lemma makes precise how sums over the points of the point process ρ can be estimated in expectation by spatial integrals.

Lemma 5.A.5. *Let $k \ge 1$ and $L_1, L_2 \simeq 1$ be fixed. Let g_{L_1,L_2} be a random function $(\mathbb{R}^d)^{k+1} \to [0,\infty]$ (which may also depend on the point process ρ), such that there is a positive constant $c \simeq 1$ for which we have almost surely for all $(x_i)_{i=0}^k, (y_i)_{i=0}^k \in \mathbb{R}^{k+1}$ with $\max_i |x_i - y_i| \le 1$,*

$$g_{L_1,L_2}(y_0,\ldots,y_k) \lesssim g_{L_1+c,L_2+c}(x_0,\ldots,x_k). \qquad (5.126)$$

Then for all $s \ge 1$ and $r > 1$ we have

$$I_s := \mathbb{E}\Big[\Big(\sum_{\substack{n_0,\ldots,n_k\\ \textit{distinct}}} g_{L_1,L_2}(q_{n_0},\ldots,q_{n_k})\Big)^s\Big]^{1/s} \lesssim_{s,r} \int_{(\mathbb{R}^d)^{k+1}} \mathbb{E}\left[g_{L_1+2c,L_2+2c}(x_0,\ldots,x_k)^{rs}\right]^{\frac{1}{rs}} dx_0\ldots dx_k.$$

◊

Proof. Let $\varepsilon \in (0,1)$ and $s \geq 1$ be fixed. Let $\mathbb{E}_{\|\rho}$ denote the conditional expectation given the point process ρ. Using the triangle inequality and sampling the point process on discrete cells of size $\gamma := 2/\sqrt{d}$, we find

$$I_s^\rho := \mathbb{E}_{\|\rho}\bigg[\Big(\sum_{\substack{n_0,\ldots,n_k\\ \text{distinct}}} g_{L_1,L_2}(q_{n_0},\ldots,q_{n_k})\Big)^s\bigg]^{1/s} \leq \sum_{\substack{n_0,\ldots,n_k\\ \text{distinct}}} \mathbb{E}_{\|\rho}[g_{L_1,L_2}(q_{n_0},\ldots,q_{n_k})^s]^{1/s}$$
$$\leq \sum_{x_0,\ldots,x_k\in\gamma\mathbb{Z}^d}\ \sum_{\substack{n_0,\ldots,n_k\\ \text{distinct}}} \mathbb{1}_{q_{n_0}\in x_0+\gamma Q,\ldots,q_{n_k}\in x_k+\gamma Q}\ \mathbb{E}_{\|\rho}[g_{L_1,L_2}(q_{n_0},\ldots,q_{n_k})^s]^{1/s},$$

As $q_{n_i} \in x_i + \gamma Q$ implies $|q_{n_i} - x_i| \leq 1$, assumption (5.126) then leads almost surely to

$$I_s^\rho \leq \sum_{x_0,\ldots,x_k\in\gamma\mathbb{Z}^d} \mathbb{E}_{\|\rho}[g_{L_1+c,L_2+c}(x_0,\ldots,x_k)^s]^{1/s} \sum_{\substack{n_0,\ldots,n_k\\ \text{distinct}}} \mathbb{1}_{q_{n_0}\in x_0+\gamma Q,\ldots,q_{n_k}\in x_k+\gamma Q}$$
$$\leq \sum_{x_0,\ldots,x_k\in\gamma\mathbb{Z}^d} \mathbb{E}_{\|\rho}[g_{L_1+c,L_2+c}(x_0,\ldots,x_k)^s]^{1/s} \prod_{j=0}^k \rho(x_j+\gamma Q).$$

Taking the expectation, using the triangle and the Hölder inequalities, and using the stationarity of the point process ρ, we obtain for all $r > 1$,

$$I_s = \mathbb{E}[(I_s^\rho)^s]^{1/s} \leq \sum_{x_0,\ldots,x_k\in\gamma\mathbb{Z}^d} \mathbb{E}\bigg[\mathbb{E}_{\|\rho}[g_{L_1+c,L_2+c}(x_0,\ldots,x_k)^s] \prod_{j=0}^k \rho(x_j+\gamma Q)^s\bigg]^{1/s}$$
$$\leq \sum_{x_0,\ldots,x_k\in\gamma\mathbb{Z}^d} \mathbb{E}\big[g_{L_1+c,L_2+c}(x_0,\ldots,x_k)^{rs}\big]^{\frac{1}{rs}}\ \mathbb{E}\big[\rho(\gamma Q)^{sr'(k+1)}\big]^{\frac{1}{r's}},$$

and hence, by assumption (5.122),

$$I_s \lesssim_{s,r,k} \sum_{x_0,\ldots,x_k\in\gamma\mathbb{Z}^d} \mathbb{E}[g_{L_1+c,L_2+c}(x_0,\ldots,x_k)^{rs}]^{\frac{1}{rs}}.$$

Taking local averages and using again assumption (5.126), the conclusion follows in the form

$$I_s \lesssim_{s,r,k} \sum_{x_0,\ldots,x_k\in\gamma\mathbb{Z}^d} \fint_{(\gamma Q)^{k+1}} \mathbb{E}\left[g_{L_1+2c,L_2+2c}(x_0+y_0,\ldots,x_k+y_k)^{rs}\right]^{\frac{1}{rs}} dy_0\ldots dy_k$$
$$\lesssim \int_{(\mathbb{R}^d)^{k+1}} \mathbb{E}[g_{L_1+2c,L_2+2c}^{rs}(y_0,\ldots,y_k)]^{\frac{1}{rs}} dy_0\ldots dy_k. \qquad \square$$

Improved energy estimates

Applying Lemma 5.A.4 together with the optimal Green's function estimates of assumption (H_2), we deduce the following version of the improved energy estimates of Proposition 5.2.6 (in particular refining the rough estimate of Lemma 5.2.2).

Proposition 5.A.6. *For all $k \geq 1$ and all $s \geq 1$ we have (uniformly in T)*

$$S_{k,s} := \mathbb{E}\bigg[\Big(\sum_{|F|=k} (\nabla\delta^F\phi_T)_R\Big)^{2s}\bigg] \lesssim_{k,s} (1+\log T)^{2sk}. \qquad \lozenge$$

Remark 5.A.7. The above result implies in particular, for all $k \geq 1$, $0 \leq j \leq k$, $l \geq 0$,

$$T_{j,k,l} := \mathbb{E}\bigg[\rho(B_R)^l \sum_{|G|=k-j} \Big| \sum_{\substack{|F|=j \\ F\cap G=\varnothing}} \nabla \delta^{F\cup G}\phi_T\Big|^2\bigg] \lesssim_{k,l} (1+\log T)^{2k}. \tag{5.127}$$

Indeed, using stationarity and the triangle inequality, we find

$$T_{j,k,l} \leq \mathbb{E}\bigg[\rho(B_{2R})^l \fint_{B_R} \sum_{|G|=k-j} \Big| \sum_{\substack{|F|=j \\ F\cap G=\varnothing}} \nabla \delta^{F\cup G}\phi_T\Big|^2\bigg] \leq \mathbb{E}\bigg[\rho(B_{2R})^l \sum_{|G|=k-j} \Big(\sum_{\substack{|F|=j \\ F\cap G=\varnothing}} (\nabla \delta^{F\cup G}\phi_T)_R\Big)^2\bigg].$$

The discrete $\ell^1 - \ell^2$ inequality then yields

$$T_{j,k,l} \lesssim_k \mathbb{E}\bigg[\rho(B_{2R})^l \Big(\sum_{|F|=k} (\nabla \delta^{F}\phi_T)_R\Big)^2\bigg],$$

so that the result (5.127) follows from Proposition 5.A.6 together with assumption (5.122). $\diamondsuit$

Proof of Proposition 5.A.6. Applying Lemmas 5.A.4 and 5.A.5, and setting $L := 3R+1 \simeq 1$ and $x_0 := 0$, we find for all $r > 1$,

$$(S_{k,s})^{\frac{1}{2s}} \lesssim_{k,s,r} \sum_{j=0}^{k-1} \int_{(\mathbb{R}^d)^{j+1}} \mathbb{E}\bigg[(\nabla\phi_T+\xi)_L^{2sr}(x_{j+1}) \Big(\prod_{l=0}^{j} \sum_{i=l+1}^{j+1} 1 \wedge (\nabla\nabla G_T)_{L,L}^{2sr}(x_l,x_i)\Big) \\ \times \Big(\prod_{i=1}^{j+1} \rho(B_L(x_i))^{2srk}\Big)\bigg]^{\frac{1}{2sr}} dx_1 \ldots dx_{j+1}.$$

Using assumptions (5.122) and (H_1), the Hölder inequality then yields

$$(S_{k,s})^{\frac{1}{2s}} \lesssim_{k,s,r} \sum_{j=0}^{k-1} \int_{(\mathbb{R}^d)^{j+1}} \prod_{l=0}^{j} \sum_{i=l+1}^{j+1} \Big(1 \wedge \mathbb{E}\Big[1 \wedge (\nabla\nabla G_T)_{L,L}^{4jsr}(x_l,x_i)\Big]^{\frac{1}{4jsr}}\Big) dx_1 \ldots dx_{j+1},$$

and hence, by assumption (H_2), recalling that $x_0 := 0$,

$$(S_{k,s})^{\frac{1}{2s}} \lesssim_{k,s,r} \sum_{j=0}^{k-1} \int_{(\mathbb{R}^d)^{j+1}} \Big(\sum_{i=1}^{j+1} \gamma_T(x_i)\Big) \prod_{l=1}^{j} \Big(\sum_{i=l+1}^{j+1} \gamma_T(x_l - x_i)\Big) dx_1 \ldots dx_{j+1}. \tag{5.128}$$

It remains to estimate the rhs. More precisely, given a nonnegative measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$, we shall compute the following integral,

$$I_j := \int_{(\mathbb{R}^d)^{j+1}} \Big(\sum_{i=1}^{j+1} f(|x_i|)\Big) \prod_{l=1}^{j} \Big(\sum_{i=l+1}^{j+1} f(|x_l - x_i|)\Big) dx_1 \ldots dx_{j+1}. \tag{5.129}$$

For that purpose, we devise a tree argument. Let an *increasing tree* on $\{0,\ldots,j+1\}$ be defined as a collection of the form $t = \{(l,t_l)\}_{l=0}^{j}$ where $t_l > l$ for all l, and let $\mathcal{T}_j$ denote the set of such increasing trees. We can then rewrite

$$I_j = \sum_{t\in\mathcal{T}_j} \int_{(\mathbb{R}^d)^{j+1}} f(|x_{t_0}|) \prod_{l=1}^{j} f(|x_l - x_{t_l}|)\, dx_1 \ldots dx_{j+1}.$$

Let $t \in \mathcal{T}_j$ be fixed. Viewing the pairs (l, t_l) as the edges of a graph on the set $\{0, \ldots, j+1\}$, we see that t defines a tree on this set. Let T_t^1 denote the set of all leaves of t distinct from 0, and let t^1 denote the graph induced by t on the set $\{0, \ldots, j+1\} \setminus T_t^1$. Let then T_t^2 be defined similarly as T_t^1 but with t replaced by t^1, and so on, until 0 is the only vertex left. This leads to a partition $\{1, \ldots, j+1\} = T_t^1 \biguplus \ldots \biguplus T_t^s$, for some $s \geq 1$. By construction, first integrating with respect to all variables x_i with $i \in T_t^1$, then with respect to all variables x_i with $i \in T_t^2$, and so on, we obtain after obvious changes of variables

$$\int_{(\mathbb{R}^d)^{j+1}} f(|x_{t_0}|) \prod_{l=1}^{j} f(|x_l - x_{t_l}|)\, dx_1 \ldots dx_{j+1} = \left(\int_{\mathbb{R}^d} f(|x|) dx \right)^{j+1}.$$

As the cardinality of $\mathcal{T}_j$ is by definition $(j+1)!$, it follows that the integral in (5.129) is equal to

$$I_j = (j+1)! \left(\int_{\mathbb{R}^d} f(|x|) dx \right)^{j+1}.$$

Using this observation and recalling that for all $\alpha \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \gamma_T(x) dx = \int_{\mathbb{R}^d} (1+|x|)^{-d} e^{-\frac{1}{C\sqrt{T}}|x|} dx \lesssim 1 + \log T,$$

the desired result directly follows from (5.128). $\square$

Proof of Theorem 5.A.1

Consider the hardcore approximations ρ_θ of the point process ρ (see Step 1 of Section 5.3.2), and recall that ρ_θ is also assumed to satisfy assumptions (H_1)–(H_3). The result of Theorem 5.1.1 applied to ρ_θ then leads to the following: the map $p \mapsto \xi \cdot A_{T,\theta}^{(p)} \xi$ is analytic and satisfies, for all $p_0 \in [0,1]$, $k \geq 1$, $-p_0 \leq p \leq 1 - p_0$, $|p| < 1/C_{p_0,\theta}$,

$$\left| \xi \cdot A_{T,\theta}^{(p_0+p)} \xi - \xi \cdot A_{T,\theta}^{(p_0)} \xi - \sum_{j=1}^{k} p^j \Delta_{T,\theta}^{(p_0),j} \right| \leq (p C_{p_0,\theta})^{k+1},$$

for some constant $C_{p_0,\theta} \simeq_{p_0,\theta} 1$, where the derivatives $\Delta_{T,\theta}^{(p_0),j}$ are equivalently given by the argument of the limits (5.23), (5.24), and (5.25). Now let us examine the uniform bounds of Proposition 5.3.2 for the derivatives $\Delta_{T,\theta}^k := \Delta_{T,\theta}^{(0),k}$ at $p_0 = 0$. Decomposing $\Delta_{T,\theta}^k = \Delta_{T,\theta,1}^k + \Delta_{T,\theta,2}^k$ as in the proof of Proposition 5.3.2, and noting that we have in the present general situation (instead of (5.78))

$$\sum_{|S| \leq k} \mathbb{1}_{J_S}(0) \leq \sum_{j=0}^{k} \binom{\rho(B_R)}{j} \lesssim \rho(B_R)^k, \tag{5.130}$$

we easily deduce

$$|\Delta_{T,\theta,1}^k| \lesssim C^k \sum_{j=0}^{k} \mathbb{E}\Bigg[\rho(B_R)^{2k} \sum_{|U|=j} |\nabla \delta_\xi^U \phi_{T,\theta}|^2 \Bigg] + C^k \sum_{j=0}^{k} \sum_{i=0}^{j} \mathbb{E}\Bigg[\rho(B_R)^{2k} \sum_{|U|=j-i} \Bigg| \sum_{\substack{|G|=i \\ G \cap U = \varnothing}} \nabla \delta_\xi^{G \cup U} \phi_{T,\theta} \Bigg|^2 \Bigg]. \tag{5.131}$$

Now applying (5.127) (which holds independently of θ), we deduce $|\Delta_{T,\theta,1}^k| \lesssim_k (1 + \log T)^{2k}$ (uniformly in θ). A similar argument holds for $\Delta_{T,\theta,2}^k$, so that we may conclude (uniformly in θ)

$$|\Delta_{T,\theta}^k| \lesssim_k (1 + \log T)^{2k}. \tag{5.132}$$

By the perturbation trick, arguing as in Step 3 of Section 5.3.2, this bound is upgraded as follows: for all $p_0 \in [0,1]$ we have (uniformly in p_0, θ)

$$|\Delta_{T,\theta}^{(p_0),k}| \lesssim_k (1+\log T)^{2k}.$$

As $p \mapsto \xi \cdot A_{T,\theta}^{(p)}\xi$ is analytic, a Taylor expansion around p_0 up to order $k \geq 1$ then gives by Lagrange's remainder theorem (uniformly in p_0, θ)

$$\Big|\xi \cdot A_{T,\theta}^{(p_0+p)}\xi - \xi \cdot A_{T,\theta}^{(p_0)}\xi - \sum_{j=1}^k p^j \Delta_{T,\theta}^{(p_0),j}\Big| \leq p^{k+1} \sup_{u\in[0,1]} |\Delta_{T,\theta}^{(p_0+up),k+1}| \lesssim_k p^{k+1}(1+\log T)^{2(k+1)}.$$

Arguing as in Step 2 of Section 5.3.2, we deduce that the limits $\lim_\theta \Delta_{T,\theta}^{(p_0),j}$ all exist, and, passing to the limit in the above estimate yields, for all $k \geq 1$,

$$\Big|\xi \cdot A_{T}^{(p_0+p)}\xi - \xi \cdot A_{T}^{(p_0)}\xi - \sum_{j=1}^k p^j \lim_\theta \Delta_{T,\theta}^{(p_0),j}\Big| \lesssim_k p^{k+1}(1+\log T)^{2(k+1)}. \tag{5.133}$$

It remains to pass to the limit $T \uparrow \infty$ in (5.133). This is made possible by assumption (H_3), which asserts that the approximate homogenized coefficients $A_T^{(p_0)}$ converge with an algebraic rate $\gamma(T) \leq T^{-\varepsilon}$. Indeed, combining this rate with (5.133) yields for all $k \geq 1$,

$$\Big|\sum_{j=1}^k p^j (\lim_\theta \Delta_{T,\theta}^{(p_0),j} - \lim_\theta \Delta_{2T,\theta}^{(p_0),j})\Big| \lesssim_k \gamma(T) + p^{k+1}(1+\log T)^{2(k+1)},$$

which implies by induction for all $j \geq 1$ (arguing similarly as in (5.121)),

$$\big|\lim_\theta \Delta_{2T,\theta}^{(p_0),j} - \lim_\theta \Delta_{T,\theta}^{(p_0),j}\big| \lesssim_j \gamma(T)^{2^{-j}}(1+\log T)^{2(j+1)} \leq T^{-\varepsilon 2^{-j}}(1+\log T)^{2(j+1)}.$$

In particular, the limits $\Delta^{(p_0),j} := \lim_T \lim_\theta \Delta_{T,\theta}^{(p_0),j}$ all exist in $\mathbb{R}$, and for all $j \geq 1$ we have

$$\big|\lim_\theta \Delta_{T,\theta}^{(p_0),j} - \Delta^{(p_0),j}\big| \lesssim_j T^{-\varepsilon 2^{-j}}(1+\log T)^{2(j+1)}.$$

Now combining this with (5.133) and applying property (H_3) once again, we find for all $k \geq 1$,

$$\Big|\xi \cdot A^{(p_0+p)}\xi - \xi \cdot A^{(p_0)}\xi - \sum_{j=1}^k p^j \Delta^{(p_0),j}\Big| \lesssim_k p^{k+1}(1+\log T)^{2(k+1)} + \sum_{j=0}^k p^j T^{-\varepsilon 2^{-j}}(1+\log T)^{2(j+1)}.$$

Choosing $T := p^{-\frac{1}{\varepsilon}2^k}$, we conclude for all $k \geq 1$,

$$\Big|\xi \cdot A^{(p_0+p)}\xi - \xi \cdot A^{(p_0)}\xi - \sum_{j=1}^k p^j \Delta^{(p_0),j}\Big| \lesssim_k p^{k+1}(1+|\log p|)^{2(k+1)}.$$

As this estimate holds for all $k \geq 1$, the error can of course a posteriori be improved into $O(p^{k+1})$. This concludes the proof of the smoothness statement in Theorem 5.A.1.

We now show that at $p_0 = 0$ we may omit the θ-regularizations in the formulas (5.23), (5.24), and (5.25) for the derivatives. We focus on formula (5.23), although the argument is the same for (5.24) and (5.25). Given $k \geq 1$, formula (5.23) at $p_0 = 0$ takes the form

$$\xi \cdot A_{\mathrm{hom}}^{(0),k}\xi = k! \lim_{T\uparrow\infty} \lim_{\theta\uparrow\infty} \sum_{|F|=k} \sum_{G\subsetneq F} (-1)^{|F\setminus G|+1} \mathbb{E}\left[\nabla \delta_\xi^G \phi_{T,\theta,\xi} \cdot (C_\theta)_{F\setminus G\|G}(\nabla \phi_{T,\theta,\xi}^F + \xi)\right],$$

or equivalently, recalling that the inclusion $\rho_\theta \subset \rho$ holds by construction,

$$\xi \cdot A_{\mathrm{hom}}^{(0),k}\xi = k! \lim_{T\uparrow\infty} \lim_{\theta\uparrow\infty} \mathbb{E}\bigg[\sum_{|F|=k} \mathbb{1}_{q_n\in\rho_\theta,\forall n\in F} \sum_{G\subsetneq F} (-1)^{|F\setminus G|+1} \nabla \delta_\xi^G \phi_{T,\xi} \cdot C_{F\setminus G\|G}(\nabla \phi_{T,\xi}^F + \xi) \bigg].$$

Adapting the proof of Proposition 5.3.2 as for (5.132) above, but now with $\theta = \infty$, we find

$$\mathbb{E}\bigg[\sum_{|F|=k} \Big| \sum_{G\subsetneq F} (-1)^{|F\setminus G|+1} \nabla \delta_\xi^G \phi_{T,\xi} \cdot C_{F\setminus G\|G}(\nabla \phi_{T,\xi}^F + \xi) \Big| \bigg] \lesssim_k (1+\log T)^{2k} < \infty,$$

and we conclude by dominated convergence

$$\xi \cdot A_{\mathrm{hom}}^{(0),k}\xi = k! \lim_{T\uparrow\infty} \sum_{|F|=k} \sum_{G\subsetneq F} (-1)^{|F\setminus G|+1} \mathbb{E}\left[\nabla \delta_\xi^G \phi_{T,\xi} \cdot C_{F\setminus G\|G}(\nabla \phi_{T,\xi}^F + \xi) \right], \tag{5.134}$$

as claimed.

It remains to check that the first-order invariance principle of Corollary 5.1.3 is still valid, as well as the electric and elastic Clausius-Mossotti formulas of Corollaries 5.1.4 and 5.1.5. Unlike the situation in Step 5 of Section 5.3.2, it is no longer clear here whether we may omit the T-regularization in the formula for $A_{\mathrm{hom}}^{(0),1}$. Nevertheless, the proof of Corollary 5.1.3 in Section 5.3.3 remains unchanged when using the formula with T-regularizations. We now turn to the validity of the electric Clausius-Mossotti formula. Assume that the inclusions are balls $J_n = B_R(q_n)$ of radius $R > 0$, and that the reference coefficients are constant $A = \alpha\,\mathrm{Id}$ and $A' = \beta\,\mathrm{Id}$. We shall compute explicitly the first derivative $\xi \cdot A_{\mathrm{hom}}^{(0),1}\xi$ of the perturbed homogenized coefficient at 0, as given by formula (5.134). On the one hand, since A is constant, the unique solution $\phi_{T,\xi}$ of $\frac{1}{T}\phi_{T,\xi} - \nabla \cdot A(\nabla\phi_{T,\xi} + \xi) = 0$ is clearly $\phi_{T,\xi} = 0$. On the other hand, for all n, the unique solution $\phi_{T,\xi}^{\{n\}} \in H^1_{\mathrm{loc}} \cap \mathrm{L}^\infty(\mathbb{R}^d)$ of $\frac{1}{T}\phi_{T,\xi}^{\{n\}} - \nabla \cdot A^{\{n\}}(\nabla\phi_{T,\xi}^{\{n\}} + \xi) = 0$ is of the form $\phi_{T,\xi}^{\{n\}}(x) = \psi_{T,\xi}(x-q_n)$ for some function $\psi_{T,\xi}$ independent of n, and it satisfies $\nabla\psi_{T,\xi} \rightharpoonup \nabla\psi_\xi$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$. Formula (5.134) with $n = 1$ then yields

$$\begin{aligned}\xi \cdot A_{\mathrm{hom}}^{(0),1}\xi &= \lim_{T\uparrow\infty} \sum_n \mathbb{E}\left[(\nabla\phi_{T,\xi} + \xi) \cdot C^{\{n\}}(\nabla\phi_{T,\xi}^{\{n\}} + \xi) \right] \\ &= (\beta-\alpha) \lim_{T\uparrow\infty} \sum_n \mathbb{E}\left[\mathbb{1}_{|q_n|<R}\, \xi \cdot (\nabla\psi_{T,\xi}(-q_n) + \xi) \right].\end{aligned}$$

As in the proof of Corollary 5.1.3 in Section 5.3.3, we have $\mathbb{E}\left[\sum_n f(q_n)\right] = \sigma \int f(x)dx$ for all continuous integrable function $f : \mathbb{R}^d \to \mathbb{R}$. Hence,

$$\xi \cdot A_{\mathrm{hom}}^{(0),1}\xi = \sigma(\beta-\alpha) \lim_{T\uparrow\infty} \int_{B_R} \xi \cdot (\nabla\psi_{T,\xi}(-x) + \xi)\, dx,$$

so that the weak convergence $\nabla\psi_{T,\xi} \rightharpoonup \nabla\psi_\xi$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ implies

$$\xi \cdot A_{\mathrm{hom}}^{(0),1}\xi = \sigma(\beta-\alpha) \int_{B_R} \xi \cdot (\nabla\psi_\xi(-x) + \xi)\, dx.$$

Using the explicit formula (5.116)–(5.117) for $\nabla\psi_\xi$, we deduce

$$\xi \cdot A_{\mathrm{hom}}^{(0),1}\xi = \sigma|B_R|(\beta-\alpha)(1+C) = \sigma|B_R| \frac{\alpha d(\beta-\alpha)}{\beta + \alpha(d-1)},$$

and the conclusion follows as in the proof of Corollary 5.1.4. The elastic counterpart is similarly easily recovered in the present setting. ☐

5.B Appendix: Corresponding results on effective fluctuation tensor

This appendix is devoted to the proof of Theorem 5.1.8 concerning the first-order expansion of the perturbed effective fluctuation tensor $\mathcal{Q}^{(p)}$, based on the quantitative theory of stochastic homogenization. More precisely, let the same assumptions hold as in Section 5.1.2 (including for simplicity the finite penetrability condition 5.4). Let in addition the assumptions (5.122) and (H_1)–(H_2) of Section 5.A hold (without θ-regularization). Also assume that the convergence in the definition (5.32) of the effective fluctuation tensor holds with an algebraic rate (cf. Proposition 3.A.1): there exists some $\varepsilon > 0$ such that for all $T > 0$ and $p \in [0,1]$,

$$|\mathcal{Q}_T^{(p)} - \mathcal{Q}^{(p)}| \lesssim T^{-\varepsilon}. \tag{5.135}$$

Further assume that the perturbed coefficients $A^{(p)}$ satisfy a standard covariance inequality (∂^{osc}-CI) uniformly in $p \in [0,1]$. In this appendix, we establish Theorem 5.1.8 under these precise assumptions.

Note that the finite penetrability condition (5.4) is believed to be inessential here. Also, the proof is immediately adapted to the case of a weighted covariance inequality with superalgebraic weight (say), and also holds when the oscillation is replaced by the functional derivative.

Sketch of the proof of Theorem 5.1.8. Recall the notation of Section 5.1.3, $\mathcal{Q}^{(p)} := \lim_{T\uparrow\infty} \mathcal{Q}_T^{(p)}$ with

$$\xi_1 \otimes \xi_2 : \mathcal{Q}_T^{(p)} : \xi_1 \otimes \xi_2 := \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\xi_2 \otimes \xi_1 : \Xi_T^{(p)}(x)\right)\left(\xi_2 \otimes \xi_1 : \Xi_T^{(p)}(0)\right)\right] dx,$$
$$\xi_1 \otimes \xi_2 : \Xi_T^{(p)} := \xi_2 \cdot (A^{(p)} - A_{\mathrm{hom}}^{(p)})(\nabla \phi_{T,\xi_1}^{(p)} + \xi_1),$$

where the integral in the definition of $\mathcal{Q}_T^{(p)}$ is absolutely convergent (as a consequence of the covariance inequality (∂^{osc}-CI), cf. e.g. the proof of Theorem 3.A.1(i) in Chapter 3). Given $|\xi_2| = |\xi_1| = 1$, we set for simplicity

$$\tilde{\mathcal{Q}}^{(p)} := \xi_1 \otimes \xi_2 : \mathcal{Q}^{(p)}, \qquad \tilde{\mathcal{Q}}_T^{(p)} := \xi_1 \otimes \xi_2 : \mathcal{Q}_T^{(p)}, \qquad \tilde{\Xi}_T^{(p)} := \xi_1 \otimes \xi_2 : \Xi_T^{(p)}.$$

We split the proof into five steps.

Step 1. Alternative formula for $\mathcal{Q}_T^{(p)}$.

In this step, we prove that for all $T > 0$ and $p \in [0,1]$ the approximate effective fluctuation tensor $\mathcal{Q}_T^{(p)}$ is equivalently given by

$$\tilde{\mathcal{Q}}_T^{(p)} = O(T^{-\frac12}) + \int_{\mathbb{R}^d} \mathrm{Cov}\Big[(\nabla\phi_{T,\xi_2}^{(p)} + \xi_2) \cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)} + \xi_1)(x)\ ;$$
$$(\nabla\phi_{T,\xi_2}^{(p)} + \xi_2) \cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)} + \xi_1)(0)\Big] dx.$$

By definition of $\tilde{\mathcal{Q}}_T^{(p)}$, it suffices to check that

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\xi_2 \cdot A_{\mathrm{hom}}^{(p)} \nabla\phi_{T,\xi_1}^{(p)}(x)\, \tilde{\Xi}_T^{(p)}(0)\right] dx = 0, \tag{5.136}$$

$$\int_{\mathbb{R}^d} \mathrm{Cov}\left[\nabla\phi_{T,\xi_2}^{(p)} \cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)} + \xi_1)(x); \tilde{\Xi}_T^{(p)}(0)\right] dx = O(T^{-\frac12}\log T), \tag{5.137}$$

$$\int_{\mathbb{R}^d} \mathbb{E}\left[(\nabla\phi_{T,\xi_2}^{(p)} + \xi_2) \cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)} + \xi_1)(x)\, \xi_2 \cdot A_{\mathrm{hom}}^{(p)} \nabla\phi_{T,\xi_1}^{(p)}(0)\right] dx = 0,$$

$$\int_{\mathbb{R}^d} \mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}^{(p)} + \xi_2) \cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)} + \xi_1)(x); \nabla\phi_{T,\xi_2}^{(p)} \cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)} + \xi_1)(0)\right] dx = O(T^{-\frac12}\log T).$$

We focus on (5.136) and (5.137), as the last two identities are obtained similarly. We start with (5.136). Writing by stationarity

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\xi_2 \cdot A^{(p)}_{\mathrm{hom}} \nabla \phi^{(p)}_{T,\xi_1}(x)\, \tilde{\Xi}^{(p)}_T(0)\right] dx = \int_{\mathbb{R}^d} \mathbb{E}\left[\xi_2 \cdot A^{(p)}_{\mathrm{hom}} \nabla \phi^{(p)}_{T,\xi_1}(x) \int_B (\tilde{\Xi}^{(p)}_T - \mathbb{E}[\tilde{\Xi}^{(p)}_T])\right] dx,$$

and noting that the integrand is absolutely integrable (as a consequence of the covariance inequality (∂^{osc}-CI), cf. e.g. the proof of Proposition 3.A.1(i) in Chapter 3), we deduce by integration by parts that (5.136) is a direct consequence of

$$\lim_{R\uparrow\infty} \frac{1}{R} \int_{B_R} \left| \mathrm{Cov}\left[\phi^{(p)}_{T,\xi_1}(x); \int_B \tilde{\Xi}^{(p)}_T\right] \right| dx = 0. \tag{5.138}$$

The covariance inequality (∂^{osc}-CI) yields

$$\left| \mathrm{Cov}\left[\phi^{(p)}_{T,\xi_2}(x); \int_B \tilde{\Xi}^{(p)}_T\right] \right| \lesssim \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A^{(p)},B(z)}\, \phi^{(p)}_{T,\xi_2}(x)\right)^2\right]^{\frac12} \mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A^{(p)},B(z)} \int_B \tilde{\Xi}^{(p)}_T\right)^2\right]^{\frac12} dz.$$

Arguing as e.g. in [212] to estimate the vertical derivatives, using the corrector estimates and the optimal annealed estimates on the Green's functions (cf. (H_1)–(H_2)), we may estimate

$$\begin{aligned}
\mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A^{(p)},B(z)}\, \phi^{(p)}_{T,\xi_2}(x)\right)^2\right]^{\frac12} &\lesssim (1+|x-z|)^{1-d} e^{-\frac{1}{C\sqrt{T}}|x-z|},\\
\mathbb{E}\left[\left(\partial^{\mathrm{osc}}_{A^{(p)},B(z)} \int_B \tilde{\Xi}^{(p)}_T\right)^2\right]^{\frac12} &\lesssim (1+|z|)^{-d} e^{-\frac{1}{C\sqrt{T}}|z|}.
\end{aligned}$$

Injecting this into the above leads to

$$\begin{aligned}
\left| \mathrm{Cov}\left[\phi^{(p)}_{T,\xi_2}(x); \int_B \tilde{\Xi}^{(p)}_T\right] \right| &\lesssim \int_{\mathbb{R}^d} (1+|x-z|)^{1-d} e^{-\frac{1}{C\sqrt{T}}|x-z|} (1+|z|)^{-d} e^{-\frac{1}{C\sqrt{T}}|z|} dz\\
&\lesssim (1+|x|)^{1-d} e^{-\frac{1}{C\sqrt{T}}|x|},
\end{aligned}$$

and hence

$$\frac{1}{R} \int_{B_R} \left| \mathrm{Cov}\left[\phi^{(p)}_{T,\xi_2}(x); \int_B \tilde{\Xi}^{(p)}_T\right] \right| dx \lesssim \frac{R \wedge \sqrt{T}}{R},$$

which indeed goes to 0 as $R\uparrow\infty$. We now turn to (5.137). Using the corrector equation for $\phi^{(p)}_{T,\xi_1}$, we may decompose

$$\nabla \phi^{(p)}_{T,\xi_2} \cdot A^{(p)}(\nabla \phi^{(p)}_{T,\xi_1} + \xi_1) = \nabla \cdot \left(\phi^{(p)}_{T,\xi_2} A^{(p)}(\nabla \phi^{(p)}_{T,\xi_1} + \xi_1)\right) - \frac{1}{T} \phi^{(p)}_{T,\xi_2} \phi^{(p)}_{T,\xi_1}.$$

Arguing as above with the covariance inequality, we easily check that

$$\int_{\mathbb{R}^d} \mathrm{Cov}\left[\nabla \cdot \left(\phi^{(p)}_{T,\xi_2} A^{(p)}(\nabla \phi^{(p)}_{T,\xi_1} + \xi_1)\right)(x); \tilde{\Xi}^{(p)}_T(0)\right] dx = 0,$$

and also

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \mathrm{Cov}\left[\frac{1}{T} \phi^{(p)}_{T,\xi_2} \phi^{(p)}_{T,\xi_1}(x); \tilde{\Xi}^{(p)}_T(0)\right] dx \right|\\
&\qquad \lesssim \frac{1}{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1+|x-z|)^{1-d} e^{-\frac{1}{C\sqrt{T}}|x-z|} (1+|z|)^{-d} e^{-\frac{1}{C\sqrt{T}}|z|} dz dx \;\lesssim T^{-\frac12} \log T,
\end{aligned}$$

which implies (5.137).

Step 2. Suitable decomposition of $\tilde{\mathcal{Q}}_T^{(p)}$.

In this step, we prove the following decomposition of $\tilde{\mathcal{Q}}_T^{(p)}$,

$$\tilde{\mathcal{Q}}_T^{(p)} - \tilde{\mathcal{Q}}_T^{(0)} = p\sum_{j=1}^{3} M_T^j + \sum_{j=1}^{7} E_T^{(p),j} + O(T^{-\frac{1}{2}}), \tag{5.139}$$

where the three main terms are

$$\begin{aligned}
M_T^1 &:= \int_{\mathbb{R}^d} \mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot\sum_n C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(x)\ ;\ \xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(0)\right]dx,\\
M_T^2 &:= \int_{\mathbb{R}^d} \mathrm{Cov}\left[\xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(x)\ ;\ (\nabla\phi_{T,\xi_2}+\xi_2)\cdot\sum_n C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(0)\right]dx,\\
M_T^3 &:= \sum_n\int_{\mathbb{R}^d} \mathbb{E}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(x)\,(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(0)\right]dx,
\end{aligned}$$

while the seven error terms are given by

$$\begin{aligned}
E_T^{(p),1} &:= \sum_{k,l=1}^{\Gamma}(-1)^{k+l}\sum_{|F|=k}\sum_{|G|=l}p^{|F\cup G|}\,\mathbb{1}_{F\cap G\neq\varnothing}\mathbb{1}_{|F\cup G|\geq 2}\\
&\qquad\times\int_{\mathbb{R}^d}\mathbb{E}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(x)\,(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(0)\right]dx,\\
E_T^{(p),2} &:= -\sum_{k,l=1}^{\Gamma}(-1)^{k+l}p^{k+l}\sum_{|F|=k}\sum_{|G|=l}\mathbb{1}_{F\cap G\neq\varnothing}\\
&\qquad\times\int_{\mathbb{R}^d}\mathbb{E}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F(\nabla\phi_{T,\xi_1}^{F\cup E^{(p)}}+\xi_1)(x)\,(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G(\nabla\phi_{T,\xi_1}^{G\cup E^{(p)}}+\xi_1)(0)\right]dx,\\
E_T^{(p),3} &:= \sum_{k,l=1}^{\Gamma}(-1)^{k+l}p^{k+l}\sum_{|F|=k}\sum_{|G|=l}\mathbb{1}_{F\cap G=\varnothing}\\
&\times\int_{\mathbb{R}^d}\Bigg(\mathbb{E}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F\nabla(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}-\phi_{T,\xi_1}^{F\cup E^{(p)}})(x)\,(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(0)\right]\\
&\qquad+\mathbb{E}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F(\nabla\phi_{T,\xi_1}^{F\cup E^{(p)}}+\xi_1)(x)\,(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G\nabla(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}-\phi_{T,\xi_1}^{G\cup E^{(p)}})(0)\right]\Bigg)dx,\\
E_T^{(p),4} &:= \sum_{k,l=1}^{\Gamma}(-1)^{k+l}p^{k+l}\int_{\mathbb{R}^d}\mathrm{Cov}\Bigg[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot\sum_{|F|=k}C_F(\nabla\phi_{T,\xi_1}^{F\cup E^{(p)}}+\xi_1)\,(x)\ ;\\
&\qquad\qquad(\nabla\phi_{T,\xi_2}+\xi_2)\cdot\sum_{|G|=l}C_G(\nabla\phi_{T,\xi_1}^{G\cup E^{(p)}}+\xi_1)\,(0)\Bigg]dx,\\
E_T^{(p),5} &:= \sum_{k=2}^{\Gamma}(-1)^{k+1}p^k\int_{\mathbb{R}^d}\Bigg(\mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot\sum_{|F|=k}C_F(\nabla\phi_{T,\xi_1}^{F\cup E^{(p)}}+\xi_1)(x);\xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(0)\right]\\
&\qquad+\mathrm{Cov}\left[\xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(x);(\nabla\phi_{T,\xi_2}+\xi_2)\cdot\sum_{|F|=k}C_F(\nabla\phi_{T,\xi_1}^{F\cup E^{(p)}}+\xi_1)(0)\right]\Bigg)dx,
\end{aligned}$$

$$E_T^{(p),6} := p\sum_n \int_{\mathbb{R}^d}$$
$$\times\Bigg(\mathbb{E}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n\nabla(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})(x)\,(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}+\xi_1)(0)\Big]$$
$$+\mathbb{E}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(x)\,(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n\nabla(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})(0)\Big]\Bigg)dx,$$
$$E_T^{(p),7} := p\int_{\mathbb{R}^d}\Bigg(\mathrm{Cov}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot\sum_n C^n\nabla(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})(x);\xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(0)\Big]$$
$$+\mathrm{Cov}\Big[\xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(x);(\nabla\phi_{T,\xi_2}+\xi_2)\cdot\sum_n C^n\nabla(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})(0)\Big]\Bigg)dx.$$

The formula of Step 1 first yields

$$\begin{aligned}
\tilde{\mathcal{Q}}_T^{(p)}-\tilde{\mathcal{Q}}_T^{(0)} &= O(T^{-\frac12}) \\
&\quad+\int_{\mathbb{R}^d}\mathrm{Cov}\Big[(\nabla\phi_{T,\xi_2}^{(p)}+\xi_2)\cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(x)-(\nabla\phi_{T,\xi_2}+\xi_2)\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(x)\ ; \\
&\qquad\qquad (\nabla\phi_{T,\xi_2}^{(p)}+\xi_2)\cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)\Big]\,dx \\
&\quad+\int_{\mathbb{R}^d}\mathrm{Cov}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(x)\ ; \\
&\qquad\qquad (\nabla\phi_{T,\xi_2}^{(p)}+\xi_2)\cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)-(\nabla\phi_{T,\xi_2}+\xi_2)\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(0)\Big]\,dx.
\end{aligned}\tag{5.140}$$

Using the corrector equations for $\phi_T^{(p)}$ and ϕ_T, we may decompose

$$\begin{aligned}
&(\nabla\phi_{T,\xi_2}^{(p)}+\xi_2)\cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)-(\nabla\phi_{T,\xi_2}+\xi_2)\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)\\
&\quad=\ \nabla(\phi_{T,\xi_2}^{(p)}-\phi_{T,\xi_2})\cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)+(\nabla\phi_{T,\xi_2}+\xi_2)\cdot A\nabla(\phi_{T,\xi_1}^{(p)}-\phi_{T,\xi_1})\\
&\qquad\qquad +(\nabla\phi_{T,\xi_2}+\xi_2)\cdot(A^{(p)}-A)(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)\\
&\quad=\ \nabla\cdot\big((\phi_{T,\xi_2}^{(p)}-\phi_{T,\xi_2})A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)\big)+\nabla\cdot\big((\nabla\phi_{T,\xi_2}+\xi_2)\cdot A(\phi_{T,\xi_1}^{(p)}-\phi_{T,\xi_1})\big) \qquad (5.141)\\
&\qquad -\frac1T(\phi_{T,\xi_2}^{(p)}-\phi_{T,\xi_2})\phi_{T,\xi_1}^{(p)}-\frac1T\phi_{T,\xi_2}^{(p)}(\phi_{T,\xi_1}^{(p)}-\phi_{T,\xi_1})+(\nabla\phi_{T,\xi_2}+\xi_2)\cdot(A^{(p)}-A)(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1).
\end{aligned}$$

Now arguing as in Step 1 with the covariance inequality we easily check that

$$\int_{\mathbb{R}^d}\mathrm{Cov}\Big[\nabla\cdot\big((\phi_{T,\xi_2}^{(p)}-\phi_{T,\xi_2})A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)\big)(x);(\nabla\phi_{T,\xi_2}^{(p)}+\xi_2)\cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)\Big]\,dx=0,$$
$$\int_{\mathbb{R}^d}\mathrm{Cov}\Big[\frac1T(\phi_{T,\xi_2}^{(p)}-\phi_{T,\xi_2})\phi_{T,\xi_1}^{(p)}(x);(\nabla\phi_{T,\xi_2}^{(p)}+\xi_2)\cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)\Big]\,dx=O(T^{-\frac12}\log T).$$

Using such estimates, and combining (5.140) with identity (5.141), we obtain

$$\begin{aligned}
&\tilde{\mathcal{Q}}_T^{(p)}-\tilde{\mathcal{Q}}_T^{(0)}+O(T^{-\frac12}\log T)\\
&\quad=\int_{\mathbb{R}^d}\mathrm{Cov}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(x)\ ;\ (\nabla\phi_{T,\xi_2}^{(p)}+\xi_2)\cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)\Big]\,dx\\
&\qquad+\int_{\mathbb{R}^d}\mathrm{Cov}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(x)\ ;\ (\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)\Big]\,dx,
\end{aligned}$$

and hence, further similarly decomposing the first right-hand side term,

$$\begin{aligned}
&\tilde{\mathcal{Q}}_T^{(p)} - \tilde{\mathcal{Q}}_T^{(0)} + O(T^{-\frac{1}{2}}\log T)\\
&\quad= \int_{\mathbb{R}^d} \mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(x)\ ;\ (\nabla\phi_{T,\xi_2}+\xi_2)\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(0)\right] dx\\
&\quad+ \int_{\mathbb{R}^d} \mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(x)\ ;\ (\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)\right] dx\\
&\quad+ \int_{\mathbb{R}^d} \mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(x)\ ;\ (\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)\right] dx.
\end{aligned}$$

Again using the corrector equation for $\phi_T^{(p)}$ in the form

$$\begin{aligned}
&(\nabla\phi_{T,\xi_2}^{(p)}+\xi_2)\cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)\\
&\qquad= \xi_2\cdot A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1) + \nabla\cdot\left(\phi_{T,\xi_2}^{(p)}A^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)\right) - \frac{1}{T}\phi_{T,\xi_2}^{(p)}\phi_{T,\xi_1}^{(p)},
\end{aligned}$$

and arguing as above, we are led to

$$\begin{aligned}
&\tilde{\mathcal{Q}}_T^{(p)} - \tilde{\mathcal{Q}}_T^{(0)} + O(T^{-\frac{1}{2}}\log T)\\
&\qquad= \int_{\mathbb{R}^d} \mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(x)\ ;\ \xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(0)\right] dx\\
&\qquad+ \int_{\mathbb{R}^d} \mathrm{Cov}\left[\xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(x)\ ;\ (\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)\right] dx\\
&+ \int_{\mathbb{R}^d} \mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(x)\ ;\ (\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)\right] dx. \qquad (5.142)
\end{aligned}$$

Appealing to the inclusion-exclusion formula (5.17) together with assumption (5.4) and with independence (5.6), the first right-hand side term becomes

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(x)\ ;\ \xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(0)\right] dx\\
&= \int_{\mathbb{R}^d} \sum_{k=1}^{\Gamma} (-1)^{k+1}p^k \sum_{|F|=k} \mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F(\nabla\phi_{T,\xi_1}^{F\cup E^{(p)}}+\xi_1)(x)\ ;\ \xi_2\cdot A(\nabla\phi_{T,\xi_1}+\xi_1)(0)\right] dx,
\end{aligned}$$

and the second right-hand side term in (5.142) is rewritten similarly, while the last right-hand side term takes the form

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{Cov}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(x)\ ;\ (\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^{(p)}(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)(0)\right] dx\\
&\quad= \int_{\mathbb{R}^d} \sum_{k,l=1}^{\Gamma} (-1)^{k+l} \sum_{|F|=k}\sum_{|G|=l}\\
&\qquad\times\bigg(p^{|F\cup G|}\,\mathbb{E}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(x)\,(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(0)\right]\\
&\qquad\quad -p^{|F|+|G|}\,\mathbb{E}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F(\nabla\phi_{T,\xi_1}^{F\cup E^{(p)}}+\xi_1)\right]\,\mathbb{E}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G(\nabla\phi_{T,\xi_1}^{G\cup E^{(p)}}+\xi_1)\right]\bigg]\bigg) dx.
\end{aligned}$$

Injecting these identities into (5.142) and suitably reorganizing the terms, the conclusion (5.139) follows.

Step 3. Estimation of the main terms $(M_T^j)_{j=1}^3$ and of the error terms $(E_T^{(p),j})_{j=1}^5$.

In this step, we prove that

$$\sum_{j=1}^{3} |M_T^j| \lesssim \log T, \qquad \text{and} \qquad \sum_{j=1}^{5} |E_T^{(p),j}| \lesssim p^2 \log T. \tag{5.143}$$

We start with the error term $E_T^{(p),1}$. Given $F,G \subset \mathbb{N}$ with $F \cap G \neq \varnothing$, we have by definition $|C_F(x)||C_G(y)| \lesssim \mathbb{1}_{J_F}(x)\mathbb{1}_{J_G}(y)\mathbb{1}_{|x-y|\le 2R}$. By stationarity, we then find

$$\begin{aligned}
&\sum_{F,G\subset\mathbb{N}} \mathbb{1}_{F\cap G\neq\varnothing}\bigg|\int_{\mathbb{R}^d}\mathbb{E}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(x)\\
&\qquad\qquad\qquad\qquad\times(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(0)\Big]dx\bigg|\\
&\lesssim \int_{B_{3R}}\int_{B_{3R}}\sum_{F,G\subset\mathbb{N}}\mathbb{E}\Big[\mathbb{1}_{J_F}(x)\mathbb{1}_{J_G}(y)\,|(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(x)|\,|(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(y)|\\
&\qquad\qquad\qquad\qquad\times|(\nabla\phi_{T,\xi_2}+\xi_2)(x)||(\nabla\phi_{T,\xi_2}+\xi_2)(y)|\Big]dxdy\\
&\lesssim \mathbb{E}\bigg[\sum_{F,G\subset\mathbb{N}}\mathbb{1}_{J_F\cap B_{3R}\neq\varnothing}\mathbb{1}_{J_G\cap B_{3R}\neq\varnothing}\,(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)_{3R}^2(0)(\nabla\phi_{T,\xi_2}+\xi_2)_{3R}^2(0)\bigg].
\end{aligned}$$

Decomposing $\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1 = \nabla(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}-\phi_{T,\xi_1}^{(p)}) + (\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)$, and noting that an a priori estimate yields

$$\int_{\mathbb{R}^d}\big|\nabla\big(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}-\phi_{T,\xi_1}^{(p)}\big)\big|^2 \lesssim \int_{J^{F\cup G}}|\nabla\phi_{T,\xi_1}^{(p)}+\xi_1|^2, \tag{5.144}$$

we easily deduce

$$\begin{aligned}
&\sum_{F,G\subset\mathbb{N}} \mathbb{1}_{F\cap G\neq\varnothing}\bigg|\int_{\mathbb{R}^d}\mathbb{E}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(x)\\
&\qquad\qquad\qquad\qquad\times(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(0)\Big]dx\bigg|\\
&\lesssim \mathbb{E}\bigg[\sum_{F,G\subset\mathbb{N}}\mathbb{1}_{J_F\cap B_{3R}\neq\varnothing}\mathbb{1}_{J_G\cap B_{3R}\neq\varnothing}\,(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)_{5R}^2(0)(\nabla\phi_{T,\xi_2}+\xi_2)_{5R}^2(0)\bigg],
\end{aligned}$$

and hence, noting that assumption (5.4) implies $\sum_{F\subset\mathbb{N}}\mathbb{1}_{J_F\cap B_{3R}\neq\varnothing} \le \sum_{k=1}^{\Gamma}\binom{\rho(B_{5R})}{k} \lesssim \rho(B_{5R})^\Gamma$, and using assumption (5.122) and the corrector estimates (H_1),

$$\begin{aligned}
&\sum_{F,G\subset\mathbb{N}} \mathbb{1}_{F\cap G\neq\varnothing}\bigg|\int_{\mathbb{R}^d}\mathbb{E}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(x)\\
&\qquad\qquad\qquad\times(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(0)\Big]dx\bigg| \lesssim 1.
\end{aligned}$$

This proves that $E_T^{(p),1} \lesssim p^2$. The same argument yields $E_T^{(p),2} \lesssim p^2$, as well as $|M_T^3| \lesssim 1$.

We now turn to the third term $E_T^{(p),3}$. By stationarity, we may estimate

$$\begin{aligned}
\sum_{F,G\subset\mathbb{N}} \int_{\mathbb{R}^d} \Big| \mathbb{E}\Big[& (\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F \nabla(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}} - \phi_{T,\xi_1}^{F\cup E^{(p)}})(x) \\
& \qquad \times (\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(0)\Big]\Big|\,dx \\
\lesssim \int_{\mathbb{R}^d} \sum_{F,G\subset\mathbb{N}} \mathbb{E}\Big[& \mathbb{1}_{J_F\cap B_R(x)\neq\varnothing}\mathbb{1}_{J_G\cap B_R(0)\neq\varnothing}\big(\nabla(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}} - \phi_{T,\xi_1}^{F\cup E^{(p)}})\big)_R(x) \\
& \times\big(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1\big)_R(0)(\nabla\phi_{T,\xi_2}(x)+\xi_2)_R(x)(\nabla\phi_{T,\xi_2}(x)+\xi_2)_R(0)\Big]\,dx.
\end{aligned} \tag{5.145}$$

Given $J_F\cap B_R(x)\neq\varnothing$ and $J_G\cap B_R(0)\neq\varnothing$, a priori estimates as in (5.144) easily yield

$$\begin{aligned}
\big(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1\big)_{3R}(x) &\lesssim \big(\nabla\phi_{T,\xi_1}^{F\cup E^{(p)}}+\xi_1\big)_{3R}(x)+\big(\nabla\phi_{T,\xi_1}^{F\cup E^{(p)}}+\xi_1\big)_{3R}(0) \\
&\lesssim (\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)_{3R}(x)+\big(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1\big)_{3R}(0),
\end{aligned} \tag{5.146}$$

and similarly

$$\big(\nabla(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}} - \phi_{T,\xi_1}^{F\cup E^{(p)}})\big)_R(x) \lesssim (\nabla\phi_{T,\xi_1}^{(p)}+\xi_1)_{3R}(x)+\big(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1\big)_{3R}(0). \tag{5.147}$$

As the function $\phi_T^{F\cup G\cup E^{(p)}} - \phi_T^{F\cup E^{(p)}}$ satisfies

$$\Big(\frac{1}{T}-\nabla\cdot A^{F\cup E^{(p)}}\nabla\Big)\big(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}} - \phi_{T,\xi_1}^{F\cup E^{(p)}}\big)=\nabla\cdot\big(A^{F\cup G\cup E^{(p)}} - A^{F\cup E^{(p)}}\big)\big(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1\big),$$

the Green's representation formula gives, for $J_F\cap B_R(x)\neq\varnothing$ and $J_G\cap B_R(0)\neq\varnothing$,

$$\big|\nabla\big(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}} - \phi_{T,\xi_1}^{F\cup E^{(p)}}\big)(x)\big| \lesssim \int_{B_{3R}} \big|\nabla_1\nabla_2 G_T^{F\cup E^{(p)}}(x,y)\big|\big|\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}(y)+\xi_1\big|\,dy.$$

Together with (5.146) and (5.147), this leads to

$$\begin{aligned}
&\big(\nabla(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}} - \phi_{T,\xi_1}^{F\cup E^{(p)}})\big)_R(x) \\
&\qquad\lesssim \Big(1\wedge\big(\nabla_1\nabla_2 G_T^{F\cup E^{(p)}}\big)_{3R,3R}(x,0)\Big)\Big(\big(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1\big)_{3R}(0)+\big(\nabla\phi_{T,\xi_1}^{(p)}+\xi_1\big)_{3R}(x)\Big).
\end{aligned}$$

Injecting this and (5.146) into (5.145), applying Lemma 5.A.3, making use of the corrector estimates and of the optimal annealed estimates on the Green's functions (cf. (H_1)–(H_2)), and again using that $\sum_{F\subset\mathbb{N}}\mathbb{1}_{J_F\cap B_R\neq\varnothing}\lesssim\rho(B_{3R})^\Gamma$, we conclude

$$\begin{aligned}
\sum_{F,G\subset\mathbb{N}} \int_{\mathbb{R}^d} \Big| \mathbb{E}\Big[& (\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_F \nabla(\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}} - \phi_{T,\xi_1}^{F\cup E^{(p)}})(x) \\
& \times(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C_G(\nabla\phi_{T,\xi_1}^{F\cup G\cup E^{(p)}}+\xi_1)(0)\Big]\Big|dx \;\lesssim \int_{\mathbb{R}^d}(1+|x|)^{-d}e^{-\frac{1}{C\sqrt{T}}|x|}dx \;\lesssim \log T.
\end{aligned}$$

This implies the bound $E_T^{(p),3}\lesssim p^2\log T$.

We now turn to the fourth term $E_T^{(p),4}$. Using the covariance inequality, arguing as e.g. in [212] to estimate the vertical derivatives, using the corrector estimates and the optimal annealed estimates on the Green's functions (cf. (H_1)–(H_2)), applying Lemma 5.A.3, and again using assumptions (5.4) and (5.122), we obtain

$$\Bigg|\mathrm{Cov}\Bigg[(\nabla\phi_{T,\xi_2}+\xi_2)(x)\cdot\sum_{|F|=k}C_F(\nabla\phi_{T,\xi_1}^{F\cup E^{(p)}}+\xi_1)\,(x)\ ;\\ (\nabla\phi_{T,\xi_2}+\xi_2)(0)\cdot\sum_{|G|=l}C_G(\nabla\phi_{T,\xi_1}^{G\cup E^{(p)}}+\xi_1)\,(0)\Bigg]\,dx\Bigg|\lesssim(1+|x|)^{-d}e^{-\frac{1}{C\sqrt{T}}|x|},$$

which directly leads to $|E_T^{(p),4}|\lesssim p^2\log T$. A similar argument implies $|E_T^{(p),5}|\lesssim p^2\log T$, as well as $|M_T^1|+|M_T^2|\lesssim\log T$.

Step 4. Estimation of the error terms $(E_T^{(p),j})_{j=6}^7$.

In this step, we prove that

$$\sum_{j=6}^{7}|E_T^{(p),j}|\lesssim p^2\log T.$$

We start with the term $E_T^{(p),6}$. Successively using equations

$$\Big(\frac{1}{T}-\nabla\cdot A^n\nabla\Big)(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})=\nabla\cdot C^n(\nabla\phi_{T,\xi_2}+\xi_2),$$

and

$$\Big(\frac{1}{T}-\nabla\cdot A^n\nabla\Big)(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})=\nabla\cdot C_{\|n}^{(p)}(\nabla\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}+\xi_1),$$

we obtain the identity

$$\begin{aligned}
&(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n\nabla(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})\\
&=\ \nabla\cdot\Big((\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})+\nabla(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})\cdot A^n(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})\Big)\\
&\qquad-\frac{1}{T}(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})-\nabla(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})\cdot A^n\nabla(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}}))\\
&=\ \nabla\cdot\Big((\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})+\nabla(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})\cdot A^n(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})\Big)\\
&\qquad-\nabla\cdot\Big((\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})A^n\nabla(\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}-\phi_{T,\xi_1}^{\{n\}})+(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})C_{\|n}^{(p)}(\nabla\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}+\xi_1)\Big)\\
&\qquad+\nabla(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})C_{\|n}^{(p)}(\nabla\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}+\xi_1).
\end{aligned}\tag{5.148}$$

Arguing as for (5.136) in Step 1, we deduce

$$\begin{aligned}
E_T^{(p),6}=p\sum_n\int_{\mathbb{R}^d}\Bigg(&\mathbb{E}\Big[\nabla(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})\cdot C_{\|n}^{(p)}(\nabla\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}+\xi_1)(x)\\
&\qquad\qquad\times(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}+\xi_1)(0)\Big]\\
&+\mathbb{E}\Big[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(x)\,\nabla(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})\cdot C_{\|n}^{(p)}(\nabla\phi_{T,\xi_1}^{\{n\}\cup E^{(p)}}+\xi_1)(0)\Big]\Bigg)dx.
\end{aligned}$$

Appealing to the inclusion-exclusion formula (5.20) together with assumption (5.4) and with independence (5.6), we find

$$E_T^{(p),6} = \sum_{k=1}^{\Gamma}(-1)^{k+1}p^{k+1}\sum_n\sum_{|F|=k}\int_{\mathbb{R}^d}\bigg(\mathbb{E}\Big[\nabla(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})\cdot C_{F\|n}(\nabla\phi_{T,\xi_1}^{\{n\}\cup F\cup E^{(p)}}+\xi_1)(x)$$
$$\times(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}\cup F\cup E^{(p)}}+\xi_1)(0)\Big]$$
$$+\,\mathbb{E}\left[(\nabla\phi_{T,\xi_2}+\xi_2)\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(x)\,\nabla(\phi_{T,\xi_2}^{\{n\}}-\phi_{T,\xi_2})\cdot C_{F\|n}(\nabla\phi_{T,\xi_1}^{\{n\}\cup F\cup E^{(p)}}+\xi_1)(0)\right]\bigg)dx.$$

Now arguing as for the term $E_T^{(p),3}$, we conclude $|E_T^{(p),6}|\lesssim p^2\log T$. A similar argument leads to $|E_T^{(p),7}|\lesssim p^2\log T$.

Step 5. Conclusion.

Combining the results of Steps 2–4, we find

$$\Big|\tilde{\mathcal{Q}}_T^{(p)}-\tilde{\mathcal{Q}}_T^{(0)}-p\sum_{j=1}^{3}M_T^j\Big|\lesssim T^{-\frac12}+p^2\log T.$$

Using assumption (5.135), it easily follows that the limit $\tilde{\mathcal{Q}}^{(0),1}:=\lim_{T\uparrow\infty}\sum_{j=1}^3 M_T^j$ exists in $\mathbb{R}$ and that there holds

$$\big|\tilde{\mathcal{Q}}^{(p)}-\tilde{\mathcal{Q}}^{(0)}-p\tilde{\mathcal{Q}}^{(0),1}\big|\lesssim p^2|\log p|.$$

Further using the perturbation argument of Step 3 of Section 5.3.2, the main part of the statement follows.

It remains to perform an explicit computation of $\tilde{\mathcal{Q}}^{(0),1}$ in the case of spherical inclusions $J_n:=B_R(q_n)$ of radius $R>0$, with constant reference coefficients $A:=\alpha\,\mathrm{Id}$ and $A':=\beta\,\mathrm{Id}$. Since A is constant, the unique decaying solution ϕ_{T,ξ_2} of $\frac1T\phi_{T,\xi_2}-\nabla\cdot A(\nabla\phi_{T,\xi_2}+\xi_2)=0$ is clearly $\phi_{T,\xi_2}=0$, and the formula (5.34) is then reduced to

$$\xi_1\otimes\xi_2:\mathcal{Q}^{(0),1}:\xi_1\otimes\xi_2=\lim_{T\uparrow\infty}\sum_n\int_{\mathbb{R}^d}\mathbb{E}\left[\xi_2\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(x)\,\xi_2\cdot C^n(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)(0)\right]dx.$$

For all n the unique solution $\phi_{T,\xi_1}^{\{n\}}\in H^1_{\mathrm{loc}}\cap\mathrm{L}^\infty(\mathbb{R}^d)$ of $\frac1T\phi_{T,\xi_1}^{\{n\}}-\nabla\cdot A^{\{n\}}(\nabla\phi_{T,\xi_1}^{\{n\}}+\xi_1)=0$ is of the form $\phi_{T,\xi_1}^{\{n\}}(x)=\psi_{T,\xi_1}(x-q_n)$ for some function ψ_{T,ξ_1} independent of n, and it admits a weak limit $\nabla\psi_{T,\xi_1}\rightharpoonup\nabla\psi_{\xi_1}$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover, as in the proof of Corollary 5.1.3 in Section 5.3.3, we have $\mathbb{E}\left[\sum_n f(q_n)\right]=\sigma\int f(x)dx$ for all continuous integrable function $f:\mathbb{R}^d\to\mathbb{R}$. The above then takes the form

$$\begin{aligned}&\xi_1\otimes\xi_2:\mathcal{Q}^{(0),1}:\xi_1\otimes\xi_2\\ &\quad=\quad\sigma(\beta-\alpha)^2\lim_{T\uparrow\infty}\int_{B_R}\int_{B_R(y)}\xi_2\cdot(\nabla\psi_{T,\xi_1}(x-y)+\xi_1)\,\xi_2\cdot(\nabla\psi_{T,\xi_1}(-y)+\xi_1)\,dxdy\\ &\quad=\quad\sigma(\beta-\alpha)^2\int_{B_R}\int_{B_R(y)}\xi_2\cdot(\nabla\psi_{\xi_1}(x-y)+\xi_1)\,\xi_2\cdot(\nabla\psi_{\xi_1}(-y)+\xi_1)\,dxdy.\end{aligned}$$

Using the explicit formula (5.116)–(5.117) for $\nabla\psi_\xi$, we deduce

$$\xi_1\otimes\xi_2:\mathcal{Q}^{(0),1}:\xi_1\otimes\xi_2=\sigma|B_R|^2(\xi_1\cdot\xi_2)^2\Big(\frac{\alpha d(\beta-\alpha)}{\beta+\alpha(d-1)}\Big)^2,$$

and the conclusion follows as in the proof of Corollary 5.1.4. The elastic counterpart is similarly easily obtained based on the computations in the proof of Corollary 5.1.5. □

Part II

Dynamics of Ginzburg-Landau vortices in disordered media

Chapter 6

Mean-field limits for Riesz interaction gradient flows

Inspired by the work of Serfaty [395] in the context of the Ginzburg-Landau vortices, we explain how a modulated energy method can be used to prove mean-field limit results for the gradient flow evolution of particle systems with Coulomb-like pairwise interactions when the number of particles tends to infinity. More precisely, we consider repulsive Riesz pairwise interactions, and we establish a mean-field limit result in dimensions 1 and 2 in some cases for which this problem was still open.

This chapter corresponds to the article [158], to which various remarks have been added as well as a detailed section on previous works on the subject.

Contents

6.1 Introduction

6.1.1 General overview

We consider the energy of a system of N particles in the Euclidean space $\mathbb{R}^d$ ($d \geq 1$) interacting via pairwise interactions,

$$H_N(x_1, \ldots, x_N) := \sum_{i \neq j}^{N} g(x_i - x_j) + N \sum_{i=1}^{N} \Phi(x_i),$$

where $x_1, \ldots, x_N \in \mathbb{R}^d$ denote the positions of the particles, where the interaction potential g is continuous on $\mathbb{R}^d \setminus \{0\}$, and where Φ is a smooth external potential on $\mathbb{R}^d$. The corresponding Newton's equations of motion then take the form

$$\partial_t^2 x_{i,N}^t = -\frac{1}{N} \nabla_i H_N(x_{1,N}^t, \ldots, x_{N,N}^t), \qquad i = 1, \ldots, N, \tag{6.1}$$

or more explicitly,

$$\partial_t^2 x_{i,N}^t = -\frac{1}{N} \sum_{j:j \neq i} \nabla g(x_{i,N}^t - x_{j,N}^t) - \nabla \Phi(x_{i,N}^t), \qquad i = 1, \ldots, N,$$

with given initial data $(x_{i,N}^t, \partial_t x_{i,N}^t)|_{t=0} = (x_{i,N}^\circ, v_{i,N}^\circ)$. If the number N of particles is very large, it quickly becomes infeasible to exactly solve the above large system of ODEs and to describe the individual trajectories. Nevertheless, in many practical cases, the detail of the dynamics is no longer relevant and we are only concerned with the "averaged" evolution of the set of particles. The idea of mean-field limit theory is then the following: the effect of all the other particles on any given particle should be approximated by a single averaged effect.

In other words, in the large N limit, we would like to pass to a continuum description of the system, in terms of the particle density distribution. For that purpose, we define the phase-space empirical measure associated with the particle dynamics: setting $v_{i,N}^t := \partial_t x_{i,N}^t$, we define

$$f_N^t := \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_{i,N}^t, v_{i,N}^t)} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d), \tag{6.2}$$

and the question is then to understand the limit of f_N^t as $N \uparrow \infty$. More precisely, assuming convergence at initial time

$$f_N^\circ := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_{i,N}^\circ, v_{i,N}^\circ} \xrightarrow{*} f^\circ, \qquad \text{as } N \uparrow \infty,$$

formal computations lead us to expect under fairly general assumptions $f_N^t \xrightarrow{*} f^t$ for all $t \geq 0$, where f^t is a solution of the following Vlasov equation, which is a nonlocal nonlinear PDE on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\partial_t f^t + v \cdot \nabla_x f^t = (\nabla g * \mu^t) \cdot \nabla_v f^t + \nabla \Phi \cdot \nabla_v f^t, \qquad \mu^t(x) := \int f^t(x,v) dv, \qquad f^t|_{t=0} = f^\circ. \tag{6.3}$$

The first rigorous discussion of such a mean-field limit result seems to be due to Neunzert and Wick [343] in the 1970s. In the case of a smooth interaction potential $g \in C_b^{1,1}(\mathbb{R}^d)$, the mean-field result was first established by Braun and Hepp [82] by a weak compactness argument, while Dobrushin [148] was the first to give a quantitative proof in 1-Wasserstein distance. Dobrushin's proof is based on the crucial observation that the empirical measure f_N already satisfies the limiting

Vlasov equation (6.3) up to diagonal terms: for smooth interaction potential g, we may write in the distributional sense,

$$\partial_t f_N^t + v \cdot \nabla_x f_N^t = (\nabla g * \mu_N^t) \cdot \nabla_v f_N^t + \nabla\Phi \cdot \nabla_v f_N^t - \frac{1}{N}\nabla g(0) \cdot \nabla_v f_N^t,$$
$$\mu_N^t(x) := \int f_N^t(x,v)dv,$$

where the error term $\frac{1}{N}\nabla g(0) \cdot \nabla_v f_N^t$ is of order $O(1/N)$, so that the mean-field result simply follows from stability estimates for the Vlasov equation (see e.g. [214] for a general overview of the subject). Nevertheless, the Coulomb and the gravitational potentials (resp. $g(x) \propto |x|^{2-d}$ and $\propto -|x|^{2-d}$ in dimension $d \geq 3$), which are so ubiquitous in nature, are examples of interaction potentials that are singular at the origin, in which case the classical Dobrushin theory obviously fails. In recent years great progress has been made in the direction of mean-field limit results for singular potentials [256, 235, 271, 70, 287, 288], but establishing a complete result in the Coulomb or gravitational case in dimension $d \geq 3$ remains a major open problem in the field.

In the present chapter, we focus on *repulsive* Coulomb-like interaction potentials (thus avoiding delicate blow-up issues) and consider the whole family of Riesz potentials,

$$g_s(x) := \begin{cases} c_{d,s}^{-1}|x|^{-s}, & \text{if } 0 < s < d; \\ -c_{d,0}^{-1}\log(|x|), & \text{if } s = 0; \end{cases} \tag{6.4}$$

with $c_{d,s} > 0$ some normalization constants. The Coulomb case corresponds to the choice $s = d-2$, $d \geq 2$. As the external potential Φ adds no difficulty to the problem, we set $\Phi := 0$ and focus on the interaction part. In addition, in order to simplify the delicate mean-field limit question, we replace Newton's equations (6.1) by the corresponding gradient-flow evolution, that is,

$$H_N(x_1,\ldots,x_N) := \sum_{i\neq j}^N g(x_i - x_j),$$
$$\partial_t x_{i,N}^t = -\frac{1}{N}\sum_{j:j\neq i} \nabla g(x_{i,N}^t - x_{j,N}^t), \qquad x_{i,N}^t|_{t=0} = x_{i,N}^\circ, \qquad i = 1,\ldots,N, \tag{6.5}$$

where $(x_{i,N}^\circ)_{i=1}^N$ is a sequence of N distinct initial positions. This gradient-flow evolution has the advantage of being dissipative, which will help us greatly in our arguments. Note that the trajectories $\{t \mapsto x_{i,N}^t\}_{i=1}^N$ are obviously smooth and well-defined on the whole of $\mathbb{R}^+$: indeed, since energy can only decrease in time and since the interaction is repulsive, particles cannot collide, and moreover it is easily seen that a particle cannot escape to infinity in finite time, so that the conclusion follows from the Picard-Lindelöf theorem. Particle systems with Riesz interactions as considered here are extensively motivated in the physics literature (see e.g. [48, 318]), as well as in the context of approximation theory with the study of Fekete points (see [231] and references therein). In the static case, a detailed description of such particle systems beyond the mean-field limit has been obtained in [362], and also in [290] for non-zero temperature.

At a formal level, the mean-field limit of the gradient flow evolution (6.5) as the number N of particles tends to infinity is again easy to guess: defining the empirical measure associated with the particle dynamics,

$$\mu_N^t := \frac{1}{N}\sum_{i=1}^N \delta_{x_{i,N}^t}, \tag{6.6}$$

and assuming convergence at initial time $\mu_N^\circ \xrightharpoonup{*} \mu^\circ$ as $N \uparrow \infty$, we expect under fairly general assumptions $\mu_N^t \xrightharpoonup{*} \mu^t$ for all $t \ge 0$, where μ^t is a solution to the following nonlocal nonlinear PDE on $\mathbb{R}^+ \times \mathbb{R}^d$,

$$\partial_t \mu^t = \operatorname{div}(\mu^t \nabla h^t), \qquad h^t := g * \mu^t, \qquad \mu^t|_{t=0} = \mu^\circ. \tag{6.7}$$

This equation in the weak sense just means the following: there hold $\mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^d))$, $g * \mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d))$, $\mu \nabla g * \mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^d))$, and for all $\phi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \mu^t(x)(\partial_t\phi(t,x) - \nabla\phi(t,x)\cdot\nabla g * \mu^t(x))dxdt + \int_{\mathbb{R}^d}\mu^\circ(x)\phi(0,x)dx = 0.$$

In the case of a Riesz potential $g = g_s$, this equation (6.7) is sometimes called a *fractional porous medium equation* (see e.g. [94, 93] for $d-2 < s < d$, $s \ge 0$, [304, 157, 18, 398] for $s = d-2$, $d \ge 2$, and [102] for $0 \le s < d-2$). Although expected to be much easier than in the conservative case (6.1)–(6.3), the justification of this mean-field limit result has remained an open problem whenever $s \ge d-2$, $s > 0$, $d \ge 2$. In the present chapter, we devise a modulated energy approach inspired by the work of Serfaty [395], and we establish the mean-field limit result for $0 \le s < 1$ in dimensions $d = 1$ and 2.

6.1.2 Previous works

Similarly as in the conservative case (6.1)–(6.3) discussed above, the classical theory for the mean-field limit result for the gradient-flow system (6.5)–(6.7) holds for smooth interaction potentials $g \in C^1_b(\mathbb{R}^d)$: in that case, we indeed check that the empirical measure μ_N satisfies in the distributional sense,

$$\partial_t\mu_N^t = \operatorname{div}(\mu_N^t\nabla h_N^t) - \frac{1}{N}\operatorname{div}(\mu_N^t\nabla g(0)), \qquad h_N^t := g * \mu_N^t, \qquad \mu_N^t|_{t=0} = \mu_N^\circ,$$

so that the desired convergence result directly follows from a weak compactness argument. If in addition $g \in C^{1,1}_b(\mathbb{R}^d)$, then we may also take advantage of the stability properties of the limiting equation (6.7) in 2-Wasserstein distance, and the following quantitative convergence result is easily obtained.

Proposition 6.1.1. *Let $d \ge 1$. Given an interaction potential $g \in C^{1,1}_b(\mathbb{R}^d)$, let $\mu_N \in \mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}(\mathbb{R}^d))$ be as in (6.5)–(6.6), and let $\mu \in \mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}\cap\mathrm{L}^1(\mathbb{R}^d))$ be the weak solution of (6.7) on $[0,T)\times\mathbb{R}^d$ with some initial data $\mu^\circ \in \mathcal{P}\cap\mathrm{L}^1(\mathbb{R}^d)$. Then for all $t \ge 0$,*

$$W_2(\mu_N^t,\mu^t) \le e^{2t\|\nabla^2 g\|_{\mathrm{L}^\infty}}\Big(W_2(\mu_N^\circ,\mu^\circ) + \frac{2t}{N}|\nabla g(0)|\Big). \qquad \Diamond$$

Proof. As μ^t is absolutely continuous, there exists an optimal transportation map T_N^t between μ^t and μ_N^t, $(T_N^t)_*\mu^t = \mu_N^t$. We may then easily estimate the right derivative

$$\begin{aligned}
\partial_t^+ W_2(\mu_N^t,\mu^t)^2 &= \partial_t^+\int_{\mathbb{R}^d}|x - T_N^t x|^2 d\mu^t(x) \\
&\le -2\int_{\mathbb{R}^d}(x - T_N^t x)\cdot\Big(\nabla g * \mu^t(x) - \frac{1}{N}\sum_{j:x_j\ne T_N^t x}^N \nabla g(T_N^t x - x_j)\Big)d\mu^t(x) \\
&= -2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(x - T_N^t x)\cdot\big(\nabla g(x-z) - \nabla g(T_N^t x - T_N^t z)\mathbb{1}_{T_N^t x\ne T_N^t z}\big)d\mu^t(x)d\mu^t(z),
\end{aligned}$$

Now note that by definition the cell $C_N^t(x) := \{z \in \mathbb{R}^d : T_N^t z = T_N^t x\}$ satisfies $\mu^t(C_N^t(x)) = N^{-1}$. This yields

$$\begin{aligned}\partial_t^+ W_2(\mu_N^t,\mu^t)^2 &\le -2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(x-T_N^t x)\cdot\big(\nabla g(x-z)-\nabla g(T_N^t x - T_N^t z)\big)d\mu^t(x)d\mu^t(z)\\ &\qquad -\frac{2}{N}\nabla g(0)\cdot\int_{\mathbb{R}^d}(x-T_N^t x)d\mu^t(x)\\ &\le 2\|\nabla^2 g\|_{\mathrm{L}^\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x-T_N^t x||x-z-T_N^t x+T_N^t z|d\mu^t(x)d\mu^t(z)\\ &\qquad +\frac{2}{N}|\nabla g(0)|\int_{\mathbb{R}^d}|x-T_N^t x|d\mu^t(x)\\ &\le 4\|\nabla^2 g\|_{\mathrm{L}^\infty}W_2(\mu_N^t,\mu^t)^2+\frac{2}{N}|\nabla g(0)|W_2(\mu_N^t,\mu^t),\end{aligned}$$

and the result follows from the Grönwall inequality. ☐

We now turn to the case of singular interaction potentials g, for which diagonal terms can no longer be neglected and for which W_2-stability of the limiting equation (6.7) fails. In the early 1990s, in the context of point-vortex numerical methods for the Euler equation, Goodman, Hou and Lowengrub [215, 246] proposed a way to prove strong mean-field limit results for Coulomb interactions in dimensions $d = 2$ and 3 in the case of initial data μ_N° concentrated on a grid. This method, strongly relying on the symmetry of the potential g, only holds for very symmetric initial particle positions. As the admissible initial data are not statistically generic, these results are however not really relevant to our concern.

In the mid-1990s, Schochet [390] studied the case of logarithmic interactions $g = g_0$ (in arbitrary dimension d) and established the expected mean-field limit result [1], based on his simplification [389] of the proof of Delort's theorem [143] on existence of weak solutions to the 2D Euler equation with vortex-sheet initial data (that is, with nonnegative initial vorticity in H^{-1}). The key idea, which only holds for logarithmic interactions, consists in exploiting some logarithmic gain of integrability to find uniform bounds on the number of close particles, which allows to directly pass to the limit in the equation and conclude by a compactness argument. However, due to a possible lack of uniqueness of L^1 "very weak" solutions to equation (6.7), this proof only establishes that the empirical measure μ_N^t converges up to a subsequence to *some* solution of (6.7). As it is instructive for the sequel of the discussion, a short proof is included.

Theorem 6.1.2 (Schochet). *Let $d \ge 1$. Let $\mu_N \in \mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}(\mathbb{R}^d))$ be as in* (6.5)–(6.6) *with logarithmic interaction potential $g = g_0$, and assume the convergence of initial data $\mu_N^\circ \overset{*}{\rightharpoonup} \mu^\circ$ for some $\mu^\circ \in \mathcal{P}\cap\mathrm{L}^\infty(\mathbb{R}^d)$. Further assume*

$$\limsup_{N\uparrow\infty}\int_{\mathbb{R}^d}|x|^2 d\mu_N^\circ(x) < \infty, \qquad \limsup_{N\uparrow\infty}\frac{1}{N^2}\sum_{i\ne j}g_0(x_{i,N}^\circ - x_{j,N}^\circ) < \infty.$$

Then up to a subsequence we have $\mu_N \overset{*}{\rightharpoonup} \mu$ in $\mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}(\mathbb{R}^d))$, where μ has finite energy

$$\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}g_0(x-y)d\mu^t(x)d\mu^t(y)dt < \infty, \qquad \textit{for all } T>0,$$

1. Schochet's original paper [390] was actually only concerned with the mean-field limit for a particle approximation of the 2D Euler equation, but the same argument directly applies to the present gradient-flow setting, as shown below.

and satisfies the limiting equation (6.7) *(with $g = g_0$) in the following very weak sense: for all $\phi \in C^\infty(\mathbb{R}^+; C_c^\infty(\mathbb{R}^d))$ such that $\phi(t,\cdot) = 0$ for all $t > 0$ large enough,*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \mu^t(x)\partial_t\phi(t,x)dxdt + \int_{\mathbb{R}^d} \mu^\circ(x)\phi(0,x)dx \\ = \frac{1}{2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\nabla\phi(t,x) - \nabla\phi(t,y)) \cdot \nabla g_0(x-y)\mu^t(x)\mu^t(y)dydxdt. \quad \Diamond$$

Proof. For all $T > 0$, we easily check that $(d\mu_N^t dt|_{[0,T]\times\mathbb{R}^d})_N$ is tight. Up to extraction of a subsequence, the Prokhorov theorem then gives $d\mu_N^t dt|_{[0,T]\times\mathbb{R}^d} \overset{*}{\rightharpoonup} d\nu^t dt|_{[0,T]\times\mathbb{R}^d}$ for all $T > 0$, for some $\nu \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^d))$. Let $\phi \in C^\infty(\mathbb{R}^+; C_c^\infty(\mathbb{R}^d))$ such that $\phi(t,\cdot) = 0$ for all $t > T$, for some $T > 0$. Using an integration by parts as well as the equations (6.5) satisfied by the trajectories, we find

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \partial_t\phi(t,x)d\mu_N^t(x)dt + \int_{\mathbb{R}^d}\phi(0,x)d\mu_N^\circ(x) = \frac{1}{N^2}\sum_{i\neq j}^{N}\int_{\mathbb{R}^+} \nabla\phi(t,x_{i,N}^t)\cdot\nabla g_0(x_{i,N}^t - x_{j,N}^t)dt,$$

or equivalently, by symmetry of the potential,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \partial_t\phi(t,x)d\mu_N^t(x)dt + \int_{\mathbb{R}^d}\phi(0,x)d\mu_N^\circ(x) \\ = \frac{1}{2}\int_{\mathbb{R}^+}\iint_{D^c} (\nabla\phi(t,x) - \nabla\phi(t,y))\cdot\nabla g_0(x-y)d\mu_N^t(x)d\mu_N^t(y)dt, \quad (6.8)$$

where $D := \{(x,x) : x \in \mathbb{R}^d\}$ denotes the diagonal. We may then pass to the limit $N \uparrow \infty$ (along the given subsequence),

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \partial_t\phi(t,x)d\nu^t(x)dt + \int_{\mathbb{R}^d}\phi(0,x)d\mu^\circ(x) \\ = \frac{1}{2}\lim_{N\uparrow\infty}\int_{\mathbb{R}^+}\iint_{D^c} (\nabla\phi(t,x) - \nabla\phi(t,y))\cdot\nabla g_0(x-y)d\mu_N^t(x)d\mu_N^t(y)dt. \quad (6.9)$$

It remains to compute to the limit in the right-hand side. Since the integrand is continuous on D^c, we find for all $r > 0$,

$$\lim_{N\uparrow\infty}\int_{\mathbb{R}^+}\iint_{|x-y|>r} (\nabla\phi(t,x) - \nabla\phi(t,y))\cdot\nabla g_0(x-y)d\mu_N^t(x)d\mu_N^t(y)dt \\ = \int_{\mathbb{R}^+}\iint_{|x-y|>r} (\nabla\phi(t,x) - \nabla\phi(t,y))\cdot\nabla g_0(x-y)d\nu^t(x)d\nu^t(y)dt. \quad (6.10)$$

Noting that

$$|(\nabla\phi(t,x) - \nabla\phi(t,y))\cdot\nabla g_0(x-y)| \le \|\nabla^2\phi\|_{\mathrm{L}^\infty},$$

and that $g_0(x) \ge -\log r = |\log r|$ for all $|x| \le r$ with $r \in (0,1)$, we also obtain

$$\begin{aligned} &\limsup_{r\downarrow 0}\limsup_{N\uparrow\infty}\left|\int_{\mathbb{R}^+}\iint_{|x-y|\le r} (\nabla\phi(t,x) - \nabla\phi(t,y))\cdot\nabla g_0(x-y)d\mu_N^t(x)d\mu_N^t(y)dt\right| \\ &\le \|\nabla^2\phi\|_{\mathrm{L}^\infty}\limsup_{r\downarrow 0}\limsup_{N\uparrow\infty}\int_0^T\iint_{|x-y|\le r} d\mu_N^t(x)d\mu_N^t(y)dt \\ &\le \|\nabla^2\phi\|_{\mathrm{L}^\infty}\limsup_{r\downarrow 0}\limsup_{N\uparrow\infty}|\log r|^{-1}\int_0^T\iint_{|x-y|\le r} g_0(x-y)d\mu_N^t(x)d\mu_N^t(y)dt. \qquad (6.11)\end{aligned}$$

A direct computation yields

$$\partial_t \iint_{D^c} |x-y|^2 d\mu_N^t(x) d\mu_N^t(y) = \frac{2}{N}\sum_{i=1}^N \partial_t |x_{i,N}^t|^2 - 2\partial_t \Big|\frac{1}{N}\sum_i^N x_{i,N}^t\Big|^2$$
$$= -\frac{4}{N^2}\sum_{i\neq j}^N x_{i,N}^t \cdot \nabla g_0(x_{i,N}^t - x_{j,N}^t) = -\frac{2}{N^2}\sum_{i\neq j}^N (x_{i,N}^t - x_{j,N}^t)\cdot \nabla g_0(x_{i,N}^t - x_{j,N}^t) \ \le 2,$$

and also

$$\partial_t \iint_{D^c} g_0(x-y) d\mu_N^t(x) d\mu_N^t(y) = -2\int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d\setminus\{x\}} \nabla g_0(x-y) d\mu_N^t(y)\Big|^2 d\mu_N^t(x) \le 0.$$

By assumption, these two estimates imply

$$\limsup_{N\uparrow\infty} \int_0^T \iint_{D^c} (g_0(x-y) + |x-y|^2) d\mu_N^t(x) d\mu_N^t(y) dt$$
$$\le T^2 + T \limsup_{N\uparrow\infty} \iint_{D^c} (g_0(x-y)+|x-y|^2) d\mu_N^\circ(x) d\mu_N^\circ(y) \ < \infty.$$

Combining this with (6.11) and noting that $g_0(x) + |x|^2 \ge 0$, we may conclude

$$\limsup_{r\downarrow 0}\limsup_{N\uparrow\infty} \Big| \int_{\mathbb{R}^+} \iint_{|x-y|\le r} (\nabla\phi(t,x) - \nabla\phi(t,y))\cdot \nabla g_0(x-y) d\mu_N^t(x) d\mu_N^t(y) dt \Big|$$
$$\le \|\nabla^2\phi\|_{\mathrm{L}^\infty} \limsup_{r\downarrow 0} |\log r|^{-1} \limsup_{N\uparrow\infty} \int_0^T \iint_{D^c} (g_0(x-y)+|x-y|^2) d\mu_N^t(x) d\mu_N^t(y) dt \ = 0. \quad (6.12)$$

It remains to pass to the limit $r \downarrow 0$ also in the right-hand side of (6.10). For that purpose, we note that

$$\begin{aligned}
&\limsup_{r\downarrow 0} \Big| \int_{\mathbb{R}^+} \iint_{|x-y|\le r} (\nabla\phi(t,x) - \nabla\phi(t,y))\cdot\nabla g_0(x-y) d\nu^t(x) d\nu^t(y)\Big| \\
&\quad \le \ \|\nabla^2\phi\|_{\mathrm{L}^\infty} \limsup_{r\downarrow 0} \int_0^T \iint_{|x-y|\le r} d\nu^t(x) d\nu^t(y) \\
&\quad = \ \|\nabla^2\phi\|_{\mathrm{L}^\infty} \limsup_{r\downarrow 0}\limsup_{N\uparrow\infty} \int_0^T \iint_{|x-y|\le r} d\mu_N^t(x) d\mu_N^t(y) = 0, \qquad (6.13)
\end{aligned}$$

where the last equality follows from the combination of (6.9), (6.11) and (6.12). Now combining (6.10), (6.12) and (6.13), we may conclude

$$\lim_{N\uparrow\infty} \frac{1}{2}\int_{\mathbb{R}^+} \iint_{D^c} (\nabla\phi(t,x) - \nabla\phi(t,y))\cdot \nabla g_0(x-y) d\mu_N^t(x) d\mu_N^t(y) dt$$
$$= \frac{1}{2}\int_{\mathbb{R}^+} \iint_{D^c} (\nabla\phi(t,x) - \nabla\phi(t,y))\cdot\nabla g_0(x-y) d\nu^t(x) d\nu^t(y) dt,$$

and the result follows. ☐

This proof by weak compactness clearly fails for more singular interaction potentials $g = g_s$, $s > 0$, as we emphasize now. Recall the symmetrized weak formulation (6.8),

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \partial_t\phi(t,x) d\mu_N^t(x) dt + \int_{\mathbb{R}^d} \phi(0,x) d\mu_N^\circ(x)$$
$$= \frac{1}{2}\int_{\mathbb{R}^+} \iint_{D^c} (\nabla\phi(t,x) - \nabla\phi(t,y))\cdot \nabla g_s(x-y) d\mu_N^t(x) d\mu_N^t(y) dt. \quad (6.14)$$

The singularity of the integrand in the right-hand side prevents us from simply passing to the weak limit: an argument is needed to uniformly neglect the contribution of the integral near the diagonal D, that is, the contribution due to very close particles. In the case $s > 0$, using $|x||\nabla g_s(x)| \le s c_{d,s}^{-1}|x|^{-s}$, the right-hand side is bounded by

$$\frac{1}{2}\int_{\mathbb{R}^+}\iint_{D^c}(\nabla\phi(t,x)-\nabla\phi(t,y))\cdot\nabla g_s(x-y)d\mu_N^t(x)d\mu_N^t(y)dt \\ \le \frac{s}{2}\|\nabla^2\phi\|_{\mathrm{L}^\infty}\int_0^T\iint_{D^c} g_s(x-y)d\mu_N^t(x)d\mu_N^t(y)dt,$$

that is, by the energy. Boundedness of the energy or other basic information about the flow therefore only ensures that the right-hand side is uniformly bounded, but is not enough to neglect the near-diagonal contribution. In the logarithmic case $s = 0$, the miracle is that the integrand in the right-hand side of (6.14) is actually bounded (but discontinuous) at the diagonal, hence is less singular than the interaction potential. Then comparing with the boundedness of the energy, this logarithmic gain of integrability at the diagonal clearly gives enough information to neglect the near-diagonal contribution. More generally, if the interaction potential g is even (i.e., $g(x) = g(-x)$) and satisfies $g(x) + C|x|^2 \ge 0$ for some constant $C > 0$, then this argument by Schochet works whenever the interaction potential g is nonnegative and has a subalgebraic blow-up at the origin in the sense of $|x||\nabla g(x)| \ll |g(x)|$ as $|x| \ll 1$. Note that an adaptation of this argument in the framework of the Sandier-Serfaty Γ-convergence of gradient flows [380, 393] is obtained in [418, pp.152–154].

In order to get beyond the logarithmic case, more information thus seems to be needed about the distribution of close particles along the flow. The first result in that direction was obtained a decade ago by Hauray and Jabin [234, 233] (see also [102]), and allows in our setting to treat $g = g_s$ for all $0 \le s < d-2$, $d \ge 3$, hence just missing the Coulomb case. The strategy consists in considering the infinite Wasserstein distance W_∞, which indeed allows to take advantage of the localization of the singularity of the interaction potential g_s, and noting that we may control both $A^t := W_\infty(\mu_N^t, \mu^t)$ and the minimal distance between the particles $B^t := \min_{i\neq j}|x_{i,N}^t - x_{j,N}^t|$ via a combined Grönwall inequality,

$$\partial_t^+ A^t \le CA^t\Big(1+\frac{(A^t)^d}{(B^t)^{s+2}}\Big)\|\mu\|_{\mathrm{L}_t^\infty(\mathrm{L}^1\cap\mathrm{L}^\infty)}, \qquad \partial_t B^t \ge -CB^t\Big(1+\frac{(A^t)^d}{(B^t)^{s+2}}\Big)\|\mu\|_{\mathrm{L}_t^\infty(\mathrm{L}^1\cap\mathrm{L}^\infty)}.$$

(We refer to [102] for more general results that are obtained with this method, including the attractive case.)

Theorem 6.1.3 (Hauray, Jabin)**.** *Let $0 \le s < d-2$, $d \ge 3$. Let $\mu_N \in \mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}(\mathbb{R}^d))$ be as in* (6.5)–(6.6) *with interaction potential $g = g_s$, and assume the convergence of initial data $W_\infty(\mu_N^\circ,\mu^\circ) \to 0$ for some $\mu^\circ \in \mathcal{P}\cap\mathrm{L}^\infty(\mathbb{R}^d)$. Further assume that the initial data are well-prepared in the sense*

$$\lim_{N\uparrow\infty}\frac{W_\infty(\mu_N^\circ,\mu^\circ)^d}{\min_{i\neq j}|x_{i,N}^\circ - x_{j,N}^\circ|^{s+2}} = 0. \tag{6.15}$$

Let $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{P}\cap\mathrm{L}^\infty(\mathbb{R}^d))\cap C_b([0,T);\mathcal{P}(\mathbb{R}^d))$ be a solution of (6.7) *with initial data μ°. Then we have $W_\infty(\mu_N^t,\mu^t) \to 0$ for all $t \ge 0$.* $\Diamond$

As shown in [233], if the initial positions $(x_{i,N}^\circ)_{i=1}^N$ are chosen i.i.d. with law μ°, the well-posedness condition (6.15) is expected to hold only for $s < \frac{d}{2}-2$, $d \ge 5$. Nevertheless, for smooth μ°, a suitable discretization method easily allows to construct initial data such that $W_\infty(\mu_N^\circ,\mu^\circ) \sim \min_{i\neq j}|x_{i,N}^\circ - x_{j,N}^\circ| \sim N^{-1/d}$, in which case the condition (6.15) is satisfied for the whole considered range $s < d-2$.

Before closing this section on previous works, we briefly explain why the 1D case is actually much easier than higher dimensions. The key observations are that the interaction potentials g_s are convex on both $\mathbb{R}^+$ and $\mathbb{R}^-$, and that the particle order in 1D is obviously unchanged along the flow. Therefore in 1D the interaction has a purely convex structure, which can nicely be exploited e.g. in terms of W_2-stability, as we show now. This has actually been realized only very recently by Berman and Önnheim [54], who further treat in a similar spirit the case with thermal noise. (A completely different 1D approach has also been proposed in [185] using the theory of viscosity solutions for non-local Hamilton-Jacobi equations.)

Theorem 6.1.4 (λ-convex interaction potential and 1D case).

(i) Let $d \geq 1$, and let $g \in C^1(\mathbb{R}^d)$ be a λ-convex symmetric interaction potential, that is,

$$\nabla g(-x) = -\nabla g(x), \qquad (x-y)\cdot(\nabla g(x) - \nabla g(y)) \geq -\lambda|x-y|^2, \qquad \text{for all } x, y \in \mathbb{R}^d,$$

and assume that $-x \cdot \nabla g(x) \leq C$ holds for all $x \in \mathbb{R}^d$. Let $\mu_N \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^d))$ be as in (6.5)–(6.6), and let $\mu \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^1(\mathbb{R}^d))$ be the weak solution of (6.7) with some initial data $\mu^\circ \in \mathcal{P} \cap \mathrm{L}^1(\mathbb{R}^d)$. Then for all $t \geq 0$,

$$W_2(\mu_N^t, \mu^t) \leq e^{\lambda t}\Big(W_2(\mu_N^\circ, \mu^\circ) + \frac{(Ct)^{1/2}}{\sqrt{N}}\Big).$$

(ii) Let $d = 1$, and let $g \in C^1(\mathbb{R} \setminus \{0\})$ be a symmetric interaction potential with λ-convex restriction on $\mathbb{R}^+$, that is,

$$g'(-x) = -g'(x), \qquad (x-y)(g'(x) - g'(y)) \geq -\lambda|x-y|^2, \qquad \text{for all } x, y \geq 0,$$

and assume that for some $0 \leq s < 1$ we have $g(x) \sim |x|^{-s}$ at the origin in the following sense,

$$\sup_{x \geq \varepsilon}(-xg'(x)) \leq C\varepsilon^{-s}, \qquad \int_{|x|<\varepsilon} |xg'(x)| \leq C\varepsilon^{1-s}, \qquad \text{for all } \varepsilon > 0.$$

Let $\mu_N \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^d))$ be as in (6.5)–(6.6), and let $\mu \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^d))$ be the weak solution of (6.7) with some initial data $\mu^\circ \in \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^d)$. Then for all $t \geq 0$,

$$W_2(\mu_N^t, \mu^t) \leq e^{\lambda t}\Big(W_2(\mu_N^\circ, \mu^\circ) + \frac{(2Ct)^{1/2}\|\mu\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}^{s/2}}{\sqrt{N}^{1-s}}\Big). \qquad \Diamond$$

Proof. We start with the proof of item (i). As μ^t is absolutely continuous, there exists an optimal transportation map T_N^t between μ^t and μ_N^t, $(T_N^t)_*\mu^t = \mu_N^t$. As in the proof of Proposition 6.1.1, we may then easily estimate the right derivative

$$\partial_t^+ W_2(\mu_N^t, \mu^t)^2 \leq -2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(x - T_N^t x)\cdot\big(\nabla g(x-z) - \nabla g(T_N^t x - T_N^t z)\mathbb{1}_{T_N^t x \neq T_N^t z}\big)d\mu^t(x)d\mu^t(z),$$

and hence, using the symmetry and the λ-convexity assumptions,

$$\begin{aligned}
&\partial_t^+ W_2(\mu_N^t, \mu^t)^2\\
&\leq -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big((x-z) - (T_N^t x - T_N^t z)\big)\cdot\big(\nabla g(x-z) - \nabla g(T_N^t x - T_N^t z)\mathbb{1}_{T_N^t x \neq T_N^t z}\big)d\mu^t(x)d\mu^t(z)\\
&= -\iint_{T_N^t x \neq T_N^t z}\big((x-z) - (T_N^t x - T_N^t z)\big)\cdot\big(\nabla g(x-z) - \nabla g(T_N^t x - T_N^t z)\big)d\mu^t(x)d\mu^t(z)\\
&\qquad\qquad - \iint_{T_N^t x = T_N^t z}(x-z)\cdot\nabla g(x-z)d\mu^t(x)d\mu^t(z)\\
&\leq 2\lambda W_2(\mu_N^t, \mu^t)^2 - \iint_{T_N^t x = T_N^t z}(x-z)\cdot\nabla g(x-z)d\mu^t(x)d\mu^t(z).
\end{aligned} \tag{6.16}$$

Now note that by definition the cell $C_N^t(x) := \{z \in \mathbb{R}^d : T_N^t z = T_N^t x\}$ satisfies $\mu^t(C_N^t(x)) = N^{-1}$. The assumption $-x \cdot \nabla g(x) \leq C$ then leads to

$$\partial_t^+ W_2(\mu_N^t, \mu^t)^2 \leq 2\lambda W_2(\mu_N^t, \mu^t)^2 + CN^{-1},$$

and the conclusion (i) follows from the Grönwall inequality.

It remains to deduce the result of item (ii). Let $d = 1$. We note that the optimal transportation map $T_N^t : \mathbb{R} \to \mathbb{R}$ is then monotone, that is, $\operatorname{sgn}(T_N^t x - T_N^t z) = \operatorname{sgn}(x - z)$ for all x, z. Therefore, in (6.16), we only need to use the inequality $(x-y)(g'(x)-g'(y)) \geq -\lambda|x-y|^2$ for all $x, y \in \mathbb{R}^+$ (which implies by symmetry the same inequality for all $x, y \in \mathbb{R}^-$). In other words, only the λ-convexity of the restriction $g|_{\mathbb{R}^+}$ is needed, and we again obtain

$$\partial_t^+ W_2(\mu_N^t, \mu^t)^2 \leq 2\lambda W_2(\mu_N^t, \mu^t)^2 - \iint_{T_N^t x = T_N^t z} (x - z)\, g'(x - z)\, d\mu^t(x) d\mu^t(z).$$

Recalling that by definition the cell $C_N^t(x) := \{z \in \mathbb{R}^d : T_N^t z = T_N^t x\}$ satisfies $\mu^t(C_N^t(x)) = N^{-1}$, and using the new assumptions on $xg'(x)$, we obtain for all $\varepsilon > 0$,

$$\begin{aligned}
\partial_t^+ W_2(\mu_N^t, \mu^t)^2 \leq 2\lambda W_2(\mu_N^t, \mu^t)^2 &+ \|\mu^t\|_{\mathrm{L}^\infty} \iint_{|x-z|<\varepsilon} \big((x - z)g'(x - z)\big)_- dz d\mu^t(x) \\
&\qquad + C\varepsilon^{-s} \int_{\mathbb{R}^d} \mu^t(C_N^t(x)) d\mu^t(x) \\
\leq 2\lambda W_2(\mu_N^t, \mu^t)^2 &+ C\varepsilon^{1-s} \|\mu^t\|_{\mathrm{L}^\infty} + C\varepsilon^{-s} N^{-1},
\end{aligned}$$

and hence, optimizing in $\varepsilon > 0$,

$$\partial_t^+ W_2(\mu_N^t, \mu^t)^2 \leq 2\lambda W_2(\mu_N^t, \mu^t)^2 + 2CN^{-(1-s)} \|\mu^t\|_{\mathrm{L}^\infty}^s,$$

so that the conclusion (ii) follows from the Grönwall inequality. □

6.1.3 Main result

In the context of the 2D Gross-Pitaevskii and parabolic Ginzburg-Landau equations, Serfaty [395] recently proposed a new way of proving such mean-field limits [2] based on a so-called "modulated energy" technique, which is similar in spirit to the relative entropy method first designed by DiPerna [145] and Dafermos [131, 132] to establish weak-strong stability principles for some hyperbolic systems. This relative entropy method was later rediscovered by Yau [426] for the hydrodynamic limit of the Ginzburg-Landau lattice model, was introduced in kinetic theory by Golse [73] for the convergence of suitably scaled solutions of the Boltzmann equation towards solutions of the incompressible Euler equations (see e.g. [378] for the many recent developments on the topic), and first took the form of a modulated *energy* method in the work by Brenier [86] on the quasi-neutral limit of the Vlasov-Poisson system. Rather than studying the mean-field limit question in an arbitrary fixed metric like W_2 or W_∞, the idea of such methods is to devise a better adapted metric modeled on the available entropy or energy structure, and to expect that this new metric is much better behaved along the flow and may lead to stronger stability results. More precisely, if $\mu \mapsto H(\mu)$ is an energy functional for the system, then a natural notion of distance on the state space is given by the associated Bregman divergence [84], called in this context "modulated energy",

$$H(\mu_1|\mu_2) := H(\mu_1) - H(\mu_2) - \Big\langle \frac{\delta H}{\delta \mu}(\mu_2)\,,\, \mu_1 - \mu_2 \Big\rangle.$$

2. In [395], the questions are different in nature, since they consist in passing to the limit in PDE evolutions, but are similar in spirit since one wishes to understand the limit dynamics of point vortices which essentially behave like Coulomb-interacting particles.

We are able to apply such a method in the present context only in dimensions 1 and 2 and for interaction potentials $g = g_s$ with s not too large. More precisely, we treat in 1D the whole range $0 \le s < 1$ as in [54] (see also Theorem 6.1.4(ii) above), and in 2D we treat the range $0 \le s < 1$ which is new and in particular completes Schochet's partial result [390] in the logarithmic case (cf. Theorem 6.1.2 above). The argument is essentially based on a Grönwall inequality for a suitable modulated energy, which is seen as an adapted measure of the distance between the empirical measure and the (postulated) limit: it strongly exploits the stability properties of the limiting equation (6.7) as well as the regularity of its solution. The advantage of this method is to be completely global, bypassing the need for a precise understanding of the particle dynamics, but this is also its limitation: as explained in Remark 6.2.12 below, in order to get beyond the restriction on s, it is expected that the method should be combined with further nontrivial information on the particle dynamics. Our main result is as follows.

Theorem 6.1.5. *Let $d = 1$ or 2, and $0 \le s < 1$. Let $\mu_N \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^d))$ be as in (6.5)–(6.6) with $g = g_s$. Let $\mu^\circ \in \mathcal{P}(\mathbb{R}^d)$, and assume that equation (6.7) with $g = g_s$ and initial data μ° admits a solution μ that belongs to $\mathrm{L}^\infty([0,T]; C_b^\sigma(\mathbb{R}^d))$ for some $T > 0$ and some $\sigma > 2 - d + s$. In the case $s = 0$, $d = 1$, also assume $\nabla\mu \in \mathrm{L}^\infty([0,T]; \mathrm{L}^p(\mathbb{R}^d))$ for some $p < \infty$. Assume the initial convergence $\mu_N^\circ \overset{*}{\rightharpoonup} \mu^\circ$, as well as the convergence of the initial energy*

$$\lim_{N\uparrow\infty} \frac{1}{N^2} H_N(x_{1,N}^\circ, \ldots, x_{N,N}^\circ) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y)d\mu^\circ(x)d\mu^\circ(y) < \infty, \tag{6.17}$$

and in the case $s = 0$ also assume that

$$\lim_{R\uparrow\infty} \limsup_{N\uparrow\infty} \int_{|x|>R} \log(2+|x|)d\mu_N^\circ(x) = 0. \tag{6.18}$$

Then μ is the only weak solution of (6.7) in $\mathrm{L}^\infty([0,T]; \mathrm{L}^\infty(\mathbb{R}^d))$, and for all $t \in [0,T]$ we have $\mu_N^t \overset{*}{\rightharpoonup} \mu^t$, as well as the convergence of the energy

$$\lim_{N\uparrow\infty} \frac{1}{N^2} H_N(x_{1,N}^t, \ldots, x_{N,N}^t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y)d\mu^t(x)d\mu^t(y) < \infty. \qquad \Diamond$$

Remarks 6.1.6.

(a) *Regularity assumption.* For a compactly supported probability measure $\mu^\circ \in \mathrm{L}^\infty(\mathbb{R}^d)$, the limiting equation (6.7) with $g = g_s$ always admits a solution in $\mathrm{L}^\infty(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^d))$, which remains a compactly supported probability measure for all times (see Proposition 6.2.4 below). As far as the additional regularity assumption is concerned, as explained at the end of Section 6.2.2, it has been proven to hold with $T = \infty$ in the case $0 \le s \le d-2$, $d \ge 2$, and at least up to some time $T > 0$ in the case $d - 2 < s \le d - 1$, $s \ge 0$, for sufficiently smooth initial data μ°. All other cases remain unsolved, although this additional regularity is crucially needed in the proof of Theorem 6.1.5. The assumptions may thus be clarified at least in the following two cases:
- In the 2D Coulomb case $s = 0$, $d = 2$, using the global regularity result of [304], the conclusion of Theorem 6.1.5 holds with $T = \infty$ (that is, with $\mathrm{L}^\infty([0,T]; \cdot)$ replaced by $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \cdot)$), whenever the initial data μ° belongs to $\mathcal{P} \cap C_b^\sigma(\mathbb{R}^2)$ for some $\sigma > 0$. This completes Schochet's partial result [390].
- In the case $s = 0$, $d = 1$, and in the case $0 < s < 1$, $d = 2$, using the local regularity result of [424], the conclusion of Theorem 6.1.5 holds for some $T > 0$ (depending on initial data), whenever the initial data μ° in the 1D case belongs to $\mathcal{P} \cap H^\sigma(\mathbb{R})$ for some $\sigma > \frac{3}{2}$, and in the 2D case belongs to $\mathcal{P} \cap C_b^\sigma(\mathbb{R}^2)$ for some $\sigma > 1$.

(b) *Assumption* (6.18). As can be seen in the proof, if assumption (6.18) is replaced by the weaker assumption $\int_{\mathbb{R}^d} \log(2+|x|)d\mu^\circ(x) < \infty$, then the same result is proven to hold except possibly the conclusion about the convergence of the energy.

(c) *Quantitative statement.* A closer look at the proof actually shows the following quantitative statement, where the distance between μ_N^t and μ^t is measured in terms of the modulated energy: for all $t \in [0, T]$,

$$\mathcal{E}_N(t) := \iint_{x \neq y} g_s(x-y) d(\mu_N^t - \mu^t)(x) d(\mu_N^t - \mu^t)(y) \\ \lesssim_t \iint_{x \neq y} g_s(x-y) d(\mu_N^\circ - \mu^\circ)(x) d(\mu_N^\circ - \mu^\circ)(y) + \begin{cases} N^{-\frac{(1-s)(1-\sigma)}{1+s-\sigma}}, & \text{if } s > 0; \\ N^{-1} \log N, & \text{if } s = 0. \end{cases}$$

Note that as a consequence of Lemma 6.2.9 the initial modulated energy in the right-hand side indeed converges to 0 under the assumptions $\mu_N^\circ \xrightharpoonup{*} \mu^\circ$ and (6.17).

(d) *Propagation of chaos.* As first formalized by Kac [267], letting $f_N^t(x_1, \dots, x_N)$ denote the image by the particle dynamics (6.5) of the "chaotic" initial law $(\mu^\circ)^{\otimes N} \in \mathcal{P}((\mathbb{R}^d)^N)$, propagation of chaos is said to hold if for all $t \geq 0$ and all $k \geq 1$ the k-th marginal

$$f_{N,(k)}^t(x_1, \dots, x_k) := \int_{(\mathbb{R}^d)^{N-k}} f_N^t(x_1, \dots, x_N) dx_{k+1} \dots dx_N$$

satisfies

$$f_{N,(k)}^t \xrightharpoonup{*} (\mu^t)^{\otimes k}, \qquad \text{as } N \uparrow \infty.$$

As a consequence of the so-called Grunbaum lemma, this notion of propagation of chaos actually follows from the weak convergence of empirical measures $\mu_N^t \xrightharpoonup{*} \mu^t$ for all $t \geq 0$ when the initial particle positions $(x_{i,N}^\circ)_{i=1}^N$ are chosen to be i.i.d. with law μ° (see e.g. [236]). It is thus enough to check that the well-preparedness assumption (6.17) in Theorem 6.1.5 is statistically generic for the initial positions $(x_{i,N}^\circ)_{i=1}^N$, in the sense that it is a.s. satisfied for i.i.d. initial positions. This easily follows from the strong law of large numbers, together with the bound (for $s > 0$)

$$\iint |g_s(x-y)| d\mu^\circ(x) d\mu^\circ(y) \\ \lesssim \iint_{|x-y| \leq 1} |x-y|^{-s} d\mu^\circ(x) d\mu^\circ(y) + \iint_{|x-y|>1} d\mu^\circ(x) d\mu^\circ(y) \lesssim \|\mu^\circ\|_{\mathrm{L}^\infty} + 1.$$

(e) *External potential.* As already mentioned, we may also add to the energy (6.5) a potential Φ. If $\Phi \in C^2(\mathbb{R}^d)$ satisfies $\|\nabla^2 \Phi\|_{\mathrm{L}^\infty} < \infty$, then all the arguments may be directly adapted, as long as the corresponding limiting equation

$$\partial_t \mu^t = \operatorname{div}(\mu^t \nabla(h^t + \Phi)), \qquad h^t := g_s * \mu^t, \qquad \mu^t|_{t=0} = \mu^\circ,$$

admits a regular enough solution.

(f) *Mixed-flow case.* In 2D we may also consider a mix between the gradient flow (6.5) and its conservative counterpart, that is, replace (6.5) by the following system of ODEs, for $i = 1, \dots, N$,

$$\partial_t x_{i,N}^t = -\frac{\alpha}{N} \sum_{j:j \neq i} \nabla g(x_{i,N}^t - x_{j,N}^t) - \frac{\beta}{N} \sum_{j:j \neq i} \nabla^\perp g(x_{i,N}^t - x_{j,N}^t) - \nabla \Phi(x_{i,N}^t), \quad x_{i,N}^t|_{t=0} = x_{i,N}^\circ. \tag{6.19}$$

If $\alpha > 0$ is fixed, then all the arguments may again be directly adapted, as long as the corresponding limiting equation

$$\partial_t \mu^t = \operatorname{div}(\mu^t(\alpha \nabla h^t + \beta \nabla^\perp h^t + \nabla \Phi)), \qquad h^t := g_s * \mu^t, \qquad \mu^t|_{t=0} = \mu^\circ,$$

admits a regular enough solution. Note that the same proof no longer works for the choice $\alpha = 0$, since in Step 2 of the proof of Proposition 6.2.11 below some term cannot be estimated directly and needs instead to be absorbed in the negative dissipation term, which vanishes in the case $\alpha = 0$. Nevertheless, a closer inspection of the proof shows that for the choice $\alpha = \alpha_N$ with $N^{-\frac{1}{s}(1-s)(1-\sigma)} \ll \alpha_N \ll 1$ the same mean-field limit result holds with the corresponding conservative limiting equation

$$\partial_t \mu^t = \operatorname{div}(\mu^t(\beta \nabla^\perp h^t + \nabla \Phi)), \qquad h^t := g_s * \mu^t, \qquad \mu^t|_{t=0} = \mu^\circ. \qquad \Diamond$$

6.1.4 Strategy of the proof

Translating the idea of Serfaty [395] to the present setting, the key observation behind the proof of Theorem 6.1.5 is the following weak-strong stability estimate, which we first present in the simpler Coulomb case. As explained, the stability is measured in terms of the modulated energy: since the limiting equation (6.7) can be seen as a Wasserstein gradient flow for the energy functional $\mu \mapsto \int_{\mathbb{R}^d} |\nabla(-\triangle)^{-1}\mu|^2$, the modulated energy between two measures μ_1 and μ_2 takes the form $\int_{\mathbb{R}^d} |\nabla(-\triangle)^{-1}(\mu_1 - \mu_2)|^2 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_{d-2}(x-y)d(\mu_1-\mu_2)(x)d(\mu_1-\mu_2)(y)$, which coincides here with the $\dot{H}^{-1}$-distance between μ_1 and μ_2. (For a discussion of the link between this metric and more classical Wasserstein metrics, we refer to [307, 365, 106].)

Lemma 6.1.7 (Stability — Coulomb case)**.** *Let $s = d-2, d \geq 2$. Let $\mu_1^\circ, \mu_2^\circ \in \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^d)$, and in the case $d = 2$ also assume $\int_{\mathbb{R}^2} \log(2+|x|)d(\mu_1^\circ + \mu_2^\circ)(x) < \infty$. For $i = 1, 2$, let μ_i be a weak solution of equation* (6.7) *with $g = g_{d-2}$ and initial data μ_i°, denote $h_i^t := g_{d-2} * \mu_i^t$, and assume $\mu_1, \mu_2 \in \mathrm{L}^\infty([0,T]; \mathrm{L}^\infty(\mathbb{R}^d))$ and $\nabla^2 h_2 \in \mathrm{L}^1([0,T]; \mathrm{L}^\infty(\mathbb{R}^d))$ for some $T > 0$. Then for all $t \in [0,T]$,*

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_{d-2}(x-y)d(\mu_1^t-\mu_2^t)(x)d(\mu_1^t-\mu_2^t)(y) \\
&\qquad \leq \exp\Big(C\int_0^t \|\nabla^2 h_2^u\|_{\mathrm{L}^\infty}du\Big)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_{d-2}(x-y)d(\mu_1^\circ-\mu_2^\circ)(x)d(\mu_1^\circ-\mu_2^\circ)(y). \quad (6.20)
\end{aligned}$$

$\Diamond$

Proof. Proposition 6.2.4(ii) below yields $\nabla(h_1 - h_2) \in \mathrm{L}^\infty([0,T]; \mathrm{L}^2(\mathbb{R}^d))$. Combining this with the additional boundedness assumptions, all integration by parts arguments in the sequel may be justified. Using the equations for μ_1 and μ_2, the time derivative of the left-hand side of (6.20) is computed as follows,

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_{d-2}(x-y)d(\mu_1^t-\mu_2^t)(x)d(\mu_1^t-\mu_2^t)(y) \\
&= 2\int_{\mathbb{R}^d}(h_1^t - h_2^t)(\partial_t\mu_1^t - \partial_t\mu_2^t) = -2\int_{\mathbb{R}^d}\nabla(h_1^t-h_2^t)(\mu_1^t\nabla h_1^t - \mu_2^t\nabla h_2^t) \\
&= -2\int_{\mathbb{R}^d}|\nabla(h_1^t-h_2^t)|^2\mu_1^t - 2\int_{\mathbb{R}^d}\nabla h_2^t\cdot\nabla(h_1^t-h_2^t)\,(\mu_1^t-\mu_2^t). \qquad (6.21)
\end{aligned}$$

The first term in the right-hand side is a modulated dissipation term and is nonpositive, so it suffices to estimate the second term. Using the relations $-\Delta h_i^t = \mu_i^t$, $i = 1,2$ (which indeed hold with a unit factor for the suitable choice of the normalization constant $c_{d,d-2} > 0$ in (6.4)), the product $\nabla(h_1^t - h_2^t)\,(\mu_1^t - \mu_2^t)$ may be rewritten à la Delort using the stress-energy tensor:

$$\begin{aligned}
-2\nabla(h_1^t-h_2^t)\,(\mu_1^t-\mu_2^t) &= 2\nabla(h_1^t-h_2^t)\,\Delta(h_1^t-h_2^t) \\
&= \operatorname{div}\big(2\nabla(h_1^t-h_2^t)\otimes\nabla(h_1^t-h_2^t) - \mathrm{Id}\,|\nabla(h_1^t-h_2^t)|^2\big), \qquad (6.22)
\end{aligned}$$

where we recall that the divergence of a 2-tensor is defined as the vector whose coordinates are the divergences of the corresponding columns of the tensor. Combining this with an integration by parts, we find

$$\begin{aligned}2\int_{\mathbb{R}^d} \nabla h_2^t \cdot \nabla(h_1^t - h_2^t)\,(\mu_1^t - \mu_2^t) &= -\int_{\mathbb{R}^d} \operatorname{div}\,\big(2\nabla(h_1^t - h_2^t) \otimes \nabla(h_1^t - h_2^t) - \mathrm{Id}\,|\nabla(h_1^t - h_2^t)|^2\big) \cdot \nabla h_2^t \\ &= \int_{\mathbb{R}^d} \big(2\nabla(h_1^t - h_2^t) \otimes \nabla(h_1^t - h_2^t) - \mathrm{Id}\,|\nabla(h_1^t - h_2^t)|^2\big) : \nabla^2 h_2^t.\end{aligned}$$

The inequality $2|ab| \le a^2 + b^2$ and an integration by parts then yield

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \nabla h_2^t \cdot \nabla(h_1^t - h_2^t)\,(\mu_1^t - \mu_2^t)\right| &\lesssim \|\nabla^2 h_2^t\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^d} |\nabla(h_1^t - h_2^t)|^2 \\ &= \|\nabla^2 h_2^t\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^d} (h_1^t - h_2^t)(\mu_1^t - \mu_2^t) \\ &= \|\nabla^2 h_2^t\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_{d-2}(x-y) d(\mu_1^t - \mu_2^t)(x) d(\mu_1^t - \mu_2^t)(y),\end{aligned} \tag{6.23}$$

so that the result (6.20) follows from (6.21) and a Grönwall argument. □

In the case of the other Riesz potentials $g = g_s$ with $d-2 < s < d$, $s \ge 0$, we need to use the extension method popularized by Caffarelli and Silvestre [92] (see also [362]) in order to find a similar Delort-type formula as in (6.22), and then repeat the same integration by parts argument, thus circumventing the fact that the Riesz kernels are no longer convolution kernels of local operators. This trick allows to prove the same estimate (6.20) in all cases $0 \vee (d-2) \le s < d$ with g_{d-2} replaced by g_s (cf. Lemma 6.2.1 below).

This weak-strong stability estimate provides a control of the $\dot H^{-1}$-distance (or the $\dot H^{-(d-s)/2}$-distance, for general $0 \le s < d$) between μ_1^t and μ_2^t in terms of the initial distance, up to a prefactor that *only* depends on the regularity of μ_2^t in the form of $\|\nabla^2 h_2^t\|_{\mathrm{L}^\infty}$ — hence the naming "weak-strong". We would then like to replace μ_2 by the smooth solution μ and to replace μ_1 by the empirical measure μ_N. Nevertheless, the corresponding $\dot H^{-(d-s)/2}$-distance would then be infinite due to the presence of Dirac masses, and moreover μ_N does not exactly satisfy the limiting equation (6.7) (diagonal terms have to be removed). The strategy of the proof of Theorem 6.1.5 is to look for a suitable way to adapt the above argument to that setting.

First, a natural way to give a meaning to this divergent $\dot H^{-(d-s)/2}$-distance between μ_N^t and μ^t simply consists in excluding the diagonal terms, thus considering the "renormalized" *modulated energy*

$$\mathcal{E}_N(t) = \iint_{x\neq y} g_s(x-y) d(\mu_N^t - \mu^t)(x) d(\mu_N^t - \mu^t)(y).$$

The goal is then to compute the time derivative $\partial_t \mathcal{E}_N(t)$, and to adapt the proof of the above stability estimate to bound it by $C\mathcal{E}_N(t)$ for some constant $C > 0$, up to a vanishing additive error. Nevertheless, at the end of the proof above, the use of the inequality $2|ab| \le a^2 + b^2$ is clearly not compatible with the removal of the diagonal terms. To solve this main issue, we draw inspiration from Serfaty's strategy [395] in the context of the Ginzburg-Landau vortices: regularizing the Dirac masses at a tiny scale η so that the diagonal terms become well-defined and diverge only as $\eta \downarrow 0$, it suffices to construct around the particle locations small balls that contain most of the divergent η-approximate energy, so that excluding diagonal terms essentially amounts to restricting the η-approximate integrals to outside these small balls. Using the same approximation argument as in [395] to be allowed to restrict all integrals to outside these balls, the end of the above proof is then easily adapted, using the inequality $2|ab| \le a^2 + b^2$ only on the restricted domain. In this way, for all $0 \vee (d-2) \le s < d$, we manage to prove $\partial_t \mathcal{E}_N(t) \leq C\mathcal{E}_N(t) + o(1)$ under some mesoscopic regularity assumption on the distribution of particles in time (cf. Proposition 6.2.11 below). Finally, in the case $s < 1$ (hence our limitation to that regime), these conditions can be directly checked using a modification of the ball construction introduced by Jerrard and Sandier [260, 379] for the analysis of the Ginzburg-Landau vortices (cf. Section 6.2.6 below).

6.1.5 Perspectives and open questions

The regularity question for the fractional porous medium equation (that is, (6.7) with $g = g_s$) remains an open problem in the range $d-1 < s < d$, $s \geq 0$, and only a local-in-time regularity result is available at the moment in the range $d-2 < s < d-1$, $s \geq 0$. Improving this regularity would clarify the assumptions in Theorem 6.1.5 (cf. Remark 6.1.6(a)).

As explained, our modulated energy argument is limited to the range $0 \leq s < 1$ (cf. Remark 6.2.12): going beyond this restriction would require more precise microscopic information on the particle dynamics and is left as a completely open question. Note that even the limiting case $s = 1$ would be particularly interesting, as in dimension $d = 3$ it corresponds to the Coulomb case.

As our modulated energy argument is limited to the range $0 \leq s < 1$ in any dimension, and as the mean-field result by Hauray and Jabin (cf. Theorem 6.1.3) already contains the range $0 \leq s < d-2$, $d \geq 3$, it was natural to restrict attention in Theorem 6.1.5 to the simpler case of dimensions $d = 1$ and 2. However, the result by Hauray and Jabin requires a well-preparedness assumption (6.15) that is only expected to be statistically generic in the range $0 \leq s < \frac{d}{2} - 2$, $d \geq 5$, while our modulated energy argument only requires convergence of initial energies, which is always statistically generic (cf. Remark 6.1.6(d)). It could therefore be interesting to extend our approach also to higher dimensions $d \geq 3$, as it would improve the result by Hauray and Jabin at least in the ranges $0 \leq s < 1$, $1 \leq d \leq 4$ and $\frac{1}{2} \leq s < 1$, $d = 5$. The main difficulty is to suitably iterate the extension method of Caffarelli and Silvestre [92] (see e.g. [425, 103]) in order to find a similar Delort-type formula as in (6.22), and could be addressed in a future work.

Our modulated energy proof of Theorem 6.1.5 makes a crucial use of the dissipation generated by the gradient flow structure: the dissipation term is indeed essential to absorb some error terms in our analysis. This prevents us from considering conservative versions of the flow (6.5) (such as e.g. (6.19) with $\alpha = 0$ in dimension $d = 2$), as well as the corresponding (second-order) Hamiltonian system (6.1). The use of modulated energy methods in these cases is thus let as an open question. In contrast, note that the methods of proof by Schochet [390] and by Hauray and Jabin [234, 233] (cf. Section 6.1.2) can both be adapted to these conservative cases.

There is also strong interest in the mean-field limit problem for the corresponding particle system with thermal noise, that is, when (6.5) is replaced by the following system of coupled SDEs,

$$dx_{i,N}^t = \sqrt{2}\sigma dB_{i,N}^t - \frac{1}{N}\sum_{j:j\neq i}^N \nabla g_s(x_{i,N}^t - x_{j,N}^t)dt, \qquad x_{i,N}^t|_{t=0} = x_{i,N}^\circ, \qquad i = 1,\ldots,N, \tag{6.24}$$

where $\sigma > 0$ is some fixed constant, where $(x_{i,N}^\circ)_{i=1}^N$ is a sequence of N distinct initial positions, and where $(B_{i,N}^t)_{i=1}^N$ is a sequence of N independent Brownian motions (defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$). Note that in the 1D logarithmic case this particle dynamics coincides with the so-called Dyson's Brownian motion studied in random matrix theory (see e.g. [227, Chapter 12]). Although clear in higher dimensions [352, 72], the well-posedness of this dynamics requires some more work in 1D to deal with possible collisions [105]. Assuming convergence at initial time $\mu_N^\circ \xrightarrow{*} \mu^\circ$, formal computations lead us to expect $\mu_N^t \xrightarrow{*} \mu^t$ a.s. for all $t \geq 0$, where μ is the deterministic solution of the following viscous version of (6.7),

$$\partial_t \mu^t = \sigma^2 \triangle \mu^t + \operatorname{div}(\mu^t \nabla h^t), \qquad h^t := g_s * \mu^t, \qquad \mu^t|_{t=0} = \mu^\circ.$$

Except in the 1D case [107, 370, 105, 184, 54] and in the 2D Coulomb case [186, 72] (see also [255]), this mean-field problem is still open. The modulated energy methods developed in this chapter unfortunately seem useless in this context, as there is strong suggestion that the expectation of the modulated energy $\mathbb{E}[\mathcal{E}_N(t)]$ should not remain small at any positive time $t > 0$. The Keller-Segel model studied in [187, 104] is the analogue of (6.24) with an attractive interaction.

Another interesting open question concerns the mean-field limit of the particle system (6.5) in the case of a hypersingular Riesz interaction potential $g(x) := g_s(x) := |x|^{-s}$ with $s > d$. It is not difficult to check in that case that the rescaled particle energy $\tilde{H}_N(x_1, \ldots, x_N) := N^{-1-s/d} \sum_{i \neq j}^N g_s(x_i - x_j)$ Γ-converges (with respect to the weak-* convergence of empirical measures) towards the continuum energy $\tilde{H}(\mu) = K_s \int_{\mathbb{R}^d} \mu^{1+s/d}$, where $K_s > 0$ is some suitable geometric constant (see e.g. [232, 230]). It is therefore expected that the mean-field limit of the particle system (6.5) with suitable time rescaling should be described by the corresponding slow diffusion equation $\partial_t \mu^t = K_s \triangle((\mu^t)^{1+s/d})$. Few results are known about this delicate mean-field limit problem [350], and its complete understanding remains open — as it seems to require a very precise monitoring of the particle geometry along the flow.

A last open question that we would like to mention concerns the mean-field limit for the gradient-flow evolution of interacting particles with ± 1 charges: given a sequence of N distinct initial positions $(x_{i,N}^\circ)_{i=1}^N$ and of corresponding charges $(m_{i,N}^\circ)_{i=1}^N \subset \{-1, +1\}$, we consider the following system of coupled ODEs, instead of (6.5),

$$\partial_t x_{i,N}^t = -\frac{1}{N} \sum_{j:j \neq i}^N m_{i,N}^t m_{j,N}^t \nabla g_s(x_{i,N}^t - x_{j,N}^t), \qquad x_{i,N}^t|_{t=0} = x_{i,N}^\circ, \qquad i = 1, \ldots, N, \tag{6.25}$$

where for all i the charge $m_{i,N}^t$ remains constant in time until the first collision with a particle of opposite charge, and takes the value 0 at all later times. We are then interested in the mean-field evolution of the (signed!) empirical measure

$$\mu_N^t := \frac{1}{N} \sum_{i=1}^N m_{i,N}^t \delta_{x_{i,N}^t} \quad \in \mathcal{M}(\mathbb{R}^d).$$

Assuming convergence at initial time $\mu_N^\circ \overset{*}{\rightharpoonup} \mu^\circ$ as $N \uparrow \infty$, formal computations lead us to expect under fairly general assumptions $\mu_N^t \overset{*}{\rightharpoonup} \mu^t$ for all $t \geq 0$, where μ^t is a solution of the following equation on $\mathbb{R}^+ \times \mathbb{R}^d$,

$$\partial_t \mu^t = \operatorname{div}(|\mu^t| \nabla h^t), \qquad h^t := g_s * \mu^t, \qquad \mu^t|_{t=0} = \mu^\circ.$$

In the 2D Coulomb case, in the context of Ginzburg-Landau vortices, this mean-field model is the so-called Chapman-Rubinstein-Schatzman-E equation [173, 111]. No result is known at all for this mean-field limit problem, even for logarithmic interaction $s = 0$, and even in dimension $d = 1$. (Let us mention that some related simplified problems are studied in [417].)

6.2 Proof of the main result

6.2.1 Extension representation for fractional Laplacian

We recall here the extension representation for the fractional Laplacian popularized by Caffarelli and Silvestre [92] (we follow notation of [362, Section 1.2]). Let $0 \vee (d-2) < s < d$ be fixed. For a finite Borel measure μ on $\mathbb{R}^d$, the associated Riesz potential $(-\Delta)^{-(d-s)/2}\mu$ can be written (up to a constant) as

$$h^\mu(x) := c_{d,s}^{-1} \int_{\mathbb{R}^d} |x-z|^{-s} d\mu(z) = g_s * \mu(x). \tag{6.26}$$

Now denote the coordinates in $\mathbb{R}^d \times \mathbb{R}$ by (x,ξ), and let $\mu\delta_{\mathbb{R}^d\times\{0\}}$ denote the Borel measure on $\mathbb{R}^d \times \mathbb{R}$ defined as follows, for all $\phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R})$,

$$\int_{\mathbb{R}^d\times\mathbb{R}} \phi(x,\xi) d(\mu\delta_{\mathbb{R}^d\times\{0\}})(x,\xi) := \int_{\mathbb{R}^d} \phi(x,0)d\mu(x).$$

Extending h^μ to $\mathbb{R}^d \times \mathbb{R}$ via

$$h^\mu(x,\xi) := c_{d,s}^{-1} \int_{\mathbb{R}^d} |(x,\xi)-(z,0)|^{-s} d\mu(z) = g_s * (\mu\delta_{\mathbb{R}^d\times\{0\}})(x,\xi),$$

where we abusively denote $g_s(x,\xi) = c_{d,s}^{-1}|(x,\xi)|^{-s}$ on $\mathbb{R}^d \times \mathbb{R}$, and choosing $\gamma := s+1-d \in (-1,1)$, the extended function h^μ on $\mathbb{R}^d \times \mathbb{R}$ satisfies in the distributional sense

$$-\operatorname{div}(|\xi|^\gamma \nabla h^\mu) = \mu\delta_{\mathbb{R}^d\times\{0\}}.$$

The function g_s is indeed a fundamental solution of the operator $-\operatorname{div}(|\xi|^\gamma\nabla)$ on $\mathbb{R}^d \times \mathbb{R}$, in the sense that $-\operatorname{div}(|\xi|^\gamma \nabla g_s) = \delta_0$ on $\mathbb{R}^d \times \mathbb{R}$. The normalization constant $c_{d,s}$ is chosen exactly to satisfy this property with a unit factor (for an explicit value, see Step 1 of the proof of Lemma 6.2.14 below).

In the case $s=0$, $d=1$, denoting $g_0(x,\xi) = -c_{d,0}^{-1}\log(|(x,\xi)|)$, we have $-\Delta g_0 = \delta_0$ on the extended space $\mathbb{R}\times\mathbb{R}$, for the suitable choice of the normalization constant $c_{1,0}$, so the above again holds with $\gamma = 0 = s+1-d$. (In the Coulomb case $s=d-2$, $d \geq 2$, no extension is needed, and the normalization $c_{d,d-2}$ is simply chosen such that $-\Delta g_{d-2} = \delta_0$ on $\mathbb{R}^d$.)

Using this extension representation, we may now directly adapt the weak-strong stability estimate of Lemma 6.1.7 to the non-Coulomb case.

Lemma 6.2.1 (Stability — Riesz case)**.** *Let $0 \vee (d-2) \leq s < d$. Let $\mu_1^\circ, \mu_2^\circ \in \mathcal{P}(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)$, and in the case $s=0$ also assume $\int_{\mathbb{R}^d} \log(2+|x|)d(\mu_1^\circ+\mu_2^\circ)(x) < \infty$. For $i=1,2$, let μ_i be a weak solution of equation (6.7) with $g=g_s$ and initial data μ_i°, denote $h_i^t := g_s * \mu_i^t$, and assume $\mu_1,\mu_2 \in \mathrm{L}^\infty([0,T];\mathrm{L}^\infty(\mathbb{R}^d))$ and $\nabla^2 h_2 \in \mathrm{L}^1([0,T];\mathrm{L}^\infty(\mathbb{R}^d))$ for some $T>0$. Then for all $t \in [0,T]$,*

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} & g_s(x-y)d(\mu_1^t-\mu_2^t)(x)d(\mu_1^t-\mu_2^t)(y) \\ &\leq \exp\Big(C\int_0^t \|\nabla^2 h_2^u\|_{\mathrm{L}^\infty}du\Big)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y)d(\mu_1^\circ-\mu_2^\circ)(x)d(\mu_1^\circ-\mu_2^\circ)(y).\end{aligned} \tag{6.27}$$

$\Diamond$

Proof. By Lemma 6.1.7, we only need to consider the case $d-2 < s < d$, $s \geq 0$. Proposition 6.2.4(ii) below yields $\nabla(h_1-h_2) \in \mathrm{L}^\infty(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}, |\xi|^\gamma dxd\xi))$. Combining this with the boundedness assumptions, all integration by parts arguments in the sequel may be justified. Just as in (6.21), the time derivative of the left-hand side of (6.27) is

$$\partial_t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_s(x-y) d(\mu_1^t - \mu_2^t)(x) d(\mu_1^t - \mu_2^t)(y)$$
$$= -2 \int_{\mathbb{R}^d} |\nabla(h_1^t - h_2^t)|^2 \mu_1^t - 2 \int_{\mathbb{R}^d} \nabla h_2^t \cdot \nabla(h_1^t - h_2^t)\, (\mu_1^t - \mu_2^t). \quad (6.28)$$

The first term in the right-hand side is nonpositive, so it suffices to estimate the second one. Using the relations $-\operatorname{div}(|\xi|^\gamma \nabla h_i^t) = \mu_i^t \delta_{\mathbb{R}^d \times \{0\}}$, for $i = 1, 2$, we find the following proxy for the Delort-type formula (6.22): for all $1 \le k \le d$,

$$-2\partial_k (h_1^t - h_2^t)\, (\mu_1^t \delta_{\mathbb{R}^d \times \{0\}} - \mu_2^t \delta_{\mathbb{R}^d \times \{0\}}) = 2\partial_k (h_1^t - h_2^t) \operatorname{div}(|\xi|^\gamma \nabla (h_1^t - h_2^t))$$
$$= \sum_{l=1}^{d+1} \partial_l \left(2|\xi|^\gamma \partial_k (h_1^t - h_2^t) \partial_l (h_1^t - h_2^t) - \delta_{kl} |\xi|^\gamma |\nabla(h_1^t - h_2^t)|^2\right).$$

Combining this with an integration by parts, we obtain

$$2 \int_{\mathbb{R}^d} \nabla h_2^t \cdot \nabla(h_1^t - h_2^t)\, (\mu_1^t - \mu_2^t)$$
$$= -\sum_{k=1}^{d} \sum_{l=1}^{d+1} \int_{\mathbb{R}^d \times \mathbb{R}} \partial_l \left(2|\xi|^\gamma \partial_k (h_1^t - h_2^t) \partial_l (h_1^t - h_2^t) - \delta_{kl} |\xi|^\gamma |\nabla(h_1^t - h_2^t)|^2\right) \partial_k h_2^t$$
$$= \sum_{k=1}^{d} \sum_{l=1}^{d+1} \int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\gamma \left(2\partial_k (h_1^t - h_2^t) \partial_l (h_1^t - h_2^t) - \delta_{kl} |\nabla(h_1^t - h_2^t)|^2\right) \partial_{kl} h_2^t.$$

Hence, arguing as in Lemma 6.1.7, an integration by parts leads to

$$\left| \int_{\mathbb{R}^d} \nabla h_2^t \cdot \nabla(h_1^t - h_2^t)\, (\mu_1^t - \mu_2^t) \right| \lesssim \|\nabla^2 h_2^t\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\gamma |\nabla(h_1^t - h_2^t)|^2$$
$$= \|\nabla^2 h_2^t\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^d \times \mathbb{R}} (h_1^t - h_2^t)(\mu_1^t \delta_{\mathbb{R}^d \times \{0\}} - \mu_2^t \delta_{\mathbb{R}^d \times \{0\}})$$
$$= \|\nabla^2 h_2^t\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^d} (h_1^t - h_2^t)(\mu_1^t - \mu_2^t)$$
$$= \|\nabla^2 h_2^t\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_s(x-y) d(\mu_1^t - \mu_2^t)(x) d(\mu_1^t - \mu_2^t)(y), \quad (6.29)$$

and the result (6.27) follows. □

Remark 6.2.2 (Beyond Riesz potentials)**.** The extension representation popularized by Caffarelli and Silvestre is unfortunately limited to the case of pure powers $|\cdot|^{-s}$ with $0 \vee (d-2) \le s < d$, hence it does a priori not allow to extend the above weak-strong stability estimate to other non-Riesz interaction potentials g. Note however that we may at least consider any convex combination of Riesz powers: if g is of the form $g(x) := \int_{0\vee(d-2)}^d |x|^{-s} da(s)$ for some (nonnegative!) $a \in \mathcal{P}([0 \vee (d-2), d] \setminus \{0\})$, then we find for $h_i := g * \mu_i$, $i = 1, 2$,

$$\partial_t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x-y) d(\mu_1^t - \mu_2^t)(x) d(\mu_1^t - \mu_2^t)(y) \le -2 \int_{\mathbb{R}^d} \nabla h_2^t \cdot \nabla(h_1^t - h_2^t)(\mu_1^t - \mu_2^t)$$
$$= -2 \int_{0\vee(d-2)}^d \left(\int_{\mathbb{R}^d} \nabla h_2^t \cdot (\mu_1^t - \mu_2^t) \nabla g_s * (\mu_1^t - \mu_2^t) \right) da(s),$$

hence, arguing as in the above proof with a Delort-type identity for each s, and then reconstructing g, we obtain

$$\partial_t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x-y)d(\mu_1^t-\mu_2^t)(x)d(\mu_1^t-\mu_2^t)(y) \lesssim \|\nabla^2 h_2^t\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x-y)d(\mu_1^t-\mu_2^t)(x)d(\mu_1^t-\mu_2^t)(y),$$

and the conclusion follows. Regarding the case of smaller pure powers $|\cdot|^{-s}$ with $0 \le s < d-2$, although not detailed here, it could be treated using a suitable iteration of the usual extension representation (see e.g. [425, 103]), and we believe that in this way a similar weak-strong stability estimate can be established as in the above lemma. ◊

Remark 6.2.3 (Similar stability for Hamiltonian dynamics)**.** Similar weak-strong stability results have been recently established in [197] for various Hamiltonian dynamics including the Euler-Poisson system (that is, the monokinetic form of the Vlasov equation (6.3) with Coulomb interaction $g = g_{d-2}$), where the modulated kinetic part of the energy then needs to be considered as well (in the spirit of e.g. [86]). The very same use of the extension representation as above then allows to immediately generalize these stability results to the corresponding Riesz setting. ◊

6.2.2 Properties of the fractional porous medium equation

Existence

We start with an existence result and some basic properties of weak solutions of (6.7) with interaction $g = g_s$. We refer to [94, 93] for $d-2 < s < d$, $s \ge 0$, to [304, 18, 398, 55] for $s = d-2$, $d \ge 2$, and to [102, Section 4] for $0 \le s < d-2$.[3]

Proposition 6.2.4 (Existence for FPME)**.** *Let $0 \le s < d$.*

(i) Existence: Let $\mu^\circ \in \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^d)$, and in the case $d-2 < s < d$, $s \ge 0$ also assume that $|\mu^\circ(x)| \le Ae^{-a|x|}$ holds for some $a, A > 0$. Then there exists a (global) weak solution $\mu \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^d))$ of (6.7) with $g = g_s$ and initial data μ°, which is unique in this class in the case $0 \le s \le d-2$, $d \ge 2$.

(ii) General properties: Let $\mu^\circ \in \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^d)$, and in the case $s = 0$ also assume $\int_{\mathbb{R}^d} \log(2+|x|)d\mu^\circ(x) < \infty$. Any weak solution μ of (6.7) with $g = g_s$ and initial data then satisfies for all $t \ge 0$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y)d\mu^t(x)d\mu^t(y) \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y)d\mu^\circ(x)d\mu^\circ(y),$$

where in particular the left-hand side remains finite. Moreover, for all $t \ge 0$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y)d(\mu^t-\mu^\circ)(x)d(\mu^t-\mu^\circ)(y) = \begin{cases} \int_{\mathbb{R}^d} |\nabla(h^t-h^\circ)|^2, & \text{if } s = d-2,\ d \ge 2; \\ \int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma |\nabla(h^t-h^\circ)|^2, & \text{if } d-2 < s < d,\ s \ge 0; \end{cases}$$

where both sides remain finite. In addition, if μ° is compactly supported, then μ^t remains compactly supported for all $t \ge 0$. ◊

3. Although in [102, Section 4] existence is established only locally in time, this result can easily be extended globally in the present repulsive context, using that no blow-up can occur in finite time. This follows from the observation that for $0 \le s < d-2$ we have $\Delta g_s(x) = -s(d-2-s)c_{d,s}^{-1}|x|^{-s-2}$, and hence for all $p \ge 1$ we find (formally) along solutions

$$\partial_t \int (\mu^t)^p = (p-1)\int (\mu^t)^p \Delta h^t \le 0.$$

Uniqueness

In the case $d-2<s<d$, $s\geq 0$, uniqueness remains an open problem: it has been established in dimension 1 by [61] (integrating the equation with respect to x and then considering viscosity solutions), but in higher dimensions no uniqueness result is known [94, 93]. Nevertheless, as a consequence of the weak-strong stability result of Lemma 6.2.1, we easily find that the uniqueness of bounded weak solutions always follows from the existence of a smooth enough strong solution, so that the uniqueness problem is somehow reduced to a regularity question.

Corollary 6.2.5 (Weak-strong uniqueness for FPME)**.** *Let $0\vee(d-2)\leq s<d$. Let $\mu^\circ\in\mathcal{P}\cap\mathrm{L}^\infty(\mathbb{R}^d)$, and in the case $s=0$ also assume $\int_{\mathbb{R}^d}\log(2+|x|)d\mu^\circ(x)<\infty$. Assume that equation* (6.7) *with $g=g_s$ and initial data μ° admits a weak solution μ such that $\mu,\nabla^2h\in\mathrm{L}^\infty([0,T];\mathrm{L}^\infty(\mathbb{R}^d))$ for some $T>0$. Then, μ is the unique weak solution of* (6.7) *with $g=g_s$ and initial data μ° up to time T in the class* $\mathrm{L}^\infty([0,T];\mathrm{L}^\infty(\mathbb{R}^d))$. ♢

Proof. Let μ be a weak solution of (6.7) as in the statement, and let $\nu\in\mathrm{L}^\infty([0,T];\mathrm{L}^\infty(\mathbb{R}^d))$ denote another weak solution of (6.7). By Lemma 6.2.1, we may then conclude

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}g_s(x-y)d(\mu^t-\nu^t)(x)d(\mu^t-\nu^t)(y)\leq 0, \tag{6.30}$$

for all $t\in[0,T]$. For $d-2<s<d$, $s\geq 0$, Proposition 6.2.4(ii) gives $\nabla g_s*(\mu^t-\nu^t)\in\mathrm{L}^2(\mathbb{R}^d,|\xi|^\gamma dxd\xi)$, so that (6.30) becomes by integration by parts

$$\int_{\mathbb{R}^d\times\mathbb{R}}|\xi|^\gamma|\nabla g_s*(\mu^t-\nu^t)|^2\leq 0.$$

This proves $\nabla g_s*\mu^t=\nabla g_s*\nu^t$, and hence, applying the operator $-\operatorname{div}(|\xi|^\gamma\cdot)$ to both sides, $\mu^t=\nu^t$ for all t. We may argue similarly in the Coulomb case $s=d-2$, $d\geq 2$. ☐

As the following lemma shows, the required boundedness of ∇^2h^t is implied by a sufficient amount of Hölder regularity for μ^t.

Lemma 6.2.6. *Let $0\vee(d-2)\leq s<d$. Let $\mu\in\mathcal{P}\cap C^\sigma_b(\mathbb{R}^d)$ for some $\sigma>2-d+s$, and denote by $h^\mu:=g_s*\mu$ the associated Riesz potential* (6.26) *on $\mathbb{R}^d$. If $s=d-1$, we further assume $\nabla\mu\in\mathrm{L}^{p_0}(\mathbb{R}^d)$ for some $p_0<\infty$. Then, we have*

$$\|(\nabla h^\mu,\nabla^2h^\mu)\|_{\mathrm{L}^\infty}\lesssim\|\mu\|_{\mathrm{L}^1}+\|\mu\|_{C^\sigma}. \tag{6.31}$$

Moreover, if $s=d-2\geq 0$ we have $\|\nabla^2h^\mu\|_{\mathrm{L}^p}\lesssim_p\|\mu\|_{\mathrm{L}^p}$ for all $1<p<\infty$, and if $s=d-1$ we have $\|\nabla^2h^\mu\|_{\mathrm{L}^p}\lesssim_p\|\nabla\mu\|_{\mathrm{L}^p}$ for all $p_0\leq p<\infty$, $p>1$. ♢

Proof. Without loss of generality we may assume $\mu\in C^\infty_c(\mathbb{R}^d)$, as the claimed result then follows by an obvious approximation argument. We first prove that for any $\mu\in C^\infty_c(\mathbb{R}^d)$ the Riesz potential h^μ satisfies (6.31). We only argue for the second gradient $\|\nabla^2h^\mu\|_{\mathrm{L}^\infty}$, the other part being similar and easier. Let $\chi\in C^\infty_c(\mathbb{R}^d)$ be symmetric around 0, with $\chi=1$ in B_1 and $\chi=0$ outside B_2. If $d-2\leq s<d-1$, decomposing

$$\begin{aligned}\nabla^2h^\mu(x)&=\int_{\mathbb{R}^d}g_s(x-y)\nabla^2\mu(y)dy\\&=\int_{\mathbb{R}^d}g_s(x-y)(1-\chi(x-y))\nabla^2\mu(y)dy+\int_{\mathbb{R}^d}g_s(x-y)\chi(x-y)\nabla^2_y(\mu(y)-\mu(x))dy,\end{aligned}$$

we find by multiple integrations by parts

$$\nabla^2 h^\mu(x) = \int_{\mathbb{R}^d} \nabla^2 g_s(x-y)(1-\chi(x-y))\mu(y)dy + \int_{\mathbb{R}^d} \nabla^2 g_s(x-y)\chi(x-y)(\mu(y)-\mu(x))dy$$
$$- \mu(x)\int_{\mathbb{R}^d} g_s(x-y)\nabla^2\chi(x-y)dy,$$

and hence, for any $2-d+s<\sigma<1$,

$$|\nabla^2 h^\mu(x)| \lesssim \int_{|x-y|\geq 1} |x-y|^{-s-2}|\mu(y)|dy + \|\mu\|_{C^\sigma}\int_{|x-y|\leq 2} |x-y|^{\sigma-s-2}dy$$
$$+ \|\mu\|_{\mathrm{L}^\infty}\int_{|x-y|\leq 2}|x-y|^{-s}dy \;\lesssim\; \|\mu\|_{\mathrm{L}^1} + \|\mu\|_{C^\sigma},$$

that is (6.31). If $d-1\leq s<d$, rather decomposing

$$\begin{aligned}\nabla^2 h^\mu(x) &= \frac{1}{2}\int_{\mathbb{R}^d} \nabla g_s(x-y)\otimes(\nabla\mu(y)-\nabla\mu(2x-y))dy\\ &= \int_{\mathbb{R}^d}\nabla^2 g_s(x-y)(1-\chi(x-y))\mu(y)dy - \int_{\mathbb{R}^d}\nabla g_s(x-y)\otimes\nabla\chi(x-y)\mu(y)dy\\ &\qquad + \frac{1}{2}\int_{\mathbb{R}^d}\nabla g_s(x-y)\otimes(\nabla\mu(y)-\nabla\mu(2x-y))\chi(x-y)dy,\end{aligned}$$

the result (6.31) again follows from a direct computation. As far as the additional L^p-boundedness is concerned, it is a direct consequence of the L^p-boundedness of Riesz transforms for $1<p<\infty$, simply noting that we have $\nabla^2 h^\mu \simeq \nabla^2(-\Delta)^{-1}\mu$ for $s=d-2\geq 0$, and $\nabla^2 h^\mu \simeq \nabla(-\Delta)^{-1/2}\nabla\mu$ for $s=d-1$. □

Regularity

Motivated by these considerations on uniqueness, we wish to show that the regularity of the initial data μ° is preserved along the flow (6.7), so that in particular the required boundedness of $\nabla^2 h^t$ would be ensured by the above lemma for sufficiently smooth μ°. In the Coulomb case $s=d-2$, $d\geq 2$, such a result was obtained by Lin and Zhang [304, Theorem 1] (their 2D proof is indeed easily rewritten in any dimension), and a similar but easier argument leads to the corresponding result in the case $0\leq s<d-2$, $d\geq 3$.

Proposition 6.2.7 (Lin, Zhang)**.** *In the case $0\leq s\leq d-2$, $d\geq 2$, given $\mu^\circ\in\mathcal{P}\cap H^\sigma(\mathbb{R}^d)$ with $\sigma>\frac{d}{2}$, there exists a unique solution $\mu\in\mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}\cap H^\sigma(\mathbb{R}^d))$ of (6.7) with $g=g_s$ and initial data μ°. Moreover, given $\mu^\circ\in\mathcal{P}\cap C^\sigma_b(\mathbb{R}^d)$ with non-integer $\sigma>0$, there exists a unique solution $\mu\in\mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}\cap C^\sigma_b(\mathbb{R}^d))$.* ◊

In the case $d-2<s\leq d-1$, $s\geq 0$, some recent results [430, 424] establish that regularity is propagated at least locally in time. [4]

Proposition 6.2.8 (Xiao, Zheng, Zhou)**.** *In the case $d-2<s\leq d-1$, $s\geq 0$, $d\geq 1$, given $\mu^\circ\in H^\sigma(\mathbb{R}^d)$ nonnegative with $\sigma>\frac{d}{2}+1$, there exists $T>0$ and a unique local solution $\mu\in\mathrm{L}^\infty([0,T];H^\sigma(\mathbb{R}^d))$ of (6.7) with $g=g_s$ and initial data μ°. Moreover, given $\mu^\circ\in C^\sigma_b(\mathbb{R}^d)$ nonnegative with non-integer $\sigma>1$, there exists a unique local solution $\mu\in\mathrm{L}^\infty([0,T];C^\sigma_b(\mathbb{R}^d))$.* ◊

4. Note that the paper [430] is however flawed: a crucial term is missing in the integration by parts in their equation (4.5). This term cannot be estimated easily, and after discussion with the authors it appeared that their whole iterative argument had to be redone differently. In [424] the authors have then later proposed a suitably corrected iterative argument and used it to establish local Sobolev regularity, while it can also be applied to correct the proof in their original paper [430] on local C^σ regularity. The key ingredients are the Kato-Ponce commutator estimates [269] and Córdoba's pointwise estimates for fractional derivatives [127, Proposition 2.3].

In the case $d-1<s<d$, in contrast, even the validity of such a local-in-time regularity result remains an open problem [94, 93, 419]. This explains the need for the additional regularity assumptions in the general statement of Theorem 6.1.5.

6.2.3 Modulated energy and elementary properties

Let $0 \le s < d$, and let $\mu^\circ, \mu_N^\circ, \mu, \mu_N$ be as in the statement of Theorem 6.1.5, for some $T \in (0,\infty)$. Note that in the case $s=0$ assumption (6.18) easily entails that μ° satisfies $\int_{\mathbb{R}^d} \log(2+|x|)d\mu^\circ(x) < \infty$.

Let $N \ge 1$. Since μ_N° is assumed to have bounded energy, and since the energy is decreasing along the flow, we find

$$\sup_{t\in[0,T]} \frac{1}{N^2}\sum_{i\neq j}^N g_s(x_{i,N}^t - x_{j,N}^t) \le \frac{1}{N^2}\sum_{i\neq j}^N g_s(x_{i,N}^\circ - x_{j,N}^\circ) < \infty.$$

For $0<s<d$, since g_s is nonnegative, this proves

$$\eta_N := \min_{i\neq j}^N \inf_{t\in[0,T]} |x_{i,N}^t - x_{j,N}^t| > 0. \tag{6.32}$$

For $s=0$, g_0 changes sign and an additional argument is then needed: noting that by symmetry

$$\partial_t \frac{1}{N}\sum_{i=1}^N x_{i,N}^t = -\frac{1}{N^2}\sum_{i\neq j}^N \nabla g_0(x_{i,N}^t - x_{j,N}^t) = \frac{c_{d,0}^{-1}}{N^2}\sum_{i\neq j}^N \frac{x_{i,N}^t - x_{j,N}^t}{|x_{i,N}^t - x_{j,N}^t|^2} = 0,$$

a direct computation yields

$$\begin{aligned}
\partial_t \frac{1}{N^2}\sum_{i\neq j}^N |x_{i,N}^t - x_{j,N}^t|^2 &= \partial_t \frac{2}{N}\sum_{i=1}^N |x_{i,N}^t|^2 - \partial_t \frac{2}{N^2}\Big|\sum_{i=1}^N x_{i,N}^t\Big|^2 \\
&= \frac{4c_{d,0}^{-1}}{N^2}\sum_{i\neq j}^N x_{i,N}^t \cdot \frac{x_{i,N}^t - x_{j,N}^t}{|x_{i,N}^t - x_{j,N}^t|^2} \\
&= \frac{2c_{d,0}^{-1}}{N^2}\sum_{i\neq j}^N (x_{i,N}^t - x_{j,N}^t)\cdot \frac{x_{i,N}^t - x_{j,N}^t}{|x_{i,N}^t - x_{j,N}^t|^2} = \frac{2c_{d,0}^{-1}(N-1)}{N},
\end{aligned}$$

and hence

$$\begin{aligned}
&\sup_{t\in[0,T]} \frac{1}{N^2}\sum_{i\neq j}^N \big(g_0(x_{i,N}^t - x_{j,N}^t) + c_{d,0}^{-1}|x_{i,N}^t - x_{j,N}^t|^2\big) \\
&\qquad \le \frac{1}{N^2}\sum_{i\neq j}^N \big(g_0(x_{i,N}^\circ - x_{j,N}^\circ) + c_{d,0}^{-1}|x_{i,N}^\circ - x_{j,N}^\circ|^2\big) + 2Tc_{d,0}^{-2}\frac{N-1}{N} < \infty.
\end{aligned}$$

As $g_0(x) + c_{d,0}^{-1}|x|^2 \ge 0$, this proves that (6.32) also holds in the case $s=0$ as well.

Next, we recall the truncation procedure introduced in [362], which serves to make energies finite without removing the diagonal. For fixed $N \ge 1$, let $\eta > 0$ be small enough such that $2\eta < 1 \wedge \eta_N$, and define

$$\mu_{N,\eta}^t := \frac{1}{N}\sum_{i=1}^N \delta_{x_{i,N}^t}^{(\eta)} \in \mathcal{P}(\mathbb{R}^d),$$

where $\delta_z^{(\eta)}$ denotes the uniform unit Dirac mass on the sphere $\partial B_\eta(z)$. Define

$$h^t := g_s * \mu^t, \qquad h_N^t := g_s * \mu_N^t, \qquad h_{N,\eta}^t := g_s * \mu_{N,\eta}^t,$$

and use the same notation for their extensions to $\mathbb{R}^d \times \mathbb{R}$ as in Section 6.2.1. Also define $g_{s,\eta} := g_s(\eta) \wedge g_s$. Noting that by symmetry

$$-\operatorname{div}(|\xi|^\gamma \nabla g_{s,\eta}) = \delta_0^{(\eta)} \delta_{\mathbb{R}^d \times \{0\}},$$

where $\delta_0^{(\eta)} \delta_{\mathbb{R}^d \times \{0\}}$ denotes the unit Dirac mass on $\partial B_\eta \times \{0\}$, we find

$$h_{N,\eta}^t(x,\xi) = \frac{1}{N} \sum_{i=1}^N g_{s,\eta}(x - x_{i,N}^t, \xi). \tag{6.33}$$

Let us now introduce our notation for the small balls around the particle locations, which we will be crucially using in the proof: for all $R > 0$, let $\mathcal{B}_N^t(R)$ denote a union of disjoint balls

$$\mathcal{B}_N^t(R) := \bigcup_{m=1}^{M_N^t(R)} B(y_{m,N}^t, r_{m,N}^t), \tag{6.34}$$

with total radius $R = \sum_m r_{m,N}^t$ and such that $x_{i,N}^t \in \mathcal{B}_N^t(R)$ for all $1 \le i \le N$. These balls will be carefully chosen in Section 6.2.6 below.

As already announced, for all $N \ge 1$, we will consider the following *modulated energy*

$$\mathcal{E}_N(t) := \iint_{D^c} g_s(x-y) d(\mu_N^t - \mu^t)(x) d(\mu_N^t - \mu^t)(y), \tag{6.35}$$

where $D := \{(x,x) : x \in \mathbb{R}^d\}$ denotes the diagonal. This quantity can be thought of as a natural renormalization of the $\dot{H}^{-(d-s)/2}$-distance in the presence of Dirac masses. Its main property is as follows.

Lemma 6.2.9 (Modulated energy). *Given μ, μ_N as in the statement of Theorem 6.1.5, if for some $t \ge 0$ the sequence $(\mu_N^t)_N$ is tight, then the following two conditions are equivalent:*

(i) $\limsup_{N\uparrow\infty} \mathcal{E}_N(t) \le 0$;

(ii) $\mu_N^t \overset{*}{\rightharpoonup} \mu^t$ *and* $\iint_{D^c} g_s(x-y) d\mu_N^t(x) d\mu_N^t(y) \to \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_s(x-y) d\mu^t(x) d\mu^t(y)$. ◊

Proof. Property (ii) clearly implies (i) (and even $\mathcal{E}_N(t) \to 0$), so it suffices to check the converse. Assume that $\limsup_N \mathcal{E}_N(t) \le 0$ holds. By tightness, up to extraction of a subsequence, the Prokhorov theorem gives $\mu_N^t \overset{*}{\rightharpoonup} \nu^t$ for some $\nu^t \in \mathcal{P}(\mathbb{R}^d)$. For any $K > 0$, we may write

$$\begin{aligned}
\iint_{D^c} g_s(x-y) d\mu_N^t(x) d\mu_N^t(y) &\ge \iint_{D^c} K \wedge g_s(x-y) d\mu_N^t(x) d\mu_N^t(y) \\
&= -\frac{K}{N} + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K \wedge g_s(x-y) d\mu_N^t(x) d\mu_N^t(y),
\end{aligned} \tag{6.36}$$

and hence, successively passing to the limits $N \uparrow \infty$ and $K \uparrow \infty$, we find in the case $0 < s < d$,

$$\liminf_{N\uparrow\infty} \iint_{D^c} g_s(x-y) d\mu_N^t(x) d\mu_N^t(y) \ge \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_s(x-y) d\nu^t(x) d\nu^t(y). \tag{6.37}$$

We now argue that the same result (6.37) holds in the case $s = 0$. Using that in the logarithmic case the particle dynamics satisfies $|\partial_t x_{i,N}^t| \le c_{d,0}^{-1}$, we may compute for all $R \ge 1$,

$$\begin{aligned}
&\left|\partial_t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(1 - e^{-|x-y|/R}\big) \log(2 + |x-y|) d\mu_N^t(x) d\mu_N^t(y)\right| \\
&\quad \le \; 2c_{d,0}^{-1} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1 - e^{-|x-y|/R}}{2 + |x-y|} d\mu_N^t(x) d\mu_N^t(y) \\
&\qquad\qquad + 2c_{d,0}^{-1} R^{-1} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \log(2 + |x-y|) e^{-|x-y|/R} d\mu_N^t(x) d\mu_N^t(y) \\
&\quad \le \; 2c_{d,0}^{-1} R^{-1} (2 + \log R),
\end{aligned}$$

and the assumption (6.18) then easily leads to

$$\begin{aligned}
&\limsup_{R\uparrow\infty} \limsup_{N\uparrow\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(1 - e^{-|x-y|/R}\big) \log(2 + |x-y|) d\mu_N^t(x) d\mu_N^t(y) \\
&\qquad \le \lim_{R\uparrow\infty} \limsup_{N\uparrow\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(1 - e^{-|x-y|/R}\big) \log(2 + |x-y|) d\mu_N^\circ(x) d\mu_N^\circ(y) \; = 0,
\end{aligned}$$

which in particular also implies

$$\limsup_{R\uparrow\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(1 - e^{-|x-y|/R}\big) \log(2 + |x-y|) d\nu^t(x) d\nu^t(y) \le 0.$$

Decomposing

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K \wedge g_0(x-y) d\mu_N^t(x) d\mu_N^t(y) &\ge \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (K \wedge g_0(x-y))\, e^{-|x-y|/R}\, d\mu_N^t(x) d\mu_N^t(y) \\
&\quad - (c_{d,0}^{-1} + K) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(1 - e^{-|x-y|/R}\big) \log(e + |x-y|) d\mu_N^t(x) d\mu_N^t(y),
\end{aligned}$$

and noting that $x \mapsto (K \wedge g_0(x))\, e^{-|x|/R}$ is now continuous and bounded, we deduce from the above

$$\begin{aligned}
\lim_{N\uparrow\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K \wedge g_0(x-y) d\mu_N^t(x) d\mu_N^t(y) &\ge \lim_{R\uparrow\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (K \wedge g_0(x-y))\, e^{-|x-y|/R}\, d\nu^t(x) d\nu^t(y) \\
&\ge \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K \wedge g_0(x-y) d\nu^t(x) d\nu^t(y).
\end{aligned}$$

Injecting this result into (6.36) finally entails that (6.37) holds in the case $s = 0$ as well. Combining (6.37) with the convergence $\mu_N^t \overset{*}{\rightharpoonup} \nu^t$ and with the assumption $\limsup_N \mathcal{E}_N(t) \le 0$, we obtain

$$\begin{aligned}
0 &\ge \limsup_{N\uparrow\infty} \iint_{D^c} g_s(x-y) d\mu_N^t(x) d\mu_N^t(y) - 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y) d\nu^t(x) d\mu^t(y) \\
&\qquad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y) d\mu^t(x) d\mu^t(y) \\
&\ge \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y) d\nu^t(x) d\nu^t(y) - 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y) d\nu^t(x) d\mu^t(y) \\
&\qquad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y) d\mu^t(x) d\mu^t(y) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y) d(\nu^t - \mu^t)(x) d(\nu^t - \mu^t)(y).
\end{aligned}$$

The conclusion then follows, noting that μ^t has bounded energy by Proposition 6.2.4, that ν^t has bounded energy by (6.37), and noting that for any two Radon measures μ, ν with finite energy we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y)d(\nu-\mu)(x)d(\nu-\mu)(y) \geq 0,$$

with equality if only if $\mu = \nu$ (see e.g. [300, Theorem 9.8] for $0 < s < d$, and [377, Lemma 1.8] for $s = 0$). ☐

In the case of bounded weak solutions μ_1, μ_2 of (6.7) as given by Proposition 6.2.4, the following identity follows from an integration by parts and was crucially used in the proofs of Lemmas 6.1.7 and 6.2.1 (cf. (6.23) and (6.29)),

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y)d(\mu_1^t-\mu_2^t)(x)d(\mu_1^t-\mu_2^t)(y) \\ = \begin{cases} \int_{\mathbb{R}^d} |\nabla(h_1^t - h_2^t)|^2, & \text{if } s = d-2,\ d \geq 2; \\ \int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma |\nabla(h_1^t - h_2^t)|^2, & \text{if } d-2 < s < d,\ s \geq 0. \end{cases} \tag{6.38}$$

Now we would need a corresponding identity in the context of the modulated energy $\mathcal{E}_N$. Since ∇h_N^t cannot belong to $\mathrm{L}^2(\mathbb{R}^d)$ or $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{R}, |\xi|^\gamma dx d\xi)$, a regularization is needed. Besides the modulated energy $\mathcal{E}_N$, we thus define the following η-approximation, based on the truncation (6.33) introduced above,

$$\mathcal{E}_{N,\eta}(t) := \begin{cases} \int_{\mathbb{R}^d} |\nabla(h_{N,\eta}^t - h^t)|^2, & \text{if } s = d-2,\ d \geq 2; \\ \int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma |\nabla(h_{N,\eta}^t - h^t)|^2, & \text{if } d-2 < s < d,\ s \geq 0. \end{cases}$$

An integration by parts then yields the following proxy for identity (6.38), showing that the difference between the modulated energy $\mathcal{E}_N(t)$ and its approximation $\mathcal{E}_{N,\eta}(t)$ just comes from the diagonal terms (which are indeed excluded in $\mathcal{E}_N(t)$ but not in $\mathcal{E}_{N,\eta}(t)$). We refer to [362, Section 2.1] for a detailed proof.

Lemma 6.2.10 (Approximate modulated energy). *Let $0 \vee (d-2) \leq s < d$. For all $t \geq 0$, $N \geq 1$ and $\eta > 0$,*

$$\mathcal{E}_{N,\eta}(t) = \mathcal{E}_N(t) + \frac{g_s(\eta)}{N} + o_N^{(\eta)}(1),$$

where for any fixed N we have $o_N^{(\eta)}(1) \to 0$ as $\eta \downarrow 0$. ◊

6.2.4 Grönwall argument on the modulated energy

By Lemma 6.2.9, in order to prove the convergence $\mu_N^t \overset{*}{\rightharpoonup} \mu^t$ and the convergence of energies, up to tightness issues, it suffices to check that $\limsup_N \mathcal{E}_N(t) \leq 0$. This is achieved by a Grönwall argument. From now on we focus on the Riesz case $d-2 < s < d$, $s \geq 0$. The Coulomb case $s = d-2$, $d \geq 2$ can be treated in exactly the same way, but is actually easier since it does not require to use the extension representation for the fractional Laplacian introduced in Section 6.2.1.

Proposition 6.2.11. *Let $d-2 < s < d$, $s \geq 0$. Let $\mu_N \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^d))$ be as in (6.5)–(6.6) with $g = g_s$. Let $\mu^\circ \in \mathcal{P}(\mathbb{R}^d)$, and in the case $s = 0$, $d = 1$ also assume $\int_{\mathbb{R}} \log(2+|x|)d\mu^\circ(x) < \infty$. Assume that equation (6.7) with $g = g_s$ and initial data μ° admits a local solution $\mu \in \mathrm{L}^\infty([0,T]; \mathcal{P} \cap C_b^\sigma(\mathbb{R}^d))$ for some $T > 0$ and some $\sigma > 2-d+s$. In the case $s = 0$, $d = 1$, also assume that $\nabla\mu \in \mathrm{L}^\infty([0,T]; \mathrm{L}^p(\mathbb{R}))$ for some $p < \infty$. Assume that the initial data satisfy $\mu_N^\circ \overset{*}{\rightharpoonup} \mu^\circ$ and*

$$\limsup_{N\uparrow\infty} \iint_{x\neq y} g_s(x-y)d\mu_N^\circ(x)d\mu_N^\circ(y) < \infty, \tag{6.39}$$

Assume that for all $t \in [0,T]$ the collection $\mathcal{B}_N^t(R_N^t)$ can be chosen with $R_N^t \to 0$ in such a way that

$$\liminf_{N\uparrow\infty} \liminf_{\eta\downarrow 0} \bigg(\int_{\mathcal{B}_N^t(R_N^t)\times\mathbb{R}} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 - \frac{g_s(\eta)}{N} \bigg) \geq 0, \tag{6.40}$$

and, denoting $g_s^+(t) := c_{d,s}^{-1} t^{-s}$ for $s > 0$ and $g_0^+(t) := c_{d,0}^{-1}(-\log t) \vee 0$ for $s = 0$,

$$\lim_{N\uparrow\infty} \frac{1}{N^2} \sum_{i=1}^N g_s^+(d(x_{i,N}^t, \partial\mathcal{B}_N^t(R_N^t))) = 0. \tag{6.41}$$

Then for all $t \in [0,T]$ we have $\mathcal{E}_N(t) \lesssim_t \mathcal{E}_N(0) + o(1)$ as $N \uparrow \infty$. ♢

Remark 6.2.12. We briefly examine the conditions (6.40) and (6.41). In the ideal case when all particles remain well-separated, that is, with a minimal distance $\eta_N \simeq N^{-1/d}$, then taking $\mathcal{B}_N^t(R_N^t)$ to be the union of balls of radius R_N^t/N centered at the points $x_{i,N}^t$ with $R_N^t/N \ll N^{-1/d}$, condition (6.41) simply becomes $g_s(R_N^t/N)/N \ll 1$. On the other hand, neglecting interactions between particles, hence focusing on the (divergent) self-interactions, we formally find

$$\begin{aligned} \int_{\mathcal{B}_N^t(R_N^t)\times\mathbb{R}} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 &= \frac{1}{N^2} \sum_{i=1}^N \int_{|x-x_{i,N}^t|<R_N^t/N} |\xi|^\gamma |\nabla g_{s,\eta}(x - x_{i,N}^t, \xi)|^2 + \ldots \\ &= \frac{1}{N} \int_{\eta<|x|<R_N^t/N} |\xi|^\gamma |\nabla g_s(x,\xi)|^2 + \ldots \\ &= \frac{1}{N}(g_s(\eta) - g_s(R_N^t/N)) + \ldots, \end{aligned}$$

so that condition (6.40) would amount to requiring $g_s(R_N^t/N)/N \ll 1$, which is thus just the same as condition (6.41). In other words, for $s > 0$, both conditions would take the form $R_N^t \gg N^{-(1-s)/s}$, which is compatible with the condition $R_N^t \to 0$ only if $s < 1$. In Section 6.2.6 below, we prove that a consistent choice of the small balls $\mathcal{B}_N^t(R_N^t)$ is indeed possible whenever $0 \leq s < 1$.
To go beyond the restriction $s < 1$ via this approach, we would need to modify Proposition 6.2.11 in order to relax the smallness condition for the total radius $R_N^t \to 0$. For that purpose, it would be necessary to refine the blind approximation argument used in Step 2 of the proof below: this approximation argument is indeed based on a worst-case scenario and could be improved with precise microscopic information on the particle dynamics. Getting a handle on such information seems to be a very difficult task and is not pursued here. Note that even an optimal bound on the minimal distance between particles would be of no help to improve this approximation argument, and that it is not clear how to formulate the needed geometric information. ♢

Proof. By the regularity assumption for μ, Lemma 6.2.6 ensures that we have $\|(\nabla_x h^t, \nabla_x^2 h^t)\|_{\mathrm{L}^\infty} \lesssim_t 1$, and also, in the case $s = 0$, $d = 1$, $\|\nabla_x^2 h^t\|_{\mathrm{L}^p} \lesssim_t 1$ for some $p < \infty$. We split the proof into four steps.

Step 1. Time derivative of $\mathcal{E}_N(t)$ and modulated stress-energy tensor.

In this step, we prove

$$\begin{aligned} \partial_t \mathcal{E}_N(t) = &- \int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma \nabla_x^2 h^t(x) : T_N^t(x,\xi) dx d\xi \\ &- 2 \int_{\mathbb{R}^d} \bigg| \mathrm{p.v.} \int_{\mathbb{R}^d\setminus\{x\}} \nabla g_s(x-y) d(\mu_N^t - \mu^t)(y) \bigg|^2 d\mu_N^t(x), \end{aligned} \tag{6.42}$$

where we use the usual principal value symbol

$$\mathrm{p.v.} \int_{\mathbb{R}^d\setminus\{x\}} := \lim_{r\downarrow 0} \int_{\mathbb{R}^d\setminus B(x,r)}$$

and where the modulated stress-energy tensor $T_N^t = (T_N^{t;kl})_{k,l=1}^{d+1}$ is defined as follows: for all $1 \le k, l \le d+1$,

$$T_N^{t;kl}(x,\xi) := 2\iint_{D^c} \partial_k g_s(x-y,\xi)\partial_l g_s(x-z,\xi)d(\mu_N-\mu)(y)d(\mu_N-\mu)(z)$$
$$-\delta_{kl}\iint_{D^c} \nabla g_s(x-y,\xi)\cdot\nabla g_s(x-z,\xi)d(\mu_N-\mu)(y)d(\mu_N-\mu)(z). \quad (6.43)$$

Moreover, as checked at the end of this step, the integrals in (6.42) are summable: more precisely, we prove that $|T_N^t|$ belongs to $\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}, |\xi|^\gamma dxd\xi)$ if $s>0$, and that $|\nabla_x^2 h^t(x)||T_N^t(x,\xi)|$ belongs to $\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}, |\xi|^\gamma dxd\xi)$ if $s=0$, $d=1$. Although the second term in the right-hand side of (6.42) is nonpositive, we do not bound it by 0 yet, contrarily to what is done in the proof of Lemmas 6.1.7 and 6.2.1, since it will be useful in Step 2 below to absorb some other error terms.

Using the equations satisfied by μ and by the trajectories $x_{i,N}$, and noting that the gradient ∇h^t is given by

$$\nabla h^t(x) = \mathrm{p.v.}\int_{\mathbb{R}^d\setminus\{x\}} \nabla g_s(x-y)d\mu^t(y),$$

where the principal value may only be omitted for $s<d-1$, we find the following expression for the time derivative of the modulated energy $\mathcal{E}_N(t)$ defined in (6.35),

$$\begin{aligned}\partial_t\mathcal{E}_N(t) &= \partial_t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(x-y)d\mu^t(x)d\mu^t(y)\\
&\qquad + \partial_t\frac{1}{N^2}\sum_{i\neq j}^N g_s(x_{i,N}^t - x_{j,N}^t) - \partial_t\frac{2}{N}\sum_{i=1}^N\int_{\mathbb{R}^d} g_s(x_{i,N}^t-y)d\mu^t(y)\\
&= -2\int_{\mathbb{R}^d}\nabla h^t(x)\cdot\mathrm{p.v.}\int_{\mathbb{R}^d\setminus\{x\}}\nabla g_s(x-y)d\mu^t(y)d\mu^t(x)\\
&\qquad -\frac{2}{N}\sum_{i=1}^N\Big|\frac{1}{N}\sum_{j,j\neq i}\nabla g_s(x_{i,N}^t-x_{j,N}^t)\Big|^2 + \frac{2}{N^2}\sum_{i\neq j}^N \nabla h^t(x_{i,N}^t)\cdot\nabla g_s(x_{i,N}^t-x_{j,N}^t)\\
&\qquad + \frac{2}{N}\sum_{i=1}^N \mathrm{p.v.}\int_{\mathbb{R}^d\setminus\{x_{i,N}^t\}}\nabla h^t(x)\cdot\nabla g_s(x-x_{i,N}^t)d\mu^t(x).\end{aligned}$$

Let us rearrange the terms as follows,

$$\begin{aligned}\partial_t\mathcal{E}_N(t) &= -2\int_{\mathbb{R}^d}\Big|\mathrm{p.v.}\int_{\mathbb{R}^d\setminus\{x\}}\nabla g_s(x-y)d(\mu_N^t-\mu^t)(y)\Big|^2 d\mu_N^t(x)\\
&\qquad - 2\int_{\mathbb{R}^d}\nabla h^t(x)\cdot\mathrm{p.v.}\int_{\mathbb{R}^d\setminus\{x\}}\nabla g_s(x-y)d\mu^t(y)d\mu^t(x)\\
&\qquad + 2\int_{\mathbb{R}^d}\nabla h^t(x)\cdot\mathrm{p.v.}\int_{\mathbb{R}^d\setminus\{x\}}\nabla g_s(x-y)d\mu^t(y)d\mu_N^t(x)\\
&\qquad - 2\int_{\mathbb{R}^d}\nabla h^t(x)\cdot\int_{\mathbb{R}^d\setminus\{x\}}\nabla g_s(x-y)d\mu_N^t(y)d\mu_N^t(x)\\
&\qquad + 2\int_{\mathbb{R}^d}\mathrm{p.v.}\int_{\mathbb{R}^d\setminus\{y\}}\nabla h^t(x)\cdot\nabla g_s(x-y)\mu^t(x)\mu_N^t(y),\end{aligned}$$

and note that the last four terms in the right-hand side may be combined to yield the following simpler expression,

$$\partial_t \mathcal{E}_N(t) = -2 \int_{\mathbb{R}^d} \left| \mathrm{p.v.} \int_{\mathbb{R}^d \setminus \{x\}} \nabla g_s(x-y) d(\mu_N^t - \mu^t)(y) \right|^2 d\mu_N^t(x)$$
$$- \underbrace{\iint_{D^c} (\nabla h^t(x) - \nabla h^t(y)) \cdot \nabla g_s(x-y) d(\mu_N^t - \mu^t)(y) d(\mu_N^t - \mu^t)(x)}_{=: I_N(t)}. \quad (6.44)$$

In the distributional sense on $\mathbb{R}^d$, using canonical regularizations, we may alternatively write

$$I_N(t) = \langle S_N^t; \nabla h^t \rangle = \sum_{k=1}^d \langle S_N^{t;k}; \partial_k h^t \rangle, \quad (6.45)$$

where $S_N^t = (S_N^{t;k})_{k=1}^d$ is given by

$$S_N^{t;k}(x) := 2(\mu_N^t - \mu^t)(x)\, \mathrm{p.v.} \int_{\mathbb{R}^d \setminus \{x\}} \partial_k g_s(x-y) d(\mu_N^t - \mu^t)(y).$$

Since $-\operatorname{div}(|\xi|^\gamma \nabla g_s(x - x_0, \xi)) = \delta_{x_0}(x) \delta_{\mathbb{R}^d \times \{0\}}(x, \xi)$ for all $x_0 \in \mathbb{R}^d$, we have in the distributional sense on $\mathbb{R}^d \times \mathbb{R}$,

$$S_N^{t,k}(x) \delta_{\mathbb{R}^d \times \{0\}}(x,\xi) = -2\, \mathrm{p.v.} \iint_{D^c} \operatorname{div}(|\xi|^\gamma \nabla g_s(x-z,\xi)) \partial_k g_s(x-y,\xi) d(\mu_N^t - \mu^t)(z) d(\mu_N^t - \mu^t)(y)$$
$$= -\, \mathrm{p.v.} \iint_{D^c} \big(\operatorname{div}(|\xi|^\gamma \nabla g_s(x-z,\xi)) \partial_k g_s(x-y,\xi) + \operatorname{div}(|\xi|^\gamma \nabla g_s(x-y,\xi)) \partial_k g_s(x-z,\xi) \big)$$
$$\times\, d(\mu_N^t - \mu^t)(z) d(\mu_N^t - \mu^t)(y).$$

Now note the following algebraic identity in the distributional sense on $\mathbb{R}^d \times \mathbb{R}$: for all $1 \le k \le d$,

$$\operatorname{div}(|\xi|^\gamma \nabla g_s(x-y,\xi)) \partial_k g_s(x-z,\xi) + \operatorname{div}(|\xi|^\gamma \nabla g_s(x-z,\xi)) \partial_k g_s(x-y,\xi)$$
$$= \frac{1}{2} \sum_{l=1}^{d+1} \big(\partial_l (|\xi|^\gamma G_s^{lk}(x,\xi;y,z)) + \partial_l (|\xi|^\gamma G_s^{lk}(x,\xi;z,y)) \big),$$

where we have set

$$G_s^{lk}(x,\xi;y,z) := 2\partial_l g_s(x-y,\xi) \partial_k g_s(x-z,\xi) - \delta_{lk} \sum_{m=1}^{d+1} \partial_m g_s(x-y,\xi) \partial_m g_s(x-z,\xi). \quad (6.46)$$

This proves the (Delort-type) identity

$$S_N^{t;k}(x) \delta_{\mathbb{R}^d \times \{0\}}(x,\xi) = -\sum_{l=1}^{d+1} \partial_l (|\xi|^\gamma T_N^{t;lk}(x,\xi)) \quad (6.47)$$

for all $1 \le k \le d$, and the conclusion (6.42) then follows from (6.44), (6.45), and an integration by parts.

We now turn to the claimed integrability of the modulated stress-energy tensor T_N^t. We first consider the case $d-2 < s < d$, $s > 0$. For that purpose, we start with the bound

$$\int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\gamma |T_N^t| \lesssim \int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\gamma \iint_{D^c} |(x-y,\xi)|^{-s-1} |(x-z,\xi)|^{-s-1} d(\mu_N^t + \mu^t)(y) d(\mu_N^t + \mu^t)(z) dx d\xi$$
$$= \iint_{D^c} \left(\int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\gamma |(x-y,\xi)|^{-s-1} |(x-z,\xi)|^{-s-1} dx d\xi \right) d(\mu_N^t + \mu^t)(y) d(\mu_N^t + \mu^t)(z). \quad (6.48)$$

Let us compute the integral over $\mathbb{R}^d \times \mathbb{R}$. Denoting for simplicity $c_{yz} := (y+z)/2$ and $q := s+1$, we decompose, for all $y \neq z$,

$$\int_{\mathbb{R}^d} (|x-y|^2+1)^{-q/2}(|x-z|^2+1)^{-q/2}dx = I^1_{yz} + I^2_{yz} + I^3_{yz} + I^4_{yz},$$

where

$$\begin{aligned}
I^1_{yz} &:= \int_{|x-y|\le\frac{1}{2}|y-z|} (|x-y|^2+1)^{-q/2}(|x-z|^2+1)^{-q/2}dx,\\
I^2_{yz} &:= \int_{|x-z|\le\frac{1}{2}|y-z|} (|x-y|^2+1)^{-q/2}(|x-z|^2+1)^{-q/2}dx,\\
I^3_{yz} &:= \int_{\substack{|x-y|,|x-z|>\frac{1}{2}|y-z|\\ |x-c_{yz}|\le|y-z|}} (|x-y|^2+1)^{-q/2}(|x-z|^2+1)^{-q/2}dx,\\
I^4_{yz} &:= \int_{|x-c_{yz}|>|y-z|} (|x-y|^2+1)^{-q/2}(|x-z|^2+1)^{-q/2}dx,
\end{aligned}$$

Using that $|x-y| \le \frac{1}{2}|y-z|$ implies $|x-z| \ge \frac{1}{2}|y-z|$, we may estimate

$$I^1_{yz} \le (|y-z|^2/4+1)^{-q/2} \int_{|x-y|\le\frac{1}{2}|y-z|} (|x-y|^2+1)^{-q/2}dx \lesssim (|y-z|/2+1)^{d-2q},$$

and similarly for I^2_{yz}. Moreover,

$$I^3_{yz} \le (|y-z|^2/4+1)^{-q} \int_{|x-c_{yz}|\le|y-z|} dx \lesssim (|y-z|^2/4+1)^{-q}|y-z|^d \lesssim (|y-z|/2+1)^{d-2q},$$

and also, since $d-2q<0$ follows from the choice $s>d-2$, $s\ge 0$,

$$\begin{aligned}
I^4_{yz} &\lesssim \int_{|x-c_{yz}|>|y-z|} (|x-y|+1)^{-q}(|x-z|+1)^{-q}dx\\
&\le \int_{|x-c_{yz}|>|y-z|} (|x-c_{yz}|-|y-z|/2+1)^{-2q}dx \lesssim (|y-z|/2+1)^{d-2q}.
\end{aligned}$$

This proves, for all $y \neq z$,

$$\int_{\mathbb{R}^d} (|x-y|^2+1)^{-q/2}(|x-z|^2+1)^{-q/2}dx \lesssim (|y-z|/2+1)^{d-2q},$$

and hence by scaling

$$\int_{\mathbb{R}^d} |(x-y,\xi)|^{-q}|(x-z,\xi)|^{-q}dx \lesssim (|y-z|/2+|\xi|)^{d-2q},$$

so that we obtain, with by definition $\gamma = q-d \in (-1,1)$,

$$\int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma|(x-y,\xi)|^{-q}|(x-z,\xi)|^{-q}dxd\xi \lesssim \int_{\mathbb{R}} |\xi|^{q-d}(|y-z|+|\xi|)^{d-2q}d\xi,$$

Splitting the integrals over ξ into the part where $|\xi| \le |y-z|$ and that where $|\xi| > |y-z|$, and noting that $q>1$ follows from $s>0$, we find

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma|(x-y,\xi)|^{-q}|(x-z,\xi)|^{-q}dxd\xi\\
&\qquad \lesssim |y-z|^{d-2q}\int_{|\xi|\le|y-z|} |\xi|^{q-d}d\xi + \int_{|\xi|>|y-z|} |\xi|^{q-d}|\xi|^{d-2q}d\xi \;\lesssim |y-z|^{1-q} = |y-z|^{-s}.
\end{aligned}$$

Combining this with (6.48) finally yields

$$\int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma |T_N^t| \lesssim \iint_{D^c} |y-z|^{-s} d(\mu_N^t+\mu^t)(y) d(\mu_N^t+\mu^t)(z),$$

and hence, by assumption (6.39), since both the particle and the mean-field energies are decreasing along the flow (see Proposition 6.2.4 for the mean-field energy),

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma |T_N^t| &\lesssim \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(y-z)d\mu^t(y)d\mu^t(z) + \iint_{D^c} g_s(y-z)d\mu_N^t(y)d\mu_N^t(z) + 2\int h^t d\mu_N^t \\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_s(y-z)d\mu^\circ(y)d\mu^\circ(z) + \iint_{D^c} g_s(y-z)d\mu_N^\circ(y)d\mu_N^\circ(z) + 2\|h^t\|_{\mathrm{L}^\infty} \lesssim_t 1.\end{aligned}$$

It remains to consider the case $s=0$, $d=1$ (hence $\gamma=0$, $q=1$). Let $1<p<\infty$ be such that $\|\nabla^2 h^t\|_{\mathrm{L}^p} \lesssim_t 1$. Arguing as above, we obtain

$$\int_{\mathbb{R}} |\nabla^2 h^t(x)||(x-y,\xi)|^{-1}|(x-z,\xi)|^{-1}dx \lesssim \|\nabla^2 h^t\|_{\mathrm{L}^\infty}(|y-z|+|\xi|)^{-1},$$

and similarly, by the Hölder inequality, for $\frac{1}{p}+\frac{1}{p'}=1$, $p'>1$,

$$\begin{aligned}\int_{\mathbb{R}} |\nabla^2 h^t(x)||(x-y,\xi)|^{-1}|(x-z,\xi)|^{-1}dx &\lesssim \|\nabla^2 h^t\|_{\mathrm{L}^p}\bigg(\int_{\mathbb{R}} |(x-y,\xi)|^{-p'}|(x-z,\xi)|^{-p'}dx\bigg)^{1/p'} \\ &\lesssim \|\nabla^2 h^t\|_{\mathrm{L}^p}(|y-z|+|\xi|)^{\frac{1}{p'}-2}.\end{aligned}$$

Splitting the integral over ξ into the part where $|\xi|\le |y-z|\vee 1$ and that where $|\xi|>|y-z|\vee 1$, we may then estimate

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{R}} |\nabla^2 h^t(x)||(x-y,\xi)|^{-1}|(x-z,\xi)|^{-1}dxd\xi \\ &\lesssim \quad \|\nabla^2 h^t\|_{\mathrm{L}^\infty}\int_{|\xi|\le|y-z|\vee 1}(|y-z|+|\xi|)^{-1}d\xi + \|\nabla^2 h^t\|_{\mathrm{L}^p}\int_{|\xi|>|y-z|\vee 1}(|y-z|+|\xi|)^{\frac{1}{p'}-2}d\xi \\ &\lesssim_t \quad 1-0\wedge\log(|y-z|) = 1+0\vee g_0(y-z),\end{aligned}$$

and the conclusion now easily follows just as in the case $s>0$.

Step 2. Approximation argument.

For all $t\ge 0$ and all $R\in(0,1)$, applying [382, Proposition 9.6], there exists a smooth approximation v^t of the function $\nabla h^t \in C_b^{0,1}(\mathbb{R}^d)^d$ such that v^t is constant on each ball of the collection $\mathcal{B}_N^t(R)$, and satisfies for all $\alpha\in[0,1]$,

$$\|v^t-\nabla h^t\|_{C^\alpha} \le CR^{1-\alpha}\|\nabla^2 h^t\|_{\mathrm{L}^\infty} \le C_t R^{1-\alpha}, \tag{6.49}$$

and in addition $\|\nabla v^t\|_{\mathrm{L}^p}\lesssim_t 1$ for some $p<\infty$ in the case $s=0$, $d=1$. In this step, we prove the following estimate,

$$\partial_t \mathcal{E}_N(t) \le -\int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma \nabla v^t : T_N^t + C_t\, o^{(R)}(1), \tag{6.50}$$

where $o^{(R)}(1)$ denotes a quantity that goes to 0 as $R\downarrow 0$.

Using relation (6.47) as well as the integrability properties of T_N^t, we may decompose the first term in the right-hand side of (6.42) as follows,

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma \nabla_x^2 h^t(x) : T_N^t(x,\xi)dxd\xi = \langle S_N^t; \nabla h^t\rangle &= \langle S_N^t; v^t\rangle + \langle S_N^t; \nabla h^t - v^t\rangle \\ &= \langle S_N^t \delta_{\mathbb{R}^d\times\{0\}}; v^t\rangle + \langle S_N^t; \nabla h^t - v^t\rangle \\ &= \int_{\mathbb{R}^d\times\mathbb{R}} |\xi|^\gamma \nabla v^t : T_N^t + \langle S_N^t; \nabla h^t - v^t\rangle. \end{aligned} \tag{6.51}$$

It remains to estimate the last term in the right-hand side of (6.51). Denoting for simplicity $w^t := \nabla h^t - v^t$, we may decompose by symmetry

$$\begin{aligned}\langle S_N^t; \nabla h^t - v^t\rangle &= \iint_{D^c} (w^t(x) - w^t(y))\cdot\nabla g_s(x-y)d(\mu_N^t - \mu^t)(y)d(\mu_N^t-\mu^t)(x) \\ &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w^t(x)\cdot\nabla g_s(x-y)d\mu^t(y)d\mu^t(x) + 2\iint_{D^c} w^t(x)\cdot\nabla g_s(x-y)d\mu_N^t(y)d\mu_N^t(x) \\ &\qquad - 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(w^t(x)-w^t(y))\cdot\nabla g_s(x-y)d\mu_N^t(y)d\mu^t(x). \end{aligned} \tag{6.52}$$

For the first right-hand side term, we simply have by (6.49),

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w^t(x)\cdot\nabla g_s(x-y)d\mu^t(y)d\mu^t(x)\right| = \left|\int_{\mathbb{R}^d} w^t\cdot\nabla h^t d\mu^t\right| \le \|w^t\|_{\mathrm{L}^\infty}\|\nabla h^t\|_{\mathrm{L}^\infty} \le C_t R.$$

Regarding the third right-hand side term in (6.52), choosing $\sigma > s+1-d$, $0\le\sigma<1$, and recalling that μ^t remains bounded by assumption, we find by (6.49),

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(w^t(x)-w^t(y))\cdot\nabla g_s(x-y)d\mu_N^t(y)d\mu^t(x)\right| \\ &\quad\lesssim \|w^t\|_{C^\sigma}\sup_{y\in\mathbb{R}^d}\int|x-y|^{-s-1+\sigma}d\mu^t(x) \\ &\quad\le \|w^t\|_{C^\sigma}\sup_{y\in\mathbb{R}^d}\left(\|\mu^t\|_{\mathrm{L}^\infty}\int_{|x-y|\le1}|x-y|^{-s-1+\sigma}dx + \int_{|x-y|>1}d\mu^t(x)\right) \\ &\quad\le \|w^t\|_{C^\sigma}(1+\|\mu^t\|_{\mathrm{L}^\infty}) \le C_t R^{1-\sigma}.\end{aligned}$$

Injecting these two estimates in (6.52), and using (6.49) once again, we obtain, for $R\downarrow 0$,

$$\begin{aligned}|\langle S_N^t; \nabla h^t - v^t\rangle| &\lesssim_t o^{(R)}(1) + R\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d\setminus\{x\}}\nabla g_s(x-y)d\mu_N^t(y)\right|d\mu_N^t(x) \\ &\lesssim_t o^{(R)}(1) + R\int_{\mathbb{R}^d}\left|\mathrm{p.\,v.}\int_{\mathbb{R}^d\setminus\{x\}}\nabla g_s(x-y)d(\mu_N^t-\mu^t)(y)\right|d\mu_N^t(x) \\ &\qquad + R\int_{\mathbb{R}^d}\left|\mathrm{p.\,v.}\int_{\mathbb{R}^d\setminus\{x\}}\nabla g_s(x-y)d\mu^t(y)\right|d\mu_N^t(x),\end{aligned}$$

and thus, noting that

$$\int_{\mathbb{R}^d}\left|\mathrm{p.\,v.}\int_{\mathbb{R}^d\setminus\{x\}}\nabla g_s(x-y)d\mu^t(y)\right|d\mu_N^t(x) \le \|\nabla h^t\|_{\mathrm{L}^\infty}\le C_t,$$

we find

$$|\langle S_N^t; \nabla h^t - v^t\rangle| \lesssim_t o^{(R)}(1) + R\int_{\mathbb{R}^d}\left|\mathrm{p.\,v.}\int_{\mathbb{R}^d\setminus\{x\}}\nabla g_s(x-y)d(\mu_N^t-\mu^t)(y)\right|d\mu_N^t(x).$$

Hence, for all $\varepsilon \in (0,1)$, using the inequality $R|a| \le \varepsilon a^2 + (4\varepsilon)^{-1}R^2$, we obtain

$$|\langle S_N^t; \nabla h^t - v^t\rangle| \lesssim_t \varepsilon^{-1} o^{(R)}(1) + \varepsilon \int_{\mathbb{R}^d} \left| \mathrm{p.\,v.} \int_{\mathbb{R}^d\setminus\{x\}} \nabla g_s(x-y) d(\mu_N^t - \mu^t)(y) \right|^2 d\mu_N^t(x),$$

and the result (6.50) then follows from (6.51) and (6.42), choosing $\varepsilon > 0$ small enough (depending on t).

Step 3. Modification with η-approximations.

In the definition (6.43) of T_N^t, the diagonal terms were excluded. In order to apply inequality $2|ab| \le a^2 + b^2$ to T_N^t as in the proof of Lemmas 6.1.7 and 6.2.1, we would need to add these diagonal terms explicitly. Then η-approximations become needed to avoid the divergence of the corresponding diagonal terms that will appear after application of the above-mentioned inequality. More precisely, we prove in this step

$$\begin{aligned}\partial_t \mathcal{E}_N(t) \lesssim_t &\int_{(\mathbb{R}^d\setminus\mathcal{B}_N^t(R))\times\mathbb{R}} |\xi|^\gamma |\nabla(h_{N,\eta}^t - h^t)|^2 \\ &+ \frac{1}{N^2}\sum_{i=1}^N \int_{(\mathbb{R}^d\setminus\mathcal{B}_N^t(R))\times\mathbb{R}} |\xi|^\gamma |\nabla v^t(x)| |\nabla g_s(x - x_{i,N}^t, \xi)|^2 dx d\xi + o^{(R)}(1) + o_{N,R}^{(\eta)}(1). \quad (6.53)\end{aligned}$$

By the choice of v^t to be constant on each ball of the collection $\mathcal{B}_N^t(R)$, and by the bound on ∇v^t, equation (6.50) takes the form

$$\partial_t \mathcal{E}_N(t) \lesssim_t \int_{(\mathbb{R}^d\setminus\mathcal{B}_N^t(R))\times\mathbb{R}} |\xi|^\gamma |\nabla v^t(x)| |T_N^t| + o_R(1). \tag{6.54}$$

Denote for simplicity

$$H_N^t(x,\xi) := (h_N^t - h^t)(x,\xi), \qquad H_{N,\eta}^t(x,\xi) := (h_{N,\eta}^t - h^t)(x,\xi),$$

and for all $1 \le k, l \le d+1$ define

$$T_{N,\eta}^{t;kl}(x,\xi) := 2\partial_k H_{N,\eta}(x,\xi)\partial_l H_{N,\eta}(x,\xi) - \delta_{kl}|\nabla H_{N,\eta}^t(x,\xi)|^2.$$

For all x with $d(x, \{x_{i,N}^t\}_{i=1}^N) > \eta$, we note that

$$\nabla H_{N,\eta}^t(x,\xi) = \frac{1}{N}\sum_{i=1}^N \nabla g_{s,\eta}(x - x_{i,N}^t) - \nabla h^t(x) = \frac{1}{N}\sum_{i=1}^N \nabla g_s(x - x_{i,N}^t) - \nabla h^t(x) = \nabla H_N^t(x,\xi). \tag{6.55}$$

Also noting that the definition (6.46) may be rewritten as

$$\begin{aligned}\frac{1}{N^2}\sum_{i=1}^N G_s^{kl}(x,\xi; x_{i,N}^t, x_{i,N}^t) = 2 &\iint_D \partial_k g_s(x-y;\xi)\partial_l g_s(x-z;\xi) d(\mu_N^t - \mu^t)(y) d(\mu_N^t - \mu^t)(z) \\ &- \delta_{kl} \iint_D \nabla g_s(x-y;\xi)\cdot\nabla g_s(x-z;\xi) d(\mu_N^t - \mu^t)(y) d(\mu_N^t - \mu^t)(z),\end{aligned}$$

the definition (6.43) yields

$$
\begin{aligned}
&T_N^{t;kl}(x,\xi)+\frac{1}{N^2}\sum_{i=1}^N G_s^{kl}(x,\xi;x_{i,N}^t,x_{i,N}^t)\\
&=2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_k g_s(x-y;\xi)\partial_l g_s(x-z;\xi)d(\mu_N^t-\mu^t)(y)d(\mu_N^t-\mu^t)(z)\\
&\qquad-\delta_{kl}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\nabla g_s(x-y;\xi)\cdot\nabla g_s(x-z;\xi)d(\mu_N^t-\mu^t)(y)d(\mu_N^t-\mu^t)(z)\\
&=2\partial_k H_N^t(x,\xi)\partial_l H_N^t(x,\xi)-\delta_{kl}|\nabla H_N^t(x,\xi)|^2.
\end{aligned}
$$

Combining this with (6.55) yields, for all $1\le k,l\le d+1$ and all x with $d(x,\{x_{i,N}^t\}_{i=1}^N)>\eta$,

$$
\begin{aligned}
&T_N^{t;kl}(x,\xi)+\frac{1}{N^2}\sum_{i=1}^N G_s^{kl}(x,\xi;x_{i,N}^t,x_{i,N}^t)\\
&\qquad\qquad=2\partial_k H_{N,\eta}^t(x,\xi)\partial_l H_{N,\eta}^t(x,\xi)-\delta_{kl}|\nabla H_{N,\eta}^t(x,\xi)|^2=T_{N,\eta}^{t;kl}(x,\xi).
\end{aligned}
$$

From (6.54), we then deduce, for all $\eta>0$ small enough such that $\bigcup_{i=1}^N B(x_{i,N}^t,\eta)\subset\mathcal{B}_N^t(R)$,

$$
\begin{aligned}
\partial_t\mathcal{E}_N(t)\lesssim_t&\int_{(\mathbb{R}^d\setminus\mathcal{B}_N^t(R))\times\mathbb{R}}|\xi|^\gamma|T_{N,\eta}^t|\\
&+\frac{1}{N^2}\sum_{i=1}^N\int_{(\mathbb{R}^d\setminus\mathcal{B}_N^t(R))\times\mathbb{R}}|\xi|^\gamma|\nabla v^t(x)||G_s(x,\xi;x_{i,N}^t,x_{i,N}^t)|+o^{(R)}(1).
\end{aligned}
$$

The result (6.53) then follows, using inequality $2|ab|\le a^2+b^2$ in the form of

$$
|T_{N,\eta}^t|\lesssim|\nabla(h_{N,\eta}^t-h^t)|^2,\qquad\text{and}\qquad |G_s(x,\xi;x_{i,N}^t,x_{i,N}^t)|\lesssim|\nabla g_s(x-x_{i,N}^t,\xi)|^2.
$$

Step 4. Conclusion.

In this step, we show that

$$
\begin{aligned}
\partial_t\mathcal{E}_N(t)\lesssim_t\mathcal{E}_N(t)+\frac{g_s(\eta)}{N}-&\int_{\mathcal{B}_N^t(R)\times\mathbb{R}}|\xi|^\gamma|\nabla h_{N,\eta}^t|^2\\
&+\frac{1}{N^2}\sum_{i=1}^N g_s^+(d(x_{i,N}^t,\partial\mathcal{B}_N^t(R)))+o^{(R)}(1)+o^{(N)}(1)+o_{N,R}^{(\eta)}(1).
\end{aligned}\tag{6.56}
$$

The statement of Proposition 6.2.11 immediately follows from this inequality and the suitable choice of $R=R_N^t$, together with a Grönwall argument.

By Lemma 6.2.10, inequality (6.53) may be rewritten as follows,

$$
\begin{aligned}
\partial_t\mathcal{E}_N(t)\lesssim_t\mathcal{E}_N(t)+\frac{g_s(\eta)}{N}-&\int_{\mathcal{B}_N^t(R)\times\mathbb{R}}|\xi|^\gamma|\nabla(h_{N,\eta}^t-h^t)|^2\\
&+\frac{1}{N^2}\sum_{i=1}^N\int_{(\mathbb{R}^d\setminus\mathcal{B}_N^t(R))\times\mathbb{R}}|\xi|^\gamma|\nabla v^t(x)||\nabla g_s(x-x_{i,N}^t,\xi)|^2dxd\xi+o^{(R)}(1)+o_{N,R}^{(\eta)}(1),
\end{aligned}
$$

or equivalently, expanding the square,

$$\partial_t\mathcal{E}_N(t) \lesssim_t \mathcal{E}_N(t)+\frac{g_s(\eta)}{N}-\int_{\mathcal{B}_N^t(R)\times\mathbb{R}}|\xi|^\gamma|\nabla h_{N,\eta}^t|^2-\int_{\mathcal{B}_N^t(R)\times\mathbb{R}}|\xi|^\gamma|\nabla h^t|^2+2\int_{\mathcal{B}_N^t(R)\times\mathbb{R}}|\xi|^\gamma\nabla h_{N,\eta}^t\cdot\nabla h^t$$
$$+\frac{1}{N^2}\sum_{i=1}^N\int_{(\mathbb{R}^d\setminus\mathcal{B}_N^t(R))\times\mathbb{R}}|\xi|^\gamma|\nabla v^t(x)||\nabla g_s(x-x_{i,N}^t,\xi)|^2dxd\xi+o^{(R)}(1)+o_{N,R}^{(\eta)}(1). \quad (6.57)$$

The last term in the first line is easily estimated as follows, using the notation (6.34) for the union $\mathcal{B}_N^t(R)$ of small balls,

$$\begin{aligned}\left|\int_{\mathcal{B}_N^t(R)\times\mathbb{R}}|\xi|^\gamma\nabla h_{N,\eta}^t\cdot\nabla h^t\right| &\lesssim \|\nabla h^t\|_{\mathrm{L}^\infty}\frac{1}{N}\sum_{i=1}^N\int_{\mathcal{B}_N^t(R)\times\mathbb{R}}|\xi|^\gamma|(x-x_{i,N}^t,\xi)|^{-s-1}dxd\xi\\ &\lesssim_t \frac{1}{N}\sum_{i=1}^N\int_{\mathcal{B}_N^t(R)}|x-x_{i,N}^t|^{1-d}dx\\ &\lesssim \sum_{m=1}^{M_N^t}\int_{|x|\le 2r_{m,N}^t}|x|^{1-d}dx\lesssim\sum_{m=1}^{M_N^t}r_{m,N}^t=R,\end{aligned} \quad (6.58)$$

while the term in the second line of (6.57) is estimated as follows, in the case $s>0$,

$$\begin{aligned}&\frac{1}{N^2}\sum_{i=1}^N\int_{(\mathbb{R}^d\setminus\mathcal{B}_N^t(R))\times\mathbb{R}}|\xi|^\gamma|\nabla v^t(x)||\nabla g_s(x-x_{i,N}^t,\xi)|^2dxd\xi\\ &\lesssim_t \frac{1}{N^2}\sum_{i=1}^N\int_{(\mathbb{R}^d\setminus\mathcal{B}_N^t(R))\times\mathbb{R}}|\xi|^\gamma|(x-x_{i,N}^t,\xi)|^{-2(s+1)}dxd\xi\\ &\lesssim \frac{1}{N^2}\sum_{i=1}^N\int_{\mathbb{R}^d\setminus\mathcal{B}_N^t(R)}|x-x_{i,N}^t|^{-d-s}dx\lesssim\frac{1}{N^2}\sum_{i=1}^N d(x_{i,N}^t,\partial\mathcal{B}_N^t(R))^{-s}.\end{aligned} \quad (6.59)$$

In the case $s=0$, $d=1$ (hence $\gamma=0$), denoting $\rho_{i,N}^t:=d(x_{i,N}^t,\partial\mathcal{B}_N^t(R))$, and applying the Hölder inequality with $\frac{1}{p}+\frac{1}{p'}=1$ and with $p<\infty$ chosen such that $\|\nabla v^t\|_{\mathrm{L}^p}\lesssim_t 1$, we rather estimate

$$\begin{aligned}&\frac{1}{N^2}\sum_{i=1}^N\int_{(\mathbb{R}\setminus\mathcal{B}_N^t(R))\times\mathbb{R}}|\nabla v^t(x)||\nabla g_0(x-x_{i,N}^t,\xi)|^2dxd\xi\\ &\lesssim \frac{1}{N^2}\sum_{i=1}^N\int_{\mathbb{R}\setminus\mathcal{B}_N^t(R)}|\nabla v^t(x)||x-x_{i,N}^t|^{-1}dx\\ &\lesssim \frac{1}{N^2}\sum_{i=1}^N\int_{\rho_{i,N}^t<|x-x_{i,N}^t|\le 1}|\nabla v^t(x)||x-x_{i,N}^t|^{-1}dx+\frac{1}{N^2}\sum_{i=1}^N\int_{|x-x_{i,N}^t|>1}|\nabla v^t(x)||x-x_{i,N}^t|^{-1}dx\\ &\lesssim_t \frac{1}{N^2}\sum_{i=1}^N\int_{\rho_{i,N}^t<|x-x_{i,N}^t|\le 1}|x-x_{i,N}^t|^{-1}dx+\frac{1}{N^2}\sum_{i=1}^N\bigg(\int_{|x-x_{i,N}^t|>1}|x-x_{i,N}^t|^{-p'}dx\bigg)^{1/p'},\end{aligned}$$

and hence, by the choice $p'>1$,

$$\frac{1}{N^2}\sum_{i=1}^N\int_{(\mathbb{R}\setminus\mathcal{B}_N^t(R))\times\mathbb{R}}|\nabla v^t(x)||\nabla g_0(x-x_{i,N}^t,\xi)|^2dxd\xi\lesssim_t\frac{1}{N^2}\sum_{i=1}^N(-0\wedge\log\rho_{i,N}^t)+N^{-1}. \quad (6.60)$$

The result (6.56) then follows from inequality (6.57) together with (6.58) and with (6.59) or (6.60). □

6.2.5 Bypass of tightness issues

Assuming that $\mathcal{E}_N(0) \le o(1)$ as $N \uparrow \infty$, Proposition 6.2.11 yields $\mathcal{E}_N(t) \le o_t(1)$ for all $t \in [0,T]$. If in addition the sequence $(\mu_N^t)_N$ is known to be tight, then Lemma 6.2.9 allows us to conclude with the desired convergence $\mu_N^t \overset{*}{\rightharpoonup} \mu^t$. Tightness can for example be directly checked under the additional assumption that the initial measures μ_N° are well localized in the sense of $\limsup_N \int |x|^2 d\mu_N^\circ < \infty$. However, in the spirit of [395, Section 4.3.5], the following refinement of Lemma 6.2.9 shows that much more information can be directly extracted from the fact that $\mathcal{E}_N(t) \le o_t(1)$, so that in particular tightness is obtained a posteriori without any additional assumption.

Corollary 6.2.13. *Let the assumptions of Proposition 6.2.11 prevail. Also assume that $\mathcal{E}_N(0) \le o(1)$ as $N \uparrow \infty$. Then for all $t \in [0,T]$ we have $\nabla h_N^t \to \nabla h^t$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d; \mathrm{L}^2(\mathbb{R}, |\xi|^\gamma d\xi))$ for all $1 \le p < 2d/(s+d)$, and hence $\mu_N^t \overset{*}{\rightharpoonup} \mu^t$. In particular, $(\mu_N^t)_N$ is tight and Lemma 6.2.9 thus also implies the convergence of the energy under the assumptions of Theorem 6.1.5.* $\Diamond$

Proof. By assumption, Proposition 6.2.11 yields $\mathcal{E}_N(t) \lesssim_t o_N(1)$ as $N \uparrow \infty$. We split the proof into three steps.

Step 1. Strong convergence outside small balls.

In this step, we prove

$$\iint_{(\mathbb{R}^d \setminus \mathcal{B}_N^t)\times\mathbb{R}} |\xi|^\gamma |\nabla(h_N^t - h^t)|^2 \lesssim_t o_N(1), \tag{6.61}$$

and hence for any $1 \le p \le 2$ the Hölder inequality implies for all $R > 0$,

$$\int_{B_R \setminus \mathcal{B}_N^t} \Big(\int_{\mathbb{R}} |\xi|^\gamma |\nabla(h_N^t - h^t)|^2 \Big)^{p/2} \lesssim R^{d(1-p/2)} \Big(\iint_{(\mathbb{R}^d \setminus \mathcal{B}_N^t)\times\mathbb{R}} |\xi|^\gamma |\nabla(h_N^t - h^t)|^2 \Big)^{p/2} \lesssim_{R,t} o_N(1). \tag{6.62}$$

Applying Lemma 6.2.10 and expanding the square, the $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{R}, |\xi|^\gamma dx d\xi)$-norm of $\nabla(h_{N,\eta}^t - h^t)$ outside the small balls $\mathcal{B}_N^t$ can be decomposed as follows,

$$\begin{aligned}
\iint_{(\mathbb{R}^d \setminus \mathcal{B}_N^t)\times\mathbb{R}} |\xi|^\gamma |\nabla(h_{N,\eta}^t - h^t)|^2 &= \mathcal{E}_{N,\eta}(t) - \iint_{\mathcal{B}_N^t \times \mathbb{R}} |\xi|^\gamma |\nabla(h_{N,\eta}^t - h^t)|^2 \\
&= \mathcal{E}_N(t) + \frac{g_s(\eta)}{N} - \iint_{\mathcal{B}_N^t \times \mathbb{R}} |\xi|^\gamma |\nabla(h_{N,\eta}^t - h^t)|^2 + o_\eta^{(N)}(1) \\
&= \mathcal{E}_N(t) + \frac{g_s(\eta)}{N} - \iint_{\mathcal{B}_N^t \times \mathbb{R}} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 - \iint_{\mathcal{B}_N^t \times \mathbb{R}} |\xi|^\gamma |\nabla h^t|^2 \\
&\qquad + 2 \iint_{\mathcal{B}_N^t \times \mathbb{R}} |\xi|^\gamma \nabla h_{N,\eta}^t \cdot \nabla h^t + o_\eta^{(N)}(1).
\end{aligned}$$

Applying Proposition 6.2.11 in the form of $\mathcal{E}_N(t) \lesssim_t o_N(1)$, and using assumption (6.40), this turns into

$$\iint_{(\mathbb{R}^d \setminus \mathcal{B}_N^t)\times\mathbb{R}} |\xi|^\gamma |\nabla(h_{N,\eta}^t - h^t)|^2 \le 2 \iint_{\mathcal{B}_N^t \times \mathbb{R}} |\xi|^\gamma \nabla h_{N,\eta}^t \cdot \nabla h^t + o_N^{(t)}(1) + o_\eta^{(N)}(1).$$

Now arguing just as in (6.58), we find

$$\Big| \iint_{\mathcal{B}_N^t \times \mathbb{R}} |\xi|^\gamma \nabla h_{N,\eta}^t \cdot \nabla h^t \Big| \lesssim_t R_N^t \lesssim_t o_N(1),$$

and hence

$$\iint_{(\mathbb{R}^d\setminus\mathcal{B}_N^t)\times\mathbb{R}} |\xi|^\gamma|\nabla(h_{N,\eta}^t-h^t)|^2 \le o_N^{(t)}(1)+o_\eta^{(N,t)}(1).$$

Passing to the limit $\eta\downarrow 0$ in this inequality, and noting that $\nabla h_{N,\eta}^t\to\nabla h_N^t$ holds in the distributional sense, the result (6.61) follows.

Step 2. Neglecting the contribution inside small balls.

The contribution inside the small balls $\mathcal{B}_N^t$ is of course infinite, since ∇h_N^t does not belong to $\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R},|\xi|^\gamma dxd\xi)$. However, we show that it is small in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d;\mathrm{L}^2(\mathbb{R},|\xi|^\gamma d\xi))$ for p small enough. More precisely, for any $1\le p<2d/(s+d)$, we show that we have for all $R>0$

$$\int_{B_R\cap\mathcal{B}_N^t}\Big(\int_{\mathbb{R}}|\xi|^\gamma|\nabla(h_N^t-h^t)|^2\Big)^{p/2}\lesssim_{R,t} o_N(1). \tag{6.63}$$

Decomposing $\nabla h_N^t(x)=\frac1N\sum_{i=1}^N\nabla g_s(x-x_{i,N}^t,\xi)$, the triangle inequality yields

$$\bigg(\int_{B_R\cap\mathcal{B}_N^t}\Big(\int_{\mathbb{R}}|\xi|^\gamma|\nabla h_N^t|^2\Big)^{p/2}\bigg)^{1/p}\lesssim\frac1N\sum_{i=1}^N\bigg(\int_{B_R}\Big(\int_{\mathbb{R}}|\xi|^\gamma|(x-x_{i,N}^t,\xi)|^{-2(s+1)}d\xi\Big)^{p/2}dx\bigg)^{1/p}.$$

A direct computation of the integral over ξ yields

$$\bigg(\int_{B_R\cap\mathcal{B}_N^t}\Big(\int_{\mathbb{R}}|\xi|^\gamma|\nabla h_N^t|^2\Big)^{p/2}\bigg)^{1/p}\lesssim\frac1N\sum_{i=1}^N\bigg(\int_{B_R}|x-x_{i,N}^t|^{\frac p2(\gamma+1-2(s+1))}dx\bigg)^{1/p}.$$

As for each i the integral over $x\in B_R$ is clearly bounded above by the same integral over $x\in B_R(x_{i,N}^t)$, we obtain

$$\bigg(\int_{B_R\cap\mathcal{B}_N^t}\Big(\int_{\mathbb{R}}|\xi|^\gamma|\nabla h_N^t|^2\Big)^{p/2}\bigg)^{1/p}\lesssim\bigg(\int_{B_R}|x|^{\frac p2(\gamma+1-2(s+1))}dx\bigg)^{1/p},$$

and hence for any $1\le p<2d/(s+d)$,

$$\int_{B_R\cap\mathcal{B}_N^t}\Big(\int_{\mathbb{R}}|\xi|^\gamma|\nabla h_N^t|^2\Big)^{p/2}\lesssim\int_{B_R}|x|^{\frac p2(\gamma+1-2(s+1))}dx=\int_{B_R}|x|^{-(s+d)p/2}dx\lesssim_R 1,$$

Now, for any $1\le p<2d/(s+d)$, choosing any $p<q<2d/(s+d)$, the Hölder inequality yields

$$\begin{aligned}\int_{B_R\cap\mathcal{B}_N^t}\Big(\int_{\mathbb{R}}|\xi|^\gamma|\nabla h_N^t|^2\Big)^{p/2}&\le|\mathcal{B}_N^t|^{1-p/q}\bigg(\int_{B_R\cap\mathcal{B}_N^t}\Big(\int_{\mathbb{R}}|\xi|^\gamma|\nabla h_N^t|^2\Big)^{q/2}\bigg)^{p/q}\\&\lesssim_R|\mathcal{B}_N^t|^{1-p/q}\lesssim_t o_N(1).\end{aligned}$$

The result (6.63) follows from this and from the Hölder inequality in the form

$$\int_{B_R\cap\mathcal{B}_N^t}\Big(\int_{\mathbb{R}}|\xi|^\gamma|\nabla h^t|^2\Big)^{p/2}\le|\mathcal{B}_N^t|^{1-p/2}\Big(\iint_{B_R\times\mathbb{R}}|\xi|^\gamma|\nabla h^t|^2\Big)^{p/2}\lesssim_{R,t} o_N(1).$$

Step 3. Conclusion.

Combining (6.62) and (6.63) for any $1\le p<2d/(s+d)$, we conclude for all $R>0$,

$$\int_{B_R}\Big(\int_{\mathbb{R}}|\xi|^\gamma|\nabla(h_N^t-h^t)|^2\Big)^{p/2}\lesssim_{R,t} o_N(1).$$

This proves $\nabla h_N^t\to\nabla h^t$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d;\mathrm{L}^2(\mathbb{R},|\xi|^\gamma d\xi))$ for any $1\le p<2d/(s+d)$. Applying the operator $-\operatorname{div}(|\xi|^\gamma\cdot)$ to both sides, we deduce $\mu_N^t\to\mu^t$ in the distributional sense on $\mathbb{R}^d\times\mathbb{R}$, and the conclusion follows. $\square$

6.2.6 Ball construction

In this section, we make the heuristics of Remark 6.2.12 rigorous, showing that for $0 \le s < 1$ the collection of small balls $\mathcal{B}_N^t(R_N^t)$ can indeed be chosen with total radius $R_N^t \to 0$ in such a way that both conditions (6.40) and (6.41) are satisfied.

Let us describe the Jerrard-Sandier ball construction, which was first introduced in [260, 379] for the analysis of the Ginzburg-Landau vortices (see also [382, Chapter 4]). We consider N disjoint small balls centered at the points $x_{i,N}^t$ with equal radii (smaller than $\eta_N/2$), and we grow their radii by the same multiplicative factor. At some point during this growth process, two (or more) balls may become tangent to one another. We then merge them into a larger ball: if tangent balls are of the form $B(a_i, r_i)$, we merge them into $B(\sum_i a_i r_i / \sum_i r_i, \sum_i r_i)$. If the resulting ball intersects other balls, we proceed to another similar merging, and so on, until all the balls are again disjoint. Then again we grow all the resulting radii by a multiplicative factor, etc., and we stop when the total radius R is the one desired.

As we will see, condition (6.41) is easily checked as a direct consequence of the above ball construction, so that we may focus on the validity of condition (6.40). For that purpose we need to study integrals of the form $\int_{\mathcal{B}_N^t(R)\times\mathbb{R}} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2$ with $R > 0$. The basic tool is the following lower bound, which is a refinement of [362, Lemma 2.2]. In the sequel, for $x \in \mathbb{R}^d$ and $t > 0$, we denote by $B'(x,t)$ the ball of radius t centered at $(x, 0)$ in the extended space $\mathbb{R}^d \times \mathbb{R}$, and we set $B_t' := B'(0,t)$.

Lemma 6.2.14 (Embryo of a lower bound)**.** *Let $d-2 < s < d$, $s \ge 0$, let $R > r$, let $(z_i)_{i=1}^k$ be a collection of points inside the ball B_r, and let $(z_{k+i})_{i=1}^l$ be a collection of points outside the ball B_R. Then*

$$\int_{B_R'\setminus B_r'} |\xi|^\gamma \Big| \sum_{i=1}^{k+l} \nabla g_s(x - z_i, \xi) \Big|^2 dx d\xi \ge k^2 (g_s(r) - g_s(R)). \tag{6.64}$$

The same remains true if point charges are smeared out on small spheres around them, that is, if g_s is replaced by $g_{s,\eta}$ with $\eta < d(\{z_i\}_{i=1}^{k+l}, B_R \setminus B_r)$. In particular, for any z_1, $R > \eta > 0$, and any collection $(z_i)_{i=2}^{1+l}$ of points outside the ball $B(z_1, R+\eta)$,

$$\int_{B'(z_1,R)} |\xi|^\gamma \Big| \sum_{i=1}^{1+l} \nabla g_{s,\eta}(x - z_i, \xi) \Big|^2 dx d\xi \ge g_s(\eta) - g_s(R). \tag{6.65}$$

$\Diamond$

Proof. We split the proof into two steps.

Step 1. Explicit value of $c_{d,s}$.

We claim that the normalization constant $c_{d,s}$ for the Riesz kernel g_s is given by the following formula, in terms of the beta function $\mathrm{B}(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ and of the measure ω_{d-1} of the unit sphere of dimension $d-1$,

$$c_{d,s} = s\omega_{d-1}\,\mathrm{B}\left(\frac{s+2-d}{2}, \frac{d}{2}\right). \tag{6.66}$$

Integrating the equality $-\operatorname{div}(|\xi|^\gamma \nabla g_s) = \delta_0$ on the infinite cylinder $C_0 := B_1 \times \mathbb{R}$ in $\mathbb{R}^d \times \mathbb{R}$, we find by integration by parts

$$\begin{aligned}-1 = \int_{C_0} \operatorname{div}(|\xi|^\gamma \nabla g_s) &= \int_{\partial C_0} |\xi|^\gamma n \cdot \nabla g_s \\ &= \int_{-\infty}^{\infty} \int_{\partial B_1} |\xi|^\gamma \partial_r g_s(u,\xi) d\sigma(u) d\xi = 2\omega_{d-1} \int_0^\infty \xi^\gamma \partial_r g_s(1,\xi) d\xi.\end{aligned}$$

Since by definition $g_s(x,\xi) = c_{d,s}^{-1}(|x|^2 + |\xi|^2)^{-s/2}$, computing the radial derivative yields

$$c_{d,s} = 2s\omega_{d-1}\int_0^\infty \xi^\gamma(1+\xi^2)^{-s/2-1}d\xi = s\omega_{d-1}\int_0^\infty \xi^{(\gamma-1)/2}(1+\xi)^{-s/2-1}d\xi.$$

The result (6.66) then easily follows using the formula $\mathrm{B}(a,b) = \int_0^\infty t^{a-1}(1+t)^{-a-b}dt$ for all $a,b>0$.

Step 2. Conclusion.

Set $\mu_{k,l} := \sum_{i=1}^{k+l}\delta_{z_i}$. The Cauchy-Schwarz inequality yields

$$\begin{aligned}\int_{B'_R\setminus B'_r}|\xi|^\gamma|\nabla g_s*\mu_{k,l}|^2 &= \int_r^R dt\int_{\partial B'_t}|\xi|^\gamma|\nabla g_s*\mu_{k,l}|^2\\ &\ge \int_r^R dt\Big(\int_{\partial B'_t}|\xi|^\gamma\Big)^{-1}\Big(\int_{\partial B'_t}|\xi|^\gamma n\cdot\nabla g_s*\mu_{k,l}\Big)^2,\end{aligned}$$

where for all $r\le t\le R$ an integration by parts then leads to

$$\int_{\partial B'_t}|\xi|^\gamma n\cdot\nabla g_s*\mu_{k,l} = \int_{B'_t}\mathrm{div}\,(|\xi|^\gamma\nabla g_s*\mu_{k,l}) = -\mu_{k,l}(B_t) = -k,$$

hence

$$\int_{B'_R\setminus B'_r}|\xi|^\gamma|\nabla g_s*\mu_{k,l}|^2 \ge k^2\int_r^R dt\Big(\int_{\partial B'_t}|\xi|^\gamma\Big)^{-1}.$$

In order to compute this last integral, we use spherical coordinates,

$$\begin{aligned}\int_{\partial B'_t}|\xi|^\gamma &= t^{s+1}\omega_{d-1}\int_0^\pi(\sin\theta)^{d-1}|\cos\theta|^\gamma d\theta = t^{s+1}\omega_{d-1}\int_{-1}^1(1-u^2)^{(d-2)/2}|u|^\gamma du\\ &= t^{s+1}\omega_{d-1}\int_0^1(1-u)^{(d-2)/2}u^{(\gamma-1)/2}du = t^{s+1}\omega_{d-1}\,\mathrm{B}\left(\frac{s+2-d}{2},\frac{d}{2}\right),\end{aligned}$$

where the last equality follows from the formula $\mathrm{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ for all $a,b>0$. By Step 1, this last expression is nothing but $t^{s+1}c_{d,s}/s$, so that we may conclude

$$\int_{B'_R\setminus B'_r}|\xi|^\gamma|\nabla g_s*\mu_{k,l}|^2 \ge k^2sc_{d,s}^{-1}\int_r^R t^{-s-1}dt = k^2c_{d,s}^{-1}(r^{-s}-R^{-s}) = k^2(g_s(r)-g_s(R)). \qquad \square$$

With this result at hand, arguing as in [382, Chapter 4], we may now deduce the following lower bound for the energy on the balls of the collection $\mathcal{B}_N^t(R)$. For logarithmic interactions (thus in particular for the Ginzburg-Landau vortices, as treated in [382, Chapter 4]), a particularly simple additive structure shows up, simplifying computations a lot; here we show that the same result still holds for all $s\le 1$.

Proposition 6.2.15 (Lower bound)**.** *Let $d-2<s<d$, $s\ge 0$, and let $R>0$. If $s\le 1$, then, for all $0<\eta<\eta_N\wedge(R/N)$,*

$$\int_{\mathcal{B}_N^t(R)\times\mathbb{R}}|\xi|^\gamma|\nabla h_{N,\eta}^t|^2 \ge \frac{1}{N}(g_s(\eta)-g_s(R/N)). \tag{6.67}$$

$\lozenge$

Proof. For all $R > 0$, we prove the following: if $B(y, r)$ is a ball belonging to the collection $\mathcal{B}_N^t(R)$ and containing n of the particles $(x_{i,N}^t)_{i=1}^N$, then

$$\int_{B'(y,r)} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 \geq \frac{n}{N^2}(g_s(\eta) - g_s(R/N)). \tag{6.68}$$

The desired result (6.67) follows by summing the inequalities (6.68) associated with each ball $B(y, r)$ of the collection $\mathcal{B}_N^t(R)$, and noting that $B'(y, r) \subset B(y, r) \times \mathbb{R}$. We prove (6.68) by induction: we first show that it holds when $B(y, r)$ contains only one particle $x_{i,N}^t$, and then that it is preserved through the growth process.

First, suppose that $B(y, r)$ is a ball of $\mathcal{B}_N^t(R)$ and contains only one particle $x_{i,N}^t$. By definition we must have $B(y, r) = B(x_{i,N}^t, r)$ and $x_{j,N}^t \notin B(y, r + \eta)$ for all $j \neq i$. Lemma 6.2.14 in the form of (6.65) then yields

$$\int_{B'(y,r)} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 \geq \frac{1}{N^2}(g_s(\eta) - g_s(r)).$$

This proves (6.68) when $B(y, r)$ contains only one particle $x_{i,N}^t$, since in that case we have by definition $r = R/N$.

Now we need to prove that (6.68) is preserved by the growth process, i.e. that it remains true through both expansion and merging of balls. On the one hand, suppose that, for some $R > 0$, $B(y, r)$ is a ball of $\mathcal{B}_N^t(R)$ for which (6.68) holds, and suppose that $B(y, r)$ inflates into $B(y, \alpha r)$ without merging when passing from $\mathcal{B}_N^t(R)$ to $\mathcal{B}_N^t(\alpha R)$ for some $\alpha > 1$. Let n denote the number of particles in $B(y, r)$. By definition, $B(y, \alpha r)$ contains the same number of particles, and the choice of η small enough ensures that no particle may lie in the annulus $B(y, \alpha r + \eta) \setminus B(y, \alpha r)$. Hence, combining (6.68) for $B(y, r)$ and Lemma 6.2.14 in the form (6.64), we obtain

$$\begin{aligned}\int_{B'(y,\alpha r)} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 &= \int_{B'(y,r)} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 + \int_{B'(y,\alpha r)\setminus B'(y,r)} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 \\ &\geq \frac{n}{N^2}(g_s(\eta) - g_s(R/N)) + \frac{n^2}{N^2}(g_s(r) - g_s(\alpha r)).\end{aligned}$$

Noting that by definition $r = nR/N$, using the choice $s \leq 1$, and noting that $g_s(R/N) - g_s(\alpha R/N) \geq 0$, we deduce

$$\begin{aligned}\int_{B'(y,\alpha r)} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 &\geq \frac{n}{N^2}(g_s(\eta) - g_s(R/N)) + \frac{n^{2-s}}{N^2}(g_s(R/N) - g_s(\alpha R/N)) \\ &\geq \frac{n}{N^2}(g_s(\eta) - g_s(R/N)) + \frac{n}{N^2}(g_s(R/N) - g_s(\alpha R/N)) \\ &= \frac{n}{N^2}(g_s(\eta) - g_s(\alpha R/N)),\end{aligned}$$

so that $B(y, \alpha r)$ also satisfies (6.68).

On the other hand, suppose that $B(y_i, r_i)$, $i = 1, \ldots, k$, are k disjoint balls of $\mathcal{B}_N^t(R^-)$ for some $R > 0$, suppose that each of them satisfies (6.68), and suppose that these balls are merged by the growth process into a larger ball $B(y, r)$, which is then disjoint of all other balls of the collection $\mathcal{B}_N^t(R)$. Denoting by n_i the number of points in $B(y_i, r_i)$, we then find

$$\int_{B'(y,r)} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 \geq \sum_{i=1}^k \int_{B'(y_i,r_i)} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 \geq \frac{1}{N^2}\left(\sum_{i=1}^k n_i\right)(g_s(\eta) - g_s(R/N)),$$

so that $B(y, r)$ also satisfies (6.68). This completes the proof. □

We are now in position to prove that both conditions (6.40) and (6.41) can be satisfied whenever $s < 1$, thus finishing the proof of Theorem 6.1.5.

Corollary 6.2.16 (Checking conditions (6.40) and (6.41))**.** *Let $d-2 < s < d$, $s \geq 0$ with $s < 1$. If $R \mapsto \mathcal{B}_N^t(R)$ is constructed as above, then the conditions* (6.40) *and* (6.41) *are automatically satisfied for any choice $N^{-(1-s)/s} \ll R_N^t \ll 1$ if $0 < s < 1$, and for any choice $e^{-No_N(1)} \lesssim_t R_N^t \ll 1$ if $s = 0$.* $\Diamond$

Proof. On the one hand, Proposition 6.2.15 gives

$$\lim_{\eta \downarrow 0} \left(\int_{\mathcal{B}_N^t(R) \times \mathbb{R}} |\xi|^\gamma |\nabla h_{N,\eta}^t|^2 - \frac{1}{N} g_s(\eta) \right) \geq -\frac{1}{N} g_s(R/N).$$

On the other hand, since by definition $\bigcup_{i=1}^N B(x_{i,N}^t, R/N) \subset \mathcal{B}_N^t(R)$, we may estimate

$$\frac{1}{N^2} \sum_{i=1}^N g_s^+(d(x_{i,N}^t, \partial \mathcal{B}_N^t(R))) \lesssim \frac{1}{N} g_s^+(R/N).$$

Therefore, both conditions (6.40) and (6.41) are satisfied if we choose R_N^t such that $\frac{1}{N} g_s^+(R_N^t/N) \ll 1$, and the conclusion follows. $\square$

Chapter 7

Well-posedness for mean-field evolutions arising in superconductivity

In Chapter 8 below, we establish mean-field limit results for the evolution of the supercurrent density in a (2D section of a) type-II superconductor with pinning and with imposed electric current. Since in certain regimes the corresponding mean-field equations appear to be new in the literature, we establish in the present chapter a complete well-posedness theory: we prove global existence results, consider general vortex-sheet initial data, and investigate the uniqueness and regularity properties of the solution. For some choice of parameters, the equation under investigation coincides with the so-called lake equation from 2D shallow water fluid dynamics, and our analysis then leads to a new existence result for rough initial data.

This chapter corresponds to the article [159], to the exception of the global results for the degenerate case in Section 7.6, which have been obtained in collaboration with Julian Fischer.

Contents

7.1 Introduction

7.1.1 General overview

We study the well-posedness of the following two evolution models coming from the mean-field limit equations of Ginzburg-Landau vortices: first, for $\alpha \geq 0$, $\beta \in \mathbb{R}$, we consider the "incompressible" flow

$$\partial_t \mathrm{v} = \nabla \mathrm{p} - \alpha(\Psi + \mathrm{v}) \operatorname{curl} \mathrm{v} + \beta(\Psi + \mathrm{v})^\perp \operatorname{curl} \mathrm{v}, \qquad \operatorname{div}(a\mathrm{v}) = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{7.1}$$

and second, for $0 \leq \lambda < \infty$, $\alpha > 0$, $\beta \in \mathbb{R}$, we consider the "compressible" flow

$$\partial_t \mathrm{v} = \lambda \nabla(a^{-1} \operatorname{div}(a\mathrm{v})) - \alpha(\Psi + \mathrm{v}) \operatorname{curl} \mathrm{v} + \beta(\Psi + \mathrm{v})^\perp \operatorname{curl} \mathrm{v}, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{7.2}$$

with $\mathrm{v} : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\operatorname{curl} \mathrm{v} \geq 0$, where $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a given forcing vector field, and where the weight $a := e^h$ is determined by a given "pinning potential" $h : \mathbb{R}^2 \to \mathbb{R}$. More precisely, we investigate existence, uniqueness, and regularity, both locally and globally in time, for the associated Cauchy problems; we also consider vortex-sheet initial data, and we study the degenerate case $\lambda = 0$ as well. Note that the incompressible model (7.1) can be seen as the limiting case $\lambda = \infty$ of the family of compressible models (7.2). As established in Chapter 8 below, these equations are obtained in certain regimes as the mean-field evolution of the supercurrent density in a (2D section of a) type-II superconductor described by the 2D Ginzburg-Landau equation with pinning and with imposed electric current — but without gauge and in whole space, for simplicity. In this context, the cases $\lambda = \infty$, $0 < \lambda < \infty$, and $\lambda = 0$ correspond respectively to a low, an intermediate, and a high vortex density regime. Note that in the parabolic case $\alpha > 0$, $\beta = 0$, the incompressible model (7.1) can be seen as a Wasserstein gradient flow for the vorticity $\operatorname{curl} \mathrm{v}$, but a common gradient flow structure seems to be missing for the whole family of equations (7.2) with $\lambda \in [0, \infty]$. In the conservative case $\alpha = 0$ with $\Psi = 0$, the incompressible model (7.1) takes the form of the so-called lake equation from 2D shallow water fluid dynamics [217, p.235] (see also [96, 97]), which reduces to the usual 2D Euler equation if the weight a is constant.

7.1.2 Relation to previous works

In the simpler parabolic case without pinning and forcing, $\alpha = 1$, $\beta = 0$, $a = 1$, $\Psi = 0$, equation (7.1) for the mean-field supercurrent density v takes on the following guise,

$$\partial_t \mathrm{v} = \nabla \mathrm{p} - \mathrm{v} \operatorname{curl} \mathrm{v}, \qquad \operatorname{div} \mathrm{v} = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{7.3}$$

or alternatively, in terms of the mean-field vortex density $\mathrm{m} := \operatorname{curl} \mathrm{v} \geq 0$, noting that the incompressibility constraint $\operatorname{div} \mathrm{v} = 0$ allows to write $\mathrm{v} = \nabla^\perp \triangle^{-1} \mathrm{m}$,

$$\partial_t \mathrm{m} = \operatorname{div}(\mathrm{m} \nabla (-\triangle)^{-1} \mathrm{m}), \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2. \tag{7.4}$$

This simplified model actually describes the mean-field limit of the gradient-flow evolution of any particle system with Coulomb interactions (see indeed (6.7) in Chapter 6). As such, it is related to nonlocal aggregation and swarming models, which have attracted a lot of mathematical interest in recent years (see e.g. [55, 102] and the references therein); these models consist in replacing the Coulomb potential $(-\triangle)^{-1}$ by a convolution with a more general kernel corresponding to an attractive (rather than repulsive) nonlocal interaction. Equation (7.4) was first studied by Lin and Zhang [304], who established global existence for vortex-sheet initial data $\mathrm{m}|_{t=0} \in \mathcal{P}(\mathbb{R}^2)$, and uniqueness in some Zygmund space. To prove global existence for such rough initial data, they proceed by regularization of the data, then pass to the limit in the equation using the compactness given by some very strong a priori estimates obtained by ODE type arguments. As our main source of inspiration, their approach is described in more detail in the sequel. When viewing (7.4) as a mean-field model for the motion of the 2D Ginzburg-Landau vortices in a superconductor, there is also interest in sign-changing solutions and the correct model then rather takes on the form of the following Chapman-Rubinstein-Schatzman-E equation [173, 111] (see also the discussions in Sections 6.1.3 and 7.1.5),

$$\partial_t \mathrm{m} = \operatorname{div}(|\mathrm{m}|\nabla(-\triangle)^{-1}\mathrm{m}), \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{7.5}$$

for which global existence and uniqueness have been investigated in [157, 316]. In [18, 17], using an energy approach where the equation is seen as a formal gradient flow in the Wasserstein space of probability measures à la Otto [353], made rigorous by the minimizing movement approach of Jordan, Kinderlehrer, and Otto [266] (see also [16]), analogues of equations (7.4)–(7.5) were studied in a 2D bounded domain, taking into account the possibility of mass entering or exiting the domain. In the case of nonnegative vorticity $\mathrm{m} \geq 0$, essentially the same existence and uniqueness results as those by Lin and Zhang are established in that setting in [18]. In the case $\mathrm{m} \geq 0$ on the whole plane, still a different approach was developed by Serfaty and Vázquez [398], where equation (7.4) is obtained as a limit of nonlocal diffusions, and where uniqueness is further established for bounded solutions using transport arguments à la Loeper [307]. Note that no uniqueness is expected to hold for general measure solutions of (7.4) (see [18, Section 8]). In the sequel, we focus on the case $\mathrm{m} \geq 0$ on the whole plane $\mathbb{R}^2$.

In the context of superfluidity, a conservative counterpart of the usual parabolic Ginzburg-Landau equation is used as a mesoscopic model, and there is also strong physical interest in rather considering the corresponding "mixed-flow" (or "complex") Ginzburg-Landau equation. The above mean-field equation for the supercurrent density (7.3) is then replaced by the following, for $\alpha \geq 0$, $\beta \in \mathbb{R}$,

$$\partial_t \mathrm{v} = \nabla \mathrm{p} - \alpha \mathrm{v} \operatorname{curl} \mathrm{v} + \beta \mathrm{v}^\perp \operatorname{curl} \mathrm{v}, \qquad \operatorname{div} \mathrm{v} = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2. \tag{7.6}$$

Note that in the conservative case $\alpha = 0$, this equation is equivalent to the 2D Euler equation, as is clear from the identity $\mathrm{v}^\perp \operatorname{curl} \mathrm{v} = (\mathrm{v} \cdot \nabla)\mathrm{v} - \frac{1}{2}\nabla|\mathrm{v}|^2$. This model (7.6) is thus seen as a linear interpolation between the gradient-flow equation (7.4) (obtained for $\alpha = 1$, $\beta = 0$) and its conservative counterpart that is the 2D Euler equation (obtained for $\alpha = 0$, $\beta = 1$). The theory for the 2D Euler equation has been well-developed for a long time: global existence for vortex-sheet initial data is due to Delort [143], while the only known uniqueness result, due to Yudovich [427], holds in the class of bounded vorticity (see also [45] and the references therein). Regarding the general model (7.6), global existence and uniqueness results for smooth solutions are easily obtained by standard methods (see e.g. [116]). Although not surprising, global existence for this model is further established here for vortex-sheet initial data, as well as uniqueness in the class of bounded vorticity.

The first rigorous deductions of such (macroscopic) mean-field models (7.6) from the (mesoscopic) 2D Ginzburg-Landau equation are due to [281, 263, 395]. However, as discovered by Serfaty [395], in some regimes with $\alpha > 0$, the limiting model (7.6) is no longer correct, and must be replaced by the following "compressible" flow,

$$\partial_t \mathrm{v} = \lambda \nabla(\operatorname{div} \mathrm{v}) - \alpha \mathrm{v} \operatorname{curl} \mathrm{v} + \beta \mathrm{v}^\perp \operatorname{curl} \mathrm{v}, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{7.7}$$

for some $\lambda > 0$. We further show in Chapter 8 that the degenerate case $\lambda = 0$ also appears as the correct mean-field evolution in some other regimes. To our knowledge, this compressible model is completely new in the literature. In [395, Appendix B], only local-in-time existence and uniqueness of smooth solutions are proven in the non-degenerate case $\lambda > 0$, using a standard iterative method. In the present chapter, in the parabolic regime $\alpha = 1$, $\beta = 0$, global existence with vortex-sheet data is further established in the non-degenerate case $\lambda > 0$, while in the degenerate case $\lambda = 0$ global existence with bounded data is obtained by exploiting the particular scalar structure of the corresponding equation.

The general equations (7.1)–(7.2) are derived in some regimes in Chapter 8 as the mean-field evolution of the supercurrent density v in the 2D Ginzburg-Landau model with pinning and with imposed electric current, where the forcing Ψ is then decomposed as $\Psi := F^\perp - \nabla^\perp h$ in terms of the pinning force $-\nabla h$ and of some vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ related to the imposed electric current. These equations (7.1)–(7.2) are seen as inhomogeneous versions of (7.6)–(7.7) with forcing. Since they are new in the literature (except in the conservative incompressible case discussed below), we wish to provide in the present chapter a detailed discussion of local and global existence, uniqueness, and regularity issues.

In the conservative regime $\alpha = 0$, $\beta = 1$, the incompressible model (7.1) takes the form of the following inhomogeneous version of the 2D Euler equation: using the identity $\mathrm{v}^\perp \operatorname{curl} \mathrm{v} = (\mathrm{v} \cdot \nabla)\,\mathrm{v} - \frac{1}{2}\nabla|\mathrm{v}|^2$, and setting $\tilde{\mathrm{p}} := \mathrm{p} - \frac{1}{2}|\mathrm{v}|^2$,

$$\partial_t \mathrm{v} = \nabla \tilde{\mathrm{p}} + \Psi^\perp \operatorname{curl} \mathrm{v} + (\mathrm{v} \cdot \nabla)\,\mathrm{v}, \qquad \operatorname{div}(a\mathrm{v}) = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2. \tag{7.8}$$

In the context of 2D fluid mechanics, this conservative equation is known as the lake equation [217, p.235] (see also [96, 97]): the pinning weight a corresponds to the effect of a varying depth in shallow water [351], while the forcing Ψ is similar to a background flow. This equation has been studied in a bounded domain by Levermore, Oliver, and Titi [297, 298, 351] (see also [87]), who established global existence for L^2 initial vorticity, as well as uniqueness in the class of bounded vorticity. In the present paper, we improve on these previous results by establishing for equation (7.8) on the whole plane $\mathbb{R}^2$ a global existence result for initial data in $\mathrm{L}^q(\mathbb{R}^2)$ with $q > 1$. It should be clear from the Delort type identity (7.11) below that inhomogeneities give rise to important difficulties: indeed, for h non-constant, the first term $-\frac{1}{2}|\mathrm{v}|^2\nabla^\perp h$ in (7.11) does not vanish and is clearly not weakly continuous as a function of v (although the second term is, as in Delort's classical theory for the 2D Euler equation [143]). Because of this difficulty and of the lack of strong enough a priori estimates for the conservative equation (7.8), we do not manage to reach vortex-sheet initial data in that case, as opposed to the simpler situation of the 2D Euler equation.

7.1.3 Notions of weak solutions

We first introduce the vorticity formulation of equations (7.1) and (7.2), which will be more convenient to work with. Setting $\mathrm{m} := \operatorname{curl} \mathrm{v}$ and $\mathrm{d} := \operatorname{div}(a\mathrm{v})$, each of these equations may be rewritten as a nonlinear nonlocal transport equation for the vorticity m,

$$\partial_t \mathrm{m} = \operatorname{div}\Big(\mathrm{m}\,\big(\alpha(\Psi + \mathrm{v})^\perp + \beta(\Psi + \mathrm{v})\big)\Big), \quad \operatorname{curl} \mathrm{v} = \mathrm{m}, \quad \operatorname{div}(a\mathrm{v}) = \mathrm{d}, \tag{7.9}$$

where in the incompressible case (7.1) we have $\mathrm{d} := 0$, while in the compressible case (7.2) d is the solution of the following transport-diffusion equation (which is highly degenerate as $\lambda = 0$),

$$\partial_t \mathrm{d} - \lambda \triangle \mathrm{d} + \lambda \operatorname{div}(\mathrm{d}\nabla h) = \operatorname{div}\Big(a\mathrm{m}\,\big(-\alpha(\Psi + \mathrm{v}) + \beta(\Psi + \mathrm{v})^\perp\big)\Big). \tag{7.10}$$

Let us now precisely define our notions of weak solutions for (7.1) and (7.2).

Definition 7.1.1. Let $h, \Psi \in \mathrm{L}^\infty(\mathbb{R}^2)$, $T > 0$, and set $a := e^h$.

(a) Given $\mathrm{v}^\circ \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$ with $\mathrm{m}^\circ = \operatorname{curl} \mathrm{v}^\circ \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2)$ and $\mathrm{d}^\circ := \operatorname{div}(a\mathrm{v}^\circ) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$, we say that v is a *weak solution of* (7.2) on $[0, T) \times \mathbb{R}^2$ with initial data v°, if $\mathrm{v} \in \mathrm{L}^2_{\mathrm{loc}}([0, T) \times \mathbb{R}^2)^2$ satisfies $\mathrm{m} := \operatorname{curl} \mathrm{v} \in \mathrm{L}^1_{\mathrm{loc}}([0, T); \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2))$, $\mathrm{d} := \operatorname{div}(av) \in \mathrm{L}^2_{\mathrm{loc}}([0, T); \mathrm{L}^2(\mathbb{R}^2))$, $|\mathrm{v}|^2\,\mathrm{m} \in \mathrm{L}^1_{\mathrm{loc}}([0, T); \mathrm{L}^1(\mathbb{R}^2))$ (hence also $\mathrm{mv} \in \mathrm{L}^1_{\mathrm{loc}}([0, T) \times \mathbb{R}^2)^2$), and satisfies (7.2) in the distributional sense, that is, for all $\psi \in C_c^\infty([0,T)\times\mathbb{R}^2)^2$,

$$\int_{\mathbb{R}^d} \psi(0,\cdot)\cdot \mathrm{v}^\circ + \iint_{\mathbb{R}^+\times\mathbb{R}^d} v\cdot\partial_t\psi$$
$$= \lambda \iint_{\mathbb{R}^+\times\mathbb{R}^d} a^{-1}\,\mathrm{d}\,\mathrm{div}\,\psi + \iint_{\mathbb{R}^+\times\mathbb{R}^d} \psi\cdot\big(\alpha(\Psi+\mathrm{v}) - \beta(\Psi+\mathrm{v})^\perp\big)\,\mathrm{m}\,.$$

(b) Given $\mathrm{v}^\circ \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$ with $\mathrm{m}^\circ := \mathrm{curl}\,\mathrm{v}^\circ \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2)$ and $\mathrm{div}\,(a\mathrm{v}^\circ) = 0$, we say that v is a *weak solution of* (7.1) on $[0,T)\times\mathbb{R}^2$ with initial data v°, if $\mathrm{v} \in \mathrm{L}^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)^2$ satisfies $\mathrm{m} := \mathrm{curl}\,\mathrm{v} \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2))$, $|\mathrm{v}|^2\,\mathrm{m} \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathrm{L}^1(\mathbb{R}^2)^2)$ (hence also $\mathrm{mv} \in \mathrm{L}^1_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)^2)$, $\mathrm{div}\,(a\mathrm{v}) = 0$ in the distributional sense, and satisfies the vorticity formulation (7.9) in the distributional sense, that is, for all $\psi \in C_c^\infty([0,T)\times\mathbb{R}^2)$,

$$\int_{\mathbb{R}^d} \psi(0,\cdot)\,\mathrm{m}^\circ + \iint_{\mathbb{R}^+\times\mathbb{R}^d} \mathrm{m}\,\partial_t\psi = \iint_{\mathbb{R}^+\times\mathbb{R}^d} \nabla\psi\cdot\big(\alpha(\Psi+\mathrm{v})^\perp + \beta(\Psi+\mathrm{v})\big)\,\mathrm{m}\,.$$

(c) Given $\mathrm{v}^\circ \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$ with $\mathrm{m}^\circ := \mathrm{curl}\,\mathrm{v}^\circ \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2)$ and $\mathrm{div}\,(a\mathrm{v}^\circ) = 0$, we say that v is a *very weak solution of* (7.1) on $[0,T)\times\mathbb{R}^2$ with initial data v°, if $\mathrm{v} \in \mathrm{L}^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)^2$ satisfies $\mathrm{m} := \mathrm{curl}\,\mathrm{v} \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2))$, $\mathrm{div}\,(a\mathrm{v}) = 0$ in the distributional sense, and satisfies, for all $\psi \in C_c^\infty([0,T)\times\mathbb{R}^2)$,

$$\int_{\mathbb{R}^d} \psi(0,\cdot)\,\mathrm{m}^\circ + \iint_{\mathbb{R}^+\times\mathbb{R}^d} \mathrm{m}\,\partial_t\psi = \iint_{\mathbb{R}^+\times\mathbb{R}^d} (\alpha\nabla\psi + \beta\nabla^\perp\psi)\cdot\Big(\Psi^\perp\,\mathrm{m} + \frac{1}{2}|\mathrm{v}|^2\nabla h\Big)$$
$$- \iint_{\mathbb{R}^+\times\mathbb{R}^d} aS_{\mathrm{v}} : \nabla\big(a^{-1}(\alpha\nabla\psi + \beta\nabla^\perp\psi)\big),$$

in terms of the stress-energy tensor $S_{\mathrm{v}} := \mathrm{v}\otimes\mathrm{v} - \frac{1}{2}\,\mathrm{Id}\,|\mathrm{v}|^2$. ◊

Remarks 7.1.2.

(i) Weak solutions of (7.2) are defined directly from (7.2), and satisfy in particular the vorticity formulation (7.9)–(7.10) in the distributional sense. Regarding weak solutions of (7.1), they are rather defined in terms of the vorticity formulation (7.9), in order to avoid compactness and regularity issues related to the pressure p. Nevertheless, if v is a weak solution of (7.1) in the above sense, then under mild regularity assumptions we may use the formula $\mathrm{v} = a^{-1}\nabla^\perp(\mathrm{div}\,a^{-1}\nabla)^{-1}\,\mathrm{m}$ to deduce that v actually satisfies (7.1) in the distributional sense on $[0,T)\times\mathbb{R}^2$ for some distribution p (cf. Lemma 7.2.8 below for detail).

(ii) The definition (c) of a very weak solution of (7.1) is motivated as follows (see also the notion of "general weak solutions" of (7.4) in [304]). In the purely conservative case $\alpha = 0$, there are too few a priori estimates to make sense of the product mv. As is now common in 2D fluid mechanics (see e.g. [116]), the idea is to reinterpret this product in terms of the stress-energy tensor S_{v}, using the following identity: given $\mathrm{div}\,(a\mathrm{v}) = 0$, we have for smooth enough fields

$$\mathrm{mv} = -\frac{1}{2}|\mathrm{v}|^2\nabla^\perp h - a^{-1}(\mathrm{div}\,(aS_{\mathrm{v}}))^\perp, \tag{7.11}$$

where the right-hand side now makes sense in $\mathrm{L}^1_{\mathrm{loc}}([0,T);W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$ whenever $\mathrm{v} \in \mathrm{L}^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)^2$. In particular, if $\mathrm{m} \in \mathrm{L}^p_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)$ and $\mathrm{v} \in \mathrm{L}^{p'}_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)$ for some $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, then the product mv makes perfect sense and the above identity (7.11) holds in the distributional sense, hence in that case v is a weak solution of (7.1) whenever it is a very weak solution. In reference to Delort's work [143], identity (7.11) is henceforth called an "(inhomogeneous) Delort type identity". ◊

7.1.4 Main results

Global existence and regularity results are summarized in the following theorem. Our approach relies on proving a priori estimates for the vorticity m in $\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$. For the compressible model (7.2), such estimates are only obtained in the parabolic regime, hence our limitation to that setting. In parabolic cases, particularly strong estimates are available, and existence is then established even for vortex-sheet initial data, thus completely extending the known theory for (7.4) (see [304, 398]). Note that the additional exponential growth in the boundedness effect (7.12) below is only due to the forcing Ψ. In the conservative incompressible case, the situation is the most delicate because of a lack of strong enough a priori estimates, and only existence of very weak solutions is expected and proven. As is standard in 2D fluid mechanics (see e.g. [116]), the natural space for the solution v is $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar{\mathrm{v}}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ for a given smooth reference field $\bar{\mathrm{v}}^\circ:\mathbb{R}^2\to\mathbb{R}^2$.

Theorem 7.1.3 (Global existence). *Let $\lambda>0$, $\alpha\ge 0$, $\beta\in\mathbb{R}$, $h,\Psi\in W^{1,\infty}(\mathbb{R}^2)^2$, and set $a:=e^h$. Let $\bar{\mathrm{v}}^\circ\in W^{1,\infty}(\mathbb{R}^2)^2$ be some reference map with $\bar{\mathrm{m}}^\circ:=\operatorname{curl}\bar{\mathrm{v}}^\circ\in\mathcal{P}\cap H^{s_0}(\mathbb{R}^2)$ for some $s_0>1$, and with either $\operatorname{div}(a\bar{\mathrm{v}}^\circ)=0$ in the case (7.1), or $\bar{\mathrm{d}}^\circ:=\operatorname{div}(a\bar{\mathrm{v}}^\circ)\in H^{s_0}(\mathbb{R}^2)$ in the case (7.2). Let $\mathrm{v}^\circ\in\bar{\mathrm{v}}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2$, with $\mathrm{m}^\circ:=\operatorname{curl}\mathrm{v}^\circ\in\mathcal{P}(\mathbb{R}^2)$, and with either $\operatorname{div}(a\mathrm{v}^\circ)=0$ in the case (7.1), or $\mathrm{d}^\circ:=\operatorname{div}(a\mathrm{v}^\circ)\in\mathrm{L}^2(\mathbb{R}^2)$ in the case (7.2). The following hold:*

(i) Parabolic compressible case (that is, (7.2) with $\alpha>0$, $\beta=0$):
There exists a weak solution $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar{\mathrm{v}}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data v°, with $\mathrm{m}:=\operatorname{curl}\mathrm{v}\in\mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}(\mathbb{R}^2))$ and $\mathrm{d}:=\operatorname{div}(a\mathrm{v})\in\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^2))$, and with

$$\|\mathrm{m}^t\|_{\mathrm{L}^\infty}\le(\alpha t)^{-1}+C\alpha^{-1}e^{Ct},\qquad\text{for all }t>0,\tag{7.12}$$

where the constant $C>0$ depends only on an upper bound on α, $|\beta|$, and $\|(h,\Psi)\|_{W^{1,\infty}}$. Moreover, if $\mathrm{m}^\circ\in\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$, then $\mathrm{m}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^q(\mathbb{R}^2))$.

(ii) Parabolic incompressible case (that is, (7.1) with $\alpha>0$, $\beta=0$, or with $\alpha>0$, $\beta\in\mathbb{R}$, h constant):
There exists a weak solution $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar{\mathrm{v}}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data v°, with $\mathrm{m}:=\operatorname{curl}\mathrm{v}\in\mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}(\mathbb{R}^2))$, and with the boundedness effect (7.12). Moreover, if $\mathrm{m}^\circ\in\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$, then $\mathrm{m}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^q(\mathbb{R}^2))\cap\mathrm{L}^{q+1}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{q+1}(\mathbb{R}^2))$.

(iii) Mixed-flow incompressible case (that is, (7.1) with $\alpha>0$, $\beta\in\mathbb{R}$):
If $\mathrm{m}^\circ\in\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$, there exists a weak solution $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar{\mathrm{v}}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data v°, and with $\mathrm{m}:=\operatorname{curl}\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{P}\cap\mathrm{L}^q(\mathbb{R}^2))\cap\mathrm{L}^{q+1}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{q+1}(\mathbb{R}^2))$.

(iv) Conservative incompressible case (that is, (7.1) with $\alpha=0$, $\beta\in\mathbb{R}$):
If $\mathrm{m}^\circ\in\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$, there exists a very weak solution $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar{\mathrm{v}}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data v°, and with $\mathrm{m}:=\operatorname{curl}\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{P}\cap\mathrm{L}^q(\mathbb{R}^2))$. This is a weak solution whenever $q\ge 4/3$.

We set $\mathrm{d}^\circ,\bar{\mathrm{d}}^\circ,\mathrm{d}:=0$ in the incompressible case (7.1). If in addition $\mathrm{m}^\circ,\mathrm{d}^\circ\in\mathrm{L}^\infty(\mathbb{R}^2)$, then we further have $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^2)^2)$, $\mathrm{m}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^2))$, and $\mathrm{d}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2\cap\mathrm{L}^\infty(\mathbb{R}^2))$. If h, Ψ, $\bar{\mathrm{v}}^\circ\in W^{s+1,\infty}(\mathbb{R}^2)^2$ and m°, $\bar{\mathrm{m}}^\circ$, d°, $\bar{\mathrm{d}}^\circ\in H^s(\mathbb{R}^2)$ for some $s>1$, then $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar{\mathrm{v}}^\circ+H^{s+1}(\mathbb{R}^2)^2)$ and $\mathrm{m},\mathrm{d}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{R}^2))$. If h, Ψ, $\mathrm{v}^\circ\in C^{s+1}(\mathbb{R}^2)^2$ for some non-integer $s>0$, then $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;C^{s+1}(\mathbb{R}^2)^2)$. ◊

Regarding the regimes that are not described in the above (that is, the mixed-flow compressible case as well as the a priori unphysical case $\alpha<0$), only local-in-time existence is proven for smooth enough initial data (stated here in Sobolev spaces). Note that for the mixed-flow degenerate case $\lambda=0$, $\alpha>0$, $\beta\neq 0$, even local-in-time existence remains an open problem.

Theorem 7.1.4 (Local existence). *Given some $s>1$, let $h,\Psi,\bar{\mathrm{v}}^\circ\in W^{s+1,\infty}(\mathbb{R}^2)^2$, set $a:=e^h$, and let $\mathrm{v}^\circ\in\bar{\mathrm{v}}^\circ+H^{s+1}(\mathbb{R}^2)^2$ with $\mathrm{m}^\circ:=\operatorname{curl}\mathrm{v}^\circ$, $\bar{\mathrm{m}}^\circ:=\operatorname{curl}\bar{\mathrm{v}}^\circ\in H^s(\mathbb{R}^2)$, and with either $\operatorname{div}(a\mathrm{v}^\circ)=$*

$\operatorname{div}(a\bar{\mathrm{v}}^\circ) = 0$ *in the case* (7.1), *or* $\mathrm{d}^\circ := \operatorname{div}(a\mathrm{v}^\circ)$, $\bar{\mathrm{d}}^\circ := \operatorname{div}(a\bar{\mathrm{v}}^\circ) \in H^s(\mathbb{R}^2)$ *in the case* (7.2). *The following hold:*

(i) Incompressible case (that is, (7.1) with $\alpha, \beta \in \mathbb{R}$):
There exists $T > 0$ *and a weak solution* $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$ *on* $[0,T) \times \mathbb{R}^2$ *with initial data* v°.

(ii) Non-degenerate compressible case (that is, (7.2) with $\alpha, \beta \in \mathbb{R}$, $\lambda > 0$):
There exists $T > 0$ *and a weak solution* $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$ *on* $[0,T) \times \mathbb{R}^2$ *with initial data* v°.

(iii) Degenerate parabolic compressible case (that is, (7.2) with $\alpha \in \mathbb{R}$, $\beta = \lambda = 0$):
If Ψ, $\bar{\mathrm{v}}^\circ \in W^{s+2,\infty}(\mathbb{R}^2)^2$ *and* m°, $\bar{\mathrm{m}}^\circ \in H^{s+1}(\mathbb{R}^2)$, *there exists* $T > 0$ *and a weak solution* $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$ *on* $[0,T) \times \mathbb{R}^2$ *with initial data* v°, *and with* $\mathrm{m} := \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); H^{s+1}(\mathbb{R}^2))$. $\Diamond$

We now turn to uniqueness issues. No uniqueness is expected to hold for general weak measure solutions of (7.1), as it is already known to fail for the 2D Euler equation (see e.g. [45] and the references therein), and as it is also expected to fail for equation (7.4) (see [18, Section 8]). In both cases, as already explained, the only known uniqueness results are in the class of bounded vorticity. For the general incompressible model (7.1), similar arguments are still available and the same uniqueness result holds. For the compressible model (7.2), we only obtain uniqueness in a class with stronger regularity, as a consequence of a weak-strong principle stated in Proposition 7.5.1.

Theorem 7.1.5 (Uniqueness). *Let* $\lambda \geq 0$, $\alpha, \beta \in \mathbb{R}$, $T > 0$, $h, \Psi \in W^{1,\infty}(\mathbb{R}^2)$, *and set* $a := e^h$. *Let* $\mathrm{v}^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ *with* $\operatorname{curl} \mathrm{v}^\circ \in \mathcal{P}(\mathbb{R}^2)$, *and with either* $\operatorname{div}(a\mathrm{v}^\circ) = 0$ *in the case* (7.1), *or* $\operatorname{div}(a\mathrm{v}^\circ) \in \mathrm{L}^2(\mathbb{R}^2)$ *in the case* (7.2).

(i) Incompressible case (that is, (7.1) with $\alpha, \beta \in \mathbb{R}$):
There exists at most a unique weak solution v *on* $[0,T) \times \mathbb{R}^2$ *with initial data* v°, *in the class of all* w*'s such that* $\operatorname{curl} w \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2))$.

(ii) Non-degenerate compressible case (that is, (7.2) with $\alpha, \beta \in \mathbb{R}$, $\lambda > 0$):
There exists at most a unique weak solution v *on* $[0,T) \times \mathbb{R}^2$ *with initial data* v°, *in the class in the class* $\mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2) \cap \mathrm{L}^\infty_{\mathrm{loc}}([0,T); W^{1,\infty}(\mathbb{R}^2)^2)$. $\Diamond$

In the degenerate parabolic case $\lambda = 0$, $\alpha = 1$, $\beta = 0$, we obtain the following global well-posedness result in collaboration with Julian Fischer. The proof is of a very different nature from the other cases, exploiting the explicit scalar structure of the solution v.

Theorem 7.1.6 (Degenerate parabolic compressible case). *Let* $\lambda = \beta = 0$, $\alpha = 1$, *let* $v^\circ, \Psi \in W^{1,\infty}(\mathbb{R}^2)^2$ *with* $\operatorname{curl} v^\circ \in \mathcal{P}(\mathbb{R}^2)$. *Then there exists a global strong solution* $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^2)) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; v^\circ + \mathrm{L}^1(\mathbb{R}^2))$ *of* (7.2) *on* $\mathbb{R}^+ \times \mathbb{R}^2$ *with initial data* v° *and with* $\operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^2))$. *This solution* v *is unique in the class of all* w*'s in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ *with* $\operatorname{curl} w \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2))$. *If in addition for some* $s \geq 0$ *we have* $v^\circ, \Psi \in W^{1\vee s,\infty}(\mathbb{R}^2)^2$ *and* $\operatorname{curl} v^\circ, \operatorname{curl} \Psi \in W^{s,\infty}(\mathbb{R}^2)$, *then* $v \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; W^{s,\infty}(\mathbb{R}^2)^2)$. *If for some* $s \geq 1$ *we further have* $v^\circ, \Psi \in W^{s,\infty}(\mathbb{R}^2)^2$, $\operatorname{curl} v^\circ \in H^s \cap W^{s,\infty}(\mathbb{R}^2)$, *and* $\operatorname{curl} \Psi \in W^{s,\infty}(\mathbb{R}^2)$, *then* $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; v^\circ + H^s \cap W^{s,\infty}(\mathbb{R}^2)^2)$. $\Diamond$

7.1.5 Perspectives and open questions

In the mixed-flow compressible case, only short-time well-posedness is proved in this chapter for smooth initial data, although it seems very likely that a global existence result should hold as well and that it should even be valid for vortex-sheet initial data. We are however unable to establish L^q a priori estimates for the vorticity in that case (cf. Lemmas 7.4.2 and 7.4.3), which is precisely the missing ingredient for a global result in our analysis. On the other hand, in the mixed-flow incompressible case, global existence is only established for L^q initial data with $q > 1$, while we believe that vortex-sheet initial data could be considered as well. Again, the missing ingredient is some strong enough a priori estimate on the vorticity (cf. Lemma 7.4.3(iii)). This is left as an open problem.

Another interesting open question concerns the mixed-flow degenerate case (that is, (7.2) with $\lambda = 0$, $\alpha > 0$, $\beta \neq 0$), for which we have not even obtained any local existence result. Note that the particular scalar structure of the parabolic degenerate equation obviously breaks down in this mixed-flow case. Let us briefly explain why even the proof of local existence fails (in the homogeneous case $a \equiv 1$, for simplicity). The vector field driving the vorticity m takes the form $-\alpha(\Psi + \mathrm{v})^\perp - \beta(\Psi + \mathrm{v})$ and has divergence $\alpha\mathrm{m} - \beta\mathrm{d} + O(1)$. In the mixed-flow case $\beta \neq 0$, the H^s-norm of the vorticity m for large s is therefore a priori only controlled by the H^s-norm of d (in short time), while on the other hand the degenerate equation (7.10) for the divergence d has no regularizing effect and the H^s-norm of d is only controlled by the H^{s+1} norm of m (in short time). This loss of derivative prevents us from concluding any iterative scheme, and it is unclear to us how to get around this difficulty. In contrast, in the parabolic case $\beta = 0$, the divergence of the vector field driving the vorticity takes the form $\alpha\mathrm{m} + O(1)$, which only involves the vorticity itself.

For the incompressible model (7.1), we have established uniqueness only in the class of bounded vorticity — in parallel to the known results for the 2D Euler equation [427] and for the corresponding gradient flow (7.4) [398]. Proving uniqueness for the 2D Euler equation with initial vorticity in $\mathcal{P} \cap \mathrm{L}^p(\mathbb{R}^2)$ with $p > 1$ is indeed a major open problem in the field (see e.g. [312, Chapter 8]), and the corresponding uniqueness problem for (7.4) is open as well.

There is also strong interest in the corresponding mean-field models for a signed vorticity, that is, equations (7.1) and (7.2) with $\mathrm{m} = \operatorname{curl} \mathrm{v}$ replaced by $|\mathrm{m}| = |\operatorname{curl} \mathrm{v}|$ (as in (7.5)). Even for the simpler case of the Chapman-Rubinstein-Schatzman-E equation (7.5) the understanding is very partial: (global) well-posedness is only known for initial vorticity either in $\mathrm{L}^1 \cap W^{1,p}(\mathbb{R}^2)$ with $2 < p < \infty$, or in $\mathrm{L}^1 \cap C_b^\alpha(\mathbb{R}^2)$ with $0 < \alpha < 1$ (cf. [157, 316]), although existence is actually expected to hold for any initial vorticity in $\mathrm{L}^1 \cap (\mathrm{L}^{4/3} \log \mathrm{L})(\mathbb{R}^2)$. Let us mention [17] as an inconclusive attempt to prove such an L^p existence result for this equation. A priori L^p bounds for smooth solutions of (7.5) are easily established, but the main difficulty comes from the fact that this equation is not well-adapted to weak convergence methods since the absolute value $|\cdot|$ is not continuous with respect to the weak-L^p topology. Since only a little compactness is missing, it seems that the existence result could be reached for any initial vorticity in $\mathrm{L}^1 \cap W^{\varepsilon,4/3}(\mathbb{R}^2)$ for any $\varepsilon > 0$; this improvement of the available results [316] could be pursued in a future work. We believe that a deeper L^p understanding of this equation would be of great help to solve the corresponding mean-field limit problem (6.25).

In the context of (edge) dislocations, it is also relevant to consider a (± 1) charged particle system where particles of opposite charge must not annihilate but can somehow cross each other. It leads to a simpler version of equation (7.5) where the evolution of the densities of positive and of negative charges split. This is sometimes called the Groma-Balogh model [219, 220], and well-posedness questions are studied in [311], where existence is established for initial densities in $\mathrm{L}^1 \cap \mathrm{L}^4(\mathbb{R}^2)$, and uniqueness for bounded densities. It seems to us that an existence result in the optimal space $\mathrm{L}^1 \cap (\mathrm{L}^{4/3} \log \mathrm{L})(\mathbb{R}^2)$ is within reach and can be further generalized to the case of other Coulomb-like potentials, which could be the object of a future work. Understanding this equation does however not lead to any progress at the level of the original equation (7.5), for which the main problem remains the understanding of the structure of the mass sink.

Finally note that in Chapter 8 the mean-field models (7.1)–(7.2) are derived from a mesoscopic 2D Ginzburg-Landau model for which we neglect for simplicity the coupling with electromagnetism. Rather starting from the physically relevant version with magnetic gauge, no new difficulty is expected to occur at the level of the mean-field limit result, but the corresponding mean-field equations then need to be modified and take the form given in Section 8.2.3. Global well-posedness for these more complex mean-field models is then a natural question that we do not pursue here.

7.1.6 Roadmap to the proof of the main results

To ease the presentation, various independent PDE results needed in the proofs are isolated in Section 7.2, including general a priori estimates for transport and transport-diffusion equations, some global elliptic regularity results, as well as critical potential theory estimates. The interest of such estimates for our purposes should be already clear from a quick look at the vorticity formulation (7.9)–(7.10). To the best of our knowledge, most of these PDE results are not standard and cannot be found in this form in the literature.

We start in Section 7.3 with the local existence of smooth solutions, summarized in Theorem 7.1.4 above. In the non-degenerate case $\lambda > 0$, the proof follows from a standard iterative scheme as in [395, Appendix B]. It is performed here in Sobolev spaces, but could be done in Hölder spaces as well. In the degenerate parabolic case $\lambda = \beta = 0$, $\alpha > 0$, a similar argument holds, but requires a more careful analysis of the iterative scheme.

In Section 7.4 we then turn to global existence. In order to pass from local to global existence, we prove estimates for the Sobolev and Hölder norms of solutions through the norm of their initial data. As shown in Section 7.4.2, these estimates essentially follow from an a priori control of the vorticity in $\mathrm{L}^\infty(\mathbb{R}^2)$. In the work by Lin and Zhang [304] on the simpler model (7.4), such an a priori estimate for the vorticity is achieved by a direct ODE argument, using that for (7.4) the evolution of the vorticity along characteristics can be integrated explicitly. This explicit structure is lost in the more sophisticated models (7.1) and (7.2), but in the parabolic case we still manage to design suitable ODE type arguments (cf. Lemma 7.4.3(iii)). This leads to the nice boundedness effect (7.12) for the vorticity (depending on the initial mass $\int \mathrm{m}^\circ = 1$ only!), which of course differs from [304] by the additional exponential growth due to the forcing Ψ, and which is at the core of the existence results for vortex-sheet initial data. In the mixed-flow case for the incompressible model (7.1), such ODE arguments are no longer available, and only a weaker estimate is obtained, controlling the L^q-norm of the solution (as well as its space-time L^{q+1}-norm if $\alpha > 0$) through the L^q-norm of the data for all $1 < q \leq \infty$ (cf. Lemma 7.4.2). This is proven by a careful examination of the evolution of L^q-norms of the vorticity.

In order to handle rough initial data, we regularize the data and then pass to the limit in the equation, using the compactness given by the available a priori estimates. As already noticed, for h non-constant, the usual Delort's argument [143] fails (due to the first right-hand side term in (7.11)), so that stronger compactness is needed to pass to the limit in the nonlinearity mv than in the simpler case of the 2D Euler equation. While energy estimates only give bounds for v in $\bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$ and for d in $\mathrm{L}^2(\mathbb{R}^2)$ (cf. Lemma 7.4.1), the additional estimates for the vorticity in $\mathrm{L}^q(\mathbb{R}^2)$, $q > 1$, turn out to be crucial. As in [304], we need to make use of some compactness result due to Lions [305] in the context of the compressible Navier-Stokes equations. The model (7.1) in the conservative case $\alpha = 0$ is however more subtle because of a lack of strong enough a priori estimates: only very weak solutions are then expected and obtained (for initial vorticity in $\mathrm{L}^q(\mathbb{R}^2)$ with $q > 1$), and compactness is in that case carefully proven by hand, which is one of the main achievements in this paper (cf. Proposition 7.4.10(iv)).

Uniqueness issues are addressed in Section 7.5. Similarly as in [395, Appendix B] and in Lemma 6.1.7 in Chapter 6, weak-strong uniqueness principles for both (7.1) and (7.2) are established by energy methods in the non-degenerate case $\lambda > 0$. In the degenerate parabolic case $\lambda = \beta = 0$, these energy methods fail: an additional term needs to be added to the usual energy, and on this basis a different weak-strong uniqueness principle is obtained. Following the modulated energy strategy developed by Serfaty [395] and exemplified in Chapter 6 in the context of Coulomb-like interaction gradient flows, these weak-strong principles are the key to the mean-field limit results for Ginzburg-Landau vortices in Chapter 8. For the incompressible model (7.1), uniqueness in the class of bounded vorticity is further obtained using the approach by Serfaty and Vázquez [398] for the simpler model (7.4), which consists in adapting the corresponding uniqueness result for the 2D Euler equation due to Yudovich [427] together with a transport argument à la Loeper [307].

Finally, the degenerate parabolic case $\lambda = 0$, $\alpha = 1$, $\beta = 0$ is treated in Section 7.6 in collaboration with Julian Fischer. The proof consists in exploiting the scalar structure of the solution v to reduce the equation to a Burgers type equation with additional quadratic damping and forcing terms, and with unit initial data. Suitable ODE type arguments then allow to explicitly integrate this equation, and the desired properties of the solution easily follow.

7.2 Preliminary results

In this section, we establish various PDE results that are needed in the sequel and are of independent interest. As most of them do not depend on the choice of space dimension 2, they are stated here in general dimension $d \geq 1$. We first recall the following useful proxy for a fractional Leibniz rule, which is essentially due to Kato and Ponce [269] based on ideas by Coifman and Meyer [121, 122] (see e.g. [228, Theorem 1.4]).

Lemma 7.2.1 (Kato-Ponce inequality). *Let $d \geq 1$, $s \geq 0$, $p \in (1, \infty)$, and let $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{p}$ with $p_i, q_i \in (1, \infty]$ for $i = 1, 2$. Then for $f, g \in C_c^\infty(\mathbb{R}^d)$ we have*

$$\|fg\|_{W^{s,p}} \lesssim \|f\|_{\mathrm{L}^{p_1}} \|g\|_{W^{s,q_1}} + \|g\|_{\mathrm{L}^{p_2}} \|f\|_{W^{s,q_2}}. \qquad \diamondsuit$$

The following gives a general estimate for the evolution of the Sobolev norms of the solutions of transport equations (see also [304, equation (7)] for a simpler version), which will be useful in the sequel since the vorticity m indeed satisfies an equation of this form (7.9).

Lemma 7.2.2 (A priori estimate for transport equations). *Let $d \geq 1$, $s \geq 0$, $T > 0$. Given a vector field $w \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); W^{1,\infty}(\mathbb{R}^d)^d)$ with $w - W \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); H^{s+1}(\mathbb{R}^d)^d)$ for some reference map $W \in W^{s+1,\infty}(\mathbb{R}^d)^d$, let $\rho \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); H^s(\mathbb{R}^d))$ satisfy the transport equation $\partial_t \rho = \operatorname{div}(\rho w)$ in the distributional sense on $[0, T) \times \mathbb{R}^d$. Then for all $t \in [0, T)$ we have*

$$\begin{aligned} \partial_t \|\rho^t\|_{H^s} \lesssim_s \|(\nabla w^t, \nabla W)\|_{\mathrm{L}^\infty} \|\rho^t\|_{H^s} + \|W\|_{W^{s+1,\infty}} \|\rho^t\|_{\mathrm{L}^2} \\ + \min\big\{ \|\rho^t\|_{\mathrm{L}^\infty} \|\operatorname{div}(w^t - W)\|_{H^s} + \|\rho^t\|_{W^{1,\infty}} \|w^t - W\|_{H^s} \,;\, \|\rho^t\|_{\mathrm{L}^\infty} \|w^t - W\|_{H^{s+1}} \big\}, \end{aligned} \tag{7.13}$$

where we use the notation $\|(\nabla w^t, \nabla W)\|_{\mathrm{L}^\infty} := \|\nabla w^t\|_{W^{1,\infty}} \vee \|\nabla W\|_{W^{1,\infty}}$. Moreover, for all $t \in [0, T)$,

$$\|\rho^t - \rho^\circ\|_{\dot{H}^{-1}} \leq \|\rho\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \|w\|_{\mathrm{L}^1_t \mathrm{L}^\infty}. \tag{7.14}$$

$\diamondsuit$

Proof. We split the proof into two steps: we first prove (7.13) as a corollary of the celebrated Kato-Ponce commutator estimate, and then we check estimate (7.14), which is but a straightforward observation.

Step 1. Proof of (7.13).

Let $s \geq 0$. The time derivative of the H^s-norm of the solution ρ can be computed as follows, using the notation $\langle \nabla \rangle := (1 + |\nabla|^2)^{1/2}$,

$$\begin{aligned} \partial_t \|\rho^t\|_{H^s}^2 = 2 \int (\langle \nabla \rangle^s \rho^t)(\langle \nabla \rangle^s \operatorname{div}(\rho^t w^t)) &= 2 \int (\langle \nabla \rangle^s \rho^t) [\langle \nabla \rangle^s \operatorname{div}, w^t] \rho^t + 2 \int (\langle \nabla \rangle^s \rho^t)(w^t \cdot \nabla \langle \nabla \rangle^s \rho^t) \\ &= 2 \int (\langle \nabla \rangle^s \rho^t) [\langle \nabla \rangle^s \operatorname{div}, w^t] \rho^t - \int |\langle \nabla \rangle^s \rho^t|^2 \operatorname{div} w^t \\ &\leq 2 \|\rho^t\|_{H^s} \|[\langle \nabla \rangle^s \operatorname{div}, w^t] \rho^t\|_{\mathrm{L}^2} + \|(\operatorname{div} w^t)_-\|_{\mathrm{L}^\infty} \|\rho^t\|_{H^s}^2, \end{aligned}$$

and hence,

$$\partial_t \|\rho^t\|_{H^s} \le \|[\langle\nabla\rangle^s \operatorname{div}, w^t - W]\rho^t\|_{\mathrm{L}^2} + \|[\langle\nabla\rangle^s \operatorname{div}, W]\rho^t\|_{\mathrm{L}^2} + \frac{1}{2}\|(\operatorname{div} w^t)_-\|_{\mathrm{L}^\infty}\|\rho^t\|_{H^s}. \qquad (7.15)$$

Now we recall the following forms of the Kato-Ponce commutator estimate [269, Lemma X1] (see e.g. [299]): given $p \in (1,\infty)$, and $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{p}$ with $p_i, q_i \in (1,\infty]$ for $i = 1,2$, we have for all $f, g \in C_c^\infty(\mathbb{R}^d)$,

$$\|[\langle\nabla\rangle^s \nabla, f]g\|_{\mathrm{L}^p} \lesssim_{s,p,p_1,p_2} \|f\|_{W^{s+1,q_1}}\|g\|_{\mathrm{L}^{p_1}} + \|\nabla f\|_{\mathrm{L}^{p_2}}\|g\|_{W^{s,q_2}},$$

and also

$$\|[\langle\nabla\rangle^s, f]\nabla g\|_{\mathrm{L}^p} \lesssim_{s,p,p_1,p_2} \|f\|_{W^{s,q_1}}\|g\|_{W^{1,p_1}} + \mathbb{1}_{s\ge 1}\|\nabla f\|_{\mathrm{L}^{p_2}}\|g\|_{W^{s,q_2}}. \qquad (7.16)$$

Together with the Kato-Ponce inequality of Lemma 7.2.1, these estimates yield on the one hand

$$\|[\langle\nabla\rangle^s \operatorname{div}, W]\rho^t\|_{\mathrm{L}^2} \lesssim_s \|W\|_{W^{s+1,\infty}}\|\rho^t\|_{\mathrm{L}^2} + \|\nabla W\|_{\mathrm{L}^\infty}\|\rho^t\|_{H^s},$$

and

$$\|[\langle\nabla\rangle^s \operatorname{div}, w^t - W]\rho^t\|_{\mathrm{L}^2} \lesssim_s \|\rho^t\|_{\mathrm{L}^\infty}\|w^t - W\|_{H^{s+1}} + \|\nabla(w^t - W)\|_{\mathrm{L}^\infty}\|\rho^t\|_{H^s},$$

and on the other hand,

$$\begin{aligned}\|[\langle\nabla\rangle^s \operatorname{div}, w^t - W]\rho^t\|_{\mathrm{L}^2} &\le \|\rho^t \operatorname{div}(w^t - W)\|_{H^s} + \|[\langle\nabla\rangle^s, (w^t - W)\cdot\,]\,\nabla\rho^t\|_{\mathrm{L}^2}\\ &\lesssim_s \|\nabla(w^t - W)\|_{\mathrm{L}^\infty}\|\rho^t\|_{H^s} + \|\rho^t\|_{\mathrm{L}^\infty}\|\operatorname{div}(w^t - W)\|_{H^s} + \|\rho^t\|_{W^{1,\infty}}\|w^t - W\|_{H^s}.\end{aligned}$$

Injecting these estimates into (7.15), the result (7.13) follows.

Step 2. Proof of (7.14).

Let $\varepsilon > 0$. We denote by $\hat{u}$ the Fourier transform of a function u on $\mathbb{R}^d$. Set $G := \rho w$, so that the equation for ρ takes the form $\partial_t \rho = \operatorname{div} G$. Rewriting this equation in Fourier space and testing it against $(\varepsilon + |\xi|)^{-2}(\hat\rho^t - \hat\rho^\circ)(\xi)$, we find

$$\begin{aligned}\partial_t \int (\varepsilon + |\xi|)^{-2}|\hat\rho^t(\xi) - \hat\rho^\circ(\xi)|^2 d\xi &= 2i \int (\varepsilon + |\xi|)^{-2}\xi\cdot \hat G^t(\xi)\left(\overline{\hat\rho^t(\xi) - \hat\rho^\circ(\xi)}\right)d\xi\\ &\le 2\int (\varepsilon + |\xi|)^{-1}|\hat\rho^t(\xi) - \hat\rho^\circ(\xi)||\hat G^t(\xi)|d\xi,\end{aligned}$$

and hence, by the Cauchy-Schwarz inequality,

$$\partial_t\bigg(\int (\varepsilon + |\xi|)^{-2}|\hat\rho^t(\xi) - \hat\rho^\circ(\xi)|^2 d\xi\bigg)^{1/2} \le \bigg(\int |\hat G^t(\xi)|^2 d\xi\bigg)^{1/2}.$$

Integrating in time and letting $\varepsilon \downarrow 0$, we obtain

$$\|\rho^t - \rho^\circ\|_{\dot H^{-1}} \le \|G\|_{\mathrm{L}^1_t \mathrm{L}^2} \le \|\rho\|_{\mathrm{L}^\infty_t \mathrm{L}^2}\|w\|_{\mathrm{L}^1_t \mathrm{L}^\infty},$$

that is, (7.14). □

As the evolution of the divergence d in the compressible model (7.2) is given by the transport-diffusion equation (7.10), the following parabolic regularity results will be needed. While items (i) and (ii) are classical, item (iii) is less standard (see however [40, Section 3.4] for a variant of this estimate).

Lemma 7.2.3 (A priori estimates for transport-diffusion equations). *Let $d \geq 1$, $T > 0$. Let $g \in \mathrm{L}^1_{\mathrm{loc}}([0,T) \times \mathbb{R}^d)^d$, and let w satisfy $\partial_t w - \triangle w + \operatorname{div}(w \nabla h) = \operatorname{div} g$ in the distributional sense on $[0,T) \times \mathbb{R}^d$ with initial data w°. The following hold:*

(i) for all $s \geq 0$, if $\nabla h \in W^{s,\infty}(\mathbb{R}^d)^d$, $w \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); H^s(\mathbb{R}^d))$, and $g \in \mathrm{L}^2_{\mathrm{loc}}([0,T); H^s(\mathbb{R}^d)^d)$, then we have for all $t \in [0,T)$,

$$\|w^t\|_{H^s} \leq Ce^{Ct}(\|w^\circ\|_{H^s} + \|g\|_{\mathrm{L}^2_t H^s}),$$

where the constant C depends only on an upper bound on s and $\|\nabla h\|_{W^{s,\infty}}$;

(ii) if $\nabla h \in \mathrm{L}^\infty(\mathbb{R}^d)$, $w^\circ \in \mathrm{L}^2(\mathbb{R}^d)$, $w \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^d))$, and $g \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^d))$, then we have for all $t \in [0,T)$,

$$\|w^t - w^\circ\|_{\dot H^{-1} \cap \mathrm{L}^2} \leq Ce^{Ct}(\|w^\circ\|_{\mathrm{L}^2} + \|g\|_{\mathrm{L}^2_t \mathrm{L}^2}),$$

where the constant C depends only on an upper bound on $\|\nabla h\|_{\mathrm{L}^\infty}$;

(iii) for all $1 \leq p, q \leq \infty$, and all $\frac{dq}{d+q} < s \leq q$, $s \geq 1$, if $\nabla h \in \mathrm{L}^\infty(\mathbb{R}^d)$, $w \in \mathrm{L}^p_{\mathrm{loc}}([0,T); \mathrm{L}^q(\mathbb{R}^d))$, and $g \in \mathrm{L}^p_{\mathrm{loc}}([0,T); \mathrm{L}^s(\mathbb{R}^d))$, then we have for all $t \in [0,T)$,

$$\|w\|_{\mathrm{L}^p_t \mathrm{L}^q} \leq C(\|w^\circ\|_{\mathrm{L}^q} + \kappa^{-1} t^\kappa \|g\|_{\mathrm{L}^p_t \mathrm{L}^s}) \exp\Big(\inf_{2<r<\infty} r^{-1}\big(1 + (r-2)^{-r/2}\big)(Ct)^{r/2}\Big).$$

where $\kappa := \frac{d}{2}(\frac{1}{d} + \frac{1}{q} - \frac{1}{s}) > 0$, and where the constant C depends only on an upper bound on $\|\nabla h\|_{\mathrm{L}^\infty}$. ◊

Proof. We split the proof into three steps, proving items (i), (ii), and (iii) separately.

Step 1. Proof of (i).

Denote $G := g - w\nabla h$, so that w satisfies $\partial_t w - \triangle w = \operatorname{div} G$. Set $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$, and let $\hat u$ denote the Fourier transform of a function u on $\mathbb{R}^d$. Let $s \geq 0$ be fixed, and assume that $\nabla h, w, g$ are as in the statement of (i) (which implies $G \in \mathrm{L}^2_{\mathrm{loc}}([0,T); H^s(\mathbb{R}^d))$ as shown below). In this step, we use the notation $\lesssim$ for $\leq$ up to a constant C as in the statement. For all $\varepsilon > 0$, rewriting the equation for w in Fourier space and testing it against $(\varepsilon + |\xi|)^{-2} \langle \xi \rangle^{2s} \partial_t \hat w(\xi)$, we obtain

$$\int (\varepsilon + |\xi|)^{-2} \langle \xi \rangle^{2s} |\partial_t \hat w^t(\xi)|^2 d\xi + \frac{1}{2} \int \frac{|\xi|^2}{(\varepsilon + |\xi|)^2} \langle \xi \rangle^{2s} \partial_t |\hat w^t(\xi)|^2 d\xi \\ = i \int (\varepsilon + |\xi|)^{-2} \langle \xi \rangle^{2s} \xi \cdot \hat G^t(\xi) \overline{\partial_t \hat w^t(\xi)} d\xi,$$

and hence, integrating over $[0,t]$, and using the inequality $2xy \leq x^2 + y^2$,

$$\begin{aligned} &\int_0^t \int (\varepsilon + |\xi|)^{-2} \langle \xi \rangle^{2s} |\partial_u \hat w^u(\xi)|^2 d\xi du + \frac{1}{2} \int \frac{|\xi|^2}{(\varepsilon + |\xi|)^2} \langle \xi \rangle^{2s} |\hat w^t(\xi)|^2 d\xi \\ &= \frac{1}{2} \int \frac{|\xi|^2}{(\varepsilon + |\xi|)^2} \langle \xi \rangle^{2s} |\hat w^\circ(\xi)|^2 d\xi + i \int_0^t \int (\varepsilon + |\xi|)^{-2} \langle \xi \rangle^{2s} \xi \cdot \hat G^u(\xi) \overline{\partial_u \hat w^u(\xi)} d\xi du \\ &\leq \frac{1}{2} \int \langle \xi \rangle^{2s} |\hat w^\circ(\xi)|^2 d\xi + \frac{1}{2} \int_0^t \int \langle \xi \rangle^{2s} |\hat G^u(\xi)|^2 d\xi du + \frac{1}{2} \int_0^t \int (\varepsilon + |\xi|)^{-2} \langle \xi \rangle^{2s} |\partial_u \hat w^u(\xi)|^2 d\xi du. \end{aligned}$$

Absorbing in the left-hand side the last right-hand side term, and letting $\varepsilon \downarrow 0$, it follows that

$$\int \langle \xi \rangle^{2s} |\hat w^t(\xi)|^2 d\xi \leq \int \langle \xi \rangle^{2s} |\hat w^\circ(\xi)|^2 d\xi + \int_0^t \int \langle \xi \rangle^{2s} |\hat G^u(\xi)|^2 d\xi du,$$

or equivalently

$$\|w^t\|_{H^s} \le \|w^\circ\|_{H^s} + \|G\|_{\mathrm{L}^2_t H^s}.$$

Lemma 7.2.1 yields

$$\begin{aligned}\|G\|_{\mathrm{L}^2_t H^s} \le \|g\|_{\mathrm{L}^2_t H^s} + \|w\nabla h\|_{\mathrm{L}^2_t H^s} &\lesssim \|g\|_{\mathrm{L}^2_t H^s} + \|\nabla h\|_{W^{s,\infty}}\|w\|_{\mathrm{L}^2_t \mathrm{L}^2} + \|\nabla h\|_{\mathrm{L}^\infty}\|w\|_{\mathrm{L}^2_t H^s}\\ &\lesssim \|g\|_{\mathrm{L}^2_t H^s} + \|w\|_{\mathrm{L}^2_t H^s},\end{aligned}$$

so that we obtain

$$\|w^t\|^2_{H^s} \lesssim \|w^\circ\|^2_{H^s} + \|g\|^2_{\mathrm{L}^2_t H^s} + \int_0^t \|w^u\|^2_{H^s} du,$$

and item (i) now follows from the Grönwall inequality.

Step 2. Proof of (ii).

Set again $G := g - w\nabla h$, and let $\nabla h, w^\circ, w, g$ be as in the statement of (ii). For all $\varepsilon > 0$, rewriting the equation for w in Fourier space and then integrating it against $(\varepsilon + |\xi|)^{-2}(\hat w^t - \hat w^\circ)(\xi)$, we may estimate

$$\begin{aligned}\partial_t \int (\varepsilon+|\xi|)^{-2}|(\hat w^t-\hat w^\circ)(\xi)|^2 d\xi &= 2\int (\varepsilon+|\xi|)^{-2}\overline{(\hat w^t-\hat w^\circ)(\xi)}\partial_t \hat w^t(\xi)d\xi\\
\le\ -2\int \frac{|\xi|^2}{(\varepsilon+|\xi|)^2}|(\hat w^t-\hat w^\circ)(\xi)|^2 &+ 2\int \frac{|\xi|^2}{(\varepsilon+|\xi|)^2}|(\hat w^t-\hat w^\circ)(\xi)||\hat w^\circ(\xi)|\\
&+2\int(\varepsilon+|\xi|)^{-1}|(\hat w^t-\hat w^\circ)(\xi)||\hat G^t(\xi)|d\xi\\
\le \int \frac{|\xi|^2}{(\varepsilon+|\xi|)^2}|\hat w^\circ(\xi)|^2 &+ \int(\varepsilon+|\xi|)^{-2}|(\hat w^t-\hat w^\circ)(\xi)|^2d\xi + \int(1+|\xi|^2)^{-1}|\hat G^t(\xi)|^2d\xi,\end{aligned}$$

that is

$$\partial_t \int (\varepsilon+|\xi|)^{-2}|(\hat w^t-\hat w^\circ)(\xi)|^2 d\xi \le \int (\varepsilon+|\xi|)^{-2}|(\hat w^t-\hat w^\circ)(\xi)|^2 d\xi + \|w^\circ\|^2_{\mathrm{L}^2} + \|G^t\|^2_{H^{-1}},$$

and hence by the Grönwall inequality,

$$\int (\varepsilon+|\xi|)^{-2}|(\hat w^t-\hat w^\circ)(\xi)|^2 d\xi \le e^t\big(t\|w^\circ\|^2_{\mathrm{L}^2} + \|G\|^2_{\mathrm{L}^2_t H^{-1}}\big).$$

Letting $\varepsilon \downarrow 0$, it follows that $w^t - w^\circ \in \dot H^{-1}(\mathbb{R}^2)$ with

$$\|w^t - w^\circ\|_{\dot H^{-1}} \le e^{Ct}(\|w^\circ\|_{\mathrm{L}^2} + \|G\|_{\mathrm{L}^2_t H^{-1}}) \le e^{Ct}(\|w^\circ\|_{\mathrm{L}^2} + \|g\|_{\mathrm{L}^2_t H^{-1}} + \|\nabla h\|_{\mathrm{L}^\infty}\|w\|_{\mathrm{L}^2_t \mathrm{L}^2}).$$

Combining this with (i) for $s = 0$, item (ii) follows.

Step 3. Proof of (iii).

Let $1 \le p, q \le \infty$, and assume that $w \in \mathrm{L}^p([0,T);\mathrm{L}^q(\mathbb{R}^d))$, $\nabla h \in \mathrm{L}^\infty(\mathbb{R}^d)$, and $g \in \mathrm{L}^p([0,T);\mathrm{L}^q(\mathbb{R}^d))$. In this step, we use the notation $\lesssim$ for $\le$ up to a constant C as in the statement. Denoting by $\Gamma^t(x) := Ct^{-d/2}e^{-|x|^2/(2t)}$ the heat kernel, Duhamel's representation formula yields

$$w^t(x) = \Gamma^t * w^\circ(x) + \phi^t_g(x) - \int_0^t \int \nabla\Gamma^u(y)\cdot\nabla h(x-y)\, w^{t-u}(x-y)dydu,$$

where we have set

$$\phi^t_g(x) := \int_0^t \int \nabla\Gamma^u(y)\cdot g^{t-u}(x-y)dydu.$$

We find by the triangle inequality

$$\|w^t\|_{\mathrm{L}^q} \le \|w^\circ\|_{\mathrm{L}^q} \int |\Gamma^t(y)|dy + \|\phi_g^t\|_{\mathrm{L}^q} + \|\nabla h\|_{\mathrm{L}^\infty} \int_0^t \|w^{t-u}\|_{\mathrm{L}^q} \int |\nabla \Gamma^u(y)|dydu,$$

hence by a direct computation

$$\|w^t\|_{\mathrm{L}^q} \lesssim \|w^\circ\|_{\mathrm{L}^q} + \|\phi_g^t\|_{\mathrm{L}^q} + \int_0^t \|w^{t-u}\|_{\mathrm{L}^q} u^{-1/2} du.$$

Integrating with respect to t, and using the triangle and the Hölder inequalities, we find

$$\begin{aligned}
\|w\|_{\mathrm{L}_t^p \mathrm{L}^q} &\lesssim t^{1/p}\|w^\circ\|_{\mathrm{L}^q} + \|\phi_g\|_{\mathrm{L}_t^p \mathrm{L}^q} + \bigg(\int_0^t \Big(\int_0^t \mathbb{1}_{u<v} \|w^{v-u}\|_{\mathrm{L}^q} u^{-1/2} du \Big)^p dv \bigg)^{1/p} \\
&\lesssim t^{1/p}\|w^\circ\|_{\mathrm{L}^q} + \|\phi_g\|_{\mathrm{L}_t^p \mathrm{L}^q} + \int_0^t \|w\|_{\mathrm{L}_u^p \mathrm{L}^q} (t-u)^{-1/2} du \\
&\lesssim t^{1/p}\|w^\circ\|_{\mathrm{L}^q} + \|\phi_g\|_{\mathrm{L}_t^p \mathrm{L}^q} + (1-r'/2)^{-1/r'} t^{\frac12 - \frac1r} \bigg(\int_0^t \|w\|_{\mathrm{L}_u^p \mathrm{L}^q}^r du \bigg)^{1/r},
\end{aligned}$$

for all $r > 2$. Noting that $(1-r'/2)^{-1/r'} \lesssim 1 + (r-2)^{-1/2}$, and optimizing in r, the Grönwall inequality then leads to

$$\|w\|_{\mathrm{L}_t^p \mathrm{L}^q} \lesssim (t^{1/p}\|w^\circ\|_{\mathrm{L}^q} + \|\phi_g\|_{\mathrm{L}_t^p \mathrm{L}^q}) \exp\Big(\inf_{2<r<\infty} \frac{C^r}{r} (1 + (r-2)^{-r/2})\, t^{r/2} \Big). \tag{7.17}$$

Now it remains to estimate the norm of ϕ_g. A similar computation as above yields $\|\phi_g\|_{\mathrm{L}_t^p \mathrm{L}^q} \lesssim t^{1/2}\|g\|_{\mathrm{L}_t^p \mathrm{L}^q}$, but a more careful estimate is needed. For $1 \le s \le q$, we may estimate by the Hölder inequality

$$|\phi_g^t(x)| \le \int_0^t \bigg(\int |\nabla \Gamma^u|^{s'/2} \bigg)^{1/s'} \bigg(\int |\nabla \Gamma^u(x-y)|^{s/2} |g^{t-u}(y)|^s dy \bigg)^{1/s} du,$$

and hence, by the triangle inequality,

$$\|\phi_g^t\|_{\mathrm{L}^q} \le \int_0^t \bigg(\int |\nabla \Gamma^u|^{s'/2} \bigg)^{1/s'} \bigg(\int |\nabla \Gamma^u|^{q/2} \bigg)^{1/q} \bigg(\int |g^{t-u}|^s \bigg)^{1/s} du.$$

Assuming that $\kappa := \frac{d}{2}\big(\frac1d + \frac1q - \frac1s\big) > 0$ (note that $\kappa \le 1/2$ follows from the choice $s \le q$), a direct computation then yields

$$\|\phi_g^t\|_{\mathrm{L}^q} \lesssim \int_0^t u^{\kappa-1} \|g^{t-u}\|_{\mathrm{L}^s} du.$$

Integrating with respect to t, we find by the triangle inequality

$$\|\phi_g\|_{\mathrm{L}_t^p \mathrm{L}^q} \lesssim \int_0^t u^{\kappa-1} \bigg(\int_0^{t-u} \|g^v\|_{\mathrm{L}^s}^p dv \bigg)^{1/p} du \ \lesssim \kappa^{-1} t^\kappa \|g^v\|_{\mathrm{L}_t^p \mathrm{L}^s},$$

and the result (iii) follows from this together with (7.17). □

Another ingredient that we need is the following string of critical potential theory estimates. The Sobolev embedding for $W^{1,d}(\mathbb{R}^d)$ gives that $\|\nabla \triangle^{-1} w\|_{\mathrm{L}^\infty}$ is *almost* bounded by the $\mathrm{L}^d(\mathbb{R}^d)$-norm of w, while the Calderón-Zygmund theory gives that $\|\nabla^2 \triangle^{-1} w\|_{\mathrm{L}^\infty}$ is *almost* bounded by the $\mathrm{L}^\infty(\mathbb{R}^d)$-norm of w. The following result makes these assertions precise in a quantitative way in the spirit of Brézis and Gallouët [88]. Item (iii) can be found e.g. in [304, Appendix] in a slightly different form, but we were unable to find items (i) and (ii) in the literature. (By $-\triangle^{-1}$ we henceforth mean the convolution with the Green kernel.)

Lemma 7.2.4 (Potential estimates in L^∞)**.** *Let $d \ge 2$. For all $w \in C_c^\infty(\mathbb{R}^d)$ the following hold:*[1]

(i) for all $1 \le p < d < q \le \infty$, choosing $\theta \in (0,1)$ such that $\frac{1}{d} = \frac{\theta}{p} + \frac{1-\theta}{q}$, we have

$$\|\nabla \triangle^{-1} w\|_{\mathrm{L}^\infty} \lesssim \big((1-d/q) \wedge (1-p/d)\big)^{-1+1/d} \|w\|_{\mathrm{L}^d} \bigg(1 + \log \frac{\|w\|_{\mathrm{L}^p}^\theta \|w\|_{\mathrm{L}^q}^{1-\theta}}{\|w\|_{\mathrm{L}^d}}\bigg)^{1-1/d};$$

(ii) if $w = \operatorname{div} \xi$ for $\xi \in C_c^\infty(\mathbb{R}^d)^d$, then, for all $d < q \le \infty$ and $1 \le p < \infty$, we have

$$\|\nabla \triangle^{-1} w\|_{\mathrm{L}^\infty} \lesssim (1-d/q)^{-1+1/d} \|w\|_{\mathrm{L}^d} \bigg(1 + \log^+ \frac{\|w\|_{\mathrm{L}^q}}{\|w\|_{\mathrm{L}^d}}\bigg)^{1-1/d} + p\|\xi\|_{\mathrm{L}^p};$$

(iii) for all $0 < s \le 1$ and $1 \le p < \infty$, we have

$$\|\nabla^2 \triangle^{-1} w\|_{\mathrm{L}^\infty} \lesssim s^{-1} \|w\|_{\mathrm{L}^\infty} \bigg(1 + \log \frac{\|w\|_{C^s}}{\|w\|_{\mathrm{L}^\infty}}\bigg) + p\|w\|_{\mathrm{L}^p}. \qquad \Diamond$$

Proof. Recall that $-\triangle^{-1} w = g_d * w$, where we define $g_d(x) := c_d |x|^{2-d}$ if $d > 2$ and $g_2(x) := -c_2 \log |x|$ if $d = 2$. The stated results are based on suitable decompositions of this Green's integral. We split the proof into three steps, separately proving items (i), (ii) and (iii).

Step 1. Proof of (i).

Let $0 < \gamma \le \Gamma < \infty$. The obvious estimate $|\nabla \triangle^{-1} w(x)| \lesssim \int |x-y|^{1-d} |w(y)| dy$ may be decomposed as

$$|\nabla \triangle^{-1} w(x)| \lesssim \int_{|x-y|<\gamma} |x-y|^{1-d} |w(y)| dy \\ + \int_{\gamma<|x-y|<\Gamma} |x-y|^{1-d} |w(y)| dy + \int_{|x-y|>\Gamma} |x-y|^{1-d} |w(y)| dy.$$

Let $1 \le p < d < q \le \infty$. We use the Hölder inequality with exponents $(q/(q-1), q)$ for the first term, $(d/(d-1), d)$ for the second, and $(p/(p-1), p)$ for the third, which yields after straightforward computations

$$|\nabla \triangle^{-1} w(x)| \lesssim (q'(1-d/q))^{-1/q'} \gamma^{1-d/q} \|w\|_{\mathrm{L}^q} \\ + (\log(\Gamma/\gamma))^{(d-1)/d} \|w\|_{\mathrm{L}^d} + (p'(d/p-1))^{-1/p'} \Gamma^{1-d/p} \|w\|_{\mathrm{L}^p}.$$

Item (i) now easily follows, choosing $\gamma^{1-d/q} = \|w\|_{\mathrm{L}^d}/\|w\|_{\mathrm{L}^q}$ and $\Gamma^{d/p-1} = \|w\|_{\mathrm{L}^p}/\|w\|_{\mathrm{L}^d}$, noting that $\gamma \le \Gamma$ follows from interpolation of L^d between L^p and L^∞, and observing that

$$(q'(1-d/q))^{-1/q'} \lesssim (1-d/q)^{-1+1/d}, \qquad (p'(d/p-1))^{-1/p'} \lesssim (1-p/d)^{-1+1/d}.$$

Step 2. Proof of (ii).

Let $0 < \gamma \le 1 \le \Gamma < \infty$, and let χ_Γ denote a cut-off function with $\chi_\Gamma = 0$ on B_Γ, $\chi_\Gamma = 1$ outside $B_{\Gamma+1}$, and $|\nabla \chi_\Gamma| \le 2$. We may then decompose

$$-\nabla \triangle^{-1} w(x) = \int_{|x-y|<\gamma} \nabla g_d(x-y) w(y) dy + \int_{\gamma \le |x-y| \le \Gamma} \nabla g_d(x-y) w(y) dy \\ + \int_{\Gamma \le |x-y| \le \Gamma+1} \nabla g_d(x-y)(1-\chi_\Gamma(x-y)) w(y) dy + \int_{|x-y| \ge \Gamma} \nabla g_d(x-y) \chi_\Gamma(x-y) w(y) dy.$$

1. A direct adaptation of the proof further shows that in parts (i) and (ii) the L^∞-norms in the left-hand sides could be replaced by Hölder C^ε-norms with $\varepsilon \in [0,1)$: the exponents d in the right-hand sides then need to be all replaced by $d/(1-\varepsilon) > d$, and an additional multiplicative prefactor $(1-\varepsilon)^{-1}$ is further needed.

Using $w = \mathrm{div}\,\xi$ and integrating by parts, the last term becomes

$$\int \nabla g_d(x-y)\chi_\Gamma(x-y)w(y)dy$$
$$= -\int \nabla g_d(x-y) \otimes \nabla\chi_\Gamma(x-y)\cdot\xi(y)dy - \int \chi_\Gamma(x-y)\nabla^2 g_d(x-y)\cdot\xi(y)dy.$$

Choosing $\Gamma = 1$, we may then estimate

$$|\nabla\triangle^{-1}w(x)| \lesssim \int_{|x-y|<\gamma} |x-y|^{1-d}|w(y)|dy$$
$$+ \int_{\gamma\le|x-y|\le 2} |x-y|^{1-d}|w(y)|dy + \int_{|x-y|\ge 1} |x-y|^{-d}|\xi(y)|dy.$$

Using the Hölder inequality just as in Step 1 for the first two terms, with $d < q \le \infty$, and using the Hölder inequality with exponents $(p/(p-1), p)$ for the last term, we obtain, for any $1 \le p < \infty$,

$$|\nabla\triangle^{-1}w(x)| \lesssim (q'(1-d/q))^{-1/q'}\gamma^{1-d/q}\|w\|_{\mathrm{L}^q} + (\log(2/\gamma))^{(d-1)/d}\|w\|_{\mathrm{L}^d} + (d(p'-1))^{-1/p'}\|\xi\|_{\mathrm{L}^p},$$

so that item (ii) follows from the choice $\gamma^{1-d/q} = 1 \wedge (\|w\|_{\mathrm{L}^d}/\|w\|_{\mathrm{L}^q})$, noting that $(d(p'-1))^{-1/p'} \le p$.

Step 3. Proof of (iii).

Given $0 < \gamma \le 1$, using the integration by parts

$$\int_{|x-y|<\gamma} \nabla^2 g_d(x-y)dy = \int_{|x-y|=\gamma} n \otimes \nabla g_d(x-y)dy,$$

we may decompose

$$|\nabla^2\triangle^{-1}w(x)| \lesssim \left|\int_{|x-y|<\gamma} \frac{(x-y)^{\otimes 2}}{|x-y|^{d+2}}w(y)dy\right|$$
$$+ \left|\int_{\gamma\le|x-y|<1} \frac{(x-y)^{\otimes 2}}{|x-y|^{d+2}}w(y)dy\right| + \left|\int_{|x-y|\ge 1} \frac{(x-y)^{\otimes 2}}{|x-y|^{d+2}}w(y)dy\right|$$
$$\lesssim \left|\int_{|x-y|<\gamma} \frac{(x-y)^{\otimes 2}}{|x-y|^{d+2}}(w(x)-w(y))dy\right| + |w(x)|\left|\int_{|x-y|=\gamma} \frac{x-y}{|x-y|^{d}}dy\right|$$
$$+ \left|\int_{\gamma\le|x-y|<1} \frac{(x-y)^{\otimes 2}}{|x-y|^{d+2}}w(y)dy\right| + \left|\int_{|x-y|\ge 1} \frac{(x-y)^{\otimes 2}}{|x-y|^{d+2}}w(y)dy\right|.$$

Let $0 < s \le 1$ and $1 \le p < \infty$. Using the inequality $|w(x) - w(y)| \le |x-y|^s|w|_{C^s}$, and then applying the Hölder inequality with exponents $(1,\infty)$ for the first three terms, and $(p/(p-1), p)$ for the last one, we obtain after straightforward computations

$$|\nabla^2\triangle^{-1}w(x)| \lesssim s^{-1}\gamma^s|w|_{C^s} + \|w\|_{\mathrm{L}^\infty} + |\log\gamma|\|w\|_{\mathrm{L}^\infty} + (d(p'-1))^{-1/p'}\|w\|_{\mathrm{L}^p}.$$

Item (iii) then follows for the choice $\gamma^s = \|w\|_{\mathrm{L}^\infty}/\|w\|_{C^s} \le 1$. $\square$

In addition to the Sobolev regularity of solutions of (7.1)–(7.2), we study in the sequel their Hölder regularity as well, in the framework of the usual Besov spaces $C^s_*(\mathbb{R}^d) := B^s_{\infty,\infty}(\mathbb{R}^d)$ (see e.g. [40]). These spaces actually coincide with the usual Hölder spaces $C^s_b(\mathbb{R}^d)$ only for non-integer $s \ge 0$ (for integer $s \ge 0$ they are strictly larger than $W^{s,\infty}(\mathbb{R}^d) \supset C^s_b(\mathbb{R}^d)$ and coincide with the corresponding Zygmund spaces). The following potential theory estimates are then needed both in Sobolev and in Hölder-Zygmund spaces. As we were unable to find item (ii) stated in the literature, a short proof is included below.

Lemma 7.2.5 (Potential estimates in Sobolev and Hölder-Zygmund spaces). *Let $d \geq 2$. For all $w \in C_c^\infty(\mathbb{R}^d)$, the following hold:*

(i) for all $s \geq 0$,

$$\|\nabla\triangle^{-1}w\|_{H^s} \lesssim \|w\|_{\dot{H}^{-1}\cap H^{s-1}}, \qquad \|\nabla^2\triangle^{-1}w\|_{H^s} \lesssim \|w\|_{H^s};$$

(ii) for all $s \in \mathbb{R}$,

$$\|\nabla\triangle^{-1}w\|_{C_*^s} \lesssim_s \|w\|_{\dot{H}^{-1}\cap C_*^{s-1}}, \qquad \|\nabla^2\triangle^{-1}w\|_{C_*^s} \lesssim_s \|w\|_{\dot{H}^{-1}\cap C_*^s},$$

and for all $1 \leq p < d$ and $1 \leq q < \infty$,

$$\|\nabla\triangle^{-1}w\|_{C_*^s} \lesssim_{p,s} \|w\|_{\mathrm{L}^p\cap\mathrm{L}^\infty\cap C_*^{s-1}}, \qquad \|\nabla^2\triangle^{-1}w\|_{C_*^s} \lesssim_{q,s} \|w\|_{\mathrm{L}^q\cap C_*^s},$$

where the subscripts s, p, q indicate the additional dependence of the multiplicative constants on an upper bound on s, $(d-p)^{-1}$, and q, respectively. $\diamondsuit$

Proof. As item (i) is obvious via Fourier transform, we focus on item (ii). Let $s \in \mathbb{R}$, let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a fixed even function with $\chi = 1$ in a neighborhood of the origin, and let $\chi(\nabla)$ denote the corresponding pseudo-differential operator. Applying [40, Proposition 2.78] to the operator $(1-\chi(\nabla))\nabla\triangle^{-1}$, we find

$$\|\nabla\triangle^{-1}w\|_{C_*^s} \leq \|(1-\chi(\nabla))\nabla\triangle^{-1}w\|_{C_*^s} + \|\chi(\nabla)\nabla\triangle^{-1}w\|_{C_*^s} \lesssim_s \|w\|_{C_*^{s-1}} + \|\chi(\nabla)\nabla\triangle^{-1}w\|_{C_*^s}.$$

Let k denote the smallest nonnegative integer $\geq s$. Noting that $\|v\|_{C_*^s} \lesssim \sum_{j=0}^k \|\nabla^j v\|_{\mathrm{L}^\infty}$ holds for all v, we deduce

$$\|\nabla\triangle^{-1}w\|_{C_*^s} \lesssim \|w\|_{C_*^{s-1}} + \sum_{j=0}^k \|\nabla^j\chi(\nabla)\nabla\triangle^{-1}w\|_{\mathrm{L}^\infty},$$

and similarly

$$\|\nabla^2\triangle^{-1}w\|_{C_*^s} \lesssim \|w\|_{C_*^s} + \sum_{j=0}^k \|\nabla^j\chi(\nabla)\nabla^2\triangle^{-1}w\|_{\mathrm{L}^\infty}.$$

Writing $\nabla^j\chi(\nabla)\nabla\triangle^{-1}w = \nabla^j\hat{\chi} * \nabla\triangle^{-1}w$, we find

$$\|\nabla^j\chi(\nabla)\nabla\triangle^{-1}w\|_{\mathrm{L}^\infty} \leq \|\nabla^j\hat{\chi}\|_{\mathrm{L}^2}\|\nabla\triangle^{-1}w\|_{\mathrm{L}^2} = \|\nabla^j\hat{\chi}\|_{\mathrm{L}^2}\|w\|_{\dot{H}^{-1}},$$

and the first two estimates in item (ii) follow. Rather writing $\nabla^j\chi(\nabla)\nabla\triangle^{-1}w = \nabla\triangle^{-1}(\nabla^j\hat{\chi} * w)$, and using the estimate $|\nabla\triangle^{-1}v(x)| \lesssim \int |x-y|^{1-d}|v(y)|dy$ as in the proof of Lemma 7.2.4, we find for all $1 \leq p < d$,

$$\begin{aligned}\|\nabla^j\chi(\nabla)\nabla\triangle^{-1}w\|_{\mathrm{L}^\infty} &\lesssim \sup_x \int_{|x-y|\leq 1} |x-y|^{1-d}|\nabla^j\hat{\chi}*w(y)|dy + \sup_x \int_{|x-y|>1} |x-y|^{1-d}|\nabla^j\hat{\chi}*w(y)|dy \\ &\lesssim_p \|\nabla^j\hat{\chi}*w\|_{\mathrm{L}^p\cap\mathrm{L}^\infty} \leq \|\nabla^j\hat{\chi}\|_{\mathrm{L}^1}\|w\|_{\mathrm{L}^p\cap\mathrm{L}^\infty},\end{aligned}$$

and the third estimate in item (ii) follows. The last estimate in item (ii) is now easily obtained, arguing similarly as in the proof of Lemma 7.2.4(iii). $\square$

We now state some global elliptic regularity results for the operator $-\operatorname{div}(b\nabla)$ on the whole plane $\mathbb{R}^2$. Considering both the case of a right-hand side f and the case of a right-hand side in divergence form $\operatorname{div} g$, we compare the properties of the corresponding solutions in terms of assumptions on (f,g). As no reference was found in the literature for this 2D setting, a detailed proof is included.

Lemma 7.2.6 (2D global elliptic regularity). *Let $b \in W^{1,\infty}(\mathbb{R}^2)^{2\times 2}$ be uniformly elliptic, that is, $\mathrm{Id} \le b \le \Lambda\,\mathrm{Id}$ for some $\Lambda < \infty$. Given $f \in C_c^\infty(\mathbb{R}^2)$ and $g \in C_c^\infty(\mathbb{R}^2)^2$, we consider the decaying solutions u and v of the following equations in $\mathbb{R}^2$,*

$$-\operatorname{div}(b\nabla u) = f, \qquad \textit{and} \qquad -\operatorname{div}(b\nabla v) = \operatorname{div} g.$$

The following properties hold.

(i) Meyers type estimates: *There exists $2 < p_0, q_0, r_0 < \infty$ (depending only on an upper bound on Λ) such that for all $2 < p \le p_0$, all $q_0 \le q < \infty$, and all $r_0' \le r \le r_0$ with $\frac{1}{r_0} + \frac{1}{r_0'} = 1$,*

$$\|\nabla u\|_{\mathrm{L}^p} \le C_p \|f\|_{\mathrm{L}^{2p/(p+2)}}, \qquad \|v\|_{\mathrm{L}^q} \le C_q \|g\|_{\mathrm{L}^{2q/(q+2)}}, \qquad \|\nabla v\|_{\mathrm{L}^r} \le C\|g\|_{\mathrm{L}^r},$$

for some constant C depending only on an upper bound on Λ, and for constants C_p and C_q further depending on an upper bound on $(p-2)^{-1}$ and q, respectively.

(ii) Sobolev regularity: *For all $s \ge 0$ we have*

$$\|\nabla u\|_{H^s} \le C_s \|f\|_{\dot H^{-1}\cap H^{s-1}}, \qquad \|\nabla v\|_{H^s} \le C_s \|g\|_{H^s},$$

where the constant C_s depends only on an upper bound on s and on $\|b\|_{W^{s,\infty}}$.

(iii) Schauder type estimate: *For all $s \in (0,1)$ we have*

$$|\nabla u|_{C^s} \le C_s \|f\|_{\mathrm{L}^{2/(1-s)}}, \qquad |v|_{C^s} \le C_s' \|g\|_{\mathrm{L}^{2/(1-s)}},$$

where the constant C_s (resp. C_s') depends only on s and on an upper bound on $\|b\|_{W^{s,\infty}}$ (resp. on s and on the modulus of continuity of b).

In particular, we have

$$\|\nabla u\|_{\mathrm{L}^\infty} \le C\|f\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}, \qquad \|v\|_{\mathrm{L}^\infty} \le C'\|g\|_{\mathrm{L}^1\cap\mathrm{L}^\infty},$$

where the constant C (resp. C') depends only on an upper bound on $\|b\|_{W^{1,\infty}}$ (resp. Λ). ◊

Proof. We split the proof into three steps, first proving (i) as a consequence of Meyers' perturbative argument, then turning to the Sobolev regularity (ii), and finally to the Schauder type estimate (iii). The additional L^∞-estimate for v directly follows from item (i) and the Sobolev embedding, while the corresponding estimate for ∇u follows from items (i) and (iii) by interpolation: for $2 < p \le p_0$ and $s \in (0,1)$, we indeed find

$$\|\nabla u\|_{\mathrm{L}^\infty} \lesssim \|\nabla u\|_{\mathrm{L}^p} + |\nabla u|_{C^s} \le C_p\|f\|_{\mathrm{L}^{2p/(p+2)}} + C_s\|f\|_{\mathrm{L}^{2/(1-s)}} \le C_{p,s}\|f\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}.$$

In the proof below, we use the notation $\lesssim$ for $\le$ up to a constant $C > 0$ that depends only on an upper bound on Λ, and we add subscripts to indicate dependence on further parameters.

Step 1. Proof of (i).

We start with the norm of v. By Meyers' perturbative argument [322], there exists some $1 < r_0 < 2$ (depending only on Λ) such that $\|\nabla v\|_{\mathrm{L}^r} \lesssim \|g\|_{\mathrm{L}^r}$ holds for all $r_0 \le r \le r_0'$, $\frac{1}{r_0} + \frac{1}{r_0'} = 1$. On the other hand, decomposing the equation for v as

$$-\triangle v = \operatorname{div}(g + (b-1)\nabla v),$$

we deduce from Riesz potential theory that for all $1 < r < 2$

$$\|v\|_{\mathrm{L}^{2r/(2-r)}} \lesssim_r \|g + (b-1)\nabla v\|_{\mathrm{L}^r} \lesssim \|g\|_{\mathrm{L}^r} + \|\nabla v\|_{\mathrm{L}^r},$$

and hence $\|v\|_{\mathrm{L}^{2r/(2-r)}} \lesssim_r \|g\|_{\mathrm{L}^r}$ for all $r_0 \le r < 2$, that is, $\|v\|_{\mathrm{L}^q} \lesssim_q \|g\|_{\mathrm{L}^{2q/(q+2)}}$ for all $\frac{2r_0}{2-r_0} \le q < \infty$.

We now turn to the norm of ∇u. The proof follows from a suitable adaptation of Meyers' perturbative argument [322], again combined with Riesz potential theory. For the reader's convenience a complete proof is included. First recall that the Calderón-Zygmund theory yields $\|\nabla^2 \triangle w\|_{\mathrm{L}^p} \le K_p \|w\|_{\mathrm{L}^p}$ for all $1 < p < \infty$ and all $w \in C_c^\infty(\mathbb{R}^2)$, where the constants K_p's moreover satisfy $\limsup_{p\to 2} K_p \le K_2$, while a simple energy estimate allows to choose $K_2 = 1$. Now rewriting the equation for u as

$$-\triangle u = \frac{2}{\Lambda+1} f + \operatorname{div}\left(\frac{2}{\Lambda+1}\Big(b - \frac{\Lambda+1}{2}\Big)\nabla u\right),$$

we deduce from Riesz potential theory and from the Calderón-Zygmund theory (applied to the first and to the second right-hand side term, respectively), for all $2 < p < \infty$,

$$\begin{aligned}
\|\nabla u\|_{\mathrm{L}^p} &\le \frac{2}{\Lambda+1}\|\nabla\triangle^{-1} f\|_{\mathrm{L}^p} + \left\|\nabla\triangle^{-1}\operatorname{div}\left(\frac{2}{\Lambda+1}\Big(b-\frac{\Lambda+1}{2}\Big)\nabla u\right)\right\|_{\mathrm{L}^p}\\
&\le \frac{2C_p}{\Lambda+1}\|f\|_{\mathrm{L}^{2p/(p+2)}} + \frac{2K_p}{\Lambda+1}\left\|\Big(b-\frac{\Lambda+1}{2}\Big)\nabla u\right\|_{\mathrm{L}^p}\\
&\le \frac{2C_p}{\Lambda+1}\|f\|_{\mathrm{L}^{2p/(p+2)}} + \frac{K_p(\Lambda-1)}{\Lambda+1}\|\nabla u\|_{\mathrm{L}^p},
\end{aligned}$$

where the last inequality follows from $\mathrm{Id} \le b \le \Lambda\,\mathrm{Id}$. Since we have $\frac{\Lambda-1}{\Lambda+1} < 1$ and $\limsup_{p\to 2} K_p \le K_2 = 1$, we may choose $\mathrm{p}_0 > 2$ close enough to 2 such that $\frac{K_p(\Lambda-1)}{\Lambda+1} < 1$ holds for all $2 \le p \le p_0$. This allows to absorb the last right-hand side term, and to conclude $\|\nabla u\|_{\mathrm{L}^p} \lesssim_p \|f\|_{\mathrm{L}^{2p/(p+2)}}$ for all $2 < p \le p_0$.

Step 2. Proof of (ii).

We focus on the result for u, as the argument for v is very similar. A simple energy estimate yields

$$\int |\nabla u|^2 \le \int \nabla u \cdot b\nabla u = \int fu \le \|f\|_{\dot H^{-1}} \|\nabla u\|_{\mathrm{L}^2},$$

hence $\|\nabla u\|_{\mathrm{L}^2} \le \|f\|_{\dot H^{-1}}$, that is, (ii) with $s = 0$. The result (ii) for any integer $s \ge 0$ is then deduced by induction, successively differentiating the equation. It remains to consider the case of fractional values $s \ge 0$. We only display the argument for $0 < s < 1$, while the other cases are similarly obtained after differentiation of the equation. Let $0 < s < 1$ be fixed. We use the following finite difference characterization of the fractional Sobolev space $H^s(\mathbb{R}^2)$: a function $w \in \mathrm{L}^2(\mathbb{R}^2)$ belongs to $H^s(\mathbb{R}^2)$, if and only if it satisfies $\|w - w(\cdot+h)\|_{\mathrm{L}^2} \le K|h|^s$ for all $h \in \mathbb{R}^2$, for some $K > 0$, and we then have $\|w\|_{\dot H^s} \le K$. This characterization is easily checked, using e.g. the identity $\|w - w(\cdot+h)\|^2_{\mathrm{L}^2} \simeq \int |1 - e^{i\xi\cdot h}|^2 |\hat w(\xi)|^2 d\xi$, where $\hat w$ denotes the Fourier transform of w, and noting that $|1 - e^{ia}| \le 2 \wedge |a|$ holds for all $a \in \mathbb{R}$. Now applying finite difference to the equation for u, we find for all $h \in \mathbb{R}^2$,

$$-\operatorname{div}\big(b(\cdot+h)(\nabla u - \nabla u(\cdot+h))\big) = \operatorname{div}\big((b - b(\cdot+h))\nabla u\big) + f - f(\cdot+h),$$

and hence, testing against $u - u(\cdot+h)$,

$$\begin{aligned}
\int |\nabla u - \nabla u(\cdot+h)|^2 &\le -\int (\nabla u - \nabla u(\cdot+h))\cdot(b - b(\cdot+h))\nabla u + \int (u - u(\cdot+h))(f - f(\cdot+h))\\
&\le |h|^s |b|_{C^s} \|\nabla u\|_{\mathrm{L}^2} \|\nabla u - \nabla u(\cdot+h)\|_{\mathrm{L}^2} + \|f - f(\cdot+h)\|_{\dot H^{-1}} \|\nabla u - \nabla u(\cdot+h)\|_{\mathrm{L}^2},
\end{aligned}$$

where we compute by means of Fourier transforms

$$\|f - f(\cdot+h)\|^2_{\dot H^{-1}} \simeq \int |\xi|^{-2} |1 - e^{i\xi\cdot h}|^2 |\hat f(\xi)|^2 d\xi \lesssim \int |\xi|^{-2} |\xi\cdot h|^{2s} |\hat f(\xi)|^2 d\xi \lesssim |h|^{2s} \|f\|^2_{\dot H^{-1}\cap H^{s-1}}.$$

Further combining this with the L^2-estimate for ∇u proven at the beginning of this step, we conclude

$$\|\nabla u-\nabla u(\cdot+h)\|_{\mathrm{L}^2}\lesssim|h|^s(|b|_{C^s}\|\nabla u\|_{\mathrm{L}^2}+\|f\|_{\dot H^{-1}\cap H^{s-1}})\lesssim|h|^s(1+|b|_{C^s})\|f\|_{\dot H^{-1}\cap H^{s-1}},$$

and the result follows from the above stated characterization of $H^s(\mathbb{R}^2)$.

Step 3. Proof of (iii).

We focus on the result for u, while that for v is easily obtained as an adaptation of [229, Theorem 3.8]. Let $x_0\in\mathbb{R}^2$ be fixed. The equation for u may be rewritten as

$$-\operatorname{div}(b(x_0)\nabla u)=f+\operatorname{div}((b-b(x_0))\nabla u).$$

For all $r>0$, let $w_r\in u+H^1_0(B(x_0,r))$ be the unique solution of $-\operatorname{div}(b(x_0)\nabla w_r)=0$ in $B(x_0,r)$. The difference $v_r:=u-w_r\in H^1_0(B(x_0,r))$ then satisfies in $B(x_0,r)$

$$-\operatorname{div}(b(x_0)\nabla v_r)=f+\operatorname{div}((b-b(x_0))\nabla u).$$

Testing this equation against v_r itself, we obtain

$$\int|\nabla v_r|^2\le\left|\int_{B(x_0,r)}fv_r\right|+\int_{B(x_0,r)}|b-b(x_0)||\nabla u||\nabla v_r|\le\left|\int_{B(x_0,r)}fv_r\right|+r^s|b|_{C^s}\|\nabla u\|_{\mathrm{L}^2(B(x_0,r))}\|\nabla v_r\|_{\mathrm{L}^2}.$$

We estimate the first term as follows

$$\left|\int_{B(x_0,r)}fv_r\right|=\left|\int_{B(x_0,r)}\nabla v_r\cdot\nabla\triangle^{-1}(\mathbb{1}_{B(x_0,r)}f)\right|\le\|\nabla v_r\|_{\mathrm{L}^{p'}(B(x_0,r))}\|\nabla\triangle^{-1}(\mathbb{1}_{B(x_0,r)}f)\|_{\mathrm{L}^p},$$

and hence by Riesz potential theory, for all $2<p<\infty$,

$$\left|\int_{B(x_0,r)}fv_r\right|\lesssim_p\|\nabla v_r\|_{\mathrm{L}^{p'}(B(x_0,r))}\|f\|_{\mathrm{L}^{2p/(p+2)}(B(x_0,r))}.$$

The Hölder inequality then yields, choosing $q:=\frac{2}{1-s}>2$,

$$\left|\int_{B(x_0,r)}fv_r\right|\lesssim_p r^{\frac{2}{p'}-1}\|\nabla v_r\|_{\mathrm{L}^2}\,r^{1+\frac2p-\frac2q}\|f\|_{\mathrm{L}^q}=r^{2(1-\frac1q)}\|\nabla v_r\|_{\mathrm{L}^2}\|f\|_{\mathrm{L}^q}=r^{1+s}\|\nabla v_r\|_{\mathrm{L}^2}\|f\|_{\mathrm{L}^{2/(1-s)}}.$$

Combining the above estimates, we deduce

$$\int|\nabla v_r|^2\lesssim r^{2(1+s)}\|f\|^2_{\mathrm{L}^{2/(1-s)}}+r^{2s}|b|^2_{C^s}\|\nabla u\|^2_{\mathrm{L}^2(B(x_0,r))}.$$

We are now in position to conclude exactly as in the classical proof of the Schauder estimates (see e.g. [229, Theorem 3.13]). ☐

The interaction force v in equation (7.9) is defined by the values of $\operatorname{curl}\mathrm{v}$ and $\operatorname{div}(a\mathrm{v})$. The following result shows how v is controlled by such specifications.

Lemma 7.2.7. *Let $a,a^{-1}\in\mathrm{L}^\infty(\mathbb{R}^2)$. For all $\delta\mathrm{m},\delta\mathrm{d}\in\dot H^{-1}(\mathbb{R}^2)$, there exists a unique $\delta\mathrm{v}\in\mathrm{L}^2(\mathbb{R}^2)^2$ such that $\operatorname{curl}\delta\mathrm{v}=\delta\mathrm{m}$ and $\operatorname{div}(a\delta\mathrm{v})=\delta\mathrm{d}$. Moreover, for all $s\ge0$, if $a,a^{-1}\in W^{s,\infty}(\mathbb{R}^2)$ and $\delta\mathrm{m},\delta\mathrm{d}\in\dot H^{-1}\cap H^{s-1}(\mathbb{R}^2)$, we have*

$$\|\delta\mathrm{v}\|_{H^s}\le C\|\delta\mathrm{m}\|_{\dot H^{-1}\cap H^{s-1}}+C\|\delta\mathrm{d}\|_{\dot H^{-1}\cap H^{s-1}},$$

where the constant C depends only on an upper bound on s and $\|(a,a^{-1})\|_{W^{s,\infty}}$. ◊

Proof. We split the proof into two steps.

Step 1. Uniqueness.

We prove that at most one function $\delta\mathrm{v} \in \mathrm{L}^2(\mathbb{R}^2)^2$ can be associated with a given couple $(\delta\mathrm{m}, \delta\mathrm{d})$. For that purpose, we assume that $\delta\mathrm{v} \in \mathrm{L}^2(\mathbb{R}^2)^2$ satisfies $\operatorname{curl} \delta\mathrm{v} = 0$ and $\operatorname{div}(a\delta\mathrm{v}) = 0$, and we deduce $\delta\mathrm{v} = 0$. By the Hodge decomposition in $\mathrm{L}^2(\mathbb{R}^2)^2$, there exist functions $\phi, \psi \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that $a\delta\mathrm{v} = \nabla\phi + \nabla^\perp\psi$ with $\nabla\phi, \nabla\psi \in \mathrm{L}^2(\mathbb{R}^2)^2$. Now note that $\triangle\phi = \operatorname{div}(a\delta\mathrm{v}) = 0$ and $\operatorname{div}(a^{-1}\nabla\psi) + \operatorname{curl}(a^{-1}\nabla\phi) = \operatorname{curl}\delta\mathrm{v} = 0$, which implies $\nabla\phi = 0$ and $\nabla\psi = 0$, hence $\delta\mathrm{v} = 0$.

Step 2. Existence.

Given $\delta\mathrm{m}, \delta\mathrm{d} \in \dot H^{-1}(\mathbb{R}^2)$, we observe that $\nabla(\operatorname{div} a^{-1}\nabla)^{-1}\delta\mathrm{m}$ and $\nabla(\operatorname{div} a\nabla)^{-1}\delta\mathrm{d}$ are well-defined in $\mathrm{L}^2(\mathbb{R}^2)^2$. The vector field

$$\delta\mathrm{v} := a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\delta\mathrm{m} + \nabla(\operatorname{div} a\nabla)^{-1}\delta\mathrm{d}$$

is thus well-defined in $\mathrm{L}^2(\mathbb{R}^2)^2$, and trivially satisfies $\operatorname{curl}\delta\mathrm{v} = \delta\mathrm{m}$, $\operatorname{div}(a\delta\mathrm{v}) = \delta\mathrm{d}$. The additional estimate follows from Lemmas 7.2.1 and 7.2.6(ii). ☐

As emphasized in Remark 7.1.2(i), weak solutions of the incompressible model (7.1) are rather defined via the vorticity formulation (7.9) in order to avoid compactness issues related to the pressure p. Although this will not be used in the sequel, we quickly check that under mild regularity assumptions a weak solution v of (7.1) automatically also satisfies equation (7.1) in the distributional sense on $[0,T) \times \mathbb{R}^2$ for some pressure $\mathrm{p} : \mathbb{R}^2 \to \mathbb{R}$.

Lemma 7.2.8 (Control on the pressure)**.** *Let $\alpha, \beta \in \mathbb{R}$, $T > 0$, $h \in W^{1,\infty}(\mathbb{R}^2)$, and $\Psi, \bar{\mathrm{v}}^\circ \in \mathrm{L}^\infty(\mathbb{R}^2)^2$. There exists some $2 < q_0 \lesssim 1$ large enough (depending only on an upper bound on $\|h\|_{\mathrm{L}^\infty}$) such that the following holds: If $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ is a weak solution of (7.1) on $[0,T) \times \mathbb{R}^2$ with $\mathrm{m} := \operatorname{curl}\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathcal{P} \cap \mathrm{L}^{q_0}(\mathbb{R}^2))$, then v satisfies (7.1) in the distributional sense on $[0,T) \times \mathbb{R}^2$ for some pressure $\mathrm{p} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^{q_0}(\mathbb{R}^2))$.* ◊

Proof. In this proof, we use the notation $\lesssim$ for $\le$ up to a constant C depending only on an upper bound on $\|(h, \Psi, \bar{\mathrm{v}}^\circ)\|_{\mathrm{L}^\infty}$. Let $2 < p_0, q_0 \lesssim 1$ and $r_0 = p_0$ be as in Lemma 7.2.6(i) (with b replaced by a or a^{-1}), and note that q_0 can be chosen large enough such that $\frac{1}{p_0} + \frac{1}{q_0} \le \frac{1}{2}$. Assume that $\mathrm{m} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathcal{P} \cap \mathrm{L}^{q_0}(\mathbb{R}^2))$ holds for this choice of the exponent q_0. By Lemma 7.2.6(i), the function

$$\mathrm{p} := (-\operatorname{div} a\nabla)^{-1}\operatorname{div}\big(a\mathrm{m}(-\alpha(\Psi + \mathrm{v}) + \beta(\Psi + \mathrm{v})^\perp)\big)$$

is well-defined in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^{q_0}(\mathbb{R}^2))$ and satisfies for all $t \ge 0$,

$$\begin{aligned}
\|\mathrm{p}^t\|_{\mathrm{L}^{q_0}} &\lesssim \|a\mathrm{m}^t(-\alpha(\Psi + \mathrm{v}^t) + \beta(\Psi + \mathrm{v}^t)^\perp)\|_{\mathrm{L}^{2q_0/(2+q_0)}} \\
&\lesssim \|\Psi + \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty}\|\mathrm{m}^t\|_{\mathrm{L}^{2q_0/(2+q_0)}} + \|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}\|\mathrm{m}^t\|_{\mathrm{L}^{q_0}} \\
&\lesssim (1 + \|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2})\|\mathrm{m}^t\|_{\mathrm{L}^1 \cap \mathrm{L}^{q_0}}.
\end{aligned}$$

Now note that the following Helmholtz-Leray type identity follows from the proof of Lemma 7.2.7: for any vector field $F \in C^\infty_c(\mathbb{R}^2)^2$,

$$F = a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\operatorname{curl} F + \nabla(\operatorname{div} a\nabla)^{-1}\operatorname{div}(aF). \tag{7.18}$$

This implies in particular, for the choice $F = \mathrm{m}\big(-\alpha(\Psi + \mathrm{v}) + \beta(\Psi + \mathrm{v})^\perp\big)$,

$$\begin{aligned}
&a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\operatorname{div}\big(\mathrm{m}(\alpha(\Psi + \mathrm{v})^\perp + \beta(\Psi + \mathrm{v}))\big) \\
&\quad= a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\operatorname{curl}\big(\mathrm{m}(-\alpha(\Psi + \mathrm{v}) + \beta(\Psi + \mathrm{v})^\perp)\big) \\
&\quad= \mathrm{m}\big(-\alpha(\Psi + \mathrm{v}) + \beta(\Psi + \mathrm{v})^\perp\big) + \nabla\mathrm{p}\,.
\end{aligned} \tag{7.19}$$

For $\phi \in C_c^\infty([0,T)\times\mathbb{R}^2)^2$, it follows from Lemma 7.2.6(i) that $(\operatorname{div} a^{-1}\nabla)^{-1}\operatorname{curl}(a^{-1}\phi)$ belongs to $C_c^\infty([0,T);\mathrm{L}^{q_0}(\mathbb{R}^2))$ and that $\nabla(\operatorname{div} a^{-1}\nabla)^{-1}\operatorname{curl}(a^{-1}\phi)$ belongs to $C_c^\infty([0,T);\mathrm{L}^2\cap\mathrm{L}^{p_0}(\mathbb{R}^2))$. With the choice $\frac{1}{p_0}+\frac{1}{q_0}\le\frac12$, the L^{q_0}-regularity of m then allows to test the weak formulation of (7.9) (which defines weak solutions of (7.1), cf. Definition 7.1.1(b)) against $(\operatorname{div} a^{-1}\nabla)^{-1}\operatorname{curl}(a^{-1}\phi)$, to the effect of

$$\int \mathrm{m}^\circ(\operatorname{div} a^{-1}\nabla)^{-1}\operatorname{curl}(a^{-1}\phi(0,\cdot))+\iint \mathrm{m}(\operatorname{div} a^{-1}\nabla)^{-1}\operatorname{curl}(a^{-1}\partial_t\phi)$$
$$=\iint \mathrm{m}(\alpha(\Psi+\mathrm{v})^\perp+\beta(\Psi+\mathrm{v}))\cdot\nabla(\operatorname{div} a^{-1}\nabla)^{-1}\operatorname{curl}(a^{-1}\phi).$$

Since by (7.18) the constraint $\operatorname{div}(a\mathrm{v})=0$ implies $\mathrm{v}=a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\,\mathrm{m}$ and similarly $\mathrm{v}^\circ=a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\,\mathrm{m}^\circ$, and since by definition $\mathrm{m}\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^1\cap\mathrm{L}^2(\mathbb{R}^2))$, Lemma 7.2.6(i) implies $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^{p_0}(\mathbb{R}^2)^2)$. We may then integrate by parts in the weak formulation above, which yields

$$\int\phi(0,\cdot)\cdot\mathrm{v}^\circ+\iint\partial_t\phi\cdot\mathrm{v}=-\iint a^{-1}\phi\cdot\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\operatorname{div}(\mathrm{m}(\alpha(\Psi+\mathrm{v})^\perp+\beta(\Psi+\mathrm{v}))),$$

and the result now directly follows from the decomposition (7.19). $\square$

7.3 Local-in-time existence of smooth solutions

In this section, we prove the local-in-time existence of smooth solutions of (7.1)–(7.2) as summarized in Theorem 7.1.4. Note that we choose to work here in the framework of Sobolev spaces, but the results could easily be adapted to Hölder spaces (compare indeed with Lemma 7.4.7). We start with the non-degenerate case $\lambda>0$, using a standard iterative scheme as e.g. in [395, Appendix B].

Proposition 7.3.1 (Local existence, non-degenerate case)**.** *Let $\alpha,\beta\in\mathbb{R}$, $\lambda>0$. Let $s>1$, and let $h,\Psi,\bar{\mathrm{v}}^\circ\in W^{s+1,\infty}(\mathbb{R}^2)^2$. Let $\mathrm{v}^\circ\in\bar{\mathrm{v}}^\circ+H^{s+1}(\mathbb{R}^2)^2$ with $\mathrm{m}^\circ:=\operatorname{curl}\mathrm{v}^\circ$, $\bar{\mathrm{m}}^\circ:=\operatorname{curl}\bar{\mathrm{v}}^\circ\in H^s(\mathbb{R}^2)$, and with either* $\operatorname{div}(a\mathrm{v}^\circ)=\operatorname{div}(a\bar{\mathrm{v}}^\circ)=0$ *in the case* (7.1)*, or* $\mathrm{d}^\circ:=\operatorname{div}(a\mathrm{v}^\circ)$, $\bar{\mathrm{d}}^\circ:=\operatorname{div}(a\bar{\mathrm{v}}^\circ)\in H^s(\mathbb{R}^2)$ *in the case* (7.2)*. Then there exists $T>0$ and a weak solution* $\mathrm{v}\in\mathrm{L}^\infty([0,T);\bar{\mathrm{v}}^\circ+H^{s+1}(\mathbb{R}^2)^2)$ *of* (7.1) *or of* (7.2) *on $[0,T)\times\mathbb{R}^2$ with initial data v°. Moreover, T depends only on an upper bound on $|\alpha|$, $|\beta|$, λ, λ^{-1}, s, $(s-1)^{-1}$, $\|(h,\Psi,\bar{\mathrm{v}}^\circ)\|_{W^{s+1,\infty}}$, $\|\mathrm{v}^\circ-\bar{\mathrm{v}}^\circ\|_{H^{s+1}}$, $\|(\mathrm{m}^\circ,\bar{\mathrm{m}}^\circ,\mathrm{d}^\circ,\bar{\mathrm{d}}^\circ)\|_{H^s}$.* $\Diamond$

Proof. We focus on the compressible case (7.2), the situation being similar and simpler in the incompressible case (7.1). Let $s>1$. We set up the following iterative scheme: let $\mathrm{v}_0:=\mathrm{v}^\circ$, $\mathrm{m}_0:=\mathrm{m}^\circ=\operatorname{curl}\mathrm{v}^\circ$ and $\mathrm{d}_0:=\mathrm{d}^\circ=\operatorname{div}(a\mathrm{v}^\circ)$, and for all $n\ge0$ given v_n, $\mathrm{m}_n:=\operatorname{curl}\mathrm{v}_n$, and $\mathrm{d}_n:=\operatorname{div}(a\mathrm{v}_n)$ we let m_{n+1} and d_{n+1} solve on $\mathbb{R}^+\times\mathbb{R}^2$ the linear equations

$$\partial_t\mathrm{m}_{n+1}=\operatorname{div}(\mathrm{m}_{n+1}(\alpha(\Psi+\mathrm{v}_n)^\perp+\beta(\Psi+\mathrm{v}_n))),\quad \mathrm{m}_{n+1}|_{t=0}=\mathrm{m}^\circ,\tag{7.20}$$
$$\partial_t\mathrm{d}_{n+1}=\lambda\triangle\mathrm{d}_{n+1}-\lambda\operatorname{div}(\mathrm{d}_{n+1}\nabla h)+\operatorname{div}(a\mathrm{m}_n(-\alpha(\Psi+\mathrm{v}_n)+\beta(\Psi+\mathrm{v}_n)^\perp)),\quad \mathrm{d}_{n+1}|_{t=0}=\mathrm{d}^\circ,\tag{7.21}$$

and we let v_{n+1} satisfy $\operatorname{curl}\mathrm{v}_{n+1}=\mathrm{m}_{n+1}$ and $\operatorname{div}(a\mathrm{v}_{n+1})=\mathrm{d}_{n+1}$. For all $n\ge0$, let also

$$t_n:=\sup\Big\{t\ge0:\|(\mathrm{m}_n^t,\mathrm{d}_n^t)\|_{H^s}+\|\mathrm{v}_n^t-\bar{\mathrm{v}}^\circ\|_{H^{s+1}}\le C_0\Big\},$$

for some $C_0\ge1$ to be suitably chosen (depending on the initial data), and let $T_0:=\inf_n t_n$. We show that this iterative scheme is well-defined with $T_0>0$, and that it converges to a solution of equation (7.2) on $[0,T_0)\times\mathbb{R}^2$.

We split the proof into two steps. In this proof, we use the notation $\lesssim$ for $\le$ up to a constant $C>0$ that depends only on an upper bound on $|\alpha|$, $|\beta|$, λ, λ^{-1}, s, $(s-1)^{-1}$, $\|(h,\Psi,\bar{\mathrm{v}}^\circ)\|_{W^{s+1,\infty}}$, $\|\mathrm{v}^\circ-\bar{\mathrm{v}}^\circ\|_{H^{s+1}}$, $\|(\mathrm{d}^\circ,\bar{\mathrm{d}}^\circ)\|_{H^s}$, and $\|(\mathrm{m}^\circ,\bar{\mathrm{m}}^\circ)\|_{H^s}$.

Step 1. The iterative scheme is well-defined.

In this step, we show that for all $n \geq 0$ the system (7.20)–(7.21) admits a unique solution $(\mathrm{m}_{n+1}, \mathrm{d}_{n+1}, \mathrm{v}_{n+1})$ with $\mathrm{m}_{n+1} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^2))$, $\mathrm{d}_{n+1} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^2))$, and $\mathrm{v}_{n+1} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$, and that moreover for a suitable choice of $1 \leq C_0 \lesssim 1$ we have $T_0 \geq C_0^{-4} > 0$. We argue by induction. Let $n \geq 0$ be fixed, and assume that $(\mathrm{m}_n, \mathrm{d}_n, \mathrm{v}_n)$ is well-defined with $\mathrm{m}_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^2))$, $\mathrm{d}_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^2))$, and $\mathrm{v}_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$. (For $n = 0$, this is indeed trivial by assumption.)

We first study the equation for m_{n+1}. By the Sobolev embedding with $s > 1$, v_n is Lipschitz-continuous, and by assumption Ψ is also Lipschitz-continuous, hence the transport equation (7.20) admits a unique continuous solution m_{n+1}, which automatically belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{m}^\circ + \dot{H}^{-1} \cap H^s(\mathbb{R}^2))$ by Lemma 7.2.2. More precisely, for all $t \geq 0$, Lemma 7.2.2 together with the Sobolev embedding for $s > 1$ yields

$$\begin{aligned}\partial_t \|\mathrm{m}^t_{n+1}\|_{H^s} &\leq C(1 + \|\mathrm{v}^t_n\|_{W^{1,\infty}})\|\mathrm{m}^t_{n+1}\|_{H^s} + C\|\mathrm{m}^t_{n+1}\|_{\mathrm{L}^\infty}\|\mathrm{v}^t_n - \bar{\mathrm{v}}^\circ\|_{H^{s+1}} \\ &\leq C(1 + \|\mathrm{v}^t_n - \bar{\mathrm{v}}^\circ\|_{H^{s+1}})\|\mathrm{m}^t_{n+1}\|_{H^s}.\end{aligned}$$

Hence, for all $t \in [0, t_n]$, we obtain $\partial_t \|\mathrm{m}^t_{n+1}\|_{H^s} \leq CC_0 \|\mathrm{m}^t_{n+1}\|_{H^s}$, which proves

$$\|\mathrm{m}^t_{n+1}\|_{H^s} \leq e^{CC_0 t}\|\mathrm{m}^\circ\|_{H^s} \leq Ce^{CC_0 t}.$$

Noting that

$$\|\mathrm{m}^\circ - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1}} \leq \|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2} \leq C,$$

Lemma 7.2.2 together with the Sobolev embedding for $s > 1$ also gives for all $t \geq 0$,

$$\begin{aligned}\|\mathrm{m}^t_{n+1} - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1}} \leq C + \|\mathrm{m}^t_{n+1} - \mathrm{m}^\circ\|_{\dot{H}^{-1}} &\leq C + Ct\|\mathrm{m}_{n+1}\|_{\mathrm{L}^\infty_t \mathrm{L}^2}(1 + \|\mathrm{v}_n\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}) \\ &\leq C + Ct\|\mathrm{m}_{n+1}\|_{\mathrm{L}^\infty_t H^s}(1 + \|\mathrm{v}_n - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty_t H^s}),\end{aligned}$$

and hence, for all $t \in [0, t_n]$,

$$\|\mathrm{m}^t_{n+1} - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1}} \leq C(1 + tC_0)e^{CC_0 t}.$$

We now turn to d_{n+1}. Equation (7.21) (with $\lambda > 0$) is a transport-diffusion equation and admits a unique solution d_{n+1}, which belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{d}^\circ + \dot{H}^{-1} \cap H^s(\mathbb{R}^2))$ by Lemma 7.2.3(i)–(ii). More precisely, for all $t \geq 0$, Lemma 7.2.3(i) yields for $s > 1$

$$\begin{aligned}\|\mathrm{d}^t_{n+1}\|_{H^s} &\leq Ce^{Ct}\big(\|\mathrm{d}^\circ\|_{H^s} + \|a\mathrm{m}_n(\alpha(\Psi + \mathrm{v}_n)^\perp + \beta(\Psi + \mathrm{v}_n))\|_{\mathrm{L}^2_t H^s}\big) \\ &\leq Ce^{Ct}\big(1 + t^{1/2}\|\mathrm{m}_n\|_{\mathrm{L}^\infty_t H^s}(1 + \|\mathrm{v}_n - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty_t H^s})\big), \qquad (7.22)\end{aligned}$$

where we have used Lemma 7.2.1 together with the Sobolev embedding to estimate the terms. Noting that

$$\|\mathrm{d}^\circ - \bar{\mathrm{d}}^\circ\|_{\dot{H}^{-1}} \leq \|a\mathrm{v}^\circ - a\bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2} \leq C,$$

Lemma 7.2.3(ii) together with the Sobolev embedding for $s > 1$ also gives for all $t \geq 0$,

$$\begin{aligned}\|\mathrm{d}^t_{n+1} - \bar{\mathrm{d}}^\circ\|_{\dot{H}^{-1}} \leq C + \|\mathrm{d}^t_{n+1} - \mathrm{d}^\circ\|_{\dot{H}^{-1}} &\leq C + Ce^{Ct}(\|\mathrm{d}^\circ\|_{\mathrm{L}^2} + \|a\mathrm{m}_n(\alpha(\Psi + \mathrm{v}_n)^\perp + \beta(\Psi + \mathrm{v}_n))\|_{\mathrm{L}^2_t \mathrm{L}^2}) \\ &\leq Ce^{Ct}(1 + t^{1/2}\|\mathrm{m}_n\|_{\mathrm{L}^\infty_t H^s}(1 + \|\mathrm{v}_n - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty_t H^s}).\end{aligned}$$

Combining this with (7.22) yields for all $t \in [0, t_n]$,

$$\|\mathrm{d}^t_{n+1}\|_{H^s} + \|\mathrm{d}^t_{n+1} - \bar{\mathrm{d}}^\circ\|_{\dot{H}^{-1}} \leq Ce^{Ct}\big(1 + t^{1/2}C_0(1 + C_0)\big) \leq C(1 + t^{1/2}C_0^2)e^{Ct}.$$

We finally turn to v_{n+1}. By the above properties of m_{n+1} and d_{n+1}, Lemma 7.2.7 ensures that v_{n+1} is uniquely defined in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$ with $\operatorname{curl}(\mathrm{v}^t_{n+1} - \bar{\mathrm{v}}^\circ) = \mathrm{m}^t_{n+1} - \bar{\mathrm{m}}^\circ$ and $\operatorname{div}(a(\mathrm{v}^t_{n+1} - \bar{\mathrm{v}}^\circ)) = \mathrm{d}^t_{n+1} - \bar{\mathrm{d}}^\circ$ for all $t \geq 0$. More precisely, Lemma 7.2.7 gives for all $t \in [0, t_n]$,

$$\begin{aligned}
\|\mathrm{v}^t_{n+1} - \bar{\mathrm{v}}^\circ\|_{H^{s+1}} &\leq C\|\mathrm{m}^t_{n+1} - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1}\cap H^s} + C\|\mathrm{d}^t_{n+1} - \bar{\mathrm{d}}^\circ\|_{\dot{H}^{-1}\cap H^s} \\
&\leq C + C\|\mathrm{m}^t_{n+1} - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1}} + C\|\mathrm{m}^t_{n+1}\|_{H^s} + C\|\mathrm{d}^t_{n+1} - \bar{\mathrm{d}}^\circ\|_{\dot{H}^{-1}} + C\|\mathrm{d}^t_{n+1}\|_{H^s} \\
&\leq C(1 + tC_0 + t^{1/2}C_0^2)e^{CC_0 t}.
\end{aligned}$$

Hence, we have proven that $(\mathrm{m}_{n+1}, \mathrm{d}_{n+1}, \mathrm{v}_{n+1})$ is well-defined in the correct space, and moreover, combining all the previous estimates, we find for all $t \in [0, t_n]$,

$$\|(\mathrm{m}^t_{n+1}, \mathrm{d}^t_{n+1})\|_{H^s} + \|\mathrm{v}^t_{n+1} - \bar{\mathrm{v}}^\circ\|_{H^{s+1}} \leq C(1 + tC_0 + t^{1/2}C_0^2)e^{CC_0 t}.$$

Therefore, choosing $C_0 = 1 + 3Ce^C \lesssim 1$, we obtain for all $t \leq t_n \wedge C_0^{-4}$,

$$\|(\mathrm{m}^t_{n+1}, \mathrm{d}^t_{n+1})\|_{H^s} + \|\mathrm{v}^t_{n+1} - \bar{\mathrm{v}}^\circ\|_{H^{s+1}} \leq C_0,$$

and thus $t_{n+1} \geq t_n \wedge C_0^{-4}$. The result follows by induction.

Step 2. Passing to the limit in the scheme.

In this step, we show that up to an extraction the iterative scheme $(\mathrm{m}_n, \mathrm{d}_n, \mathrm{v}_n)$ converges to a weak solution of equation (7.2) on $[0, T_0) \times \mathbb{R}^2$.

By Step 1, the sequences $(\mathrm{m}_n)_n$ and $(\mathrm{d}_n)_n$ are bounded in $\mathrm{L}^\infty([0, T_0]; H^s(\mathbb{R}^2)^2)$, and the sequence $(\mathrm{v}_n)_n$ is bounded in $\mathrm{L}^\infty([0, T_0]; \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$. Up to an extraction, we thus have $\mathrm{m}_n \overset{*}{\rightharpoonup} \mathrm{m}$, $\mathrm{d}_n \overset{*}{\rightharpoonup} \mathrm{d}$ in $\mathrm{L}^\infty([0, T_0]; H^s(\mathbb{R}^2))$, and $\mathrm{v}_n \overset{*}{\rightharpoonup} \mathrm{v}$ in $\mathrm{L}^\infty([0, T_0]; \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$. Comparing with equation (7.20), we deduce that $(\partial_t \mathrm{m}_n)_n$ is bounded in $\mathrm{L}^\infty([0, T_0]; H^{s-1}(\mathbb{R}^2))$. Since by the Rellich theorem the space $H^s(U)$ is compactly embedded in $H^{s-1}(U)$ for any bounded domain $U \subset \mathbb{R}^2$, the Aubin-Simon lemma ensures that we have $\mathrm{m}_n \to \mathrm{m}$ strongly in $C^0([0, T_0]; H^{s-1}_{\mathrm{loc}}(\mathbb{R}^2))$. This implies in particular $\mathrm{m}_n \mathrm{v}_n \to \mathrm{m}\mathrm{v}$ in the distributional sense, and hence we may pass to the limit in the weak formulation of equations (7.20)–(7.21), which yields $\operatorname{curl} \mathrm{v} = \mathrm{m}$, $\operatorname{div}(a\mathrm{v}) = \mathrm{d}$, with m and d satisfying in the distributional sense on $[0, T_0) \times \mathbb{R}^2$,

$$\begin{aligned}
\partial_t \mathrm{m} &= \operatorname{div}(\mathrm{m}(\alpha(\Psi + \mathrm{v})^\perp + \beta(\Psi + \mathrm{v}))), \quad \mathrm{m}|_{t=0} = \mathrm{m}^\circ, \\
\partial_t \mathrm{d} &= \lambda \triangle \mathrm{d} - \lambda \operatorname{div}(\mathrm{d}\nabla h) + \operatorname{div}(a\mathrm{m}(-\alpha(\Psi + \mathrm{v}) + \beta(\Psi + \mathrm{v})^\perp)), \quad \mathrm{d}|_{t=0} = \mathrm{d}^\circ,
\end{aligned}$$

that is, the vorticity formulation (7.9)–(7.10). Let us quickly deduce that v is a weak solution of (7.2). From the above equations, we deduce $\partial_t \mathrm{m} \in \mathrm{L}^\infty([0, T_0]; \dot{H}^{-1} \cap H^{s-1}(\mathbb{R}^2))$ and $\partial_t \mathrm{d} \in \mathrm{L}^\infty([0, T_0]; \dot{H}^{-1} \cap H^{s-2}(\mathbb{R}^2))$. Lemma 7.2.7 then implies $\partial_t \mathrm{v} \in \mathrm{L}^\infty([0, T_0]; H^{s-1}(\mathbb{R}^2)^2)$. We may then deduce that the quantity

$$V := \partial_t \mathrm{v} - \lambda \nabla(a^{-1}\mathrm{d}) + \alpha(\Psi + \mathrm{v})\,\mathrm{m} - \beta(\Psi + \mathrm{v})^\perp\,\mathrm{m}$$

belongs to $\mathrm{L}^\infty([0, T_0]; \mathrm{L}^2(\mathbb{R}^2)^2)$ and satisfies $\operatorname{curl} V = \operatorname{div}(aV) = 0$ in the distributional sense. Using the Hodge decomposition in $\mathrm{L}^2(\mathbb{R}^2)^2$, we easily conclude $V = 0$, hence $\mathrm{v} \in \mathrm{L}^\infty([0, T_0]; \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$ is indeed a weak solution of (7.2) on $[0, T_0) \times \mathbb{R}^2$. ☐

We turn to the local-in-time existence of smooth solutions of (7.2) in the degenerate case $\lambda = 0$. The analysis of the iterative scheme needs to be carefully adapted in this case, as m and v are now on an equal footing with regard to regularity. Note that the proof only holds in the parabolic regime $\beta = 0$.

Proposition 7.3.2 (Local existence, degenerate case)**.** *Let $\alpha \in \mathbb{R}$, $\beta = \lambda = 0$. Let $s > 2$, and let $h \in W^{s,\infty}(\mathbb{R}^2)$, $\Psi, \bar{\mathrm{v}}^\circ \in W^{s+1,\infty}(\mathbb{R}^2)^2$. Let $\mathrm{v}^\circ \in \bar{\mathrm{v}}^\circ + H^s(\mathbb{R}^2)^2$ with $\mathrm{m}^\circ := \operatorname{curl} \mathrm{v}^\circ$, $\bar{\mathrm{m}}^\circ := \operatorname{curl} \bar{\mathrm{v}}^\circ \in H^s(\mathbb{R}^2)$ and $\mathrm{d}^\circ := \operatorname{div}(a\mathrm{v}^\circ)$, $\bar{\mathrm{d}}^\circ := \operatorname{div}(a\bar{\mathrm{v}}^\circ) \in H^{s-1}(\mathbb{R}^2)$. Then, there exists $T > 0$ and a weak solution $\mathrm{v} \in \mathrm{L}^\infty([0,T); \bar{\mathrm{v}}^\circ + H^s(\mathbb{R}^2)^2)$ of (7.2) on $[0,T) \times \mathbb{R}^2$, with initial data v°. Moreover, T depends only on an upper bound on $|\alpha|$, s, $(s-2)^{-1}$, $\|h\|_{W^{s,\infty}}$, $\|(\Psi, \bar{\mathrm{v}}^\circ)\|_{W^{s+1,\infty}}$, $\|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ\|_{H^s}$, $\|(\mathrm{m}^\circ, \bar{\mathrm{m}}^\circ)\|_{H^s}$, and $\|(\mathrm{d}^\circ, \bar{\mathrm{d}}^\circ)\|_{H^{s-1}}$.* $\Diamond$

Proof. We consider the same iterative scheme $(\mathrm{m}_n, \mathrm{d}_n, \mathrm{v}_n)$ as in the proof of Proposition 7.3.1, but with $\lambda = \beta = 0$. Let $s > 2$. For all $n \geq 0$, let

$$t_n := \sup \Big\{ t \geq 0 : \|\mathrm{m}_n^t\|_{H^s} + \|\mathrm{d}_n^t\|_{H^{s-1}} + \|\mathrm{v}_n^t - \bar{\mathrm{v}}^\circ\|_{H^s} \leq C_0 \Big\},$$

for some $C_0 \geq 1$ to be suitably chosen (depending on initial data), and let $T_0 := \inf_n t_n$. In this proof, we use the notation $\lesssim$ for $\leq$ up to a constant $C > 0$ that depends only on an upper bound on $|\alpha|$, s, $(s-2)^{-1}$, $\|h\|_{W^{s,\infty}}$, $\|(\Psi, \bar{\mathrm{v}}^\circ)\|_{W^{s+1,\infty}}$, $\|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ\|_{H^s}$, $\|(\mathrm{d}^\circ, \bar{\mathrm{d}}^\circ)\|_{H^{s-1}}$, and $\|(\mathrm{m}^\circ, \bar{\mathrm{m}}^\circ)\|_{H^s}$.

Just as in the proof of Proposition 7.3.1, we first need to show that this iterative scheme is well-defined and that $T_0 > 0$. We proceed by induction: let $n \geq 0$ be fixed, and assume that $(\mathrm{m}_n, \mathrm{d}_n, \mathrm{v}_n)$ is well-defined with $\mathrm{m}_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^2))$, $\mathrm{d}_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s-1}(\mathbb{R}^2))$, and $\mathrm{v}_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + H^s(\mathbb{R}^2)^2)$. (For $n = 0$ this is indeed trivial by assumption.)

We first study d_{n+1}. As $\lambda = 0$, equation (7.21) takes the form $\partial_t \mathrm{d}_{n+1} = -\alpha \operatorname{div}(a\mathrm{m}_n(\Psi + \mathrm{v}_n))$. Integrating this equation in time then yields

$$\|\mathrm{d}_{n+1}^t\|_{H^{s-1}} \leq \|\mathrm{d}^\circ\|_{H^{s-1}} + |\alpha| \int_0^t \|\mathrm{m}_n^u(\Psi + \mathrm{v}_n^u)\|_{H^s} du \lesssim 1 + t(1 + \|\mathrm{v}_n - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty_t H^s}) \|\mathrm{m}_n\|_{\mathrm{L}^\infty_t H^s}.$$

where we have used Lemma 7.2.1 together with the Sobolev embedding to estimate the last term. Similarly, noting that $\|\mathrm{d}^\circ - \bar{\mathrm{d}}^\circ\|_{\dot{H}^{-1}} \leq \|a\mathrm{v}^\circ - a\bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2} \leq C$, we find for $s > 1$,

$$\begin{aligned} \|\mathrm{d}_{n+1}^t - \bar{\mathrm{d}}^\circ\|_{\dot{H}^{-1}} \leq C + \|\mathrm{d}_{n+1}^t - \mathrm{d}^\circ\|_{\dot{H}^{-1}} &\leq \|\mathrm{d}^\circ\|_{H^{s-1}} + |\alpha| \int_0^t \|\mathrm{m}_n^u(\Psi + \mathrm{v}_n^u)\|_{\mathrm{L}^2} du \\ &\lesssim 1 + t(1 + \|\mathrm{v}_n - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty_t H^s}) \|\mathrm{m}_n\|_{\mathrm{L}^\infty_t H^s}. \end{aligned}$$

Hence we obtain for all $t \in [0, t_n]$,

$$\|\mathrm{d}_{n+1}^t\|_{H^{s-1}} + \|\mathrm{d}_{n+1}^t - \bar{\mathrm{d}}^\circ\|_{\dot{H}^{-1}} \leq C + Ct(1 + C_0)C_0 \leq C(1 + tC_0^2).$$

We now turn to the study of m_{n+1}. As $\beta = 0$, equation (7.20) takes the form $\partial_t \mathrm{m}_{n+1} = \alpha \operatorname{div}(\mathrm{m}_{n+1}(\Psi + \mathrm{v}_n)^\perp)$. For all $t \geq 0$, Lemma 7.2.2 together with the Sobolev embedding for $s > 2$ then yields (here the choice $\beta = 0$ is crucial, since otherwise the higher norm $\|\mathrm{v}_n^t - \bar{\mathrm{v}}^\circ\|_{H^{s+1}}$ would appear in the right-hand side!)

$$\begin{aligned} \partial_t \|\mathrm{m}_{n+1}^t\|_{H^s} &\lesssim (1 + \|\mathrm{v}_n^t\|_{W^{1,\infty}}) \|\mathrm{m}_{n+1}^t\|_{H^s} + \|\mathrm{m}_{n+1}^t\|_{\mathrm{L}^\infty} \|\operatorname{curl}(\mathrm{v}_n^t - \bar{\mathrm{v}}^\circ)\|_{H^s} + \|\mathrm{m}_{n+1}^t\|_{W^{1,\infty}} \|\mathrm{v}_n^t - \bar{\mathrm{v}}^\circ\|_{H^s} \\ &\lesssim (1 + \|\mathrm{m}_n^t\|_{H^s} + \|\mathrm{v}_n^t - \bar{\mathrm{v}}^\circ\|_{H^s}) \|\mathrm{m}_{n+1}^t\|_{H^s}. \end{aligned}$$

For all $t \in [0, t_n]$, this implies $\partial_t \|\mathrm{m}_{n+1}^t\|_{H^s} \leq C(1 + 2C_0) \|\mathrm{m}_{n+1}^t\|_{H^s}$, and thus

$$\|\mathrm{m}_{n+1}^t\|_{H^s} \leq \|\mathrm{m}^\circ\|_{H^s} e^{C(1+2C_0)t} \leq Ce^{CC_0 t}.$$

Moreover, noting that $\|\mathrm{m}^\circ - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1}} \leq \|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2} \leq C$, and applying Lemma 7.2.2 together with the Sobolev embedding, we obtain

$$\begin{aligned} \|\mathrm{m}_{n+1}^t - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1}} &\leq C + \|\mathrm{m}_{n+1}^t - \mathrm{m}^\circ\|_{\dot{H}^{-1}} \\ &\leq C + Ct(1 + \|\mathrm{v}_n\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}) \|\mathrm{m}_{n+1}\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \\ &\leq C + Ct(1 + \|\mathrm{v}_n - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty_t H^s}) \|\mathrm{m}_{n+1}\|_{\mathrm{L}^\infty_t \mathrm{L}^2}, \end{aligned}$$

hence for all $t \in [0, t_n]$

$$\|\mathrm{m}_{n+1}^t - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1}} \le C + Ct(1+C_0)\|\mathrm{m}_{n+1}\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \le C + CC_0 t e^{CC_0 t}.$$

We finally turn to v_{n+1}. By the above properties of m_{n+1} and d_{n+1}, Lemma 7.2.7 ensures that v_{n+1} is uniquely defined in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + H^s(\mathbb{R}^2)^2)$, and for all $t \in [0, t_n]$ we have

$$\begin{aligned}
\|\mathrm{v}_{n+1}^t - \bar{\mathrm{v}}^\circ\|_{H^s} &\le C\|\mathrm{m}_{n+1}^t - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1}\cap H^{s-1}} + C\|\mathrm{d}_{n+1}^t - \bar{\mathrm{d}}^\circ\|_{\dot{H}^{-1}\cap H^{s-1}} \\
&\le C + C\|\mathrm{m}_{n+1}^t - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1}} + C\|\mathrm{m}_{n+1}^t\|_{H^s} + C\|\mathrm{d}_{n+1}^t - \bar{\mathrm{d}}^\circ\|_{\dot{H}^{-1}} + C\|\mathrm{d}_{n+1}^t\|_{H^{s-1}} \\
&\le C(1 + tC_0^2)e^{CC_0 t}.
\end{aligned}$$

Hence, we have proven that $(\mathrm{m}_{n+1}, \mathrm{d}_{n+1}, \mathrm{v}_{n+1})$ is well-defined in the correct space, and moreover, combining all the previous estimates, we find for all $t \in [0, t_n]$

$$\|\mathrm{m}_{n+1}^t\|_{H^s} + \|\mathrm{d}_{n+1}^t\|_{H^{s-1}} + \|\mathrm{v}_{n+1}^t - \bar{\mathrm{v}}^\circ\|_{H^s} \le C(1 + tC_0^2)e^{CC_0 t}.$$

Therefore, choosing $C_0 = 1 + 2Ce^C \lesssim 1$, we obtain for all $t \le t_n \wedge C_0^{-2}$

$$\|\mathrm{m}_{n+1}^t\|_{H^s} + \|\mathrm{d}_{n+1}^t\|_{H^{s-1}} + \|\mathrm{v}_{n+1}^t - \bar{\mathrm{v}}^\circ\|_{H^s} \le C_0,$$

and thus $t_{n+1} \ge t_n \wedge C_0^{-2}$. The conclusion now follows just as in the proof of Proposition 7.3.1. □

7.4 Global existence

As local existence is proven above in the framework of Sobolev spaces, the strategy for global existence consists in looking for a priori estimates on the Sobolev norms. Since we are also interested in Hölder regularity of solutions, we establish a priori estimates on Hölder-Zygmund norms as well. As we will see, the key ingredient is given by some a priori estimates for the vorticity m in $\mathrm{L}^\infty(\mathbb{R}^2)$.

7.4.1 A priori estimates

We start with the following elementary energy estimates. Note that in the degenerate case $\lambda = 0$, the a priori estimate for d in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2))$ disappears, which is the main difficulty to establish a global result in that case. Although we stick in the sequel to the framework of item (iii), a priori estimates in slightly more general spaces are obtained in item (ii) for the compressible model (7.2).

Lemma 7.4.1 (Energy estimates). *Let $\lambda \ge 0$, $\alpha \ge 0$, $\beta \in \mathbb{R}$, $T > 0$ and $\Psi \in W^{1,\infty}(\mathbb{R}^2)$. Let $\mathrm{v}^\circ \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$ be such that $\mathrm{m}^\circ := \operatorname{curl} \mathrm{v}^\circ \in \mathcal{P} \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$, and such that either $\operatorname{div}(a\mathrm{v}^\circ) = 0$ in the case (7.1), or $\mathrm{d}^\circ := \operatorname{div}(a\mathrm{v}^\circ) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ in the case (7.2). Let $\mathrm{v} \in \mathrm{L}^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^2)^2$ be a weak solution of (7.1) or of (7.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°. Set $\mathrm{d} := 0$ in the case (7.1). Then the following properties hold.*

(i) For all $t \in [0,T)$, we have $\mathrm{m}^t \in \mathcal{P}(\mathbb{R}^2)$.

(ii) Localized energy estimate for (7.2): *If $\mathrm{v} \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ satisfies $\mathrm{m} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2))$ and $\mathrm{d} \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2))$, then we have for all $t \in [0,T)$,*

$$\|\mathrm{v}^t\|^2_{\mathrm{L}^2_{\mathrm{uloc}}} + \alpha \||\mathrm{v}|^2 \mathrm{m}\|_{\mathrm{L}^1_t \mathrm{L}^1_{\mathrm{uloc}}} + \lambda \|\mathrm{d}\|^2_{\mathrm{L}^2_t \mathrm{L}^2_{\mathrm{uloc}}} \le \begin{cases} Ce^{C(1+\lambda^{-1})t}\|\mathrm{v}^\circ\|^2_{\mathrm{L}^2_{\mathrm{uloc}}}, & \text{if } \alpha = 0,\ \lambda > 0; \\ C\alpha^{-1}\lambda^{-1}(e^{\lambda t} - 1) + Ce^{\lambda t}\|\mathrm{v}^\circ\|^2_{\mathrm{L}^2_{\mathrm{uloc}}}, & \text{if } \alpha > 0,\ \lambda > 0; \\ C\alpha^{-1} t + C\|\mathrm{v}^\circ\|^2_{\mathrm{L}^2_{\mathrm{uloc}}}, & \text{if } \alpha > 0,\ \lambda = 0; \end{cases}$$

where the constant C depends only on an upper bound on α, $|\beta|$, λ, $\|h\|_{W^{1,\infty}}$, $\|\Psi\|_{\mathrm{L}^\infty}$, and additionally on $\|\nabla\Psi\|_{\mathrm{L}^\infty}$ in the case $\alpha = 0$.

(iii) Relative energy estimate for (7.1) and (7.2): *If there is some* $\bar{\mathrm{v}}^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$ *such that* $\mathrm{v}^\circ \in \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$, $\bar{\mathrm{m}}^\circ := \operatorname{curl} \bar{\mathrm{v}}^\circ \in \mathrm{L}^2(\mathbb{R}^2)$, *and such that either* $\operatorname{div}(a\bar{\mathrm{v}}^\circ) = 0$ *in the case* (7.1), *or* $\bar{\mathrm{d}}^\circ := \operatorname{div}(a\bar{\mathrm{v}}^\circ) \in \mathrm{L}^2(\mathbb{R}^2)$ *in the case* (7.2), *and if* $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2))$, $\mathrm{m} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2))$, $\mathrm{d} \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^2))$, *then we have for all* $t \in [0,T)$,

$$\int_{\mathbb{R}^2} a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2 + \alpha \int_0^t du \int_{\mathbb{R}^2} a|\mathrm{v}^u - \bar{\mathrm{v}}^\circ|^2 \, \mathrm{m}^u + \lambda \int_0^t du \int_{\mathbb{R}^2} a^{-1} |\mathrm{d}^u|^2$$
$$\le \begin{cases} Ct(1+\alpha^{-1}) + \int_{\mathbb{R}^2} a|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ|^2, & \textit{in both cases (7.1) and (7.2), with } \alpha > 0; \\ e^{Ct}\big(1 + \int_{\mathbb{R}^2} a|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ|^2\big), & \textit{in the case (7.1), with } \alpha = 0 \\ C(e^{C(1+\lambda^{-1})t} - 1) + e^{C(1+\lambda^{-1})t} \int_{\mathbb{R}^2} a|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ|^2, & \textit{in the case (7.2), with } \alpha = 0,\ \lambda > 0; \end{cases}$$

where the constant C *depends only on an upper bound on* α, $|\beta|$, λ, $\|h\|_{W^{1,\infty}}$, $\|(\Psi, \bar{\mathrm{v}}^\circ)\|_{\mathrm{L}^\infty}$, $\|\bar{\mathrm{d}}^\circ\|_{\mathrm{L}^2}$, *and additionally on* $\|\bar{\mathrm{m}}^\circ\|_{\mathrm{L}^2}$ *and* $\|(\nabla\Psi, \nabla\bar{\mathrm{v}}^\circ)\|_{\mathrm{L}^\infty}$ *in the case* $\alpha = 0$. ♢

Proof. Item (i) is a standard consequence of the fact that m satisfies a transport equation (7.9). It thus remains to check items (ii) and (iii). We split the proof into three steps.

Step 1. Proof of (ii).

Let v be a weak solution of the compressible equation (7.2) as in the statement, and let also $C > 0$ denote any constant as in the statement. We prove more precisely, for all $t \in [0,T)$ and $x_0 \in \mathbb{R}^2$,

$$\int ae^{-|x-x_0|}|\mathrm{v}^t|^2 + \alpha \int_0^t du \int ae^{-|x-x_0|}|\mathrm{v}^u|^2 \, \mathrm{m}^u + \lambda \int_0^t du \int a^{-1} e^{-|x-x_0|} |\mathrm{d}^u|^2 \tag{7.23}$$
$$\le \begin{cases} e^{C(1+\lambda^{-1})t} \int ae^{-|x-x_0|}|\mathrm{v}^\circ|^2, & \text{if } \alpha = 0,\ \lambda > 0; \\ C\alpha^{-1}\lambda^{-1}(e^{\lambda t} - 1) + e^{\lambda t} \int ae^{-|x-x_0|}|\mathrm{v}^\circ|^2, & \text{if } \alpha > 0,\ \lambda > 0; \\ C\alpha^{-1} t + \int ae^{-|x-x_0|}|\mathrm{v}^\circ|^2, & \text{if } \alpha > 0,\ \lambda = 0. \end{cases}$$

Item (ii) directly follows from this, noting that

$$\|f\|^p_{\mathrm{L}^p_{\mathrm{uloc}}} \simeq \sup_{x_0 \in \mathbb{R}^2} \int e^{-|x-x_0|} |f(x)|^p dx$$

holds for all $1 \le p < \infty$. So it suffices to prove (7.23). Let $x_0 \in \mathbb{R}^2$ be fixed, and denote by $\chi(x) := e^{-|x-x_0|}$ the exponential cut-off function centered at x_0. From equation (7.2) we compute the following time derivative

$$\partial_t \int a\chi|\mathrm{v}^t|^2 = 2 \int a\chi\big(\lambda \nabla(a^{-1}\mathrm{d}^t) - \alpha(\Psi + \mathrm{v}^t)\,\mathrm{m}^t + \beta(\Psi + \mathrm{v}^t)^\perp \mathrm{m}^t\,\big) \cdot \mathrm{v}^t,$$

and hence, by integration by parts with $|\nabla\chi| \le \chi$,

$$\begin{aligned} \partial_t \int a\chi|\mathrm{v}^t|^2 &= -2\lambda \int a^{-1}\chi|\mathrm{d}^t|^2 - 2\lambda \int \nabla\chi \cdot \mathrm{v}^t \,\mathrm{d}^t - 2\alpha \int a\chi|\mathrm{v}^t|^2 \,\mathrm{m}^t + 2 \int a\chi(-\alpha\Psi + \beta\Psi^\perp) \cdot \mathrm{v}^t\mathrm{m}^t \\ &\le -2\lambda \int a^{-1}\chi|\mathrm{d}^t|^2 + 2\lambda \int \chi|\mathrm{d}^t||\mathrm{v}^t| - 2\alpha \int a\chi|\mathrm{v}^t|^2 \,\mathrm{m}^t + 2 \int a\chi(-\alpha\Psi + \beta\Psi^\perp) \cdot \mathrm{v}^t\mathrm{m}^t. \end{aligned} \tag{7.24}$$

First consider the case $\alpha > 0$. We may then bound the terms as follows, using the inequality $2xy \le x^2 + y^2$,

$$\begin{aligned} \partial_t \int a\chi|\mathrm{v}^t|^2 &\le -2\lambda \int a^{-1}\chi|\mathrm{d}^t|^2 + 2\lambda \int \chi|\mathrm{d}^t||\mathrm{v}^t| - 2\alpha \int a\chi|\mathrm{v}^t|^2 \,\mathrm{m}^t + 2C \int a\chi|\mathrm{v}^t| \,\mathrm{m}^t \\ &\le -\lambda \int a^{-1}\chi|\mathrm{d}^t|^2 + \lambda \int a\chi|\mathrm{v}^t|^2 - \alpha \int a\chi|\mathrm{v}^t|^2 \,\mathrm{m}^t + C\alpha^{-1} \underbrace{\int a\chi \,\mathrm{m}^t}_{\le C}. \end{aligned}$$

As m^t is nonnegative by item (i), the first and third right-hand side terms are nonpositive, and the Grönwall inequality yields $\int a\chi|\mathrm{v}^t|^2 \le C\alpha^{-1}\lambda^{-1}(e^{\lambda t}-1)+e^{\lambda t}\int a\chi|\mathrm{v}^\circ|^2$ (or $\int a\chi|\mathrm{v}^t|^2 \le C\alpha^{-1}t + \int a\chi|\mathrm{v}^\circ|^2$ if $\lambda=0$). The above estimate may then be rewritten as follows,

$$\begin{aligned}\alpha\int a\chi|\mathrm{v}^t|^2\,\mathrm{m}^t+\lambda\int a^{-1}\chi|\mathrm{d}^t|^2 &\le C\alpha^{-1}+\lambda\int a\chi|\mathrm{v}^t|^2-\partial_t\int a\chi|\mathrm{v}^t|^2\\ &\le C\alpha^{-1}e^{\lambda t}+\lambda e^{\lambda t}\int a\chi|\mathrm{v}^\circ|^2-\partial_t\int a\chi|\mathrm{v}^t|^2.\end{aligned}$$

Integrating in time yields

$$\alpha\int_0^t du\int a\chi|\mathrm{v}^t|^2\,\mathrm{m}^u+\lambda\int_0^t du\int a^{-1}\chi|\mathrm{d}^u|^2\le C\alpha^{-1}\lambda^{-1}(e^{-\lambda t}-1)+e^{\lambda t}\int a\chi|\mathrm{v}^\circ|^2-\int a\chi|\mathrm{v}^t|^2,$$

so that (7.23) is proven for $\alpha>0$. We now turn to the case $\alpha=0$, $\lambda>0$. In that case, using the following Delort type identity, which holds here in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$,

$$\mathrm{m}\mathrm{v}=a^{-1}\,\mathrm{d}\mathrm{v}^\perp-\frac12|\mathrm{v}|^2\nabla^\perp h-a^{-1}(\operatorname{div}(aS_{\mathrm{v}}))^\perp,\qquad S_{\mathrm{v}}:=\mathrm{v}\otimes\mathrm{v}-\frac12|\mathrm{v}|^2\,\mathrm{Id},$$

the estimate (7.24) becomes, by integration by parts with $|\nabla\chi|\le\chi$,

$$\begin{aligned}\partial_t\int a\chi|\mathrm{v}^t|^2&\le-2\lambda\int a^{-1}\chi|\mathrm{d}^t|^2+2\lambda\int\chi|\mathrm{d}^t||\mathrm{v}^t|-2\alpha\int a\chi|\mathrm{v}^t|^2\,\mathrm{m}^t+2\int\chi(-\alpha\Psi+\beta\Psi^\perp)\cdot(\mathrm{v}^t)^\perp\,\mathrm{d}^t\\ &\quad-\int a\chi(-\alpha\Psi+\beta\Psi^\perp)\cdot\nabla^\perp h|\mathrm{v}^t|^2+2\int a\chi(\alpha\nabla\Psi^\perp+\beta\nabla\Psi):S_{\mathrm{v}^t}+2\int a\chi|\alpha\Psi^\perp+\beta\Psi||S_{\mathrm{v}^t}|,\end{aligned}$$

and hence, noting that $|S_{\mathrm{v}^t}|\le C|\mathrm{v}^t|^2$, and using the inequality $2xy\le x^2+y^2$,

$$\begin{aligned}\partial_t\int a\chi|\mathrm{v}^t|^2&\le-2\lambda\int a^{-1}\chi|\mathrm{d}^t|^2+2C\int\chi|\mathrm{d}^t||\mathrm{v}^t|-2\alpha\int a\chi|\mathrm{v}^t|^2\,\mathrm{m}^t+C\int a\chi|\mathrm{v}^t|^2\\ &\le-\lambda\int a^{-1}\chi|\mathrm{d}^t|^2+C(1+\lambda^{-1})\int a\chi|\mathrm{v}^t|^2.\end{aligned}$$

The Grönwall inequality yields $\int a\chi|\mathrm{v}^t|^2\le e^{C(1+\lambda^{-1})t}\int a\chi|\mathrm{v}^\circ|^2$. The above estimate may then be rewritten as follows,

$$\begin{aligned}\lambda\int a^{-1}\chi|\mathrm{d}^t|^2&\le C(1+\lambda^{-1})\int a\chi|\mathrm{v}^t|^2-\partial_t\int a\chi|\mathrm{v}^t|^2\\ &\le C(1+\lambda^{-1})e^{C(1+\lambda^{-1})t}\int a\chi|\mathrm{v}^\circ|^2-\partial_t\int a\chi|\mathrm{v}^t|^2.\end{aligned}$$

Integrating in time, the result (7.23) is proven for $\alpha=0$. (Note that this proof cannot be adapted to the incompressible case (7.1), due to the lack of a sufficiently good control on the pressure p in (7.1) in general.)

Step 2. Proof of (iii) for (7.2).

We denote by C any positive constant as in the statement of item (iii). From equation (7.2), we compute the following time derivative,

$$\partial_t\int a|\mathrm{v}^t-\bar{\mathrm{v}}^\circ|^2=2\int a(\lambda\nabla(a^{-1}\,\mathrm{d}^t)-\alpha(\Psi+\mathrm{v}^t)\,\mathrm{m}^t+\beta(\Psi+\mathrm{v}^t)^\perp\,\mathrm{m}^t)\cdot(\mathrm{v}^t-\bar{\mathrm{v}}^\circ),$$

or equivalently, integrating by parts and suitably regrouping the terms,

$$\partial_t \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2 = -2\lambda \int a^{-1}|\mathrm{d}^t|^2 + 2\lambda \int a^{-1}\,\mathrm{d}^t\,\bar{\mathrm{d}}^\circ - 2\alpha \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2\,\mathrm{m}^t \\ + 2\int a(-\alpha(\Psi + \bar{\mathrm{v}}^\circ) + \beta(\Psi + \bar{\mathrm{v}}^\circ)^\perp) \cdot (\mathrm{v}^t - \bar{\mathrm{v}}^\circ)\,\mathrm{m}^t. \quad (7.25)$$

First consider the case $\alpha > 0$. We may then bound the terms as follows, using the inequality $2xy \le x^2 + y^2$,

$$\partial_t \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2 \le -2\lambda \int a^{-1}|\mathrm{d}^t|^2 + 2\lambda \int a^{-1}\,\mathrm{d}^t\,\bar{\mathrm{d}}^\circ - 2\alpha \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2\,\mathrm{m}^t + 2C\int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|\,\mathrm{m}^t \\ \le -\lambda \int a^{-1}|\mathrm{d}^t|^2 + \lambda \int a^{-1}|\bar{\mathrm{d}}^\circ|^2 - \alpha \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2\,\mathrm{m}^t + C\alpha^{-1},$$

and the result of item (iii) in the case $\alpha > 0$ follows by integration. We now turn to the case $\alpha = 0$, $\lambda > 0$. In that case, we rather rewrite (7.25) in the form

$$\partial_t \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2 = -2\lambda \int a^{-1}|\mathrm{d}^t|^2 + 2\lambda \int a^{-1}\,\mathrm{d}^t\,\bar{\mathrm{d}}^\circ - 2\alpha \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2\,\mathrm{m}^t \\ +2\int a(-\alpha(\Psi+\bar{\mathrm{v}}^\circ)+\beta(\Psi+\bar{\mathrm{v}}^\circ)^\perp)\cdot(\mathrm{v}^t-\bar{\mathrm{v}}^\circ)(\mathrm{m}^t-\bar{\mathrm{m}}^\circ)+2\int a(-\alpha(\Psi+\bar{\mathrm{v}}^\circ)+\beta(\Psi+\bar{\mathrm{v}}^\circ)^\perp)\cdot(\mathrm{v}^t-\bar{\mathrm{v}}^\circ)\bar{\mathrm{m}}^\circ,$$

so that, using the following Delort type identity, which holds here in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$,

$$(\mathrm{m} - \bar{\mathrm{m}}^\circ)(\mathrm{v} - \bar{\mathrm{v}}^\circ) = a^{-1}(\mathrm{d} - \bar{\mathrm{d}}^\circ)(\mathrm{v} - \bar{\mathrm{v}}^\circ)^\perp - \frac{1}{2}|\mathrm{v} - \bar{\mathrm{v}}^\circ|^2\nabla^\perp h - a^{-1}(\operatorname{div}(aS_{v-\bar{\mathrm{v}}^\circ}))^\perp,$$

we find by integration by parts

$$\partial_t \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2 = -2\lambda \int a^{-1}|\mathrm{d}^t|^2 + 2\lambda \int a^{-1}\,\mathrm{d}^t\,\bar{\mathrm{d}}^\circ - 2\alpha \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2\,\mathrm{m}^t \\ +2\int(-\alpha(\Psi+\bar{\mathrm{v}}^\circ)+\beta(\Psi+\bar{\mathrm{v}}^\circ)^\perp)\cdot(\mathrm{v}^t-\bar{\mathrm{v}}^\circ)^\perp(\mathrm{d}^t-\bar{\mathrm{d}}^\circ) - \int a(-\alpha(\Psi+\bar{\mathrm{v}}^\circ)+\beta(\Psi+\bar{\mathrm{v}}^\circ)^\perp)\cdot\nabla^\perp h|\mathrm{v}^t-\bar{\mathrm{v}}^\circ|^2 \\ + 2\int a\nabla(\alpha(\Psi+\bar{\mathrm{v}}^\circ)^\perp + \beta(\Psi+\bar{\mathrm{v}}^\circ)) : S_{\mathrm{v}^t-\bar{\mathrm{v}}^\circ} + 2\int a(-\alpha(\Psi+\bar{\mathrm{v}}^\circ) + \beta(\Psi+\bar{\mathrm{v}}^\circ)^\perp)\cdot(\mathrm{v}^t - \bar{\mathrm{v}}^\circ)\,\bar{\mathrm{m}}^\circ.$$

We may then bound the terms as follows, using the inequality $2xy \le x^2 + y^2$,

$$\partial_t \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2 \le -2\lambda \int a^{-1}|\mathrm{d}^t|^2 + 2\lambda \int a^{-1}|\mathrm{d}^t||\bar{\mathrm{d}}^\circ| - 2\alpha \int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2\,\mathrm{m}^t \\ + C\int |\mathrm{v}^t - \bar{\mathrm{v}}^\circ|\,|\mathrm{d}^t| + C\int |\mathrm{v}^t - \bar{\mathrm{v}}^\circ|\,|\bar{\mathrm{d}}^\circ| + C\int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2 + C\int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|\bar{\mathrm{m}}^\circ \\ \le -\lambda \int a^{-1}|\mathrm{d}^t|^2 + C\int a^{-1}|\bar{\mathrm{d}}^\circ|^2 + C\int |\bar{\mathrm{m}}^\circ|^2 + C(1+\lambda^{-1})\int a|\mathrm{v}^t - \bar{\mathrm{v}}^\circ|^2.$$

Item (iii) in the case $\alpha = 0$ then easily follows from the Grönwall inequality.

Step 3. Proof of (iii) for (7.1).

We denote by C any positive constant as in the statement of item (iii). Noting that the identity $\mathrm{v} - \bar{\mathrm{v}}^\circ = a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}(\mathrm{m} - \bar{\mathrm{m}}^\circ)$ follows from (7.18) together with the constraint $\operatorname{div}(a\mathrm{v}) = \operatorname{div}(a\bar{\mathrm{v}}^\circ) = 0$, and recalling that by assumption $\mathrm{v} - \bar{\mathrm{v}}^\circ \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^2)^2)$, we deduce $\mathrm{m} - \bar{\mathrm{m}}^\circ \in$

$\mathrm{L}^2_{\mathrm{loc}}([0,T);\dot H^{-1}(\mathbb{R}^2))$ and $(\operatorname{div} a^{-1}\nabla)^{-1}(\mathrm{m}-\bar{\mathrm{m}}^\circ)\in\mathrm{L}^2_{\mathrm{loc}}([0,T);\dot H^{1}(\mathbb{R}^2))$. In particular, this implies by integration by parts

$$\int a|\mathrm{v}-\bar{\mathrm{v}}^\circ|^2=\int a^{-1}|\nabla(\operatorname{div} a^{-1}\nabla)^{-1}(\mathrm{m}-\bar{\mathrm{m}}^\circ)|^2=\int(\mathrm{m}-\bar{\mathrm{m}}^\circ)(-\operatorname{div} a^{-1}\nabla)^{-1}(\mathrm{m}-\bar{\mathrm{m}}^\circ). \tag{7.26}$$

From equation (7.9), we compute the following time derivative

$$\begin{aligned}
&\partial_t\int(\mathrm{m}-\bar{\mathrm{m}}^\circ)(-\operatorname{div} a^{-1}\nabla)^{-1}(\mathrm{m}-\bar{\mathrm{m}}^\circ)\\
&=\;2\int\nabla(\operatorname{div} a^{-1}\nabla)^{-1}(\mathrm{m}-\bar{\mathrm{m}}^\circ)\cdot(\alpha(\Psi+\mathrm{v})^\perp+\beta(\Psi+\mathrm{v}))\,\mathrm{m}\\
&=\;-2\int a(\mathrm{v}-\bar{\mathrm{v}}^\circ)^\perp\cdot\Big(\alpha(\mathrm{v}-\bar{\mathrm{v}}^\circ)^\perp+\beta(\mathrm{v}-\bar{\mathrm{v}}^\circ)+\alpha(\Psi+\bar{\mathrm{v}}^\circ)^\perp+\beta(\Psi+\bar{\mathrm{v}}^\circ)\Big)\,\mathrm{m}\\
&=\;-2\alpha\int a|\mathrm{v}-\bar{\mathrm{v}}^\circ|^2\,\mathrm{m}-2\int a\mathrm{m}(\mathrm{v}-\bar{\mathrm{v}}^\circ)^\perp\cdot(\alpha(\Psi+\bar{\mathrm{v}}^\circ)^\perp+\beta(\Psi+\bar{\mathrm{v}}^\circ)).
\end{aligned}$$

Combining this with identity (7.26), we are now in position to conclude exactly as in Step 2 after equation (7.25) (but with here $\mathrm{d},\bar{\mathrm{d}}^\circ=0$). ☐

The energy estimates given by Lemma 7.4.1 above are not strong enough to deduce global existence, and the key is to find an additional a priori L^p-estimate for the vorticity m with $p>1$. We start with the following new result, based on a careful examination of the evolution of L^p-norms of the vorticity. The argument can unfortunately not be adapted to the mixed-flow compressible case (that is, (7.2) with $\alpha\ge0$, $\beta\ne0$), as it would require a too strong additional control on the norm $\|\mathrm{d}^t\|_{\mathrm{L}^{p+1}}$; this is why this case is excluded from our global results in Theorem 7.1.3.

Lemma 7.4.2 (L^p-estimates for vorticity)**.** *Let $\lambda,\alpha\ge0$, $\beta\in\mathbb{R}$, $T>0$, $h,\Psi\in W^{1,\infty}(\mathbb{R}^2)$, $\bar{\mathrm{v}}^\circ\in\mathrm{L}^\infty(\mathbb{R}^2)^2$, and $\mathrm{v}^\circ\in\bar{\mathrm{v}}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2$, with $\mathrm{m}^\circ:=\operatorname{curl}\mathrm{v}^\circ\in\mathcal{P}(\mathbb{R}^2)$, $\bar{\mathrm{m}}^\circ:=\operatorname{curl}\bar{\mathrm{v}}^\circ\in\mathcal{P}\cap\mathrm{L}^\infty(\mathbb{R}^2)$. In the case (7.1), also assume $\operatorname{div}(a\mathrm{v}^\circ)=\operatorname{div}(a\bar{\mathrm{v}}^\circ)=0$. Let $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\bar{\mathrm{v}}^\circ+\mathrm{L}^2\cap\mathrm{L}^\infty(\mathbb{R}^2)^2)$ be a weak solution of (7.1) or of (7.2) on $[0,T)\times\mathbb{R}^2$ with initial data v°, and with $\mathrm{m}:=\operatorname{curl}\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathcal{P}\cap\mathrm{L}^\infty(\mathbb{R}^2))$. For all $1<p\le\infty$ and $t\in[0,T)$,*

(i) in the case (7.1) with $\alpha>0$, $\beta\in\mathbb{R}$, we have

$$\left(\frac{\alpha(p-1)}{2}\right)^{1/p}\|\mathrm{m}\|^{1+1/p}_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}+\|\mathrm{m}^t\|_{\mathrm{L}^p}\le\|\mathrm{m}^\circ\|_{\mathrm{L}^p}+C_p, \tag{7.27}$$

where the constant C_p depends only on an upper bound on $(p-1)^{-1}$, α, α^{-1}, $|\beta|$, T, $\|(h,\Psi)\|_{W^{1,\infty}}$, $\|(\bar{\mathrm{v}}^\circ,\bar{\mathrm{m}}^\circ)\|_{\mathrm{L}^\infty}$, and on $\|\mathrm{v}^\circ-\bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}$;

(ii) in both cases (7.1) and (7.2) with $\alpha\ge0$, $\beta=0$, $\lambda\ge0$, the same estimate (7.27) holds, where the constant $C_p=C$ depends only on an upper bound on α, T, and on $\|(\operatorname{curl}\Psi)_-\|_{\mathrm{L}^\infty}$. ◊

Proof. It is sufficient to prove the result for all $1<p<\infty$. In this proof, we use the notation $\lesssim$ for $\le$ up to a constant $C>0$ as in the statement but independent of p. As explained at the end of Step 1, we may focus on item (i), the other being much simpler. We split the proof into three steps. Set $\bar\theta^\circ:=\operatorname{div}\bar{\mathrm{v}}^\circ$, $\theta:=\operatorname{div}v$. In the sequel, we repeatedly use the a priori estimate of Lemma 7.4.1(i) in the following interpolated form: for all $s\le q$ and $t\in[0,T)$,

$$\|\mathrm{m}^t\|_{\mathrm{L}^s}\le\|\mathrm{m}^t\|^{q'/s'}_{\mathrm{L}^q}\|\mathrm{m}^t\|^{1-q'/s'}_{\mathrm{L}^1}=\|\mathrm{m}^t\|^{q'/s'}_{\mathrm{L}^q}. \tag{7.28}$$

Step 1. Preliminary estimate for m (in case (i)): for all $1<p<\infty$ and all $t\in[0,T)$,

$$\alpha(p-1)\|\mathrm{m}\|^{p+1}_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}+\|\mathrm{m}^t\|^p_{\mathrm{L}^p}\le\|\mathrm{m}^\circ\|^p_{\mathrm{L}^p}+C(p-1)(t^{1/p}+\|\mathrm{v}\|_{\mathrm{L}^p_t\mathrm{L}^\infty})\|\mathrm{m}\|^{p-1/p}_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}. \tag{7.29}$$

Using equation (7.9) and integrating by parts we may compute

$$\begin{aligned}\partial_t \int (\mathrm{m}^t)^p &= p \int (\mathrm{m}^t)^{p-1} \operatorname{div}\left(\mathrm{m}^t(\alpha(\Psi+\mathrm{v}^t)^\perp + \beta(\Psi+\mathrm{v}^t))\right)\\ &= -p(p-1)\int (\mathrm{m}^t)^{p-1}\nabla \mathrm{m}^t \cdot(\alpha(\Psi+\mathrm{v}^t)^\perp + \beta(\Psi+\mathrm{v}^t))\\ &= -(p-1)\int \nabla(\mathrm{m}^t)^p\cdot(\alpha(\Psi+\mathrm{v}^t)^\perp + \beta(\Psi+\mathrm{v}^t))\\ &= (p-1)\int (\mathrm{m}^t)^p \operatorname{div}(\alpha(\Psi+\mathrm{v}^t)^\perp + \beta(\Psi+\mathrm{v}^t)).\end{aligned}$$

In case (i), using the constraint $\operatorname{div}(a\mathrm{v}) = 0$ to compute $\operatorname{div}(\alpha\mathrm{v}^\perp + \beta\mathrm{v}) = -\alpha\mathrm{m} + \beta \operatorname{div}\mathrm{v} = -\alpha\mathrm{m} - \beta\nabla h\cdot \mathrm{v}$, we find

$$(p-1)^{-1}\partial_t \int (\mathrm{m}^t)^p \le -\alpha\int (\mathrm{m}^t)^{p+1} + C\int (\mathrm{m}^t)^p(1+|\mathrm{v}^t|) \le -\alpha\int(\mathrm{m}^t)^{p+1} + C(1+\|\mathrm{v}^t\|_{\mathrm{L}^\infty})\int (\mathrm{m}^t)^p.$$

By interpolation (7.28), we obtain

$$\alpha\int (\mathrm{m}^t)^{p+1} + (p-1)^{-1}\partial_t\int(\mathrm{m}^t)^p \le C(1+\|\mathrm{v}^t\|_{\mathrm{L}^\infty})\|\mathrm{m}^t\|_{\mathrm{L}^{p+1}}^{p-1/p},$$

and the result (7.29) directly follows by integration with respect to t and by the Hölder inequality. In case (ii) we rather have $\operatorname{div}(\alpha(\Psi+\mathrm{v})^\perp + \beta(\Psi+\mathrm{v})) = -\alpha(\operatorname{curl}\Psi + \mathrm{m})$, and hence

$$\alpha\int(\mathrm{m}^t)^{p+1} + (p-1)^{-1}\partial_t\int(\mathrm{m}^t)^p \le \alpha\|(\operatorname{curl}\Psi)_-\|_{\mathrm{L}^\infty}\int(\mathrm{m}^t)^p \le \alpha\|(\operatorname{curl}\Psi)_-\|_{\mathrm{L}^\infty}\Big(\int(\mathrm{m}^t)^{p+1}\Big)^{1-1/p},$$

from which the conclusion (ii) already follows.

Step 2. Preliminary estimate for v (in case (i)): for all $2 < q \le \infty$ and $t \in [0,T)$,

$$\|\mathrm{v}^t\|_{\mathrm{L}^\infty} \lesssim 1 + (1-2/q)^{-1/2}\|\mathrm{m}^t\|_{\mathrm{L}^q}^{q'/2}\log^{1/2}(2+\|\mathrm{m}^t\|_{\mathrm{L}^q}). \tag{7.30}$$

Let $2 < q \le \infty$. Note that $\mathrm{v}^t - \bar{\mathrm{v}}^\circ = \nabla^\perp\triangle^{-1}(\mathrm{m}^t - \mathrm{m}^\circ) + \nabla\triangle^{-1}(\theta^t - \bar\theta^\circ)$. By Lemma 7.2.4(i) for $w := \mathrm{m}^t - \bar{\mathrm{m}}^\circ$ and Lemma 7.2.4(ii) for $w := \theta^t - \bar\theta^\circ = \operatorname{div}(\mathrm{v}^t - \bar{\mathrm{v}}^\circ)$, we find

$$\begin{aligned}\|\mathrm{v}^t\|_{\mathrm{L}^\infty} &\le \|\bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty} + \|\nabla\triangle^{-1}(\mathrm{m}^t - \bar{\mathrm{m}}^\circ)\|_{\mathrm{L}^\infty} + \|\nabla\triangle^{-1}(\theta^t - \bar\theta^\circ)\|_{\mathrm{L}^\infty}\\ &\lesssim 1 + (1-2/q)^{-1/2}\|\mathrm{m}^t - \bar{\mathrm{m}}^\circ\|_{\mathrm{L}^2}\log^{1/2}(2+\|\mathrm{m}^t - \bar{\mathrm{m}}^\circ\|_{\mathrm{L}^1\cap\mathrm{L}^q})\\ &\qquad + \|\theta^t - \bar\theta^\circ\|_{\mathrm{L}^2}\log^{1/2}(2+\|\theta^t - \bar\theta^\circ\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}) + \|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}.\end{aligned}$$

Noting that $\theta^t - \bar\theta^\circ = -\nabla h\cdot(\mathrm{v}^t - \bar{\mathrm{v}}^\circ)$, using interpolation (7.28) in the form $\|\mathrm{m}^t\|_{\mathrm{L}^2} \lesssim \|\mathrm{m}^t\|_{\mathrm{L}^q}^{q'/2}$, and using the a priori estimates of Lemma 7.4.1 in the form $\|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2} + \|\mathrm{m}^t\|_{\mathrm{L}^1} \lesssim 1$, we obtain

$$\|\mathrm{v}^t\|_{\mathrm{L}^\infty} \lesssim (1-2/q)^{-1/2}\|\mathrm{m}^t\|_{\mathrm{L}^q}^{q'/2}\log^{1/2}(2+\|\mathrm{m}^t\|_{\mathrm{L}^q}) + \log^{1/2}(2+\|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty}),$$

and the result follows, absorbing in the left-hand side the last norm of v.

Step 3. Conclusion.

Let $1 < p < \infty$. From (7.30) with $q = p+1$, we deduce in particular

$$\|\mathrm{v}^t\|_{\mathrm{L}^\infty} \lesssim 1 + (1-1/p)^{-1/2}\|\mathrm{m}^t\|_{\mathrm{L}^{p+1}}^{\frac12(1+1/p)}\log^{1/2}(2+\|\mathrm{m}^t\|_{\mathrm{L}^{p+1}}) \lesssim (1-1/p)^{-1/2}\big(1+\|\mathrm{m}^t\|_{\mathrm{L}^{p+1}}^{\frac34(1+1/p)}\big),$$

and hence, integrating with respect to t and combining with (7.29),

$$\begin{aligned}\alpha(p-1)\|\mathrm{m}\|^{p+1}_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}+\|\mathrm{m}^t\|^p_{\mathrm{L}^p}&\le\|\mathrm{m}^\circ\|^p_{\mathrm{L}^p}+Cp\big(1+\|\mathrm{m}\|^{\frac34(1+1/p)}_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}\big)\|\mathrm{m}\|^{p-1/p}_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}\\&\le\|\mathrm{m}^\circ\|^p_{\mathrm{L}^p}+Cp\|\mathrm{m}\|^{p-1/p}_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}+Cp\|\mathrm{m}\|^{p+\frac34}_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}.\end{aligned}$$

We may now absorb in the left-hand side the last two terms, to the effect of

$$\frac{\alpha(p-1)}{2}\|\mathrm{m}\|^{p+1}_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}+\|\mathrm{m}^t\|^p_{\mathrm{L}^p}\le\|\mathrm{m}^\circ\|^p_{\mathrm{L}^p}+C_p^p,$$

where the constant C_p further depends on an upper bound on $(p-1)^{-1}$, and the conclusion follows. □

The following result partially improves and completes the results of Lemma 7.4.2 above in the case (7.1) with either $\alpha=0$ or h constant (cf. item (ii)), and in both cases (7.1) and (7.2) with $\alpha>0$ and $\beta=0$ (cf. item (iii)). For that purpose, inspired by the work of Lin and Zhang [304], we exploit by ODE arguments the very particular structure of the transport equation (7.9). In the parabolic case $\alpha>0$, $\beta=0$, note that we establish an a priori L^p-estimate for the vorticity m through its initial L^1-norm only (cf. item (iii)), which is the key for global existence results with vortex-sheet initial data. While in [304] for the simpler model (7.4) such an a priori estimate is achieved by explicitly integrating the evolution of the vorticity along characteristics, this explicit structure is lost for the more sophisticated models (7.1) and (7.2), and a more subtle argument is required.

Lemma 7.4.3 (L^p-estimates for vorticity, cont'd)**.** *Let $\lambda\ge0$, $\alpha\ge0$, $\beta\in\mathbb{R}$, $T>0$, and $h,\Psi,\mathrm{v}^\circ\in W^{1,\infty}(\mathbb{R}^2)^2$, with $\mathrm{m}^\circ:=\operatorname{curl}\mathrm{v}^\circ\in\mathcal{P}\cap C^0(\mathbb{R}^2)$. Set $\mathrm{d}^\circ:=\operatorname{div}(a\mathrm{v}^\circ)$, and in the case* (7.1) *assume that* $\operatorname{div}(a\mathrm{v}^\circ)=0$. *Let $\mathrm{v}\in W^{1,\infty}_{\mathrm{loc}}([0,T);W^{1,\infty}(\mathbb{R}^2)^2)$ be a weak solution of* (7.1) *or of* (7.2) *on $[0,T)\times\mathbb{R}^2$ with initial data v°. For all $1\le p\le\infty$ and $t\in[0,T)$, the following properties hold,*

(i) in both cases (7.1) *and* (7.2)*, without restriction on the parameters,*

$$\begin{aligned}\|\mathrm{m}^t\|_{\mathrm{L}^p}\le\|\mathrm{m}^\circ\|_{\mathrm{L}^p}\min\bigg\{&\exp\Big(\frac{p-1}{p}\big(Ct+C|\beta|\|\mathrm{d}\|_{\mathrm{L}^1_t\mathrm{L}^\infty}+C|\beta|\|\nabla h\|_{\mathrm{L}^\infty}\|\mathrm{v}\|_{\mathrm{L}^1_t\mathrm{L}^\infty}\big)\Big);\\&\exp\Big(\frac{p-1}{p}\big(C+Ct+C|\beta|\|\mathrm{d}\|_{\mathrm{L}^1_t\mathrm{L}^\infty}+C\alpha\|\nabla h\|_{\mathrm{L}^\infty}\|\mathrm{v}\|_{\mathrm{L}^1_t\mathrm{L}^\infty}\big)\Big)\bigg\};\end{aligned}$$

(ii) in the case (7.1) *with either $\beta=0$ or $\alpha=0$ or h constant, and in the case* (7.2) *with $\beta=0$, we have*

$$\|\mathrm{m}^t\|_{\mathrm{L}^p}\le Ce^{Ct}\|\mathrm{m}^\circ\|_{\mathrm{L}^p};$$

(iii) given $\alpha>0$, in the case (7.1) *with either $\beta=0$ or h constant, and in the case* (7.2) *with $\beta=0$, we have*

$$\|\mathrm{m}^t\|_{\mathrm{L}^p}\le\Big((\alpha t)^{-1}+C\alpha^{-1}e^{Ct}\Big)^{1-1/p};$$

where the constant C depends only on an upper bound on α, $|\beta|$, and on $\|(h,\Psi)\|_{W^{1,\infty}}$. ◊

Remark 7.4.4. In the context of item (iii), if we further assume $\Psi\equiv0$ (i.e. no forcing), then the constant C in Step 2 of the proof below may then be set to 0, so that we simply obtain, for all $1\le p<\infty$ and all $t>0$,

$$\|\mathrm{m}^t\|_{\mathrm{L}^p}\le\bigg(\int|\mathrm{m}^\circ|^p(1+\alpha t\,\mathrm{m}^\circ)^{1-p}\bigg)^{1/p}\le(\alpha t)^{-(1-1/p)},$$

without additional exponential growth. ◊

Proof. We split the proof into two steps, and we use the notation $\lesssim$ for $\le$ up to a constant $C>0$ as in the statement.

Step 1. General bounds.

In this step, we prove (i) (from which (ii) directly follows, noting that choosing a constant implies $\nabla h\equiv 0$). Let us consider the flow

$$\partial_t\psi^t(x)=-\alpha(\Psi+\mathrm{v}^t)^\perp(\psi^t(x))-\beta(\Psi+\mathrm{v}^t)(\psi^t(x)),\qquad \psi^t(x)|_{t=0}=x.$$

The Lipschitz assumptions ensure that ψ is well-defined in $W^{1,\infty}_{\mathrm{loc}}([0,T);W^{1,\infty}(\mathbb{R}^2)^2)$. As m satisfies the transport equation (7.9) with initial data $\mathrm{m}^\circ\in C^0(\mathbb{R}^2)$, the method of propagation along characteristics yields

$$\mathrm{m}^t(x)=\mathrm{m}^\circ((\psi^t)^{-1}(x))|\det\nabla(\psi^t)^{-1}(x)|=\mathrm{m}^\circ((\psi^t)^{-1}(x))|\det\nabla\psi^t((\psi^t)^{-1}(x))|^{-1},$$

and hence for all $1\le p<\infty$ we have

$$\int|\mathrm{m}^t|^p=\int|\mathrm{m}^\circ((\psi^t)^{-1}(x))|^p|\det\nabla\psi^t((\psi^t)^{-1}(x))|^{-p}dx=\int|\mathrm{m}^\circ(x)|^p|\det\nabla\psi^t(x)|^{1-p}dx,\quad(7.31)$$

while for $\mathrm{p}=\infty$,

$$\|\mathrm{m}^t\|_{\mathrm{L}^\infty}\le\|\mathrm{m}^\circ\|_{\mathrm{L}^\infty}\|(\det\nabla\psi^t)^{-1}\|_{\mathrm{L}^\infty}.$$

Now let us examine this determinant more closely. By the Liouville-Ostrogradski formula,

$$|\det\nabla\psi^t(x)|^{-1}=\exp\bigg(\int_0^t\operatorname{div}\Big(\alpha(\Psi+\mathrm{v}^u)^\perp+\beta(\Psi+\mathrm{v}^u)\Big)(\psi^u(x))du\bigg).\quad(7.32)$$

A simple computation gives

$$\operatorname{div}(\alpha(\mathrm{v}^t)^\perp+\beta\mathrm{v}^t)=-\alpha\operatorname{curl}\mathrm{v}^t+\beta\operatorname{div}\mathrm{v}^t=-\alpha\mathrm{m}^t+\beta a^{-1}\,\mathrm{d}^t-\beta\nabla h\cdot\mathrm{v}^t,\quad(7.33)$$

hence by non-negativity of m,

$$\operatorname{div}(\alpha(\mathrm{v}^t)^\perp+\beta\mathrm{v}^t)\le|\beta|\|a^{-1}\|_{\mathrm{L}^\infty}\|\mathrm{d}^t\|_{\mathrm{L}^\infty}+|\beta|\|\nabla h\|_{\mathrm{L}^\infty}\|\mathrm{v}^t\|_{\mathrm{L}^\infty}.$$

We then deduce from (7.32),

$$|\det\nabla\psi^t(x)|^{-1}\le\exp\big(t\alpha\|\operatorname{curl}\Psi\|_{\mathrm{L}^\infty}+t|\beta|\|\operatorname{div}\,\Psi\|_{\mathrm{L}^\infty}+|\beta|\|a^{-1}\|_{\mathrm{L}^\infty}\|\mathrm{d}\|_{\mathrm{L}^1_t\mathrm{L}^\infty}+|\beta|\|\nabla h\|_{\mathrm{L}^\infty}\|\mathrm{v}\|_{\mathrm{L}^1_t\mathrm{L}^\infty}\big),$$

and thus, combined with (7.31), for all $1\le p\le\infty$,

$$\begin{aligned}\|\mathrm{m}^t\|_{\mathrm{L}^p}\le\|\mathrm{m}^\circ\|_{\mathrm{L}^p}\exp\bigg(\frac{p-1}{p}\big(&t\alpha\|\operatorname{curl}\Psi\|_{\mathrm{L}^\infty}+t|\beta|\|\operatorname{div}\,\Psi\|_{\mathrm{L}^\infty}\\&+|\beta|\|a^{-1}\|_{\mathrm{L}^\infty}\|\mathrm{d}\|_{\mathrm{L}^1_t\mathrm{L}^\infty}+|\beta|\|\nabla h\|_{\mathrm{L}^\infty}\|\mathrm{v}\|_{\mathrm{L}^1_t\mathrm{L}^\infty}\big)\bigg).\end{aligned}\quad(7.34)$$

On the other hand, noting that

$$\partial_th(\psi^t(x))=-\nabla h(\psi^t(x))\cdot(\alpha(\Psi+\mathrm{v}^t)^\perp+\beta(\Psi+\mathrm{v}^t))(\psi^t(x)),$$

we may alternatively rewrite

$$\begin{aligned}\operatorname{div}(\alpha(\mathrm{v}^t)^\perp+\beta\mathrm{v}^t)(\psi^t(x))&=\big(-\alpha\mathrm{m}^t+\beta a^{-1}\,\mathrm{d}^t-\beta\nabla h\cdot\mathrm{v}^t\big)(\psi^t(x))\\&=\partial_th(\psi^t(x))+\big(-\alpha\mathrm{m}^t+\beta a^{-1}\,\mathrm{d}^t-\alpha\nabla^\perp h\cdot\mathrm{v}^t+\nabla h\cdot(\alpha\Psi^\perp+\beta\Psi)\big)(\psi^t(x)).\end{aligned}$$

Integrating this identity with respect to t and using again the same formula for $|\det \nabla\psi^t|^{-1}$, we obtain

$$\|\mathrm{m}^t\|_{\mathrm{L}^p} \le \|\mathrm{m}^\circ\|_{\mathrm{L}^p} \exp\bigg(\frac{p-1}{p}\big(t\alpha\|\mathrm{curl}\,\Psi\|_{\mathrm{L}^\infty} + t|\beta|\|\operatorname{div}\,\Psi\|_{\mathrm{L}^\infty} + |\beta|\|a^{-1}\|_{\mathrm{L}^\infty}\|\mathrm{d}\|_{\mathrm{L}^1_t\mathrm{L}^\infty} \\ + 2\|h\|_{\mathrm{L}^\infty} + t(\alpha+|\beta|)\|\nabla h\|_{\mathrm{L}^\infty}\|\Psi\|_{\mathrm{L}^\infty} + \alpha\|\nabla h\|_{\mathrm{L}^\infty}\|v\|_{\mathrm{L}^1_t\mathrm{L}^\infty}\big)\bigg). \tag{7.35}$$

Combining (7.34) and (7.35), the conclusion (i) follows.

Step 2. Proof of (iii).

It suffices to prove the result for any $1 < p < \infty$. Let such a p be fixed. Assuming either $\beta = 0$, or $\mathrm{d} \equiv 0$ and a constant, we deduce from (7.31), (7.32), and (7.33),

$$\int |\mathrm{m}^t|^p = \int |\mathrm{m}^\circ(x)|^p \exp\Big((p-1)\int_0^t \operatorname{div}\,\big(\alpha(\Psi+\mathrm{v}^u)^\perp + \beta(\Psi+\mathrm{v}^u)\big)(\psi^u(x))du\Big)dx \\ \le e^{C(p-1)t}\int |\mathrm{m}^\circ(x)|^p \exp\Big(-\alpha(p-1)\int_0^t \mathrm{m}^u(\psi^u(x))du\Big)dx. \tag{7.36}$$

Let x be momentarily fixed, and set $f_x(t) := \mathrm{m}^t(\psi^t(x))$. We need to estimate the integral $\int_0^t f_x(u)du$. For that purpose, we first compute $\partial_t f_x$: again using (7.33) (with either $\beta = 0$, or $\mathrm{d} \equiv 0$ and a constant), we find

$$\partial_t f_x(t) = \operatorname{div}\,\big(\mathrm{m}^t(\alpha(\Psi+\mathrm{v}^t)^\perp + \beta(\Psi+\mathrm{v}^t))\big)(\psi^t(x)) - \nabla\mathrm{m}^t(\psi^t(x))\cdot\big(\alpha(\Psi+\mathrm{v}^t)^\perp + \beta(\Psi+\mathrm{v}^t)\big)(\psi^t(x)) \\ = \mathrm{m}^t(\psi^t(x))\operatorname{div}\,\big(\alpha(\Psi+\mathrm{v}^t)^\perp + \beta(\Psi+\mathrm{v}^t))\big)(\psi^t(x)) \\ = -\alpha(\mathrm{m}^t(\psi^t(x)))^2 + \big(-\alpha\mathrm{m}^t\,\mathrm{curl}\,\Psi + \beta\mathrm{m}^t\operatorname{div}\,\Psi\big)(\psi^t(x)),$$

and hence

$$\partial_t f_x \ge -\alpha f_x^2 - Cf_x.$$

We may then deduce $f_x \ge g_x$ pointwise, where g_x satisfies

$$\partial_t g_x = -\alpha g_x^2 - Cg_x, \qquad g_x(0) = f_x(0) = \mathrm{m}^\circ(x).$$

A direct computation yields

$$g_x(t) = \frac{Ce^{-Ct}\,\mathrm{m}^\circ(x)}{C+\alpha(1-e^{-Ct})\,\mathrm{m}^\circ(x)},$$

and hence

$$\int_0^t f_x(u)du \ge \int_0^t g_x(u)du = \alpha^{-1}\log\Big(1+\alpha C^{-1}(1-e^{-Ct})\,\mathrm{m}^\circ(x)\Big).$$

Inserting this into (7.36), we obtain for all $t > 0$

$$\int |\mathrm{m}^t|^p \le e^{C(p-1)t}\int |\,\mathrm{m}^\circ(x)|^p\Big(1+\alpha C^{-1}(1-e^{-Ct})\,\mathrm{m}^\circ(x)\Big)^{1-p}dx \\ \le \bigg(\frac{C\alpha^{-1}e^{Ct}}{1-e^{-Ct}}\bigg)^{p-1}\int |\,\mathrm{m}^\circ(x)|dx = \bigg(\frac{C\alpha^{-1}e^{Ct}}{1-e^{-Ct}}\bigg)^{p-1}.$$

The result (iii) then follows from the obvious inequality $e^{Ct}(1-e^{-Ct})^{-1} \le e^{Ct}+1+(Ct)^{-1}$ for all $t > 0$. □

The previous two lemmas establish uniform bounds on the vorticity m in various regimes. As a preliminary to the propagation of regularity, we now show that this bound on m implies similar uniform bounds on v and on the divergence d. In the incompressible case (7.1), this already follows from Step 2 of the proof of Lemma 7.4.2 above, but more analysis is needed in the compressible case (7.2).

Lemma 7.4.5 (Relative L^p-estimates). *Let $\lambda > 0$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $T > 0$, $h, \Psi, \bar{\mathrm{v}}^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$, and $\mathrm{v}^\circ \in \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$, with $\mathrm{m}^\circ := \operatorname{curl} \mathrm{v}^\circ \in \mathcal{P}(\mathbb{R}^2)$, $\bar{\mathrm{m}}^\circ := \operatorname{curl} \bar{\mathrm{v}}^\circ \in \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^2)$, and with either $\operatorname{div}(a\mathrm{v}^\circ) = \operatorname{div}(a\bar{\mathrm{v}}^\circ) = 0$ in the case (7.1), or $\mathrm{d}^\circ := \operatorname{div}(a\mathrm{v}^\circ)$, $\bar{\mathrm{d}}^\circ := \operatorname{div}(a\bar{\mathrm{v}}^\circ) \in \mathrm{L}^2 \cap \mathrm{L}^\infty(\mathbb{R}^2)$ in the case (7.2). Let $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ be a weak solution of (7.1) or of (7.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°, and with $\mathrm{m} := \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty([0,T]; \mathrm{L}^\infty(\mathbb{R}^2))$. Then we have for all $t \in [0,T)$*

$$\|\mathrm{d}^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \leq C, \qquad \|\operatorname{div}(\mathrm{v}^t - \bar{\mathrm{v}}^\circ)\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \leq C, \qquad \|\mathrm{v}^t\|_{\mathrm{L}^\infty} \leq C,$$

where the constant C depends only on an upper bound on α, $|\beta|$, λ, λ^{-1}, T, $\|h\|_{W^{1,\infty}}$, $\|(\Psi, \bar{\mathrm{v}}^\circ)\|_{\mathrm{L}^\infty}$, $\|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}$, $\|\bar{\mathrm{m}}^\circ\|_{\mathrm{L}^1 \cap \mathrm{L}^\infty}$, $\|(\mathrm{d}^\circ, \bar{\mathrm{d}}^\circ)\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}$, $\|\mathrm{m}\|_{\mathrm{L}^\infty_T \mathrm{L}^\infty}$, and additionally on $\|(\nabla\Psi, \nabla\bar{\mathrm{v}}^\circ)\|_{\mathrm{L}^\infty}$ (resp. on α^{-1}) in the case $\alpha = 0$ (resp. $\alpha > 0$). $\Diamond$

Proof. In this proof, we use the notation $\lesssim$ for $\leq$ up to a constant $C > 0$ as in the statement, and we also set $\theta := \operatorname{div} \mathrm{v}$ and $\bar{\theta}^\circ := \operatorname{div} \bar{\mathrm{v}}^\circ$. In the incompressible case (7.1) the conclusion follows from Step 2 of the proof of Lemma 7.4.2 together with the identity $\operatorname{div} \mathrm{v} = -\nabla h \cdot \mathrm{v}$. We may thus focus on the case of the compressible equation (7.2). We split the proof into three steps.

Step 1. Preliminary estimate for v: for all $t \in [0,T)$,

$$\|\mathrm{v}^t\|_{\mathrm{L}^\infty} \lesssim 1 + \|\theta^t - \bar{\theta}^\circ\|_{\mathrm{L}^2} \log^{1/2}(2 + \|\theta^t - \bar{\theta}^\circ\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}). \tag{7.37}$$

Note that $\mathrm{v}^t - \bar{\mathrm{v}}^\circ = \nabla^\perp \triangle^{-1}(\mathrm{m}^t - \bar{\mathrm{m}}^\circ) + \nabla\triangle^{-1}(\theta^t - \bar{\theta}^\circ)$. By Lemma 7.2.4(i)–(ii), we may then estimate

$$\begin{aligned}
&\|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty} \leq \|\nabla\triangle^{-1}(\mathrm{m}^t - \bar{\mathrm{m}}^\circ)\|_{\mathrm{L}^\infty} + \|\nabla\triangle^{-1}(\theta^t - \bar{\theta}^\circ)\|_{\mathrm{L}^\infty} \\
&\lesssim \|\mathrm{m}^t - \bar{\mathrm{m}}^\circ\|_{\mathrm{L}^2} \log^{1/2}(2 + \|\mathrm{m}^t - \bar{\mathrm{m}}^\circ\|_{\mathrm{L}^1 \cap \mathrm{L}^\infty}) + \|\theta^t - \bar{\theta}^\circ\|_{\mathrm{L}^2} \log^{1/2}(2 + \|\theta^t - \bar{\theta}^\circ\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}) + \|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2},
\end{aligned}$$

so that (7.37) follows from the a priori estimates of Lemma 7.4.1 (in the form $\|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2} + \|\mathrm{m}^t\|_{\mathrm{L}^1} \lesssim 1$) and from the boundedness assumption $\|\mathrm{m}\|_{\mathrm{L}^\infty_T \mathrm{L}^\infty} \lesssim 1$.

Step 2. Boundedness of θ: we prove $\|\theta^t - \bar{\theta}^\circ\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim 1$ for all $t \in [0,T)$.

We start with the L^2-estimate. As d satisfies the transport-diffusion equation (7.10), Lemma 7.2.3(i) with $s = 0$ leads to

$$\begin{aligned}
\|\mathrm{d}^t\|_{\mathrm{L}^2} &\lesssim \|\mathrm{d}^\circ\|_{\mathrm{L}^2} + \|a\mathrm{m}(-\alpha(\Psi + \mathrm{v}) + \beta(\Psi + \mathrm{v})^\perp)\|_{\mathrm{L}^2_t \mathrm{L}^2} \\
&\lesssim 1 + \|\mathrm{m}\|_{\mathrm{L}^2_t \mathrm{L}^\infty} \|\mathrm{v} - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty_t \mathrm{L}^2} + \|\mathrm{m}\|_{\mathrm{L}^2_t \mathrm{L}^2} \|(\Psi, \bar{\mathrm{v}}^\circ)\|_{\mathrm{L}^\infty},
\end{aligned}$$

and hence $\|\mathrm{d}^t\|_{\mathrm{L}^2} \lesssim 1$ follows from the a priori estimates of Lemma 7.4.1 (in the form $\|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2} + \|\mathrm{m}^t\|_{\mathrm{L}^1} \lesssim 1$) and the boundedness assumption for m. Similarly, for $\theta^t = a^{-1}\mathrm{d}^t - \nabla h \cdot \mathrm{v}^t$, we deduce $\|\theta^t - \bar{\theta}^\circ\|_{\mathrm{L}^2} \lesssim 1$. We now turn to the L^∞-estimate. Lemma 7.2.3(iii) with $\mathrm{p} = q = s = \infty$ gives

$$\|\mathrm{d}^t\|_{\mathrm{L}^\infty} \lesssim \|\mathrm{d}^\circ\|_{\mathrm{L}^\infty} + \|a\mathrm{m}(-\alpha(\Psi + \mathrm{v}) + \beta(\Psi + \mathrm{v})^\perp)\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty} \lesssim 1 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}(1 + \|\mathrm{v}\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}), \tag{7.38}$$

or alternatively, for $\theta^t = a^{-1}\mathrm{d}^t - \nabla h \cdot \mathrm{v}^t$,

$$\|\theta^t\|_{\mathrm{L}^\infty} \lesssim 1 + \|\mathrm{v}^t\|_{\mathrm{L}^\infty} + \|\mathrm{m}\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}(1 + \|v\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}).$$

Combining this estimate with the result of Step 1 yields

$$\begin{aligned}
\|\theta^t\|_{\mathrm{L}^\infty} &\lesssim 1 + \|\theta^t - \bar{\theta}^\circ\|_{\mathrm{L}^2} \log^{1/2}(2 + \|\theta^t - \bar{\theta}^\circ\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}) \\
&\qquad + \|\mathrm{m}\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}(1 + \|\theta - \bar{\theta}^\circ\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \log^{1/2}(2 + \|\theta - \bar{\theta}^\circ\|_{\mathrm{L}^\infty_t(\mathrm{L}^2 \cap \mathrm{L}^\infty)})).
\end{aligned}$$

Now the boundedness assumption on m and the L^2-estimate for θ proven above reduce this expression to

$$\|\theta^t\|_{\mathrm{L}^\infty} \lesssim \log^{1/2}(2 + \|\theta\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}).$$

Taking the supremum with respect to t, we may then conclude $\|\theta^t\|_{\mathrm{L}^\infty} \lesssim 1$ for all $t \in [0, T)$.

Step 3. Conclusion.

By the result of Step 2, the estimate (7.37) of Step 1 takes the form $\|\mathrm{v}^t\|_{\mathrm{L}^\infty} \lesssim 1$. The estimate (7.38) of Step 2 then yields $\|\mathrm{d}^t\|_{\mathrm{L}^\infty} \lesssim 1$, while the L^2-estimate for d is already established in Step 2. ☐

7.4.2 Propagation of regularity

Since local existence is established in Section 7.3 only for smooth enough data, it is necessary for the global existence result to first prove propagation of regularity along the flow. In this section, we show that propagation of regularity is a consequence of the boundedness of the vorticity m, which was indeed proven to hold in various regimes in Lemmas 7.4.2 and 7.4.3 above. We start with the propagation of Sobolev H^s-regularity.

Lemma 7.4.6 (Sobolev regularity)**.** *Let $s > 1$. Let $\lambda > 0$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $T > 0$, $h, \Psi, \bar{\mathrm{v}}^\circ \in W^{s+1,\infty}(\mathbb{R}^2)^2$, and $\mathrm{v}^\circ \in \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$, with $\mathrm{m}^\circ := \operatorname{curl} \mathrm{v}^\circ, \bar{\mathrm{m}}^\circ := \operatorname{curl} \bar{\mathrm{v}}^\circ \in \mathcal{P} \cap H^s(\mathbb{R}^2)$, and with either* $\operatorname{div}(a\mathrm{v}^\circ) = \operatorname{div}(a\bar{\mathrm{v}}^\circ) = 0$ *in the case* (7.1)*, or* $\mathrm{d}^\circ := \operatorname{div}(a\mathrm{v}^\circ), \bar{\mathrm{d}}^\circ := \operatorname{div}(a\bar{\mathrm{v}}^\circ) \in H^s(\mathbb{R}^2)$ *in the case* (7.2)*. Let* $\mathrm{v} \in \mathrm{L}^\infty([0, T]; \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$ *be a weak solution of* (7.1) *or of* (7.2) *on* $[0, T) \times \mathbb{R}^2$ *with initial data* v°*. Then for all* $t \in [0, T)$ *we have*

$$\|\mathrm{m}^t\|_{H^s} \leq C, \qquad \|\mathrm{d}^t\|_{H^s} \leq C, \qquad \|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{H^{s+1}} \leq C, \qquad \|\nabla \mathrm{v}^t\|_{\mathrm{L}^\infty} \leq C,$$

where the constant C depends only on an upper bound on s, $(s-1)^{-1}$, α, $|\beta|$, λ, λ^{-1}, T, $\|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}$, $\|(h, \Psi, \bar{\mathrm{v}}^\circ)\|_{W^{s+1,\infty}}$, $\|(\mathrm{m}^\circ, \bar{\mathrm{m}}^\circ, \mathrm{d}^\circ, \bar{\mathrm{d}}^\circ)\|_{H^s}$, $\|\mathrm{m}\|_{\mathrm{L}^\infty_T \mathrm{L}^\infty}$, and additionally on α^{-1} in the case $\alpha > 0$. ◊

Proof. We set $\theta := \operatorname{div} \mathrm{v}$, $\bar{\theta}^\circ := \operatorname{div} \bar{\mathrm{v}}^\circ$. In this proof, we use the notation $\lesssim$ for $\leq$ up to a constant $C > 0$ as in the statement. We focus on the compressible case (7.2), the other case being similar and simpler. We split the proof into four steps.

Step 1. Time derivative of $\|\mathrm{m}\|_{H^s}$: for all $s \geq 0$ and $t \in [0, T)$,

$$\partial_t \|\mathrm{m}^t\|_{H^s} \lesssim (1 + \|\nabla \mathrm{v}^t\|_{\mathrm{L}^\infty})(1 + \|\mathrm{m}^t\|_{H^s}) + \|\theta^t - \bar{\theta}^\circ\|_{H^s}.$$

Lemma 7.2.2 with $\rho = \mathrm{m}$, $w = \alpha(\Psi + \mathrm{v})^\perp + \beta(\Psi + \mathrm{v})$, and $W = \alpha(\Psi + \bar{\mathrm{v}}^\circ)^\perp + \beta(\Psi + \bar{\mathrm{v}}^\circ)$ yields

$$\partial_t \|\mathrm{m}^t\|_{H^s} \lesssim (1 + \|\nabla \mathrm{v}^t\|_{\mathrm{L}^\infty})\|\mathrm{m}^t\|_{H^s} + \|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{H^{s+1}} \|\mathrm{m}^t\|_{\mathrm{L}^\infty}. \tag{7.39}$$

Using Lemma 7.2.7, noting that $\|(\mathrm{m}^t - \bar{\mathrm{m}}^\circ, \theta^t - \bar{\theta}^\circ)\|_{\dot{H}^{-1}} \lesssim \|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}$, and using Lemma 7.4.1(iii) in the form $\|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2} \lesssim 1$, we obtain

$$\|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{H^{s+1}} \lesssim \|\mathrm{m}^t - \bar{\mathrm{m}}^\circ\|_{\dot{H}^{-1} \cap H^s} + \|\theta^t - \bar{\theta}^\circ\|_{\dot{H}^{-1} \cap H^s} \lesssim 1 + \|\mathrm{m}^t - \bar{\mathrm{m}}^\circ\|_{H^s} + \|\theta^t - \bar{\theta}^\circ\|_{H^s}.$$

Injecting this into (7.39), the claim follows from Lemma 7.4.5 and the boundedness assumption $\|\mathrm{m}\|_{\mathrm{L}^\infty_T \mathrm{L}^\infty} \lesssim 1$.

Step 2. Lipschitz estimate for v: for all $s > 1$ and $t \in [0, T)$,

$$\|\nabla \mathrm{v}^t\|_{\mathrm{L}^\infty} \lesssim \log(2 + \|\mathrm{m}^t\|_{H^s} + \|\theta^t - \bar{\theta}^\circ\|_{H^s}). \tag{7.40}$$

Since $\mathrm{v}^t - \bar{\mathrm{v}}^\circ = \nabla^\perp \triangle^{-1}(\mathrm{m}^t - \bar{\mathrm{m}}^\circ) + \nabla\triangle^{-1}(\theta^t - \bar\theta^\circ)$, Lemma 7.2.4(iii) yields, together with the Sobolev embedding of H^s into a Hölder space for all $s > 1$,

$$\begin{aligned}\|\nabla(\mathrm{v}^t - \bar{\mathrm{v}}^\circ)\|_{\mathrm{L}^\infty} &\le \|\nabla^2\triangle^{-1}(\mathrm{m}^t - \bar{\mathrm{m}}^\circ)\|_{\mathrm{L}^\infty} + \|\nabla^2\triangle^{-1}(\theta^t - \bar\theta^\circ)\|_{\mathrm{L}^\infty}\\ &\lesssim \|\mathrm{m}^t - \bar{\mathrm{m}}^\circ\|_{\mathrm{L}^\infty}\log(2 + \|\mathrm{m}^t - \bar{\mathrm{m}}^\circ\|_{H^s}) + \|\mathrm{m}^t - \bar{\mathrm{m}}^\circ\|_{\mathrm{L}^1}\\ &\qquad + \|\theta^t - \bar\theta^\circ\|_{\mathrm{L}^\infty}\log(2 + \|\theta^t - \bar\theta^\circ\|_{H^s}) + \|\theta^t - \bar\theta^\circ\|_{\mathrm{L}^2},\end{aligned}$$

and the claim (7.40) then follows from Lemma 7.4.1(i), Lemma 7.4.5, and the boundedness assumption on m.

Step 3. Sobolev estimate for θ: for all $s \ge 0$ and $t \in [0, T)$,

$$\|\theta^t - \bar\theta^\circ\|_{H^s} \lesssim 1 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s}. \tag{7.41}$$

As d satisfies the transport-diffusion equation (7.10), Lemma 7.2.3(i) gives for all $s \ge 0$,

$$\|\mathrm{d}^t\|_{H^s} \lesssim \|\mathrm{d}^\circ\|_{H^s} + \|a\mathrm{m}(-\alpha(\Psi + \mathrm{v}) + \beta(\Psi + \mathrm{v})^\perp)\|_{\mathrm{L}^2_t H^s}.$$

Using Lemma 7.2.1 to estimate the right-hand side, we find for all $s \ge 0$,

$$\begin{aligned}\|\mathrm{d}^t\|_{H^s} &\lesssim 1 + \|a\mathrm{m}(-\alpha(\mathrm{v} - \bar{\mathrm{v}}^\circ) + \beta(\mathrm{v} - \bar{\mathrm{v}}^\circ)^\perp)\|_{\mathrm{L}^2_t H^s} + \|a\mathrm{m}(-\alpha(\Psi + \bar{\mathrm{v}}^\circ) + \beta(\Psi + \bar{\mathrm{v}}^\circ)^\perp)\|_{\mathrm{L}^2_t H^s}\\ &\lesssim 1 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}\|\mathrm{v} - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2_t H^s} + \|\mathrm{m}\|_{\mathrm{L}^2_t H^s}\|\mathrm{v} - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}\\ &\qquad + \|\mathrm{m}\|_{\mathrm{L}^2_t \mathrm{L}^2}(1 + \|\bar{\mathrm{v}}^\circ\|_{W^{s,\infty}}) + \|\mathrm{m}\|_{\mathrm{L}^2_t H^s}(1 + \|\bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty}),\end{aligned}$$

and hence, by Lemma 7.4.5 and the boundedness assumption on m,

$$\|\mathrm{d}^t\|_{H^s} \lesssim 1 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s} + \|\mathrm{v} - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^\infty_t H^s}. \tag{7.42}$$

Lemma 7.2.7 then yields for all $s \ge 0$,

$$\|\mathrm{d}^t\|_{H^s} \lesssim 1 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s} + \|\mathrm{m} - \bar{\mathrm{m}}^\circ\|_{\mathrm{L}^\infty_t(\dot H^{-1}\cap H^{s-1})} + \|\mathrm{d} - \bar{\mathrm{d}}^\circ\|_{\mathrm{L}^\infty_t(\dot H^{-1}\cap H^{s-1})}.$$

Noting that $\|(\mathrm{m} - \bar{\mathrm{m}}^\circ, \mathrm{d} - \bar{\mathrm{d}}^\circ)\|_{\dot H^{-1}} \lesssim \|\mathrm{v} - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}$, and using Lemma 7.4.1(iii) in the form $\|\mathrm{v} - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2} \lesssim 1$, we deduce

$$\|\mathrm{d}^t\|_{H^s} \lesssim 1 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s} + \|\mathrm{d}\|_{\mathrm{L}^\infty_t H^{s-1}}.$$

Taking the supremum in time, we find by induction $\|\mathrm{d}\|_{\mathrm{L}^\infty_t H^s} \lesssim 1 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s} + \|\mathrm{d}\|_{\mathrm{L}^\infty_t \mathrm{L}^2}$ for all $s \ge 0$. Recalling that Lemma 7.4.5 gives $\|\theta^t - \bar\theta^\circ\|_{\mathrm{L}^2} \lesssim 1$, and using the identity $\theta^t = a^{-1}\mathrm{d}^t - \nabla h \cdot \mathrm{v}^t$, the claim (7.41) directly follows.

Step 4. Conclusion.

Combining the results of the three previous steps yields, for all $s > 1$,

$$\begin{aligned}\partial_t\|\mathrm{m}^t\|_{H^s} &\lesssim (1 + \|\mathrm{m}^t\|_{H^s})\log(2 + \|\mathrm{m}^t\|_{H^s} + \|\theta^t - \bar\theta^\circ\|_{H^s}) + \|\theta^t - \bar\theta^\circ\|_{H^s}\\ &\lesssim (1 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s})\log(2 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s}),\end{aligned}$$

hence

$$\partial_t\|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s} \le \sup_{[0,t]}\partial_t\|\mathrm{m}\|_{H^s} \lesssim (1 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s})\log(2 + \|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s}),$$

and the Grönwall inequality then gives $\|\mathrm{m}\|_{\mathrm{L}^\infty_t H^s} \lesssim 1$. Combining this with (7.40), (7.41) and (7.42), and recalling the identity $\mathrm{v}^t - \bar{\mathrm{v}}^\circ = \nabla^\perp\triangle^{-1}(\mathrm{m}^t - \bar{\mathrm{m}}^\circ) + \nabla\triangle^{-1}(\theta^t - \bar\theta^\circ)$, the conclusion follows. □

We now turn to the propagation of Hölder regularity. More precisely, we consider the Besov spaces $C^s_*(\mathbb{R}^2) := B^s_{\infty,\infty}(\mathbb{R}^2)$. Recall that these spaces coincide with the usual Hölder spaces $C^s_b(\mathbb{R}^2)$ only for non-integer $s \geq 0$ (for integer $s > 0$, they are strictly larger and coincide with the corresponding Zygmund spaces).

Lemma 7.4.7 (Hölder-Zygmund regularity)**.** *Let $s > 0$. Let $\lambda > 0$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $T > 0$, and $h, \Psi, \mathrm{v}^\circ \in C^{s+1}_*(\mathbb{R}^2)^2$ with $\mathrm{m}^\circ := \operatorname{curl} \mathrm{v}^\circ \in \mathcal{P}(\mathbb{R}^2)$, and with either $\operatorname{div}(a\mathrm{v}^\circ) = 0$ in the case (7.1), or $\mathrm{d}^\circ := \operatorname{div}(a\mathrm{v}^\circ) \in \mathrm{L}^2(\mathbb{R}^2)$ in the case (7.2). Let $\mathrm{v} \in \mathrm{L}^\infty([0,T]; C^{s+1}_*(\mathbb{R}^2)^2)$ be a weak solution of (7.1) or of (7.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°. Then we have for all $t \in [0,T)$,*

$$\|\mathrm{m}^t\|_{C^s_*} \leq C, \qquad \|\mathrm{d}^t\|_{C^s_*} \leq C, \qquad \|\mathrm{v}^t\|_{C^{s+1}_*} \leq C,$$

where the constant C depends only on an upper bound on s, s^{-1}, α, $|\beta|$, λ, λ^{-1}, T, $\|(h, \Psi, \mathrm{v}^\circ)\|_{C^{s+1}_*}$, $\|\mathrm{d}^\circ\|_{\mathrm{L}^2}$, $\|\mathrm{m}\|_{\mathrm{L}^\infty_T \mathrm{L}^\infty}$, and additionally on α^{-1} in the case $\alpha > 0$. $\Diamond$

Proof. We set $\theta := \operatorname{div} \mathrm{v}$. In this proof, we use the notation $\lesssim$ for $\leq$ up to a constant $C > 0$ as in the statement. We may focus on the compressible equation (7.2), the other case being similar and simpler. We split the proof into four steps, and make a systematic use of the standard Besov machinery as presented in [40].

Step 1. Time derivative of $\|\mathrm{m}^t\|_{C^s_*}$: for all $s > 0$ and $t \in [0,T)$,

$$\partial_t \|\mathrm{m}^t\|_{C^s_*} \lesssim (1 + \|\mathrm{m}^t\|_{C^s_*})(1 + \|\nabla \mathrm{v}^t\|_{\mathrm{L}^\infty \cap C^{s-1}_*}) + \|\theta^t\|_{C^s_*}.$$

The transport equation (7.9) has the form $\partial_t \mathrm{m}^t = \operatorname{div}(\mathrm{m}^t w^t)$ with $w^t = \alpha(\Psi + \mathrm{v}^t)^\perp + \beta(\Psi + \mathrm{v}^t)$. Arguing as in [40, Chapter 3.2] (that is, similarly as in the proof of Lemma 7.2.2, but using the corresponding commutator estimates in Besov spaces [40, Lemma 2.100]), we obtain for all $s > 0$,

$$\partial_t \|\mathrm{m}^t\|_{C^s_*} \lesssim \|\mathrm{m}^t\|_{C^s_*} \|\nabla w^t\|_{\mathrm{L}^\infty \cap C^{s-1}_*} + \|\mathrm{m}^t \operatorname{div} w^t\|_{C^s_*}.$$

Using the usual product rules [40, Corollary 2.86] for all $s > 0$,

$$\begin{aligned}\partial_t \|\mathrm{m}^t\|_{C^s_*} &\lesssim \|\mathrm{m}^t\|_{C^s_*} \|\nabla w^t\|_{\mathrm{L}^\infty \cap C^{s-1}_*} + \|\mathrm{m}^t\|_{\mathrm{L}^\infty} \|\operatorname{div} w^t\|_{C^s_*} + \|\mathrm{m}^t\|_{C^s_*} \|\operatorname{div} w^t\|_{\mathrm{L}^\infty} \\ &\lesssim \|\mathrm{m}^t\|_{C^s_*}(1 + \|\nabla \mathrm{v}^t\|_{\mathrm{L}^\infty \cap C^{s-1}_*}) + \|\mathrm{m}^t\|_{\mathrm{L}^\infty}(1 + \|\mathrm{m}^t\|_{C^s_*} + \|\theta^t\|_{C^s_*}),\end{aligned}$$

and the result follows from the boundedness assumption $\|\mathrm{m}\|_{\mathrm{L}^\infty_T \mathrm{L}^\infty} \lesssim 1$.

Step 2. Lipschitz estimate for v: for all $s > 0$ and $t \in [0,T)$,

$$\|\nabla \mathrm{v}^t\|_{\mathrm{L}^\infty \cap C^{s-1}_*} \lesssim \|\mathrm{m}^t\|_{C^{s-1}_*} + \|\theta^t\|_{C^{s-1}_*} + \log(2 + \|\mathrm{m}^t\|_{C^s_*} + \|\theta^t\|_{C^s_*}).$$

Since $\mathrm{v}^t - \mathrm{v}^\circ = \nabla^\perp \triangle^{-1}(\mathrm{m}^t - \mathrm{m}^\circ) + \nabla \triangle^{-1}(\theta^t - \theta^\circ)$, Lemma 7.2.5(ii) yields for all $s \in \mathbb{R}$,

$$\|\nabla \mathrm{v}^t\|_{C^{s-1}_*} \lesssim 1 + \|\mathrm{m}^t - \mathrm{m}^\circ\|_{\dot{H}^{-1} \cap C^{s-1}_*} + \|\theta^t - \theta^\circ\|_{\dot{H}^{-1} \cap C^{s-1}_*},$$

and thus, noting that $\|(\mathrm{m} - \mathrm{m}^\circ, \theta - \theta^\circ)\|_{\dot{H}^{-1}} \lesssim \|\mathrm{v} - \mathrm{v}^\circ\|_{\mathrm{L}^2}$, and using Lemma 7.4.1(iii) in the form $\|\mathrm{v} - \mathrm{v}^\circ\|_{\mathrm{L}^2} \lesssim 1$,

$$\|\nabla \mathrm{v}^t\|_{C^{s-1}_*} \lesssim 1 + \|\mathrm{m}^t\|_{C^{s-1}_*} + \|\theta^t\|_{C^{s-1}_*}.$$

Arguing as in Step 2 of the proof of Lemma 7.4.6 further yields for all $s > 0$,

$$\|\nabla \mathrm{v}^t\|_{\mathrm{L}^\infty} \lesssim \log(2 + \|\mathrm{m}^t\|_{C^s_*} + \|\theta^t - \theta^\circ\|_{C^s_*}),$$

and the result follows.

Step 3. Estimate for θ: for all $s>0$ and $t\in[0,T)$,

$$\|\theta^t\|_{C^s_*}\lesssim 1+\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^{s-1}_*}.$$

As d satisfies the transport-diffusion equation (7.10), we obtain for all $s>0$, arguing as in [40, Chapter 3.4],

$$\|\mathrm{d}^t\|_{C^s_*}\lesssim\|\mathrm{d}^\circ\|_{C^s_*}+\|a\mathrm{m}(-\alpha(\Psi+\mathrm{v})+\beta(\Psi+\mathrm{v})^\perp)\|_{\mathrm{L}^\infty_t C^{s-1}_*},$$

and thus, by the usual product rules [40, Corollary 2.86], the boundedness assumption on m, and Lemma 7.4.5, we deduce for all $s>0$,

$$\begin{aligned}\|\mathrm{d}^t\|_{C^s_*}&\lesssim 1+\|\mathrm{m}\|_{\mathrm{L}^\infty_t(\mathrm{L}^\infty\cap C^{s-1}_*)}(1+\|\mathrm{v}\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})+\|\mathrm{m}\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}(1+\|\mathrm{v}\|_{\mathrm{L}^\infty_t(\mathrm{L}^\infty\cap C^{s-1}_*)})\\&\lesssim 1+\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^{s-1}_*}+\|\mathrm{v}\|_{\mathrm{L}^\infty_t C^{s-1}_*},\end{aligned}$$

or alternatively, in terms of $\theta^t=a^{-1}\,\mathrm{d}^t-\nabla h\cdot\mathrm{v}^t$,

$$\|\theta^t\|_{C^s_*}\lesssim\|\mathrm{d}^t\|_{\mathrm{L}^\infty\cap C^s_*}+\|\mathrm{v}^t\|_{\mathrm{L}^\infty\cap C^s_*}\lesssim 1+\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^{s-1}_*}+\|\mathrm{v}\|_{\mathrm{L}^\infty_t C^s_*}.$$

Decomposing $\mathrm{v}^t-\mathrm{v}^\circ=\nabla^\perp\triangle^{-1}(\mathrm{m}^t-\mathrm{m}^\circ)+\nabla\triangle^{-1}(\theta^t-\theta^\circ)$, using Lemma 7.2.5(ii), and again Lemma 7.4.1(iii) in the form $\|(\mathrm{m}-\mathrm{m}^\circ,\theta-\theta^\circ)\|_{\dot H^{-1}}\lesssim\|\mathrm{v}-\mathrm{v}^\circ\|_{\mathrm{L}^2}\lesssim 1$, we find

$$\|\mathrm{v}^t\|_{C^s_*}\lesssim 1+\|\mathrm{m}^t-\mathrm{m}^\circ\|_{\dot H^{-1}\cap C^{s-1}_*}+\|\theta^t-\theta^\circ\|_{\dot H^{-1}\cap C^{s-1}_*}\lesssim 1+\|\mathrm{m}^t\|_{C^{s-1}_*}+\|\theta^t\|_{C^{s-1}_*},$$

and hence

$$\|\theta\|_{\mathrm{L}^\infty_t C^s_*}\lesssim 1+\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^{s-1}_*}+\|\theta\|_{\mathrm{L}^\infty_t C^{s-1}_*}.$$

If $s\le 1$, then we have $\|\cdot\|_{C^{s-1}_*}\lesssim\|\cdot\|_{\mathrm{L}^\infty}$, so that the above estimate, the boundedness assumption on m, and Lemma 7.4.5 yield $\|\theta\|_{\mathrm{L}^\infty_t C^s_*}\lesssim 1$. The result for $s>1$ then follows by induction.

Step 4. Conclusion.

Combining the results of the three previous steps yields, for all $s>0$,

$$\begin{aligned}\partial_t\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^s_*}&\le\sup_{[0,t]}\partial_t\|\mathrm{m}\|_{C^s_*}\\&\lesssim\ (1+\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^s_*})\big(\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^{s-1}_*}+\|\theta\|_{\mathrm{L}^\infty_t C^{s-1}_*}+\log(2+\|\mathrm{m}^t\|_{C^s_*}+\|\theta^t\|_{C^s_*})\big)+\|\theta\|_{\mathrm{L}^\infty_t C^s_*}\\&\lesssim\ (1+\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^s_*})\big(\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^{s-1}_*}+\log(2+\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^s_*})\big).\end{aligned}$$

If $s\le 1$, then we have $\|\cdot\|_{C^{s-1}_*}\lesssim\|\cdot\|_{\mathrm{L}^\infty}$, so that the above estimate and the boundedness assumption on m yield $\partial_t\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^s_*}\lesssim(1+\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^s_*})\log(2+\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^s_*})$, hence $\|\mathrm{m}\|_{\mathrm{L}^\infty_t C^s_*}\lesssim 1$ by the Grönwall inequality. The conclusion for $s>1$ then follows by induction. $\square$

7.4.3 Global existence of solutions

With Lemmas 7.4.6 and 7.4.7 at hand, together with the a priori bounds of Lemmas 7.4.2 and 7.4.3, it is straightforward to deduce the following global existence result from the local existence statement of Proposition 7.3.1.

Corollary 7.4.8 (Global existence of smooth solutions)**.** *Let $s>1$. Let $\lambda>0$, $\alpha\ge 0$, $\beta\in\mathbb{R}$, $h,\Psi,\bar{\mathrm{v}}^\circ\in W^{s+1,\infty}(\mathbb{R}^2)^2$, and $\mathrm{v}^\circ\in\bar{\mathrm{v}}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2$, with $\mathrm{m}^\circ:=\operatorname{curl}\mathrm{v}^\circ,\bar{\mathrm{m}}^\circ:=\operatorname{curl}\bar{\mathrm{v}}^\circ\in\mathcal{P}\cap H^s(\mathbb{R}^2)$, and with either $\operatorname{div}(a\mathrm{v}^\circ)=\operatorname{div}(a\bar{\mathrm{v}}^\circ)=0$ in the case* (7.1), *or $\mathrm{d}^\circ:=\operatorname{div}(a\mathrm{v}^\circ),\bar{\mathrm{d}}^\circ:=\operatorname{div}(a\bar{\mathrm{v}}^\circ)\in H^s(\mathbb{R}^2)$ in the case* (7.2). *Then,* *(i) there exists a global weak solution* $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$ *of* (7.1) *on* $\mathbb{R}^+ \times \mathbb{R}^2$ *with initial data* v°*, and with* $\mathrm{m} := \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap H^s(\mathbb{R}^2))$*;*

(ii) if $\beta = 0$*, there exists a global weak solution* $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2)$ *of* (7.2) *on* $\mathbb{R}^+ \times \mathbb{R}^2$ *with initial data* v°*, and with* $\mathrm{m} := \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap H^s(\mathbb{R}^2))$ *and* $\mathrm{d} := \operatorname{div}(a\mathrm{v}) \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^2))$*.* $\Diamond$

Proof. We may focus on item (ii), the first item being completely similar. In this proof we use the notation $\simeq$ and $\lesssim$ for $=$ and $\le$ up to positive constants that depend only on an upper bound on α, α^{-1}, $|\beta|$, λ, λ^{-1}, s, $(s-1)^{-1}$, $\|(h, \Psi, \bar{\mathrm{v}}^\circ)\|_{W^{s+1,\infty}}$, $\|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}$, $\|(\mathrm{m}^\circ, \bar{\mathrm{m}}^\circ, \mathrm{d}^\circ, \bar{\mathrm{d}}^\circ)\|_{H^s}$.

Given $\bar{\mathrm{v}}^\circ \in W^{s+1,\infty}(\mathbb{R}^2)^2$ and $\mathrm{v}^\circ \in \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$ with $\mathrm{m}^\circ, \bar{\mathrm{m}}^\circ \in \mathcal{P} \cap H^s(\mathbb{R}^2)$ and $\mathrm{d}^\circ, \bar{\mathrm{d}}^\circ \in H^s(\mathbb{R}^2)$, Proposition 7.3.1 gives a time $T > 0$, $T \simeq 1$, such that there exists a weak solution $\mathrm{v} \in \mathrm{L}^\infty([0,T); \bar{\mathrm{v}}^\circ + H^s(\mathbb{R}^2)^2)$ of (7.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°. For all $t \in [0,T)$, Lemma 7.4.3(ii) (with $\beta = 0$) then gives $\|\mathrm{m}^t\|_{\mathrm{L}^\infty} \lesssim 1$, which implies by Lemma 7.4.6,

$$\|\mathrm{m}^t\|_{H^s} + \|\mathrm{d}^t\|_{H^s} + \|\mathrm{v}^t - \bar{\mathrm{v}}^\circ\|_{H^{s+1}} \lesssim 1,$$

and moreover by Lemma 7.4.1(i) we have $\mathrm{m}^t \in \mathcal{P}(\mathbb{R}^2)$ for all $t \in [0,T)$. These a priori estimates show that the solution v can be extended globally in time. $\square$

We now extend this global existence result beyond the setting of smooth initial data. We start with the following result for L^2-data, which is easily deduced by approximation.

Corollary 7.4.9 (Global existence for L^2-data)**.** *Let* $\lambda > 0$*,* $\alpha \ge 0$*,* $\beta \in \mathbb{R}$*,* $h, \Psi \in W^{1,\infty}(\mathbb{R}^2)^2$*. Let* $\bar{\mathrm{v}}^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$ *be some reference map with* $\bar{\mathrm{m}}^\circ := \operatorname{curl} \bar{\mathrm{v}}^\circ \in \mathcal{P} \cap H^s(\mathbb{R}^2)$ *for some* $s > 1$*, and with either* $\operatorname{div}(a\bar{\mathrm{v}}^\circ) = 0$ *in the case* (7.1)*, or* $\bar{\mathrm{d}}^\circ := \operatorname{div}(a\bar{\mathrm{v}}^\circ) \in H^s(\mathbb{R}^2)$ *in the case* (7.2)*. Let* $\mathrm{v}^\circ \in \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$*, with* $\mathrm{m}^\circ := \operatorname{curl} \mathrm{v}^\circ \in \mathcal{P} \cap \mathrm{L}^2(\mathbb{R}^2)$*, and with either* $\operatorname{div}(a\mathrm{v}^\circ) = 0$ *in the case* (7.1)*, or* $\mathrm{d}^\circ := \operatorname{div}(a\mathrm{v}^\circ) \in \mathrm{L}^2(\mathbb{R}^2)$ *in the case* (7.2)*. Then,*

(i) there exists a global weak solution $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ *of* (7.1) *on* $\mathbb{R}^+ \times \mathbb{R}^2$ *with initial data* v°*, and with* $\mathrm{v} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + H^1(\mathbb{R}^2)^2)$ *and* $\mathrm{m} := \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^2(\mathbb{R}^2))$*;*

(ii) if $\beta = 0$*, there exists a global weak solution* $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ *of* (7.2) *on* $\mathbb{R}^+ \times \mathbb{R}^2$ *with initial data* v°*, and with* $\mathrm{v} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + H^1(\mathbb{R}^2)^2)$*,* $\mathrm{m} := \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^2(\mathbb{R}^2))$ *and* $\mathrm{d} := \operatorname{div}(a\mathrm{v}) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2))$*.* $\Diamond$

Proof. We may focus on the case (ii) (with $\beta = 0$), the other case being exactly similar. In this proof we use the notation $\lesssim$ for $\le$ up to a positive constant that depends only on an upper bound on α, α^{-1}, λ, $(s-1)^{-1}$, $\|(h, \Psi, \bar{\mathrm{v}}^\circ)\|_{W^{1,\infty}}$, $\|(\bar{\mathrm{m}}^\circ, \bar{\mathrm{d}}^\circ)\|_{H^s}$, $\|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}$, and $\|(\mathrm{m}^\circ, \mathrm{d}^\circ)\|_{\mathrm{L}^2}$. We use the notation $\lesssim_t$ if it further depends on an upper bound on time t.

Let $\rho \in C^\infty_c(\mathbb{R}^2)$ with $\rho \ge 0$, $\int \rho = 1$, and $\rho(0) = 1$. Define $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(x/\varepsilon)$ for all $\varepsilon > 0$, and set $\mathrm{m}^\circ_\varepsilon := \rho_\varepsilon * \mathrm{m}^\circ$, $\bar{\mathrm{m}}^\circ_\varepsilon := \rho_\varepsilon * \bar{\mathrm{m}}^\circ$, $\mathrm{d}^\circ_\varepsilon := \rho_\varepsilon * \mathrm{d}^\circ$, $\bar{\mathrm{d}}^\circ_\varepsilon := \rho_\varepsilon * \bar{\mathrm{d}}^\circ$, $a_\varepsilon := \rho_\varepsilon * a$ and $\Psi_\varepsilon := \rho_\varepsilon * \Psi$. For all $\varepsilon > 0$, we have $\mathrm{m}^\circ_\varepsilon, \bar{\mathrm{m}}^\circ_\varepsilon \in \mathcal{P} \cap H^\infty(\mathbb{R}^2)$, $\mathrm{d}^\circ_\varepsilon, \bar{\mathrm{d}}^\circ_\varepsilon \in H^\infty(\mathbb{R}^2)$, and $a_\varepsilon, a_\varepsilon^{-1}, \Psi_\varepsilon \in C^\infty_b(\mathbb{R}^2)^2$. By construction, we have $a_\varepsilon \to a$, $a_\varepsilon^{-1} \to a^{-1}$, $\Psi_\varepsilon \to \Psi$ in $W^{1,\infty}(\mathbb{R}^2)$, $\bar{\mathrm{m}}^\circ_\varepsilon - \bar{\mathrm{m}}^\circ, \bar{\mathrm{d}}^\circ_\varepsilon - \bar{\mathrm{d}}^\circ \to 0$ in $\dot{H}^{-1} \cap H^s(\mathbb{R}^2)$, and $\mathrm{m}^\circ_\varepsilon - \mathrm{m}^\circ, \mathrm{d}^\circ_\varepsilon - \mathrm{d}^\circ \to 0$ in $\dot{H}^{-1} \cap \mathrm{L}^2(\mathbb{R}^2)$. The additional convergence in $\dot{H}^{-1}(\mathbb{R}^2)$ indeed follows from the following computation with Fourier transforms,

$$\|\mathrm{m}^\circ_\varepsilon - \mathrm{m}^\circ\|^2_{\dot{H}^{-1}} = \int |\xi|^{-2} |\hat{\rho}(\varepsilon\xi) - 1|^2 |\hat{\mathrm{m}}^\circ(\xi)|^2 d\xi \le \varepsilon^2 \|\nabla\hat{\rho}\|^2_{\mathrm{L}^\infty} \|\mathrm{m}^\circ\|^2_{\mathrm{L}^2},$$

and similarly for $\bar{\mathrm{m}}^\circ_\varepsilon$, $\mathrm{d}^\circ_\varepsilon$, and $\bar{\mathrm{d}}^\circ_\varepsilon$. Lemma 7.2.7 then gives a unique $\mathrm{v}^\circ_\varepsilon \in \mathrm{v}^\circ + H^1(\mathbb{R}^2)^2$ and a unique $\bar{\mathrm{v}}^\circ_\varepsilon \in \bar{\mathrm{v}}^\circ + H^{s+1}(\mathbb{R}^2)^2$ such that $\operatorname{curl} \mathrm{v}^\circ_\varepsilon = \mathrm{m}^\circ_\varepsilon$, $\operatorname{curl} \bar{\mathrm{v}}^\circ_\varepsilon = \bar{\mathrm{m}}^\circ_\varepsilon$, $\operatorname{div}(a_\varepsilon \mathrm{v}^\circ_\varepsilon) = \mathrm{d}^\circ_\varepsilon$, $\operatorname{div}(a_\varepsilon \bar{\mathrm{v}}^\circ_\varepsilon) = \bar{\mathrm{d}}^\circ_\varepsilon$, and we have $\mathrm{v}^\circ_\varepsilon - \mathrm{v}^\circ \to 0$ in $H^1(\mathbb{R}^2)^2$ and $\bar{\mathrm{v}}^\circ_\varepsilon - \bar{\mathrm{v}}^\circ \to 0$ in $H^{s+1}(\mathbb{R}^2)^2$. In particular, the assumption $\bar{\mathrm{v}}^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$ yields by the Sobolev embedding with $s > 1$, for $\varepsilon > 0$ small enough,

$$\|\bar{\mathrm{v}}^\circ_\varepsilon\|_{W^{1,\infty}} \lesssim \|\bar{\mathrm{v}}^\circ_\varepsilon - \bar{\mathrm{v}}^\circ\|_{H^{s+1}} + \|\bar{\mathrm{v}}^\circ\|_{W^{1,\infty}} \lesssim 1,$$

and the assumption $\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ \in \mathrm{L}^2(\mathbb{R}^2)^2$ implies

$$\| \mathrm{v}^\circ_\varepsilon - \bar{\mathrm{v}}^\circ_\varepsilon \|_{\mathrm{L}^2} \le \| \mathrm{v}^\circ_\varepsilon - \mathrm{v}^\circ \|_{\mathrm{L}^2} + \| \mathrm{v}^\circ - \bar{\mathrm{v}}^\circ \|_{\mathrm{L}^2} + \| \bar{\mathrm{v}}^\circ_\varepsilon - \bar{\mathrm{v}}^\circ \|_{\mathrm{L}^2} \lesssim 1.$$

Corollary 7.4.8 then gives a solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ_\varepsilon + H^\infty(\mathbb{R}^2)^2)$ of (7.2) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data $\mathrm{v}^\circ_\varepsilon$, and with (a, Ψ) replaced by $(a_\varepsilon, \Psi_\varepsilon)$. Lemma 7.4.1(iii) and Lemma 7.4.3(ii) (with $\beta = 0$) give for all $t \ge 0$,

$$\| \mathrm{v}_\varepsilon - \bar{\mathrm{v}}^\circ_\varepsilon \|_{\mathrm{L}^\infty_t \mathrm{L}^2} + \| \mathrm{d}_\varepsilon \|_{\mathrm{L}^2_t \mathrm{L}^2} + \| \mathrm{m}_\varepsilon \|_{\mathrm{L}^\infty_t \mathrm{L}^2} \lesssim_t 1,$$

hence by Lemma 7.2.7, together with the obvious estimate $\| (\mathrm{m}_\varepsilon - \bar{\mathrm{m}}^\circ_\varepsilon, \mathrm{d}_\varepsilon - \bar{\mathrm{d}}^\circ_\varepsilon) \|_{\dot H^{-1}} \lesssim \| \mathrm{v}_\varepsilon - \bar{\mathrm{v}}^\circ_\varepsilon \|_{\mathrm{L}^2}$,

$$\| \mathrm{v}_\varepsilon - \bar{\mathrm{v}}^\circ_\varepsilon \|_{\mathrm{L}^2_t H^1} \lesssim \| \mathrm{v}_\varepsilon - \bar{\mathrm{v}}^\circ_\varepsilon \|_{\mathrm{L}^2_t \mathrm{L}^2} + \| \mathrm{d}_\varepsilon - \bar{\mathrm{d}}^\circ_\varepsilon \|_{\mathrm{L}^2_t \mathrm{L}^2} + \| \mathrm{m}_\varepsilon - \bar{\mathrm{m}}^\circ_\varepsilon \|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim_t 1.$$

As $\bar{\mathrm{v}}^\circ_\varepsilon$ is bounded in $H^1_{\mathrm{loc}}(\mathbb{R}^2)^2$, we deduce up to an extraction $\mathrm{v}_\varepsilon \rightharpoonup \mathrm{v}$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{loc}}(\mathbb{R}^2)^2)$, and also $\mathrm{m}_\varepsilon \rightharpoonup \mathrm{m}$, $\mathrm{d}_\varepsilon \rightharpoonup \mathrm{d}$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2))$, for some functions $\mathrm{v}, \mathrm{m}, \mathrm{d}$. Comparing equation (7.9) with the above estimates, we deduce that $(\partial_t \mathrm{m}_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2))$. Since by the Rellich theorem the space $\mathrm{L}^2(U)$ is compactly embedded in $H^{-1}(U) \subset W^{-1,1}(U)$ for any bounded domain $U \subset \mathbb{R}^2$, the Aubin-Simon lemma ensures that we have $\mathrm{m}_\varepsilon \to \mathrm{m}$ strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^{-1}_{\mathrm{loc}}(\mathbb{R}^2))$. This implies $\mathrm{m}_\varepsilon \mathrm{v}_\varepsilon \to \mathrm{m}\mathrm{v}$ in the distributional sense. We may then pass to the limit in the weak formulation of equation (7.2), and the result follows. □

We turn to the case of rougher initial data. Using the a priori estimates of Lemmas 7.4.2 and 7.4.3(ii), we establish global existence for L^q-data with $q > 1$. In the parabolic regime $\alpha > 0$, $\beta = 0$, the finer a priori estimates of Lemma 7.4.3(iii) further imply global existence for vortex-sheet data $\mathrm{m}^\circ \in \mathcal{P}(\mathbb{R}^2)$. Arguing by approximation, the main work consists in passing to the limit in the nonlinear term mv. For that purpose, as in [304], we make a crucial use of some compactness result due to Lions [305] in the context of the compressible Navier-Stokes equations. The conservative regime (iv) below is however more subtle due to a lack of strong enough a priori estimates: only very weak solutions are then expected and obtained in that case, and compactness is carefully proven by hand.

Proposition 7.4.10 (Global existence for general data). *Let $\lambda > 0$, $\alpha \ge 0$, $\beta \in \mathbb{R}$, and $h, \Psi \in W^{1,\infty}(\mathbb{R}^2)^2$. Let $\bar{\mathrm{v}}^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$ be some reference map with $\bar{\mathrm{m}}^\circ := \operatorname{curl} \bar{\mathrm{v}}^\circ \in \mathcal{P} \cap H^s(\mathbb{R}^2)$ for some $s > 1$, and with either $\operatorname{div}(a\bar{\mathrm{v}}^\circ) = 0$ in the case (7.1), or $\bar{\mathrm{d}}^\circ := \operatorname{div}(a\bar{\mathrm{v}}^\circ) \in H^s(\mathbb{R}^2)$ in the case (7.2). Let $\mathrm{v}^\circ \in \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$ with $\mathrm{m}^\circ = \operatorname{curl} \mathrm{v}^\circ \in \mathcal{P}(\mathbb{R}^2)$, and with either $\operatorname{div}(a\mathrm{v}^\circ) = 0$ in the case (7.1), or $\mathrm{d}^\circ := \operatorname{div}(a\mathrm{v}^\circ) \in \mathrm{L}^2(\mathbb{R}^2)$ in the case (7.2). Then the following hold.*

(i) Case (7.2) with $\alpha > 0$, $\beta = 0$: *There exists a weak solution $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°, and with $\mathrm{m} = \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$ and $\mathrm{d} = \operatorname{div}(a\mathrm{v}) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2))$.*

(ii) Case (7.1) with $\alpha > 0$, and either $\beta = 0$ or a constant: *There exists a weak solution $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°, and with $\mathrm{m} = \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$.*

(iii) Case (7.1) with $\alpha > 0$: *If $\mathrm{m}^\circ \in \mathrm{L}^q(\mathbb{R}^2)$ for some $q > 1$, there exists a weak solution $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°, and with $\mathrm{m} = \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^q(\mathbb{R}^2))$.*

(iv) Case (7.1) with $\alpha = 0$: *If $\mathrm{m}^\circ \in \mathrm{L}^q(\mathbb{R}^2)$ for some $q > 1$, there exists a very weak solution $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°, and with $\mathrm{m} = \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^q(\mathbb{R}^2))$. This is a weak solution whenever $q \ge 4/3$.* ◊

Proof. We split the proof into three steps, first proving item (i), then explaining how the argument has to be adapted to prove items (ii) and (iii), and finally turning to item (iv).

Step 1. Proof of (i).

In this step, we use the notation $\lesssim$ for $\le$ up to a positive constant that depends only on an upper bound on α, α^{-1}, λ, $\|(h,\Psi,\bar{\mathrm{v}}^\circ)\|_{W^{1,\infty}}$, $\|(\bar{\mathrm{m}}^\circ,\bar{\mathrm{d}}^\circ)\|_{H^s}$, $\|\mathrm{v}^\circ-\bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}$, and $\|\mathrm{d}^\circ\|_{\mathrm{L}^2}$. We use the notation $\lesssim_t$ (resp. $\lesssim_{t,U}$) if it further depends on an upper bound on time t (resp. and on the size of $U\subset\mathbb{R}^2$).

Let $\rho\in C_c^\infty(\mathbb{R}^2)$ with $\rho\ge0$, $\int\rho=1$, $\rho(0)=1$, and $\rho|_{\mathbb{R}^2\setminus B_1}=0$, define $\rho_\varepsilon(x):=\varepsilon^{-d}\rho(x/\varepsilon)$ for all $\varepsilon>0$, and set $\mathrm{m}_\varepsilon^\circ:=\rho_\varepsilon*\mathrm{m}^\circ$, $\bar{\mathrm{m}}_\varepsilon^\circ:=\rho_\varepsilon*\bar{\mathrm{m}}^\circ$, $\mathrm{d}_\varepsilon^\circ:=\rho_\varepsilon*\mathrm{d}^\circ$, $\bar{\mathrm{d}}_\varepsilon^\circ:=\rho_\varepsilon*\bar{\mathrm{d}}^\circ$. For all $\varepsilon>0$, we have $\mathrm{m}_\varepsilon^\circ,\bar{\mathrm{m}}_\varepsilon^\circ\in\mathcal{P}\cap H^\infty(\mathbb{R}^2)$, $\mathrm{d}_\varepsilon^\circ,\bar{\mathrm{d}}_\varepsilon^\circ\in H^\infty(\mathbb{R}^2)$. As in the proof of Corollary 7.4.9, we have by construction $\bar{\mathrm{m}}_\varepsilon^\circ-\bar{\mathrm{m}}^\circ$, $\bar{\mathrm{d}}_\varepsilon^\circ-\bar{\mathrm{d}}^\circ\to0$ in $\dot H^{-1}\cap H^s(\mathbb{R}^2)$, and $\mathrm{d}_\varepsilon^\circ-\mathrm{d}^\circ\to0$ in $\dot H^{-1}\cap\mathrm{L}^2(\mathbb{R}^2)$. The assumption $\mathrm{v}^\circ-\bar{\mathrm{v}}^\circ\in\mathrm{L}^2(\mathbb{R}^2)^2$ further yields $\mathrm{m}^\circ-\bar{\mathrm{m}}^\circ\in\dot H^{-1}(\mathbb{R}^2)$, which implies $\mathrm{m}_\varepsilon^\circ-\bar{\mathrm{m}}_\varepsilon^\circ\to\mathrm{m}^\circ-\bar{\mathrm{m}}^\circ$, hence $\mathrm{m}_\varepsilon^\circ-\mathrm{m}^\circ\to0$, in $\dot H^{-1}(\mathbb{R}^2)$. Lemma 7.2.7 then gives a unique $\mathrm{v}_\varepsilon^\circ\in\mathrm{v}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2$ and a unique $\bar{\mathrm{v}}_\varepsilon^\circ\in\bar{\mathrm{v}}^\circ+H^{s+1}(\mathbb{R}^2)^2$ such that $\operatorname{curl}\mathrm{v}_\varepsilon^\circ=\mathrm{m}_\varepsilon^\circ$, $\operatorname{curl}\bar{\mathrm{v}}_\varepsilon^\circ=\bar{\mathrm{m}}_\varepsilon^\circ$, $\operatorname{div}(a_\varepsilon\mathrm{v}_\varepsilon^\circ)=\mathrm{d}_\varepsilon^\circ$, $\operatorname{div}(a_\varepsilon\bar{\mathrm{v}}_\varepsilon^\circ)=\bar{\mathrm{d}}_\varepsilon^\circ$, and we have $\mathrm{v}_\varepsilon^\circ-\mathrm{v}^\circ\to0$ in $\mathrm{L}^2(\mathbb{R}^2)^2$ and $\bar{\mathrm{v}}_\varepsilon^\circ-\bar{\mathrm{v}}^\circ\to0$ in $H^{s+1}(\mathbb{R}^2)^2$. In particular, arguing as in the proof of Corollary 7.4.9, the assumption $\bar{\mathrm{v}}^\circ\in W^{1,\infty}(\mathbb{R}^2)^2$ yields $\|\bar{\mathrm{v}}_\varepsilon^\circ\|_{W^{1,\infty}}\lesssim1$ by the Sobolev embedding with $s>1$, and the assumption $\mathrm{v}^\circ-\bar{\mathrm{v}}^\circ\in\mathrm{L}^2(\mathbb{R}^2)^2$ implies $\|\mathrm{v}_\varepsilon^\circ-\bar{\mathrm{v}}_\varepsilon^\circ\|_{\mathrm{L}^2}\lesssim1$.

Corollary 7.4.9 then gives a global weak solution $\mathrm{v}_\varepsilon\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar{\mathrm{v}}_\varepsilon^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ of (7.2) on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data $\mathrm{v}_\varepsilon^\circ$, and Lemma 7.4.1(iii) yields for all $t\ge0$,

$$\|\mathrm{v}_\varepsilon-\bar{\mathrm{v}}_\varepsilon^\circ\|_{\mathrm{L}^\infty_t\mathrm{L}^2}+\|\mathrm{d}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2}\lesssim_t1,\tag{7.43}$$

while Lemma 7.4.3(iii) (with $\beta=0$) yields after time integration for all $1\le p<2$,

$$\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^p_t\mathrm{L}^p}\lesssim\bigg(\int_0^t\big(u^{1-p}+e^{Cu}\big)\,du\bigg)^{1/p}\lesssim_t(2-p)^{-1/p}.$$

Using this last estimate for p $=3/2$ and $11/6$, and combining it with Lemma 7.4.1(i) in the form $\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^1}\le1$, we deduce by interpolation

$$\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^2_t(\mathrm{L}^{4/3}\cap\mathrm{L}^{12/7})}\lesssim_t1.$$

Now we need to prove more precise estimates on v_ε. First recall the identity

$$\mathrm{v}_\varepsilon=\mathrm{v}_{\varepsilon,1}+\mathrm{v}_{\varepsilon,2},\qquad\mathrm{v}_{\varepsilon,1}:=\nabla^\perp\triangle^{-1}\mathrm{m}_\varepsilon,\qquad\mathrm{v}_{\varepsilon,2}:=\nabla\triangle^{-1}\operatorname{div}\mathrm{v}_\varepsilon\,.\tag{7.44}$$

On the one hand, as m_ε is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{4/3}\cap\mathrm{L}^{12/7}(\mathbb{R}^2))$, we deduce from Riesz potential theory that $\mathrm{v}_{\varepsilon,1}$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^4\cap\mathrm{L}^{12}(\mathbb{R}^2)^2)$, and we deduce from the Calderón-Zygmund theory that $\nabla\mathrm{v}_{\varepsilon,1}$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{4/3}(\mathbb{R}^2))$. On the other hand, decomposing

$$\mathrm{v}_{\varepsilon,2}=\nabla\triangle^{-1}\operatorname{div}(\mathrm{v}_\varepsilon-\bar{\mathrm{v}}_\varepsilon^\circ)+\bar{\mathrm{v}}_\varepsilon^\circ-\nabla^\perp\triangle^{-1}\bar{\mathrm{m}}_\varepsilon^\circ,$$

noting that $\mathrm{v}_\varepsilon-\bar{\mathrm{v}}_\varepsilon^\circ$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^2)^2)$ (cf. (7.43)), that $\bar{\mathrm{v}}_\varepsilon^\circ$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$, and that $\|\nabla\triangle^{-1}\bar{\mathrm{m}}_\varepsilon^\circ\|_{\mathrm{L}^2}\lesssim\|\bar{\mathrm{m}}_\varepsilon^\circ\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}\lesssim1$ (cf. Lemma 7.2.4), we deduce that $\mathrm{v}_{\varepsilon,2}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2)$. Further, decomposing

$$\mathrm{v}_{\varepsilon,2}=\nabla\triangle^{-1}(a^{-1}(\mathrm{d}_\varepsilon-\bar{\mathrm{d}}_\varepsilon^\circ))-\nabla\triangle^{-1}(\nabla h\cdot(\mathrm{v}_\varepsilon-\bar{\mathrm{v}}_\varepsilon^\circ))+\bar{\mathrm{v}}_\varepsilon^\circ-\nabla^\perp\triangle^{-1}\bar{\mathrm{m}}_\varepsilon^\circ,$$

we easily check that $\nabla\mathrm{v}_{\varepsilon,2}$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2)$. We then conclude from the Sobolev embedding that $\mathrm{v}_{\varepsilon,2}$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^2)^2)$ for all $q<\infty$. For our purposes it is enough to choose $q=4$ and 12. In particular, we have proven that for all bounded subset $U\subset\mathbb{R}^2$,

$$\begin{aligned}&\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^{4/3}}+\|\mathrm{d}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2}+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^2(U)}\\&\qquad+\|\mathrm{v}_{\varepsilon,1}\|_{\mathrm{L}^2_t(\mathrm{L}^4\cap\mathrm{L}^{12})}+\|\nabla\mathrm{v}_{\varepsilon,1}\|_{\mathrm{L}^2_t\mathrm{L}^{4/3}}+\|\mathrm{v}_{\varepsilon,2}\|_{\mathrm{L}^2_t(\mathrm{L}^4\cap\mathrm{L}^{12}(U))}+\|\nabla\mathrm{v}_{\varepsilon,2}\|_{\mathrm{L}^2_t\mathrm{L}^2(U)}\lesssim_{t,U}1.\end{aligned}\tag{7.45}$$

Therefore we have up to an extraction $\mathrm{m}_\varepsilon \rightharpoonup \mathrm{m}$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{4/3}(\mathbb{R}^2))$, $\mathrm{d}_\varepsilon \rightharpoonup \mathrm{d}$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^2))$, $\mathrm{v}_{\varepsilon,1} \rightharpoonup \mathrm{v}_1$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^4(\mathbb{R}^2)^2)$, and $\mathrm{v}_{\varepsilon,2} \rightharpoonup \mathrm{v}_2$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^4_{\mathrm{loc}}(\mathbb{R}^2)^2)$, for some functions $\mathrm{m},\mathrm{d},\mathrm{v}_1,\mathrm{v}_2$. Comparing the above estimates with (7.9), we deduce that $(\partial_t \mathrm{m}_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2))$. Moreover, we find by interpolation for all $|\xi|<1$ and all bounded domain $U\subset\mathbb{R}^2$, denoting by $U^1 := U + B_1$ its 1-fattening,

$$\begin{aligned}
&\|\mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon(\cdot+\xi)\|_{\mathrm{L}^2_t\mathrm{L}^4(U)} \le \|\mathrm{v}_{\varepsilon,1} - \mathrm{v}_{\varepsilon,1}(\cdot+\xi)\|_{\mathrm{L}^2_t\mathrm{L}^4(U)} + \|\mathrm{v}_{\varepsilon,2} - \mathrm{v}_{\varepsilon,2}(\cdot+\xi)\|_{\mathrm{L}^2_t\mathrm{L}^4(U)}\\
&\quad\le\ \|\mathrm{v}_{\varepsilon,1} - \mathrm{v}_{\varepsilon,1}(\cdot+\xi)\|^{1/4}_{\mathrm{L}^2_t\mathrm{L}^{4/3}(U)}\|\mathrm{v}_{\varepsilon,1} - \mathrm{v}_{\varepsilon,1}(\cdot+\xi)\|^{3/4}_{\mathrm{L}^2_t\mathrm{L}^{12}(U)}\\
&\qquad\qquad\qquad\qquad+\|\mathrm{v}_{\varepsilon,2} - \mathrm{v}_{\varepsilon,2}(\cdot+\xi)\|^{2/5}_{\mathrm{L}^2_t\mathrm{L}^2(U)}\|\mathrm{v}_{\varepsilon,2} - \mathrm{v}_{\varepsilon,2}(\cdot+\xi)\|^{3/5}_{\mathrm{L}^2_t\mathrm{L}^{12}(U)}\\
&\quad\le\ 2\|\mathrm{v}_{\varepsilon,1} - \mathrm{v}_{\varepsilon,1}(\cdot+\xi)\|^{1/4}_{\mathrm{L}^2_t\mathrm{L}^{4/3}(U)}\|\mathrm{v}_{\varepsilon,1}\|^{3/4}_{\mathrm{L}^2_t\mathrm{L}^{12}(U^1)} + 2\|\mathrm{v}_{\varepsilon,2} - \mathrm{v}_{\varepsilon,2}(\cdot+\xi)\|^{2/5}_{\mathrm{L}^2_t\mathrm{L}^2(U)}\|\mathrm{v}_{\varepsilon,2}\|^{3/5}_{\mathrm{L}^2_t\mathrm{L}^{12}(U^1)}\\
&\quad\le\ 2|\xi|^{1/4}\|\nabla\mathrm{v}_{\varepsilon,1}\|^{1/4}_{\mathrm{L}^2_t\mathrm{L}^{4/3}(U^1)}\|\mathrm{v}_{\varepsilon,1}\|^{3/4}_{\mathrm{L}^2_t\mathrm{L}^{12}(U^1)} + 2|\xi|^{2/5}\|\nabla\mathrm{v}_{\varepsilon,2}\|^{2/5}_{\mathrm{L}^2_t\mathrm{L}^2(U^1)}\|\mathrm{v}_{\varepsilon,2}\|^{3/5}_{\mathrm{L}^2_t\mathrm{L}^{12}(U^1)},
\end{aligned}$$

and hence by (7.45),

$$\|\mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon(\cdot+\xi)\|_{\mathrm{L}^2_t\mathrm{L}^4(U)} \lesssim_{t,U} |\xi|^{1/4} + |\xi|^{2/5}.$$

Let us summarize the previous observations: up to an extraction, setting $\mathrm{v} := \mathrm{v}_1 + \mathrm{v}_2$, we have

$$\begin{gathered}
\mathrm{m}_\varepsilon \rightharpoonup \mathrm{m} \text{ in } \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{4/3}(\mathbb{R}^2)),\quad \mathrm{v}_\varepsilon \rightharpoonup \mathrm{v} \text{ in } \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^4_{\mathrm{loc}}(\mathbb{R}^2)^2),\\
(\partial_t\mathrm{m}_\varepsilon)_\varepsilon \text{ bounded in } \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)),\\
\sup_{\varepsilon>0}\|\mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon(\cdot+\xi)\|_{\mathrm{L}^2_t\mathrm{L}^4(U)} \to 0 \ \text{ as } |\xi|\to 0, \text{ for all } t\ge 0 \text{ and all bounded subset } U\subset\mathbb{R}^2.
\end{gathered}$$

We may then apply [305, Lemma 5.1], which ensures that $\mathrm{m}_\varepsilon\mathrm{v}_\varepsilon \to \mathrm{m}\mathrm{v}$ holds in the distributional sense. This allows to pass to the limit in the weak formulation of equation (7.2), and the result follows.

Step 2. Proof of (ii) and (iii).

The proof of item (ii) is again based on Lemma 7.4.3(iii), and is completely analogous to the proof of item (i) above. Regarding item (iii), Lemma 7.4.3(iii) does no longer apply in that case, but, since we further assume $\mathrm{m}^\circ \in \mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$, Lemma 7.4.2 gives the following a priori estimate: for all $t\ge 0$

$$\|\mathrm{m}\|_{\mathrm{L}^{q+1}_t\mathrm{L}^{q+1}} + \|\mathrm{m}\|_{\mathrm{L}^\infty_t\mathrm{L}^q} \lesssim_t 1, \tag{7.46}$$

hence in particular by interpolation $\|\mathrm{m}\|_{\mathrm{L}^p_t\mathrm{L}^p} \lesssim_t 1$ for all $1\le p\le 2$. (Here we use the notation $\lesssim_t$ for $\le$ up to a constant that depends only on an upper bound on t, $(q-1)^{-1}$, α, α^{-1}, $|\beta|$, $\|(h,\Psi)\|_{W^{1,\infty}}$, $\|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}$, and $\|\mathrm{m}^\circ\|_{\mathrm{L}^q}$.) The conclusion follows from a similar argument as in Step 1.

Step 3. Proof of (iv).

We finally turn to the incompressible equation (7.1) in the conservative regime $\alpha=0$. Let $q>1$ be such that $\mathrm{m}^\circ\in\mathrm{L}^q(\mathbb{R}^2)$. Lemma 7.4.2 or 7.4.3(ii) ensures that m_ε is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1\cap\mathrm{L}^q(\mathbb{R}^2))$, and hence, for $q>4/3$, replacing the exponents $4/3$ and $12/7$ of Step 1 by $4/3$ and q, the argument of Step 1 can be immediately adapted to this case, for which we thus obtain global existence of a weak solution. In the remaining case $1<q<4/3$, the product $\mathrm{m}\,\nabla\Delta^{-1}\,\mathrm{m}$ (hence the product mv, cf. (7.44)) does not make sense any more for $\mathrm{m}\in\mathrm{L}^q(\mathbb{R}^2)$. Since in the conservative regime $\alpha=0$ no additional regularity is available (in particular, (7.46) does not hold), we do not expect the existence of a weak solution, and we need to turn to the notion of very weak solutions as defined in Definition 7.1.1(c), where the product mv is reinterpreted à la Delort. Let $1<q\le 4/3$. We establish the global existence

of a very weak solution. (For the critical exponent $q = 4/3$, the integrability of v found below directly implies by Remark 7.1.2(ii) that the constructed very weak solution is automatically a weak solution.) In this step, we use the notation $\lesssim$ for $\le$ up to a constant C that depends only on an upper bound on $(q-1)^{-1}$, $|\beta|$, $\|(h, \Psi, \bar{\mathrm{v}}^\circ)\|_{W^{1,\infty}}$, $\|\mathrm{v}^\circ - \bar{\mathrm{v}}^\circ\|_{\mathrm{L}^2}$, $\|\bar{\mathrm{m}}^\circ\|_{\mathrm{L}^2}$, and $\|\mathrm{m}^\circ\|_{\mathrm{L}^q}$, and we use the notation $\lesssim_t$ (resp. $\lesssim_{t,U}$) if it further depends on an upper bound on time t (resp. on t and on the size of $U \subset \mathbb{R}^2$).

Let $\mathrm{m}_\varepsilon^\circ$, $\bar{\mathrm{m}}_\varepsilon^\circ$, $\mathrm{v}_\varepsilon^\circ$, $\bar{\mathrm{v}}_\varepsilon^\circ$ be defined as in Step 1 (with of course $\mathrm{d}_\varepsilon^\circ = \bar{\mathrm{d}}_\varepsilon^\circ = 0$), and let $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{\mathrm{v}}_\varepsilon^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ be a global weak solution of (7.1) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data $\mathrm{v}_\varepsilon^\circ$, as given by Corollary 7.4.9. Lemmas 7.4.1(iii) and 7.4.3(ii) then give for all $t \ge 0$,

$$\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t(\mathrm{L}^1 \cap \mathrm{L}^q)} + \|\mathrm{v}_\varepsilon - \bar{\mathrm{v}}_\varepsilon^\circ\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \lesssim_t 1. \tag{7.47}$$

As $\bar{\mathrm{v}}_\varepsilon^\circ$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$, we deduce in particular that v_ε is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2))$. Moreover, using the Delort type identity

$$\mathrm{m}_\varepsilon \mathrm{v}_\varepsilon = -\frac{1}{2}|\mathrm{v}_\varepsilon|^2 \nabla^\perp h - a^{-1}(\mathrm{div}\,(a S_{\mathrm{v}_\varepsilon}))^\perp,$$

we then deduce that $\mathrm{m}_\varepsilon \mathrm{v}_\varepsilon$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$. Let us now recall the following useful decomposition,

$$\mathrm{v}_\varepsilon = \mathrm{v}_{\varepsilon,1} + \mathrm{v}_{\varepsilon,2}, \qquad \mathrm{v}_{\varepsilon,1} := \nabla^\perp \triangle^{-1} \mathrm{m}_\varepsilon, \qquad \mathrm{v}_{\varepsilon,2} := \nabla \triangle^{-1} \mathrm{div}\, \mathrm{v}_\varepsilon\,. \tag{7.48}$$

By Riesz potential theory $\mathrm{v}_{\varepsilon,1}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^p(\mathbb{R}^2)^2)$ for all $2 < p \le \frac{2q}{2-q}$, while as in Step 1 we check that $\mathrm{v}_{\varepsilon,2}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{loc}}(\mathbb{R}^2)^2)$. Hence by the Sobolev embedding, for all bounded domain $U \subset \mathbb{R}^2$ and all $t \ge 0$,

$$\|(\mathrm{v}_\varepsilon, \mathrm{v}_{\varepsilon,1})\|_{\mathrm{L}^\infty_t \mathrm{L}^{2q/(2-q)}(U)} \lesssim_{t,U} 1. \tag{7.49}$$

Up to an extraction we then have $\mathrm{v}_\varepsilon \overset{*}{\rightharpoonup} \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2)$ and $\mathrm{m}_\varepsilon \overset{*}{\rightharpoonup} \mathrm{m}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^q(\mathbb{R}^2))$, for some functions v, m, with necessarily $\mathrm{m} = \mathrm{curl}\,\mathrm{v}$ and $\mathrm{div}\,(a\mathrm{v}) = 0$.

We now need to pass to the limit in the nonlinearity $\mathrm{m}_\varepsilon \mathrm{v}_\varepsilon$. For that purpose, for all $\eta > 0$, we set $\mathrm{v}_{\varepsilon,\eta} := \rho_\eta * v_\varepsilon$ and $\mathrm{m}_{\varepsilon,\eta} := \rho_\eta * \mathrm{m}_\varepsilon = \mathrm{curl}\,\mathrm{v}_{\varepsilon,\eta}$, where $\rho_\eta(x) := \eta^{-d}\rho(x/\eta)$ is the regularization kernel defined in Step 1, and we then decompose the nonlinearity as follows,

$$\mathrm{m}_\varepsilon \mathrm{v}_\varepsilon = (\mathrm{m}_{\varepsilon,\eta} - \mathrm{m}_\varepsilon)(\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon) - \mathrm{m}_{\varepsilon,\eta}\, \mathrm{v}_{\varepsilon,\eta} + \mathrm{m}_{\varepsilon,\eta} \mathrm{v}_\varepsilon + \mathrm{m}_\varepsilon \mathrm{v}_{\varepsilon,\eta}\,.$$

We study each right-hand side term separately, and split the proof into four further substeps.

Substep 3.1. We prove that $(\mathrm{m}_{\varepsilon,\eta} - \mathrm{m}_\varepsilon)(\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon) \to 0$ holds in the distributional sense (and even strongly in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$) as $\eta \downarrow 0$, uniformly in $\varepsilon > 0$.

For that purpose, we use the Delort type identity

$$(\mathrm{m}_{\varepsilon,\eta} - \mathrm{m}_\varepsilon)(\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon) = a^{-1}(\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon)\,\mathrm{div}\,(a(\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon)) - \frac{1}{2}|\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon|^2 \nabla^\perp h - a^{-1}(\mathrm{div}\,(a S_{\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon}))^\perp.$$

Noting that the constraint $0 = a^{-1}\,\mathrm{div}\,(a\mathrm{v}_\varepsilon) = \nabla h \cdot \mathrm{v}_\varepsilon + \mathrm{div}\,\mathrm{v}_\varepsilon$ yields

$$a^{-1}\,\mathrm{div}\,(a(\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon)) = \nabla h \cdot \mathrm{v}_{\varepsilon,\eta} + \mathrm{div}\ \mathrm{v}_{\varepsilon,\eta} = \nabla h \cdot (\rho_\eta * \mathrm{v}_\varepsilon) + \rho_\eta * \mathrm{div}\ \mathrm{v}_\varepsilon = \nabla h \cdot (\rho_\eta * \mathrm{v}_\varepsilon) - \rho_\eta * (\nabla h \cdot \mathrm{v}_\varepsilon),$$

the above identity becomes

$$\begin{aligned}(\mathrm{m}_{\varepsilon,\eta} - \mathrm{m}_\varepsilon)(\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon) = (\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon)\big(\nabla h \cdot (\rho_\eta * \mathrm{v}_\varepsilon) - \rho_\eta * (\nabla h \cdot \mathrm{v}_\varepsilon)\big)&\\ - \frac{1}{2}|\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon|^2 \nabla^\perp h - a^{-1}(\mathrm{div}\,(a S_{\mathrm{v}_{\varepsilon,\eta} - \mathrm{v}_\varepsilon}))^\perp&.\end{aligned}$$

First, using the boundedness of v_ε (hence of $\mathrm{v}_{\varepsilon,\eta}$) in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2)$, we may estimate, for all bounded domain $U\subset\mathbb{R}^2$, denoting by $U^\eta:=U+B_\eta$ its η-fattening,

$$\begin{aligned}
&\int_U \big|(\mathrm{v}_{\varepsilon,\eta}-\mathrm{v}_\varepsilon)\big(\nabla h\cdot(\rho_\eta*\mathrm{v}_\varepsilon)-\rho_\eta*(\nabla h\cdot\mathrm{v}_\varepsilon)\big)\big|\\
&\quad\le\ \|(\mathrm{v}_\varepsilon,\mathrm{v}_{\varepsilon,\eta})\|_{\mathrm{L}^2(U)}\bigg(\int_U\Big(\int\rho_\eta(y)|\nabla h(x)-\nabla h(x-y)||\mathrm{v}_\varepsilon(x-y)|dy\Big)^2dx\bigg)^{1/2}\\
&\quad\lesssim\ \|(\mathrm{v}_\varepsilon,\mathrm{v}_{\varepsilon,\eta})\|^2_{\mathrm{L}^2(U^\eta)}\Big(\int\rho_\eta(y)\int_U|\nabla h(x)-\nabla h(x-y)|^2dxdy\Big)^{1/2},
\end{aligned}$$

where the right-hand side converges to 0 as $\eta\downarrow 0$, uniformly in ε. Second, using the decomposition (7.48), and setting $\mathrm{v}_{\varepsilon,\eta,1}:=\rho_\eta*\mathrm{v}_{\varepsilon,1}$, $\mathrm{v}_{\varepsilon,\eta,2}:=\rho_\eta*\mathrm{v}_{\varepsilon,2}$, the Hölder inequality yields for all bounded domain $U\subset\mathbb{R}^2$,

$$\begin{aligned}
\int_U|(\mathrm{v}_\varepsilon-\mathrm{v}_{\varepsilon,\eta})\otimes(\mathrm{v}_\varepsilon-\mathrm{v}_{\varepsilon,\eta})|&\le\int_U|\mathrm{v}_\varepsilon-\mathrm{v}_{\varepsilon,\eta}||\mathrm{v}_{\varepsilon,1}-\mathrm{v}_{\varepsilon,\eta,1}|+\int_U|\mathrm{v}_\varepsilon-\mathrm{v}_{\varepsilon,\eta}||\mathrm{v}_{\varepsilon,2}-\mathrm{v}_{\varepsilon,\eta,2}|\\
&\le\|(\mathrm{v}_\varepsilon,\mathrm{v}_{\varepsilon,\eta})\|_{\mathrm{L}^{2q/(2-q)}(U)}\|\mathrm{v}_{\varepsilon,1}-\mathrm{v}_{\varepsilon,\eta,1}\|_{\mathrm{L}^{2q/(3q-2)}(U)}+\|(\mathrm{v}_\varepsilon,\mathrm{v}_{\varepsilon,\eta})\|_{\mathrm{L}^2(U)}\|\mathrm{v}_{\varepsilon,2}-\mathrm{v}_{\varepsilon,\eta,2}\|_{\mathrm{L}^2(U)}.
\end{aligned}$$

Recalling the choice $1<q\le 4/3$, we find by interpolation

$$\begin{aligned}
\|\mathrm{v}_{\varepsilon,1}-\mathrm{v}_{\varepsilon,\eta,1}\|_{\mathrm{L}^{2q/(3q-2)}(U)}&\le\|\mathrm{v}_{\varepsilon,1}-\mathrm{v}_{\varepsilon,\eta,1}\|^{\frac{4-3q}{2-q}}_{\mathrm{L}^2(U)}\|\mathrm{v}_{\varepsilon,1}-\mathrm{v}_{\varepsilon,\eta,1}\|^{2\frac{q-1}{2-q}}_{\mathrm{L}^q(U)}\\
&\le\eta^{2\frac{q-1}{2-q}}\|(\mathrm{v}_{\varepsilon,1},\mathrm{v}_{\varepsilon,\eta,1})\|^{\frac{4-3q}{2-q}}_{\mathrm{L}^2(U)}\|\nabla\mathrm{v}_{\varepsilon,1}\|^{2\frac{q-1}{2-q}}_{\mathrm{L}^q},
\end{aligned}$$

and hence by the Calderón-Zygmund theory,

$$\|\mathrm{v}_{\varepsilon,1}-\mathrm{v}_{\varepsilon,\eta,1}\|_{\mathrm{L}^{2q/(3q-2)}(U)}\lesssim\eta^{2\frac{q-1}{2-q}}\|(\mathrm{v}_{\varepsilon,1},\mathrm{v}_{\varepsilon,\eta,1})\|^{\frac{4-3q}{2-q}}_{\mathrm{L}^2(U)}\|\mathrm{m}_\varepsilon\|^{2\frac{q-1}{2-q}}_{\mathrm{L}^q},$$

while as in Step 1 we find

$$\|\mathrm{v}_{\varepsilon,2}-\mathrm{v}_{\varepsilon,\eta,2}\|_{\mathrm{L}^2_t\mathrm{L}^2(U)}\le\eta\|\nabla\mathrm{v}_{\varepsilon,2}\|_{\mathrm{L}^2_t\mathrm{L}^2(U^\eta)}\lesssim_U\eta.$$

Combining this with the a priori estimate (7.49), we may conclude

$$\int_0^t\int_U|(\mathrm{v}_\varepsilon-\mathrm{v}_{\varepsilon,\eta})\otimes(\mathrm{v}_\varepsilon-\mathrm{v}_{\varepsilon,\eta})|\lesssim_{t,U}\eta^{2\frac{q-1}{2-q}}+\eta,$$

and the claim follows.

Substep 3.2. We set $\mathrm{v}_\eta:=\rho_\eta*\mathrm{v}$, $\mathrm{m}_\eta:=\rho_\eta*\mathrm{m}=\operatorname{curl}\mathrm{v}_\eta$, and we prove that $-\mathrm{m}_{\varepsilon,\eta}\mathrm{v}_{\varepsilon,\eta}+\mathrm{m}_{\varepsilon,\eta}\mathrm{v}_\varepsilon+\mathrm{m}_\varepsilon\mathrm{v}_{\varepsilon,\eta}\to-\mathrm{m}_\eta\mathrm{v}_\eta+\mathrm{m}_\eta\mathrm{v}+\mathrm{m}\mathrm{v}_\eta$ in the distributional sense as $\varepsilon\downarrow 0$, for any fixed $\eta>0$.

As $q<2<q'$, the weak convergences $\mathrm{v}_\varepsilon\overset{*}{\rightharpoonup}\mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2)$ and $\mathrm{m}_\varepsilon\overset{*}{\rightharpoonup}\mathrm{m}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^q(\mathbb{R}^2))$ imply for instance $\mathrm{v}_{\varepsilon,\eta}\overset{*}{\rightharpoonup}\mathrm{v}_\eta$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;W^{1,q'}_{\mathrm{loc}}(\mathbb{R}^2)^2)$ and $\mathrm{m}_{\varepsilon,\eta}\overset{*}{\rightharpoonup}\mathrm{m}_\eta$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{R}^2))$ as $\varepsilon\downarrow 0$, for any fixed $\eta>0$ (note that these are still only weak-* convergences because no regularization occurs with respect to the time variable t). Moreover, examining equation (7.9) together with the a priori estimates obtained at the beginning of this step, we observe that $\partial_t\mathrm{m}_\varepsilon$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;W^{-2,1}_{\mathrm{loc}}(\mathbb{R}^2))$, hence $\partial_t\mathrm{m}_{\varepsilon,\eta}=\rho_\eta*\partial_t\mathrm{m}_\varepsilon$ is also bounded in the same space. Since by the Rellich theorem the space $\mathrm{L}^q(U)$ is compactly embedded in $W^{-1,q}(U)\subset W^{-2,1}(U)$ for all bounded domain $U\subset\mathbb{R}^2$, the Aubin-Simon lemma ensures that we have $\mathrm{m}_\varepsilon\to\mathrm{m}$ strongly in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;W^{-1,q}_{\mathrm{loc}}(\mathbb{R}^2))$, and similarly, since $H^1(U)$ is compactly embedded in $\mathrm{L}^2(U)\subset W^{-2,1}(U)$, we also deduce $\mathrm{m}_{\varepsilon,\eta}\to\mathrm{m}_\eta$ strongly in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2))$. This proves the claim.

Substep 3.3. We prove that $-\mathrm{m}_\eta \mathrm{v}_\eta + \mathrm{m}_\eta \mathrm{v} + \mathrm{m} \mathrm{v}_\eta \to -\frac{1}{2}|\mathrm{v}|^2 \nabla^\perp h - a^{-1}(\operatorname{div}(aS_{\mathrm{v}}))^\perp$ holds in the distributional sense as $\eta \downarrow 0$.

For that purpose, we use the following Delort type identity,

$$\begin{aligned}-\mathrm{m}_\eta \mathrm{v}_\eta + \mathrm{m}_\eta \mathrm{v} + \mathrm{m} \mathrm{v}_\eta = -a^{-1}(\mathrm{v}_\eta - \mathrm{v}) \operatorname{div}(a(\mathrm{v}_\eta - \mathrm{v})) + \frac{1}{2}|\mathrm{v}_\eta - \mathrm{v}|^2 \nabla^\perp h + a^{-1}(\operatorname{div}(aS_{\mathrm{v}_\eta - \mathrm{v}}))^\perp \\ + a^{-1}\mathrm{v} \operatorname{div}(a\mathrm{v}) - \frac{1}{2}|\mathrm{v}|^2 \nabla^\perp h - a^{-1}(\operatorname{div}(aS_{\mathrm{v}}))^\perp.\end{aligned}$$

Noting that the limiting constraint $0 = a^{-1} \operatorname{div}(a\mathrm{v}) = \nabla h \cdot \mathrm{v} + \operatorname{div} \mathrm{v}$ gives

$$a^{-1} \operatorname{div}(a(\mathrm{v}_\eta - \mathrm{v})) = \nabla h \cdot \mathrm{v}_\eta + \operatorname{div} \mathrm{v}_\eta = \nabla h \cdot (\rho_\eta * \mathrm{v}) + \rho_\eta * \operatorname{div} \mathrm{v} = \nabla h \cdot (\rho_\eta * \mathrm{v}) - \rho_\eta * (\nabla h \cdot \mathrm{v}),$$

the above identity takes the form

$$\begin{aligned}-\mathrm{m}_\eta \mathrm{v}_\eta + \mathrm{m}_\eta \mathrm{v} + \mathrm{m} \mathrm{v}_\eta = -a^{-1}(\mathrm{v}_\eta - \mathrm{v})\big(\nabla h \cdot (\rho_\eta * \mathrm{v}) - \rho_\eta * (\nabla h \cdot \mathrm{v})\big) \\ + \frac{1}{2}|\mathrm{v}_\eta - \mathrm{v}|^2 \nabla^\perp h + a^{-1}(\operatorname{div}(aS_{\mathrm{v}_\eta - \mathrm{v}}))^\perp - \frac{1}{2}|\mathrm{v}|^2 \nabla^\perp h - a^{-1}(\operatorname{div}(aS_{\mathrm{v}}))^\perp,\end{aligned}$$

and it is thus sufficient to prove that the first three right-hand side terms tend to 0 in the distributional sense as $\eta \downarrow 0$. This is proven just as in Substep 3.1 above, with $\mathrm{v}_{\varepsilon,\eta}, \mathrm{v}_\varepsilon$ replaced by $\mathrm{v}_\eta, \mathrm{v}$.

Substep 3.4. Conclusion.

Combining the three previous substeps yields $\mathrm{m}_\varepsilon \mathrm{v}_\varepsilon \to -\frac{1}{2}|\mathrm{v}|^2 \nabla^\perp h - a^{-1}(\operatorname{div}(aS_{\mathrm{v}}))^\perp$ in the distributional sense as $\varepsilon \downarrow 0$. Passing to the limit in the very weak formulation of equation (7.9), the conclusion follows. □

7.5 Uniqueness

We turn to the uniqueness results stated in Theorem 7.1.5. Using similar energy arguments as in the proof of Lemma 7.4.1, in the spirit of [395, Appendix B], we prove a general weak-strong uniqueness principle. Note that in the degenerate case $\lambda = 0$ an additional term needs to be added to the usual energy, in link with the fact that m and v are then on an equal footing with regard to regularity. In the incompressible case, we further prove uniqueness in the class of bounded vorticity based on transport arguments à la Loeper [307] (see also [398]), but these tools are not available in the compressible case.

Proposition 7.5.1 (Uniqueness). *Let $\alpha, \beta \in \mathbb{R}$, $\lambda \geq 0$, $T > 0$, and $h, \Psi \in W^{1,\infty}(\mathbb{R}^2)^2$. Let $\mathrm{v}^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ with $\mathrm{m}^\circ := \operatorname{curl} \mathrm{v}^\circ \in \mathcal{P}(\mathbb{R}^2)$, and in the incompressible case* (7.1) *further assume that* $\operatorname{div}(a\mathrm{v}^\circ) = 0$.

(i) Weak-strong uniqueness principle for (7.1) and (7.2) in the non-degenerate case $\lambda > 0$, $\alpha \geq 0$: *If* (7.1) *or* (7.2) *admits a weak solution* $\mathrm{v} \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2) \cap \mathrm{L}^\infty_{\mathrm{loc}}([0,T); W^{1,\infty}(\mathbb{R}^2)^2)$ *on* $[0,T) \times \mathbb{R}^2$ *with initial data* v°*, then it is the unique weak solution of* (7.1) *or of* (7.2) *on* $[0,T) \times \mathbb{R}^2$ *in the class* $\mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ *with initial data* v°.

(ii) Weak-strong uniqueness principle for (7.2) in the degenerate parabolic case $\lambda = \beta = 0$, $\alpha \geq 0$: *Let E^2_{T,v° denote the class of all $w \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ with $\operatorname{curl} w \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^2))$. If* (7.2) *admits a weak solution* $\mathrm{v} \in E^2_{T,\mathrm{v}^\circ} \cap \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2)^2)$ *on* $[0,T) \times \mathbb{R}^2$ *with initial data* v°*, and with* $\mathrm{m} := \operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); W^{1,\infty}(\mathbb{R}^2))$*, then it is the unique weak solution of* (7.2) *on* $[0,T) \times \mathbb{R}^2$ *in the class* E^2_{T,v° *with initial data* v°.

(iii) Uniqueness for (7.1) with bounded vorticity, $\alpha, \beta \in \mathbb{R}$: *There exists at most a unique weak solution* v *of* (7.1) *on* $[0,T) \times \mathbb{R}^2$ *with initial data* v°*, in the class of all w's such that* $\operatorname{curl} w \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2))$.

Moreover, in items (i)–(ii), the condition $\alpha \ge 0$ may be dropped if we further restrict to weak solutions v *such that* $\operatorname{curl}\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^\infty(\mathbb{R}^2))$. $\Diamond$

Proof. In this proof, we use the notation $\lesssim$ for $\le$ up to a constant $C > 0$ that depends only on an upper bound on α, $|\beta|$, λ, λ^{-1}, and $\|(h,\Psi)\|_{W^{1,\infty}}$, and we add subscripts to indicate dependence on further parameters. We split the proof into four steps, first proving item (i) in the case (7.1), then in the case (7.2), and finally turning to items (ii) and (iii).

Step 1. Proof of (i) in the case (7.1).

Let $\alpha \ge 0$, $\beta \in \mathbb{R}$, and let $\mathrm{v}_1, \mathrm{v}_2 \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ be two weak solutions of (7.1) on $[0,T) \times \mathbb{R}^2$ with initial data v°, and assume $\mathrm{v}_2 \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T);W^{1,\infty}(\mathbb{R}^2)^2)$. Set $\delta\mathrm{v} := \mathrm{v}_1 - \mathrm{v}_2$ and $\delta\mathrm{m} := \mathrm{m}_1 - \mathrm{m}_2$. As the constraint $\operatorname{div}(a\delta\mathrm{v}) = 0$ yields $\delta\mathrm{v} = a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\delta\mathrm{m}$, and as by assumption $\delta\mathrm{v} \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{L}^2(\mathbb{R}^2)^2)$, we deduce $\delta\mathrm{m} \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\dot H^{-1}(\mathbb{R}^2))$ and $(\operatorname{div} a^{-1}\nabla)^{-1}\delta\mathrm{m} \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\dot H^1(\mathbb{R}^2))$. Moreover, the definition of a weak solution ensures that $\mathrm{m}_i := \operatorname{curl}\mathrm{v}_i \in \mathrm{L}^\infty([0,T);\mathcal{P}(\mathbb{R}^2))$ (cf. Lemma 7.4.1(i)), and $|\mathrm{v}_i|^2\,\mathrm{m}_i \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathrm{L}^1(\mathbb{R}^2))$, for $i = 1,2$, so that all the integrations by parts below are directly justified. From equation (7.9), we compute the following time derivative

$$\begin{aligned}\partial_t \int \delta\mathrm{m}(-\operatorname{div} a^{-1}\nabla)^{-1}\delta\mathrm{m} &= 2\int \nabla(\operatorname{div} a^{-1}\nabla)^{-1}\delta\mathrm{m}\cdot\Big((\alpha(\Psi+\mathrm{v}_1)^\perp + \beta(\Psi+\mathrm{v}_1))\,\mathrm{m}_1 \\ &\qquad\qquad - (\alpha(\Psi+\mathrm{v}_2)^\perp + \beta(\Psi+\mathrm{v}_2))\,\mathrm{m}_2\Big) \\ &= -2\int a\delta\mathrm{v}^\perp\cdot\Big((\alpha(\delta\mathrm{v})^\perp + \beta\delta\mathrm{v})\,\mathrm{m}_1 + (\alpha(\Psi+\mathrm{v}_2)^\perp + \beta(\Psi+\mathrm{v}_2))\delta\mathrm{m}\Big) \\ &= -2\alpha\int a|\delta\mathrm{v}|^2\,\mathrm{m}_1 - 2\int a\delta\mathrm{m}\,\delta\mathrm{v}^\perp\cdot(\alpha(\Psi+\mathrm{v}_2)^\perp + \beta(\Psi+\mathrm{v}_2)). \qquad (7.50)\end{aligned}$$

Since v_2 is Lipschitz-continuous, and since the definition of a weak solution ensures that $\mathrm{m}_1\mathrm{v}_1 \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathrm{L}^1(\mathbb{R}^2)^2)$, the following Delort type identity holds in $\mathrm{L}^1_{\mathrm{loc}}([0,T);W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$,

$$\delta\mathrm{m}\,\delta\mathrm{v}^\perp = \frac{1}{2}|\delta\mathrm{v}|^2\nabla h + a^{-1}\operatorname{div}(aS_{\delta\mathrm{v}}).$$

Combining this with (7.50) and the non-negativity of $\alpha\mathrm{m}_1$ yields

$$\begin{aligned}\partial_t \int \delta\mathrm{m}(-\operatorname{div} a^{-1}\nabla)^{-1}\delta\mathrm{m} &\le -\int a|\delta\mathrm{v}|^2\nabla h\cdot(\alpha(\Psi+\mathrm{v}_2)^\perp + \beta(\Psi+\mathrm{v}_2)) \\ &\qquad + 2\int aS_{\delta\mathrm{v}} : \nabla(\alpha(\Psi+\mathrm{v}_2)^\perp + \beta(\Psi+\mathrm{v}_2)) \\ &\le C(1 + \|\mathrm{v}_2\|_{W^{1,\infty}})\int a|\delta\mathrm{v}|^2.\end{aligned}$$

The uniqueness result $\delta\mathrm{v} = 0$ then follows from the Grönwall inequality, since by integration by parts

$$\int a|\delta\mathrm{v}|^2 = \int a^{-1}|\nabla(\operatorname{div} a^{-1}\nabla)^{-1}\delta\mathrm{m}|^2 = \int \delta\mathrm{m}(-\operatorname{div} a^{-1}\nabla)^{-1}\delta\mathrm{m}\,.$$

Note that if we further assume $\mathrm{m}_1 \in \mathrm{L}^\infty([0,T);\mathrm{L}^\infty(\mathbb{R}^2))$, then the non-negativity of α can be dropped: it indeed suffices to estimate in that case $-2\alpha\int a|\delta\mathrm{v}|^2\,\mathrm{m}_1 \le C\|\mathrm{m}_1\|_{\mathrm{L}^\infty}\int a|\delta\mathrm{v}|^2$, and the result then follows as above. A similar observation also holds in the context of item (ii).

Step 2. Proof of (i) in the case (7.2).

Let $\alpha \ge 0$, $\beta \in \mathbb{R}$, $\lambda > 0$, and let $\mathrm{v}_1, \mathrm{v}_2 \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ be two weak solutions of (7.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°, and assume $\mathrm{v}_2 \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T);W^{1,\infty}(\mathbb{R}^2)^2)$. The definition of a weak solution ensures that $\mathrm{m}_i := \operatorname{curl} \mathrm{v}_i \in \mathrm{L}^\infty([0,T);\mathcal{P}(\mathbb{R}^2))$ (cf. Lemma 7.4.1(i)), $\mathrm{d}_i := \operatorname{div}(a\mathrm{v}_i) \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{L}^2(\mathbb{R}^2))$, and $|\mathrm{v}_i|^2\,\mathrm{m}_i \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathrm{L}^1(\mathbb{R}^2))$, for $i=1,2$, and hence the integrations by parts below are directly justified. Set $\delta\mathrm{v} := \mathrm{v}_1 - \mathrm{v}_2$, $\delta\mathrm{m} := \mathrm{m}_1 - \mathrm{m}_2$, and $\delta\mathrm{d} := \mathrm{d}_1 - \mathrm{d}_2$. From equation (7.2), we compute the following time derivative

$$\begin{aligned}\partial_t \int a|\delta\mathrm{v}|^2 &= 2\int a\delta\mathrm{v}\cdot\Big(\lambda\nabla(a^{-1}\delta\mathrm{d}) - \alpha(\Psi+\mathrm{v}_1)\,\mathrm{m}_1 + \beta(\Psi+\mathrm{v}_1)^\perp\,\mathrm{m}_1\\ &\qquad\qquad + \alpha(\Psi+\mathrm{v}_2)\,\mathrm{m}_2 - \beta(\Psi+\mathrm{v}_2)^\perp\,\mathrm{m}_2\Big)\\ &= -2\lambda\int a^{-1}|\delta\mathrm{d}\,|^2 - 2\alpha\int a|\delta\mathrm{v}|^2\,\mathrm{m}_1 + 2\int a\delta\mathrm{m}\,\delta\mathrm{v}\cdot\big(\alpha(\Psi+\mathrm{v}_2) - \beta(\Psi+\mathrm{v}_2)^\perp\big).\end{aligned}$$

As v_2 is Lipschitz-continuous, and as the definition of a weak solution implies $\mathrm{m}_1\mathrm{v}_1 \in \mathrm{L}^1_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)^2$, the following Delort type identity holds in $\mathrm{L}^1_{\mathrm{loc}}([0,T);W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$,

$$\delta\mathrm{m}\,\delta\mathrm{v} = a^{-1}\delta\mathrm{d}\,\delta\mathrm{v}^\perp - \frac{1}{2}|\delta\mathrm{v}|^2\nabla^\perp h - a^{-1}(\operatorname{div}(aS_{\delta\mathrm{v}}))^\perp.$$

The above may then be estimated as follows, after integration by parts,

$$\begin{aligned}\partial_t\int a|\delta\mathrm{v}|^2 &\le -2\lambda\int a^{-1}|\delta\mathrm{d}|^2 - 2\alpha\int a|\delta\mathrm{v}|^2\,\mathrm{m}_1\\ &\qquad + C(1+\|\mathrm{v}_2\|_{\mathrm{L}^\infty})\int|\delta\mathrm{d}||\delta\mathrm{v}| + C(1+\|\mathrm{v}_2\|_{W^{1,\infty}})\int a|\delta\mathrm{v}|^2,\end{aligned}$$

and thus, using the choice $\lambda > 0$, the inequality $2xy \le x^2+y^2$, and the non-negativity of $\alpha\mathrm{m}_1$,

$$\partial_t\int a|\delta\mathrm{v}|^2 \le C(1+\lambda_\varepsilon^{-1})(1+\|\mathrm{v}_2\|^2_{W^{1,\infty}})\int a|\delta\mathrm{v}|^2.$$

The Grönwall inequality then implies uniqueness, $\delta\mathrm{v} = 0$.

Step 3. Proof of (ii).

Let $\lambda = \beta = 0$, $\alpha = 1$, and let $\mathrm{v}_1, \mathrm{v}_2 \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ be two weak solutions of (7.2) on $[0,T)\times\mathbb{R}^2$ with initial data v°, and with $\mathrm{m}_i := \operatorname{curl}\mathrm{v}_i \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{L}^2(\mathbb{R}^2))$ for $i=1,2$, and further assume $\mathrm{v}_2 \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^\infty(\mathbb{R}^2)^2)$ and $\mathrm{m}_2 \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T);W^{1,\infty}(\mathbb{R}^2))$. The definition of a weak solution ensures that $\mathrm{m}_i := \operatorname{curl}\mathrm{v}_i \in \mathrm{L}^\infty([0,T);\mathcal{P}(\mathbb{R}^2))$ (cf. Lemma 7.4.1(i)), $\mathrm{d}_i := \operatorname{div}(a\mathrm{v}_i) \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{L}^2(\mathbb{R}^2))$, and $|\mathrm{v}_i|^2\,\mathrm{m}_i \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathrm{L}^1(\mathbb{R}^2))$, for $i=1,2$, and hence the integrations by parts below are directly justified. Denoting $\delta\mathrm{v} := \mathrm{v}_1 - \mathrm{v}_2$ and $\delta\mathrm{m} := \mathrm{m}_1 - \mathrm{m}_2$, equation (7.2) yields

$$\partial_t\delta\mathrm{v} = -(\Psi+\mathrm{v}_2)\delta\mathrm{m} - \mathrm{m}_1\,\delta\mathrm{v}, \tag{7.51}$$

while equation (7.9) takes the form

$$\begin{aligned}\partial_t\delta\mathrm{m} &= \operatorname{div}((\Psi+\mathrm{v}_2)^\perp\delta\mathrm{m}) + \operatorname{div}(\mathrm{m}_1\,\delta\mathrm{v}^\perp)\\ &= \operatorname{div}((\Psi+\mathrm{v}_2)^\perp\delta\mathrm{m}) + \nabla\mathrm{m}_1\cdot\delta\mathrm{v}^\perp - \mathrm{m}_1\,\delta\mathrm{m}\\ &= \operatorname{div}((\Psi+\mathrm{v}_2)^\perp\delta\mathrm{m}) + \nabla\mathrm{m}_2\cdot\delta\mathrm{v}^\perp + \nabla\delta\mathrm{m}\cdot\delta\mathrm{v}^\perp - \mathrm{m}_1\,\delta\mathrm{m}\,.\end{aligned} \tag{7.52}$$

Testing equation (7.51) against $\delta\mathrm{v}$ yields, by non-negativity of m_1,

$$\partial_t\int|\delta\mathrm{v}|^2 = -2\int|\delta\mathrm{v}|^2\,\mathrm{m}_1 - 2\int\delta\mathrm{v}\cdot(\Psi+\mathrm{v}_2)\delta\mathrm{m} \le C(1+\|\mathrm{v}_2\|_{\mathrm{L}^\infty})\int|\delta\mathrm{v}||\delta\mathrm{m}|.$$

Testing equation (7.52) against $\delta\mathrm{m}$ and integrating by parts yields, by non-negativity of m_1 and m_2,

$$\begin{aligned}\partial_t\int|\delta\mathrm{m}|^2&=-\int\nabla|\delta\mathrm{m}|^2\cdot(\Psi+\mathrm{v}_2)^\perp+2\int\delta\mathrm{m}\,\nabla\mathrm{m}_2\cdot\delta\mathrm{v}^\perp+\int\nabla|\delta\mathrm{m}|^2\cdot\delta\mathrm{v}^\perp-2\int|\delta\mathrm{m}|^2\,\mathrm{m}_1\\&=-\int|\delta\mathrm{m}|^2(\operatorname{curl}\Psi+\mathrm{m}_2)+2\int\delta\mathrm{m}\,\nabla\mathrm{m}_2\cdot\delta\mathrm{v}^\perp+\int|\delta\mathrm{m}|^2(\mathrm{m}_1-\mathrm{m}_2)-2\int|\delta\mathrm{m}|^2\,\mathrm{m}_1\\&\le C\int|\delta\mathrm{m}|^2+2\|\nabla\mathrm{m}_2\|_{\mathrm{L}^\infty}\int|\delta\mathrm{v}||\delta\mathrm{m}|.\end{aligned}$$

Combining the above two estimates and using the inequality $2xy\le x^2+y^2$, we find

$$\partial_t\int(|\delta\mathrm{v}|^2+|\delta\mathrm{m}|^2)\le C(1+\|(\mathrm{v}_2,\nabla\mathrm{m}_2)\|_{\mathrm{L}^\infty})\int(|\delta\mathrm{v}|^2+|\delta\mathrm{m}|^2),$$

and the uniqueness result follows from the Grönwall inequality.

Step 4. Proof of (iii).

Let $\alpha,\beta\in\mathbb{R}$, and let $\mathrm{v}_1,\mathrm{v}_2$ denote two solutions of (7.1) on $[0,T)\times\mathbb{R}^2$ with initial data v°, and with $\mathrm{m}_1,\mathrm{m}_2\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^\infty(\mathbb{R}^2))$. First we prove that $\mathrm{v}_1^t,\mathrm{v}_2^t$ are log-Lipschitz for all $t\in[0,T)$ (compare with the easier situation in [398, Lemma 4.1]). For $i=1,2$, using the identity $\mathrm{v}_i^t=\nabla^\perp\triangle^{-1}\mathrm{m}_i^t+\nabla\triangle^{-1}\operatorname{div}\mathrm{v}_i^t$ with $\operatorname{div}\mathrm{v}_i^t=-\nabla h\cdot\mathrm{v}_i^t$, we may decompose for all x,y,

$$|\mathrm{v}_i^t(x)-\mathrm{v}_i^t(y)|\le|\nabla\triangle^{-1}\mathrm{m}_i^t(x)-\nabla\triangle^{-1}\mathrm{m}_i^t(y)|+|\nabla\triangle^{-1}(\nabla h\cdot\mathrm{v}_i^t)(x)-\nabla\triangle^{-1}(\nabla h\cdot\mathrm{v}_i^t)(y)|.$$

By the embedding of the Zygmund space $C^1_*(\mathbb{R}^2)=B^1_{\infty,\infty}(\mathbb{R}^2)$ into the space of log-Lipschitz functions (see e.g. [40, Proposition 2.107]), we may estimate

$$|\mathrm{v}_i^t(x)-\mathrm{v}_i^t(y)|\lesssim\big(\|\nabla^2\triangle^{-1}\mathrm{m}_i^t\|_{C^0_*}+\|\nabla^2\triangle^{-1}(\nabla h\cdot\mathrm{v}_i^t)\|_{C^0_*}\big)|x-y|(1+\log_-(|x-y|)),$$

and hence, applying Lemma 7.2.5(ii) and recalling that $\mathrm{L}^\infty(\mathbb{R}^2)$ is embedded in $C^0_*(\mathbb{R}^2)=B^0_{\infty,\infty}(\mathbb{R}^2)$, we find for all $1\le p<\infty$,

$$\begin{aligned}|\mathrm{v}_i^t(x)-\mathrm{v}_i^t(y)|&\lesssim_p\big(\|\mathrm{m}_i^t\|_{\mathrm{L}^1\cap C^0_*}+\|\nabla h\cdot\mathrm{v}_i^t\|_{\mathrm{L}^p\cap C^0_*}\big)|x-y|(1+\log_-(|x-y|))\\&\lesssim\big(\|\mathrm{m}_i^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}+\|\mathrm{v}_i^t\|_{\mathrm{L}^p\cap\mathrm{L}^\infty}\big)|x-y|(1+\log_-(|x-y|)).\end{aligned}$$

Noting that $\mathrm{v}_i^t=a^{-1}\nabla^\perp(\operatorname{div}a^{-1}\nabla)^{-1}\mathrm{m}_i^t$, the elliptic estimates of Lemma 7.2.6 yield $\|\mathrm{v}_i^t\|_{\mathrm{L}^{p_0}\cap\mathrm{L}^\infty}\lesssim\|\mathrm{m}_i^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}$ for some exponent $2<p_0\lesssim1$. For the choice $\mathrm{p}=p_0$, the above thus takes the following form,

$$\begin{aligned}|\mathrm{v}_i^t(x)-\mathrm{v}_i^t(y)|&\lesssim\|\mathrm{m}_i^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}|x-y|(1+\log_-(|x-y|))\\&\qquad\le(1+\|\mathrm{m}_i^t\|_{\mathrm{L}^\infty})|x-y|(1+\log_-(|x-y|)),\end{aligned}\tag{7.53}$$

which proves that $\mathrm{v}_1^t,\mathrm{v}_2^t$ are log-Lipschitz for all $t\in[0,T)$.

For $i=1,2$, as the vector field $\alpha(\Psi+\mathrm{v}_i)+\beta(\Psi+\mathrm{v}_i)^\perp$ is log-Lipschitz in space, the associated flow $\psi_i:[0,T)\times\mathbb{R}^2\to\mathbb{R}^2$ is well-defined globally,

$$\partial_t\psi_i(x)=-(\alpha(\Psi+\mathrm{v}_i)+\beta(\Psi+\mathrm{v}_i)^\perp)(\psi_i(x)).$$

As the transport equation (7.9) ensures that $\mathrm{m}_i^t=(\psi_i^t)_*\mathrm{m}^\circ$ for $i=1,2$, the 2-Wasserstein distance between the solutions $\mathrm{m}_1^t,\mathrm{m}_2^t\in\mathcal{P}(\mathbb{R}^2)$ is bounded by

$$W_2(\mathrm{m}_1^t,\mathrm{m}_2^t)^2\le Q^t:=\int|\psi_1^t(x)-\psi_2^t(x)|^2\,\mathrm{m}^\circ(x)dx.\tag{7.54}$$

Now the time derivative of Q is estimated by

$$\begin{aligned}\partial_t Q^t &= -2\int (\psi_1^t(x)-\psi_2^t(x))\cdot\big((\alpha\Psi+\beta\Psi^\perp)(\psi_1^t(x))-(\alpha\Psi+\beta\Psi^\perp)(\psi_2^t(x))\big)\,\mathrm{m}^\circ(x)dx\\&\qquad -2\int (\psi_1^t(x)-\psi_2^t(x))\cdot\big((\alpha\mathrm{v}_1^t+\beta(\mathrm{v}_1^t)^\perp)(\psi_1^t(x))-(\alpha\mathrm{v}_2^t+\beta(\mathrm{v}_2^t)^\perp)(\psi_2^t(x))\big)\,\mathrm{m}^\circ(x)dx\\&\le CQ^t+C(Q^t)^{1/2}\Big(\int|\mathrm{v}_1^t(\psi_1^t(x))-\mathrm{v}_2^t(\psi_2^t(x))|^2\,\mathrm{m}^\circ(x)dx\Big)^{1/2}\\&\le CQ^t+C(Q^t)^{1/2}(T_1^t+T_2^t)^{1/2},\end{aligned}$$

where we have set

$$T_1^t:=\int|(\mathrm{v}_1^t-\mathrm{v}_2^t)(\psi_2^t(x))|^2\,\mathrm{m}^\circ(x)dx,\qquad T_2^t:=\int|\mathrm{v}_1^t(\psi_1^t(x))-\mathrm{v}_1^t(\psi_2^t(x))|^2\,\mathrm{m}^\circ(x)dx.$$

We first study T_1. Using that $\mathrm{v}_i=a^{-1}\nabla^\perp(\operatorname{div}\,a^{-1}\nabla)^{-1}\,\mathrm{m}_i$, we find

$$\begin{aligned}T_1^t=\int|\mathrm{v}_1^t-\mathrm{v}_2^t|^2\,\mathrm{m}_2^t\le\|\mathrm{m}_2^t\|_{\mathrm{L}^\infty}\int|\mathrm{v}_1^t-\mathrm{v}_2^t|^2&=\|\mathrm{m}_2^t\|_{\mathrm{L}^\infty}\int|\nabla(\operatorname{div}\,a^{-1}\nabla)^{-1}(\mathrm{m}_1^t-\mathrm{m}_2^t)|^2\\&\lesssim\|\mathrm{m}_2^t\|_{\mathrm{L}^\infty}\int|\nabla\triangle^{-1}(\mathrm{m}_1^t-\mathrm{m}_2^t)|^2.\end{aligned}$$

(Here, we use the fact that if $-\operatorname{div}(a^{-1}\nabla u_1)=-\triangle u_2$ with $u_1,u_2\in H^1(\mathbb{R}^2)$, then $\int a^{-1}|\nabla u_1|^2=\int\nabla u_1\cdot\nabla u_2\le\frac12\int a^{-1}|\nabla u_1|^2+\frac12\int a|\nabla u_2|^2$, hence $\int a^{-1}|\nabla u_1|^2\le\int a|\nabla u_2|^2$.) Loeper's inequality [307, Proposition 3.1] and the bound (7.54) then imply

$$T_1^t\le\|\mathrm{m}_2^t\|_{\mathrm{L}^\infty}(\|\mathrm{m}_1^t\|_{\mathrm{L}^\infty}\vee\|\mathrm{m}_2^t\|_{\mathrm{L}^\infty})W_2(\mathrm{m}_1^t,\mathrm{m}_2^t)^2\le\|(\mathrm{m}_1^t,\mathrm{m}_2^t)\|_{\mathrm{L}^\infty}^2Q^t.$$

We finally turn to T_2. Using the log-Lipschitz property (7.53) and the concavity of the function $x\mapsto x(1+\log_- x)^2$, we obtain by Jensen's inequality,

$$\begin{aligned}T_2^t&\lesssim\|\mathrm{m}_1^t\|_{\mathrm{L}^\infty}^2\int(1+\log_-(|\psi_1^t-\psi_2^t|))^2|\psi_1^t-\psi_2^t|^2\,\mathrm{m}^\circ\\&\le\|\mathrm{m}_1^t\|_{\mathrm{L}^\infty}^2\Big(1+\log_-\int|\psi_1^t-\psi_2^t|^2\,\mathrm{m}^\circ\Big)^2\int|\psi_1^t-\psi_2^t|^2\,\mathrm{m}^\circ\\&\lesssim\|\mathrm{m}_1^t\|_{\mathrm{L}^\infty}^2(1+\log_- Q^t)^2\,Q^t.\end{aligned}$$

We may thus conclude $\partial_t Q\lesssim(1+\|(\mathrm{m}_1,\mathrm{m}_2)\|_{\mathrm{L}^\infty})(1+\log_- Q)\,Q$, and the uniqueness result follows from a Grönwall argument. ☐

7.6 Degenerate parabolic case

We now turn to the study of the compressible equation (7.2) in the degenerate parabolic case $\lambda=\beta=0$, $\alpha=1$, that is,

$$\partial_t\mathrm{v}=-(\Psi+\mathrm{v})\operatorname{curl}\mathrm{v},\qquad\text{in }\mathbb{R}^+\times\mathbb{R}^2,\tag{7.55}$$

with initial data $\mathrm{v}|_{t=0}=\mathrm{v}^\circ$. A local existence result is already established in Proposition 7.3.2 above, and uniqueness is obtained in Proposition 7.5.1(ii), but the absence of strong enough a priori estimates on the divergence $\operatorname{div}\mathrm{v}$ due to the degeneracy of the equation make the question of global existence delicate. In the present section, we show how to exploit the particular scalar structure of the solution v to establish global existence and finer uniqueness results. More precisely, we establish the following, which in particular implies Theorem 7.1.6. This result is a joint work with Julian Fischer.

Theorem 7.6.1. *Let $\lambda = 0$, $\alpha = 1$, $\beta = 0$, let $\mathrm{v}^\circ, \Psi \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)^2$ with $\operatorname{curl} \mathrm{v}^\circ, \operatorname{curl} \Psi \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ and $\operatorname{curl} \mathrm{v}^\circ \geq 0$, and assume that v°, Ψ are* log*-Lipschitz, that is, for all x, y,*

$$|\mathrm{v}^\circ(x) - \mathrm{v}^\circ(y)| + |\Psi(x) - \Psi(y)| \leq C|x-y|(1 + \log_-(|x-y|)), \qquad \text{for all } x, y.$$

There exists a unique global strong solution $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ of (7.55) with $\operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ and $\operatorname{curl} \mathrm{v} \geq 0$. Moreover the following hold:

(i) if $\mathrm{v}^\circ, \Psi \in W^{1,\infty}(\mathbb{R}^2)^2$, then the solution v satisfies $\operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^2))$, and if in addition $\operatorname{curl} \mathrm{v}^\circ \in \mathcal{P}(\mathbb{R}^2)$, then there holds $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}^\circ + \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^2)^2)$ and $\operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^2))$;

(ii) if for some $s \geq 0$ we have $\mathrm{v}^\circ, \Psi \in W^{s\vee 1,\infty}(\mathbb{R}^2)^2$ and $\operatorname{curl} \mathrm{v}^\circ, \operatorname{curl} \Psi \in W^{s,\infty}(\mathbb{R}^2)$, then for all $0 \leq u \leq s$ the solution v belongs to $W^{u+1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; W^{s-u,\infty}(\mathbb{R}^2)^2)$;

(iii) if for some $s \geq 1$ we have $\mathrm{v}^\circ, \Psi \in W^{s,\infty}(\mathbb{R}^2)^2$, $\operatorname{curl} \mathrm{v}^\circ \in H^s \cap W^{s,\infty}(\mathbb{R}^2)$, and $\operatorname{curl} \Psi \in W^{s,\infty}(\mathbb{R}^2)$, then the solution v belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}^\circ + H^s \cap W^{s,\infty}(\mathbb{R}^2)^2)$. $\Diamond$

We start with a suitable reduction of equation (7.55), making its scalar structure appear. Assume that $\mathrm{v} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2))$ is a strong solution of (7.55) with $\operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$. Since the forcing vector field Ψ is time-independent, equation (7.55) for v can be rewritten as follows,

$$\partial_t(\Psi + \mathrm{v}) = -(\Psi + \mathrm{v})\operatorname{curl} \mathrm{v}, \qquad (\Psi + \mathrm{v})|_{t=0} = \Psi + \mathrm{v}^\circ,$$

which implies for all $x \in \mathbb{R}^2$ and $t \geq 0$,

$$(\Psi + \mathrm{v}^t)(x) = \kappa^t(x)(\Psi + \mathrm{v}^\circ)(x), \qquad \kappa^t(x) := \exp\Big(-\int_0^t \operatorname{curl} \mathrm{v}^s(x)\, ds\Big), \tag{7.56}$$

together with the following scalar equation for κ,

$$\partial_t \kappa = -\kappa \operatorname{curl} \mathrm{v}, \qquad \kappa|_{t=0} = 1.$$

Assuming $\operatorname{curl} \Psi \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)$, the definition (7.56) of κ in the form $\mathrm{v} = -\Psi + \kappa(\Psi + \mathrm{v}^\circ)$ and the assumption $\operatorname{curl} \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ ensure that the directional derivative $((\Psi + \mathrm{v}^\circ)^\perp \cdot \nabla)\kappa$ is well-defined in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$, and the above scalar equation for κ turns into

$$\partial_t \kappa = \kappa\,((\Psi + \mathrm{v}^\circ)^\perp \cdot \nabla)\kappa - \kappa^2 \operatorname{curl} \mathrm{v}^\circ + \kappa(1-\kappa)\operatorname{curl} \Psi, \qquad \kappa|_{t=0} = 1. \tag{7.57}$$

Along the characteristic curves of the vector field $(\Psi + \mathrm{v}^\circ)^\perp$, this equation takes the form of a Burgers' equation with additional quadratic damping and forcing terms. Although such a Burgers' equation may in general develop discontinuities in finite time (shock waves), we show that it cannot happen for constant initial data $\kappa|_{t=0} = 1$ as considered here. Recall that we focus here on the case with signed vorticity $\operatorname{curl} \mathrm{v}^\circ \geq 0$.

Lemma 7.6.2. *Let $W \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)^2$ be* log*-Lipschitz (that is, $|W(x) - W(y)| \leq C|x-y|(1 + \log_-(|x-y|))$ for all x, y), and let $f, g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ with $f \geq 0$. We consider the following Cauchy problem on $\mathbb{R}^+ \times \mathbb{R}^2$,*

$$\partial_t \kappa = \kappa\,(W \cdot \nabla)\kappa - \kappa^2 f + \kappa(1-\kappa)g, \qquad \kappa|_{t=0} = 1. \tag{7.58}$$

There exists a global strong solution $\kappa \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)) \cap \mathrm{L}^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ with $\frac{1}{\kappa}, (W \cdot \nabla)\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$. This solution is unique in the class

$$\mathcal{C} := \big\{\kappa \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)) : (W \cdot \nabla)\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)\big\}.$$

Moreover the following hold:

(i) if $f,g \in \mathrm{L}^\infty(\mathbb{R}^2)$ and $W \in W^{1,\infty}(\mathbb{R}^2)^2$, then the solution κ satisfies $\frac{1}{\kappa}, (W\cdot\nabla)\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^2))$, and if in addition $f \in \mathrm{L}^1(\mathbb{R}^2)$, then there holds $1-\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^2))$;

(ii) if for some $s \ge 0$ we have $W \in W^{s\vee 1,\infty}(\mathbb{R}^2)^2$ and $f,g \in W^{s,\infty}(\mathbb{R}^2)$, then for all $0 \le u \le s$ the solution κ belongs to $W^{u+1,\infty}_{\mathrm{loc}}(\mathbb{R}^+;W^{s-u,\infty}(\mathbb{R}^2))$;

(iii) if for some $s \ge 1$ we have $f \in H^s \cap W^{s,\infty}(\mathbb{R}^2)$, $W \in W^{s,\infty}(\mathbb{R}^2)^2$, and $g \in W^{s,\infty}(\mathbb{R}^2)$, then the solution κ satisfies $1-\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{R}^2))$. ◊

Proof. Let $W \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)^2$ be log-Lipschitz, and let $f,g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ with $f \ge 0$. Then the flow $\psi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2 : (s,x) \mapsto \psi^s_x$ associated with the vector field $-W$ is well-defined globally on $\mathbb{R}\times\mathbb{R}^2$,

$$\partial_s \psi^s_x = -W(\psi^s_x), \qquad \psi^s_x|_{s=0} = x.$$

We have $\psi \in C^1(\mathbb{R};C(\mathbb{R}^2))$, and for all $s \in \mathbb{R}$ the map $\psi^s : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism with inverse ψ^{-s}. More precisely, since W is log-Lipschitz, the map ψ^s is a Hölder homeomorphism in the following sense: we have for all s,x,y,

$$e^{-e^{C|s|}}(1\wedge|x-y|)^{e^{C|s|}} \le 1\wedge|\psi^s_x-\psi^s_y| \le e(1\wedge|x-y|)^{e^{-C|s|}}.$$

We split the proof into three steps.

Step 1. Uniqueness.

In this step, we show that for all $x \in \mathbb{R}^d$ and $\sigma^\circ \in \mathbb{R}$ there exists a unique global solution $\sigma_x(\sigma^\circ) : \mathbb{R}^+ \to \mathbb{R} : t \mapsto \sigma^t_x(\sigma^\circ)$ of

$$\partial_t\sigma_x(\sigma^\circ) = 1 - \int_{\sigma^\circ}^{\sigma_x(\sigma^\circ)} f(\psi^s_x)\exp\Big(-\int_s^{\sigma_x(\sigma^\circ)}(f+g)(\psi^u_x)\,du\Big)ds, \qquad \sigma_x(\sigma^\circ)|_{t=0} = \sigma^\circ, \tag{7.59}$$

and that the corresponding map $\sigma^t_x : \mathbb{R} \to \mathbb{R}$ is invertible on $\mathbb{R}$. In addition, assuming that for some $T > 0$ there exists a local strong solution $\kappa \in W^{1,\infty}_{\mathrm{loc}}([0,T);\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2))$ of (7.58) on $[0,T)\times\mathbb{R}^2$ with $(W\cdot\nabla)\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)$, we show that such a solution κ is necessarily given by the following explicit formula,

$$\kappa^t(x) = 1 - \int_{(\sigma^t_x)^{-1}(0)}^0 f(\psi^s_x)\exp\Big(-\int_s^0(f+g)(\psi^u_x)du\Big)ds. \tag{7.60}$$

This implies the stated uniqueness result.

Setting $\hat\kappa^t_x(s) := \kappa^t(\psi^s_x)$, and noting that $\partial_s\hat\kappa^t_x(s) = -(W\cdot\nabla\kappa^t)(\psi^s_x)$, we deduce by assumption $\hat\kappa_x \in W^{1,\infty}_{\mathrm{loc}}([0,T)\times\mathbb{R})$ for almost all x. Picard's existence theorem then ensures the local existence and uniqueness of the flow σ_x on $\mathbb{R}$ associated with the vector field $\hat\kappa_x$: for almost all x, for all σ°, there exists $0 < T_x(\sigma^\circ) \le T$ and a unique local solution $\sigma_x(\sigma^\circ) \in C^1([0,T_x(\sigma^\circ)))$ of the Cauchy problem

$$\partial_t\sigma^t_x(\sigma^\circ) = \hat\kappa^t_x(\sigma^t_x(\sigma^\circ)), \qquad \sigma^t_x(\sigma^\circ)|_{t=0} = \sigma^\circ. \tag{7.61}$$

Now note that by definition the function $t \mapsto \hat\kappa^t_x(\sigma^t_x(\sigma^\circ))$ belongs to $W^{1,\infty}_{\mathrm{loc}}([0,T_x(\sigma^\circ)))$ and satisfies

$$\partial_t\big(\hat\kappa^t_x(\sigma^t_x(\sigma^\circ))\big) = -\big(\hat\kappa^t_x(\sigma^t_x(\sigma^\circ))\big)^2 f(\psi_x^{\sigma^t_x(\sigma^\circ)}) + \hat\kappa^t_x(\sigma^t_x(\sigma^\circ))\big(1-\hat\kappa^t_x(\sigma^t_x(\sigma^\circ))\big)\,g(\psi_x^{\sigma^t_x(\sigma^\circ)}), \tag{7.62}$$
$$\hat\kappa^t_x(\sigma^t_x(\sigma^\circ))|_{t=0} = 1.$$

For $f,g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)$, this equation admits a unique global solution in $W^{1,\infty}_{\mathrm{loc}}([0,T_x(\sigma^\circ)))$, which must be given by the explicit formula

$$\hat\kappa^t_x(\sigma^t_x(\sigma^\circ)) = 1 - \int_{\sigma^0}^{\sigma^t_x(\sigma^\circ)} f(\psi^s_x)\exp\Big(-\int_s^{\sigma^t_x(\sigma^\circ)}(f+g)(\psi^u_x)\,du\Big)ds. \tag{7.63}$$

On the one hand, since the positive part $0\vee\hat\kappa_x(\sigma_x(\sigma^\circ))$ belongs to $W^{1,\infty}_{\rm loc}([0,T_x(\sigma^\circ)))$ and also satisfies equation (7.62), we deduce by uniqueness that $\hat\kappa_x(\sigma_x(\sigma^\circ))$ must remain nonnegative. Moreover, formula (7.63) with $f\ge 0$ ensures that $\hat\kappa_x(\sigma_x(\sigma^\circ))$ remains bounded above by 1, so that it is actually $[0,1]$-valued on its domain. On the other hand, due to formula (7.63), equation (7.61) takes on the following guise,

$$\partial_t\sigma_x(\sigma^\circ)=Z(\sigma_x(\sigma^\circ),\sigma^\circ),\qquad \sigma_x(\sigma^\circ)|_{t=0}=\sigma^\circ, \tag{7.64}$$

where we have set

$$Z(\sigma,\sigma^\circ):=\max\Big\{0\,;\,1-\int_{\sigma^0}^{\sigma}f(\psi_x^s)\exp\Big(-\int_s^\sigma(f+g)(\psi_x^u)du\Big)ds\Big\}.$$

As $0\le Z(\sigma,\sigma^\circ)\le 1$, we deduce $\sigma^\circ\le\sigma_x^t(\sigma^\circ)\le\sigma^\circ+t$ for all $t\ge0$. Since in addition for $f,g\in\mathrm{L}^\infty_{\rm loc}(\mathbb{R}^2)$ we have $Z\in W^{1,\infty}_{\rm loc}(\mathbb{R}\times\mathbb{R})$, the flow $\sigma_x(\sigma^\circ)$ must exist globally. We may therefore choose $T_x(\sigma^\circ)=T$ and the representation (7.63) holds for all $0\le t<T$.

It remains to invert (7.63) and deduce the formula (7.60) for the solution κ itself. For that purpose, we need to invert the (non-decreasing) map $\sigma_x^t:\mathbb{R}\to\mathbb{R}$ globally for all $t\ge0$. Since we have shown $\hat\kappa_x^t(\sigma_x^t(\sigma^\circ))=Z(\sigma_x^t(\sigma^\circ),\sigma^\circ)\in[0,1]$ for all $t\in[0,T)$, equation (7.64) leads to

$$\begin{aligned}\partial_t\frac{\partial\sigma_x^t(\sigma^\circ)}{\partial\sigma^\circ}&=f(\psi_x^{\sigma^\circ})\exp\Big(-\int_{\sigma^\circ}^{\sigma_x^t(\sigma^\circ)}(f+g)(\psi_x^u)du\Big)\\&+\frac{\partial\sigma_x^t(\sigma^\circ)}{\partial\sigma^\circ}\Big(-f(\psi_x^{\sigma_x^t(\sigma^\circ)})+(f+g)(\psi_x^{\sigma_x^t(\sigma^\circ)})\int_{\sigma^\circ}^{\sigma_x^t(\sigma^\circ)}f(\psi_x^s)\exp\Big(-\int_s^{\sigma_x^t(\sigma^\circ)}(f+g)(\psi_x^u)du\Big)ds\Big).\end{aligned}\tag{7.65}$$

For all x,t,σ°, define the compact set $K_x^t(\sigma^\circ):=\overline{B}+\{\psi_x^s:\sigma^\circ\le s\le\sigma^\circ+t\}$, where $\overline{B}$ is the closed unit Euclidean ball at the origin in $\mathbb{R}^2$. Hence, for $f,g\in\mathrm{L}^\infty_{\rm loc}(\mathbb{R}^2)$ with $f\ge0$, we find for almost all x, for all $t\in[0,T)$,

$$\begin{aligned}\partial_t\frac{\partial\sigma_x^t(\sigma^\circ)}{\partial\sigma^\circ}&\ge-\frac{\partial\sigma_x^t(\sigma^\circ)}{\partial\sigma^\circ}\|f\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}\Big(1+\|g\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}\int_{\sigma^\circ}^{\sigma_x^t(\sigma^\circ)}e^{(\sigma_x^t(\sigma^\circ)-s)\|g\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}}ds\Big)\\&\ge-\frac{\partial\sigma_x^t(\sigma^\circ)}{\partial\sigma^\circ}\,\|f\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}\Big(1+e^{(\sigma_x^t(\sigma^\circ)-\sigma^\circ)\|g\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}}\Big)\\&\ge-2\,\frac{\partial\sigma_x^t(\sigma^\circ)}{\partial\sigma^\circ}\,\|f\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}\,e^{t\|g\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}},\end{aligned}$$

while from (7.63) we deduce

$$\begin{aligned}\partial_t\frac{\partial\sigma_x^t(\sigma^\circ)}{\partial\sigma^\circ}&=f(\psi_x^{\sigma^\circ})\exp\Big(-\int_{\sigma^\circ}^{\sigma_x^t(\sigma^\circ)}(f+g)(\psi_x^u)du\Big)\\&\qquad+\frac{\partial\sigma_x^t(\sigma^\circ)}{\partial\sigma^\circ}\Big((1-\hat\kappa_x^t(\sigma_x^t(\sigma^\circ)))\,g(\psi_x^{\sigma_x^t(\sigma^\circ)})-\hat\kappa_x^t(\sigma_x^t(\sigma^\circ)\,f(\psi_x^{\sigma_x^t(\sigma^\circ)})\Big)\\&\le e^{t\|g\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}}\|f\|_{\mathrm{L}^\infty(K_x^0(\sigma^\circ))}+\frac{\partial\sigma_x^t(\sigma^\circ)}{\partial\sigma^\circ}\|g\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}.\end{aligned}$$

For almost all x, for all $t\in[0,T)$, this implies

$$\exp\Big(-2t\|f\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}e^{t\|g\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}}\Big)\le\frac{\partial\sigma_x^t(\sigma^\circ)}{\partial\sigma^\circ}\le\big(1+t\|f\|_{\mathrm{L}^\infty(K_x^0(\sigma^\circ))}\big)e^{t\|g\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}},$$

which shows that the map $\sigma_x^t:\mathbb{R}\to\mathbb{R}$ is a Lipschitz diffeomorphism, with also

$$\big(1+t\|f\|_{\mathrm{L}^\infty(K_x^0(\sigma^\circ))}\big)^{-1}e^{-t\|g\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}}\le\frac{\partial(\sigma_x^t)^{-1}(\sigma^\circ)}{\partial\sigma^\circ}\le\exp\Big(2t\|f\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}e^{t\|g\|_{\mathrm{L}^\infty(K_x^t(\sigma^\circ))}}\Big).\tag{7.66}$$

The representation (7.63) applied to $\sigma^\circ = (\sigma_x^t)^{-1}(0)$ then yields the desired result (7.60).

Step 2. Existence.

Let κ, σ be given by (7.60)–(7.59). Using the relation

$$\partial_t(\sigma_x^t)^{-1}(0) = -\kappa^t(x)\frac{\partial(\sigma_x^t)^{-1}}{\partial\sigma^\circ}(0),$$

the definition (7.60) and the estimate (7.66) ensure that $\kappa \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2))$. We now check that $(W \cdot \nabla)\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$. For almost all x and for all t, σ°, rewriting equation (7.59) in the form

$$\partial_t \sigma_{\psi_x^r}(\sigma^\circ) = 1 - \int_{r+\sigma^\circ}^{r+\sigma_{\psi_x^r}(\sigma^\circ)} f(\psi_x^s) \exp\Big(- \int_s^{r+\sigma_{\psi_x^r}(\sigma^\circ)} (f+g)(\psi_x^u)\, du\Big) ds,$$

we easily find that the map $r \mapsto \sigma^t_{\psi_x^r}(\sigma^\circ)$ belongs to $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$. Using the relation

$$\partial_r(\sigma^t_{\psi_x^r})^{-1}(0) = -\big(\partial_r \sigma^t_{\psi_x^r}\big)\big((\sigma^t_{\psi_x^r})^{-1}(0)\big)\, \frac{\partial(\sigma^t_{\psi_x^r})^{-1}}{\partial\sigma^\circ}(0),$$

it follows that the map $r \mapsto (\sigma^t_{\psi_x^r})^{-1}(0)$ also belongs to $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$. For almost all x and for all t, writing $(W \cdot \nabla)\kappa^t(x) = -\partial_r \kappa^t(\psi_x^r)|_{r=0}$, and using the definition (7.60) in the form

$$\kappa^t(\psi_x^r) = 1 - \int_{r+(\sigma^t_{\psi_x^r})^{-1}(0)}^{r} f(\psi_x^s) \exp\Big(- \int_s^r (f+g)(\psi_x^u) du\Big) ds,$$

we then easily deduce that $(W \cdot \nabla)\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$. We now check that κ is a strong solution of the Cauchy problem (7.58). By construction, the map $t \mapsto \kappa^t(\psi_x^{\sigma_x^t(\sigma^\circ)})$ is given by (7.63) and thus satisfies

$$\partial_t\big(\kappa^t(\psi_x^{\sigma_x^t(\sigma^\circ)})\big) = -\big(\kappa^t(\psi_x^{\sigma_x^t(\sigma^\circ)})\big)^2 f(\psi_x^{\sigma_x^t(\sigma^\circ)}) + \kappa^t(\psi_x^{\sigma_x^t(\sigma^\circ)})\big(1-\kappa^t(\psi_x^{\sigma_x^t(\sigma^\circ)})\big)\, g(\psi_x^{\sigma_x^t(\sigma^\circ)}),$$

or alternatively,

$$\big(\partial_t \kappa^t - \kappa^t\, (W\cdot\nabla)\kappa^t\big)(\psi_x^{\sigma_x^t(\sigma^\circ)}) = \big(-(\kappa^t)^2 f + \kappa^t(1-\kappa^t) g\big)(\psi_x^{\sigma_x^t(\sigma^\circ)}).$$

As this holds for almost all x and for all σ°, we indeed deduce that κ is a strong solution of (7.58). It remains to check that $\frac{1}{\kappa} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$. For that purpose, we note that equation (7.58) implies

$$\big|\partial_t\big(|\kappa^t(x)|^{-1}\big)\big| \le |\kappa^t(x)|^{-1}\Big(|(W\cdot\nabla)\kappa^t(x)| + (1+|\kappa^t(x)|)|g(x)|\Big) + |f(x)|,$$

which easily implies by a Grönwall argument that $\frac{1}{\kappa} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$.

Step 3. Regularity and integrability.

The additional regularity statement (ii) in $W^{s,\infty}$ is a straightforward consequence of formulas (7.60)–(7.59), together with the identity (7.65) and the estimate (7.66). Also note that for $f, g \in \mathrm{L}^\infty(\mathbb{R}^2)$ and $W \in W^{1,\infty}(\mathbb{R}^2)$ the argument in Step 2 ensures that $\frac{1}{\kappa}, (W\cdot\nabla)\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^2))$.

We now turn to the additional integrability (i) for $1-\kappa$. Assume that $f \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^2)$, $W \in W^{1,\infty}(\mathbb{R}^2)$, and $g \in \mathrm{L}^\infty(\mathbb{R}^2)$. For all $R \ge 1$, denote by $\chi_R(x) := e^{-|x|/R}$ the exponential cut-off function at scale R. We compute

$$\partial_t \int_{\mathbb{R}^2} \chi_R |1-\kappa^t| \le \int_{\mathbb{R}^2} \chi_R \kappa^t W \cdot \nabla |1-\kappa^t| + \int_{\mathbb{R}^2} \chi_R (\kappa^t)^2 f + \int_{\mathbb{R}^2} \chi |\kappa^t g||1-\kappa^t|,$$

and hence, after integration by parts, using the property $|\nabla\chi_R| \le \chi_R$ of the exponential cut-off function, for all $R \ge 1$,

$$\begin{aligned}\partial_t \int_{\mathbb{R}^2} \chi_R|1-\kappa^t| &\le \|\kappa^t\|_{\mathrm{L}^\infty}^2 \|f\|_{\mathrm{L}^1} + (\|\chi_R^{-1}\operatorname{div}(\kappa^t\chi_R W)\|_{\mathrm{L}^\infty} + \|\kappa^t g\|_{\mathrm{L}^\infty}) \int_{\mathbb{R}^2} \chi_R|1-\kappa^t| \\ &\le \|\kappa^t\|_{\mathrm{L}^\infty}^2 \|f\|_{\mathrm{L}^1} + (\|(W\cdot\nabla)\kappa^t\|_{\mathrm{L}^\infty} + \|\kappa^t\|_{\mathrm{L}^\infty}\|W\|_{W^{1,\infty}} + \|\kappa^t\|_{\mathrm{L}^\infty}\|g\|_{\mathrm{L}^\infty}) \int_{\mathbb{R}^2} \chi_R|1-\kappa^t|.\end{aligned}$$

Applying the Grönwall inequality, and letting $R \uparrow \infty$, we deduce $1-\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^2))$.

We finally turn to the H^s-regularity. Let $s \ge 1$ be fixed. Assume that $f \in H^s \cap W^{s,\infty}(\mathbb{R}^2)$, $W \in W^{s,\infty}(\mathbb{R}^2)^2$, $g \in W^{s,\infty}(\mathbb{R}^2)$. For all $R \ge 1$, denote by $\tilde\chi_R(x) := \exp(-(1+|x|^2)^{1/2}/R)$ a smooth exponential cut-off function at scale R. We compute

$$\begin{aligned}\partial_t\|\tilde\chi_R(1-\kappa^t)\|_{H^s}^2 &= -2\int_{\mathbb{R}^2} \langle\nabla\rangle^s\big(\tilde\chi_R(1-\kappa^t)\big)\langle\nabla\rangle^s\big(\kappa^t\tilde\chi_R W\cdot\nabla\kappa^t\big) \\ &\qquad -2\int_{\mathbb{R}^2} \langle\nabla\rangle^s\big(\tilde\chi_R(1-\kappa^t)\big)\langle\nabla\rangle^s\big(-\tilde\chi_R(\kappa^t)^2 f + \tilde\chi_R\kappa^t(1-\kappa^t)g\big).\end{aligned} \tag{7.67}$$

Decomposing

$$\begin{aligned}-2\langle\nabla\rangle^s\big(\kappa^t\tilde\chi_R W\cdot\nabla\kappa^t\big) &= 2[\langle\nabla\rangle^s,\kappa^t W\cdot]\nabla(\tilde\chi_R(1-\kappa^t)) + 2\kappa^t W\cdot\nabla\langle\nabla\rangle^s(\tilde\chi_R(1-\kappa^t)) \\ &\qquad - 2\langle\nabla\rangle^s\big((1-\kappa^t)\kappa^t W\cdot\nabla\tilde\chi_R\big),\end{aligned}$$

we find, after integration by parts in the second right-hand side term,

$$\begin{aligned}\partial_t\|\tilde\chi_R(1-\kappa^t)\|_{H^s}^2 &= 2\int_{\mathbb{R}^2} \langle\nabla\rangle^s\big(\tilde\chi_R(1-\kappa^t)\big)[\langle\nabla\rangle^s,\kappa^t W\cdot]\nabla(\tilde\chi_R(1-\kappa^t)) \\ &\qquad - \int_{\mathbb{R}^2} |\langle\nabla\rangle^s(\tilde\chi_R(1-\kappa^t))|^2 \operatorname{div}(\kappa^t W) \\ &\qquad - 2\int_{\mathbb{R}^2} \langle\nabla\rangle^s\big(\tilde\chi_R(1-\kappa^t)\big)\langle\nabla\rangle^s\big((1-\kappa^t)\kappa^t W\cdot\nabla\tilde\chi_R - \tilde\chi_R(\kappa^t)^2 f + \tilde\chi_R\kappa^t(1-\kappa^t)g\big),\end{aligned}$$

and hence,

$$\begin{aligned}\partial_t\|\tilde\chi_R(1-\kappa^t)\|_{H^s} &\lesssim \|[\langle\nabla\rangle^s,\kappa^t W\cdot]\nabla(\tilde\chi_R(1-\kappa^t))\|_{\mathrm{L}^2} + \|\kappa^t\|_{W^{s,\infty}}^2\|\tilde\chi_R f\|_{H^s} \\ &\qquad + \big(\|\operatorname{div}(\kappa^t W)\|_{\mathrm{L}^\infty} + \|\tilde\chi_R^{-1}\kappa^t W\cdot\nabla\tilde\chi_R\|_{W^{s,\infty}} + \|\kappa^t g\|_{W^{s,\infty}}\big)\|\tilde\chi_R(1-\kappa^t)\|_{H^s}.\end{aligned}$$

Applying the Kato-Ponce commutator estimate [269, Lemma X1] in the form (7.16) with $s \ge 1$ in order to estimate the first right-hand side term, we find

$$\begin{aligned}\partial_t\|\tilde\chi_R(1-\kappa^t)\|_{H^s} &\lesssim \big(\|\kappa^t W\|_{W^{s,\infty}} + \|\tilde\chi_R^{-1}\kappa^t W\cdot\nabla\tilde\chi_R\|_{W^{s,\infty}} + \|\kappa^t g\|_{W^{s,\infty}}\big)\|\tilde\chi_R(1-\kappa^t)\|_{H^s} \\ &\qquad + \|\kappa^t\|_{W^{s,\infty}}^2\|\tilde\chi_R f\|_{H^s},\end{aligned}$$

and thus, for all $R \ge 1$, using the properties of the smooth exponential cut-off function $\tilde\chi_R$,

$$\partial_t\|\tilde\chi_R(1-\kappa^t)\|_{H^s} \lesssim \|\kappa^t\|_{W^{s,\infty}}\|(W,g)\|_{W^{s,\infty}}\|\tilde\chi_R(1-\kappa^t)\|_{H^s} + \|\kappa^t\|_{W^{s,\infty}}^2\|f\|_{H^s},$$

Applying the Grönwall inequality, using the regularity result for the solution κ in $W^{s,\infty}(\mathbb{R}^2)$, and letting $R \uparrow \infty$, this implies $1-\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{R}^2))$. $\square$

We may now conclude with the proof of Theorem 7.6.1.

Proof of Theorem 7.6.1. Let $\mathrm{v}^\circ, \Psi \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)^2$ be log-Lipschitz vector fields with $\operatorname{curl}\mathrm{v}^\circ, \operatorname{curl}\Psi \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ and $\operatorname{curl}\mathrm{v}^\circ \geq 0$. We start with the existence part. By Lemma 7.6.2 with $W := (\Psi+\mathrm{v}^\circ)^\perp$, $f := \operatorname{curl}\mathrm{v}^\circ$, and $g := \operatorname{curl}\Psi$, there exists a global strong solution $\kappa \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2))$ of (7.57) with $\frac{1}{\kappa}, ((\Psi+\mathrm{v}^\circ)^\perp\cdot\nabla)\kappa \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^2)$. Then the function $\mathrm{v} := -\Psi+\kappa(\Psi+\mathrm{v}^\circ) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2))$ is by construction a global strong solution of (7.55) with initial data v° and with $\operatorname{curl}\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^2)$. The additional regularity statements follow from the corresponding statements for κ in Lemma 7.6.2 together with the representation $\mathrm{v}-\mathrm{v}^\circ = -(1-\kappa)(\mathrm{v}^\circ+\Psi)$.

We now turn to the uniqueness part. Assume that $\mathrm{v}_1, \mathrm{v}_2 \in W^{1,\infty}_{\mathrm{loc}}([0,T);\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2))$ are strong solutions of (7.55) on $[0,T)\times\mathbb{R}^2$ with $\operatorname{curl}\mathrm{v}_1, \operatorname{curl}\mathrm{v}_2 \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)$ and $\operatorname{curl}\mathrm{v}_1, \operatorname{curl}\mathrm{v}_2 \geq 0$. From (7.56), it follows that for $i=1,2$ we have $\mathrm{v}_i = -\Psi+\kappa_i(\Psi+\mathrm{v}^\circ)$ where κ_i is given by

$$\kappa_i^t(x) := \exp\Big(-\int_0^t \operatorname{curl}\mathrm{v}_i^s(x)\,ds\Big).$$

As v_i is a strong solution of (7.55) on $[0,T)\times\mathbb{R}^2$, we deduce that κ_i is a strong solution of equation (7.57) on $[0,T)\times\mathbb{R}^2$, and the boundedness assumption on $\operatorname{curl}\mathrm{v}_i$ implies that κ_i belongs to $W^{1,\infty}_{\mathrm{loc}}([0,T);\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2))$ and satisfies $\frac{1}{\kappa_i}, ((\Psi+v^\circ)^\perp\cdot\nabla)\kappa_i \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)$. $\square$

Chapter 8

Mean-field dynamics of Ginzburg-Landau vortices with pinning and applied force

We consider the time-dependent 2D Ginzburg-Landau equation in the whole plane with terms modeling the applied current and the impurities in the sample. The Ginzburg-Landau vortices are then subjected to three forces: their mutual repulsive Coulomb interaction, the Lorentz-like force due to the applied current and pushing the vortices in a given direction, and the pinning force attracting them towards the impurities. The competition between the three is expected to lead to complicated glassy effects.

We first rigorously study the limit in which the number of vortices N_ε blows up as the inverse Ginzburg-Landau parameter ε goes to 0, and we derive via a modulated energy method the limiting fluid-like mean-field evolution equations. These results hold in the case of parabolic, conservative, and mixed-flow dynamics in appropriate regimes of $N_\varepsilon \uparrow \infty$. We next consider the problem of homogenization of the limiting mean-field equations when the pinning potential oscillates rapidly: we formulate a number of questions and heuristics on the appropriate limiting stick-slip equations, as well as some rigorous results on the simplest regimes.

This chapter essentially corresponds to the article [169] jointly written with Sylvia Serfaty, to the exception of the mean-field results in the superdense parabolic regime (cf. Theorem 8.1.3 and Sections 8.3.3 and 8.8).

Contents

8.1 Introduction

8.1.1 General overview

Superconductors are materials that lose their resistivity at sufficiently low temperature (or low pressure), which allows them to carry electric currents without energy dissipation. Another important property of these materials is the so-called Meissner effect: (moderate) external magnetic fields are completely expelled from the sample. If the external field is much too strong, however, the superconducting material returns to a normal state. In the case of a type-II superconductor, an intermediate regime is possible between two critical values of the external field: the material is then in a mixed state, allowing a partial penetration of the external field through "vortex filaments". This mixed state has however a major drawback: when an electric current is applied, it flows through the sample, inducing a Lorentz-like force that sets the vortices in motion, and hence, since vortices are flux filaments, their movement generates an electric field in the direction of the electric current, which dissipates energy and destroys the superconductivity property.

While ordinary superconductors need extreme cooling to achieve superconductivity, the discovery of high-temperature superconductors from the 1980s onwards has given an major boost to technological applications, as the critical temperature of such materials is now reached with only liquid nitrogen. These high-temperature superconductors happen to be in practice strongly of type II and, as such, they show vortices for a very wide range of values of the applied magnetic field. Most technological applications of superconductors therefore occur in this mixed state, and it is thus crucial to design ways to prevent vortices from moving in order to recover the desired property of dissipation-free current flow. For that purpose a common attempt consists in introducing normal impurities in the material, which are meant to destroy superconductivity locally and therefore "pin down" the vortices to their locations if the applied current is not too strong.

With these applications in mind, there is a strong interest in the physics community in understanding the precise effect of such impurities (which are typically randomly scattered around the sample) on the statics and dynamics of vortices. Of particular interest is the critical applied current needed to depin the vortices from their pinning sites, as well as the slow motion of vortices — named *creep* — in the disordered sample when the applied current has a small intensity and thermal or quantum effects are taken into consideration (see e.g. [65, 195, 369]). In the sequel, we are interested in the collective dynamics of many vortices in a (2D section of a) type-II superconductor with applied current and impurities, and wish to establish in various regimes the correct mean-field equations describing the vortex matter. The richness of the dynamic phase diagram is particularly striking for this vortex matter in terms of the different tunable parameters (see e.g. [306, 369]).

The phenomenology of superconductivity is accurately described by the (mesoscopic) Ginzburg-Landau theory. Restricting ourselves to a 2D section of a superconducting material, we rather consider the simpler 2D Ginzburg-Landau model, and vortex filaments are replaced by "point vortices". We refer e.g. to [412, 411] for further reference on these models, and to [382] for a mathematical introduction. The (mesoscopic) impurities in the material are usually modeled by introducing a pinning weight $a : \mathbb{R}^2 \to [0, 1]$, which locally lowers the energy penalty associated with the vortices [284, 109] (see also [108]): regions with $a = 1$ correspond to the pure superconducting material, while points with $a \approx 0$ define the normal inclusions. In the time-dependent 2D Ginzburg-Landau equation (which is the gradient flow for the corresponding energy), the pinning weight and the applied electric current appear as follows,

$$\begin{cases} \partial_t w_\varepsilon = \triangle w_\varepsilon + \frac{w_\varepsilon}{\varepsilon^2}(a - |w_\varepsilon|^2), & \text{in } \mathbb{R}^+ \times \Omega, \\ n \cdot \nabla w_\varepsilon = i w_\varepsilon |\log \varepsilon| n \cdot J_{\mathrm{ex}}, & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ w_\varepsilon|_{t=0} = w_\varepsilon^\circ, \end{cases} \tag{8.1}$$

where Ω is a domain of $\mathbb{R}^2$, where n is the outer unit normal on $\partial\Omega$, where $w_\varepsilon : \mathbb{R}^+ \times \Omega \to \mathbb{C}$ is the complex-valued order parameter describing superconductivity, where $|\log\varepsilon|J_{\mathrm{ex}}:\partial\Omega\to\mathbb{R}^2$ denotes the (critically-scaled) applied electric current, and where $\varepsilon>0$ is the inverse Ginzburg-Landau parameter (a characteristic of the material, which is typically very small for real-life superconductors). More precisely, as first derived by Schmid [386] and by Gor'kov and Eliashberg [216], the true Ginzburg-Landau model should further be coupled to electromagnetism, replacing the above equation by a suitable version with magnetic gauge, and in particular the imposed electric current J_{ex} should then rather appear as a boundary condition for the electric and magnetic fields themselves. Since the gauge does not introduce any significant mathematical difficulty, we however focus on the above simplified form of the model, and only briefly comment on the case with gauge in Section 8.2.3. (Note that in the simplified model the number of vortices has to be imposed artificially via the boundary condition, while in the true model it is determined by the value of the external magnetic field.) The order parameter w_ε has the following meaning: the values $|w_\varepsilon|=1$ and 0 correspond to a superconducting and to a normal phase, respectively, and the vortices are the zeroes of w_ε with non-zero topological degree. Vortices typically have a core of size of order ε. Moreover, a vortex of degree d at a point x carries a (self-interaction) energy $\pi|d|a(x)|\log\varepsilon|$, which varies with its location due to the pinning weight a and implies that vortices are indeed attracted to the minima of the weight, that is, to the normal inclusions.

An important variant of this model (8.1) is the corresponding (conservative) Schrödinger flow, with $\partial_t w_\varepsilon$ replaced by $i\partial_t w_\varepsilon$. This coincides with the so-called Gross-Pitaevskii equation, which is an example of a nonlinear Schrödinger equation and serves as a model for Bose-Einstein condensates and superfluidity [4, 376], as well as for nonlinear optics [27]. As argued e.g. in [26], there is also physical interest in the "mixed-flow" (or "complex") Ginzburg-Landau equation, which is a mix between the (parabolic) Ginzburg-Landau and the (conservative) Gross-Pitaevskii equations. Instead of (8.1) we thus turn to the following more general equation, for any $\alpha\ge 0$, $\beta\in\mathbb{R}$, $\alpha^2+\beta^2=1$,

$$\begin{cases}(\alpha+i|\log\varepsilon|\beta)\partial_t w_\varepsilon=\triangle w_\varepsilon+\frac{w_\varepsilon}{\varepsilon^2}(a-|w_\varepsilon|^2), & \text{in } \mathbb{R}^+\times\Omega,\\ n\cdot\nabla w_\varepsilon=iw_\varepsilon|\log\varepsilon|n\cdot J_{\mathrm{ex}}, & \text{on } \mathbb{R}^+\times\partial\Omega,\\ w_\varepsilon|_{t=0}=w_\varepsilon^\circ, \end{cases}\tag{8.2}$$

which indeed allows to consider by the same token both the parabolic or Ginzburg-Landau case ($\alpha>0$, $\beta=0$) and the conservative or Gross-Pitaevskii case ($\alpha=0$, $\beta\in\mathbb{R}$). The mixed-flow case with $\alpha>0$, $\beta\in\mathbb{R}$ is henceforth referred to as the dissipative case.

In this context, including both a pinning potential and an applied current, we aim to understand the dynamics of the vortices in the asymptotic regime $\varepsilon\downarrow 0$. For a fixed number N of vortices, this asymptotic regime of equation (8.2) was well-understood in the physics community since the 1990s [340, 153, 361, 110], and shortly after various rigorous studies became available in the parabolic case [302, 301, 262, 264, 380], in the conservative case [123, 303, 261, 279], as well as in the mixed-flow case [410, 397]. As seen there, vortices are subjected to three forces:

— their mutual repulsive Coulomb (logarithmic) interaction;

— the Lorentz-like force F due to the applied current of intensity J_{ex};

— the pinning force, equal to $-\nabla h$ in terms of the so-called pinning potential $h:=\log a$ defined by the pinning weight a.

Neglecting boundary effects, and assuming that all vortices have the same degree $+1$, the effective vortex dynamics is then given by a system of ODEs of the form

$$(\alpha+\mathbb{J}\beta)\partial_t x_i=-N^{-1}\nabla_{x_i}W_N(x_1,\ldots,x_N)-\nabla h(x_i)+F(x_i),\qquad 1\le i\le N,\tag{8.3}$$

$$h:=\log a,\qquad W_N(x_1,\ldots,x_N):=-\sum_{i\neq j}^N\log|x_i-x_j|,$$

where the x_i's are the macroscopic vortex trajectories, and where $\mathbb{J}$ denotes the rotation of vectors by angle $\pi/2$ in the plane. The pinning and applied force intensities are parameters which can be tuned, leading to regimes in which one or two forces dominate over the others, or all are of the same order. In [410] no pinning force is considered, and the treated regimes lead to the applied force being of the same order as the interaction. In [397] the pinning and applied forces are chosen to be of the same order, and both dominate the interaction. Finally in [279], in the conservative case, the critical scaling is considered, that is, with all forces being of the same order.

In the sequel we consider the situation when the number N_ε of vortices is not fixed but depends on ε and blows up as $\varepsilon \downarrow 0$, which is a physically more realistic situation in many regimes of applied fields and currents. We then wish to describe the evolution of the density of the corresponding vortex liquid. At least when the number N_ε of vortices does not blow up too quickly, the correct limiting equation is naturally expected to coincide with the formal mean-field limit of the discrete dynamics (8.3) (as already discussed in Chapter 6), that is, the following nonlinear nonlocal transport equation for the mean-field vorticity m,

$$\partial_t \mathrm{m} = \operatorname{div}\,\big((\alpha - \mathbb{J}\beta)(\nabla h - F - \nabla\triangle^{-1}\mathrm{m})\,\mathrm{m}\,\big), \tag{8.4}$$

or alternatively, in terms of the mean-field supercurrent density v (related to m via $\mathrm{m} = \operatorname{curl}\mathrm{v}$),

$$\partial_t \mathrm{v} = (\alpha - \beta\mathbb{J})(\nabla^\perp h - F^\perp - \mathrm{v})\operatorname{curl}\mathrm{v}, \qquad \operatorname{div}\mathrm{v} = 0. \tag{8.5}$$

In the case without pinning and applied current ($a = 1$, $J_{\mathrm{ex}} = 0$), such a mean-field dynamics has been rigorously established in a number of settings in the conservative and parabolic cases:

— for the Gross-Pitaevskii equation ($\alpha = 0$, $\beta \in \mathbb{R}$), Jerrard and Spirn [263] have shown in the regime $1 \ll N_\varepsilon \lesssim (\log|\log\varepsilon|)^{1/2}$ that the vorticity of solutions converges to the solution of (8.4), which in that case coincide with the incompressible Euler equation in vorticity form, while Serfaty [395] has shown in the regime $|\log\varepsilon| \ll N_\varepsilon \ll \varepsilon^{-1}$ that the supercurrent of solutions converges to the solution of the incompressible Euler equation (8.5);

— for the Ginzburg-Landau equation ($\alpha > 0$, $\beta = 0$), the convergence of the vorticity of solutions to the solution of (8.4), first formally derived by Chapman, Rubinstein, Schatzman, and E [111, 173], has been rigorously established by Kurzke and Spirn [281] in the regime $1 \ll N_\varepsilon \le (\log\log|\log\varepsilon|)^{1/4}$, while Serfaty [395] has shown that in the whole regime $1 \ll N_\varepsilon \ll |\log\varepsilon|$ the supercurrent further converges to the solution of (8.5) but that in the regime $N_\varepsilon \simeq |\log\varepsilon|$ it converges to a different *compressible* mean-field model.

All these results assume that the initial data is suitably "well-prepared". Note that the delicate boundary issues are neglected in [263] and [395], where the Gross-Pitaevskii and Ginzburg-Landau equations are set for simplicity on the whole plane, while in [281] Dirichlet boundary conditions on a bounded domain Ω are further considered. The results of [281] and [263] rely on a direct method and a careful study of the vortex trajectories, while the results of Serfaty [395] are based on a "modulated energy approach" and rely on the assumed regularity of the solutions of the limiting equations. The situation in all the remaining regimes is still an open question, to the exception of the regime $|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|\log|\log\varepsilon|$ for the Ginzburg-Landau equation, which is further treated in the present chapter and leads to yet another mean-field equation.

The main goal of this chapter is to adapt the modulated energy approach of Serfaty [395] to the setting with pinning and applied current, thus extending the results of [410, 397, 279] to the case with $N_\varepsilon \gg 1$ vortices — in the whole plane for simplicity. The derivation bears several complications compared to the situation of Serfaty [395], in particular due to the lack of sufficient decay at infinity of the various quantities, and also to the fact that the self-interaction energy of each vortex now varies with its location due to the pinning weight. In addition to the parabolic and conservative cases, we also consider the mixed-flow case $\alpha > 0$, $\beta \neq 0$. We establish the convergence to suitable limiting fluid-type evolution equations, which in the simplest case take the form of nonlinear nonlocal transport equations (8.4)–(8.5) but are different in some regimes, and for which global well-posedness has already been discussed in detail in Chapter 7. As described above, different regimes for the intensity of the pinning and applied current lead to different limiting equations: in particular, the mean-field equation (8.4) is reduced to a simple linear transport equation with only the pinning and applied forces remaining when these are scaled to be much stronger than the interaction.

Although we perform this derivation for a pinning force which varies at the macroscopic scale, the most interesting situation from the modeling viewpoint is to let the pinning weight oscillate quickly at some mesoscopic scale η_ε, which also tends to 0 as $\varepsilon \downarrow 0$. In real-life materials, the way in which the impurities are inserted typically leads them to be uniformly and randomly scattered in the sample. This is well modeled by the η_ε-rescaling of a typical realization of a random stationary pinning weight $a(x) = a^0(x/\eta_\varepsilon)$. For simplicity, we may focus on the periodic case. One is thus led to the question of combining the mean-field limit for the Ginzburg-Landau or Gross-Pitaevskii evolution equations with a homogenization limit. In other words, can one perform the derivation of the limiting equation as $\varepsilon \downarrow 0$, $N_\varepsilon \uparrow \infty$, and $\eta_\varepsilon \downarrow 0$, and in which regimes does it hold?

While the homogenization of the (static) Ginzburg-Landau energy functional with pinning weight has been studied in some settings [5, 23, 154], we believe that these homogenization questions in the dynamical case are particularly challenging. They are in fact already very hard for just a finite number of vortices: studying the limit as $\eta \downarrow 0$ of the discrete dynamics (8.3) with pinning potential of the form $h(x) = \hat{h}^0(x/\eta)$ with $\hat{h}^0$ periodic or stationary, is a question of homogenization of a system of nonlinear coupled ODEs and is notoriously difficult. This difficulty is due to the complexity of the collective effects of the interacting vortices, in relation to the possible "glassy" properties predicted by physicists for such systems (see e.g. [195]). In contrast, the case with no interaction term and with F constant is much simpler to analyze, and seems to be known as a "washboard" in the physics literature. When $F = 0$, a vortex is simply attracted towards the local wells of the pinning potential h. Otherwise, the constant applied force $F \neq 0$ can be absorbed into the term $-\nabla h$ by adding to the potential h an affine function, which effectively tilts the potential landscape into a washboard-shaped graph. As will be seen, beyond some positive value of the intensity $|F|$ the tilted potential has no local minimum, leading the particle to fall downwards. In the setting of a superconductor with pinning and applied current, this corresponds to the critical "depinning current" above which the vortices are depinned from their pinning locations. Note that when the applied force F is non-constant and varies at the macroscopic scale (still without interaction term) the situation is already much more subtle and only partial results are obtained in [319].

Since our modulated energy method to establish the mean-field limit results does not seem well-adapted to include homogenization effects, we will not say much about commuting the limits $\varepsilon \downarrow 0$, $N \uparrow \infty$, and $\eta \downarrow 0$, but instead we formulate a few partial results in the direction of homogenizing the derived mean-field equations of the type (8.4)–(8.5), and we formulate many open questions which we believe to be interesting both from an applied and a theoretical point of view. This topic is indeed very delicate on its own, with the same kind of difficulties as for the homogenization of the discrete system of coupled ODEs (8.3), but in the case without interaction and with F constant the problem is considerably simpler and leads to well-defined limiting stick-slip equations. Finally, in order to model thermal effects, one may replace the mean-field transport equations of the type (8.4)–(8.5) by their viscous versions, and we will give a few heuristics on the corresponding homogenization questions.

8.1.2 Mean-field limit results

Precise setting

Since the presence of the boundary creates mathematical difficulties which we do not know how to overcome (due to the possible entrance and exit of vortices), we modify the mesoscopic model (8.2) and consider a suitable version on the whole plane with boundary conditions "at infinity". As in [410, 397], the boundary conditions can be changed into a bulk force term by a suitable change of phase in the unknown function. Dividing also the unknown function by the expected density $\sqrt{a}$, we arrive at the equation

$$\begin{cases} \lambda_\varepsilon(\alpha + i|\log \varepsilon|\beta)\partial_t u_\varepsilon = \triangle u_\varepsilon + \frac{a}{\varepsilon^2} u_\varepsilon(1 - |u_\varepsilon|^2) + \nabla h \cdot \nabla u_\varepsilon + i|\log \varepsilon| F^\perp \cdot \nabla u_\varepsilon + f u_\varepsilon, \\ u_\varepsilon|_{t=0} = u_\varepsilon^\circ, \end{cases} \tag{8.6}$$

with $h := \log a$, $f : \mathbb{R}^2 \to \mathbb{R}$, and $F : \mathbb{R}^2 \to \mathbb{R}^2$, where F is an effective applied force corresponding to the Lorentz-like force generated by the applied current. The parameter λ_ε is an appropriate time rescaling to obtain a nontrivial limiting dynamics. Within the derivation of (8.6) from (8.2), the zeroth-order term f takes the following explicit form (but this is largely unimportant, and the scaling in the corresponding bounds (8.43)–(8.44) below may also be substantially relaxed),

$$f := \frac{\triangle \sqrt{a}}{\sqrt{a}} - \frac{1}{4} |\log \varepsilon|^2 |F|^2. \tag{8.7}$$

The discussion of the derivation of (8.6) from (8.2), as well as that of the boundary conditions and the assumptions at infinity, is postponed to Section 8.2.1, while the global well-posedness of (8.6) is discussed in Section 8.2.2. For simplicity we assume that the pinning weight satisfies

$$\frac{1}{C} \leq a(x) \leq 1, \qquad \text{for all } x, \tag{8.8}$$

which avoids degenerate situations: physically one would like to consider a pinning weight a that may vanish, representing true normal inclusions [109], but this is much more delicate mathematically (see e.g. [23]). Setting $F \equiv 0$, $a \equiv 1$, $h \equiv 0$, and $f \equiv 0$, we naturally retrieve the equation studied e.g. in [281, 263, 395], and our results will thus indeed be a generalization of those in [281, 395].

Given solutions of the mesoscopic model (8.6), we wish to establish the convergence of their *supercurrent*, defined by

$$j_\varepsilon := \langle \nabla u_\varepsilon, i u_\varepsilon \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{C}$ as identified with $\mathbb{R}^2$, that is, $\langle x, y \rangle = \Re(x\bar{y})$ for all $x, y \in \mathbb{C}$. The *vorticity* μ_ε is derived from the supercurrent via $\mu_\varepsilon := \operatorname{curl} j_\varepsilon$. Note that this indeed corresponds to the density of vortices, defined as zeros of u_ε weighted by their degrees, in the sense that

$$\mu_\varepsilon \sim 2\pi \sum_i d_i \delta_{x_i}, \qquad \text{as } \varepsilon \downarrow 0, \tag{8.9}$$

with $\{x_i\}_i$ the vortex locations and $\{d_i\}_i$ their degrees (this is made rigorous by the so-called Jacobian estimates, a notion to which we will come back in Section 8.5). In this setting, we wish to show that the rescaled supercurrent $N_\varepsilon^{-1} j_\varepsilon$ converges as $\varepsilon \downarrow 0$ to a velocity field v solving a limiting PDE, which as in [395] is assumed to be regular enough. The limiting equations are fluid-like equations of the form (8.5), where however the incompressibility condition can be lost when the density of vortices becomes too large. Such equations are studied in detail in Chapter 7, where solutions are shown in most cases to be global and indeed regular enough if the initial data is. A formal derivation of this mean-field limit result is included in Section 1.2.3.

In order to establish this convergence, we will adapt the modulated energy technique used by Serfaty [395], of which we have already given some account in Chapter 6 (see Section 6.1.3). In the present situation, the method consists in defining a modulated energy, which without pinning takes the form

$$\frac{1}{2} \int_{\mathbb{R}^2} \Big(|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \Big), \tag{8.10}$$

where v denotes the solution of the (postulated) limiting equation. This modulated energy thus somehow measures the distance between the supercurrent $j_\varepsilon = \langle \nabla u_\varepsilon, iu_\varepsilon \rangle$ and the postulated limit $N_\varepsilon \mathrm{v}$, in a way that is well adapted to the energy structure. Under some regularity assumptions on v, it is then proved in [395] that, thanks to the suitable limiting equation satisfied by v, this quantity (8.10) satisfies a Grönwall relation, so that if it is initially small, more precisely $o(N_\varepsilon^2)$, it remains so, yielding the desired convergence $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$. However, in the regimes where $N_\varepsilon \lesssim |\log \varepsilon|$, the modulated energy cannot be of order $o(N_\varepsilon^2)$, because each vortex of degree d carries a self-interaction energy $\pi |d| |\log \varepsilon|$. For that reason (and assuming that all vortices have positive degrees initially), we need to subtract the fixed quantity $\pi N_\varepsilon |\log \varepsilon|$ from (8.10). Note that, while the Ginzburg-Landau energy (that is, (8.10) with $\mathrm{v} = 0$) diverges for configurations u_ε with nonzero degree at infinity,

$$0 \neq \deg(u_\varepsilon) := \lim_{R \uparrow \infty} \int_{\partial B_R} \langle \nabla u_\varepsilon, iu_\varepsilon \rangle \cdot n^\perp,$$

the modulated energy may indeed converge (and does if v has the correct circulation at infinity).

In the present context with pinning weight a, the modulated energy (8.10) should naturally be changed into a weighted one,

$$\frac{1}{2} \int_{\mathbb{R}^2} a \left(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\,|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \right). \tag{8.11}$$

This leads to several additional difficulties:

- This energy does usually not remain finite along the flow because ∇h, F, and f in (8.6) are only assumed to be bounded (in order to include at least the case of a fixed applied current circulating through the sample). This leads us to consider a truncated version of (8.11). In the Gross-Pitaevskii case, we must actually assume that ∇h, F, and f decay sufficiently at infinity in order to guarantee the well-posedness of the mesoscopic model (8.6), and hence a truncation of (8.11) is no longer needed. However, in that case, due to the presence of pinning, the pressure p in the limiting equation for v is no longer square-integrable, and another truncation argument then becomes needed in order to deal with this lack of integrability.
- In some regimes, it is crucial to replace in the modulated energy (8.11) the solution v of the limiting equation by some suitable ε-dependent map $\mathrm{v}_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$, which is separately shown to converge to v. This amounts to including lower-order terms in the modulated energy. Note that in this way the difficulty is split into two parts: first we prove that $N_\varepsilon^{-1} j_\varepsilon$ is close to v_ε by means of a Grönwall argument on the modulated energy, which requires some careful vortex analysis, and then we check that v_ε indeed converges to v, which is a softer consequence of the stability of the limiting equation.
- In the present weighted setting, a vortex of degree d at a point x carries a self-interaction energy $\pi |d| a(x) |\log \varepsilon|$, so that what needs to be subtracted from the modulated energy (8.11) is no longer $\pi N_\varepsilon |\log \varepsilon|$ but rather, in view of (8.9),

  $$\pi \sum_i d_i a(x_i) |\log \varepsilon| \sim \frac{|\log \varepsilon|}{2} \int_{\mathbb{R}^2} a \mu_\varepsilon.$$

  The presence of this pinning weight leads to various complications and requires a particularly careful vortex analysis (cf. Section 8.5).

We thus consider the following truncated version of the modulated energy (8.11),

$$\mathcal{E}_{\varepsilon,R} := \int_{\mathbb{R}^2} \frac{a \chi_R}{2} \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \Big), \tag{8.12}$$

as well as the following truncated modulated energy excess,

$$\mathcal{D}_{\varepsilon,R} := \mathcal{E}_{\varepsilon,R} - \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R\mu_\varepsilon$$
$$= \int_{\mathbb{R}^2} \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big), \quad (8.13)$$

where for all $r > 0$ we set $\chi_r := \chi(\cdot/r)$ for some fixed cut-off function $\chi \in C_c^\infty(\mathbb{R}^2;[0,1])$ with $\chi|_{B_1} = 1$ and $\chi|_{\mathbb{R}^2\setminus B_2} = 0$. In the sequel, all energy integrals are thus truncated as above with the cut-off function χ_R, for some scale $R \gg 1$ to be later suitably chosen as a function of ε. We write $\mathcal{E}_\varepsilon := \mathcal{E}_{\varepsilon,\infty}$ for the corresponding quantity without the cut-off χ_R in the definition (formally $R = \infty$), and also $\mathcal{D}_\varepsilon := \sup_{R\ge 1}\mathcal{D}_{\varepsilon,R}$. Rather than the L^2-norm restricted to the ball B_R centered at the origin, our methods further allow to consider the uniform $\mathrm{L}^2_{\mathrm{loc}}$-norm at the scale R: setting $\chi_R^z := \chi_R(\cdot - z)$ for all $z \in \mathbb{R}^2$, we define

$$\mathcal{E}^*_{\varepsilon,R} := \sup_z \mathcal{E}^z_{\varepsilon,R}\,, \qquad \mathcal{E}^z_{\varepsilon,R} := \int_{\mathbb{R}^2}\frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big), \quad (8.14)$$

$$\mathcal{D}^*_{\varepsilon,R} := \sup_z \mathcal{D}^z_{\varepsilon,R}\,, \qquad \mathcal{D}^z_{\varepsilon,R} := \mathcal{E}^z_{\varepsilon,R} - \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\mu_\varepsilon, \quad (8.15)$$

where the suprema run over all lattice points $z \in R\mathbb{Z}^2$.

Assumptions

For the essential part of the proof, in the dissipative case ($\alpha > 0$), it suffices to assume $h \in W^{2,\infty}(\mathbb{R}^2)$ and $F \in W^{1,\infty}(\mathbb{R}^2)^2$ (hence $f \in \mathrm{L}^\infty(\mathbb{R}^2)$ in view of (8.7)). In the Gross-Pitaevskii case, as already explained, we need to restrict to a decaying setting in order to ensure the well-posedness of the mesoscopic model (8.6), that is, we need to further assume $\nabla h, F \in W^{1,p}(\mathbb{R}^2)^2$ for some $p < \infty$, $f \in \mathrm{L}^2(\mathbb{R}^2)$, and additionally $\operatorname{div} F = 0$. Nevertheless, in both cases, in order to ensure strong enough regularity properties of the solution v of the limiting equation, stronger assumptions on the data are needed and are listed below. Note that we do not try to optimize these regularity assumptions on the data.

Assumption 8.1.1. Let $\alpha \ge 0$, $\beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$, $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$, $u_\varepsilon^\circ : \mathbb{R}^2 \to \mathbb{C}$, and $\mathrm{v}_\varepsilon^\circ, \mathrm{v}^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ for all $\varepsilon > 0$. Assume that (8.7) and (8.8) hold, and that the initial data $(u_\varepsilon^\circ, \mathrm{v}_\varepsilon^\circ, \mathrm{v}^\circ)$ are well-prepared as $\varepsilon \downarrow 0$, in the sense

$$\mathcal{D}_\varepsilon^{*,\circ} := \sup_{R\ge1}\sup_{z\in\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon^\circ - iu_\varepsilon^\circ N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon^\circ|^2)^2 - |\log\varepsilon|\operatorname{curl}\langle\nabla u_\varepsilon^\circ, iu_\varepsilon^\circ\rangle\Big) \ll N_\varepsilon^2, \quad (8.16)$$

with $\mathrm{v}_\varepsilon^\circ \to \mathrm{v}^\circ$ in $\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2$, and with $\operatorname{curl}\mathrm{v}_\varepsilon^\circ, \operatorname{curl}\mathrm{v}^\circ \in \mathcal{P}(\mathbb{R}^2)$. Assume that $\mathrm{v}_\varepsilon^\circ$ and v° are bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$. In addition,

(a) *Dissipative case ($\alpha > 0$), general non-decaying setting:*
For some $s > 0$, assume that $u_\varepsilon^\circ \in H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, that $h \in W^{s+3,\infty}(\mathbb{R}^2)$, $F \in W^{s+2,\infty}(\mathbb{R}^2)^2$ (hence $f \in W^{1,\infty}(\mathbb{R}^2)$ in view of (8.7)), that $\mathrm{v}_\varepsilon^\circ$, v° are bounded in $W^{s+2,\infty}(\mathbb{R}^2)^2$, and that $\operatorname{curl}\mathrm{v}_\varepsilon^\circ$, $\operatorname{curl}\mathrm{v}^\circ$, $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)$ are bounded in $H^{s+1}\cap W^{s+1,\infty}(\mathbb{R}^2)$.

(b) *Gross-Pitaevskii case ($\alpha = 0$), decaying setting:*
Assume that $u_\varepsilon^\circ \in U + H^2(\mathbb{R}^2;\mathbb{C})$ for some reference map $U \in \mathrm{L}^\infty(\mathbb{R}^2;\mathbb{C})$ with $\nabla^2 U \in H^1(\mathbb{R}^2;\mathbb{C})$, $\nabla|U| \in \mathrm{L}^2(\mathbb{R}^2)$, $1-|U|^2 \in \mathrm{L}^2(\mathbb{R}^2)$, and $\nabla U \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C})$ for all $p > 2$ (typically we may choose U smooth and equal to $e^{iN_\varepsilon\theta}$ in polar coordinates outside a ball at the origin). Assume that $h \in W^{3,\infty}(\mathbb{R}^2)$, $\nabla h \in H^2(\mathbb{R}^2)^2$, $F \in H^3\cap W^{3,\infty}(\mathbb{R}^2)^2$, $f \in H^2\cap W^{2,\infty}(\mathbb{R}^2)$, and that we have $\operatorname{div} F = 0$ pointwise, and $a(x) \to 1$ uniformly as $|x| \uparrow \infty$. Assume that $\mathrm{v}_\varepsilon^\circ$, v° are bounded in $W^{2,\infty}(\mathbb{R}^2)^2$, and that $\operatorname{curl}\mathrm{v}_\varepsilon^\circ$, $\operatorname{curl}\mathrm{v}^\circ$ are bounded in $H^1(\mathbb{R}^2)$. ◊

Considered regimes

We distinguish between the following four main (critically scaled) regimes, in which the relative strengths of the pinning, the applied forces, and the interaction emerge.

(GL_1) Weighted dissipative case, small number of vortices:
$\alpha > 0$, $N_\varepsilon \ll |\log \varepsilon|$, $\lambda_\varepsilon = \frac{N_\varepsilon}{|\log \varepsilon|}$, $F = \lambda_\varepsilon \hat{F}$, $h = \lambda_\varepsilon \hat{h}$ (hence $a = \hat{a}^{\lambda_\varepsilon}$);

(GL_2) Weighted dissipative case, critical number of vortices:
$\alpha > 0$, $N_\varepsilon \simeq |\log \varepsilon|$, $\frac{N_\varepsilon}{|\log \varepsilon|} \to \lambda \in (0, \infty)$, $\lambda_\varepsilon = 1$, $F = \hat{F}$, $h = \hat{h}$ (hence $a = \hat{a}$);

(GL_3) Weighted dissipative case, large number of vortices:
$\alpha > 0$, $|\log \varepsilon| \ll N_\varepsilon \ll \varepsilon^{-1}$, $\lambda_\varepsilon = \frac{N_\varepsilon}{|\log \varepsilon|}$, $F = \lambda_\varepsilon \hat{F}$, $h = \hat{h}$ (hence $a = \hat{a}$);

(GP) Weighted Gross-Pitaevskii case, large number of vortices:
$\alpha = 0$, $\beta = 1$, $|\log \varepsilon| \ll N_\varepsilon \ll \varepsilon^{-1}$, $\lambda_\varepsilon = \frac{N_\varepsilon}{|\log \varepsilon|}$, $F = \lambda_\varepsilon \hat{F}$, $h = \hat{h}$ (hence $a = \hat{a}$);

where $\hat{h}$ and $\hat{F}$ are independent of ε, and $\hat{h} \le 0$ is bounded below. The critical threshold for the number N_ε of vortices at the order $|\log \varepsilon|$ is easily understood since in this regime the vortex energy $O(N_\varepsilon |\log \varepsilon|)$ precisely becomes of the same order as the phase energy $O(N_\varepsilon^2)$. As we will see, in the dissipative case, these regimes lead to drastically different mean-field behaviors. Another critical threshold is expected to occur when the number N_ε of vortices becomes of the order ε^{-1}, due to the overlap of the vortex cores. Note that just as in [395] the modulated energy approach does not allow us to treat the Gross-Pitaevskii case with fewer (but still unboundedly many) vortices $1 \ll N_\varepsilon \lesssim |\log \varepsilon|$, although in that case the same mean-field behavior is expected as in the case $|\log \varepsilon| \ll N_\varepsilon \ll \varepsilon^{-1}$.

Let us intuitively justify the choice of the above scalings for the pinning and the applied force. From energy considerations, we expect the pinning, the applied force, and the interaction to be of order $N_\varepsilon |\log \varepsilon| |\nabla h|$, $N_\varepsilon |\log \varepsilon| |F|$, and N_ε^2, respectively. The critical scaling (such that pinning, applied force and interactions are all of the same order) should thus amount to choosing both ∇h and F of order $N_\varepsilon / |\log \varepsilon|$. However, the non-degeneracy condition (8.8) for the pinning weight $a = e^h$ imposes for the pinning potential $h \le 0$ to remain uniformly bounded in ε, hence the particular non-critical choice in (GL_3) and in (GP) (with h of order 1 rather than $\lambda_\varepsilon \gg 1$).

In the dissipative case, we may also consider sub- or supercritical scalings, for which the pinning either dominates, or is dominated by the interaction. In these cases, the limiting equations are considerably simplified.

(GL_1') (GL_1) with subcritically scaled oscillating pinning, very weak interaction:
$\alpha > 0$, $N_\varepsilon \ll |\log \varepsilon|$, $\lambda_\varepsilon = 1$, $F = \hat{F}$, $h = \hat{h}$ (hence $a = \hat{a}$);

(GL_2') (GL_1) with subcritically scaled oscillating pinning, weak interaction:
$\alpha > 0$, $N_\varepsilon \ll |\log \varepsilon|$, $\frac{N_\varepsilon}{|\log \varepsilon|} \ll \lambda_\varepsilon \ll 1$, $F = \lambda_\varepsilon \hat{F}$, $h = \lambda_\varepsilon \hat{h}$ (hence $a = \hat{a}^{\lambda_\varepsilon}$);

(GL_3') (GL_1) with supercritically scaled oscillating pinning, strong interaction:
$\alpha > 0$, $N_\varepsilon \ll |\log \varepsilon|$, $\lambda_\varepsilon = \frac{N_\varepsilon}{|\log \varepsilon|}$, $F = \lambda_\varepsilon \hat{F}$, $h = \lambda'_\varepsilon \hat{h}$ (hence $a = \hat{a}^{\lambda'_\varepsilon}$), $\lambda'_\varepsilon \ll \lambda_\varepsilon$;

(GL_4') (GL_2) with supercritically scaled oscillating pinning, strong interaction:
$\alpha > 0$, $N_\varepsilon \simeq |\log \varepsilon|$, $\frac{N_\varepsilon}{|\log \varepsilon|} \to \lambda \in (0, \infty)$, $\lambda_\varepsilon = 1$, $F = \hat{F}$, $h = \lambda'_\varepsilon \hat{h}$, $\lambda'_\varepsilon \ll 1$;

where again $\hat{h}$ and $\hat{F}$ are independent of ε, with $\hat{h} \le 0$ bounded below. Since in the present work we are mostly interested in pinning effects, we focus on the subcritical regimes (GL_1') and (GL_2'), while for the two supercritical regimes the pinning effects vanish in the limiting equation and the situation is thus much easier and closer to [395]. For simplicity, subscripts "ε" are systematically dropped from the data a, h, F, f, the precise dependence being always implicitly chosen as above.

Statement of main results

We are now in position to state our main mean-field results. We start with the dissipative case, and first consider the critical regimes (GL_1) and (GL_2), as well as the subcritical regimes (GL_1') and (GL_2'). Note that the results are slightly finer in the parabolic case. Although all the proofs in this chapter are quantitative, we only give qualitative statements to simplify the exposition. The following result generalizes those in [281, 395] to the case with pinning and forcing. Note that the limiting mean-field equations are fluid-like of the form (8.5), except that the incompressibility condition is lost in some regimes, as first evidenced by Serfaty [395]. In the regimes (GL_1) and (GL_2'), the weight a naturally disappears from the incompressibility condition $\operatorname{div} \mathrm{v} = 0$ due to the assumption $a = \hat{a}^{\lambda_\varepsilon} \to 1$ as $\varepsilon \downarrow 0$.

Theorem 8.1.2 (Dissipative case)**.** *Let Assumption 8.1.1(a) hold, with the initial data* $(u_\varepsilon^\circ, \mathrm{v}_\varepsilon^\circ, \mathrm{v}^\circ)$ *satisfying the well-preparedness condition* (8.16)*. For all* $\varepsilon > 0$*, let* $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2; \mathbb{C}))$ *denote the unique global solution of* (8.6) *on* $\mathbb{R}^+ \times \mathbb{R}^2$*. Then, the following hold for the supercurrent density* $j_\varepsilon := \langle \nabla u_\varepsilon, iu_\varepsilon \rangle$*.*

(i) Regime (GL_1) with $\log|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|$, and $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ) = \operatorname{div}\mathrm{v}^\circ = 0$:
We have $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ *as* $\varepsilon \downarrow 0$*, where* v *is the unique global (smooth) solution of*

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v})\operatorname{curl}\mathrm{v}, \qquad \operatorname{div}\mathrm{v} = 0, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ\,. \tag{8.17}$$

In the parabolic case $\beta = 0$*, the same conclusion also holds for* $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$*.*

(ii) Regime (GL_2) with $N_\varepsilon/|\log\varepsilon| \to \lambda \in (0,\infty)$, and $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$:
For some $T > 0$*, we have* $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ *as* $\varepsilon \downarrow 0$*, where* v *is the unique local (smooth) solution of*

$$\partial_t \mathrm{v} = \alpha^{-1}\nabla(\hat{a}^{-1}\operatorname{div}(\hat{a}\mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\lambda\mathrm{v})\operatorname{curl}\mathrm{v}, \quad \mathrm{v}|_{t=0} = \mathrm{v}^\circ, \tag{8.18}$$

on $[0,T) \times \mathbb{R}^2$*. In the parabolic case* $\beta = 0$*, this solution* v *can be extended globally, and the above holds with* $T = \infty$*.*

(iii) Regime (GL_1') with $\log|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|$, and $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$:
We have $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ *as* $\varepsilon \downarrow 0$*, where* v *is the unique global (smooth) solution of*

$$\partial_t \mathrm{v} = \alpha^{-1}\nabla(\hat{a}^{-1}\operatorname{div}(\hat{a}\mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp)\operatorname{curl}\mathrm{v}, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ\,. \tag{8.19}$$

(iv) Regime (GL_2') with $\log|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|$, and $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ) = \operatorname{div}\mathrm{v}^\circ = 0$:
We have $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ *as* $\varepsilon \downarrow 0$*, where* v *is the unique global (smooth) solution of*

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp)\operatorname{curl}\mathrm{v}, \qquad \operatorname{div}\mathrm{v} = 0, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ\,. \tag{8.20}$$

In the parabolic case $\beta = 0$ *with* $N_\varepsilon/|\log\varepsilon| \ll \lambda_\varepsilon \lesssim e^{o(N_\varepsilon)}/|\log\varepsilon|$*, the same conclusion also holds for* $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$*.* ◊

We now turn to the superdense regime (GL_3). The following result is only proven to hold in the parabolic case in the moderate regime $|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|\log|\log\varepsilon|$, and gives rise to a new degenerate limiting equation, which is studied in detail in Chapter 7. This is new even in the case without pinning and forcing. The situation for the mixed-flow dissipative case or for a larger number of vortices remains an open question. In particular, in the mixed-flow dissipative case, even the correct limiting equation is unclear since the local well-posedness of the mixed-flow version of the degenerate equation (8.22) below remains unresolved (cf. Section 7.1.5).

Theorem 8.1.3 (Superdense parabolic case)**.** *Let Assumption 8.1.1(a) hold with* $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$*, and with the initial data* $(u_\varepsilon^\circ, \mathrm{v}_\varepsilon^\circ, \mathrm{v}^\circ)$ *satisfying the following stronger well-preparedness condition, for some* $\delta > 0$*,*

$$\mathcal{D}_\varepsilon^{*,\circ} := \sup_{R\geq 1}\sup_{z\in\mathbb{R}^2} \int \frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon^\circ - iu_\varepsilon^\circ N_\varepsilon \mathrm{v}^\circ|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon^\circ|^2)^2 - |\log\varepsilon|\mathrm{curl}\,\langle\nabla u_\varepsilon^\circ, iu_\varepsilon^\circ\rangle\Big) \lesssim N_\varepsilon^{2-\delta}. \tag{8.21}$$

For some $s > 3$*, assume in addition that* $h \in W^{s+2,\infty}(\mathbb{R}^2)$*,* $F \in W^{s+1,\infty}(\mathbb{R}^2)^2$*, and that* $\mathrm{v}^\circ \in W^{s+1,\infty}(\mathbb{R}^2)^2$ *with* $\mathrm{m}^\circ := \mathrm{curl}\,\mathrm{v}^\circ \in \mathcal{P}\cap H^s(\mathbb{R}^2)$ *and* $\mathrm{d}^\circ := \mathrm{div}\,(a\mathrm{v}^\circ) \in H^{s-1}(\mathbb{R}^2)$*. For all* $\varepsilon > 0$*, let* $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ *denote the unique global solution of* (8.6) *on* $\mathbb{R}^+\times\mathbb{R}^2$*. Then, in the regime* (GL_3) *with* $|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|\log|\log\varepsilon|$ *and with* $\alpha = 1$*,* $\beta = 0$*, the supercurrent density* $j_\varepsilon := \langle\nabla u_\varepsilon, iu_\varepsilon\rangle$ *satisfies* $N_\varepsilon^{-1}j_\varepsilon \to \mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ *as* $\varepsilon\downarrow 0$*, where* v *is the unique global (smooth) solution of*

$$\partial_t\mathrm{v} = -(\hat{F}^\perp + 2\mathrm{v})\,\mathrm{curl}\,\mathrm{v}, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ\,. \tag{8.22}$$

$\Diamond$

We finally turn to the Gross-Pitaevskii case in the regime (GP). Note that in the regime $N_\varepsilon \gg |\log\varepsilon|$ the well-preparedness condition (8.16) is naturally simplified, as the vortex self-interaction energy is no longer dominant. Note that the pinning force $-\nabla\hat{h}$ is absent from the limiting equation since in the regime (GP) the interaction and the applied force dominate, but the weight $a = \hat{a}$ nevertheless remains in the incompressibility condition $\mathrm{div}\,(\hat{a}\mathrm{v}) = 0$. The following result generalizes those in [395] to the case with pinning and forcing.

Theorem 8.1.4 (Gross-Pitaevskii case)**.** *Let Assumption 8.1.1(b) hold with* $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$*, and with the initial data* $(u_\varepsilon^\circ, \mathrm{v}_\varepsilon^\circ, \mathrm{v}^\circ)$ *satisfying the following simplified well-preparedness condition,*

$$\mathcal{E}_\varepsilon^\circ := \int_{\mathbb{R}^2}\frac{a}{2}\Big(|\nabla u_\varepsilon^\circ - iu_\varepsilon^\circ N_\varepsilon\mathrm{v}^\circ|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon^\circ|^2)^2\Big) \ll N_\varepsilon^2.$$

For all $\varepsilon > 0$*, let* $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^2(\mathbb{R}^2;\mathbb{C}))$ *denote the unique global solution of* (8.6) *on* $\mathbb{R}^+\times\mathbb{R}^2$*. Then, in the regime* (GP) *with* $|\log\varepsilon| \ll N_\varepsilon \ll \varepsilon^{-1}$*, we have* $N_\varepsilon^{-1}j_\varepsilon \to \mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;(\mathrm{L}^1+\mathrm{L}^2)(\mathbb{R}^2)^2)$ *as* $\varepsilon\downarrow 0$*, where* v *is the unique global (smooth) solution of*

$$\partial_t\mathrm{v} = \nabla\mathrm{p} - (\hat{F} - 2\mathrm{v}^\perp)\,\mathrm{curl}\,\mathrm{v}, \qquad \mathrm{div}\,(\hat{a}\mathrm{v}) = 0, \qquad \mathrm{v}^t|_{t=0} = \mathrm{v}^\circ\,. \tag{8.23}$$

$\Diamond$

The same mean-field limit result is actually expected to hold for $1 \ll N_\varepsilon \ll \varepsilon^{-1}$ (see indeed [263] for the other extreme regime $1 \ll N_\varepsilon \lesssim (\log|\log\varepsilon|)^{1/2}$). The restriction $N_\varepsilon \gg |\log\varepsilon|$ in the above is thus purely technical: as in [395], it is caused by the difficulty in controlling the velocity of the individual vortices because of the lack of control on $\int_{\mathbb{R}^2}|\partial_t u_\varepsilon|^2$, which is however crucially needed within the modulated energy approach. As the Gross-Pitaevskii vortex dynamics formally behaves like the conservative flow for Coulomb particles, this difficulty is strongly related to the lack of a modulated energy proof for the mean-field limit of such a discrete particle system (cf. Section 6.1.5; the only known proof is by compactness [390]).

On the other hand, the restriction $N_\varepsilon \ll \varepsilon^{-1}$ is quite natural, since for a larger number of vortices the modulus $|u_\varepsilon|$ of the order parameter should further enter the limiting equation, leading to different compressible fluid-like equations [57, 58, 56, 101].

The structure of the mean-field equations (8.17), (8.18), (8.19), (8.20), (8.22), and (8.23) is more transparent when expressed in terms of the mean-field vorticity $\mathrm{m} := \operatorname{curl} \mathrm{v}$. In the case of (8.17) (and similarly for (8.20) and (8.23)), the vorticity m satisfies a nonlinear nonlocal transport equation,

$$\begin{cases} \partial_t \mathrm{m} = \operatorname{div} \big((\alpha - \mathbb{J}\beta)(\nabla \hat{h} - \hat{F} + 2\mathrm{v}^\perp)\, \mathrm{m} \big), \\ \operatorname{curl} \mathrm{v} = \mathrm{m}, \quad \operatorname{div} \mathrm{v} = 0. \end{cases} \tag{8.24}$$

In the case of (8.18) (and similarly for (8.19)) the vorticity m satisfies a similar equation coupled with a transport-diffusion equation for the divergence $\mathrm{d} := \operatorname{div}(\hat{a}\mathrm{v})$,

$$\begin{cases} \partial_t \mathrm{m} = \operatorname{div} \big((\alpha - \mathbb{J}\beta)(\nabla \hat{h} - \hat{F} + 2\lambda \mathrm{v}^\perp)\, \mathrm{m} \big), \\ \partial_t \mathrm{d} - \alpha^{-1} \triangle\, \mathrm{d} + \alpha^{-1} \operatorname{div}(\mathrm{d} \nabla \hat{h}) = \operatorname{div} \big((\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\lambda \mathrm{v}) \hat{a} \mathrm{m} \big), \\ \operatorname{curl} \mathrm{v} = \mathrm{m}, \quad \operatorname{div}(\hat{a}\mathrm{v}) = \mathrm{d}, \end{cases} \tag{8.25}$$

while the transport-diffusion equation becomes degenerate in the case of (8.22), in terms of e.g. $\theta := \operatorname{div} \mathrm{v}$,

$$\begin{cases} \partial_t \mathrm{m} = \operatorname{div} \big((-\hat{F} + 2\lambda \mathrm{v}^\perp)\, \mathrm{m} \big), \\ \partial_t \theta = \operatorname{div} \big((-\hat{F}^\perp - 2\lambda \mathrm{v})\, \mathrm{m} \big), \\ \operatorname{curl} \mathrm{v} = \mathrm{m}, \quad \operatorname{div} \mathrm{v} = \theta. \end{cases} \tag{8.26}$$

A detailed study of these equations is provided in Chapter 7, including global existence results for rough initial data. While the limiting vorticity m satisfies strictly different equations in the critical regimes (GL_1) and (GL_2), we observe that it satisfies just the same equation in both subcritical regimes (GL_1') and (GL_2'), that is, a simple linear transport equation.

The proofs of Theorems 8.1.2, 8.1.3, and 8.1.4 follow the outline of [395], and rely on all the tools for vortex analysis developed over the years: lower bounds via the Jerrard-Sandier ball construction, "Jacobian estimate", "product estimate". In addition to the problems at infinity created by the non-decay of the forcing F that we wish to allow, the presence of the pinning weight introduces additional technical difficulties, as always in the analysis of Ginzburg-Landau. The fact that the energy of a vortex depends on its location makes it more difficult to a priori control the total number of vortices, and requires localized estimates, in particular localized ball constructions. Adapting the required tools and analysis to this setting is done in Section 8.5.

8.1.3 Homogenization results and open questions

As explained, the most interesting situation from the modeling viewpoint is to let the pinning potential h vary quickly at some mesoscale $\eta_\varepsilon \ll 1$, thus coupling the mean-field limit for the vortex density with a homogenization limit. More precisely, we set

$$\hat{h}(x) := \eta_\varepsilon \hat{h}^0(x, x/\eta_\varepsilon), \tag{8.27}$$

for some $\hat{h}^0$ independent of ε, and we will refer to η_ε as the "pin separation". For simplicity, we assume that $\hat{h}^0$ is periodic in its second variable. Since in the superdense parabolic case and in the Gross-Pitaevskii case we are anyway limited to the less interesting supercritical regimes (GL_3) and (GP) (for which the pinning force $-\nabla \hat{h}$ is indeed absent from the limiting equations (8.22) and (8.23)), we focus attention on the dissipative regimes (GL_1), (GL_2), (GL_1'), and (GL_2').

Small pin separation limit and stick-slip models

As explained in Section 8.9.3, our modulated energy methods only allow to treat a diagonal regime, that is, when the pin separation η_ε tends very slowly to 0, in which case the homogenization limit can simply be performed *after* the mean-field limit. The other regimes are left as an open question.

Corollary 8.1.5. *Let the same assumptions hold as in Theorem 8.1.2. In the regime* (GL$_2$)*, we further restrict to the parabolic case $\beta = 0$. Then there exists a sequence $\eta_{\varepsilon,0} \downarrow 0$ (depending on all the data of the problem) such that for all $\eta_{\varepsilon,0} \ll \eta_\varepsilon \ll 1$, choosing the fast oscillating pinning potential* (8.27)*, the same conclusions hold as in Theorem 8.1.2 in the form $N_\varepsilon^{-1} j_\varepsilon - \bar{\mathrm{v}}_\varepsilon \to 0$, where $\bar{\mathrm{v}}_\varepsilon$ is now the unique global (smooth) solution of the corresponding equations* (8.17)–(8.20) *with $\nabla \hat{h}(x)$ replaced by $\nabla_2 \hat{h}^0(x, x/\eta_\varepsilon)$.* $\Diamond$

In a diagonal regime, the above result thus reduces the understanding of the limiting behavior of the rescaled supercurrent $N_\varepsilon^{-1} j_\varepsilon$ to that of the solution $\bar{\mathrm{v}}_\varepsilon$ of the mean-field equations (8.17)–(8.20) with fast oscillating pinning, that is, a (periodic) homogenization problem for the mean-field equations. In more general regimes, only two minor rigorous results are obtained:

(a) For very small forcing $\|F\|_{\mathrm{L}^\infty} \ll \|\nabla h\|_{\mathrm{L}^\infty}$, in the subcritical regimes (GL$_1'$) and (GL$_2'$), the vorticity is shown to remain "stuck" in the limit, that is, to converge at all times to its initial data (cf. Proposition 8.9.13). This is a very particular case of the pinning phenomenon evidenced below in the diagonal regime.

(b) In a short timescale of order $O(\eta_\varepsilon)$, the vorticity is shown to concentrate in each (mesoscopic) periodicity cell onto the invariant measure associated with the initial vector field (cf. Proposition 8.9.2). This mesoscopic initial-boundary layer result is in clear agreement with the description of the dynamics on larger timescales obtained below in the diagonal regime, where the transport is indeed shown to take place "along" the invariant measures.

Subcritical regimes. In the subcritical regimes (GL$_1'$) and (GL$_2'$), the nonlinear interaction term vanishes in the mean-field equations (8.19)–(8.20): in terms of the vorticity $\bar{\mathrm{m}}_\varepsilon := \operatorname{curl} \bar{\mathrm{v}}_\varepsilon$ we are thus left with a (periodic) homogenization problem for a simple *linear* transport equation, but with a compressible velocity field. Such questions were first investigated in the 2D periodic case by Menon [319], and are still partially open. The situation is however much simpler if the pinning potential $\hat{h}^0(x, x/\eta_\varepsilon) := \tilde{h}^0(x/\eta_\varepsilon)$ is independent of the macroscopic variable and if the forcing is a constant vector $\hat{F} := F_0 \in \mathbb{R}^2$, that is, the so-called "washboard model". The homogenization result is then a particular case of the nonlinear setting considered in [137] (see also [172, 254] for the incompressible case, and [188, 136] for the linear Hamiltonian case), but in the present framework a more precise characterization of the asymptotic behavior of $\bar{\mathrm{m}}_\varepsilon$ is possible (cf. Theorem 8.9.8). In the simplest situation, the result is summarized as follows.

Proposition 8.1.6 (Subcritical regimes)**.** *Let $\bar{\mathrm{v}}_\varepsilon$ denote the unique global (smooth) solution of* (8.19) *or* (8.20) *with $\nabla \hat{h}(x)$ replaced by $\nabla \tilde{h}^0(x/\eta_\varepsilon)$, for $\tilde{h}^0 \in C^2_{\mathrm{per}}(Q)$ (independent of ε) and $\eta_\varepsilon \ll 1$, and with $\hat{F} := F_0 \in \mathbb{R}^2$ a constant vector. Consider the periodic vector field*

$$\Gamma^{F_0} := (\alpha - \mathbb{J}\beta)(\nabla \tilde{h}^0 - F_0) \quad : \quad Q \to \mathbb{R}^2,$$

and assume that the dynamics on the 2-torus Q associated with the vector field $-\Gamma^{F_0}$ has a unique stable invariant measure $\mu^{F_0} \in \mathcal{P}_{\mathrm{per}}(Q)$. Define the averaged vector

$$\Gamma^{F_0}_{\mathrm{hom}} := \int_Q \Gamma^{F_0} d\mu^{F_0}.$$

Then we have $\bar{\mathrm{m}}_\varepsilon := \operatorname{curl} \bar{\mathrm{v}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{m}}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$, where $\bar{\mathrm{m}}$ is the unique solution of the constant-coefficient transport equation

$$\partial_t \bar{\mathrm{m}} = \operatorname{div}(\Gamma^{F_0}_{\mathrm{hom}} \bar{\mathrm{m}}), \qquad \bar{\mathrm{m}}|_{t=0} = \operatorname{curl} \mathrm{v}^\circ. \qquad \Diamond$$

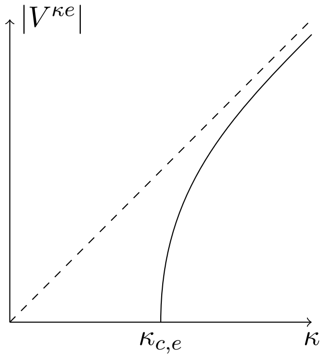


Figure 8.1 – Typical forcing-velocity characteristics exhibiting a stick-slip velocity law.

This result describes a so-called stick-slip velocity law: On the one hand, for F_0 close enough to 0, any stable invariant measure μ^{F_0} is concentrated at fixed points, that is, at minima of $\tilde{h}^0$, hence the corresponding velocity field is $V^{F_0} := -\Gamma_{\mathrm{hom}}^{F_0} = 0$, meaning that the vorticity gets stuck, as the vortices are trapped in local wells of the pinning potential. On the other hand, for F_0 large enough, the measure μ^{F_0} becomes non-trivial, hence we have $V^{F_0} \neq 0$, meaning that the vorticity is transported, but at a reduced speed due to the attraction by the local wells of the pinning potential. We further show that the velocity law $F_0 \mapsto V^{F_0} := -\Gamma_{\mathrm{hom}}^{F_0}$ is not smooth at the depinning threshold, but typically has a square-root behavior (cf. Proposition 8.9.11), denoting $\kappa := |F_0|$,

$$|V^{\kappa e}| = C(1 + o(1))(\kappa - \kappa_{c,e})^{1/2}, \qquad \text{as} \quad 0 < \kappa - \kappa_{c,e} \ll 1, \tag{8.28}$$

where $e \in \mathbb{S}^1$ is some direction and where $\kappa_{c,e}e$ ($\kappa_{c,e} \geq 0$) is the critical depinning threshold in the direction e. However, no general such result is obtained (cf. open question in Remark 8.9.12(a)). For very large $|F_0| \gg 1$, we naturally find $V^{F_0} \sim (\alpha - \mathbb{J}\beta)F_0$, that is, the system flows as if there were no disorder. The typical response of the system in this stick-slip velocity law is plotted in Figure 8.1. For more detail, we refer to Section 8.9.5. Note that a similar frictional stick-slip dynamics is observed for very different physical processes (see e.g. the Barkhausen effect for the magnetization of a domain under an applied field [222]).

Critical regimes. In the critical regimes (GL_1) and (GL_2), the nonlinear interaction term can no longer be neglected in the mean-field equations (8.17)–(8.18). A purely formal 2-scale expansion yields the following heuristics for the asymptotic behavior of $\bar{\mathrm{v}}_\varepsilon$. Note that a rigorous justification of this homogenization limit seems particularly challenging due to the nonlinear nonlocal character of the mean-field equations and to their instability as $\eta_\varepsilon \downarrow 0$, and moreover the well-posedness of the formal limiting equations (8.29)–(8.30) below is unclear (since the vector field $\Gamma_{\mathrm{hom}}[\bar{\mathrm{v}}]$ is in general not Lipschitz continuous even for smooth $\bar{\mathrm{v}}$, cf. (8.28)). Making good sense of the formal limiting equations and justifying the limit are thus both left as open questions. We refer to Section 8.9.4 and Remark 8.9.5 for detail.

Heuristics 8.1.7 (Critical regimes — formal asymptotic)**.** *For all $w : \mathbb{R}^2 \to \mathbb{R}^2$ and $x \in \mathbb{R}^2$, consider the periodic vector field*

$$\Gamma_x[w] := (\alpha - \mathbb{J}\beta)\big(\nabla_2 \hat{h}^0(x,\cdot) - \hat{F}(x) + 2w^\perp(x)\big) \quad : \quad Q \to \mathbb{R}^2,$$

and assume that the dynamics on the 2-torus Q associated with the vector field $-\Gamma_x[w]$ has a unique stable invariant measure $\mu_x[w] \in \mathcal{P}_{\mathrm{per}}(Q)$. We then define the averaged vector field

$$\Gamma_{\mathrm{hom}}[w](x) := \int_Q \Gamma_x[w](y)\, d\mu_x[w](y).$$

(i) Regime (GL_1) with fast oscillating pinning (8.27):
Let $\bar{\mathrm{v}}_\varepsilon$ *denote the unique global (smooth) solution of* (8.17) *with* $\nabla \hat{h}(x)$ *replaced by* $\nabla_2 \hat{h}^0(x, x/\eta_\varepsilon)$, $\eta_\varepsilon \ll 1$, *and with* $\hat{h}^0$ *independent of* ε. *Then we expect* $\operatorname{curl} \bar{\mathrm{v}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{m}}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$, *where* $\bar{\mathrm{m}}$ *satisfies*

$$\partial_t \bar{\mathrm{m}} = \operatorname{div}\left(\Xi_{\mathrm{hom}}[\bar{\mathrm{m}}]\, \bar{\mathrm{m}}\right), \qquad \bar{\mathrm{m}}|_{t=0} = \operatorname{curl} \mathrm{v}^\circ, \tag{8.29}$$

where the homogenized velocity is given by the following formula,

$$\Xi_{\mathrm{hom}}[\bar{\mathrm{m}}](x) := \Gamma_{\mathrm{hom}}[\nabla^\perp \triangle^{-1} \bar{\mathrm{m}}](x).$$

Similarly, $\bar{\mathrm{v}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{v}} := \nabla^\perp \triangle^{-1} \bar{\mathrm{m}}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2))$, *where* $\bar{\mathrm{v}}$ *thus satisfies*

$$\partial_t \bar{\mathrm{v}} = \nabla \bar{\mathrm{p}} + \Gamma_{\mathrm{hom}}[\bar{\mathrm{v}}]^\perp \operatorname{curl} \bar{\mathrm{v}}, \qquad \operatorname{div} \bar{\mathrm{v}} = 0, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ .$$

More precisely, we expect for all $t > 0$,

$$\int_0^t \Big(\operatorname{curl} \bar{\mathrm{v}}_\varepsilon^\tau(x) - \bar{\mathrm{m}}^\tau(x)\, \mu_x[\nabla^\perp \triangle^{-1} \bar{\mathrm{m}}^\tau](x/\eta_\varepsilon)\Big) d\tau \to 0,$$

in the strong sense of measures.

(ii) Regime (GL_2) in the parabolic case $\beta = 0$, with fast oscillating pinning (8.27):
Let $\beta = 0$, *and let* $\bar{\mathrm{v}}_\varepsilon$ *denote the unique global (smooth) solution of* (8.18) *with* $\nabla \hat{h}(x)$ *replaced by* $\nabla_2 \hat{h}^0(x, x/\eta_\varepsilon)$, $\eta_\varepsilon \ll 1$, *and with* $\hat{h}^0$ *independent of* ε. *Then we expect* $\operatorname{curl} \bar{\mathrm{v}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{m}}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$ *and* $\operatorname{div}(\hat{a} \bar{\mathrm{v}}_\varepsilon) \rightharpoonup \bar{\mathrm{d}}$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$, *where* $\bar{\mathrm{m}}$ *and* $\bar{\mathrm{d}}$ *satisfy*

$$\begin{aligned} \partial_t \bar{\mathrm{m}} &= \operatorname{div}\left(\Xi_{\mathrm{hom}}[\bar{\mathrm{m}}, \bar{\mathrm{d}}]\, \bar{\mathrm{m}}\right), & \bar{\mathrm{m}}|_{t=0} &= \operatorname{curl} \mathrm{v}^\circ, \\ \partial_t \bar{\mathrm{d}} &= \alpha^{-1} \triangle \bar{\mathrm{d}} + \operatorname{div}\left(\Xi_{\mathrm{hom}}[\bar{\mathrm{m}}, \bar{\mathrm{d}}]^\perp\, \bar{\mathrm{m}}\right), & \bar{\mathrm{d}}|_{t=0} &= \operatorname{div} \mathrm{v}^\circ, \end{aligned} \tag{8.30}$$

where the homogenized velocity is given by the following formula,

$$\Xi_{\mathrm{hom}}[\bar{\mathrm{m}}, \bar{\mathrm{d}}](x) := \Gamma_{\mathrm{hom}}[\nabla^\perp \triangle^{-1} \bar{\mathrm{m}} + \nabla \triangle^{-1} \bar{\mathrm{d}}](x).$$

Similarly, $\bar{\mathrm{v}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{v}} := \nabla^\perp \triangle^{-1} \bar{\mathrm{m}} + \nabla \triangle^{-1} \bar{\mathrm{d}}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2))$, *where* $\bar{\mathrm{v}}$ *thus satisfies*

$$\partial_t \bar{\mathrm{v}} = \alpha^{-1} \nabla \operatorname{div} \bar{\mathrm{v}} + \Gamma_{\mathrm{hom}}[\bar{\mathrm{v}}]^\perp \operatorname{curl} \bar{\mathrm{v}}, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ .$$

More precisely, we expect for all $t > 0$,

$$\int_0^t \Big(\operatorname{curl} \bar{\mathrm{v}}_\varepsilon^\tau(x) - \bar{\mathrm{m}}^\tau(x)\, \mu_x[\nabla^\perp \triangle^{-1} \bar{\mathrm{m}}^\tau + \nabla \triangle^{-1} \bar{\mathrm{d}}^\tau](x/\eta_\varepsilon)\Big) d\tau \to 0,$$

in the strong sense of measures. ◊

Due to the competition between the pinning potential and the vortex interaction, the dynamical properties of the limiting $\bar{\mathrm{v}}$ are expected to change dramatically with respect to the subcritical regimes: the interacting vortices are now expected to move as a coherent elastic object in a heterogeneous medium, yielding very particular glassy properties [195, 369]. To describe the dynamics, we again consider the forcing-velocity curve. Assume that the forcing $\hat{F} := F_0 \in \mathbb{R}^2$ is a constant vector, let $\bar{\mathrm{v}}^{F_0} := \bar{\mathrm{v}}$ denote as above the corresponding limit of $\bar{\mathrm{v}}_\varepsilon$ as $\varepsilon \downarrow 0$, and set $\bar{\mathrm{m}}^{F_0} := \operatorname{curl} \bar{\mathrm{v}}^{F_0}$. Formally, the mean velocity is then defined as

$$V^{F_0} := \lim_{t \uparrow \infty} \frac{1}{t} \int x \, d\bar{\mathrm{m}}^{F_0, t}(x). \tag{8.31}$$

Intuitively, for F_0 close enough to 0, the above heuristics predicts that the vorticity $\bar{\mathrm{m}}^{F_0}$ should spread due to the vortex repulsion, until the interaction force $\bar{\mathrm{v}}^{F_0}$ becomes small enough that the invariant measure $\mu_x^{F_0}[\bar{\mathrm{v}}^{F_0}]$ remains concentrated at a fixed point of the dynamics generated by $-\Gamma_x^{F_0}[\bar{\mathrm{v}}^{F_0}]$, in which case there holds $\Gamma_{\mathrm{hom}}^{F_0}[\bar{\mathrm{v}}^{F_0}] = 0$. Therefore, just as in the subcritical regimes, we expect to find $V^{F_0} = 0$ for all F_0 close enough to 0, $V^{F_0} \neq 0$ for F_0 large enough, and $V^{F_0} \sim (\alpha - \mathbb{J}\beta)F_0$ for very large $|F_0| \gg 1$ (cf. Figure 8.1). Nevertheless, the precise picture is expected to be very different at the depinning threshold: the velocity law $F_0 \mapsto V^{F_0}$ should still be non-smooth at this threshold, of the form

$$|V^{\kappa e}| = C(1 + o(1))(\kappa - \kappa_{c,e})^{\zeta}, \qquad \text{as} \quad 0 < \kappa - \kappa_{c,e} \ll 1, \tag{8.32}$$

in some direction $e \in \mathbb{S}^1$, but the value of the depinning threshold $\kappa_{c,e} > 0$ and of the depinning exponent $\zeta \in (0,1)$ are expected to differ completely from the case without interaction (8.28) and to be related to the glassy properties of the system, as predicted in the physics literature [336, 339, 115] (see also [195, Section 5]). A rigorous justification of this whole description is left as an open question.

Since the vortices are elastically coupled by the interaction, the problem is formally analogous to that of understanding the motion of general elastic systems in disordered media, which is the framework considered in the above-cited physics papers. In this spirit, a considerable attention has been devoted in the physics community to the simpler Quenched Edwards-Wilkinson model for elastic interface motion in disordered media [268, 83]. Note that for this interface model some rigorous mathematical understanding is available: the pinning of the interface at low forcing is proved in [146] in dimension $d \geq 2$, while the (ballistic) motion of the interface at large forcing is obtained in [128, 151] in dimension $d = 2$, and more recently in [69, 150] for various related discrete models in any dimension $d \geq 2$. These questions are also related (although again for different models) to the recent rigorous homogenization results for the forced mean curvature equation and for more general geometric Hamilton-Jacobi equations [28].

System with thermal noise

Different stochastic variants of the Ginzburg-Landau equation have been introduced in the physics literature in order to model the effect of thermal noise in type-II superconductors [387, 243, 140, 141] (see also [401, 189, 190, 407] for corresponding stochastic versions of the mixed-flow Gross-Pitaevskii equation to model thermal and quantum noise in Bose-Einstein condensates). Although we do not study here the mean-field limit problem for such models, for a finite number N of vortices, in the limit $\varepsilon \downarrow 0$, we expect the thermal noise to act on the vortices as N independent Brownian motions: more precisely, in the regime (GL_1), the limiting trajectories $(x_i)_{i=1}^N$ of the N vortices are expected to satisfy the following system of coupled SDEs (see e.g. [173, Section III.B]),

$$dx_i = (\alpha - \mathbb{J}\beta)\big(N^{-1}\nabla_{x_i} W_N(x_1, \ldots, x_N) - \nabla \hat{h}(x_i) + \hat{F}(x_i)\big)dt + \sqrt{2T}dB_i^t, \qquad 1 \leq i \leq N, \tag{8.33}$$

$$W_N(x_1, \ldots, x_N) := -\pi \sum_{i \neq j}^{N} \log|x_i - x_j|,$$

where $B_1, \ldots, B_N$ are N independent 2D Brownian motions. Such macroscopic phenomenological models, where the thermal noise acts via random Langevin kicks, are abundantly used by physicists [65, 195, 369].

In the case of a diverging number of vortices $N_\varepsilon \gg 1$, in the regime (GL_1), it is then natural to postulate that a good phenomenological model for the limiting supercurrent $\mathrm{v} := \lim_\varepsilon N_\varepsilon^{-1} j_\varepsilon$ is given by the (deterministic) mean-field limit of the particle system (8.33), that is, the following version of (8.17) with viscosity,

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v})\operatorname{curl} \mathrm{v} + T\triangle \mathrm{v}, \qquad \operatorname{div} \mathrm{v} = 0, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ, \tag{8.34}$$

while in the regime (GL_2) a natural model for the limit v is rather given by the following version of (8.18) with viscosity,

$$\partial_t \mathrm{v} = \alpha^{-1}\nabla(\hat a^{-1}\,\mathrm{div}\,(\hat a \mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat h - \hat F^\perp - 2\lambda \mathrm{v})\mathrm{curl}\,\mathrm{v} + T\triangle \mathrm{v}, \quad \mathrm{v}|_{t=0} = \mathrm{v}^\circ. \tag{8.35}$$

In the regimes (GL_1') and (GL_2'), these equations should be replaced by their versions without interaction term. Note that in [186, 187] the mean-field limit of the particle system (8.33) has indeed been rigorously proved to coincide with (8.34) (although the modulated energy method seems to fail in that case, as explained in Section 6.1.5).

In this viscous context, we may now consider the homogenization limit of the phenomenological thermal mean-field models (8.34)–(8.35) with fast oscillating pinning (8.27), or equivalently, with $\nabla \hat h(x)$ replaced by $\nabla_2 \hat h^0(x, x/\eta_\varepsilon)$. We denote by $\bar{\mathrm{v}}_\varepsilon$ the unique (smooth) solution of the corresponding equation. We naturally restrict attention to the critical scaling for the temperature, that is, $T := \eta_\varepsilon T_0$ for some fixed $T_0 > 0$.

Remark 8.1.8. On the one hand, for temperatures $T \ll \eta_\varepsilon$, the viscous term in equations (8.34)–(8.35) is expected to have no effect in the limit, yielding the same asymptotic behavior as for $T = 0$. On the other hand, for $T \gg \eta_\varepsilon$, the viscous term is so strong that the energy barriers are instantaneously overcome by the dynamics: for $T = \kappa_\varepsilon T_0$ with $\eta_\varepsilon \ll \kappa_\varepsilon \ll 1$, the limit $\bar{\mathrm{v}}$ of the solution $\bar{\mathrm{v}}_\varepsilon$ of (8.34) or (8.35) with oscillating pinning is expected to satisfy respectively (as suggested by a formal multiscale expansion)

$$\begin{aligned} &\partial_t \bar{\mathrm{v}} = \nabla \bar{\mathrm{p}} - (\alpha - \mathbb{J}\beta)(\hat F^\perp + 2\bar{\mathrm{v}})\,\mathrm{curl}\,\bar{\mathrm{v}}, \qquad \mathrm{div}\ \bar{\mathrm{v}} = 0, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ, \\ \text{or} \quad &\partial_t \bar{\mathrm{v}} = \alpha^{-1}\nabla(\mathrm{div}\ \bar{\mathrm{v}}) - (\alpha - \mathbb{J}\beta)(\hat F^\perp + 2\lambda\bar{\mathrm{v}})\,\mathrm{curl}\,\bar{\mathrm{v}}, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ, \end{aligned}$$

while for $T = T_0$ of order 1 the limit $\bar{\mathrm{v}}$ should satisfy respectively

$$\begin{aligned} &\partial_t \bar{\mathrm{v}} = \nabla \bar{\mathrm{p}} - (\alpha - \mathbb{J}\beta)(\hat F^\perp + 2\bar{\mathrm{v}})\,\mathrm{curl}\,\bar{\mathrm{v}} + T_0\triangle\bar{\mathrm{v}}, \qquad \mathrm{div}\ \bar{\mathrm{v}} = 0, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ, \\ \text{or} \quad &\partial_t \bar{\mathrm{v}} = \alpha^{-1}\nabla(\mathrm{div}\ \bar{\mathrm{v}}) - (\alpha - \mathbb{J}\beta)(\hat F^\perp + 2\lambda\bar{\mathrm{v}})\,\mathrm{curl}\,\bar{\mathrm{v}} + T_0\triangle\bar{\mathrm{v}}, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ. \end{aligned}$$

It is thus natural to restrict attention to the less trivial case of the critically scaled temperature. $\Diamond$

Subcritical regimes. In the subcritical regimes (GL_1') and (GL_2'), the thermal mean-field models take the form (8.34)–(8.35) without interaction term. In terms of the vorticity $\bar{\mathrm{m}}_\varepsilon := \mathrm{curl}\,\bar{\mathrm{v}}_\varepsilon$, with oscillating pinning, and with critically scaled temperature $T = \eta_\varepsilon T_0$, $T_0 > 0$, these equations become

$$\partial_t \bar{\mathrm{m}}_\varepsilon = \mathrm{div}\ \big((\alpha - \mathbb{J}\beta)(\nabla_2 \hat h^0(\cdot, \cdot/\eta_\varepsilon) - \hat F)\,\bar{\mathrm{m}}_\varepsilon\big) + \eta_\varepsilon T_0 \triangle \bar{\mathrm{m}}_\varepsilon, \qquad \bar{\mathrm{m}}_\varepsilon|_{t=0} = \mathrm{curl}\,\mathrm{v}^\circ. \tag{8.36}$$

The limit $\eta_\varepsilon \downarrow 0$ of this equation is a particular case of homogenization of a parabolic equation with vanishing viscosity, as studied by Dalibard [135]. Alternatively, using Nguetseng's 2-scale compactness theorem (in the form of Lemma 8.9.10, as e.g. in the proof of Theorem 8.9.8), we easily obtain the following.

Proposition 8.1.9 (Subcritical regimes with temperature)**.** *Let $\bar{\mathrm{m}}_\varepsilon$ be as above, and assume that $\hat h^0 \in C^1_b(\mathbb{R}^2; C^1_{\mathrm{per}}(Q))$ and $\hat F \in C^1_b(\mathbb{R}^2)$. Let $\tilde\mu^{T_0} \in W^{1,\infty}(\mathbb{R}^2; \mathrm{L}^\infty \cap \mathcal{P}_{\mathrm{per}}(Q))$ denote the unique weak solution of the following cell problem,*

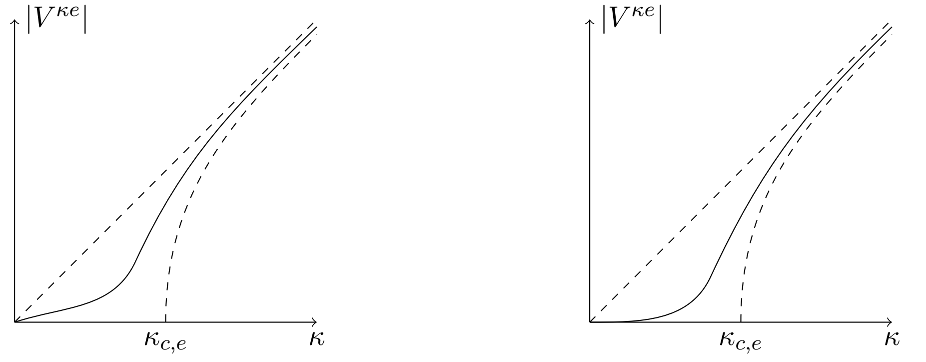
$$T_0 \triangle_y \tilde\mu^{T_0}(x,y) + \mathrm{div}_y\big((\alpha - \mathbb{J}\beta)(\nabla_2 \hat h^0(x,y) - \hat F(x))\tilde\mu^{T_0}(x,y)\big) = 0, \tag{8.37}$$

and define the following averaged vector field,

$$\Gamma^{T_0}_{\mathrm{hom}}(x) := \int_Q (\alpha - \mathbb{J}\beta)(\nabla_2 \hat h^0(x,y) - \hat F(x))\tilde\mu^{T_0}(x,y)dy. \tag{8.38}$$

Then we have $\bar{\mathrm{m}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{m}}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$, where $\bar{\mathrm{m}}$ is the unique solution of the transport equation

$$\partial_t \bar{\mathrm{m}} = \mathrm{div}\,(\Gamma^{T_0}_{\mathrm{hom}}\bar{\mathrm{m}}), \qquad \bar{\mathrm{m}}|_{t=0} = \mathrm{curl}\,\mathrm{v}^\circ. \qquad \Diamond$$



(a) Subcritical regimes: (linear) ohmic velocity law in the low-forcing limit.

(b) Critical regimes: (nonlinear) creep velocity law in the low-forcing limit.

Figure 8.2 – Typical forcing-velocity characteristics in the presence of (low) temperature.

Note that this result is very similar to that of Proposition 8.1.6, except that here the invariant measure is replaced by its viscous version (8.37). In order to describe the dynamical properties of this limiting model, we again investigate the behavior of the typical forcing-velocity curve: we consider a constant forcing vector $\hat{F} := F_0 \in \mathbb{R}^2$, we assume that $\hat{h}^0(x, x/\eta_\varepsilon) := \tilde{h}^0(x/\eta_\varepsilon)$ is independent of the macroscopic variable, we denote by $\Gamma_{\text{hom}}^{F_0,T_0} \in \mathbb{R}^2$ the corresponding averaged vector field (8.38), and we investigate the behavior of the velocity law $F_0 \mapsto V^{F_0,T_0} := -\Gamma_{\text{hom}}^{F_0,T_0}$. For large $|F_0|$, the picture is essentially the same as in the case without temperature $T_0 = 0$. However, since the viscous invariant measure $\tilde{\mu}^{F_0,T_0} \in \mathcal{P}(Q)$ does not vanish anywhere in the cell Q, we find $V^{F_0,T_0} \neq 0$ for all $F_0 \neq 0$, that is, in the presence of temperature $T_0 > 0$ the mass is always transported (at a reduced speed) and cannot get stuck forever in the local wells of the pinning potential. The precise behavior of V^{F_0,T_0} for F_0 close to 0 is then of particular interest. Heuristically, the forcing $F_0 \neq 0$ tilts the energy landscape, and the energy barriers of size $\operatorname{osc} \tilde{h}^0 := \max \tilde{h}^0 - \min \tilde{h}^0$ are then overcome by thermal activation even for small $F_0 \neq 0$. The velocity law for this so-called thermally assisted flux flow is then expected to satisfy the classical Arrhenius law from statistical thermodynamics (see e.g. [195, Section 5.1]),

$$V^{F_0,T_0} = C(1 + o(1)) \exp\Big(-\frac{C}{T_0} \operatorname{osc} \tilde{h}^0\Big) F_0, \qquad \text{as} \quad |F_0| \ll T_0 \ll 1, \tag{8.39}$$

that is, the response should be linear, but exponentially small as a function of T_0. This asymptotic law is related to the Eyring-Kramers formula, which has been rigorously established in any dimension [74, 237, 53]. Note that for the corresponding problem in dimension 1 (in the parabolic case $\beta = 0$) the averaged vector V^{F_0,T_0} can be explicitly computed, and the above law (8.39) is easily checked by hand. The typical forcing-velocity characteristics are plotted in Figure 8.2(a).

Critical regimes. In the critical regimes (GL_1) and (GL_2), the nonlinear interaction term can no longer be neglected, and we need to consider the homogenization limit of the complete thermal mean-field models (8.34)–(8.35), with $\nabla \hat{h}(x)$ replaced by $\nabla_2 \hat{h}^0(x, x/\eta_\varepsilon)$, and with critically scaled temperature $T := \eta_\varepsilon T_0$, $T_0 > 0$. In spite of the vanishing viscosity term, the rigorous justification of this homogenization limit remains very challenging due to the nonlinear nonlocal character of the mean-field models and to their instability as $\eta_\varepsilon \downarrow 0$. A purely formal 2-scale expansion yields the following heuristics for the asymptotic behavior of $\bar{\mathrm{v}}_\varepsilon$. Note that this coincides with Heuristics 8.1.7 except that here the invariant measures are replaced by viscous versions. Justifying the limit is again left as an open question. We refer to Section 8.9.4 and Remark 8.9.6 for detail.

Heuristics 8.1.10 (Critical regimes with temperature — formal asymptotics)**.** *For all* $w:\mathbb{R}^2\to\mathbb{R}^2$ *and* $x\in\mathbb{R}^2$*, consider the periodic vector field*

$$\Gamma_x[w] := (\alpha-\mathbb{J}\beta)(\nabla_2\hat h^0(x,\cdot)-\hat F(x)+2w^\perp(x)) \quad : \quad Q\to\mathbb{R}^2,$$

denote by $\tilde\mu_x^{T_0}[w]\in\mathrm{L}^\infty\cap\mathcal{P}_{\mathrm{per}}(Q)$ *the unique solution of the following equation on the* 2*-torus* Q*,*

$$T_0\triangle\tilde\mu_x^{T_0}[w]+\operatorname{div}(\Gamma_x[w]\tilde\mu_x^{T_0}[w])=0,$$

and define the averaged vector field

$$\Gamma_{\mathrm{hom}}^{T_0}[w](x) := \int_Q \Gamma_x[w](y)d\tilde\mu_x^{T_0}[w](y).$$

Let $\bar{\mathrm{v}}_\varepsilon$ *denote the unique global (smooth) solution of* (8.34) *or* (8.35) *with the vector field* $\nabla\hat h(x)$ *replaced by* $\nabla_2\hat h^0(x,x/\eta_\varepsilon)$*, and with* $T:=\eta_\varepsilon T_0$*,* $\eta_\varepsilon\ll 1$*, with* $\hat h^0$ *and* $T_0>0$ *independent of* ε*. Then the same asymptotic results should hold as in Heuristics 8.1.7, but with* $\Gamma_{\mathrm{hom}}[\cdot]$ *replaced by its better-behaved viscous version* $\Gamma_{\mathrm{hom}}^{T_0}[\cdot]$*.* ◊

Noting that the viscous invariant measures $\tilde\mu_x^{T_0}[w]$ depend smoothly on w — unlike the situation without temperature —, the well-posedness of the limiting equations for $\bar{\mathrm{v}}$ is now easily obtained. Again we are interested in the mean velocity law $F_0\mapsto V^{F_0,T_0}$ (defined as in (8.31)). The overall picture is essentially the same as in the subcritical regimes. However, as in the case without temperature, due to the competition between the pinning potential and the vortex interaction, the precise dynamical properties of $\bar{\mathrm{v}}$ are expected to change dramatically: the interacting vortices now move as a coherent whole, satisfying glassy properties [195]. The main manifestation of this difference is visible in the low-forcing low-temperature limit ($|F|\ll T_0\ll 1$), where the linear Arrhenius law (8.39) is now expected to break down, being replaced by the following so-called creep law, with stretched exponential dependence in the imposed forcing,

$$V^{F_0,T_0}=C(1+o(1))\exp\Big(-\frac{C}{T_0|F_0|^\mu}\Big),\qquad \text{as}\quad |F_0|\ll T_0\ll 1, \tag{8.40}$$

for some creep exponent $\mu>0$. This was first predicted by physicists for related elastic interface motion models [337, 251] and then adapted to vortex systems [180, 338, 196, 114, 115] (see also [195, Section 5] and references therein). The typical forcing-velocity curves are plotted in Figure 8.2(b). This particular glassy dynamical behavior is more generally expected to hold for any elastic object (here, a system of interacting vortices) that fluctuates in a heterogeneous medium, but even for simpler models no rigorous derivation is available. For an attempt at a mathematical approach to creep laws, we refer to [6]. Note that the crucial influence of the interactions on the dynamics is interestingly already exemplified in a simplified 1D model in [173, Section IV].

Infinite mobility limit and Bean's model

A further asymptotic limit may be considered in order to reduce the above limiting equations to simpler laws: let us assume that the forcing $\hat F$ is time-dependent, but varies on a much larger timescale than the vortex motion. More precisely, let us consider the following rescaling of the mean-field equations (8.34)–(8.35) for $\bar{\mathrm{v}}_\varepsilon$ with oscillating pinning potential and with critically scaled temperature $T:=\eta_\varepsilon T_0$: in the regime (GL$_1$),

$$\eta_\varepsilon\partial_t\bar{\mathrm{v}}_\varepsilon=\nabla\bar{\mathrm{p}}_\varepsilon+(\alpha-\mathbb{J}\beta)(\nabla_2^\perp\hat h^0(\cdot,\cdot/\eta_\varepsilon)-\hat F^\perp-2\bar{\mathrm{v}}_\varepsilon)\operatorname{curl}\bar{\mathrm{v}}_\varepsilon+\eta_\varepsilon T_0\triangle\bar{\mathrm{v}}_\varepsilon,\quad \operatorname{div}\bar{\mathrm{v}}_\varepsilon=0,\quad \bar{\mathrm{v}}_\varepsilon|_{t=0}=\mathrm{v}^\circ,$$

and in the regime (GL$_2$),

$$\eta_\varepsilon\partial_t\bar{\mathrm{v}}_\varepsilon=\alpha^{-1}\nabla(\hat a^{-1}\operatorname{div}(\hat a\bar{\mathrm{v}}_\varepsilon))+(\alpha-\mathbb{J}\beta)(\nabla_2^\perp\hat h^0(\cdot,\cdot/\eta_\varepsilon)-\hat F^\perp-2\bar{\mathrm{v}}_\varepsilon)\operatorname{curl}\bar{\mathrm{v}}_\varepsilon+\eta_\varepsilon T_0\triangle\bar{\mathrm{v}}_\varepsilon,\quad \bar{\mathrm{v}}_\varepsilon|_{t=0}=\mathrm{v}^\circ,$$

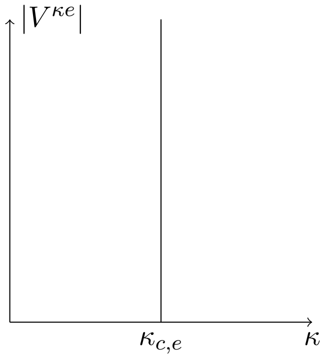


Figure 8.3 – In the Bean and Kim-Anderson models, the exact velocity law typically given by Figure 8.1 is replaced by this simplified law.

while in the subcritical regimes (GL_1')–(GL_2') we consider the corresponding equations without interaction term. In the case without temperature ($T_0 = 0$), in the timescale of variation of the forcing $\hat{F}$, as $\eta_\varepsilon \downarrow 0$, we may heuristically replace the velocity law plotted in Figure 8.1 by the simplified law pictured in Figure 8.3, meaning that the vortices have infinite mobility beyond the depinning threshold, hence rearrange themselves instantaneously. Such rate-independent limiting models are known as the Bean or the Kim-Anderson models; we refer to [108, Sections 6.3–6.4] and [388] for more detail. In the subcritical regimes (GL_1')–(GL_2'), for the model without interaction and without temperature ($T_0 = 0$), the convergence to a suitable rate-independent process is proved in any dimension in [404], while an approach to the corresponding case with temperature $T_0 > 0$ is proposed in [405]. Rigorously treating the critical regimes with interaction is much more delicate, and is not pursued here.

8.1.4 Perspectives and open questions

As explained, the modulated energy method does not make it possible to establish the mean-field limit result in the Gross-Pitaevskii case in the regime $1 \ll N_\varepsilon \lesssim |\log \varepsilon|$, nor in the parabolic Ginzburg-Landau case in the regime $N_\varepsilon \gtrsim |\log \varepsilon| \log |\log \varepsilon|$. In the first case, it seems related to the lack of a modulated energy proof for the mean-field limit of the corresponding conservative system of discrete Coulomb particles (cf. Section 6.1.5), and is left here as an open problem. In the second case, it is related to the failure of the usual weak-strong stability principle for the degenerate limiting equation (8.22) in the modulated energy metric. Another weak-strong principle is available for (8.22) in a stronger metric (cf. Proposition 7.5.1(ii)), but the possibility of using it for the desired mean-field limit result remains an open problem.

Let us now turn to open questions related to the homogenization limit. All the non-diagonal regimes beyond the scope of Corollary 8.1.5 remain open. In Proposition 8.9.13, we establish the pinning phenomenon by energy methods in the subcritical regimes with pinning force dominating the forcing; extending this to critical regimes would be quite interesting.

Another interesting question concerns the subcritical regimes and the possibility of devising a general homogenization theory for the washboard model of Proposition 8.1.6 (see also (8.257) and Theorem 8.9.8) in the ergodic stationary random setting — while here we restricted to the periodic case. For explicit Poisson-like pinning potentials, this model can actually be completely understood, and in particular the square-root power law (8.28) can be established at the depinning threshold under a simple non-degeneracy condition. We believe that the same should hold for more general ergodic stationary random pinning potentials. The simplification compared to the periodic case would result from the fact that the invariant measure is expected to be unique in the depinned regime. However, no developed theory seems to be available for invariant measures in this stationary setting, and we postpone these questions to a future investigation.

In the critical regimes, the vortex interaction can no longer be neglected, and we are left with a particularly subtle nonlinear homogenization question. The corresponding viscous setting (8.34)–(8.35) has the advantage of formally leading to a limiting PDE that is clearly well-posed (cf. Heuristics 8.1.10). However, two-scale compactness methods do not allow to solve this homogenization question even in the viscous case (cf. Remark 8.9.5(a)), and instability issues prevent any quantitative approach from succeeding.

Note that the homogenization problem is much simplified in the corresponding conservative case, that is, starting from (8.34) with $\alpha = 0$. Indeed, as explained in Remark 8.9.7, the vorticity $\bar{\mathrm{m}}_\varepsilon$ is then bounded, which leads to strong compactness of $\bar{\mathrm{v}}_\varepsilon$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)^2$ and allows to directly pass to the two-scale limit in the equation, thus proving Heuristics 8.1.10 in that case. In this simpler conservative setting, it would be interesting to consider the case without viscosity as well: the same two-scale argument can then be repeated, but the well-posedness for the obtained limiting equation (that is, (8.29) with $\alpha = 0$) remains open.

Beyond the derivation of the nonlinear homogenized equations (8.29)–(8.30), the next step would be to deduce the peculiar glassy properties (8.32)–(8.40) that they are expected to imply. This independent problem is expected to be very delicate on its own and is completely left open here.

In Section 8.1.3, following the use in physics (see e.g. [173, Section III.B]), we have proposed to phenomenologically incorporate thermal noise in the mean-field equations (8.17)–(8.20) as a viscosity effect. A natural question then consists in deriving this macroscopic viscosity from a suitable thermal variant of the mesoscopic Ginzburg-Landau model. Both in type-II superconductors [387, 243, 140, 141] and in Bose-Einstein condensates [401, 189, 190, 407], the effect of thermal and quantum noise in the Ginzburg-Landau or Gross-Pitaevskii equation are often modeled as a coupling to a heat bath, leading to the following (mixed-flow) stochastic Ginzburg-Landau equation (without pinning and forcing, for simplicity),

$$\lambda_\varepsilon(\alpha + i|\log\varepsilon|\beta)\partial_t u_\varepsilon = \Delta u_\varepsilon + \varepsilon^{-2} u_\varepsilon(1 - |u_\varepsilon|^2) + \Xi + i\xi u_\varepsilon + \Lambda u_\varepsilon, \tag{8.41}$$

where Ξ and ξ are respectively complex and real space-time white noises. More precisely, the solution u_ε of this equation is to be understood as the limit $\delta \downarrow 0$ of the solution $u_{\varepsilon,\delta}$ of a suitably renormalized equation,

$$\lambda_\varepsilon(\alpha + i|\log\varepsilon|\beta)\partial_t u_{\varepsilon,\delta} = \Delta u_{\varepsilon,\delta} + \varepsilon^{-2} u_{\varepsilon,\delta}(1 - |u_{\varepsilon,\delta}|^2) + \Xi_\delta + i\xi_\delta u_{\varepsilon,\delta} + \Lambda_{\varepsilon,\delta} u_{\varepsilon,\delta}, \tag{8.42}$$

where Ξ_δ and ξ_δ are regularizations of Ξ and ξ at the scale $\delta > 0$, and where $\Lambda_{\varepsilon,\delta}$ is a renormalization constant that suitably blows up as $\delta \downarrow 0$. Well-posedness for such models has recently been discussed in [245, 244] (see also [330]). On the other hand, in the context of Bose-Einstein condensates, a more accurate description of thermal noise is given by the so-called Zaremba-Nikuni-Griffin (ZNG) theory: thermal effects make the condensate not to be completely condensed, so that the condensate actually interacts with a non-condensed thermal cloud. The ZNG model thus consists in coupling the Gross-Pitaevskii equation for the condensate with a Boltzmann kinetic equation describing the thermal cloud (see e.g. [218, Section 3] or [257, Section II.A]). Formal derivations of the stochastic Ginzburg-Landau model (8.41) from the more accurate ZNG theory are available in the physics literature [401, 189, 190, 407].

An interesting question then concerns the understanding of the macroscopic diffusion of vortices starting from these mesoscopic thermal models. Formal derivations are given in [385, 170, 257, 372, 191], but a rigorous analysis is missing. In order to have enough regularity at our disposal to repeat modulated energy arguments, it is natural to first take the limit $\varepsilon \downarrow 0$ in (8.42) before passing to the limit in the white-noise regularization $\delta \downarrow 0$, thus considering regimes with $\varepsilon \ll \delta$. Nevertheless, it seems that in this regime the noise term does not lead to a classical brownian diffusion. For the original regime $\delta \ll \varepsilon$, regularity is missing, and clarifying these questions is left as an open question. There is also interest in directly starting from the ZNG model.

Finally, although we focus in this chapter on the vortex dynamics, we wish to briefly mention some interesting open questions in the stationary setting as well. The mean-field limit or leading-order behavior of (quasi)minimizers of the Ginzburg-Landau energy with a pinning weight (8.11) is examined in [5, 155, 154] in various settings, but we would rather like to comment on the next-order behavior, that is, on the vortex point configurations (almost) minimizing the renormalized Ginzburg-Landau energy [394], when the pin separation is of the same order as the vortex spacing. Without pinning weight, the celebrated 2D crystallization conjecture states that (quasi)minimizers should be given by Abrikosov's triangular lattice [64]. In the presence of a random pinning weight, on the other hand, the periodic lattice structure competes with the randomness of the pinning, but formal arguments by Giamarchi and Ledoussal [196] indicate that for small pinning intensity the positional crystalline order should not be lost, leading to the notion of Bragg glass (see e.g. [195, Sections 3–4] for an introduction). More recently, another observation was made by Le Thien, McDermott, Reichhardt, and Reichhardt [289]: for moderately small pinning intensity, whatever the distribution of pinning sites, simulations suggest that (almost) minimizing vortex positions always have hyperuniform statistics in the sense of Torquato and Stillinger [414, 429], that is, the variance of the number of points in a ball of size $R > 0$ is of order $o(|B_R|) = o(R^2)$ (or equivalently, the structure factor of the point configuration vanishes at 0). In other words, under the effect of random pinning, the positional crystalline order of the triangular lattice is destroyed beyond some critical value of the pinning intensity, while hyperuniformity is destroyed only beyond some higher critical value. Similarly as Bragg glasses, disordered hyperuniform matter shares both liquid-like and crystalline-like properties. Incidentally, this strong hyperuniform structure of vortex positions entails that for hyperuniform pinning sites the fraction of unoccupied sites is smaller, hence the critical current is higher, which is of great practical interest to design optimal pinning site geometries [289]. On the other hand, in the case without pinning, this discussion leads us to formulate the following much weaker version of the crystallization conjecture: stationary point processes (almost) minimizing the renormalized energy should at least be hyperuniform. Nevertheless, even this simpler version seems very difficult to establish (compare with [290, Lemma 3.10]).

8.2 Discussion of the mesoscopic model

For future reference, note that in each of the considered regimes (GL_1), (GL_2), (GL_3), (GL_1'), (GL_2'), and (GP), due to the explicit choice (8.7) of the zeroth-order term f, the following scalings hold, in the case $\eta_\varepsilon = 1$,

(a) *Dissipative case, general non-decaying setting:*

$$\|\nabla h\|_{W^{1,\infty}} \lesssim 1 \wedge \lambda_\varepsilon, \qquad \|F\|_{W^{1,\infty}} \lesssim \lambda_\varepsilon, \qquad \|f\|_{W^{1,\infty}} \lesssim 1 \wedge \lambda_\varepsilon + \lambda_\varepsilon^2 |\log \varepsilon|^2 \lesssim \lambda_\varepsilon^2 |\log \varepsilon|^2; \tag{8.43}$$

(b) *Gross-Pitaevskii case, decaying setting:*

$$\|\nabla h\|_{H^1 \cap W^{1,\infty}} \lesssim 1, \qquad \|F\|_{H^1 \cap W^{1,\infty}} \lesssim \lambda_\varepsilon, \qquad \|f\|_{H^1 \cap W^{1,\infty}} \lesssim 1 + \lambda_\varepsilon^2 |\log \varepsilon|^2 \lesssim N_\varepsilon^2. \tag{8.44}$$

8.2.1 Derivation of the modified mesoscopic model

In this section we justify the modified model (8.6) based on the 2D mixed-flow Ginzburg-Landau model (8.2) without gauge. For that purpose, as in [410, 397], we transform the rescaled order parameter $w_\varepsilon/\sqrt{a}$ in order to turn the Neumann boundary condition into a homogeneous one, which makes the applied electric current J_{ex} appear directly in the equation. For that purpose, we assume that $a = 1$ holds on the boundary $\partial\Omega$, and that the total incoming current equals the total outgoing current, that is, $\int_{\partial\Omega} n \cdot J_{\rm ex} = 0$. We then have $\int_{\partial\Omega} an \cdot J_{\rm ex} = 0$, so that there exists a unique solution $\psi \in H^1(\Omega)$ of

$$\begin{cases} \operatorname{div}(a\nabla\psi) = 0, & \text{in } \Omega, \\ n \cdot \nabla\psi = n \cdot J_{\rm ex}, & \text{on } \partial\Omega. \end{cases}$$

A straightforward computation shows that the transformed order parameter $u_\varepsilon := e^{-i|\log\varepsilon|\psi} w_\varepsilon/\sqrt{a}$ satisfies

$$\begin{cases} \lambda_\varepsilon(\alpha + i|\log\varepsilon|\beta)\partial_t u_\varepsilon = \triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2) \\ \qquad\qquad +\nabla h \cdot \nabla u_\varepsilon + i|\log\varepsilon|F^\perp \cdot \nabla u_\varepsilon + fu_\varepsilon, & \text{in } \mathbb{R}^+ \times \Omega, \\ n \cdot \nabla(u_\varepsilon\sqrt{a}) = 0, & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u_\varepsilon|_{t=0} = u_\varepsilon^\circ, \end{cases} \tag{8.45}$$

where we have set

$$h := \log a, \qquad F := -2\nabla^\perp\psi, \qquad \text{and} \qquad f := \frac{\triangle\sqrt{a}}{\sqrt{a}} - \frac{1}{4}|\log\varepsilon|^2|F|^2. \tag{8.46}$$

Note that the vector field F satisfies $\operatorname{div} F = \operatorname{curl}(aF) = 0$. In order to avoid delicate boundary issues, [1] a natural approach consists in sending the boundary $\partial\Omega$ to infinity and study the corresponding problem on the whole of $\mathbb{R}^2$. The assumption $a|_{\partial\Omega} = 1$ is then replaced by

$$a(x) \to 1 \quad (\text{that is, } h(x) \to 0), \qquad \text{and} \qquad \nabla h(x) \to 0, \qquad \text{uniformly as } |x| \uparrow \infty,$$

while F, f are simply assumed to be bounded. Noting that $2\nabla\sqrt{a} = \sqrt{a}\nabla h \to 0$ holds by assumption at infinity, the Neumann boundary condition in (8.45) formally translates into $\frac{x}{|x|} \cdot \nabla u_\varepsilon \to 0$ at infinity. Further imposing the natural condition $|u_\varepsilon| \to 1$ at infinity, we look for a global solution $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}$ of the corresponding equation (8.45) with fixed total degree $D_\varepsilon \in \mathbb{Z}$, and with

$$|u_\varepsilon| \to 1, \qquad \frac{x}{|x|} \cdot \nabla u_\varepsilon \to 0, \qquad \text{as } |x| \uparrow \infty, \qquad \text{and} \qquad \deg u_\varepsilon = D_\varepsilon.$$

In the dissipative case $\alpha > 0$, global existence and uniqueness of a solution $u_\varepsilon \in \mathrm{L}^\infty_{\rm loc}(\mathbb{R}^+; H^1_{\rm uloc}(\mathbb{R}^2;\mathbb{C}))$ is established in Appendix 8.A, as well as additional regularity, but, due to the possibly complicated advection structure at infinity caused by the non-decaying fields F, f, it is unclear whether the above properties at infinity are satisfied. In particular, it is not even clear whether the total degree of the constructed solution u_ε is well-defined. This difficulty originates in the possibility of instantaneous creation of many vortex dipoles at infinity for fixed $\varepsilon > 0$ due to forcing and pinning effects, although these dipoles are shown to necessarily disappear at infinity in the limit $\varepsilon \downarrow 0$ e.g. as a consequence of our mean-field results. Anyway, since a more precise description of u_ε at infinity is irrelevant for our purposes, it is not pursued here. Note that the global existence and uniqueness for u_ε in the uniformly locally integrable class is proved even without any decay assumption on h.

For simplicity, we may further choose to truncate the forcing F, f at infinity, thus focusing on the local behavior of the solution near the origin. In the Gross-Pitaevskii case, our results are limited to this decaying setting. Note that then at least one of the conditions $\operatorname{div} F = \operatorname{curl}(aF) = 0$ must be relaxed: we may for instance rather truncate ψ and define F via formula (8.46), so that only the

1. Another way to avoid boundary issues consists in rather considering the equation on the 2-torus. Nevertheless, the total degree of the map u_ε then necessarily vanishes, and hence, in order to describe a non-trivial vorticity with distinguished sign, we would have no other choice than working with the complete Ginzburg-Landau model with gauge. Working with the gauge actually does not change anything deep, but makes all computations even heavier, which we wanted to avoid for clarity.

condition div $F = 0$ is preserved. Since there is no advection at infinity in this setting, we prove existence and uniqueness of a solution u_ε in an affine space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U_\varepsilon + H^1(\mathbb{R}^2;\mathbb{C}))$, for some fixed smooth non-decaying "reference map" $U_\varepsilon : \mathbb{R}^2 \to \mathbb{C}$ satisfying $|U_\varepsilon| \to 1$ and $\frac{x}{|x|} \cdot \nabla U_\varepsilon \to 0$ as $|x| \uparrow \infty$. Given $D_\varepsilon \in \mathbb{Z}$, we typically choose $U_\varepsilon := U_{D_\varepsilon}$ smooth and equal to $e^{iD_\varepsilon \theta}$ (in polar coordinates) outside a neighborhood of the origin, which imposes for u_ε a fixed total degree equal to D_ε.

Remark 8.2.1. Rather than normalizing the original order parameter w_ε by the expected density $\sqrt{a}$, another natural choice was proposed by Lassoued and Mironescu [285], and consists in normalizing w_ε by a minimizer γ_ε of the weighted Ginzburg-Landau energy, that is, a nonvanishing solution of

$$\begin{cases} -\triangle \gamma_\varepsilon = \frac{\gamma_\varepsilon}{\varepsilon^2}(a - |\gamma_\varepsilon|^2), & \text{in } \Omega, \\ n \cdot \nabla \gamma_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

and setting $\tilde{u}_\varepsilon := e^{-i|\log \varepsilon|\psi} w_\varepsilon / \gamma_\varepsilon$ with ψ as before. This new order parameter $\tilde{u}_\varepsilon$ satisfies

$$\lambda_\varepsilon(\alpha + i|\log\varepsilon|\beta)\partial_t \tilde{u}_\varepsilon = \triangle \tilde{u}_\varepsilon + \frac{\gamma_\varepsilon^2 \tilde{u}_\varepsilon}{\varepsilon^2}(1 - |\tilde{u}_\varepsilon|^2) + \nabla \tilde{h} \cdot \nabla \tilde{u}_\varepsilon + i|\log\varepsilon| \tilde{F}^\perp \cdot \nabla \tilde{u}_\varepsilon + \tilde{f}\tilde{u}_\varepsilon,$$

in terms of $\tilde{h} := \log \gamma_\varepsilon^2$, $\tilde{F} := -2\nabla^\perp \psi$, and $\tilde{f} := -\frac{1}{4}|F|^2$. We are thus again reduced to a similar framework as the one above, and the results are easily adapted. $\Diamond$

8.2.2 Well-posedness of the modified mesoscopic model

In this section, we address global well-posedness of equation (8.6), both in the dissipative and in the Gross-Pitaevskii cases. In the dissipative case ($\alpha > 0$), a well-posedness result for (8.6) is established in the space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ for general non-decaying data, but no precise description of the solution at infinity is obtained, due to a possibly subtle advection structure at infinity. In particular, it is not even clear to us whether the total degree of the constructed solution is well-defined. In contrast, in the case of decaying data, no advection is allowed at infinity. As is classical since the work of Bethuel and Smets [59] (see also [323]), we then consider the existence of a solution u_ε of (8.6) in the space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U_\varepsilon + H^1(\mathbb{R}^2;\mathbb{C}))$ for some "reference map" U_ε, which is typically chosen smooth and equal (in polar coordinates) to $e^{iD_\varepsilon\theta}$ outside a ball at the origin, for some given $D_\varepsilon \in \mathbb{Z}$. Such a choice $U_\varepsilon = U_{D_\varepsilon}$ imposes a fixed total degree D_ε at infinity. More generally, we may consider the following set of "admissible" reference maps,

$$E_1(\mathbb{R}^2) := \{U \in \mathrm{L}^\infty(\mathbb{R}^2;\mathbb{C}) : \nabla^2 U \in H^1(\mathbb{R}^2;\mathbb{C}), \nabla|U| \in \mathrm{L}^2(\mathbb{R}^2), 1 - |U|^2 \in \mathrm{L}^2(\mathbb{R}^2), \\ \nabla U \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C})\ \forall p > 2\}.$$

Our global well-posedness results are summarized in the following; finer results and detailed proofs are given in Appendix 8.A, including additional regularity statements.

Proposition 8.2.2 (Well-posedness for (8.6))**.**

(i) Dissipative case ($\alpha > 0$, $\beta \in \mathbb{R}$), general non-decaying setting:
Let $h \in W^{1,\infty}(\mathbb{R}^2)$, $a := e^h$, $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, $f \in \mathrm{L}^\infty(\mathbb{R}^2)$, and $u_\varepsilon^\circ \in H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$. Then there exists a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ of (8.6) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u_ε°, and this solution satisfies $\partial_t u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$.

(ii) Gross-Pitaevskii case ($\alpha = 0$, $\beta \neq 0$), decaying setting:
Let $h \in W^{3,\infty}(\mathbb{R}^2)$, $\nabla h \in H^2(\mathbb{R}^2)^2$, $a := e^h$, $F \in H^3 \cap W^{3,\infty}(\mathbb{R}^2)^2$ with div $F = 0$*, $f \in H^2 \cap W^{2,\infty}(\mathbb{R}^2)$, and $u_\varepsilon^\circ \in U + H^2(\mathbb{R}^2;\mathbb{C})$ for some $U \in E_1(\mathbb{R}^2)$. Then there exists a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^2(\mathbb{R}^2;\mathbb{C}))$ of (8.6) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u_ε°, and this solution satisfies $\partial_t u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2;\mathbb{C}))$.* $\Diamond$

Proof. Item (i) follows from Proposition 8.A.2. We turn to item (ii). By Proposition 8.A.1(ii), the assumptions in the above statement ensure the existence of a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^2(\mathbb{R}^2;\mathbb{C}))$. This directly implies that $\triangle u_\varepsilon$, $\nabla h\cdot\nabla u_\varepsilon$, $F^\perp\cdot\nabla u_\varepsilon$, and fu_ε belong to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^2;\mathbb{C}))$. Using the Sobolev embedding of $H^1(\mathbb{R}^2)$ into $\mathrm{L}^4\cap\mathrm{L}^6(\mathbb{R}^2)$, and decomposing $u_\varepsilon(1-|u_\varepsilon|^2)$ in terms of $u_\varepsilon = U + \hat u_\varepsilon$ with $\hat u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^2;\mathbb{C}))$, we further deduce that $u_\varepsilon(1-|u_\varepsilon|^2)$ belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^2;\mathbb{C}))$. Inserting this into equation (8.6) yields the claimed integrability of $\partial_t u_\varepsilon$. □

Although a detailed proof of this well-posedness statement is included in Appendix 8.A, we include here a brief description of the strategy. In the dissipative case with decaying h, F, f, the arguments in [59, 323] are easily adapted to the present context with both pinning and forcing. The Gross-Pitaevskii regime is however more delicate, and we then use the structure of the equation to make a change of variables that usefully transforms the first-order terms into zeroth-order ones. The additional regularity assumptions in item (ii) above are precisely needed for this transformation to be well-behaved. Finally, the general result stated in item (i) for the dissipative case with non-decaying h, F, f, is deduced from the corresponding result with decaying h, F, f by a careful approximation argument in the space $H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$.

8.2.3 Case with gauge

In the dissipative case $\alpha > 0$, it is interesting to make the computations also in the case with gauge, which is the true physical model for superconductors as first derived by Schmid [386] and by Gor'kov and Eliashberg [216]. The evolution equation (8.2) is then replaced by the following, here written in the mixed-flow case, with strong (critically scaled) imposed current $|\log\varepsilon| J_{\mathrm{ex}} : \partial\Omega \to \mathbb{R}^2$ and imposed magnetic field $|\log\varepsilon| H_{\mathrm{ex}} : \partial\Omega \to \mathbb{R}$ at the boundary, and with a non-uniform pinning weight $a : \mathbb{R}^2 \to [0,1]$,

$$\begin{cases}\lambda_\varepsilon(\alpha + i|\log\varepsilon|\beta)(\partial_t w_\varepsilon - i w_\varepsilon \Psi_\varepsilon) = \nabla^2_{B_\varepsilon} w_\varepsilon + \frac{w_\varepsilon}{\varepsilon^2}(a - |w_\varepsilon|^2), & \text{in } \mathbb{R}^+\times\Omega,\\ \sigma(\partial_t B_\varepsilon - \nabla\Psi_\varepsilon) = \nabla^\perp \mathrm{curl}\, B_\varepsilon + \langle i w_\varepsilon, \nabla_{B_\varepsilon} w_\varepsilon\rangle, & \text{in } \mathbb{R}^+\times\Omega,\\ \mathrm{curl}\, B_\varepsilon = |\log\varepsilon| H_{\mathrm{ex}}, & \text{on } \mathbb{R}^+\times\partial\Omega,\\ n\cdot\nabla_{B_\varepsilon} w_\varepsilon = i w_\varepsilon |\log\varepsilon| n\cdot J_{\mathrm{ex}}, & \text{on } \mathbb{R}^+\times\partial\Omega,\\ w_\varepsilon|_{t=0} = w_\varepsilon^\circ, \end{cases}$$

where $B_\varepsilon : \mathbb{R}^+\times\mathbb{R}^2 \to \mathbb{R}^2$ now represents the gauge of the magnetic field $\mathrm{curl}\, B_\varepsilon$, where $\Psi_\varepsilon : \mathbb{R}^+\times\mathbb{R}^2 \to \mathbb{R}$ is the gauge of the electric field $-\partial_t B_\varepsilon + \nabla\Psi_\varepsilon$, where $\nabla_{B_\varepsilon} := \nabla - iB_\varepsilon$ is the usual covariant derivative, and where the real parameter $\sigma \ge 0$ characterizes the relaxation time of the magnetic field. We refer to [410] for a detailed discussion of the form of the boundary data. We are then interested in the asymptotic behavior of the supercurrent density $\langle \nabla_{B_\varepsilon}(w_\varepsilon/\sqrt{a}), i(w_\varepsilon/\sqrt{a})\rangle$, naturally obtained after rescaling the order parameter w_ε by the pinning weight. As in [410, 397], it is useful to further modify the rescaled order parameter $w_\varepsilon/\sqrt{a}$ in order to turn the boundary conditions into homogeneous ones, which then makes the imposed current and magnetic field J_{ex} and H_{ex} appear directly in the equation. Further, for simplicity, in order to avoid boundary issues, under similar assumptions on a as in Section 8.2.1, we may formally send the boundary $\partial\Omega$ to infinity and study the corresponding problem on the whole of $\mathbb{R}^2$. Without explicitly describing the transformation (which includes a choice of the gauge Ψ_ε; we refer to [397, Section 2] for detail), the transformed couple $(u_\varepsilon, A_\varepsilon)$ replacing the triplet $(w_\varepsilon, B_\varepsilon, \Psi_\varepsilon)$ then satisfies the following equation,

$$\begin{cases}\lambda_\varepsilon(\alpha + i|\log\varepsilon|\beta)\partial_t u_\varepsilon = \nabla^2_{A_\varepsilon} u_\varepsilon + \frac{a u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2) & \\ \qquad\qquad + \nabla h\cdot\nabla_{A_\varepsilon} u_\varepsilon + i|\log\varepsilon| F^\perp\cdot\nabla_{A_\varepsilon} u_\varepsilon + f u_\varepsilon, & \text{in } \mathbb{R}^+\times\Omega,\\ \sigma\partial_t A_\varepsilon = \nabla^\perp \mathrm{curl}\, A_\varepsilon + a\langle i u_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon\rangle - \frac12 |\log\varepsilon| a F^\perp(1-|u_\varepsilon|^2), & \text{in } \mathbb{R}^+\times\Omega,\\ u_\varepsilon|_{t=0} = u_\varepsilon^\circ, \end{cases}$$

where $h := \log a$, and where F and f are given explicitly in terms of a, J_{ex} and H_{ex}. Natural quantities associated with this transformed model are the gauge-invariant supercurrent and vorticity,

$$j_\varepsilon := \langle \nabla_{A_\varepsilon} u_\varepsilon, iu_\varepsilon \rangle, \qquad \mu_\varepsilon := \operatorname{curl}(j_\varepsilon + A_\varepsilon),$$

and the electric field

$$E_\varepsilon := -\partial_t A_\varepsilon.$$

We believe that the derivation of mean-field limit results from this gauged version of the model (8.6) does not cause any major difficulty, and can be achieved following the kind of computations performed in [395, Appendix C]. Formally, the corresponding results to Theorems 8.1.2 and 8.1.3 are the convergences

$$\frac{j_\varepsilon}{N_\varepsilon} \to \mathrm{v}, \qquad \frac{\mu_\varepsilon}{N_\varepsilon} \to \mathrm{m} := \operatorname{curl} \mathrm{v} + \mathrm{H}, \qquad \frac{\operatorname{curl} A_\varepsilon}{N_\varepsilon} \to \mathrm{H}, \qquad \frac{E_\varepsilon}{N_\varepsilon} \to \mathrm{E},$$

where the limiting triplet (v, H, E) satisfies, in the regime (GL_1),

$$\begin{cases} \partial_t \mathrm{v} - \mathrm{E} = \nabla \mathrm{p} + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v})\,\mathrm{m}, \\ \operatorname{div} \mathrm{v} = 0, \\ -\sigma \mathrm{E} = \mathrm{v} + \nabla^\perp \mathrm{H}, \\ \partial_t \mathrm{H} = -\operatorname{curl} \mathrm{E}, \end{cases} \tag{8.47}$$

or in the regime (GL_2),

$$\begin{cases} \partial_t \mathrm{v} - \mathrm{E} = \alpha^{-1} \nabla(\hat{a}^{-1} \operatorname{div}(\hat{a}\mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\lambda \mathrm{v})\,\mathrm{m}, \\ -\sigma \mathrm{E} = \mathrm{v} + \nabla^\perp \mathrm{H}, \\ \partial_t \mathrm{H} = -\operatorname{curl} \mathrm{E}, \end{cases} \tag{8.48}$$

or in the regime (GL_3) with $\alpha = 1$, $\beta = 0$,

$$\begin{cases} \partial_t \mathrm{v} - \mathrm{E} = -(\hat{F}^\perp + 2\lambda \mathrm{v})\,\mathrm{m}, \\ -\sigma \mathrm{E} = \mathrm{v} + \nabla^\perp \mathrm{H}, \\ \partial_t \mathrm{H} = -\operatorname{curl} \mathrm{E}, \end{cases} \tag{8.49}$$

while in the subcritical regimes (GL_1')–(GL_2') the mean-field equations are obtained from (8.47)–(8.48) by removing the nonlinear interaction terms vm. The structure of these equations is maybe more transparent at the level of the vorticity $\mathrm{m} := \operatorname{curl} \mathrm{v} + \mathrm{H}$: the system (8.47) takes the form

$$\begin{cases} \partial_t \mathrm{m} = \operatorname{div}\big((\alpha - \mathbb{J}\beta)(\nabla \hat{h} - \hat{F} + 2\mathrm{v}^\perp)\,\mathrm{m}\big), \\ \sigma \partial_t \mathrm{H} - \triangle \mathrm{H} + \mathrm{H} = \mathrm{m}, \\ \operatorname{div} \mathrm{v} = 0, \quad \operatorname{curl} \mathrm{v} = \mathrm{m} - H \end{cases}$$

while the system (8.48) becomes for $\sigma > 0$, setting in addition $\mathrm{d} := \operatorname{div}(\hat{a}\mathrm{v})$,

$$\begin{cases} \partial_t \mathrm{m} = \operatorname{div}\big((\alpha - \mathbb{J}\beta)(\nabla \hat{h} - \hat{F} + 2\mathrm{v}^\perp)\,\mathrm{m}\big), \\ \partial_t \mathrm{d} - \alpha^{-1} \triangle \mathrm{d} + \alpha^{-1} \operatorname{div}(\mathrm{d} \nabla \hat{h}) + \frac{1}{\sigma} \mathrm{d} = -\frac{1}{\sigma} \hat{a} \nabla \hat{h} \cdot \nabla^\perp H + \operatorname{div}\big((\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\lambda \mathrm{v})\hat{a}\mathrm{m}\big), \\ \sigma \partial_t \mathrm{H} - \triangle \mathrm{H} + \mathrm{H} = \mathrm{m}, \\ \operatorname{div}(\hat{a}\mathrm{v}) = \mathrm{d}, \quad \operatorname{curl} \mathrm{v} = \mathrm{m} - H, \end{cases}$$

that is a transport equation for m, coupled with a linear heat equation for H, and in the case (8.48) further coupled with a transport-diffusion equation for the divergence $\mathrm{d} := \operatorname{div}(\hat{a}\mathrm{v})$. In the case (8.49), the transport-diffusion equation becomes degenerate: in terms of $\theta := \operatorname{div} \mathrm{v}$,

$$\begin{cases} \partial_t \mathrm{m} = \operatorname{div}\big((-\hat{F} + 2\mathrm{v}^\perp)\,\mathrm{m}\big), \\ \partial_t \theta + \frac{1}{\sigma}\theta = -\operatorname{div}\big((\hat{F}^\perp + 2\lambda \mathrm{v})\hat{a}\mathrm{m}\big), \\ \sigma \partial_t \mathrm{H} - \triangle \mathrm{H} + \mathrm{H} = \mathrm{m}, \\ \operatorname{div} \mathrm{v} = \theta, \quad \operatorname{curl} \mathrm{v} = \mathrm{m} - H, \end{cases}$$

In the rest of this chapter, we focus for simplicity on the model without gauge (8.6).

8.3 Preliminaries on the limiting equations

As already explained, it is convenient to first compare the rescaled supercurrent density $j_\varepsilon/N_\varepsilon$ with an intermediate ε-dependent map $\mathrm{v}_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$, which is better adapted to the ε-dependence of the pinning potential and will in a second step be shown to converge to the correct limit v. In all considered regimes, we derive equations for v_ε of the form

$$\partial_t \mathrm{v}_\varepsilon = \nabla \mathrm{p}_\varepsilon + \Gamma_\varepsilon \operatorname{curl} \mathrm{v}_\varepsilon, \qquad \mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ, \tag{8.50}$$

for some smooth pressure $\mathrm{p}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$, and some smooth vector field $\Gamma_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$. The pressure will either be proportional to $a^{-1} \operatorname{div}(a\mathrm{v}_\varepsilon)$, or be the Lagrange multiplier associated with the constraint $\operatorname{div}(a\mathrm{v}_\varepsilon) = 0$. Before Section 8.6, we only manipulate these quantities $\mathrm{v}_\varepsilon, \mathrm{p}_\varepsilon, \Gamma_\varepsilon$ formally, while the suitable choices will be exploited later. In order to ensure that all our computations are licit, we need to work under the following integrability and smoothness assumptions.

Assumption 8.3.1.

(a) *Dissipative case ($\alpha > 0$, $\beta \in \mathbb{R}$):*
There exists some $T > 0$ such that for all $\varepsilon > 0$, all $t \in [0, T)$, and all $q > 2$,

$$\begin{gathered} \|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2 + \mathrm{L}^q) \cap \mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\operatorname{curl} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1 \cap \mathrm{L}^\infty} \lesssim_t 1, \quad \|\operatorname{div}(a\mathrm{v}_\varepsilon^t)\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim_t 1, \\ \|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{-1/2} \wedge \lambda_\varepsilon^{-1}, \quad \|\nabla \mathrm{p}_\varepsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim_t 1 \wedge \lambda_\varepsilon^{-1}, \\ \|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim_t 1 + \lambda_\varepsilon^{-1/2}, \quad \|\partial_t \mathrm{v}_\varepsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim_t 1, \quad \|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim_t \lambda_\varepsilon^{-1}, \\ \|\Gamma_\varepsilon^t\|_{W^{1,\infty}} \lesssim_t 1, \quad \|\partial_t \Gamma_\varepsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim_t 1. \end{gathered}$$

(b) *Gross-Pitaevskii case ($\alpha = 0$, $\beta \neq 0$):*
There exists some $T > 0$ such that for all $\varepsilon > 0$, all $t \in [0, T)$, and all $2 < q < \infty$,

$$\begin{gathered} \|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2 + \mathrm{L}^q) \cap \mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\operatorname{curl} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1 \cap \mathrm{L}^\infty} \lesssim_t 1 \\ \|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q \cap \mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim_t 1, \quad \|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1, \quad \|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q} \lesssim_{t,q} 1, \\ \|\Gamma_\varepsilon^t\|_{W^{1,\infty}} \lesssim_t 1, \quad \|\partial_t \Gamma_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1. \end{gathered}$$

$\diamondsuit$

In the dissipative case of Theorem 8.1.2 the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon$ is shown in Section 8.6 to remain close to the solution v_ε of the following equation,

$$\begin{gathered} \partial_t \mathrm{v}_\varepsilon = \nabla \mathrm{p}_\varepsilon + \Gamma_\varepsilon \operatorname{curl} \mathrm{v}_\varepsilon, \qquad \mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ, \\ \Gamma_\varepsilon := \lambda_\varepsilon^{-1}(\alpha - \mathbb{J}\beta)\Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log \varepsilon|}\, \mathrm{v}_\varepsilon\Big), \qquad \mathrm{p}_\varepsilon := (\lambda_\varepsilon \alpha a)^{-1} \operatorname{div}(a\mathrm{v}_\varepsilon), \end{gathered} \tag{8.51}$$

while in the superdense parabolic case of Theorem 8.1.3 the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon$ is shown in Section 8.8 to remain close to the solution v_ε of the following equation,

$$\begin{aligned} \partial_t \mathrm{v}_\varepsilon &= \nabla \mathrm{p}_\varepsilon + \Gamma_\varepsilon \operatorname{curl} \mathrm{v}_\varepsilon, \qquad \mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}^\circ, \\ \Gamma_\varepsilon &:= \lambda_\varepsilon^{-1}\Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log \varepsilon|}\mathrm{v}_\varepsilon\Big), \qquad \mathrm{p}_\varepsilon := (\lambda_\varepsilon a)^{-1} \operatorname{div}(a \mathrm{v}_\varepsilon), \end{aligned} \tag{8.52}$$

and while in the Gross-Pitaevskii case of Theorem 8.1.4 the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon$ is shown in Section 8.7 to remain close to the solution v_ε of the following equation,

$$\begin{aligned} \partial_t \mathrm{v}_\varepsilon &= \nabla \mathrm{p}_\varepsilon + \Gamma_\varepsilon \operatorname{curl} \mathrm{v}_\varepsilon, \qquad \operatorname{div}(a \mathrm{v}_\varepsilon) = 0, \qquad \mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ, \\ \Gamma_\varepsilon &:= -\lambda_\varepsilon^{-1}\Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log \varepsilon|}\mathrm{v}_\varepsilon\Big)^\perp. \end{aligned} \tag{8.53}$$

In the present section, we show that the solutions v_ε of the above equations (8.51), (8.52), and (8.53) exist and satisfy all the properties of Assumption 8.3.1. Using the choice of the scalings for $\lambda_\varepsilon, h, F$ in each regime, we further show how to pass to the limit $\varepsilon \downarrow 0$ in these equations, which is needed to conclude the proofs of Theorems 8.1.2, 8.1.3, and 8.1.4. Note that in the regimes (GL_1) and (GL_2'), as a consequence of the choice $\lambda_\varepsilon \downarrow 0$, we expect the solution v_ε of (8.51) to converge to the solution v of some incompressible equation with the constraint $\operatorname{div} \mathrm{v} = 0$. We thus naturally refer to (GL_1), (GL_2') and (GP) as the *incompressible regimes*, and to (GL_2) and (GL_1') as the *compressible regimes*. In contrast, the choice of $\lambda_\varepsilon \uparrow \infty$ in the superdense parabolic regime (GL_3) leads to a degenerate equation, so that we refer to (GL_3) as the *degenerate parabolic regime*, while the other dissipative regimes are called *non-degenerate*. (In particular, we establish in the present section for regular initial data the continuity of the solutions of the mean-field models (7.2) with respect to the parameter λ on $(0, \infty]$ in the dissipative case, and on $[0, \infty]$ in the parabolic case.)

8.3.1 Non-degenerate dissipative case

It is instructive to examine the vorticity formulation of the equation (8.51) for v_ε. In terms of $\mathrm{m}_\varepsilon := \operatorname{curl} \mathrm{v}_\varepsilon$ and $\mathrm{d}_\varepsilon := \operatorname{div}(a \mathrm{v}_\varepsilon)$, equation (8.51) may be rewritten as a nonlinear nonlocal transport equation for the vorticity m_ε, coupled with a transport-diffusion equation for the divergence d_ε,

$$\begin{cases} \partial_t \mathrm{m}_\varepsilon = -\operatorname{div}(\Gamma_\varepsilon^\perp \mathrm{m}_\varepsilon), \quad \mathrm{m}_\varepsilon|_{t=0} = \operatorname{curl} \mathrm{v}_\varepsilon^\circ, \\ \partial_t \mathrm{d}_\varepsilon - (\alpha\lambda_\varepsilon)^{-1} \triangle \mathrm{d}_\varepsilon + (\alpha\lambda_\varepsilon)^{-1} \operatorname{div}(\mathrm{d}_\varepsilon \nabla h) = \operatorname{div}(a \Gamma_\varepsilon \mathrm{m}_\varepsilon), \quad \mathrm{d}_\varepsilon|_{t=0} = \operatorname{div}(a \mathrm{v}_\varepsilon^\circ), \\ \operatorname{curl} \mathrm{v}_\varepsilon = \mathrm{m}_\varepsilon, \quad \operatorname{div}(a \mathrm{v}_\varepsilon) = \mathrm{d}_\varepsilon\,. \end{cases} \tag{8.54}$$

A detailed study of this kind of equations is performed in Chapter 7, including global existence results for vortex-sheet initial data. The following proposition in particular states that a solution v_ε always exists and satisfies the various properties of Assumption 8.3.1(a) under suitable regularity assumptions on the initial data $\mathrm{v}_\varepsilon^\circ$. Compared with Chapter 7, this result however requires some more work in the incompressible cases $\lambda_\varepsilon \downarrow 0$, since it is then needed to make clear the link with the limiting incompressible equations, in particular in order to establish global existence in the mixed-flow case.

Proposition 8.3.2. *Let $\alpha > 0$, $\beta \in \mathbb{R}$, $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, and let $\mathrm{v}_\varepsilon^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ be bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$, and satisfy $\operatorname{curl} \mathrm{v}_\varepsilon^\circ \in \mathcal{P}(\mathbb{R}^2)$. For some $s > 0$, assume that $h \in W^{s+3,\infty}(\mathbb{R}^2)$, $F \in W^{s+2,\infty}(\mathbb{R}^2)^2$, that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{s+2,\infty}(\mathbb{R}^2)^2$, and that $\operatorname{curl} \mathrm{v}_\varepsilon^\circ$ and $\operatorname{div}(a \mathrm{v}_\varepsilon^\circ)$ are bounded in $H^{s+1}(\mathbb{R}^2)$.*

(i) Non-degenerate compressible regimes $\lambda_\varepsilon \simeq 1$ (that is, (GL_2)–(GL_1')):
There exist $T > 0$ (independent of ε) and a unique (local) solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{v}_\varepsilon^\circ + H^2 \cap$

$W^{2,\infty}(\mathbb{R}^2)^2)$ *of* (8.51) *on* $[0,T)\times\mathbb{R}^2$. *Moreover, all the properties of Assumption 8.3.1(a) are satisfied, that is, for all* $t\in[0,T)$ *and all* $q>2$,

$$\|(\mathrm{v}_\varepsilon^t,\nabla\mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap\mathrm{L}^\infty}\lesssim_{t,q}1,\quad \|\mathrm{curl}\,\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}\lesssim_t 1,\quad \|\mathrm{div}\,(a\mathrm{v}_\varepsilon^t)\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim_t 1,$$
$$\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim_t 1,\quad \|\nabla\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2}\lesssim_t 1,\quad \|\partial_t\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim_t 1,\quad \|\partial_t\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2_t\mathrm{L}^2}\lesssim_t 1.$$

In the parabolic case $\beta=0$, *the solution* v_ε *can be extended globally, i.e.* $T=\infty$. *In the small-interaction regime* (GL_1'), *in the mixed-flow case* $\beta\neq0$, *the existence time* T *can be taken arbitrarily large for* $\varepsilon>0$ *small enough.*

(ii) Incompressible regimes $\lambda_\varepsilon\ll1$ (that is, (GL_1)–(GL_2')):
Further assume $\mathrm{div}\,(a\mathrm{v}_\varepsilon^\circ)=0$. *There exist* $T>0$ *(independent of* ε*) and a unique (local) solution* $\mathrm{v}_\varepsilon\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{v}_\varepsilon^\circ+H^2\cap W^{2,\infty}(\mathbb{R}^2)^2)$ *of* (8.51) *on* $\mathbb{R}^+\times\mathbb{R}^2$. *Moreover, all the properties of Assumption 8.3.1(a) are satisfied, that is, for all* $t\in[0,T)$ *and all* $q>2$,

$$\|(\mathrm{v}_\varepsilon^t,\nabla\mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap\mathrm{L}^\infty}\lesssim_{t,q}1,\quad \|\mathrm{curl}\,\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}\lesssim_t 1,\quad \|\mathrm{div}\,(a\mathrm{v}_\varepsilon^t)\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim_t 1,$$
$$\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim_t\lambda_\varepsilon^{-1/2},\quad \|\nabla\mathrm{p}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2}\lesssim_t 1,\quad \|\partial_t\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2_t\mathrm{L}^2}\lesssim_t\lambda_\varepsilon^{-1},$$
$$\|\partial_t\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim_t\lambda_\varepsilon^{-1/2},\quad \|\partial_t\mathrm{v}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2}\lesssim_t 1.$$

In the parabolic case $\beta=0$, *the solution* v_ε *can be extended globally, i.e.* $T=\infty$. *In the mixed-flow case* $\beta\neq0$, *the existence time* T *can be taken arbitrarily large for* $\varepsilon>0$ *small enough.* ◊

Proof. Item (i) is proved in Step 1 below, while the proof of (ii) is split into three further steps. The proof of the global existence for the regime (GL_1'), also stated in (i), is postponed to the last step.

Step 1. Non-degenerate compressible regimes.

Let $s>0$ be non-integer. The assumption $\|\hat h\|_{W^{s+3,\infty}},\ \|\hat F\|_{W^{s+2,\infty}}\lesssim1$ yields $\|\lambda_\varepsilon^{-1}(\nabla^\perp h-F^\perp)\|_{W^{s+2,\infty}}\lesssim1$ in the considered regimes, and also $\lambda_\varepsilon^{-1}N_\varepsilon/|\log\varepsilon|\lesssim1$ and $\lambda_\varepsilon\simeq1$. Further using the assumptions on the initial data $\mathrm{v}_\varepsilon^\circ$, Theorems 7.1.4 and 7.1.5 in Chapter 7 imply that in each of the compressible regimes (GL_2)–(GL_1') there exists a unique (local) solution $\mathrm{v}_\varepsilon\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{v}_\varepsilon^\circ+H^2\cap W^{2,\infty}(\mathbb{R}^2)^2)$ of (8.51) on $[0,T)\times\mathbb{R}^2$ with initial data $\mathrm{v}_\varepsilon^\circ$, for some $T\gtrsim1$. Moreover, it is shown in Chapter 7 that this solution satisfies for all $t\in[0,T)$,

$$\|\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ\|_{H^2\cap W^{2,\infty}}\lesssim_t1,\qquad \|(\mathrm{m}_\varepsilon^t,\mathrm{d}_\varepsilon^t)\|_{H^1\cap W^{1,\infty}}\lesssim_t1,\qquad \int_{\mathbb{R}^2}\mathrm{m}_\varepsilon^t=1,\qquad \mathrm{m}_\varepsilon^t\geq0.\tag{8.55}$$

Note that in the parabolic case ($\alpha=1$, $\beta=0$) Theorem 7.1.3 actually gives a global solution, i.e. $T=\infty$. We claim that all the desired properties of v_ε follow from (8.55). Combining (8.55) with the assumption that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q>2$, we find

$$\|(\mathrm{v}_\varepsilon^t,\nabla\mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap\mathrm{L}^\infty}\lesssim_{t,q}1.$$

The choice $\mathrm{p}_\varepsilon=(\lambda_\varepsilon\alpha a)^{-1}\,\mathrm{d}_\varepsilon$ with $\lambda_\varepsilon\simeq1$ leads to

$$\|\mathrm{p}_\varepsilon^t\|_{H^1\cap W^{1,\infty}}\lesssim\|\mathrm{d}_\varepsilon^t\|_{H^1\cap W^{1,\infty}}\lesssim_t1.$$

Inserting this information into equation (8.51), we deduce

$$\|\partial_t\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim\|\nabla\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}+\|\Gamma_\varepsilon^t\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim_t1.$$

Testing the transport-diffusion equation $\partial_t\mathrm{d}_\varepsilon-(\lambda_\varepsilon\alpha)^{-1}(\triangle\mathrm{d}_\varepsilon-\mathrm{div}\,(\mathrm{d}_\varepsilon\nabla h))=\mathrm{div}\,(a\Gamma_\varepsilon\mathrm{m}_\varepsilon)$ against $\partial_t\mathrm{d}_\varepsilon$ yields

$$\int_{\mathbb{R}^2}|\partial_t\mathrm{d}_\varepsilon|^2+\frac12(\lambda_\varepsilon\alpha)^{-1}\partial_t\int_{\mathbb{R}^2}|\nabla\mathrm{d}_\varepsilon|^2=-\int_{\mathbb{R}^2}\partial_t\mathrm{d}_\varepsilon\,\mathrm{div}\,\big((\lambda_\varepsilon\alpha)^{-1}\mathrm{d}_\varepsilon\nabla h-a\Gamma_\varepsilon\mathrm{m}_\varepsilon\big),$$

and hence, integrating in time, with $\lambda_\varepsilon \simeq 1$,

$$\|\partial_t \mathrm{d}_\varepsilon\|^2_{\mathrm{L}^2_t \mathrm{L}^2} + \frac{1}{2}(\lambda_\varepsilon \alpha)^{-1} \|\nabla \mathrm{d}_\varepsilon\|^2_{\mathrm{L}^2} \lesssim \|\nabla \mathrm{d}^\circ_\varepsilon\|^2_{\mathrm{L}^2} + \|\partial_t \mathrm{d}_\varepsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} (\|\mathrm{d}_\varepsilon\|_{\mathrm{L}^2_t H^1} + \|a\Gamma_\varepsilon\|_{\mathrm{L}^\infty_t W^{1,\infty}} \|\mathrm{m}_\varepsilon\|_{\mathrm{L}^2_t H^1})$$
$$\lesssim_t 1 + \|\partial_t \mathrm{d}_\varepsilon\|_{\mathrm{L}^2_t \mathrm{L}^2}.$$

Absorbing the last right-hand side term, we conclude

$$\|\partial_t \mathrm{p}_\varepsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim \|\partial_t \mathrm{d}_\varepsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim_t 1. \tag{8.56}$$

All the stated estimates follow.

Step 2. Estimates for transport-diffusion equations with large diffusivity.

In the incompressible regimes (GL_1) and (GL'_2), the conclusion does not follow as in Step 1, because the corresponding choice $\mathrm{p}_\varepsilon = (\lambda_\varepsilon \alpha a)^{-1} \operatorname{div}(a\mathrm{v}_\varepsilon)$ contains the large prefactor $(\lambda_\varepsilon\alpha)^{-1} \gg 1$. In particular, equation (8.54) for the divergence $\mathrm{d}_\varepsilon := \operatorname{div}(a\mathrm{v}_\varepsilon)$ now takes the form

$$\partial_t \mathrm{d}_\varepsilon - (\lambda_\varepsilon\alpha)^{-1} \triangle \mathrm{d}_\varepsilon + \alpha^{-1} \operatorname{div}(\mathrm{d}_\varepsilon \nabla \hat{h}) = \operatorname{div}(a\Gamma_\varepsilon \mathrm{m}_\varepsilon), \tag{8.57}$$

with a large prefactor $(\lambda_\varepsilon\alpha)^{-1} \gg 1$ in front of the Laplacian, and with initial data $\mathrm{d}^\circ_\varepsilon := \operatorname{div}(a\mathrm{v}^\circ_\varepsilon) = 0$. In this step, we consider the model transport-diffusion equation

$$\partial_t w - \nu \triangle w + \operatorname{div}(w \nabla \hat{h}) = \operatorname{div} g, \qquad w|_{t=0} = 0,$$

with large diffusivity $\nu \gg 1$. Using that the initial condition is chosen to be zero, a direct adaptation of Lemma 7.2.3 gives the following bounds: for all $\nu \gtrsim 1$,

(a) for all $s \ge 0$, $t \ge 0$, for some constant C depending only on an upper bound on s and $\|\nabla \hat{h}\|_{W^{s,\infty}}$,

$$\|w^t\|_{H^s} + \nu^{1/2} \|\nabla w\|_{\mathrm{L}^2_t H^s} \le C(t/\nu)^{1/2} e^{Ct/\nu} \|g\|_{\mathrm{L}^\infty_t H^s} \le Ct^{1/2} e^{Ct} \|g\|_{\mathrm{L}^\infty_t H^s};$$

(b) for some constant C depending only on an upper bound on $\|\nabla \hat{h}\|_{\mathrm{L}^\infty}$,

$$\|w^t\|_{\dot{H}^{-1}} \le Ce^{Ct} \|g\|_{\mathrm{L}^2_t \mathrm{L}^2};$$

(c) for all $1 \le p, q \le \infty$, $t \ge 0$, for some constant C depending only on an upper bound on $\|\nabla \hat{h}\|_{\mathrm{L}^\infty}$,

$$\|w\|_{\mathrm{L}^p_t \mathrm{L}^q} \le C(t/\nu)^{1/2} e^{C(t/\nu)^2} \|g\|_{\mathrm{L}^p_t \mathrm{L}^q} \le Ct^{1/2} e^{Ct^2} \|g\|_{\mathrm{L}^p_t \mathrm{L}^q}.$$

In particular, the same bounds as in Lemma 7.2.3 hold uniformly with respect to the large diffusivity $\nu \gg 1$. Further adapting the proof of (8.56) in Step 1 above, we easily find

(d) for some constant C depending only on an upper bound on $\|\nabla \hat{h}\|_{W^{1,\infty}}$,

$$\|\partial_t w\|_{\mathrm{L}^2_t \mathrm{L}^2} \le \|\nabla g\|_{\mathrm{L}^2_t \mathrm{L}^2} + C(t/\nu)^{1/2} e^{Ct/\nu} \|g\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \le Ct^{1/2} e^{Ct} \|g\|_{\mathrm{L}^\infty_t H^1}.$$

Step 3. Incompressible regimes.

In the vorticity formulation (8.54), the large prefactor $(\lambda_\varepsilon\alpha)^{-1} \gg 1$ does not affect the equation for the vorticity m_ε, but only the equation for the divergence d_ε, which now takes the form (8.57). However, for the choice $\mathrm{d}^\circ_\varepsilon = 0$, the result of Step 2 ensures that the estimates for d_ε used in Chapter 7 hold uniformly with respect to the large prefactor. Hence, as in Step 1, using the assumptions on the initial data, the proof of Theorems 7.1.4 and 7.1.5 imply that in the incompressible regimes (GL_1) and (GL'_2) there exists a unique (local) solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{v}^\circ + H^2 \cap W^{2,\infty}(\mathbb{R}^2)^2)$ of (8.51) on

$[0,T)\times\mathbb{R}^2$ with initial data v°, for some $T\gtrsim 1$. Moreover, it is shown in Chapter 7 that this solution satisfies for all $t\in[0,T)$,

$$\|\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ\|_{H^2\cap W^{2,\infty}}\lesssim_t 1,\qquad \|(\mathrm{m}_\varepsilon^t,\mathrm{d}_\varepsilon^t)\|_{H^1\cap W^{1,\infty}}\lesssim_t 1,\qquad \int_{\mathbb{R}^2}\mathrm{m}_\varepsilon^t=1,\qquad \mathrm{m}_\varepsilon^t\geq 0. \tag{8.58}$$

Note that in the parabolic case ($\alpha=1$, $\beta=0$) Theorem 7.1.3 actually gives a global solution, i.e. $T=\infty$. We claim that all the desired properties of v_ε follow from (8.58). By definition (8.51), we find $\|\Gamma_\varepsilon^t\|_{W^{1,\infty}}\lesssim_t 1$. Combining (8.58) with the assumption that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q>2$, we obtain

$$\|(\mathrm{v}_\varepsilon^t,\nabla\mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap\mathrm{L}^\infty}\lesssim_{t,q} 1.$$

Using (8.51) in the form $\mathrm{p}_\varepsilon=(\lambda_\varepsilon\alpha a)^{-1}\,\mathrm{d}_\varepsilon$, and applying items (a)–(c) of Step 2, we find

$$\|\mathrm{p}_\varepsilon^t\|_{H^1\cap W^{1,\infty}}\lesssim\lambda_\varepsilon^{-1}\|\mathrm{d}_\varepsilon^t\|_{H^1\cap W^{1,\infty}}\lesssim_t\lambda_\varepsilon^{-1/2}\|a\Gamma_\varepsilon\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty(H^1\cap W^{1,\infty})}\lesssim_t\lambda_\varepsilon^{-1/2},$$

where the last bound follows from (8.58). Similarly, using the choice $h=\lambda_\varepsilon\hat h$ in the form

$$\nabla\mathrm{p}_\varepsilon=(\lambda_\varepsilon\alpha)^{-1}\nabla(a^{-1}\mathrm{d}_\varepsilon)=(\alpha a)^{-1}(\lambda_\varepsilon^{-1}\nabla\mathrm{d}_\varepsilon-\mathrm{d}_\varepsilon\nabla\hat h),$$

item (a) of Step 2 yields

$$\|\nabla\mathrm{p}_\varepsilon\|_{\mathrm{L}_t^2\,\mathrm{L}^2}\lesssim_t\lambda_\varepsilon^{-1}\|\nabla\mathrm{d}_\varepsilon\|_{\mathrm{L}_t^2\,\mathrm{L}^2}+\|\mathrm{d}_\varepsilon\|_{\mathrm{L}_t^\infty\,\mathrm{L}^2}\lesssim_t\|a\Gamma_\varepsilon\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty\,\mathrm{L}^2}\lesssim_t 1.$$

Inserting this information into equation (8.51), we deduce

$$\|\partial_t\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim\|\nabla\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}+\|\Gamma_\varepsilon^t\|_{\mathrm{L}^\infty}\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim_t\lambda_\varepsilon^{-1/2},$$

and similarly

$$\|\partial_t\mathrm{v}_\varepsilon\|_{\mathrm{L}_t^2\,\mathrm{L}^2}\lesssim\|\nabla\mathrm{p}_\varepsilon\|_{\mathrm{L}_t^2\,\mathrm{L}^2}+\|\Gamma_\varepsilon\|_{\mathrm{L}_t^\infty\,\mathrm{L}^\infty}\|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^2\,\mathrm{L}^2}\lesssim_t 1.$$

Finally, item (d) of Step 2 yields

$$\|\partial_t\mathrm{p}_\varepsilon\|_{\mathrm{L}_t^2\,\mathrm{L}^2}\lesssim\lambda_\varepsilon^{-1}\|\partial_t\mathrm{d}_\varepsilon\|_{\mathrm{L}_t^2\,\mathrm{L}^2}\lesssim_t\lambda_\varepsilon^{-1}\|a\Gamma_\varepsilon\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty\,H^1}\lesssim_t\lambda_\varepsilon^{-1}.$$

All the stated estimates follow.

Step 4. Global existence in the (mixed-flow) incompressible regimes.

The energy estimates of Lemma 7.4.1(iii) yield

$$\|\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^2}\lesssim_t 1.$$

Using this estimate as well as $\int_{\mathbb{R}^2}|\mathrm{m}_\varepsilon^t|=1$ for all t, and arguing as in Step 1 of the proof of Lemma 7.4.5, we find

$$\begin{aligned}\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty}\lesssim_t 1+\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty}^{1/2}\log^{1/2}(2+\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty})&\\ +\|\operatorname{div}(\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2}\log^{1/2}(2+\|\operatorname{div}(\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}).&\end{aligned}\tag{8.59}$$

On the other hand, item (a) of Step 2 above yields

$$\begin{aligned}\|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^2}\lesssim_t\lambda_\varepsilon^{1/2}\|a\Gamma_\varepsilon\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty\,\mathrm{L}^2}&\lesssim_t\lambda_\varepsilon^{1/2}\|\mathrm{v}_\varepsilon-\mathrm{v}^\circ\|_{\mathrm{L}_t^\infty\,\mathrm{L}^2}\|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty\,\mathrm{L}^\infty}+\lambda_\varepsilon^{1/2}\|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty\,\mathrm{L}^2}\\ &\lesssim_t\lambda_\varepsilon^{1/2}\|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty\,\mathrm{L}^\infty}+\lambda_\varepsilon^{1/2}\|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty\,\mathrm{L}^\infty}^{1/2},\end{aligned}$$

and hence, in terms of $\operatorname{div}(\mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon^\circ) = a^{-1}\mathrm{d}_\varepsilon - \lambda_\varepsilon \nabla\hat{h}\cdot(\mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon^\circ)$,

$$\|\operatorname{div}(\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2} \lesssim_t \lambda_\varepsilon^{1/2}(1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}).$$

Inserting this into (8.59), we find

$$\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t (1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}) \log^{1/2}(2 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty} + \|\operatorname{div} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty}). \tag{8.60}$$

Item (c) of Step 2 yields

$$\|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{1/2} \|a\Gamma_\varepsilon \mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty} \lesssim \lambda_\varepsilon^{1/2}(1 + \|\mathrm{v}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty})\|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty},$$

or alternatively, in terms of $\operatorname{div} \mathrm{v}_\varepsilon = a^{-1}\mathrm{d}_\varepsilon - \lambda_\varepsilon \nabla\hat{h}\cdot \mathrm{v}_\varepsilon$,

$$\|\operatorname{div} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{1/2}(1 + \|\mathrm{v}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty})(1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}).$$

Combining this with (8.60) leads to

$$\|\operatorname{div} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{1/2}(1 + \|\mathrm{m}_\varepsilon\|^2_{\mathrm{L}_t^\infty \mathrm{L}^\infty}) \log^{1/2}(2 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty} + \|\operatorname{div} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty}),$$

and hence, using $\lambda_\varepsilon \ll 1$ and the inequality $a \log b \le b + a\log a$ for all $a, b \ge 0$, in order to absorb the term $\|\operatorname{div} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty}$ appearing in the right-hand side,

$$\|\operatorname{div} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{1/2}(1 + \|\mathrm{m}_\varepsilon\|^2_{\mathrm{L}_t^\infty \mathrm{L}^\infty}) \log(2 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}),$$

so that (8.60) finally takes the form

$$\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t (1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}) \log^{1/2}(2 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}).$$

In particular, we have proved the following estimates,

$$\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t (1 + \|\mathrm{m}_\varepsilon\|^2_{\mathrm{L}_t^\infty \mathrm{L}^\infty}), \qquad \text{and} \qquad \|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{1/2}(1 + \|\mathrm{m}_\varepsilon\|^3_{\mathrm{L}_t^\infty \mathrm{L}^\infty}).$$

The result in Lemma 7.4.3(i) then gives the following bound on the vorticity m_ε,

$$\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim \exp\Big(Ct\big(1 + \|\mathrm{d}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty} + \lambda_\varepsilon \|\mathrm{v}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}\big)\Big) \lesssim_t \exp\big(Ct\lambda_\varepsilon^{1/2}(1 + \|\mathrm{m}_\varepsilon\|^3_{\mathrm{L}_t^\infty \mathrm{L}^\infty})\big).$$

As $\lambda_\varepsilon \ll 1$, this bound easily implies that for all $T > 0$ there exists some $\varepsilon_0(T)$ such that for all $0 < \varepsilon < \varepsilon_0(T)$ the vorticity m_ε^t (if it exists) remains bounded in $\mathrm{L}^\infty(\mathbb{R}^2)$ for all $t \in [0, T]$. Then repeating the arguments in Sections 7.4.2–7.4.3, this a priori bound on the vorticity allows to deduce existence and uniqueness of a solution on the whole time interval $[0, T]$.

Step 5. Global existence in the (mixed-flow) compressible regime (GL_1').

Just as in (8.59) above, we find the bounds $\|\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^2} \lesssim_t 1$ and

$$\begin{aligned}\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1 + \|\mathrm{m}_\varepsilon^t\|^{1/2}_{\mathrm{L}^\infty} \log^{1/2}(2 + \|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty}) \\ + \|\operatorname{div}(\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2} \log^{1/2}(2 + \|\operatorname{div}(\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}).\end{aligned} \tag{8.61}$$

On the other hand, considering the equation (8.54) satisfied by d_ε, the a priori estimates in Lemma 7.2.3 yield

$$\|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1 + \|a\Gamma_\varepsilon \mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^2} \lesssim_t 1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^2} + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty} \|\mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}_t^\infty \mathrm{L}^2} \lesssim_t 1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty},$$

and also

$$\|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1 + \|a\Gamma_\varepsilon \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty} \lesssim_t 1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}).$$

As by definition $\operatorname{div}(\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ) = a^{-1}(\mathrm{d}_\varepsilon^t - \mathrm{d}_\varepsilon^\circ) - \nabla h \cdot (\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ)$, the above estimates take the following form,

$$\begin{aligned} \|\operatorname{div}(\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2} &\lesssim_t 1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}, \\ \|\operatorname{div}\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} &\lesssim_t (1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}). \end{aligned} \tag{8.62}$$

Combining these estimates with (8.61) yields

$$\begin{aligned} \|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1 + \|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty}^{1/2} \log^{1/2}(2+\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty}) \\ + (1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}) \log^{1/2}\big((1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})\big), \end{aligned}$$

and hence, using the inequality $a\log b \le b + a\log a$ for all $a,b\ge 0$, in order to absorb the term $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}$ appearing in the right-hand side,

$$\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty} \lesssim_t (1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})\log(1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}),$$

so that (8.62) finally takes the form,

$$\|\operatorname{div}\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty} \lesssim_t (1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})^2\log(1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}).$$

The result in Lemma 7.4.3(i) then gives the following bound on the vorticity m_ε, in the considered regime (GL_1'),

$$\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim \exp\left(Ct\Big(1+\frac{N_\varepsilon}{|\log\varepsilon|}\|(\mathrm{v}_\varepsilon,\operatorname{div}\mathrm{v}_\varepsilon)\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}\Big)\right) \lesssim_t \exp\left(\frac{CtN_\varepsilon}{|\log\varepsilon|}\|\mathrm{m}_\varepsilon\|^3_{\mathrm{L}^\infty_t\mathrm{L}^\infty}\right).$$

As $N_\varepsilon \ll |\log\varepsilon|$, this bound easily implies that for all $T>0$ there exists some $\varepsilon_0(T)$ such that for all $0<\varepsilon<\varepsilon_0(T)$ the vorticity m_ε^t (if it exists) remains bounded in $\mathrm{L}^\infty(\mathbb{R}^2)$ for all $t\in[0,T]$. Then repeating the arguments in Sections 7.4.2–7.4.3, existence and uniqueness of a solution on the whole time interval $[0,T]$ follows from this a priori bound. ☐

We now show how to pass to the limit in equation (8.51) as $\varepsilon\downarrow 0$, which is easily achieved e.g. by a Grönwall argument on the L^2-distance between v_ε and the solution v of the limiting equation.

Lemma 8.3.3. *Let the same assumptions hold as in Proposition 8.3.2, and let* $\mathrm{v}_\varepsilon : [0,T)\times\mathbb{R}^2\to\mathbb{R}^2$ *be the corresponding local solution of* (8.51)*, for some* $T>0$ *(independent of* ε*). Assume that* $\mathrm{v}_\varepsilon^\circ \to \mathrm{v}^\circ$ *in* $\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2$ *as* $\varepsilon\downarrow 0$*. Then the following hold.*

(i) Regime (GL_1):
We have $\mathrm{v}_\varepsilon \to \mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ *as* $\varepsilon\downarrow 0$*, where* $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{v}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ *is the unique global (smooth) solution of*

$$\partial_t\mathrm{v} = \nabla\mathrm{p} + (\alpha-\mathbb{J}\beta)(\nabla^\perp\hat h - \hat F^\perp - 2\mathrm{v})\operatorname{curl}\mathrm{v}, \qquad \operatorname{div}\mathrm{v}=0, \qquad \mathrm{v}|_{t=0}=\mathrm{v}^\circ; \tag{8.63}$$

(ii) Regime (GL_2) with $N_\varepsilon/|\log\varepsilon|\to\lambda\in(0,\infty)$ and $\mathrm{v}_\varepsilon^\circ=\mathrm{v}^\circ$:
We have $\mathrm{v}_\varepsilon\to\mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^2(\mathbb{R}^2)^2)$ *as* $\varepsilon\downarrow 0$*, where* $\mathrm{v}\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{v}^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ *is the unique local (smooth) solution of*

$$\partial_t\mathrm{v} = \alpha^{-1}\nabla(\hat a^{-1}\operatorname{div}(\hat a\mathrm{v})) + (\alpha-\mathbb{J}\beta)(\nabla^\perp\hat h - \hat F^\perp - 2\lambda\mathrm{v})\operatorname{curl}\mathrm{v}, \qquad \mathrm{v}|_{t=0}=\mathrm{v}^\circ; \tag{8.64}$$

(iii) Regime (GL_1') with $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$:
We have $\mathrm{v}_\varepsilon \to \mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^2(\mathbb{R}^2)^2)$ *as* $\varepsilon \downarrow 0$, *where* $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ *is the unique global (smooth) solution of*

$$\partial_t \mathrm{v} = \alpha^{-1}\nabla(\hat{a}^{-1}\,\mathrm{div}\,(\hat{a}\mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp)\mathrm{curl}\,\mathrm{v}, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ; \tag{8.65}$$

(iv) Regime (GL_2'):
We have $\mathrm{v}_\varepsilon \to \mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ *as* $\varepsilon \downarrow 0$, *where* $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ *is the unique global (smooth) solution of*

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp)\mathrm{curl}\,\mathrm{v}, \qquad \mathrm{div}\,\mathrm{v} = 0, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ. \tag{8.66}$$

$\Diamond$

Proof. We treat each of the four regimes separately. We denote by $\xi_R^z(x) := e^{-|x-z|/R}$ the exponential cut-off at the scale $R \geq 1$ centered at $z \in R\mathbb{Z}^2$.

Step 1. Regime (GL_1).

Using the choice of the scalings for $\lambda_\varepsilon, h, F$ in the regime (GL_1), with $\lambda_\varepsilon = N_\varepsilon/|\log\varepsilon| \ll 1$, and setting $a_\varepsilon := a = \hat{a}^{\lambda_\varepsilon}$, equation (8.51) takes on the following guise,

$$\partial_t \mathrm{v}_\varepsilon = \nabla \mathrm{p}_\varepsilon + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v}_\varepsilon)\,\mathrm{curl}\,\mathrm{v}_\varepsilon, \qquad \mathrm{p}_\varepsilon := (\lambda_\varepsilon \alpha a_\varepsilon)^{-1}\,\mathrm{div}\,(a_\varepsilon \mathrm{v}_\varepsilon),$$

with initial data $\mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ \to \mathrm{v}^\circ$ in $\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2$. As $\lambda_\varepsilon \to 0$, it is then formally clear from the vorticity formulation of this equation that v_ε should converge to the solution v of (8.63).

The existence and uniqueness of a global smooth solution $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ of (8.63) are established in Theorems 7.1.3 and 7.1.5 in Chapter 7. Moreover, the following estimates hold for all $t \geq 0$ and all $R, \theta > 0$,

$$\|\mathrm{v}^t\|_{W^{1,\infty}} \lesssim_t 1, \qquad \|(\mathrm{v}^t, \mathrm{p}^t)\|_{\mathrm{L}^2(B_R)} \lesssim_{t,\theta} R^\theta, \qquad \|\mathrm{curl}\,\mathrm{v}^t\|_{\mathrm{L}^1} = 1. \tag{8.67}$$

The bounds on v are indeed direct consequences of the results in Chapter 7 together with the regularity assumptions on the data (in particular $\mathrm{v}^\circ \in \mathrm{L}^q(\mathbb{R}^2)^2$ for all $q > 2$). It remains to check the bound on the pressure p. Taking the divergence of both sides of equation (8.63), we obtain the following equation for the pressure p^t, for all $t \geq 0$,

$$-\triangle \mathrm{p}^t = \mathrm{div}\,\big((\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v}^t)\mathrm{curl}\,\mathrm{v}^t\big).$$

By Riesz potential theory, we deduce for all $2 < q < \infty$,

$$\|\mathrm{p}^t\|_{\mathrm{L}^q} \lesssim_q (1 + \|\mathrm{v}^t\|_{\mathrm{L}^\infty})\|\mathrm{curl}\,\mathrm{v}^t\|_{\mathrm{L}^{2q/(2+q)}} \lesssim (1 + \|\mathrm{v}^t\|_{\mathrm{L}^\infty})(\|\mathrm{curl}\,\mathrm{v}^t\|_{\mathrm{L}^1} + \|\nabla \mathrm{v}^t\|_{\mathrm{L}^\infty}) \lesssim_t 1,$$

and the bound on the pressure p in (8.67) follows.

Now we turn to the Grönwall argument to prove the convergence $\mathrm{v}_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2)$. Using the equations for $\mathrm{v}_\varepsilon, \mathrm{v}$, we find

$$\begin{aligned}\partial_t \int_{\mathbb{R}^2} a_\varepsilon \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 = 2\int_{\mathbb{R}^2} a_\varepsilon \xi_R^z (\mathrm{v}_\varepsilon - \mathrm{v}) \cdot \nabla(\mathrm{p}_\varepsilon - \mathrm{p}) - 4\alpha \int_{\mathbb{R}^2} a_\varepsilon \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 \mathrm{curl}\,\mathrm{v}_\varepsilon \\ + 2\int_{\mathbb{R}^2} a_\varepsilon \xi_R^z (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v}) \cdot (\mathrm{v}_\varepsilon - \mathrm{v})\,\mathrm{curl}\,(\mathrm{v}_\varepsilon - \mathrm{v}).\end{aligned} \tag{8.68}$$

Integrating by parts in the first term, decomposing

$$\mathrm{div}\,(a_\varepsilon \xi_R^z(\mathrm{v}_\varepsilon - \mathrm{v})) = a_\varepsilon \nabla \xi_R^z \cdot (\mathrm{v}_\varepsilon - \mathrm{v}) + \lambda_\varepsilon \alpha a_\varepsilon \xi_R^z \mathrm{p}_\varepsilon - \lambda_\varepsilon a_\varepsilon \xi_R^z \nabla \hat{h} \cdot \mathrm{v},$$

noting that the second right-hand side term in (8.68) is nonpositive, and using the following weighted Delort-type identity (as abundantly used in Chapter 7),

$$\begin{aligned}
&(\mathrm{v}_\varepsilon-\mathrm{v})\operatorname{curl}(\mathrm{v}_\varepsilon-\mathrm{v})\\
&\quad= a_\varepsilon^{-1}(\mathrm{v}_\varepsilon-\mathrm{v})^\perp\operatorname{div}(a_\varepsilon(\mathrm{v}_\varepsilon-\mathrm{v}))-\frac12 a_\varepsilon^{-1}|\mathrm{v}_\varepsilon-\mathrm{v}|^2\nabla^\perp a_\varepsilon-a_\varepsilon^{-1}(\operatorname{div}(a_\varepsilon S_{\mathrm{v}_\varepsilon-\mathrm{v}}))^\perp \qquad (8.69)\\
&\quad= \lambda_\varepsilon\alpha\mathrm{p}_\varepsilon(\mathrm{v}_\varepsilon-\mathrm{v})^\perp-\lambda_\varepsilon(\nabla\hat h\cdot\mathrm{v})(\mathrm{v}_\varepsilon-\mathrm{v})^\perp-\frac{\lambda_\varepsilon}{2}|\mathrm{v}_\varepsilon-\mathrm{v}|^2\nabla^\perp\hat h-a_\varepsilon^{-1}(\operatorname{div}(a_\varepsilon S_{\mathrm{v}_\varepsilon-\mathrm{v}}))^\perp,
\end{aligned}$$

in terms of the stress-energy tensor $S_w:=w\otimes w-\frac12|w|^2\,\mathrm{Id}$, we deduce

$$\begin{aligned}
\partial_t\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon-\mathrm{v}|^2&\le-2\int_{\mathbb{R}^2}a_\varepsilon(\mathrm{p}_\varepsilon-\mathrm{p})\nabla\xi_R^z\cdot(\mathrm{v}_\varepsilon-\mathrm{v})-2\lambda_\varepsilon\alpha\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z\,\mathrm{p}_\varepsilon(\mathrm{p}_\varepsilon-\mathrm{p})\\
&+2\lambda_\varepsilon\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z(\mathrm{p}_\varepsilon-\mathrm{p})\,\mathrm{v}\cdot\nabla\hat h+2\lambda_\varepsilon\alpha\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z\mathrm{p}_\varepsilon(\alpha-\mathbb{J}\beta)(\nabla^\perp\hat h-\hat F^\perp-2\mathrm{v})\cdot(\mathrm{v}_\varepsilon-\mathrm{v})^\perp\\
&-2\lambda_\varepsilon\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z(\nabla\hat h\cdot\mathrm{v})(\alpha-\mathbb{J}\beta)(\nabla^\perp\hat h-\hat F^\perp-2\mathrm{v})\cdot(\mathrm{v}_\varepsilon-\mathrm{v})^\perp\\
&-\lambda_\varepsilon\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon-\mathrm{v}|^2(\alpha-\mathbb{J}\beta)(\nabla^\perp\hat h-\hat F^\perp-2\mathrm{v})\cdot\nabla^\perp\hat h\\
&-2\int_{\mathbb{R}^2}a_\varepsilon S_{\mathrm{v}_\varepsilon-\mathrm{v}}:\nabla\Big(\xi_R^z(\alpha\mathbb{J}+\beta)(\nabla^\perp\hat h-\hat F^\perp-2\mathrm{v})\Big),
\end{aligned}$$

and hence, using (8.67) in the form $\|\mathrm{v}^t\|_{W^{1,\infty}}\lesssim1$, the assumption $\|(\nabla\hat h,\hat F)\|_{W^{1,\infty}}\lesssim1$, the property $|\nabla\xi_R^z|\lesssim R^{-1}\xi_R^z$ of the exponential cut-off, and the pointwise estimate $|S_w|\lesssim|w|^2$, we obtain

$$\begin{aligned}
\partial_t\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon-\mathrm{v}|^2&\le(R^{-2}-\lambda_\varepsilon\alpha)\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z|\mathrm{p}_\varepsilon|^2\\
&+C_t(R^{-2}+\lambda_\varepsilon)\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z(|\mathrm{p}|^2+|\mathrm{v}|^2)+C_t\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon-\mathrm{v}|^2.
\end{aligned}$$

Choosing $R=\lambda_\varepsilon^{-n}$ for some $n\ge1$, we obtain $R^{-2}\ll\lambda_\varepsilon$, and hence, for ε small enough, using (8.67) to estimate the second term, we obtain

$$\partial_t\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon-\mathrm{v}|^2\lesssim_{t,\theta}R^{2\theta}(R^{-2}+\lambda_\varepsilon)+\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon-\mathrm{v}|^2\lesssim\lambda_\varepsilon^{1-2n\theta}+\int_{\mathbb{R}^2}a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon-\mathrm{v}|^2.$$

For $\theta>0$ small enough, the conclusion follows from the Grönwall inequality.

Step 2. Regime (GL_2).

Using the choice of the scalings for λ_ε,h,F in the regime (GL_2), equation (8.51) takes on the following guise,

$$\partial_t\mathrm{v}_\varepsilon=\alpha^{-1}\nabla(\hat a^{-1}\operatorname{div}(\hat a\mathrm{v}_\varepsilon))+\Big((\alpha-\mathbb{J}\beta)\Big(\nabla^\perp\hat h-\hat F^\perp-\frac{2N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\Big)\operatorname{curl}\mathrm{v}_\varepsilon,$$

with initial data $\mathrm{v}_\varepsilon|_{t=0}=\mathrm{v}^\circ$. As $N_\varepsilon/|\log\varepsilon|\to\lambda\in(0,\infty)$, it is formally clear that v_ε should converge to the solution v of equation (8.64). Note that the existence and uniqueness of the (local) solution v are established in Theorems 7.1.4 and 7.1.5 in Chapter 7, and we have in addition the following bounds for all $t\in[0,T)$,

$$\|(\mathrm{v}^t,\mathrm{v}_\varepsilon^t)\|_{W^{1,\infty}}\lesssim_t1,\qquad\|\operatorname{curl}\mathrm{v}^t\|_{\mathrm{L}^1}=1. \qquad (8.70)$$

Using the equations for $\mathrm{v}_\varepsilon, \mathrm{v}$, we find

$$\partial_t \int_{\mathbb{R}^2} \hat a \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 = 2\alpha^{-1} \int_{\mathbb{R}^2} \hat a \xi_R^z (\mathrm{v}_\varepsilon - \mathrm{v}) \cdot \nabla(\hat a^{-1} \operatorname{div}(\hat a(\mathrm{v}_\varepsilon - \mathrm{v}))) - \frac{4\alpha N_\varepsilon}{|\log\varepsilon|} \int_{\mathbb{R}^2} \hat a \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 \operatorname{curl} \mathrm{v}_\varepsilon$$
$$+ 2 \int_{\mathbb{R}^2} \hat a \xi_R^z \Big((\alpha - \mathbb{J}\beta) \Big(\nabla^\perp \hat h - \hat F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|} \mathrm{v} \Big) \Big) \cdot (\mathrm{v}_\varepsilon - \mathrm{v}) (\operatorname{curl} \mathrm{v}_\varepsilon - \operatorname{curl} \mathrm{v})$$
$$- 4 \Big(\frac{N_\varepsilon}{|\log\varepsilon|} - \lambda \Big) \int_{\mathbb{R}^2} \hat a \xi_R^z (\mathrm{v}_\varepsilon - \mathrm{v}) \cdot (\alpha - \mathbb{J}\beta) \mathrm{v} \operatorname{curl} \mathrm{v}.$$

Integrating by parts, using the weighted Delort-type identity (8.69) in the form

$$(\mathrm{v}_\varepsilon - \mathrm{v}) \operatorname{curl}(\mathrm{v}_\varepsilon - \mathrm{v}) = \hat a^{-1} (\mathrm{v}_\varepsilon - \mathrm{v})^\perp \operatorname{div}(\hat a(\mathrm{v}_\varepsilon - \mathrm{v})) - \frac12 |\mathrm{v}_\varepsilon - \mathrm{v}|^2 \nabla^\perp \hat h - \hat a^{-1} (\operatorname{div}(\hat a S_{\mathrm{v}_\varepsilon - \mathrm{v}}))^\perp,$$

using the properties (8.70) of $\mathrm{v}, \mathrm{v}_\varepsilon$, the assumption $\|(\nabla \hat h, \hat F)\|_{W^{1,\infty}} \lesssim 1$, and simplifying the terms as in Step 1, we easily obtain

$$\partial_t \int_{\mathbb{R}^2} \hat a \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 \le -2\alpha^{-1} \int_{\mathbb{R}^2} \hat a^{-1} \xi_R^z |\operatorname{div}(\hat a(\mathrm{v}_\varepsilon - \mathrm{v}))|^2$$
$$+ C_t \int_{\mathbb{R}^2} \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}| |\operatorname{div}(\hat a(\mathrm{v}_\varepsilon - \mathrm{v}))| + C_t \int_{\mathbb{R}^2} \hat a \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 + C_t \Big| \frac{N_\varepsilon}{|\log\varepsilon|} - \lambda \Big|,$$

hence $\partial_t \int_{\mathbb{R}^2} \hat a \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 \lesssim C_t \int_{\mathbb{R}^2} \hat a \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 + o_t(1)$, and the conclusion now follows from the Grönwall inequality, letting $R \uparrow \infty$.

Step 3. Regime (GL_1').

Using the choice of the scalings for $\lambda_\varepsilon, h, F$ in the regime (GL_1'), equation (8.51) takes on the following guise,

$$\partial_t \mathrm{v}_\varepsilon = \alpha^{-1} \nabla(\hat a^{-1} \operatorname{div}(\hat a \mathrm{v}_\varepsilon)) + (\alpha - \mathbb{J}\beta) \Big(\nabla^\perp \hat h - \hat F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|} \mathrm{v}_\varepsilon \Big) \operatorname{curl} \mathrm{v}_\varepsilon,$$

with initial data $\mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}^\circ$. As by assumption $N_\varepsilon / |\log\varepsilon| \to 0$, it is formally clear that v_ε should converge to the solution v of equation (8.65) as $\varepsilon \downarrow 0$. Existence, uniqueness, and regularity of this (global) solution v are given by Proposition 8.3.2 just as for v_ε, and the convergence result follows as in Step 2 (with $\lambda = 0$).

Step 4. Regime (GL_2').

Using the choice of the scalings for $\lambda_\varepsilon, h, F$ in the regime (GL_2'), equation (8.51) takes the following form, with $a_\varepsilon := \hat a^{\lambda_\varepsilon}$,

$$\partial_t \mathrm{v}_\varepsilon = \nabla \mathrm{p}_\varepsilon + (\alpha - \mathbb{J}\beta) \Big(\nabla^\perp \hat h - \hat F^\perp - \frac{2\lambda_\varepsilon^{-1} N_\varepsilon}{|\log\varepsilon|} \mathrm{v}_\varepsilon \Big) \operatorname{curl} \mathrm{v}_\varepsilon,$$
$$\mathrm{p}_\varepsilon := (\lambda_\varepsilon \alpha a_\varepsilon)^{-1} \operatorname{div}(a_\varepsilon \mathrm{v}_\varepsilon),$$

with initial data $\mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ \to \mathrm{v}^\circ$ in $\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2$. As by assumption $\lambda_\varepsilon^{-1} N_\varepsilon / |\log\varepsilon| \to 0$, it is formally clear that v_ε should converge to the solution v of equation (8.66) as $\varepsilon \downarrow 0$. Existence, uniqueness, and regularity of this (global) solution v are given by Proposition 8.3.2 just as for v_ε, and the convergence result follows as in Step 1. $\square$

8.3.2 Gross-Pitaevskii case

Let us first examine the vorticity formulation of equation (8.53) for v_ε. In terms of $\mathrm{m}_\varepsilon := \operatorname{curl}\mathrm{v}_\varepsilon$, equation (8.53) may be rewritten as a nonlinear nonlocal transport equation for the vorticity m_ε,

$$\begin{cases}\partial_t \mathrm{m}_\varepsilon = -\operatorname{div}(\Gamma_\varepsilon^\perp \mathrm{m}_\varepsilon), & \mathrm{m}_\varepsilon|_{t=0} = \operatorname{curl}\mathrm{v}_\varepsilon^\circ, \\ \operatorname{curl}\mathrm{v}_\varepsilon = \mathrm{m}_\varepsilon, & \operatorname{div}(a\mathrm{v}_\varepsilon) = 0.\end{cases} \tag{8.71}$$

Given the form of Γ_ε in (8.53), this equation can be seen as an "inhomogeneous" 2D Euler equation with "forcing". A detailed study of this kind of equations is performed in Chapter 7. The following proposition states in particular that a solution v_ε always exists globally and satisfies the various properties of Assumption 8.3.1(b), under suitable regularity assumptions on the initial data $\mathrm{v}_\varepsilon^\circ$.

Proposition 8.3.4. *Let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, and let $\mathrm{v}_\varepsilon^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ be bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$, and satisfy $\operatorname{curl}\mathrm{v}_\varepsilon^\circ \in \mathcal{P}(\mathbb{R}^2)$. Assume that $h \in \mathrm{L}^\infty(\mathbb{R}^2)$, $\nabla h, F \in \mathrm{L}^4 \cap W^{2,\infty}(\mathbb{R}^2)^2$, that $a(x) \to 1$ uniformly as $|x| \uparrow \infty$, that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{2,\infty}(\mathbb{R}^2)^2$ with $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ) = 0$, and that $\operatorname{curl}\mathrm{v}_\varepsilon^\circ$ is bounded in $H^1(\mathbb{R}^2)$. Let the regime (GP) hold.*
Then there exists a unique (global) solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}_\varepsilon^\circ + H^2 \cap W^{1,\infty}(\mathbb{R}^2)^2)$ of (8.53) on $\mathbb{R}^+ \times \mathbb{R}^2$. Moreover, all the properties of Assumption 8.3.1(b) are satisfied, that is, for all $t \ge 0$ and all $2 < q < \infty$,

$$\|(\mathrm{v}_\varepsilon^t, \nabla\mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap\mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\operatorname{curl}\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty} \lesssim_t 1,$$
$$\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q\cap\mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\nabla\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1, \quad \|\partial_t\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1, \quad \|\partial_t\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q} \lesssim_{t,q} 1.$$

Further, for all $\theta > 0$ and $\varrho \ge 1$, setting $\mathrm{p}_{\varepsilon,\varrho} := \chi_\varrho\,\mathrm{p}_\varepsilon$, we have for all $t \ge 0$,

$$\|\nabla(\mathrm{p}_{\varepsilon,\varrho}^t - \mathrm{p}_\varepsilon^t)\|_{\mathrm{L}^2} \lesssim_{\theta,t} \varrho^{\theta-2} + \int_{|x|>\varrho} |\operatorname{curl}\mathrm{v}_\varepsilon^\circ|^2. \tag{8.72}$$

$\Diamond$

Proof. We split the proof into three steps.

Step 1. Preliminary.

In this step, we prove the following Meyers-type elliptic regularity estimate: if $b \in \mathrm{L}^\infty(\mathbb{R}^2)$ satisfies $1/2 \le b \le 1$ pointwise, and $b(x) \to 1$ uniformly as $|x| \uparrow \infty$, then for all $g \in \mathrm{L}^1 \cap \mathrm{L}^2(\mathbb{R}^2)^2$ the decaying solution v of equation $-\operatorname{div}(b\nabla v) = \operatorname{div} g$ satisfies for all $2 < q < \infty$,

$$\|v\|_{\mathrm{L}^q} \lesssim_q \|g\|_{\mathrm{L}^{2q/(q+2)}\cap\mathrm{L}^2} \lesssim \|g\|_{\mathrm{L}^1\cap\mathrm{L}^2}.$$

Let $b \in \mathrm{L}^\infty(\mathbb{R}^2)$ be fixed with $1/2 \le b \le 1$ pointwise and $b(x) \to 1$ uniformly as $|x| \uparrow \infty$. Set $b_r := \chi_r + b(1-\chi_r)$, and decompose the equation for v as follows,

$$-\operatorname{div}(b_r\nabla v) = \operatorname{div}\big(g + (b-b_r)\nabla v\big).$$

Let $1 < p < 2$. Meyers' perturbative argument [322] gives a value $\kappa_p > 0$ such that, if $\tilde b \in \mathrm{L}^\infty(\mathbb{R}^2)$ satisfies $\kappa_p \le \tilde b \le 1$, then for all $k \in \mathrm{L}^1 \cap \mathrm{L}^2(\mathbb{R}^2)^2$ the decaying solution w of equation $-\operatorname{div}(\tilde b\nabla w) = \operatorname{div} k$ satisfies $\|\nabla w\|_{\mathrm{L}^p} \lesssim_p \|k\|_{\mathrm{L}^p}$. By definition, for r large enough, the truncated coefficient b_r satisfies $\kappa_p \le b_r \le 1$, hence

$$\|\nabla v\|_{\mathrm{L}^p} \lesssim_p \|g + (b-b_r)\nabla v\|_{\mathrm{L}^p}.$$

Using the elementary energy estimate $\|\nabla v\|_{\mathrm{L}^2} \lesssim \|g\|_{\mathrm{L}^2}$, and noting that $b_r = b$ on $\mathbb{R}^2 \setminus B_{2r}$, we find by the Hölder inequality,

$$\|\nabla v\|_{\mathrm{L}^p} \lesssim_p \|g\|_{\mathrm{L}^p} + \|\nabla v\|_{\mathrm{L}^p(B_{2r})} \lesssim \|g\|_{\mathrm{L}^p} + r^{2(\frac1p-\frac12)}\|\nabla v\|_{\mathrm{L}^2} \lesssim \|g\|_{\mathrm{L}^p} + r^{2(\frac1p-\frac12)}\|g\|_{\mathrm{L}^2}.$$

On the other hand, rather decomposing the equation for v as follows,

$$-\triangle v = \operatorname{div}(g + (b-1)\nabla v),$$

we deduce from Riesz potential theory, with $2 < q := 2p/(2-p) < \infty$,

$$\|v\|_{\mathrm{L}^q} \lesssim_q \|g\|_{\mathrm{L}^p} + \|\nabla v\|_{\mathrm{L}^p}.$$

Combining this with the above, the conclusion follows.

Step 2. Proof of Assumption 8.3.1(b).

The assumptions $\|\hat h\|_{W^{3,\infty}}$, $\|(\nabla \hat h, \hat F)\|_{\mathrm{L}^4\cap W^{2,\infty}} \lesssim 1$ yield $\|\lambda_\varepsilon^{-1}(\nabla^\perp h - F^\perp)\|_{\mathrm{L}^4\cap W^{2,\infty}} \lesssim 1$ in the considered regime, and also note that $\lambda_\varepsilon^{-1}N_\varepsilon/|\log\varepsilon| = 1$ and $\lambda_\varepsilon^{-1} \lesssim 1$. Further using the assumptions on the initial data v°, Theorems 7.1.3 and 7.1.5 in Chapter 7 imply that there exists a unique (global) solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}_\varepsilon^\circ + H^2 \cap W^{1,\infty}(\mathbb{R}^2)^2)$ of (8.53) on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data $\mathrm{v}_\varepsilon^\circ$. Moreover, it is shown in [159] that this solution satisfies in particular, for all $t \ge 0$,

$$\|\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ\|_{H^2\cap W^{1,\infty}} \lesssim_t 1, \qquad \|\mathrm{m}_\varepsilon^t\|_{H^1\cap\mathrm{L}^\infty} \lesssim_t 1, \qquad \int_{\mathbb{R}^2} \mathrm{m}_\varepsilon^t = 1, \qquad \mathrm{m}_\varepsilon^t \ge 0. \tag{8.73}$$

(As such, in order to ensure $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}_\varepsilon^\circ + H^2(\mathbb{R}^2)^2)$, the results in Chapter 7 would actually further require $\nabla h, F, \mathrm{v}^\circ \in W^{s+2,\infty}(\mathbb{R}^2)^2$ for some $s > 0$, due to the use of the Sobolev embedding for $H^{s+1}(\mathbb{R}^2)$ into $W^{s,\infty}(\mathbb{R}^2)$ in the proof of [159, Lemma 4.6]. However, this use of the Sobolev embedding is easily replaced by an a priori estimate for v_ε in $W^{s+1,\infty}(\mathbb{R}^2)^2$, for which it is already enough to assume $\nabla h, F, \mathrm{v}^\circ \in W^{2,\infty}(\mathbb{R}^2)^2$ as we do here.)

We claim that all the desired properties of v_ε follow from the bounds (8.73). Combining (8.73) with the assumption that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$, we obtain

$$\|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap\mathrm{L}^\infty} \lesssim_{t,q} 1.$$

Applying the operator $\operatorname{div}(\hat a\,\cdot)$ to both sides of equation (8.53), we find the following equation for the pressure, in the considered regime (GP),

$$-\operatorname{div}(\hat a \nabla \mathrm{p}_\varepsilon^t) = \operatorname{div}(\hat a \Gamma_\varepsilon^t \mathrm{m}_\varepsilon^t) = -\operatorname{div}\big(\hat a \mathrm{m}_\varepsilon^t(\lambda_\varepsilon^{-1}\nabla^\perp \hat h - \hat F^\perp - 2\mathrm{v}_\varepsilon^t)^\perp\big). \tag{8.74}$$

An energy estimate directly yields

$$\|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim \|\hat a \mathrm{m}_\varepsilon^t(\lambda_\varepsilon^{-1}\nabla^\perp \hat h - \hat F^\perp - 2\mathrm{v}_\varepsilon^t)^\perp\|_{\mathrm{L}^2} \lesssim_t 1, \tag{8.75}$$

and similarly, first differentiating both sides of equation (8.74),

$$\|\nabla^2 \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim \|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2} + \big\|\nabla\big(\hat a \mathrm{m}_\varepsilon^t(\lambda_\varepsilon^{-1}\nabla^\perp \hat h - \hat F^\perp - 2\mathrm{v}_\varepsilon^t)^\perp\big)\big\|_{\mathrm{L}^2} \lesssim_t 1. \tag{8.76}$$

Inserting (8.75) into equation (8.53) yields

$$\|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2} \le \|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2} + \|\Gamma_\varepsilon^t \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1.$$

Applying to equation (8.74) the Meyers-type result of Step 1, we find for all $2 < q < \infty$,

$$\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q} \lesssim_q \|\hat a \mathrm{m}_\varepsilon^t(\lambda_\varepsilon^{-1}\nabla^\perp \hat h - \hat F^\perp - 2\mathrm{v}_\varepsilon^t)^\perp\|_{\mathrm{L}^1\cap\mathrm{L}^2} \lesssim_t 1.$$

Combining this with (8.76), we deduce from the Sobolev embedding $\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q\cap\mathrm{L}^\infty} \lesssim_{q,t} 1$ for all $q > 2$. First differentiating both sides of equation (8.74) with respect to the time variable, the Meyers-type result of Step 1 further yields for all $2 < q < \infty$,

$$\begin{aligned}
\|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q} &\lesssim_q \big\|\hat a \partial_t\big(\mathrm{m}_\varepsilon^t(\lambda_\varepsilon^{-1}\nabla^\perp \hat h - \hat F^\perp - 2\mathrm{v}_\varepsilon^t)^\perp\big)\big\|_{\mathrm{L}^1\cap\mathrm{L}^2} \\
&\lesssim \|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2} + \|\Gamma_\varepsilon^t \partial_t \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^2} \\
&\lesssim_t 1 + \|\Gamma_\varepsilon^t \partial_t \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^2}.
\end{aligned}$$

Using equation (8.71) to estimate the time derivative of the vorticity, and using that $\|\lambda_\varepsilon^{-1}\nabla\hat h - \hat F\|_{\mathrm{L}^4\cap W^{1,\infty}} \lesssim 1$, we find

$$\|\Gamma_\varepsilon^t \partial_t \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^2} \lesssim \|\Gamma_\varepsilon^t\|^2_{\mathrm{L}^4\cap\mathrm{L}^\infty}\|\nabla \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2} + \|\Gamma_\varepsilon^t\|^2_{W^{1,\infty}}\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^2} \\ \lesssim_t \|\Gamma_\varepsilon^t\|^2_{\mathrm{L}^4\cap W^{1,\infty}} \lesssim 1 + \|\mathrm{v}_\varepsilon^t\|^2_{\mathrm{L}^4\cap W^{1,\infty}} \lesssim_t 1,$$

and hence $\|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q} \lesssim_{t,q} 1$. All the stated estimates follow.

Step 3. Proof of (8.72).

For all $t \geq 0$, testing equation (8.74) against $(1-\chi_\varrho)\,\mathrm{p}_\varepsilon^t$, and using $|\nabla\chi_\varrho| \lesssim \varrho^{-1}(1-\chi_\varrho)^{1/2}$ and the inequality $2xy \leq x^2 + y^2$, we find

$$\int_{\mathbb{R}^2} \hat a(1-\chi_\varrho)|\nabla \mathrm{p}_\varepsilon^t|^2 = \int_{\mathbb{R}^2} \hat a\, \mathrm{p}_\varepsilon^t\, \nabla\chi_\varrho\cdot\nabla \mathrm{p}_\varepsilon^t - \int_{\mathbb{R}^2}\hat a(1-\chi_\varrho)\nabla \mathrm{p}_\varepsilon^t\cdot\Gamma_\varepsilon^t\,\mathrm{m}_\varepsilon^t + \int_{\mathbb{R}^2}\hat a \mathrm{p}_\varepsilon^t\,\nabla\chi_\varrho\cdot\Gamma_\varepsilon^t\,\mathrm{m}_\varepsilon^t \\ \leq \frac12\int_{\mathbb{R}^2}\hat a(1-\chi_\varrho)|\nabla \mathrm{p}_\varepsilon^t|^2 + C\varrho^{-2}\int_{\varrho\leq|x|\leq 2\varrho}|\mathrm{p}_\varepsilon^t|^2 + C\int_{\mathbb{R}^2}(1-\chi_\varrho)|\Gamma_\varepsilon^t|^2|\mathrm{m}_\varepsilon^t|^2.$$

Absorbing the first right-hand side term, and recalling that Step 2 gives $\|\Gamma_\varepsilon^t\|_{\mathrm{L}^\infty}$, $\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1$, and $\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^p} \lesssim_{p,t} 1$ for all $2<p<\infty$, we obtain with the Hölder inequality,

$$\int_{\mathbb{R}^2}(1-\chi_\varrho)|\nabla \mathrm{p}_\varepsilon^t|^2 \lesssim_t \varrho^{-2}\int_{\varrho\leq|x|\leq 2\varrho}|\mathrm{p}_\varepsilon^t|^2 + \int_{\mathbb{R}^2}(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2 \lesssim_{p,t} \varrho^{-4/p} + \int_{\mathbb{R}^2}(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2,$$

and thus for all $2<p<\infty$,

$$\|\nabla(\mathrm{p}_{\varepsilon,\varrho}^t - \mathrm{p}_\varepsilon^t)\|^2_{\mathrm{L}^2} \lesssim \int_{\mathbb{R}^2}(1-\chi_\varrho)|\nabla \mathrm{p}_\varepsilon^t|^2 + \varrho^{-2}\int_{\varrho\leq|x|\leq 2\varrho}|\mathrm{p}_\varepsilon^t|^2 \lesssim_{p,t} \varrho^{-4/p} + \int_{\mathbb{R}^2}(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2.$$

It remains to estimate the last right-hand side term. For all $t\geq 0$, using again the bounds of Step 2 and the estimate $|\nabla\chi_\varrho| \lesssim \varrho^{-1}(1-\chi_\varrho)^{1/2}$, we deduce from equation (8.71),

$$\begin{aligned}
\partial_t\int_{\mathbb{R}^2}(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2 &= 2\int_{\mathbb{R}^2}(1-\chi_\varrho)\,\mathrm{m}_\varepsilon^t\,\mathrm{curl}\,(\Gamma_\varepsilon^t\mathrm{m}_\varepsilon^t)\\
&= 2\int_{\mathbb{R}^2}|\mathrm{m}_\varepsilon^t|^2\Gamma_\varepsilon^t\cdot\nabla^\perp\chi_\varrho - \int_{\mathbb{R}^2}(1-\chi_\varrho)\Gamma_\varepsilon^t\cdot\nabla^\perp|\mathrm{m}_\varepsilon^t|^2\\
&= 2\int_{\mathbb{R}^2}|\mathrm{m}_\varepsilon^t|^2\Gamma_\varepsilon^t\cdot\nabla^\perp\chi_\varrho + \int_{\mathbb{R}^2}|\mathrm{m}_\varepsilon^t|^2\mathrm{curl}\,\big((1-\chi_\varrho)\Gamma_\varepsilon^t\big)\\
&\lesssim_t \varrho^{-1}\int_{\mathbb{R}^2}(1-\chi_\varrho)^{1/2}|\mathrm{m}_\varepsilon^t|^2 + \int_{\mathbb{R}^2}(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2\\
&\lesssim_t \varrho^{-2} + \int_{\mathbb{R}^2}(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2,
\end{aligned}$$

hence by the Grönwall inequality,

$$\int_{\mathbb{R}^2}(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2 \lesssim_t \varrho^{-2} + \int_{\mathbb{R}^2}(1-\chi_\varrho)|\mathrm{curl}\,\mathrm{v}_\varepsilon^\circ|^2,$$

and the result (8.72) follows. ☐

We now show how to pass to the limit in equation (8.53) as $\varepsilon\downarrow 0$, which is easily achieved by a Grönwall argument on the L^2-distance between v_ε and the solution v of the limiting equation. Note that in the limit pinning effects remain only in the constraint.

Lemma 8.3.5. *Let the same assumptions hold as in Proposition 8.3.4, and let* $\mathrm{v}_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ *be the corresponding global solution of* (8.53). *Then, in the regime* (GP) *with* $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$*, we have* $\mathrm{v}_\varepsilon \to \mathrm{v}$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2)^2)$ *as* $\varepsilon \downarrow 0$*, where* v *is the unique global (smooth) solution of*

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (-\hat{F} + 2\mathrm{v}^\perp)\mathrm{curl}\,\mathrm{v}, \qquad \mathrm{div}\,(\hat{a}\mathrm{v}) = 0, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ\,. \tag{8.77}$$

$\Diamond$

Proof. Using the choice of the scalings for $\lambda_\varepsilon, h, F$ in the regime (GP), equation (8.53) takes on the following guise,

$$\partial_t \mathrm{v}_\varepsilon = \nabla \mathrm{p}_\varepsilon + \big(\lambda_\varepsilon^{-1}\nabla\hat{h} - \hat{F} + 2\mathrm{v}_\varepsilon^\perp\big)\mathrm{curl}\,\mathrm{v}_\varepsilon, \qquad \mathrm{div}\,(\hat{a}\mathrm{v}_\varepsilon) = 0, \qquad \mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}^\circ\,.$$

As $\lambda_\varepsilon^{-1} \to 0$, it is formally clear that v_ε should converge to the solution v of equation (8.77) as $\varepsilon \downarrow 0$. Note that the existence, uniqueness, and regularity of this solution v are given by Proposition 8.3.4 just as for v_ε, and we have in particular the following bounds for all $t \geq 0$,

$$\|(\mathrm{v}^t, \mathrm{v}^t_\varepsilon)\|_{W^{1,\infty}} \lesssim_t 1, \qquad \|\mathrm{curl}\,\mathrm{v}^t_\varepsilon\|_{\mathrm{L}^1} = 1, \qquad \|(\mathrm{p}^t, \mathrm{p}^t_\varepsilon)\|_{\mathrm{L}^\infty} \lesssim_t 1, \tag{8.78}$$

and for all $\theta > 0$,

$$\|(\mathrm{v}^t, \mathrm{v}^t_\varepsilon)\|_{\mathrm{L}^2(B_R)} \lesssim_{t,\theta} R^\theta, \quad \|(\mathrm{p}^t, \mathrm{p}^t_\varepsilon)\|_{\mathrm{L}^2(B_R)} \lesssim_{t,\theta} R^\theta. \tag{8.79}$$

We denote by $\xi_R^z(x) := e^{-|x-z|/R}$ the exponential cut-off at the scale $R \geq 1$ centered at $z \in R\mathbb{Z}^2$. Using the equations for $\mathrm{v}_\varepsilon, \mathrm{v}$, we find

$$\begin{aligned}\partial_t \int \hat{a}\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 = 2\int \hat{a}\xi_R^z(\mathrm{v}_\varepsilon - \mathrm{v})\cdot\nabla(\mathrm{p}_\varepsilon - \mathrm{p}) + 2\int \hat{a}\xi_R^z(-\hat{F} + 2\mathrm{v}^\perp)\cdot(\mathrm{v}_\varepsilon - \mathrm{v})(\mathrm{curl}\,\mathrm{v}_\varepsilon - \mathrm{curl}\,\mathrm{v})\\ + 2\lambda_\varepsilon^{-1}\int \hat{a}\xi_R^z\nabla\hat{h}\cdot(\mathrm{v}_\varepsilon - \mathrm{v})\mathrm{curl}\,\mathrm{v}_\varepsilon\,.\end{aligned}$$

Integrating by parts in the first right-hand side term with $\mathrm{div}\,(\hat{a}\xi_R^z(\mathrm{v}_\varepsilon - \mathrm{v})) = \hat{a}\nabla\xi_R^z\cdot(\mathrm{v}_\varepsilon - \mathrm{v})$, and using the weighted Delort-type identity (8.69) in the form

$$(\mathrm{v}_\varepsilon - \mathrm{v})\mathrm{curl}\,(\mathrm{v}_\varepsilon - \mathrm{v}) = -\frac{1}{2}|\mathrm{v}_\varepsilon - \mathrm{v}|^2\nabla^\perp\hat{h} - \hat{a}^{-1}(\mathrm{div}\,(\hat{a}S_{\mathrm{v}_\varepsilon - \mathrm{v}}))^\perp,$$

we deduce

$$\begin{aligned}\partial_t \int \hat{a}\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 = -2\int \hat{a}\nabla\xi_R^z\cdot(\mathrm{v}_\varepsilon - \mathrm{v})(\mathrm{p}_\varepsilon - \mathrm{p}) - \int \hat{a}\xi_R^z\nabla^\perp\hat{h}\cdot(-\hat{F} + 2\mathrm{v}^\perp)|\mathrm{v}_\varepsilon - \mathrm{v}|^2\\ + 2\int \hat{a}S_{\mathrm{v}_\varepsilon - \mathrm{v}} : \nabla(\xi_R^z(\hat{F}^\perp + 2\mathrm{v})) + 2\lambda_\varepsilon^{-1}\int \hat{a}\xi_R^z\nabla\hat{h}\cdot(\mathrm{v}_\varepsilon - \mathrm{v})\mathrm{curl}\,\mathrm{v}_\varepsilon,\end{aligned}$$

and hence, using (8.78)–(8.79), the assumption $\|(\nabla\hat{h}, \hat{F})\|_{W^{1,\infty}} \lesssim 1$, the property $|\nabla\xi_R^z| \lesssim R^{-1}\xi_R^z$ of the exponential cut-off, and the pointwise estimate $|S_w| \lesssim |w|^2$,

$$\partial_t \int \hat{a}\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 \lesssim_{t,\theta} R^{-2(1-\theta)} + \lambda_\varepsilon^{-2} + \int \hat{a}\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2.$$

Choosing $\theta = 1/2$, the Grönwall inequality yields $\sup_z \int a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 \lesssim_t R^{-1} + \lambda_\varepsilon^{-2}$, and the conclusion follows, letting $R \uparrow \infty$. $\square$

8.3.3 Degenerate parabolic case

Let us examine the vorticity formulation of equation (8.52) for v_ε. In terms of $\mathrm{m}_\varepsilon := \operatorname{curl} \mathrm{v}_\varepsilon$ and $\mathrm{d}_\varepsilon := \operatorname{div}(a\mathrm{v}_\varepsilon)$, equation (8.52) may be rewritten as a nonlinear nonlocal transport equation for the vorticity m_ε, coupled with a transport-diffusion equation for the divergence d_ε,

$$\begin{cases} \partial_t \mathrm{m}_\varepsilon = -\operatorname{div}(\Gamma_\varepsilon^\perp \mathrm{m}_\varepsilon), \quad \mathrm{m}_\varepsilon|_{t=0} = \operatorname{curl} \mathrm{v}^\circ, \\ \partial_t \mathrm{d}_\varepsilon - \lambda_\varepsilon^{-1} \triangle \mathrm{d}_\varepsilon + \lambda_\varepsilon^{-1} \operatorname{div}(\mathrm{d}_\varepsilon \nabla h) = \operatorname{div}(a \Gamma_\varepsilon \mathrm{m}_\varepsilon), \quad \mathrm{d}_\varepsilon|_{t=0} = \operatorname{div}(a\mathrm{v}^\circ), \\ \operatorname{curl} \mathrm{v}_\varepsilon = \mathrm{m}_\varepsilon, \quad \operatorname{div}(a\mathrm{v}_\varepsilon) = \mathrm{d}_\varepsilon \,. \end{cases}$$

A detailed study of this kind of equations is performed in Chapter 7, including global existence results for vortex-sheet initial data. In the present situation with $\lambda_\varepsilon \uparrow \infty$, the diffusion tends to be degenerate, and more work is thus needed to ensure the validity of uniform a priori estimates. The key consists in suitably exploiting the well-posedness of the corresponding degenerate limiting equation studied in Chapter 7. As an immediate corollary of such estimates, we also deduce that v_ε converges as $\varepsilon \downarrow 0$ to the solution v of this degenerate limiting equation.

Proposition 8.3.6. *Let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, and let $\mathrm{v}_\varepsilon^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ be bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$, and satisfy $\operatorname{curl} \mathrm{v}_\varepsilon^\circ \in \mathcal{P}(\mathbb{R}^2)$. For some $s > 0$, assume that $h \in W^{s+6,\infty}(\mathbb{R}^2)$, $F \in W^{s+5,\infty}(\mathbb{R}^2)^2$, that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{s+5,\infty}(\mathbb{R}^2)^2$, that $\operatorname{curl} \mathrm{v}_\varepsilon^\circ$ is bounded in $H^{s+4}(\mathbb{R}^2)$, and that $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)$ is bounded in $H^{s+3}(\mathbb{R}^2)$.*
Then in the regime (GL$_3$) *with $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$, there exists a unique solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}^\circ + H^{s+4}(\mathbb{R}^2)^2)$ of* (8.52) *on $\mathbb{R}^+ \times \mathbb{R}^2$. Moreover, all the properties of Assumption 8.3.1(a) are satisfied, that is, for all $t \ge 0$ and all $q > 2$, we have for all $\varepsilon > 0$ small enough (only depending on an upper bound on s, s^{-1}, $\|\hat h\|_{W^{s+6,\infty}}$, $\|(\hat F, \mathrm{v}^\circ)\|_{W^{s+5,\infty}}$, $\|\mathrm{v}^\circ\|_{W^{1,q}}$, $\|\mathrm{m}^\circ\|_{H^{s+4}}$, and $\|\mathrm{d}^\circ\|_{H^{s+3}}$),*

$$\begin{gathered} \|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2 + \mathrm{L}^q) \cap \mathrm{L}^\infty} \lesssim_{t,q} 1, \qquad \|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^1 \cap \mathrm{L}^\infty} \lesssim_t 1, \qquad \|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim_t 1, \\ \|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim_t 1, \qquad \|\nabla \mathrm{d}_\varepsilon^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim_t 1, \qquad \|\partial_t \mathrm{d}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1. \end{gathered} \tag{8.80}$$

In addition, there holds $\mathrm{v}_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}^\circ + H^{s+3}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$, where $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}^\circ + H^{s+4} \cap W^{s+4,\infty}(\mathbb{R}^2)^2)$ is the unique global solution of

$$\partial_t \mathrm{v} = -(\hat F^\perp + 2\mathrm{v}) \operatorname{curl} \mathrm{v}, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ \,. \tag{8.81}$$

$\Diamond$

Proof. Direct estimates on v_ε as in Chapter 7 are not uniform with respect to $\lambda_\varepsilon \gg 1$. As we show, however, exploiting strong a priori estimates on the limiting solution v allows to deduce the desired uniform estimates on v_ε. We split the proof into two steps.

Step 1. A priori estimates.

Let $s > 0$, and assume that $\hat h \in W^{s+3,\infty}(\mathbb{R}^2)$, $\hat F \in W^{s+2,\infty}(\mathbb{R}^2)^2$, and that there exists a unique global solution v of equation (8.81) with $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W^{s+2,\infty}(\mathbb{R}^2)^2)$ and $\mathrm{m}, \mathrm{d} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s+2}(\mathbb{R}^2))$. Also assume that there exists a unique global solution v_ε of (8.52) in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}^\circ + H^{s+2}(\mathbb{R}^2))$. In this step, we consider the regime $\lambda_\varepsilon \gg 1$, and we show that for any fixed $t \ge 0$ we have for all $\varepsilon > 0$ small enough (that is, for all λ_ε large enough),

$$\begin{gathered} \|\mathrm{v}_\varepsilon - \mathrm{v}\|_{\mathrm{L}^\infty_t H^{s+1}} + \|\mathrm{m}_\varepsilon - \mathrm{m}\|_{\mathrm{L}^\infty_t H^{s+1}} + \|\mathrm{d}_\varepsilon - \mathrm{d}\|_{\mathrm{L}^\infty_t H^s} \le C_t \lambda_\varepsilon^{-1}, \\ \|\mathrm{d}_\varepsilon - \mathrm{d}\|_{\mathrm{L}^\infty_t H^{s+1}} \le C_t \lambda_\varepsilon^{-1/2}, \end{gathered} \tag{8.82}$$

hence in particular,

$$\|\mathrm{v}_\varepsilon - \mathrm{v}^\circ\|_{\mathrm{L}^\infty_t H^{s+2}} + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}} + \|\mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}} \le C_t, \tag{8.83}$$

where the constant C_t only depends on an upper bound on λ_ε^{-1}, s, s^{-1}, $\|\hat h\|_{W^{s+3,\infty}}$, $\|\hat F\|_{W^{s+2,\infty}}$, $\|\mathrm{v}\|_{\mathrm{L}^\infty_t W^{s+2,\infty}}$, $\|(\mathrm{m},\mathrm{d})\|_{\mathrm{L}^\infty_t H^{s+2}}$, $\|\mathrm{v}-\mathrm{v}^\circ\|_{\mathrm{L}^\infty_t \mathrm{L}^2}$, and on time t. We split the proof into six further substeps. In this proof, we use the notation $\lesssim_t$ for $\le$ up to a constant $C_t>0$ as above, and we use the notation $\lesssim$ for $\le$ up to a constant that depends only on an upper bound on λ_ε^{-1}, $\|\hat h\|_{W^{s+3,\infty}}$, and on $\|\hat F\|_{W^{s+2,\infty}}$.

Substep 1.1. Notation.

Define $\delta\mathrm{v}_\varepsilon := \lambda_\varepsilon(\mathrm{v}_\varepsilon-\mathrm{v})$, $\delta\mathrm{m}_\varepsilon := \operatorname{curl}\delta\mathrm{v}_\varepsilon = \lambda_\varepsilon(\mathrm{m}_\varepsilon-\mathrm{m})$, and $\delta\mathrm{d}_\varepsilon := \operatorname{div}(\hat a\delta\mathrm{v}_\varepsilon) = \lambda_\varepsilon(\mathrm{d}_\varepsilon-\mathrm{d})$. Given the choice of the scalings, equation (8.52) for v_ε takes on the following guise,

$$\partial_t\mathrm{v}_\varepsilon = \lambda_\varepsilon^{-1}\nabla(\hat a^{-1}\mathrm{d}_\varepsilon) + \big(\lambda_\varepsilon^{-1}\nabla^\perp\hat h - \hat F^\perp - 2\mathrm{v}_\varepsilon\big)\,\mathrm{m}_\varepsilon, \tag{8.84}$$

and hence decomposing $\mathrm{v}_\varepsilon = \mathrm{v}+\lambda_\varepsilon^{-1}\delta\mathrm{v}_\varepsilon$ leads to

$$\begin{aligned}\partial_t\mathrm{v}+\lambda_\varepsilon^{-1}\partial_t\delta\mathrm{v}_\varepsilon = -(\hat F^\perp+2\mathrm{v})\,\mathrm{m}+\lambda_\varepsilon^{-1}\Big(\nabla(\hat a^{-1}\mathrm{d}) + \mathrm{m}\,\nabla^\perp\hat h - \hat F^\perp\delta\mathrm{m}_\varepsilon - 2\mathrm{v}\delta\mathrm{m}_\varepsilon - 2\mathrm{m}\delta\mathrm{v}_\varepsilon\Big)\\ + \lambda_\varepsilon^{-2}\Big(\nabla(\hat a^{-1}\delta\mathrm{d}_\varepsilon) + \delta\mathrm{m}_\varepsilon\nabla^\perp\hat h - 2\delta\mathrm{v}_\varepsilon\delta\mathrm{m}_\varepsilon\Big).\end{aligned}$$

Injecting equation (8.81) for v and multiplying both sides by λ_ε, we obtain the following equation for $\delta\mathrm{v}_\varepsilon$,

$$\partial_t\delta\mathrm{v}_\varepsilon = \lambda_\varepsilon^{-1}\nabla(\hat a^{-1}\delta\mathrm{d}_\varepsilon) + (W_\varepsilon - 2\lambda_\varepsilon^{-1}\delta\mathrm{v}_\varepsilon)\delta\mathrm{m}_\varepsilon - 2\mathrm{m}\delta\mathrm{v}_\varepsilon + G, \tag{8.85}$$

with initial data $\delta\mathrm{v}_\varepsilon|_{t=0}=0$, where we have set

$$G := \nabla(\hat a^{-1}\mathrm{d}) + \mathrm{m}\,\nabla^\perp\hat h, \qquad W_\varepsilon := \lambda_\varepsilon^{-1}\nabla^\perp\hat h - \hat F^\perp - 2\mathrm{v}\,.$$

Taking the curl of equation (8.85) leads to

$$\partial_t\delta\mathrm{m}_\varepsilon = -\operatorname{div}\big((W_\varepsilon^\perp - 2\lambda_\varepsilon^{-1}\delta\mathrm{v}_\varepsilon^\perp)\delta\mathrm{m}_\varepsilon\big) + 2\delta\mathrm{v}_\varepsilon^\perp\cdot\nabla\mathrm{m} - 2\mathrm{m}\,\delta\mathrm{m}_\varepsilon + \operatorname{curl}G, \tag{8.86}$$

while applying the operator $\operatorname{div}(\hat a\,\cdot)$ yields

$$\partial_t\delta\mathrm{d}_\varepsilon = \lambda_\varepsilon^{-1}\triangle\delta\mathrm{d}_\varepsilon - \lambda_\varepsilon^{-1}\operatorname{div}(\delta\mathrm{d}_\varepsilon\nabla\hat h) + \operatorname{div}\big(\hat a(W_\varepsilon - 2\lambda_\varepsilon^{-1}\delta\mathrm{v}_\varepsilon)\delta\mathrm{m}_\varepsilon\big) - 2\operatorname{div}(\hat a\mathrm{m}\delta\mathrm{v}_\varepsilon) + \operatorname{div}(\hat aG), \tag{8.87}$$

with initial data $\delta\mathrm{m}_\varepsilon|_{t=0}=0$ and $\delta\mathrm{d}_\varepsilon|_{t=0}=0$. Proving the result (8.82) thus amounts to establishing uniform a priori estimates for the solutions $\delta\mathrm{v}_\varepsilon$, $\delta\mathrm{m}_\varepsilon$, and $\delta\mathrm{d}_\varepsilon$ of the above equations.

Substep 1.2. L^2-estimate on $\delta\mathrm{v}_\varepsilon$ and $\delta\mathrm{m}_\varepsilon$.

In this step, we show that

$$\|\delta\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^2} + \|\delta\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t(\dot H^{-1}\cap\mathrm{L}^2)} + \|\delta\mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t\dot H^{-1}} \lesssim_t 1. \tag{8.88}$$

On the one hand, from equation (8.85), noting that $-2\lambda_\varepsilon^{-1}\delta\mathrm{v}_\varepsilon\,\delta\mathrm{m}_\varepsilon - 2\mathrm{m}\delta\mathrm{v}_\varepsilon = -2\mathrm{m}_\varepsilon\delta\mathrm{v}_\varepsilon$, we find by integration by parts,

$$\partial_t\int_{\mathbb R^2}\hat a|\delta\mathrm{v}_\varepsilon|^2 = -2\lambda_\varepsilon^{-1}\int_{\mathbb R^2}\hat a^{-1}|\delta\mathrm{d}_\varepsilon|^2 + 2\int_{\mathbb R^2}\hat a\delta\mathrm{v}_\varepsilon\cdot\big(W_\varepsilon\delta\mathrm{m}_\varepsilon - 2\mathrm{m}_\varepsilon\delta\mathrm{v}_\varepsilon + G\big) \le 2\int_{\mathbb R^2}\hat a\delta\mathrm{v}_\varepsilon\cdot\big(W_\varepsilon\delta\mathrm{m}_\varepsilon + G\big),$$

and hence, using the Cauchy-Schwarz inequality and injecting the definition of G and W_ε,

$$\begin{aligned}\partial_t\Big(\int_{\mathbb R^2}\hat a|\delta\mathrm{v}_\varepsilon^t|^2\Big)^{1/2} &\le \|W_\varepsilon^t\|_{\mathrm{L}^\infty}\Big(\int_{\mathbb R^2}\hat a|\delta\mathrm{m}_\varepsilon^t|^2\Big)^{1/2} + \Big(\int_{\mathbb R^2}\hat a|G^t|^2\Big)^{1/2}\\ &\lesssim \|(\nabla\hat h,\hat F,\mathrm{v}^t)\|_{\mathrm{L}^\infty}\|\delta\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2} + \|\operatorname{div}(\hat a\mathrm{v}^t)\|_{H^1} + \|\mathrm{m}^t\|_{\mathrm{L}^2}\\ &\lesssim_t 1 + \|\delta\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2},\end{aligned}$$

that is,

$$\|\delta \mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \lesssim_t \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^2} + 1. \tag{8.89}$$

On the other hand, equation (8.86) yields by integration by parts,

$$\partial_t \int_{\mathbb{R}^2} |\delta \mathrm{m}_\varepsilon|^2 = \int_{\mathbb{R}^2} |\delta \mathrm{m}_\varepsilon|^2 \operatorname{div}(-W_\varepsilon^\perp + 2\lambda_\varepsilon^{-1}\delta \mathrm{v}_\varepsilon^\perp) - 4\int_{\mathbb{R}^2} |\delta \mathrm{m}_\varepsilon|^2 \mathrm{m} + 2\int_{\mathbb{R}^2} \delta \mathrm{m}_\varepsilon \big(2\delta \mathrm{v}_\varepsilon^\perp \cdot \nabla \mathrm{m} + \operatorname{curl} G\big),$$

and hence, decomposing $\operatorname{div}(\lambda_\varepsilon^{-1}\delta \mathrm{v}_\varepsilon^\perp) = -\lambda_\varepsilon^{-1}\delta \mathrm{m}_\varepsilon = \mathrm{m} - \mathrm{m}_\varepsilon \le \mathrm{m}$,

$$\begin{aligned}\partial_t \int_{\mathbb{R}^2} |\delta \mathrm{m}_\varepsilon|^2 &\le \int_{\mathbb{R}^2} |\delta \mathrm{m}_\varepsilon|^2 \operatorname{curl} W_\varepsilon + 2\int_{\mathbb{R}^2} \delta \mathrm{m}_\varepsilon \big(2\delta \mathrm{v}_\varepsilon^\perp \cdot \nabla \mathrm{m} + \operatorname{curl} G\big)\\ &\le \|\nabla W_\varepsilon\|_{\mathrm{L}^\infty} \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^2}^2 + 4\|\nabla \mathrm{m}\|_{\mathrm{L}^\infty} \|\delta \mathrm{v}_\varepsilon\|_{\mathrm{L}^2} \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^2} + 2\|\operatorname{curl} G\|_{\mathrm{L}^2} \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^2}.\end{aligned}$$

Injecting the definition of G and W_ε with $\lambda_\varepsilon^{-1} \lesssim 1$, and using (8.89) to estimate the L^2-norm of $\delta \mathrm{v}_\varepsilon$ in the right-hand side, we deduce

$$\partial_t \|\delta \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t \|\delta \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2} + \|\delta \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2} + 1 \lesssim_t \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^2} + 1.$$

Combining this with (8.89) and with the obvious estimate $\|(\delta \mathrm{m}_\varepsilon, \delta \mathrm{d}_\varepsilon)\|_{\dot H^{-1}} \lesssim \|\delta \mathrm{v}_\varepsilon\|_{\mathrm{L}^2}$, the conclusion (8.88) follows from the Grönwall inequality.

Substep 1.3. H^{s+1}-estimate on $\delta \mathrm{m}_\varepsilon$.

In this step, we show that

$$\partial_t \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} \lesssim_t 1 + \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} + \|\delta \mathrm{d}_\varepsilon\|_{H^s} + \lambda_\varepsilon^{-1}\big(\|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}}^2 + \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} \|\delta \mathrm{d}_\varepsilon\|_{H^{s+1}}\big). \tag{8.90}$$

Arguing as in the proof of Lemma 7.2.2 in Chapter 7, we may compute as follows the time derivative of the H^s-norm of the vorticity $\delta \mathrm{m}_\varepsilon$, with $s > 0$,

$$\begin{aligned}\partial_t \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} &\le \frac12 \|\operatorname{div}(W_\varepsilon^\perp - 2\lambda_\varepsilon^{-1}\delta \mathrm{v}_\varepsilon^\perp)\|_{\mathrm{L}^\infty} \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} + \|[\langle \nabla \rangle^{s+1} \operatorname{div}, W_\varepsilon^\perp] \delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^2}\\ &\quad + 2\lambda_\varepsilon^{-1} \|[\langle \nabla \rangle^{s+1} \operatorname{div}, \delta \mathrm{v}_\varepsilon^\perp] \delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^2} + 2\|\delta \mathrm{v}_\varepsilon^\perp \cdot \nabla \mathrm{m}\|_{H^{s+1}} + 2\|\mathrm{m}\, \delta \mathrm{m}_\varepsilon\|_{H^{s+1}} + \|\operatorname{curl} G\|_{H^{s+1}}\\ &\lesssim (\|W_\varepsilon\|_{W^{s+2,\infty}} + \|\mathrm{m}\|_{W^{s+1,\infty}}) \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} + \|\mathrm{m}\|_{H^{s+2}} \|\delta \mathrm{v}_\varepsilon\|_{H^{s+1}}\\ &\quad + \lambda_\varepsilon^{-1}\big(\|\delta \mathrm{v}_\varepsilon\|_{W^{1,\infty}} \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} + \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty} \|\delta \mathrm{v}_\varepsilon\|_{H^{s+2}}\big) + \|\operatorname{curl} G\|_{H^{s+1}}.\end{aligned}$$

Injecting the definition of G and W_ε with $\lambda_\varepsilon^{-1} \lesssim 1$, and using the Sobolev embedding with $s > 0$, we find

$$\partial_t \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} \lesssim_t \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} + \|\delta \mathrm{v}_\varepsilon\|_{H^{s+1}} + \lambda_\varepsilon^{-1} \|\delta \mathrm{v}_\varepsilon\|_{H^{s+2}} \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} + 1. \tag{8.91}$$

Decomposing $\delta \mathrm{v}_\varepsilon = \hat a^{-1} \nabla^\perp (\operatorname{div} \hat a^{-1} \nabla)^{-1} \delta \mathrm{m}_\varepsilon + \nabla (\operatorname{div} \hat a \nabla)^{-1} \delta \mathrm{d}_\varepsilon$, we may apply Lemma 7.2.6 in the form

$$\|\delta \mathrm{v}_\varepsilon\|_{H^{r+1}} \lesssim \|\delta \mathrm{m}_\varepsilon\|_{\dot H^{-1} \cap H^r} + \|\delta \mathrm{d}_\varepsilon\|_{\dot H^{-1} \cap H^r}, \tag{8.92}$$

with $r = s$ and $r = s+1$. Injecting this into (8.91), and using the result (8.88) of Substep 1.2 in the form $\|(\delta \mathrm{m}_\varepsilon, \delta \mathrm{d}_\varepsilon)\|_{\dot H^{-1}} \lesssim_t 1$, the conclusion (8.90) follows.

Substep 1.4. H^{s+1}-estimate on $\delta \mathrm{d}_\varepsilon$ without loss of derivative.

In this step we show that

$$\lambda_\varepsilon^{-1/2} \|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}} \lesssim_t 1 + \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}} + \|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^s} + \lambda_\varepsilon^{-1} \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}^2. \tag{8.93}$$

Equation (8.87) for the divergence $\delta \mathrm{d}_\varepsilon$ takes the form $\partial_t \delta \mathrm{d}_\varepsilon = \lambda_\varepsilon^{-1} \triangle \delta \mathrm{d}_\varepsilon + \operatorname{div} H_\varepsilon$, where we have set

$$H_\varepsilon := -\lambda_\varepsilon^{-1} \delta \mathrm{d}_\varepsilon \, \nabla \hat{h} + a(W_\varepsilon - 2\lambda_\varepsilon^{-1} \delta \mathrm{v}_\varepsilon) \delta \mathrm{m}_\varepsilon - 2a\mathrm{m} \delta \mathrm{v}_\varepsilon + aG.$$

Arguing as in the proof of Lemma 7.2.3(i) in Chapter 7, testing this equation with $(-\triangle)^{-1} \langle \nabla \rangle^{2(s+1)} \partial_t \delta \mathrm{d}_\varepsilon$, we find

$$\lambda_\varepsilon^{-1} \|\delta \mathrm{d}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} \le \int_0^t \|H_\varepsilon^u\|^2_{H^{s+1}} du,$$

and hence, injecting the definitions of H_ε, G, and W_ε, with $s > 0$,

$$\lambda_\varepsilon^{-1} \|\delta \mathrm{d}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} \lesssim_t \lambda_\varepsilon^{-2} \int_0^t \|\delta \mathrm{d}_\varepsilon^u\|^2_{H^{s+1}} du + \lambda_\varepsilon^{-2} \|\delta \mathrm{v}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} \|\delta \mathrm{m}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} \\ + \|\delta \mathrm{m}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} + \|\delta \mathrm{v}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} + 1.$$

The Grönwall inequality with $\lambda_\varepsilon^{-1} \lesssim 1$ then yields

$$\lambda_\varepsilon^{-1} \|\delta \mathrm{d}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} \lesssim_t 1 + \|\delta \mathrm{m}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} + \|\delta \mathrm{v}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} + \lambda_\varepsilon^{-2} \|\delta \mathrm{v}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} \|\delta \mathrm{m}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}}.$$

The result (8.93) follows from this together with the bound (8.92) and with the result (8.88) of Substep 1.2 in the form $\|(\delta \mathrm{m}_\varepsilon, \delta \mathrm{d}_\varepsilon)\|_{\dot{H}^{-1}} \lesssim_t 1$.

Substep 1.5. H^s-estimate on $\delta \mathrm{d}_\varepsilon$ with loss of derivative.

In this step, we show that

$$\partial_t \|\delta \mathrm{d}_\varepsilon\|_{H^s} \lesssim_t 1 + \|\delta \mathrm{d}_\varepsilon\|_{H^s} + \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} + \lambda_\varepsilon^{-1} \big(\|\delta \mathrm{m}_\varepsilon\|^2_{H^{s+1}} + \|\delta \mathrm{d}_\varepsilon\|_{H^s} \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} \big). \tag{8.94}$$

Equation (8.87) for the divergence $\delta \mathrm{d}_\varepsilon$ yields after integration by parts,

$$\begin{aligned} \partial_t \|\delta \mathrm{d}_\varepsilon\|^2_{H^s} &\le -2\lambda_\varepsilon^{-1} \int_{\mathbb{R}^2} |\nabla \langle \nabla \rangle^s \delta \mathrm{d}_\varepsilon|^2 + 2\lambda_\varepsilon^{-1} \int_{\mathbb{R}^2} \langle \nabla \rangle^s (\delta \mathrm{d}_\varepsilon \nabla \hat{h}) \cdot \nabla \langle \nabla \rangle^s \delta \mathrm{d}_\varepsilon \\ &\qquad + 2 \int_{\mathbb{R}^2} \big(\langle \nabla \rangle^s \delta \mathrm{d}_\varepsilon \big) \operatorname{div} \langle \nabla \rangle^s \big(a(W_\varepsilon - 2\lambda_\varepsilon^{-1} \delta \mathrm{v}_\varepsilon) \delta \mathrm{m}_\varepsilon - 2a\mathrm{m} \delta \mathrm{v}_\varepsilon + aG \big) \\ &\le \lambda_\varepsilon^{-1} \|\delta \mathrm{d}_\varepsilon \nabla \hat{h}\|^2_{H^s} + 2\|\delta \mathrm{d}_\varepsilon\|_{H^s} \big(\|a(W_\varepsilon - 2\lambda_\varepsilon^{-1} \delta \mathrm{v}_\varepsilon) \delta \mathrm{m}_\varepsilon + aG\|_{H^{s+1}} + 2\|\mathrm{m} \delta \mathrm{d}_\varepsilon\|_{H^s} + 2\|a \delta \mathrm{v}_\varepsilon \cdot \nabla \mathrm{m}\|_{H^s} \big), \end{aligned}$$

and hence, injecting the definition of G and W_ε,

$$\partial_t \|\delta \mathrm{d}_\varepsilon\|_{H^s} \lesssim_t 1 + \|\delta \mathrm{d}_\varepsilon\|_{H^s} + \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}} + \|\delta \mathrm{v}_\varepsilon\|_{H^s} + \lambda_\varepsilon^{-1} \|\delta \mathrm{v}_\varepsilon\|_{H^{s+1}} \|\delta \mathrm{m}_\varepsilon\|_{H^{s+1}}.$$

The result (8.94) follows from this together with the bound (8.92) and with the result (8.88) of Substep 1.2 in the form $\|(\delta \mathrm{m}_\varepsilon, \delta \mathrm{d}_\varepsilon)\|_{\dot{H}^{-1}} \lesssim_t 1$.

Substep 1.6. Proof of (8.82) and (8.83).

Injecting (8.93) into (8.90) with $\lambda_\varepsilon^{-1} \lesssim 1$, we find

$$\partial_t \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}} \lesssim_t 1 + \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}} + \|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^s} \\ + \lambda_\varepsilon^{-1/2} \big(\|\delta \mathrm{m}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} + \|\delta \mathrm{d}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^s} \big) + \lambda_\varepsilon^{-3/2} \|\delta \mathrm{m}_\varepsilon\|^3_{\mathrm{L}^\infty_t H^{s+1}}.$$

Together with (8.94), this yields

$$\begin{aligned} &\partial_t \big(\|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}} + \|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^s} \big) \\ &\quad \lesssim_t \quad 1 + \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}} + \|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^s} + \lambda_\varepsilon^{-1/2} \big(\|\delta \mathrm{m}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^{s+1}} + \|\delta \mathrm{d}_\varepsilon\|^2_{\mathrm{L}^\infty_t H^s} \big) + \lambda_\varepsilon^{-3/2} \|\delta \mathrm{m}_\varepsilon\|^3_{\mathrm{L}^\infty_t H^{s+1}} \\ &\quad \lesssim_t \quad 1 + \|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}} + \|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^s} + \lambda_\varepsilon^{-3/4} (\|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}} + \|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^s})^3. \end{aligned}$$

We then deduce by integration,

$$\begin{aligned}\|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s}}&\le C_t\Big(1+\lambda_\varepsilon^{-3/4}\big(\|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s}}\big)^2\Big)^{1/2}\\&\le C_t+C_t\lambda_\varepsilon^{-3/8}\big(\|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s}}\big).\end{aligned}$$

For any time $t\ge 0$, choosing $\varepsilon>0$ small enough such that $2C_t\lambda_\varepsilon^{-3/8}\le 1$, we obtain

$$\|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s}}\lesssim_t 1.$$

Combining this with the bound (8.92) and with the result (8.88) of Substep 1.2 in the form of $\|(\delta \mathrm{m}_\varepsilon,\delta \mathrm{d}_\varepsilon)\|_{\dot H^{-1}}\lesssim_t 1$, we actually have

$$\|\delta \mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s}}\lesssim_t 1.$$

Injecting this into the result (8.93) of Substep 1.4, we find

$$\|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}\lesssim_t \lambda_\varepsilon^{1/2},$$

and the result (8.82) follows. Further decomposing $\mathrm{v}_\varepsilon=\mathrm{v}+\lambda_\varepsilon^{-1}\delta\mathrm{v}_\varepsilon$, these results yield

$$\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}\lesssim_t 1.$$

Again combining this with the bound (8.92), we obtain

$$\begin{aligned}\|\mathrm{v}_\varepsilon-\mathrm{v}^\circ\|_{\mathrm{L}^\infty_t H^{s+2}}&\lesssim\|\mathrm{m}_\varepsilon-\mathrm{m}^\circ\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\mathrm{d}_\varepsilon-\mathrm{d}^\circ\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\mathrm{v}_\varepsilon-\mathrm{v}^\circ\|_{\mathrm{L}^\infty_t \mathrm{L}^2}\\&\lesssim_t 1+\lambda_\varepsilon^{-1}\big(\|\delta \mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\delta \mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\delta \mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^2}\big)\lesssim_t 1,\end{aligned}$$

and the result (8.83) follows.

Step 2. Conclusion.

Let $s>1$, and assume that $\hat h\in W^{s+5,\infty}(\mathbb{R}^2)$, $\hat F\in W^{s+4,\infty}(\mathbb{R}^2)^2$, $\mathrm{v}^\circ\in W^{s+3,\infty}(\mathbb{R}^2)^2$, $\operatorname{curl}\mathrm{v}^\circ\in H^{s+3}\cap W^{s+3,\infty}(\mathbb{R}^2)$, and $\operatorname{div}(a\mathrm{v}^\circ)\in H^{s+2}(\mathbb{R}^2)$. In the sequel, we use the notation $\lesssim$ for $\le$ up to a constant that depends only on an upper bound on the norms of these data and on s and $(s-1)^{-1}$, and we write $\lesssim_t$ to indicate the further dependence on an upper bound on time t.

Under these assumptions we know from Theorem 7.1.6 in Chapter 7 that equation (8.81) admits a unique global solution v with $\mathrm{v}-\mathrm{v}^\circ\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{s+3}\cap W^{s+3,\infty}(\mathbb{R}^2)^2)$, which implies in particular

$$\|\mathrm{v}-\mathrm{v}^\circ\|_{\mathrm{L}^\infty_t H^{s+3}}+\|\mathrm{v}\|_{\mathrm{L}^\infty_t W^{s+3,\infty}}+\|(\mathrm{m},\mathrm{d})\|_{\mathrm{L}^\infty_t H^{s+2}}\lesssim_t 1.$$

In addition, we know from Theorem 7.1.3(i) that equation (8.52) also admits a unique global solution $\mathrm{v}_\varepsilon\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{v}^\circ+H^{s+3}(\mathbb{R}^2))$. We may thus apply the result of Step 1, which for any $t\ge0$ yields for all $\varepsilon>0$ small enough,

$$\|\mathrm{v}_\varepsilon-\mathrm{v}^\circ\|_{\mathrm{L}^\infty_t H^{s+2}}+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}+\|\mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t H^{s+1}}\lesssim_t 1.$$

As $s>1$, this implies by the Sobolev embedding,

$$\|\mathrm{v}_\varepsilon-\mathrm{v}^\circ\|_{\mathrm{L}^\infty_t (H^3\cap W^{2,\infty})}+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t (H^2\cap W^{1,\infty})}+\|\mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t (H^2\cap W^{1,\infty})}\lesssim_t 1,$$

and hence, using these bounds in equation (8.84),

$$\|\partial_t\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t (H^1\cap \mathrm{L}^\infty)}+\|\partial_t\mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^2}\lesssim_t 1.$$

The desired estimates (8.80) follow. Finally, the convergence $\mathrm{v}_\varepsilon\to\mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{v}^\circ+H^{s+2}(\mathbb{R}^2)^2)$ as $\varepsilon\downarrow 0$ is a direct consequence of the result (8.82) of Step 1 with $\lambda_\varepsilon\gg 1$. $\square$

8.4 Computations on the modulated energy

In this section, we adapt to the weighted case with pinning and forcing the computations of Serfaty [395]: we compute the time derivative of the modulated energy excess (8.15) and express it with only quadratic terms in the error instead of terms which initially appear as linear and would thus make a Grönwall argument impossible. These computations are based on purely algebraic manipulations using all the equations and appropriate quantities that we will now describe.

For simplicity, in the estimates in this section, we focus on the non-oscillating case $\eta_\varepsilon = 1$, and we consider each of the regimes (GL_1), (GL_2), (GL_3), (GP), (GL_1'), and (GL_2').

8.4.1 Modulated energy

We first recall the definitions of modulated energy and energy excess in (8.12)–(8.15). In order to prove that the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon := N_\varepsilon^{-1}\langle \nabla u_\varepsilon, iu_\varepsilon\rangle$ is close to v_ε, we follow the strategy of Serfaty [395], considering the following *modulated energy*, which is modeled on the weighted Ginzburg-Landau energy, plays the role of an adapted measure of the distance between $N_\varepsilon^{-1} j_\varepsilon$ and v_ε, and is localized by means of the cut-off function χ_R at some scale $R \gg 1$ (to be later optimized as a function of ε),

$$\mathcal{E}_{\varepsilon,R} := \int_{\mathbb{R}^2} \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).$$

As usual, this modulated energy $\mathcal{E}_{\varepsilon,R}$ further needs to be renormalized by subtracting the expected self-interaction energy of the vortices (compare with Lemma 8.5.1 below), which then yields the following *modulated energy excess*,

$$\mathcal{D}_{\varepsilon,R} := \mathcal{E}_{\varepsilon,R} - \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R\mu_\varepsilon = \int_{\mathbb{R}^2}\frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big).$$

As explained in the introduction, the cut-off χ_R is not needed in the Gross-Pitaevskii case, where we only treat the case when h and F decay at infinity. We write $\mathcal{E}_\varepsilon := \mathcal{E}_{\varepsilon,\infty}$ for the corresponding quantity without the cut-off χ_R in the definition (formally $R=\infty$), and also $\mathcal{D}_\varepsilon := \sup_{R\geq 1}\mathcal{D}_{\varepsilon,R}$.

On the one hand, rather than the L^2-norm restricted to the ball B_R centered at the origin, our methods further allow to consider the uniform $\mathrm{L}^2_{\mathrm{loc}}$-norm at the scale R: setting $\chi_R^z := \chi_R(\cdot - z)$, we define

$$\mathcal{E}^*_{\varepsilon,R} := \sup_z \mathcal{E}^z_{\varepsilon,R}, \qquad \mathcal{E}^z_{\varepsilon,R} := \int_{\mathbb{R}^2}\frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big),$$

where henceforth the supremum always implicitly runs over all lattice points $z \in R\mathbb{Z}^2$, and similarly

$$\mathcal{D}^*_{\varepsilon,R} := \sup_z \mathcal{D}^z_{\varepsilon,R}, \qquad \mathcal{D}^z_{\varepsilon,R} := \mathcal{E}^z_{\varepsilon,R} - \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\mu_\varepsilon.$$

Note that by definition we have for all $x\in\mathbb{R}^2$ and all $L>0$,

$$\|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_L(x))} + \varepsilon^{-1}\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_L(x))} \lesssim \Big(1+\frac{L}{R}\Big)^d \mathcal{E}^*_{\varepsilon,R}. \tag{8.95}$$

On the other hand, in order to simplify computations, we need as in [395] to add some suitable lower-order term, and rather consider, for some other scale $\varrho \gg 1$ (to be also later optimized as a function of ε),

$$\hat{\mathcal{E}}_{\varepsilon,\varrho,R} := \int_{\mathbb{R}^2}\frac{a}{2}\Big(\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} + f\chi_R)\Big),$$

and similarly for the modulated energy excess,

$$\begin{aligned}\hat{\mathcal{D}}_{\varepsilon,\varrho,R} &:= \hat{\mathcal{E}}_{\varepsilon,\varrho,R} - \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R\mu_\varepsilon \\ &= \int_{\mathbb{R}^2}\frac{a}{2}\Big(\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N^2\psi_{\varepsilon,\varrho,R} + f\chi_R) - |\log\varepsilon|\chi_R\mu_\varepsilon\Big),\end{aligned} \tag{8.96}$$

where the function $\psi_{\varepsilon,\varrho,R} : \mathbb{R}^2 \to \mathbb{R}$ is precisely chosen as follows,

$$\psi_{\varepsilon,\varrho,R} := -\chi_R|\mathrm{v}_\varepsilon|^2 + \frac{|\log\varepsilon|}{N_\varepsilon}\chi_R\,\mathrm{v}_\varepsilon\cdot(\nabla^\perp h - F^\perp) + \frac{\lambda_\varepsilon\beta|\log\varepsilon|}{N_\varepsilon}\chi_R\,\mathrm{p}_{\varepsilon,\varrho} - \frac{|\log\varepsilon|}{N_\varepsilon}\nabla\chi_R\cdot\mathrm{v}_\varepsilon^\perp, \tag{8.97}$$

in terms of the truncated pressure $\mathrm{p}_{\varepsilon,\varrho} := \chi_\varrho\,\mathrm{p}_\varepsilon$. This choice is motivated by the fact that it yields some crucial cancellations in the proof of Lemma 8.4.4 below. Again, replacing χ_R and $\mathrm{p}_{\varepsilon,\varrho}$ by χ_R^z and $\mathrm{p}_{\varepsilon,\varrho}^z = \chi_\varrho^z\,\mathrm{p}_\varepsilon$, we further define $\hat{\mathcal{E}}^z_{\varepsilon,\varrho,R}$ and $\hat{\mathcal{D}}^z_{\varepsilon,\varrho,R}$ for $z\in\mathbb{R}^2$, and we then set $\hat{\mathcal{E}}^*_{\varepsilon,\varrho,R} := \sup_z \hat{\mathcal{E}}^z_{\varepsilon,\varrho,R}$ and $\hat{\mathcal{D}}^*_{\varepsilon,\varrho,R} := \sup_z \hat{\mathcal{D}}^z_{\varepsilon,\varrho,R}$ (where again the suprema implicitly run over all lattice points $z\in R\mathbb{Z}^2$). The truncation scale $\rho \gg 1$ is introduced here to cure the lack of integrability of the pressure p_ε in the Gross-Pitaevskii case: indeed, the pressure p_ε does in general not belong to $\mathrm{L}^2(\mathbb{R}^2)$ (cf. Assumption 8.3.1(b) above, which is indeed optimal in that respect), while it does always in the case without pinning and forcing (cf. [395]). In the dissipative case this truncation is not needed, so that we may set $\mathrm{p}_{\varepsilon,\infty} := \mathrm{p}_\varepsilon$ with $\varrho := \infty$, and we then drop for simplicity the subscript ϱ from the notation, writing $\psi_{\varepsilon,R} := \psi_{\varepsilon,\infty,R}$, $\hat{\mathcal{E}}_{\varepsilon,R} := \hat{\mathcal{E}}_{\varepsilon,\infty,R}$, etc.

In the dissipative case, as a consequence of (8.43) and of Assumption 8.3.1(a), $\psi_{\varepsilon,R}$ is bounded uniformly with respect to R in $\mathrm{L}^p(\mathbb{R}^2)$ for all $2 < p \le \infty$ (but not in $\mathrm{L}^2(\mathbb{R}^2)$), and using the bound (8.43) we have in the considered regimes, for all $t\in[0,T)$ and $\theta>0$,

$$\|\psi^t_{\varepsilon,R}\|_{\mathrm{L}^2} \lesssim_{t,\theta} 1 + \frac{|\log\varepsilon|}{N_\varepsilon}(\lambda_\varepsilon R^\theta + 1\wedge\lambda_\varepsilon^{1/2} + R^{-1+\theta}), \qquad \|\partial_t\psi_{\varepsilon,R}\|_{\mathrm{L}^2_t\mathrm{L}^2} \lesssim_{t,\theta} 1 + \frac{|\log\varepsilon|}{N_\varepsilon}. \tag{8.98}$$

In the Gross-Pitaevskii case, in the considered regime (GP), the bound (8.44) and Assumption 8.3.1(b) rather yield, for all $t\in[0,T)$ and $\theta>0$,

$$\|\psi^t_{\varepsilon,\varrho,R}\|_{\mathrm{L}^2} + \|\partial_t\psi^t_{\varepsilon,\varrho,R}\|_{\mathrm{L}^2} \lesssim_{t,\theta} 1 + \frac{|\log\varepsilon|}{N_\varepsilon}\lambda_\varepsilon\varrho^\theta \lesssim \varrho^\theta. \tag{8.99}$$

Based on these estimates, the following lemma states that the additional terms in $\hat{\mathcal{E}}_{\varepsilon,\varrho,R}$ are indeed of lower order, so that $\hat{\mathcal{E}}_{\varepsilon,\varrho,R}$ itself controls the same quantities as the modulated energy $\mathcal{E}_{\varepsilon,R}$.

Lemma 8.4.1 (Neglecting lower-order terms)**.** *Let $h:\mathbb{R}^2\to\mathbb{R}$, $a := e^h$, $F:\mathbb{R}^2\to\mathbb{R}^2$ satisfy* (8.43) *or* (8.44)*, let $u_\varepsilon : [0,T)\times\mathbb{R}^2\to\mathbb{C}$, and let $\mathrm{v}_\varepsilon : [0,T)\times\mathbb{R}^2\to\mathbb{R}^2$ be as in Assumption 8.3.1, for some $T>0$. Further assume that $0<\varepsilon\ll 1$ and $\varrho, R\gg 1$ satisfy for some $\theta>0$, in the dissipative case,*

$$\varepsilon\big(N_\varepsilon^2 + N_\varepsilon|\log\varepsilon|(\lambda_\varepsilon R^\theta + 1\wedge\lambda_\varepsilon^{1/2} + R^{-1+\theta}) + R\lambda_\varepsilon^2|\log\varepsilon|^2\big) \ll N_\varepsilon\Big(1\wedge\frac{N_\varepsilon}{|\log\varepsilon|}\Big)^{1/2}, \tag{8.100}$$

or in the Gross-Pitaevskii case,

$$\varepsilon N_\varepsilon^2(\varrho^\theta + R) \ll N_\varepsilon\Big(1\wedge\frac{N_\varepsilon}{|\log\varepsilon|}\Big)^{1/2}. \tag{8.101}$$

Then for all $z\in\mathbb{R}^2$ we have

$$|\hat{\mathcal{E}}^{z,t}_{\varepsilon,\varrho,R} - \mathcal{E}^{z,t}_{\varepsilon,R}| = |\hat{\mathcal{D}}^{z,t}_{\varepsilon,\varrho,R} - \mathcal{D}^{z,t}_{\varepsilon,R}| \lesssim_t o(N_\varepsilon)\Big(1\wedge\frac{N_\varepsilon}{|\log\varepsilon|}\Big)^{1/2}(\mathcal{E}^{z,t}_{\varepsilon,R})^{1/2}. \qquad \Diamond$$

Proof. We focus on the dissipative case, the other case is similar. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
|\hat{\mathcal{E}}_{\varepsilon,R}^z - \mathcal{E}_{\varepsilon,R}^z| &\lesssim \int_{\mathbb{R}^2} |1-|u_\varepsilon|^2| (N_\varepsilon^2 |\psi_{\varepsilon,R}^z| + |f|\chi_R^z) \\
&\le \Big(\int_{\mathbb{R}^2} \chi_R^z (1-|u_\varepsilon|^2)^2 \Big)^{1/2} \big(N_\varepsilon^2 \|\psi_{\varepsilon,R}^z/(\chi_R^z)^{1/2}\|_{\mathrm{L}^2} + \|f\|_{\mathrm{L}^2(B_{2R}(z))} \big) \\
&\lesssim \varepsilon (\mathcal{E}_{\varepsilon,R}^z)^{1/2} \big(N_\varepsilon^2 \|\psi_{\varepsilon,R}^z/(\chi_R^z)^{1/2}\|_{\mathrm{L}^2} + R\|f\|_{\mathrm{L}^\infty} \big).
\end{aligned}$$

Arguing just as in (8.98), using (8.43), Assumption 8.3.1(a), and the fact that $|\nabla\chi_R(x)/\chi_R^{1/2}(x)| \lesssim R^{-1}\mathbb{1}_{|x|\le 2R}$, the choice (8.97) of $\psi_{\varepsilon,R}$ yields, for all $\theta > 0$,

$$\|\psi_{\varepsilon,R}/\chi_R^{1/2}\|_{\mathrm{L}^2} \lesssim_{t,\theta} 1 + \frac{|\log\varepsilon|}{N_\varepsilon}(\lambda_\varepsilon R^\theta + 1 \wedge \lambda_\varepsilon^{1/2} + R^{-1+\theta}).$$

Combined with (8.43) and with assumption (8.100), this proves the result. ☐

8.4.2 Physical quantities and identities

Next to the *supercurrent density* $j_\varepsilon := \langle \nabla u_\varepsilon, iu_\varepsilon \rangle$ and the *vorticity* $\mu_\varepsilon := \operatorname{curl} j_\varepsilon$, we define the *vortex velocity* $V_\varepsilon := 2\langle \nabla u_\varepsilon, i\partial_t u_\varepsilon\rangle$. The following identities are easily checked from these definitions:

$$\partial_t j_\varepsilon = V_\varepsilon + \nabla\langle \partial_t u_\varepsilon, iu_\varepsilon\rangle, \qquad \partial_t \mu_\varepsilon = \operatorname{curl} V_\varepsilon, \tag{8.102}$$

and also, using equation (8.6) for u_ε,

$$\operatorname{div} j_\varepsilon = \langle \triangle u_\varepsilon, iu_\varepsilon\rangle = \lambda_\varepsilon\alpha\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle - j_\varepsilon\cdot\nabla h - \frac{\lambda_\varepsilon\beta|\log\varepsilon|}{2}\partial_t(1-|u_\varepsilon|^2) + \frac{|\log\varepsilon|}{2}F^\perp\cdot\nabla(1-|u_\varepsilon|^2). \tag{8.103}$$

We then consider the *weighted energy density*

$$e_\varepsilon := \frac{a}{2}\Big(|\nabla u_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big).$$

In the same vein as when introducing the modulated energy and energy excess, we define the following *modulated vorticity* and *modulated velocity*,

$$\tilde{\mu}_\varepsilon := \operatorname{curl}(N_\varepsilon \mathrm{v}_\varepsilon + \langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon, iu_\varepsilon\rangle) = \mu_\varepsilon + \operatorname{curl}(N_\varepsilon\mathrm{v}_\varepsilon(1-|u_\varepsilon|^2)), \tag{8.104}$$

$$\tilde{V}_{\varepsilon,\varrho} := 2\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon, i(\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho})\rangle = V_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon\partial_t|u_\varepsilon|^2 + N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}\nabla|u_\varepsilon|^2. \tag{8.105}$$

For the computations, we will also need the 2×2 *stress-energy tensor* S_ε,

$$S_\varepsilon^{kl} := a\langle\partial_k u_\varepsilon, \partial_l u_\varepsilon\rangle - \frac{a}{2}\operatorname{Id}\Big(|\nabla u_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big), \tag{8.106}$$

and its modulated version $\tilde{S}_\varepsilon$,

$$\begin{aligned}
\tilde{S}_\varepsilon^{kl} := a\Big(&\langle \partial_k u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_{\varepsilon,k}, \partial_l u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_{\varepsilon,l}\rangle + N_\varepsilon^2(1-|u_\varepsilon|^2)\,\mathrm{v}_{\varepsilon,k}\mathrm{v}_{\varepsilon,l}\Big) \\
&- \frac{a}{2}\operatorname{Id}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + f)\Big).
\end{aligned} \tag{8.107}$$

We close this section with the following pointwise estimates.

Lemma 8.4.2. *We have*

$$
\begin{aligned}
|j_\varepsilon - N_\varepsilon v_\varepsilon| &\le |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon||1-|u_\varepsilon|^2| + N_\varepsilon|\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2|,\\
|\mu_\varepsilon| &\le 2|\nabla u_\varepsilon|^2 \le 4|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + 4N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + 4N_\varepsilon^2|1-|u_\varepsilon|^2||\mathrm{v}_\varepsilon|^2,\\
|V_\varepsilon| &\le 2\big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon||\partial_t u_\varepsilon| + N_\varepsilon|\mathrm{v}_\varepsilon||\partial_t u_\varepsilon| + N_\varepsilon|1-|u_\varepsilon|^2||\mathrm{v}_\varepsilon||\partial_t u_\varepsilon|\big),\\
|\tilde V_{\varepsilon,\varrho}| &\le 2|\partial_t u_\varepsilon||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + 2N_\varepsilon|\mathrm{p}_{\varepsilon,\varrho}||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + 2N_\varepsilon|\mathrm{p}_{\varepsilon,\varrho}||1-|u_\varepsilon|^2||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|,\\
|\partial_t|u_\varepsilon|| &\le |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|,\\
|\nabla|u_\varepsilon|| &\le |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|. \qquad\Diamond
\end{aligned}
$$

Proof. The first estimate is obtained as follows,

$$
\begin{aligned}
|j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon| &\le |\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon, iu_\varepsilon\rangle| + N_\varepsilon|1-|u_\varepsilon|^2||\mathrm{v}_\varepsilon|\\
&\le |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon||1-|u_\varepsilon|^2| + N_\varepsilon|\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2|,
\end{aligned}
$$

while the estimates on V_ε and $\tilde V_{\varepsilon,\varrho}$ similarly follow the definitions. The estimate on μ_ε is a direct consequence of the representation $\mu_\varepsilon = \operatorname{curl}\langle \nabla u_\varepsilon, iu_\varepsilon\rangle = 2\langle \nabla_2 u_\varepsilon, i\nabla_1 u_\varepsilon\rangle$. Finally noting that

$$
|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon\,|^2 = |\partial_t|u_\varepsilon||^2 + |u_\varepsilon|^2\Big|\partial_t\frac{u_\varepsilon}{|u_\varepsilon|} - i\frac{u_\varepsilon}{|u_\varepsilon|}N_\varepsilon \mathrm{p}_\varepsilon\,\Big|^2,
$$

the result on $\partial_t|u_\varepsilon|$ follows, and the result on $\nabla|u_\varepsilon|$ is obtained similarly. $\square$

8.4.3 Divergence of the modulated stress-energy tensor

In the following lemma we explicitly compute the divergence of the modulated stress-energy tensor: as already mentioned, it will be crucial in the sequel in order to replace some linear terms in the error by quadratic ones (cf. Step 3 of the proof of Lemma 8.4.4 below).

Lemma 8.4.3. *Let $u_\varepsilon : [0,T)\times\mathbb{R}^2 \to \mathbb{C}$ be a solution of (8.6) as in Proposition 8.2.2, and let $\mathrm{v}_\varepsilon : [0,T)\times\mathbb{R}^2\to\mathbb{R}^2$ be as in Assumption 8.3.1. Then, defining by $(\operatorname{div}\tilde S_\varepsilon)_k := \sum_l \partial_l(\tilde S_\varepsilon)_{kl}$ the divergence of the 2-tensor $\tilde S_\varepsilon$, where $(\tilde S_\varepsilon)_{kl}$ denotes the (k,l)-component of $\tilde S_\varepsilon$, we have*

$$
\begin{aligned}
\operatorname{div}\tilde S_\varepsilon = a\lambda_\varepsilon\alpha\,&\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle\\
&- a\mu_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon^\perp - |\log\varepsilon|F/2) + aN_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)^\perp \operatorname{curl}\mathrm{v}_\varepsilon\\
&+ \frac{a\lambda_\varepsilon\beta}{2}|\log\varepsilon|\tilde V_{\varepsilon,\varrho} + aN_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)(\operatorname{div}\mathrm{v}_\varepsilon + \nabla h\cdot \mathrm{v}_\varepsilon - \lambda_\varepsilon\alpha\,\mathrm{p}_{\varepsilon,\varrho}) - \frac{a}{2}(1-|u_\varepsilon|^2)\nabla f\\
&- \frac{a}{2}\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + f)\Big)\\
&+ a\lambda_\varepsilon\alpha N_\varepsilon^2 \mathrm{v}_\varepsilon \mathrm{p}_{\varepsilon,\varrho}(1-|u_\varepsilon|^2) - \frac{a\lambda_\varepsilon\beta}{2}N_\varepsilon|\log\varepsilon|\,\mathrm{p}_{\varepsilon,\varrho}\nabla|u_\varepsilon|^2 + \frac{a}{2}N_\varepsilon|\log\varepsilon|(F^\perp\cdot\nabla|u_\varepsilon|^2)\,\mathrm{v}_\varepsilon\,. \quad\Diamond
\end{aligned}
$$

Proof. A direct computation yields, for the stress-energy tensor,

$$
\begin{aligned}
\operatorname{div} S_\varepsilon = a\Big\langle \nabla u_\varepsilon, \triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2) + \nabla h\cdot\nabla u_\varepsilon + fu_\varepsilon\Big\rangle&\\
- \frac{a}{2}\nabla h\Big(|\nabla u_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big) - \frac{a}{2}(1-|u_\varepsilon|^2)\nabla f.& \quad (8.108)
\end{aligned}
$$

On the other hand, the modulated stress-energy tensor may be decomposed as

$$
\tilde S_\varepsilon = S_\varepsilon - aN_\varepsilon \mathrm{v}_\varepsilon\otimes j_\varepsilon - aN_\varepsilon j_\varepsilon\otimes \mathrm{v}_\varepsilon + aN_\varepsilon^2\mathrm{v}_\varepsilon\otimes \mathrm{v}_\varepsilon - \frac{aN_\varepsilon}{2}\operatorname{Id}\big(N_\varepsilon|\mathrm{v}_\varepsilon|^2 - 2\mathrm{v}_\varepsilon\cdot j_\varepsilon\big),
$$

which, combined with (8.108), yields

$$
\begin{aligned}
\operatorname{div}\,\tilde S_\varepsilon = {}& a\left\langle \nabla u_\varepsilon, \triangle u_\varepsilon + \frac{a u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2) + \nabla h\cdot\nabla u_\varepsilon + f u_\varepsilon\right\rangle \\
& - \frac{a}{2}\nabla h\Big(|\nabla u_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big) - \frac{a}{2}(1-|u_\varepsilon|^2)\nabla f \\
& - aN_\varepsilon\Big(j_\varepsilon\nabla h\cdot \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\nabla h\cdot j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon\nabla h\cdot\mathrm{v}_\varepsilon + \frac{1}{2}N_\varepsilon|\mathrm{v}_\varepsilon|^2\nabla h - \mathrm{v}_\varepsilon\cdot j_\varepsilon\nabla h\Big) \\
& - aN_\varepsilon j_\varepsilon\operatorname{div}\mathrm{v}_\varepsilon - aN_\varepsilon(\mathrm{v}_\varepsilon\cdot\nabla)j_\varepsilon - aN_\varepsilon\mathrm{v}_\varepsilon\operatorname{div}\,j_\varepsilon - aN_\varepsilon(j_\varepsilon\cdot\nabla)\,\mathrm{v}_\varepsilon + aN_\varepsilon^2\mathrm{v}_\varepsilon\operatorname{div}\mathrm{v}_\varepsilon + aN_\varepsilon^2(\mathrm{v}_\varepsilon\cdot\nabla)\,\mathrm{v}_\varepsilon \\
& - aN_\varepsilon^2\sum_l \mathrm{v}_{\varepsilon,l}\,\nabla\mathrm{v}_{\varepsilon,l} + aN_\varepsilon\sum_l \mathrm{v}_{\varepsilon,l}\,\nabla j_{\varepsilon,l} + aN_\varepsilon\sum_l j_{\varepsilon,l}\nabla\mathrm{v}_{\varepsilon,l},
\end{aligned}
$$

where we denote by $\mathrm{v}_{\varepsilon,l}$ and $j_{\varepsilon,l}$ the l-th component of the vector fields v_ε and j_ε, respectively. Noting that $(F\cdot\nabla)G - \sum_l F_l\nabla G_l = F^\perp\operatorname{curl}G$, and using equation (8.6) for u_ε, this becomes

$$
\begin{aligned}
\operatorname{div}\,\tilde S_\varepsilon = {}& a\lambda_\varepsilon\,\langle(\alpha+i\beta|\log\varepsilon|)\partial_t u_\varepsilon, \nabla u_\varepsilon\rangle - a|\log\varepsilon|\langle\nabla u_\varepsilon, iF^\perp\cdot\nabla u_\varepsilon\rangle \\
& - \frac{a}{2}\nabla h\Big(|\nabla u_\varepsilon|^2 + N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 - 2N_\varepsilon\mathrm{v}_\varepsilon\cdot j_\varepsilon + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big) \\
& - \frac{a}{2}(1-|u_\varepsilon|^2)\nabla f - aN_\varepsilon\big(j_\varepsilon\nabla h\cdot\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\nabla h\cdot j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon\nabla h\cdot\mathrm{v}_\varepsilon\big) \\
& + aN_\varepsilon\big(-\mathrm{v}_\varepsilon^\perp\mu_\varepsilon + (N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)^\perp\operatorname{curl}\mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon\operatorname{div}\,j_\varepsilon + (N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\operatorname{div}\mathrm{v}_\varepsilon\big). \qquad (8.109)
\end{aligned}
$$

Using identity (8.103), the first right-hand side term above may be rewritten as

$$
\begin{aligned}
& \lambda_\varepsilon\,\langle(\alpha+i\beta|\log\varepsilon|)\partial_t u_\varepsilon, \nabla u_\varepsilon\rangle \\
= {}& \lambda_\varepsilon\alpha\,\big\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\big\rangle + N_\varepsilon\lambda_\varepsilon\alpha\mathrm{v}_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle \\
& + N_\varepsilon\lambda_\varepsilon\alpha\,\mathrm{p}_{\varepsilon,\varrho}\,j_\varepsilon - N_\varepsilon^2\lambda_\varepsilon\alpha|u_\varepsilon|^2\,\mathrm{p}_{\varepsilon,\varrho}\mathrm{v}_\varepsilon + \frac{\lambda_\varepsilon\beta}{2}|\log\varepsilon|V_\varepsilon \\
= {}& \lambda_\varepsilon\alpha\,\big\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\big\rangle + N_\varepsilon\mathrm{v}_\varepsilon(\operatorname{div}\,j_\varepsilon + j_\varepsilon\cdot\nabla h) + \frac{1}{2}N_\varepsilon|\log\varepsilon|(F^\perp\cdot\nabla|u_\varepsilon|^2)\,\mathrm{v}_\varepsilon \\
& + \frac{\lambda_\varepsilon\beta}{2}N_\varepsilon|\log\varepsilon|\mathrm{v}_\varepsilon\partial_t(1-|u_\varepsilon|^2) + N_\varepsilon\lambda_\varepsilon\alpha\,\mathrm{p}_{\varepsilon,\varrho}\,j_\varepsilon - N_\varepsilon^2\lambda_\varepsilon\alpha|u_\varepsilon|^2\mathrm{p}_{\varepsilon,\varrho}\mathrm{v}_\varepsilon + \frac{\lambda_\varepsilon\beta}{2}|\log\varepsilon|V_\varepsilon.
\end{aligned}
$$

Inserting this into (8.109), recombining $|\nabla u_\varepsilon|^2 + N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 - 2N_\varepsilon\mathrm{v}_\varepsilon\cdot j_\varepsilon = |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + N_\varepsilon^2(1-|u_\varepsilon|^2)|\mathrm{v}_\varepsilon|^2$, noting that $\langle\nabla u_\varepsilon, iF^\perp\cdot\nabla u_\varepsilon\rangle = -F\mu_\varepsilon/2$, and using (8.105) to transform V_ε into $\tilde V_{\varepsilon,\varrho}$, we obtain

$$
\begin{aligned}
\operatorname{div}\,\tilde S_\varepsilon = {}& a\lambda_\varepsilon\alpha\,\big\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\big\rangle \\
& + aN_\varepsilon\mathrm{v}_\varepsilon(\operatorname{div}\,j_\varepsilon + j_\varepsilon\cdot\nabla h) + \frac{a}{2}N_\varepsilon|\log\varepsilon|(F^\perp\cdot\nabla|u_\varepsilon|^2)\,\mathrm{v}_\varepsilon \\
& + \lambda_\varepsilon\alpha aN_\varepsilon\,\mathrm{p}_{\varepsilon,\varrho}\,j_\varepsilon - aN_\varepsilon^2\lambda_\varepsilon\alpha|u_\varepsilon|^2\,\mathrm{p}_{\varepsilon,\varrho}\mathrm{v}_\varepsilon + \frac{a\lambda_\varepsilon\beta}{2}|\log\varepsilon|\tilde V_{\varepsilon,\varrho} - \frac{a\lambda_\varepsilon\beta}{2}N_\varepsilon|\log\varepsilon|\,\mathrm{p}_{\varepsilon,\varrho}\nabla|u_\varepsilon|^2 \\
& - a\mu_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon^\perp - |\log\varepsilon|F/2) - \frac{a}{2}\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + f)\Big) \\
& - \frac{a}{2}(1-|u_\varepsilon|^2)\nabla f - aN_\varepsilon\big(j_\varepsilon\nabla h\cdot\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\nabla h\cdot j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon\nabla h\cdot\mathrm{v}_\varepsilon\big) \\
& + aN_\varepsilon\big((N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)^\perp\operatorname{curl}\mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon\operatorname{div}\,j_\varepsilon + (N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\operatorname{div}\mathrm{v}_\varepsilon\big),
\end{aligned}
$$

and the result follows after straightforward simplifications. □

8.4.4 Time derivative of the modulated energy excess

In the present section, we prove the following decomposition of the time derivative of the modulated energy excess $\hat{\mathcal{D}}_{\varepsilon,\varrho,R}$. As will be seen in Sections 8.6–8.8, mean-field limit results are then essentially reduced to the estimation of the different terms in this decomposition. To simplify notation, it is stated here using truncations centered at $z = 0$, but the translated result of course also holds for all $z \in \mathbb{R}^2$.

Lemma 8.4.4. *Let $\alpha \geq 0$, $\beta \in \mathbb{R}$, and let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy* (8.43) *or* (8.44). *Let $u_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{C}$ and $\mathrm{v}_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{R}^2$ be solutions of* (8.6) *and* (8.50) *as in Proposition 8.2.2 and as in Assumption 8.3.1, respectively. Let $0 < \varepsilon \ll 1$, $\varrho, R \gg 1$, and let $\bar{\Gamma}_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{R}^2$ be a given vector field with $\|\bar{\Gamma}_\varepsilon^t\|_{W^{1,\infty}} \lesssim_t 1$. Then, we have*

$$\partial_t \hat{\mathcal{D}}_{\varepsilon,\varrho,R} = I^S_{\varepsilon,\varrho,R} + I^V_{\varepsilon,\varrho,R} + I^E_{\varepsilon,\varrho,R} + I^D_{\varepsilon,\varrho,R} + I^H_{\varepsilon,\varrho,R} + I^d_{\varepsilon,\varrho,R} + I^g_{\varepsilon,\varrho,R} + I^n_{\varepsilon,\varrho,R} + I'_{\varepsilon,\varrho,R},$$

where we have set

$$\begin{aligned}
I^S_{\varepsilon,\varrho,R} &:= -\int_{\mathbb{R}^2} \chi_R \nabla \bar{\Gamma}_\varepsilon^\perp : \tilde{S}_\varepsilon,\\
I^V_{\varepsilon,\varrho,R} &:= \int_{\mathbb{R}^2} \frac{a\chi_R|\log\varepsilon|}{2}\, \tilde{V}_{\varepsilon,\varrho} \cdot \Big(-\lambda_\varepsilon \beta \Gamma_\varepsilon^\perp + \nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon \Big),\\
I^E_{\varepsilon,\varrho,R} &:= -\int_{\mathbb{R}^2} \frac{a\chi_R|\log\varepsilon|}{2}\, \Gamma_\varepsilon \cdot \Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|}\,\mathrm{v}_\varepsilon \Big)\mu_\varepsilon,\\
I^D_{\varepsilon,\varrho,R} &:= -\int_{\mathbb{R}^2} \lambda_\varepsilon \alpha a \chi_R |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}|^2 - \int_{\mathbb{R}^2} \lambda_\varepsilon \alpha a \chi_R \Gamma_\varepsilon^\perp \cdot \big\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon \big\rangle,\\
I^H_{\varepsilon,\varrho,R} &:= \int_{\mathbb{R}^2} \frac{a\chi_R}{2} \Gamma_\varepsilon^\perp \cdot \nabla h \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon \Big),
\end{aligned}$$

and also

$$\begin{aligned}
I^d_{\varepsilon,\varrho,R} &:= \int_{\mathbb{R}^2} a\chi_R N_\varepsilon \big(\bar{\Gamma}_\varepsilon^\perp \cdot (j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon) + \langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, iu_\varepsilon \rangle \big) (\operatorname{div} \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon \cdot \nabla h - \lambda_\varepsilon \alpha\, \mathrm{p}_{\varepsilon,\varrho}),\\
I^g_{\varepsilon,\varrho,R} &:= \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon) \cdot (\Gamma_\varepsilon - \bar{\Gamma}_\varepsilon) \operatorname{curl} \mathrm{v}_\varepsilon + \int_{\mathbb{R}^2} \frac{a\chi_R}{2} \lambda_\varepsilon \beta |\log\varepsilon| \tilde{V}_{\varepsilon,\varrho} \cdot (\Gamma_\varepsilon - \bar{\Gamma}_\varepsilon)^\perp\\
&\quad + \int_{\mathbb{R}^2} \lambda_\varepsilon \alpha a \chi_R (\Gamma_\varepsilon - \bar{\Gamma}_\varepsilon)^\perp \cdot \big\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon \big\rangle\\
&\quad + \int_{\mathbb{R}^2} \frac{a\chi_R}{2} (\bar{\Gamma}_\varepsilon - \Gamma_\varepsilon)^\perp \cdot \nabla h \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 \Big)\\
&\quad + \int_{\mathbb{R}^2} a\chi_R (\bar{\Gamma}_\varepsilon - \Gamma_\varepsilon) \cdot (N_\varepsilon \mathrm{v}_\varepsilon + |\log\varepsilon| F^\perp/2)\mu_\varepsilon + \int_{\mathbb{R}^2} a\chi_R \lambda_\varepsilon \beta N_\varepsilon |\log\varepsilon| (\bar{\Gamma}_\varepsilon - \Gamma_\varepsilon)^\perp \cdot \mathrm{v}_\varepsilon\, \partial_t |u_\varepsilon|^2,\\
I^n_{\varepsilon,\varrho,R} &:= -\int_{\mathbb{R}^2} \nabla \chi_R \cdot \tilde{S}_\varepsilon \cdot \bar{\Gamma}_\varepsilon^\perp - \int_{\mathbb{R}^2} a \nabla \chi_R \cdot \Big(\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon \rangle + \frac{|\log\varepsilon|}{2} \tilde{V}_{\varepsilon,\varrho}^\perp \Big),
\end{aligned}$$

and where the error $I'_{\varepsilon,\varrho,R}$ is estimated as follows, in the dissipative case, in the considered regimes,

$$\int_0^t |I'_{\varepsilon,\varrho,R}| \lesssim_t \varepsilon R (N_\varepsilon^2 + |\log\varepsilon|^2)(\mathcal{E}^*_{\varepsilon,R})^{1/2}, \tag{8.110}$$

or in the Gross-Pitaevskii case (GP), *for all $\theta > 0$,*

$$|I'_{\varepsilon,\varrho,R}| \lesssim_{t,\theta} \varepsilon N_\varepsilon \mathcal{E}^*_{\varepsilon,R} + N_\varepsilon (\mathcal{E}^*_{\varepsilon,R})^{1/2} \|\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho})\|_{\mathrm{L}^2} + \varepsilon N_\varepsilon^2 \varrho^\theta (\mathcal{E}^*_{\varepsilon,R})^{1/2}. \tag{8.111}$$

◊

Proof. We split the proof into three steps, first computing the time derivative $\partial_t\hat{\mathcal{E}}_{\varepsilon,\varrho,R}$, then deducing an expression for $\partial_t\hat{\mathcal{D}}_{\varepsilon,\varrho,R}$, and finally introducing the modulated stress-energy tensor to replace the linear terms by quadratic ones, which are better suited for a Grönwall argument.

Step 1. Time derivative of the modulated energy.

In this step, we prove the following identity,

$$\begin{aligned}\partial_t\hat{\mathcal{E}}_{\varepsilon,\varrho,R} = -\int_{\mathbb{R}^2} a\nabla\chi_R\cdot\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle + \int_{\mathbb{R}^2}\frac{aN_\varepsilon^2}{2}\partial_t\big((1-|u_\varepsilon|^2)(\psi_{\varepsilon,\varrho,R}-\chi_R|\mathrm{v}_\varepsilon|^2)\big)\\ + \int_{\mathbb{R}^2} N_\varepsilon a\chi_R\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle(\operatorname{div}\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h)\\ + \int_{\mathbb{R}^2} a\chi_R\Big(N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon - \lambda_\varepsilon\alpha|\partial_t u_\varepsilon|^2 - N_\varepsilon\mathrm{v}_\varepsilon\cdot V_\varepsilon - \frac{|\log\varepsilon|}{2}F^\perp\cdot V_\varepsilon\Big).\end{aligned} \tag{8.112}$$

For that purpose, let us first compute the time derivative of the modulated energy density

$$\begin{aligned}&\frac{1}{2}\partial_t\Big(\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} + f\chi_R)\Big)\\ &\quad = \chi_R\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon, \nabla\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\partial_t\mathrm{v}_\varepsilon - i\partial_t u_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle - \chi_R\langle\partial_t u_\varepsilon, \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)\rangle\\ &\qquad + \frac{1}{2}\partial_t\big((1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} + f\chi_R)\big).\end{aligned} \tag{8.113}$$

Note that the first term in the right-hand side may be rewritten as

$$\begin{aligned}&\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon, \nabla\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\partial_t\mathrm{v}_\varepsilon - i\partial_t u_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle\\ &= \langle\nabla u_\varepsilon, \nabla\partial_t u_\varepsilon\rangle - N_\varepsilon\partial_t\mathrm{v}_\varepsilon\cdot j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon\cdot\langle\nabla u_\varepsilon, i\partial_t u_\varepsilon\rangle - N_\varepsilon\mathrm{v}_\varepsilon\cdot\langle iu_\varepsilon, \nabla\partial_t u_\varepsilon\rangle\\ &\qquad + \frac{N_\varepsilon^2}{2}|u_\varepsilon|^2\partial_t|\mathrm{v}_\varepsilon|^2 + \frac{N_\varepsilon^2}{2}|\mathrm{v}_\varepsilon|^2\partial_t|u_\varepsilon|^2\\ &= \operatorname{div}\langle\nabla u_\varepsilon, \partial_t u_\varepsilon\rangle - \langle\partial_t u_\varepsilon, \triangle u_\varepsilon\rangle - N_\varepsilon\partial_t\mathrm{v}_\varepsilon\cdot j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon\cdot\langle\nabla u_\varepsilon, i\partial_t u_\varepsilon\rangle\\ &\qquad - N_\varepsilon\mathrm{v}_\varepsilon\cdot(\partial_t j_\varepsilon - \langle i\partial_t u_\varepsilon, \nabla u_\varepsilon\rangle) + \frac{N_\varepsilon^2}{2}\partial_t(|u_\varepsilon|^2|\mathrm{v}_\varepsilon|^2)\\ &= \operatorname{div}\langle\nabla u_\varepsilon, \partial_t u_\varepsilon\rangle - \langle\partial_t u_\varepsilon, \triangle u_\varepsilon\rangle - N_\varepsilon\mathrm{v}_\varepsilon\cdot\partial_t j_\varepsilon - N_\varepsilon j_\varepsilon\cdot\partial_t\mathrm{v}_\varepsilon + \frac{N_\varepsilon^2}{2}\partial_t(|u_\varepsilon|^2|\mathrm{v}_\varepsilon|^2),\end{aligned} \tag{8.114}$$

where

$$\begin{aligned}\operatorname{div}\langle\nabla u_\varepsilon, \partial_t u_\varepsilon\rangle &= \operatorname{div}\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + \operatorname{div}(N_\varepsilon\mathrm{v}_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle)\\ &= \operatorname{div}\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + N_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle\operatorname{div}\mathrm{v}_\varepsilon + N_\varepsilon\mathrm{v}_\varepsilon\cdot(\partial_t j_\varepsilon - V_\varepsilon).\end{aligned} \tag{8.115}$$

Combining (8.113), (8.114) and (8.115), the time derivative of the energy density takes on the following guise, after straightforward simplifications,

$$\begin{aligned}&\frac{1}{2}\partial_t\Big(\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} + f\chi_R)\Big)\\ &= \chi_R\operatorname{div}\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + N_\varepsilon\chi_R\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle\operatorname{div}\mathrm{v}_\varepsilon - N_\varepsilon\chi_R\mathrm{v}_\varepsilon\cdot V_\varepsilon + N_\varepsilon\chi_R(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon\\ &\qquad - \chi_R\Big\langle\partial_t u_\varepsilon, \triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)\Big\rangle + \frac{1}{2}\partial_t\big((1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} - N_\varepsilon^2\chi_R|\mathrm{v}_\varepsilon|^2 + f\chi_R)\big).\end{aligned}$$

Integrating this identity in space yields

$$\partial_t \int_{\mathbb{R}^2} \frac{a}{2}\Big(\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} + f\chi_R)\Big)$$
$$= \int_{\mathbb{R}^2} a\chi_R\Big(N_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle \operatorname{div}\mathrm{v}_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon\cdot V_\varepsilon + N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon - \Big\langle \partial_t u_\varepsilon, \triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)\Big\rangle\Big)$$
$$+ \int_{\mathbb{R}^2}\frac{a}{2}\partial_t\big((1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} - N_\varepsilon^2\chi_R|\mathrm{v}_\varepsilon|^2 + f\chi_R)\big) - \int_{\mathbb{R}^2}\nabla(a\chi_R)\cdot\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle.$$

Decomposing $\nabla(a\chi_R) = a\chi_R\nabla h + a\nabla\chi_R$, and using the equation (8.6) satisfied by u_ε in the form

$$\Big\langle \partial_t u_\varepsilon, \triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2) + \nabla h\cdot\nabla u_\varepsilon\Big\rangle = \Big\langle \partial_t u_\varepsilon, \lambda_\varepsilon(\alpha + i\beta|\log\varepsilon|)\partial_t u_\varepsilon - i|\log\varepsilon|F^\perp\cdot\nabla u_\varepsilon - fu_\varepsilon\Big\rangle$$
$$= \lambda_\varepsilon\alpha|\partial_t u_\varepsilon|^2 + \frac{|\log\varepsilon|}{2}F^\perp\cdot V_\varepsilon - \frac{1}{2}f\partial_t|u_\varepsilon|^2,$$

the result (8.112) follows after straightforward simplifications.

Step 2. Time derivative of the modulated energy excess.

In this step, we prove the following identity,

$$\partial_t\hat{\mathcal{D}}_{\varepsilon,\varrho,R} = \int_{\mathbb{R}^2}\frac{a\chi_R}{2}\tilde V_{\varepsilon,\varrho}\cdot(|\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon\mathrm{v}_\varepsilon) + \int_{\mathbb{R}^2} a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\Gamma_\varepsilon\operatorname{curl}\mathrm{v}_\varepsilon$$
$$- \int_{\mathbb{R}^2}\lambda_\varepsilon\alpha a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}|^2 + \int_{\mathbb{R}^2} a\chi_R N_\varepsilon\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, iu_\varepsilon\rangle(\operatorname{div}\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h - \lambda_\varepsilon\alpha\,\mathrm{p}_{\varepsilon,\varrho})$$
$$+ \int_{\mathbb{R}^2} a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}) - \int_{\mathbb{R}^2} a\nabla\chi_R\cdot\Big(\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + \frac{|\log\varepsilon|}{2}\tilde V_{\varepsilon,\varrho}^\perp\Big)$$
$$- \int_{\mathbb{R}^2} aN_\varepsilon^2\mathrm{p}_{\varepsilon,\varrho}(1-|u_\varepsilon|^2)\big(\mathrm{v}_\varepsilon\cdot\nabla\chi_R + \chi_R(\operatorname{div}\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h)\big) + \int_{\mathbb{R}^2}\frac{aN_\varepsilon^2}{2}\partial_t\big((1-|u_\varepsilon|^2)(\psi_{\varepsilon,\varrho,R} - \chi_R|\mathrm{v}_\varepsilon|^2)\big)$$
$$+ \int_{\mathbb{R}^2}\frac{aN_\varepsilon|\log\varepsilon|}{2}\partial_t(1-|u_\varepsilon|^2)\Big(\mathrm{v}_\varepsilon^\perp\cdot\nabla\chi_R - \lambda_\varepsilon\beta\chi_R\mathrm{p}_{\varepsilon,\varrho} - \chi_R\mathrm{v}_\varepsilon\cdot\Big(\nabla^\perp h - F^\perp - 2\frac{N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\Big)$$
$$+ \int_{\mathbb{R}^2}\frac{aN_\varepsilon|\log\varepsilon|}{2}\,\mathrm{p}_{\varepsilon,\varrho}\nabla(1-|u_\varepsilon|^2)\cdot\Big(\nabla^\perp\chi_R + \chi_R\Big(\nabla^\perp h - 2F^\perp - 2\frac{N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\Big). \quad (8.116)$$

Noting that by identity (8.102) we have

$$|\log\varepsilon|\int_{\mathbb{R}^2} a\chi_R\partial_t\mu_\varepsilon = |\log\varepsilon|\int_{\mathbb{R}^2} a\chi_R\operatorname{curl}V_\varepsilon = -|\log\varepsilon|\int_{\mathbb{R}^2} a\chi_R V_\varepsilon\cdot\nabla^\perp h - |\log\varepsilon|\int_{\mathbb{R}^2} aV_\varepsilon\cdot\nabla^\perp\chi_R,$$

it is immediate to deduce from (8.112) the following identity for the time derivative of the modulated energy excess,

$$\partial_t\hat{\mathcal{D}}_{\varepsilon,\varrho,R} = \int_{\mathbb{R}^2}\frac{a\chi_R}{2}V_\varepsilon\cdot(|\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon\mathrm{v}_\varepsilon) + \int_{\mathbb{R}^2} aN_\varepsilon\chi_R\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle(\operatorname{div}\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h)$$
$$+ \int_{\mathbb{R}^2} a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon - \int_{\mathbb{R}^2}\lambda_\varepsilon\alpha a\chi_R|\partial_t u_\varepsilon|^2 + \int_{\mathbb{R}^2}\frac{aN_\varepsilon^2}{2}\partial_t\big((1-|u_\varepsilon|^2)(\psi_{\varepsilon,\varrho,R} - \chi_R|\mathrm{v}_\varepsilon|^2)\big)$$
$$- \int_{\mathbb{R}^2} a\nabla\chi_R\cdot\Big(\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + \frac{|\log\varepsilon|}{2}V_\varepsilon^\perp\Big). \quad (8.117)$$

Now using equation (8.50) for the time evolution of v_ε and an integration by parts, we find

$$\begin{aligned}
&\int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \partial_t \mathrm{v}_\varepsilon \\
&= \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \Gamma_\varepsilon \mathrm{curl}\, \mathrm{v}_\varepsilon + \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \nabla \mathrm{p}_\varepsilon \\
&= \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \Gamma_\varepsilon \mathrm{curl}\, \mathrm{v}_\varepsilon + \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \nabla (\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}) \\
&\quad - \int_{\mathbb{R}^2} a\chi_R N_\varepsilon \mathrm{p}_{\varepsilon,\varrho} (N_\varepsilon \operatorname{div} \mathrm{v}_\varepsilon - \operatorname{div}\, j_\varepsilon) - \int_{\mathbb{R}^2} a\chi_R N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\, \nabla h \cdot (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon) \\
&\qquad - \int_{\mathbb{R}^2} a N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\, \nabla \chi_R \cdot (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon).
\end{aligned}$$

Combining this with identity (8.103) yields

$$\begin{aligned}
&\int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \partial_t \mathrm{v}_\varepsilon \\
&= \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \Gamma_\varepsilon \mathrm{curl}\, \mathrm{v}_\varepsilon + \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \nabla (\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}) \\
&\quad - \int_{\mathbb{R}^2} a\chi_R N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\, \nabla h \cdot (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon) - \int_{\mathbb{R}^2} a N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\, \nabla \chi_R \cdot (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon) \\
&\quad - \int_{\mathbb{R}^2} a\chi_R N_\varepsilon \mathrm{p}_{\varepsilon,\varrho} \Big(N_\varepsilon \operatorname{div} \mathrm{v}_\varepsilon + j_\varepsilon \cdot \nabla h - \lambda_\varepsilon \alpha \langle \partial_t u_\varepsilon, i u_\varepsilon \rangle + \frac{|\log \varepsilon|}{2} F^\perp \cdot \nabla |u_\varepsilon|^2 - \frac{\lambda_\varepsilon \beta |\log \varepsilon|}{2} \partial_t |u_\varepsilon|^2 \Big) \\
&= \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \Gamma_\varepsilon \mathrm{curl}\, \mathrm{v}_\varepsilon + \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \nabla (\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}) \\
&\quad - \int_{\mathbb{R}^2} a\chi_R N_\varepsilon^2\, \mathrm{p}_{\varepsilon,\varrho} (\operatorname{div} \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon \cdot \nabla h) - \int_{\mathbb{R}^2} a N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\, \nabla \chi_R \cdot (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon) \\
&\qquad + \int_{\mathbb{R}^2} a\chi_R N_\varepsilon \mathrm{p}_{\varepsilon,\varrho} \Big(\lambda_\varepsilon \alpha \langle \partial_t u_\varepsilon, i u_\varepsilon \rangle - \frac{|\log \varepsilon|}{2} F^\perp \cdot \nabla |u_\varepsilon|^2 + \frac{\lambda_\varepsilon \beta |\log \varepsilon|}{2} \partial_t |u_\varepsilon|^2 \Big).
\end{aligned}$$

Inserting this into (8.117), we then find

$$\begin{aligned}
\partial_t \hat{\mathcal{D}}_{\varepsilon,\varrho,R} &= \int_{\mathbb{R}^2} \frac{a\chi_R}{2} V_\varepsilon \cdot \big(|\log \varepsilon| (\nabla^\perp h - F^\perp) - 2 N_\varepsilon \mathrm{v}_\varepsilon \big) \\
&\quad + \int_{\mathbb{R}^2} a\chi_R N_\varepsilon \langle \partial_t u_\varepsilon, i u_\varepsilon \rangle (\operatorname{div} \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon \cdot \nabla h + \lambda_\varepsilon \alpha\, \mathrm{p}_{\varepsilon,\varrho}) - \int_{\mathbb{R}^2} a\chi_R N_\varepsilon^2\, \mathrm{p}_{\varepsilon,\varrho} (\operatorname{div} \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon \cdot \nabla h) \\
&\qquad + \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \Gamma_\varepsilon \mathrm{curl}\, \mathrm{v}_\varepsilon + \int_{\mathbb{R}^2} a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \nabla (\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}) \\
&\quad + \int_{\mathbb{R}^2} \frac{a N_\varepsilon^2}{2} \partial_t \big((1-|u_\varepsilon|^2)(\psi_{\varepsilon,\varrho,R} - \chi_R |\mathrm{v}_\varepsilon|^2) \big) + \int_{\mathbb{R}^2} \frac{a\chi_R}{2} N_\varepsilon |\log \varepsilon| \mathrm{p}_{\varepsilon,\varrho} \big(\lambda_\varepsilon \beta \partial_t |u_\varepsilon|^2 - F^\perp \cdot \nabla |u_\varepsilon|^2 \big) \\
&\quad - \int_{\mathbb{R}^2} \lambda_\varepsilon \alpha a \chi_R |\partial_t u_\varepsilon|^2 - \int_{\mathbb{R}^2} a \nabla \chi_R \cdot \Big(\langle \partial_t u_\varepsilon, \nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon \rangle + \frac{|\log \varepsilon|}{2} V_\varepsilon^\perp + N_\varepsilon \mathrm{p}_{\varepsilon,\varrho} (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon) \Big).
\end{aligned} \tag{8.118}$$

Using identity (8.105) to transform V_ε into $\tilde{V}_{\varepsilon,\varrho}$, the first right-hand side term may be rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{a\chi_R}{2} V_\varepsilon \cdot (|\log \varepsilon| (\nabla^\perp h - F^\perp) - 2 N_\varepsilon \mathrm{v}_\varepsilon) \\
&\qquad = \int_{\mathbb{R}^2} \frac{a\chi_R}{2} \big(\tilde{V}_{\varepsilon,\varrho} - N_\varepsilon \mathrm{v}_\varepsilon \partial_t (1-|u_\varepsilon|^2) - N_\varepsilon \mathrm{p}_{\varepsilon,\varrho} \nabla |u_\varepsilon|^2 \big) \cdot \big(|\log \varepsilon| (\nabla^\perp h - F^\perp) - 2 N_\varepsilon \mathrm{v}_\varepsilon \big),
\end{aligned}$$

while the last right-hand side term of (8.118) becomes

$$\int_{\mathbb{R}^2} a\nabla\chi_R\cdot\Big(\langle\partial_t u_\varepsilon,\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle+\frac{|\log\varepsilon|}{2}V_\varepsilon^\perp+N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}(N_\varepsilon \mathrm{v}_\varepsilon-j_\varepsilon)\Big)$$
$$=\int_{\mathbb{R}^2} a\nabla\chi_R\cdot\Big(\langle\partial_t u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho},\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle+N_\varepsilon^2\,\mathrm{p}_{\varepsilon,\varrho}\mathrm{v}_\varepsilon(1-|u_\varepsilon|^2)$$
$$+\frac{|\log\varepsilon|}{2}\tilde V_{\varepsilon,\varrho}^\perp-\frac{N_\varepsilon|\log\varepsilon|}{2}\mathrm{v}_\varepsilon^\perp\partial_t(1-|u_\varepsilon|^2)-\frac{N_\varepsilon|\log\varepsilon|}{2}\,\mathrm{p}_{\varepsilon,\varrho}\nabla^\perp|u_\varepsilon|^2\Big).$$

Further decomposing

$$|\partial_t u_\varepsilon|^2=|\partial_t u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}|^2+2N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\langle\partial_t u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho},iu_\varepsilon\rangle+N_\varepsilon^2|\mathrm{p}_{\varepsilon,\varrho}|^2-(1-|u_\varepsilon|^2)N_\varepsilon^2|\mathrm{p}_{\varepsilon,\varrho}|^2,$$
$$\langle\partial_t u_\varepsilon,iu_\varepsilon\rangle=\langle\partial_t u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho},iu_\varepsilon\rangle+|u_\varepsilon|^2N_\varepsilon \mathrm{p}_{\varepsilon,\varrho},$$

the result (8.116) easily follows after straightforward simplifications.

Step 3. Conclusion.

In the right-hand side of (8.116), the term $\int_{\mathbb{R}^2} a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon-j_\varepsilon)\cdot\Gamma_\varepsilon\mathrm{curl}\,\mathrm{v}_\varepsilon$ is linear in $N_\varepsilon \mathrm{v}_\varepsilon-j_\varepsilon$, preventing a direct Grönwall argument. As already explained, just as in [395], the idea is to replace this bad term by others involving the modulated stress-energy tensor $\tilde S_\varepsilon$, which is indeed a nicer *quadratic* quantity. For that purpose, let us integrate the result of Lemma 8.4.3 in space against $\chi_R\bar\Gamma_\varepsilon^\perp$, where $\bar\Gamma_\varepsilon:[0,T)\to W^{1,\infty}(\mathbb{R}^2)^2$ is a given vector field (we would like to simply choose $\bar\Gamma_\varepsilon=\Gamma_\varepsilon$, but as we will see a suitable perturbation of it is needed), and obtain

$$\int_{\mathbb{R}^2}\chi_R\bar\Gamma_\varepsilon^\perp\cdot\mathrm{div}\,\tilde S_\varepsilon=\int_{\mathbb{R}^2}\lambda_\varepsilon\alpha a\chi_R\bar\Gamma_\varepsilon^\perp\cdot\big\langle\partial_t u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho},\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\big\rangle$$
$$-\int_{\mathbb{R}^2} a\chi_R\bar\Gamma_\varepsilon\cdot(N_\varepsilon \mathrm{v}_\varepsilon+|\log\varepsilon|F^\perp/2)\mu_\varepsilon+\int_{\mathbb{R}^2} a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon-j_\varepsilon)\cdot\bar\Gamma_\varepsilon\mathrm{curl}\,\mathrm{v}_\varepsilon$$
$$+\int_{\mathbb{R}^2}\lambda_\varepsilon\beta\frac{a\chi_R}{2}|\log\varepsilon|\bar\Gamma_\varepsilon^\perp\cdot\tilde V_{\varepsilon,\varrho}-\int_{\mathbb{R}^2}\lambda_\varepsilon\beta\frac{a\chi_R}{2}N_\varepsilon|\log\varepsilon|\,\mathrm{p}_{\varepsilon,\varrho}\,\bar\Gamma_\varepsilon^\perp\cdot\nabla|u_\varepsilon|^2$$
$$+\int_{\mathbb{R}^2} a\chi_R N_\varepsilon\bar\Gamma_\varepsilon^\perp\cdot(N_\varepsilon \mathrm{v}_\varepsilon-j_\varepsilon)(\mathrm{div}\,\mathrm{v}_\varepsilon+\nabla h\cdot \mathrm{v}_\varepsilon-\lambda_\varepsilon\alpha\,\mathrm{p}_{\varepsilon,\varrho})-\int_{\mathbb{R}^2}\frac{a\chi_R}{2}(1-|u_\varepsilon|^2)\bar\Gamma_\varepsilon^\perp\cdot\nabla f$$
$$-\int_{\mathbb{R}^2}\frac{a\chi_R}{2}\bar\Gamma_\varepsilon^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2+(1-|u_\varepsilon|^2)(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+f)\Big)$$
$$+\int_{\mathbb{R}^2}\lambda_\varepsilon\alpha a\chi_R N_\varepsilon^2\mathrm{p}_{\varepsilon,\varrho}(1-|u_\varepsilon|^2)(\bar\Gamma_\varepsilon^\perp\cdot \mathrm{v}_\varepsilon)+\int_{\mathbb{R}^2}\frac{a\chi_R}{2}N_\varepsilon|\log\varepsilon|(F^\perp\cdot\nabla|u_\varepsilon|^2)(\bar\Gamma_\varepsilon^\perp\cdot \mathrm{v}_\varepsilon).$$

In this last right-hand side, the term $\int_{\mathbb{R}^2} a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon-j_\varepsilon)\cdot\bar\Gamma_\varepsilon\mathrm{curl}\,\mathrm{v}_\varepsilon$ exactly corresponds to the bad term in the right-hand side of (8.116). Replacing it by this new expression involving the modulated stress-energy tensor, and treating as errors all the terms involving the difference $\bar\Gamma_\varepsilon-\Gamma_\varepsilon$, we find

$$\partial_t\hat{\mathcal{D}}_{\varepsilon,\varrho,R}=\sum_{j=0}^3 T_{\varepsilon,R}^j+I_{\varepsilon,\varrho,R}^g+I_{\varepsilon,\varrho,R}^n-\int_{\mathbb{R}^2}\chi_R\nabla\bar\Gamma_\varepsilon^\perp:\tilde S_\varepsilon$$
$$-\int_{\mathbb{R}^2}\lambda_\varepsilon\alpha a\chi_R\Gamma_\varepsilon^\perp\cdot\big\langle\partial_t u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho},\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\big\rangle$$
$$+\int_{\mathbb{R}^2}\frac{a\chi_R}{2}\Gamma_\varepsilon^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\int_{\mathbb{R}^2}\lambda_\varepsilon\alpha a\chi_R|\partial_t u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}|^2$$
$$+\int_{\mathbb{R}^2} a\chi_R\Gamma_\varepsilon\cdot(N_\varepsilon \mathrm{v}_\varepsilon+|\log\varepsilon|F^\perp/2)\mu_\varepsilon+\int_{\mathbb{R}^2}\frac{a\chi_R}{2}\tilde V_{\varepsilon,\varrho}\cdot(-\lambda_\varepsilon\beta|\log\varepsilon|\Gamma_\varepsilon^\perp+|\log\varepsilon|(\nabla^\perp h-F^\perp)-2N_\varepsilon \mathrm{v}_\varepsilon)$$
$$+\int_{\mathbb{R}^2} a\chi_R N_\varepsilon\big(\langle\partial_t u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho},iu_\varepsilon\rangle+\bar\Gamma_\varepsilon^\perp\cdot(j_\varepsilon-N_\varepsilon \mathrm{v}_\varepsilon)\big)(\mathrm{div}\,\mathrm{v}_\varepsilon+\mathrm{v}_\varepsilon\cdot\nabla h-\lambda_\varepsilon\alpha\,\mathrm{p}_{\varepsilon,\varrho}),$$

where $I^g_{\varepsilon,\varrho,R}$ and $I^n_{\varepsilon,\varrho,R}$ are given as in the statement, and where we have set

$$\begin{aligned}
T^0_{\varepsilon,\varrho,R} &:= \int_{\mathbb{R}^2} a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}),\\
T^1_{\varepsilon,\varrho,R} &:= \int_{\mathbb{R}^2} \frac{a\chi_R}{2}(1-|u_\varepsilon|^2)(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+f)\bar\Gamma_\varepsilon^\perp\cdot\nabla h\\
&\quad - \int_{\mathbb{R}^2} aN_\varepsilon^2 \mathrm{p}_{\varepsilon,\varrho}(1-|u_\varepsilon|^2)\big(\mathrm{v}_\varepsilon\cdot\nabla\chi_R + \chi_R(\operatorname{div}\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h)\big)\\
&\qquad + \int_{\mathbb{R}^2} \frac{a\chi_R}{2}(1-|u_\varepsilon|^2)\bar\Gamma_\varepsilon^\perp\cdot\nabla f - \int_{\mathbb{R}^2}\lambda_\varepsilon\alpha a\chi_R N_\varepsilon^2\,\mathrm{p}_{\varepsilon,\varrho}(1-|u_\varepsilon|^2)\bar\Gamma_\varepsilon^\perp\cdot\mathrm{v}_\varepsilon,\\
T^2_{\varepsilon,\varrho,R} &:= \int_{\mathbb{R}^2}\frac{aN_\varepsilon|\log\varepsilon|}{2}\,\mathrm{p}_{\varepsilon,\varrho}\nabla(1-|u_\varepsilon|^2)\cdot\bigg(\nabla^\perp\chi_R + \chi_R\Big(\nabla^\perp h - 2F^\perp - \lambda_\varepsilon\beta\bar\Gamma_\varepsilon^\perp - 2\frac{N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\bigg)\\
&\qquad + \int_{\mathbb{R}^2}\frac{a\chi_R}{2}N_\varepsilon|\log\varepsilon|\big(F^\perp\cdot\nabla(1-|u_\varepsilon|^2)\big)\bar\Gamma_\varepsilon^\perp\cdot\mathrm{v}_\varepsilon,\\
T^3_{\varepsilon,\varrho,R} &:= \int_{\mathbb{R}^2}\frac{aN_\varepsilon|\log\varepsilon|}{2}\partial_t(1-|u_\varepsilon|^2)\bigg(\mathrm{v}_\varepsilon^\perp\cdot\nabla\chi_R - \lambda_\varepsilon\beta\chi_R\mathrm{p}_{\varepsilon,\varrho} - \chi_R\mathrm{v}_\varepsilon\cdot\Big(\nabla^\perp h - F^\perp - 2\frac{N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\bigg)\\
&\qquad + \int_{\mathbb{R}^2}\frac{aN_\varepsilon^2}{2}\partial_t\big((1-|u_\varepsilon|^2)(\psi_{\varepsilon,\varrho,R}-\chi_R|\mathrm{v}_\varepsilon|^2)\big).
\end{aligned}$$

It remains to estimate these four error terms $T^i_{\varepsilon,\varrho,R}$, $0\le i\le 3$. First consider the term $T^0_{\varepsilon,\varrho,R}$. In the dissipative case we take $\varrho=\infty$, hence $T^0_{\varepsilon,\varrho,R}=0$. In the Gross-Pitaevskii case, using the pointwise estimate of Lemma 8.4.2 for $j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon$, and using Assumption 8.3.1(b), with in particular

$$\|\nabla(\mathrm{p}^t_\varepsilon - \mathrm{p}^t_{\varepsilon,\varrho})\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim \|\nabla\mathrm{p}^t_\varepsilon\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} + \varrho^{-1}\|\mathrm{p}^t_{\varepsilon,\varrho}\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1,$$

we find

$$\begin{aligned}
|T^0_{\varepsilon,\varrho,R}| &\lesssim_t N_\varepsilon\|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_{2R})}(\|\nabla(\mathrm{p}_\varepsilon-\mathrm{p}_{\varepsilon,\varrho})\|_{\mathrm{L}^2} + \|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})})\\
&\qquad\qquad + N_\varepsilon^2\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}\|\nabla(\mathrm{p}_\varepsilon-\mathrm{p}_{\varepsilon,\varrho})\|_{\mathrm{L}^2}\\
&\lesssim_t \varepsilon N_\varepsilon\mathcal{E}^*_{\varepsilon,R} + (1+\varepsilon N_\varepsilon)N_\varepsilon(\mathcal{E}^*_{\varepsilon,R})^{1/2}\|\nabla(\mathrm{p}_\varepsilon-\mathrm{p}_{\varepsilon,\varrho})\|_{\mathrm{L}^2}.
\end{aligned}$$

Second, using (8.43) or (8.44), Assumption 8.3.1, and the assumption $\|\bar\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\lesssim_t 1$, we obtain in the considered regimes, in the dissipative case,

$$|T^1_{\varepsilon,\varrho,R}| \lesssim_t \varepsilon\big(\lambda_\varepsilon^{-1/2}N_\varepsilon^2 + R\lambda_\varepsilon^2|\log\varepsilon|^2\big)(\mathcal{E}^*_{\varepsilon,R})^{1/2},$$

or in the Gross-Pitaevskii case,

$$|T^1_{\varepsilon,\varrho,R}| \lesssim_t \varepsilon(N_\varepsilon^2 + \lambda_\varepsilon^2|\log\varepsilon|^2)(\mathcal{E}^*_{\varepsilon,R})^{1/2} \lesssim \varepsilon N_\varepsilon^2(\mathcal{E}^*_{\varepsilon,R})^{1/2}.$$

Integrating by parts, $T^2_{\varepsilon,\varrho,R}$ takes the form

$$\begin{aligned}
T^2_{\varepsilon,\varrho,R} &= -\int_{\mathbb{R}^2}\frac{N_\varepsilon|\log\varepsilon|}{2}(1-|u_\varepsilon|^2)\\
&\quad\times\operatorname{div}\bigg(a\mathrm{p}_{\varepsilon,\varrho}\nabla^\perp\chi_R + a\chi_R F^\perp(\bar\Gamma_\varepsilon^\perp\cdot\mathrm{v}_\varepsilon) + a\mathrm{p}_{\varepsilon,\varrho}\chi_R\Big(\nabla^\perp h - 2F^\perp - \lambda_\varepsilon\beta\bar\Gamma_\varepsilon^\perp - 2\frac{N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\bigg),
\end{aligned}$$

and hence, again using (8.43) or (8.44), Assumption 8.3.1, and the bound $\|\bar\Gamma_\varepsilon\|_{W^{1,\infty}}\lesssim 1$, we obtain, for all $\theta>0$, in the considered regimes, in the dissipative case,

$$|T^2_{\varepsilon,\varrho,R}| \lesssim_{t,\theta} \varepsilon N_\varepsilon|\log\varepsilon|\big(1+R^{-1}\lambda_\varepsilon^{-1/2}+\lambda_\varepsilon R^\theta\big)(\mathcal{E}^*_{\varepsilon,R})^{1/2},$$

or in the Gross-Pitaevskii case,

$$|T^2_{\varepsilon,\varrho,R}| \lesssim_{t,\theta} \varepsilon N_\varepsilon |\log\varepsilon|(1+\lambda_\varepsilon \varrho^\theta)(\mathcal{E}^*_{\varepsilon,R})^{1/2} \lesssim \varepsilon N_\varepsilon^2 \varrho^\theta (\mathcal{E}^*_{\varepsilon,R})^{1/2}.$$

Finally, we observe that the choice (8.97) of $\psi_{\varepsilon,\varrho,R}$ exactly yields

$$T^3_{\varepsilon,\varrho,R} = \int_{\mathbb{R}^2} \frac{aN_\varepsilon^2}{2}(1-|u_\varepsilon|^2)\partial_t(\psi_{\varepsilon,\varrho,R} - \chi_R|\mathrm{v}_\varepsilon|^2) = \int_{\mathbb{R}^2} \frac{aN_\varepsilon^2}{2}(1-|u_\varepsilon|^2)(\partial_t\psi_{\varepsilon,\varrho,R} - 2\chi_R\,\mathrm{v}_\varepsilon\cdot\partial_t\mathrm{v}_\varepsilon),$$

and hence, using (8.98) or (8.99), and Assumption 8.3.1, in the considered regimes, we find in the dissipative case,

$$\|T^3_{\varepsilon,\varrho,R}\|_{\mathrm{L}^1_t} \lesssim \varepsilon N_\varepsilon^2 \Big(1+\frac{|\log\varepsilon|}{N_\varepsilon}\Big)(\mathcal{E}^*_{\varepsilon,R})^{1/2} \lesssim \varepsilon(N_\varepsilon^2 + N_\varepsilon|\log\varepsilon|)(\mathcal{E}^*_{\varepsilon,R})^{1/2},$$

or in the Gross-Pitaevskii case,

$$|T^3_{\varepsilon,\varrho,R}| \lesssim \varepsilon N_\varepsilon^2 \varrho^\theta (\mathcal{E}^*_{\varepsilon,R})^{1/2}.$$

The estimates (8.110) and (8.111) now follow from the above with $I'_{\varepsilon,\varrho,R} := T^0_{\varepsilon,\varrho,R} + T^1_{\varepsilon,\varrho,R} + T^2_{\varepsilon,\varrho,R} + T^3_{\varepsilon,\varrho,R}$. ☐

8.5 Vortex analysis

In this section, we first recall and revisit some standard tools for vortex analysis, which are needed in order to control the various terms appearing in the decomposition in Lemma 8.4.4. These tools will only be used in the dissipative case, and we restrict in this section to the case $N_\varepsilon \lesssim |\log\varepsilon|$. (Suitable adaptations to the case $N_\varepsilon \gg |\log\varepsilon|$ are discussed in Section 8.8.1.)

8.5.1 Ball construction lower bounds

We need a version of the ball construction lower bounds à la Jerrard-Sandier [379, 260] which is *localizable* in order to be adapted both to the weighted case and to the setting of the infinite plane with no finite energy control (hence no a priori finiteness assumption on the number of vortices), and which further yields very small errors (we need an error of order $o(N_\varepsilon^2)$, which gets very small when N_ε diverges slowly). For that purpose we use the version developed in [383], which in particular allows to cover the plane with balls centered at the points of the lattice $R\mathbb{Z}^2$, make the standard ball construction in each ball of the covering, assemble all the constructed balls, and then discard some balls from the collection so as to make it disjoint again. The error in the lower bounds given by this ball construction is essentially $N_\varepsilon|\log r|$, where r is the total radius of the balls, so that we need to take r large enough (almost as large as $O(1)$ when N_ε diverges slowly), but here the pinning weight adds again a difficulty since it may vary significantly over the size of the balls of this construction, thus perturbing the lower bound itself.

The following preliminary result describes the precise contribution of the vortices to the energy, and in particular defines the vortex "locations".

Lemma 8.5.1 (Localized lower bound)**.** *Let $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, with $1\lesssim a\le 1$, let $u_\varepsilon:\mathbb{R}^2\to\mathbb{C}$, $\mathrm{v}_\varepsilon:\mathbb{R}^2\to\mathbb{R}^2$, with $\|\mathrm{curl}\,\mathrm{v}_\varepsilon\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim 1$. Let $0<\varepsilon\ll 1$, $N_\varepsilon, R\ge 1$, and assume that $\log\mathcal{E}^*_{\varepsilon,R} \ll |\log\varepsilon|$. Then, for some $\bar r\simeq 1$, for all $\varepsilon>0$ small enough, and all $r\in(\varepsilon^{1/2},\bar r)$, there exists a locally finite union of disjoint closed balls $\mathcal{B}^r_{\varepsilon,R}$, monotone in r and covering the set $\{x: |u_\varepsilon(x)|<1/2\}$, such that for all $z\in R\mathbb{Z}^2$ the sum of the radii of the balls of the collection $\mathcal{B}^r_{\varepsilon,R}$ centered at points of $B_R(z)$ is bounded by r, and such that, letting $\mathcal{B}^r_{\varepsilon,R} = \biguplus_j B^j$, $B^j := \bar B(y_j,r_j)$, $d_j:=\deg(u_\varepsilon,\partial B^j)$, and defining the point-vortex measure $\nu^r_{\varepsilon,R} := 2\pi\sum_j d_j\delta_{y_j}$, the following properties hold,* *(i)* Localized lower bound: *for all $\phi \in W^{1,\infty}(\mathbb{R}^2)$ with $\phi \geq 0$, we have for all j,*

$$\frac{1}{2}\int_{B^j}\phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \geq \pi\phi(y_j)|d_j|\log(r/\varepsilon) - O(r_j\mathcal{E}^*_{\varepsilon,R})\|\nabla\phi\|_{\mathrm{L}^\infty} - O\bigg(r_j^2N_\varepsilon^2 + |d_j|\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg)\|\phi\|_{\mathrm{L}^\infty}, \quad (8.119)$$

and similarly, for all $\phi \in W^{1,\infty}(\mathbb{R}^2)$ supported in a ball of radius R,

$$\frac{1}{2}\int_{\mathcal{B}^r_{\varepsilon,R}}\phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \geq \frac{\log(r/\varepsilon)}{2}\int_{\mathbb{R}^2}\phi|\nu^r_{\varepsilon,R}| - O(r\mathcal{E}^*_{\varepsilon,R})\|\nabla\phi\|_{\mathrm{L}^\infty} - O\bigg(r^2N_\varepsilon^2 + \frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg)\|\phi\|_{\mathrm{L}^\infty}; \quad (8.120)$$

(ii) Number of vortices:

$$\sup_z\int_{B_R(z)}|\nu^r_{\varepsilon,R}| \lesssim \frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}; \quad (8.121)$$

(iii) Jacobian estimate: *for all $\gamma \in [0,1]$,*

$$\sup_z\|\nu^r_{\varepsilon,R} - \tilde\mu_\varepsilon\|_{(C^\gamma_c(B_R(z)))^*} \lesssim r^\gamma\frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} + \varepsilon^{\gamma/2}(\mathcal{E}^*_{\varepsilon,R}+\varepsilon^2N_\varepsilon^2). \qquad \Diamond$$

Proof. We split the proof into two steps.

Step 1. Proof of (i)–(ii).

We use the notation $\tilde{\mathcal{E}}^*_{\varepsilon,R} := \sup_z\int_{B_R(z)}\tilde e_\varepsilon$, with

$$\tilde e_\varepsilon := \frac{1}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a_{\min}}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big), \qquad a_{\min} := \inf_x a(x) \gtrsim 1.$$

Note that by assumption we have in particular $\tilde{\mathcal{E}}^*_{\varepsilon,R} \lesssim \mathcal{E}^*_{\varepsilon,R} \lesssim \varepsilon^{-1/5}$. We may apply [383, Proposition 2.1] with $\Omega_\varepsilon = \mathbb{R}^2$, $A_\varepsilon = N_\varepsilon \mathrm{v}_\varepsilon$, with ε replaced by $\varepsilon/\sqrt{a_{\min}}$, and with open cover $(U_\alpha)_\alpha = (B_R(z))_{z\in R\mathbb{Z}^2}$ (note that the argument in [383] indeed works identically on the whole space, and that the energy bound is only needed uniformly on all elements of the open cover). For some $\varepsilon_0, C_0, \bar r \simeq 1$, for all $\varepsilon < \varepsilon_0$ and all $r \in (\varepsilon^{1/2}, \bar r)$, we obtain a locally finite collection $\mathcal{B}^r_{\varepsilon,R}$ of disjoint closed balls covering the set $\{x : |u_\varepsilon(x)| < 1/2\}$, such that for all $B \in \mathcal{B}^r_{\varepsilon,R}$ we have

$$\int_B\Big(\tilde e_\varepsilon + \frac{N_\varepsilon^2}{2}|\mathrm{curl}\,\mathrm{v}_\varepsilon|^2\Big) \geq \pi|d_B|\Big(\log\frac{r}{\varepsilon\bar C_B} - C_0\Big),$$

where we have set $d_B := \deg(u_\varepsilon, \partial B)$, and where $\bar C_B$ is defined as in [383, equation (2.4)]. Moreover, the construction in [383] ensures that the collection $\mathcal{B}^r_{\varepsilon,R}$ is monotone in r, and that $B_R(z)\cap\mathcal{B}^r_{\varepsilon,R}$ has total radius bounded by r for all $z \in R\mathbb{Z}^2$. By [383, Lemma 2.1], we have $\bar C_B \leq 16|\log\varepsilon|^{-1}\tilde{\mathcal{E}}^*_{\varepsilon,R} \lesssim |\log\varepsilon|^{-1}\mathcal{E}^*_{\varepsilon,R}$, so that the above becomes, for all $B \in \mathcal{B}^r_{\varepsilon,R}$,

$$\int_B\Big(\tilde e_\varepsilon + \frac{N_\varepsilon^2}{2}|\mathrm{curl}\,\mathrm{v}_\varepsilon|^2\Big) \geq \pi|d_B|\log(r/\varepsilon) - |d_B|O\bigg(\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg). \quad (8.122)$$

Let $r \in (\varepsilon^{1/2}, \bar r)$ be fixed, and set $\mathcal{B}^r_{\varepsilon,R} = \biguplus_j B^j$, $B^j := \bar B(y_j, r_j)$, with corresponding degrees $d_j := d_{B^j}$. Noting that by assumption we have

$$\int_{B^j} |\mathrm{curl}\, \mathrm{v}_\varepsilon|^2 \lesssim |B^j| \lesssim r_j^2,$$

the result (8.122) takes the following form, for all j,

$$\int_{B^j} \tilde e_\varepsilon \ge \pi |d_j| \log(r/\varepsilon) - |d_j| O\bigg(\log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg) - O(r_j^2 N_\varepsilon^2). \tag{8.123}$$

Using the assumption $\log \mathcal{E}^*_{\varepsilon,R} \ll |\log\varepsilon|$ and the choice $r > \varepsilon^{1/2}$, the above right-hand side is bounded from below by $\frac{\pi}{2}|d_j||\log\varepsilon|(1-o(1)) - O(r_j^2 N_\varepsilon^2)$, and hence, summing over $B^j \in \mathcal{B}^r_{\varepsilon,R}$ with $y_j \in B_R(z)$, we find for all $\varepsilon > 0$ small enough,

$$\frac{\pi}{3}|\log\varepsilon| \sum_{j: y_j \in B_R(z)} |d_j| \;\le\; \int_{B_{R+1}(z)\cap \mathcal{B}^r_{\varepsilon,R}} \tilde e_\varepsilon + O(N_\varepsilon^2) \sum_{j:y_j\in B_R(z)} r_j^2 \;\lesssim\; \mathcal{E}^*_{\varepsilon,R} + r^2 N_\varepsilon^2,$$

and hence, with the choice $r \lesssim 1$,

$$\sum_{j:y_j\in B_R(z)} |d_j| \lesssim \frac{N_\varepsilon^2 + \mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}, \tag{8.124}$$

that is, item (ii). Let us now prove item (i). Let $\phi \in W^{1,\infty}(\mathbb{R}^2)$, $\phi \ge 0$. For all $B^j \in \mathcal{B}^r_{\varepsilon,R}$, we have from (8.123)

$$\begin{aligned}\int_{B^j} \phi \tilde e_\varepsilon &\ge \phi(y_j) \int_{B^j} \tilde e_\varepsilon - r_j \|\nabla\phi\|_{\mathrm{L}^\infty} \int_{B^j} \tilde e_\varepsilon \\ &\ge \pi\phi(y_j)|d_j|\log(r/\varepsilon) - \phi(y_j)|d_j|\, O\bigg(\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg) - \phi(y_j) O(r_j^2 N_\varepsilon^2) - r_j\|\nabla\phi\|_{\mathrm{L}^\infty}\int_{B^j}\tilde e_\varepsilon,\end{aligned}$$

hence

$$\int_{B^j} \phi\tilde e_\varepsilon \ge \pi\phi(y_j)|d_j|\log(r/\varepsilon) - O\bigg(r_j^2 N_\varepsilon^2 + |d_j|\log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg)\|\phi\|_{\mathrm{L}^\infty} - O(r_j \mathcal{E}^*_{\varepsilon,R})\|\nabla\phi\|_{\mathrm{L}^\infty}.$$

Further assuming that ϕ is supported in $B_R(z)$ for some $z \in R\mathbb{Z}^2$, summing the above with respect to j with $y_j \in B_R$, setting $\nu^r_{\varepsilon,R} := 2\pi\sum_j d_j \delta_{y_j}$, and using (8.124), we find

$$\int_{\mathcal{B}^r_{\varepsilon,R}} \phi\tilde e_\varepsilon \ge \frac{\log(r/\varepsilon)}{2}\int_{\mathbb{R}^2}\phi|\nu^r_{\varepsilon,R}| - O\bigg(r^2N_\varepsilon^2 + \frac{N_\varepsilon^2 + \mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg)\|\phi\|_{\mathrm{L}^\infty} - O(r\mathcal{E}^*_{\varepsilon,R})\|\nabla\phi\|_{\mathrm{L}^\infty}.$$

Item (i) then follows by definition of $\tilde e_\varepsilon$ with $a_{\min} \le a$.

Step 2. Proof of (iii).

Using item (i) and arguing just as in [395, item (5) of Proposition 4.4], for $\gamma \in [0,1]$, we obtain for all $r \in (\varepsilon^{1/2}, \bar r)$ and all $\phi \in C^\gamma_c(\mathbb{R}^2)$ supported in $B_R(z)$ for some $z \in R\mathbb{Z}^2$,

$$\begin{aligned}\bigg|\int \phi(\nu^r_{\varepsilon,R} - \tilde\mu_\varepsilon)\bigg| &\lesssim r^\gamma \|\phi\|_{C^\gamma} \sum_{j:y_j\in B_R(z)} |d_j| \\ &\qquad + \varepsilon^{\gamma/2}\|\phi\|_{C^\gamma}\int_{B_R}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} + N_\varepsilon|1-|u_\varepsilon|^2||\mathrm{curl}\,\mathrm{v}_\varepsilon|\Big) \\ &\lesssim r^\gamma \frac{N_\varepsilon^2 + \mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}|\phi|_{C^\gamma} + \bigg(\varepsilon^{\gamma/2}\mathcal{E}^*_{\varepsilon,R} + \varepsilon^{2+\gamma/2}N_\varepsilon^2\int_{B_R}|\mathrm{curl}\,\mathrm{v}_\varepsilon|^2\bigg)\|\phi\|_{C^\gamma},\end{aligned} \tag{8.125}$$

and the result follows from the assumption $\|\mathrm{curl}\,\mathrm{v}_\varepsilon\|_{\mathrm{L}^2} \lesssim 1$. ☐

In Section 8.6 below, strong estimates are proved on the time derivative of the modulated energy excess $\mathcal{D}^*_{\varepsilon,R}$, but these estimates involve the modulated energy $\mathcal{E}^*_{\varepsilon,R}$. In order to buckle the argument, it is thus crucial to independently find an optimal control on $\mathcal{E}^*_{\varepsilon,R}$, or equivalently on the number of vortices, in terms of $\mathcal{D}^*_{\varepsilon,R}$. Note that in the case without pinning and forcing no cut-off is needed and this difficulty is absent (the excess is then indeed simply defined by $\mathcal{D}_\varepsilon = \mathcal{E}_\varepsilon - \pi N_\varepsilon|\log\varepsilon|$, cf. [395]). This control of $\mathcal{E}^*_{\varepsilon,R}$ is the main content of the following result, and allows to further refine the conclusions of Lemma 8.5.1 above. Particular attention is needed in the regime $N_\varepsilon \lesssim \log|\log\varepsilon|$ to ensure an error as small as $o(N_\varepsilon^2)$ in the energy lower bound. Various useful corollaries are further included. In particular, item (vi) gives an optimal control of the energy inside the small balls, measured in L^p for any $p<2$; since this result in L^p is already enough for our purposes, it is not necessary here to adapt the more precise Lorentz estimates of [396, Corollary 1.2] to the present weighted context, and we instead use a more direct argument adapted from [403].

Proposition 8.5.2 (Refined lower bound). *Let $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, with $1\lesssim a\le 1$ and $\|\nabla h\|_{\mathrm{L}^\infty}\lesssim 1$, let $u_\varepsilon:\mathbb{R}^2\to\mathbb{C}$, $\mathrm{v}_\varepsilon:\mathbb{R}^2\to\mathbb{R}^2$, with $\|\mathrm{curl}\,\mathrm{v}_\varepsilon\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}, \|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}\lesssim 1$. Let $0<\varepsilon\ll 1$, $1\ll N_\varepsilon\lesssim|\log\varepsilon|$, and $R\ge 1$ with $|\log\varepsilon|\lesssim R\lesssim|\log\varepsilon|^n$ for some $n\ge 1$, and assume that $\mathcal{D}^*_{\varepsilon,R}\lesssim N_\varepsilon^2$. Then $\mathcal{E}^*_{\varepsilon,R}\lesssim N_\varepsilon|\log\varepsilon|$ holds for all $\varepsilon>0$ small enough. Moreover, for some $\bar r\simeq 1$, for all $\varepsilon>0$ small enough and all $r\in(\varepsilon^{1/2},\bar r)$, there exists a locally finite union of disjoint closed balls $\mathcal{B}^r_{\varepsilon,R}$, monotone in r and covering the set $\{x:|u_\varepsilon(x)|<1/2\}$, and for all $r_0\in(\varepsilon^{1/2},\bar r)$ and $r\ge r_0$ there exists a locally finite union of disjoint closed balls $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$, monotone in r and covering the set $\{x:||u_\varepsilon(x)|-1|\ge|\log\varepsilon|^{-1}\}$, such that $\mathcal{B}^{r_0}_{\varepsilon,R}\subset\tilde{\mathcal{B}}^{r_0,r_0}_{\varepsilon,R}$, such that for all $z\in R\mathbb{Z}^2$ the sum of the radii of the balls of the collection $\mathcal{B}^r_{\varepsilon,R}$ centered at points of $B_R(z)$ is bounded by r and the sum of the radii of the balls of the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ centered at points of $B_R(z)$ is bounded by Cr, and such that, letting $\mathcal{B}^r_{\varepsilon,R}=\biguplus_j B^j$, $B^j:=\bar B(y_j,r_j)$, $d_j:=\deg(u_\varepsilon,\partial B^j)$, and defining the point-vortex measure $\nu^r_{\varepsilon,R}:=2\pi\sum_j d_j\delta_{y_j}$, the following properties hold,*

(i) Lower bound: *in the regime $N_\varepsilon\gg\log|\log\varepsilon|$, we have for all $e^{-o(N_\varepsilon)}\le r\ll N_\varepsilon|\log\varepsilon|^{-1}$ and all $z\in\mathbb{R}^2$,*

$$\frac12\int_{\mathcal{B}^r_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ge\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi_R^z|\nu^r_{\varepsilon,R}|-o(N_\varepsilon^2),\qquad(8.126)$$

while in the regime $1\ll N_\varepsilon\lesssim\log|\log\varepsilon|$ we have for all $e^{-o(N_\varepsilon)}\le r\ll 1$ and all $r_0\le r$ with $\varepsilon^{1/2}<r_0\ll N_\varepsilon|\log\varepsilon|^{-1}$, for all $z\in\mathbb{R}^2$,

$$\frac12\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ge\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi_R^z\nu^{r_0}_{\varepsilon,R}-o(N_\varepsilon^2);\qquad(8.127)$$

(ii) Number of vortices: *for $\varepsilon^{1/2}<r\ll 1$,*

$$\sup_z\int_{B_R(z)}|\nu^r_{\varepsilon,R}|\ \lesssim\ N_\varepsilon,\qquad(8.128)$$

and moreover in the regime $1\ll N_\varepsilon\ll|\log\varepsilon|^{1/2}$ the measure $\nu^r_{\varepsilon,R}$ is nonnegative for all $e^{-o(1)N_\varepsilon^{-1}|\log\varepsilon|}\le r<\bar r$;

(iii) Jacobian estimate: *for $\varepsilon^{1/2}<r\ll 1$, for all $\gamma\in[0,1]$,*

$$\sup_z\|\nu^r_{\varepsilon,R}-\tilde\mu_\varepsilon\|_{(C^\gamma_c(B_R(z)))^*}\lesssim r^\gamma N_\varepsilon+\varepsilon^{\gamma/2}N_\varepsilon|\log\varepsilon|,\qquad(8.129)$$

$$\sup_z\|\mu_\varepsilon-\tilde\mu_\varepsilon\|_{(C^\gamma_c(B_R(z)))^*}\lesssim\varepsilon^\gamma N_\varepsilon|\log\varepsilon|^{n+1},\qquad(8.130)$$

hence in particular, for all $\gamma \in (0,1]$,

$$\sup_z \|\tilde\mu_\varepsilon\|_{(C_c^\gamma(B_R(z)))^*} \simeq_\gamma N_\varepsilon, \qquad \sup_z \|\mu_\varepsilon\|_{(C_c^\gamma(B_R(z)))^*} \simeq_\gamma N_\varepsilon; \tag{8.131}$$

(iv) Excess energy estimate: *for all $\phi \in W^{1,\infty}(\mathbb{R}^2)$ supported in a ball of radius R,*

$$\int_{\mathbb{R}^2} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big) \lesssim (\mathcal{D}^*_{\varepsilon,R} + o(N_\varepsilon^2))\|\phi\|_{W^{1,\infty}}; \tag{8.132}$$

(v) Energy outside small balls: *in the regime $N_\varepsilon \gg \log|\log\varepsilon|$, we have for all $e^{-o(N_\varepsilon)} \le r < \bar r$ and all $z \in \mathbb{R}^2$,*

$$\int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \le \mathcal{D}^z_{\varepsilon,R} + o(N_\varepsilon^2), \tag{8.133}$$

while in the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$ we have for all $r \ge e^{-o(N_\varepsilon)}$ and all $r_0 \le r$ with $\varepsilon^{1/2} < r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$, for all $z \in \mathbb{R}^2$,

$$\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \le \mathcal{D}^z_{\varepsilon,R} + o(N_\varepsilon^2); \tag{8.134}$$

(vi) L^p-estimate inside small balls: *in the regime $N_\varepsilon \gg \log|\log\varepsilon|$, we have for all $\varepsilon^{1/2} < r < \bar r$ and all $1 \le p < 2$,*

$$\sup_z \int_{\mathcal{B}^r_{\varepsilon,R}} \chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim_p o(N_\varepsilon^p), \tag{8.135}$$

while in the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$ we have for all $r > \varepsilon^{1/2}$ and all $r_0 \le r$ with $\varepsilon^{1/2} < r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$, for all $1 \le p < 2$,

$$\sup_z \int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} \chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim_p o(N_\varepsilon^p). \tag{8.136}$$

$\Diamond$

Proof. We split the proof into eight steps. The main work consists in checking that the assumptions imply the optimal bound on the energy $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon|$. The conclusion is obtained in Step 5 for the regime $\log|\log\varepsilon| \lesssim N_\varepsilon \lesssim |\log\varepsilon|$, but only in Step 7 for the complementary regime $1 \ll N_\varepsilon \ll \log|\log\varepsilon|$. The various other stated conclusions are then deduced in Step 8.

Step 1. Rough a priori estimate on the energy.

In this step, we prove $\mathcal{E}^*_{\varepsilon,R} \lesssim R^2|\log\varepsilon|^2$, and hence by the choice of R we deduce $\mathcal{E}^*_{\varepsilon,R} \lesssim |\log\varepsilon|^m$ for some $m \ge 4$. Decomposing $\mu_\varepsilon = N_\varepsilon \mathrm{curl}\,\mathrm{v}_\varepsilon + \mathrm{curl}\,(j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon)$, the assumption $\mathcal{D}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$ yields for all $z \in \mathbb{R}^2$,

$$\begin{aligned}\mathcal{E}^z_{\varepsilon,R} &\le \mathcal{D}^*_{\varepsilon,R} + \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\mu_\varepsilon \\ &\lesssim N_\varepsilon^2 + N_\varepsilon|\log\varepsilon|\int_{\mathbb{R}^2} a\chi_R^z|\mathrm{curl}\,\mathrm{v}_\varepsilon| + |\log\varepsilon|\int_{\mathbb{R}^2}|\nabla(a\chi_R^z)||j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon|.\end{aligned} \tag{8.137}$$

Using the pointwise estimate of Lemma 8.4.2 for $j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon$, using $|\nabla(a\chi_R^z)| \lesssim \mathbb{1}_{B_{2R}(z)}$, $\|\operatorname{curl} \mathrm{v}_\varepsilon\|_{\mathrm{L}^1} \lesssim 1$, and $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty} \lesssim 1$, we obtain

$$\begin{aligned}
\mathcal{E}_{\varepsilon,R}^z &\lesssim |\log\varepsilon|^2 + |\log\varepsilon|\Big(\int_{B_{2R}(z)} (1-|u_\varepsilon|^2)^2\Big)^{1/2}\Big(\int_{B_{2R}(z)} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2} \\
&\qquad + R|\log\varepsilon|\Big(\int_{B_{2R}(z)} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2} + RN_\varepsilon|\log\varepsilon|\Big(\int_{B_{2R}(z)} (1-|u_\varepsilon|^2)^2\Big)^{1/2} \\
&\lesssim |\log\varepsilon|^2 + \varepsilon|\log\varepsilon|\mathcal{E}_{\varepsilon,R}^* + R|\log\varepsilon|(\mathcal{E}_{\varepsilon,R}^*)^{1/2}.
\end{aligned}$$

Taking the supremum over z, and absorbing $\mathcal{E}_{\varepsilon,R}^*$ into the left-hand side, the result follows.

Step 2. Applying Lemma 8.5.1.

The result of Step 1 yields in particular $\log\mathcal{E}_{\varepsilon,R}^* \ll |\log\varepsilon|$, which allows to apply Lemma 8.5.1. For fixed $r \in (\varepsilon^{1/2}, \bar r)$, let $\mathcal{B}_{\varepsilon,R}^r = \biguplus_j B^j$ denote the union of disjoint closed balls given by Lemma 8.5.1, and let $\nu_{\varepsilon,R}^r$ denote the associated point-vortex measure. Using Lemma 8.5.1(ii) in the form

$$\int_{B_R(z)} |\nu_{\varepsilon,R}^r| = \sum_{j:y_j\in B_R(z)} |d_j| \lesssim N_\varepsilon + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}, \tag{8.138}$$

Lemma 8.5.1(i) gives, for all $\phi \in W^{1,\infty}(\mathbb{R}^2)$ with $\phi \ge 0$, if ϕ is supported in a ball of radius R,

$$\begin{aligned}
\frac12 \int_{\mathcal{B}_{\varepsilon,R}^r} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) &\ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} \phi|\nu_{\varepsilon,R}^r| - O(r\mathcal{E}_{\varepsilon,R}^*)\|\nabla\phi\|_{\mathrm{L}^\infty} \\
- O\bigg(r^2N_\varepsilon^2 + |\log r|\Big(N_\varepsilon + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}\Big) &+ \Big(N_\varepsilon + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}\Big)\log\Big(2 + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}\Big)\bigg)\|\phi\|_{\mathrm{L}^\infty}.
\end{aligned} \tag{8.139}$$

We now prove the following consequence of these bounds,

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus\mathcal{B}_{\varepsilon,R}^r} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \\
&\le \mathcal{D}_{\varepsilon,R}^z + O\bigg(r\mathcal{E}_{\varepsilon,R}^* + (|\log r| + r|\log\varepsilon|)\Big(N_\varepsilon + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}\Big) + \Big(N_\varepsilon + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}\Big)\log\Big(2 + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}\Big)\bigg).
\end{aligned} \tag{8.140}$$

First, the lower bound (8.139) applied to $\phi = a\chi_R^z$ is rewritten as follows,

$$\begin{aligned}
&\frac12\int_{\mathbb{R}^2\setminus\mathcal{B}_{\varepsilon,R}^r} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \\
&\qquad\le T_{\varepsilon,R}^{r,z} + O\bigg(r\mathcal{E}_{\varepsilon,R}^* + r^2N_\varepsilon^2 + |\log r|\Big(N_\varepsilon + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}\Big) + \Big(N_\varepsilon + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}\Big)\log\Big(2 + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}\Big)\bigg),
\end{aligned}$$

where we have set

$$T_{\varepsilon,R}^{r,z} := \frac12\int_{\mathbb{R}^2} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\nu_{\varepsilon,R}^r\Big).$$

If $\nu_{\varepsilon,R}^r$ was replaced by μ_ε in this last expression, we would recognize the definition of the excess $\mathcal{D}_{\varepsilon,R}^z$, and the result (8.140) would follow. Hence, in order to deduce (8.140), it only remains to check that for all $\phi \in W^{1,\infty}(\mathbb{R}^2)$ supported in a ball of radius R,

$$\Big|\int_{\mathbb{R}^2} \phi(\mu_\varepsilon - \nu_{\varepsilon,R}^r)\Big| \lesssim r\Big(N_\varepsilon + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log\varepsilon|}\Big)\|\phi\|_{W^{1,\infty}} + \varepsilon^{1/3}\|\phi\|_{W^{1,\infty}}. \tag{8.141}$$

Using the result of Step 1 in the form $\varepsilon^{1/6}\mathcal{E}^*_{\varepsilon,R}\lesssim 1$, Lemma 8.5.1(iii) with $\gamma=1$ yields

$$\Big|\int_{\mathbb{R}^2}\phi(\tilde\mu_\varepsilon-\nu^r_{\varepsilon,R})\Big|\lesssim r\Big(N_\varepsilon+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\|\phi\|_{W^{1,\infty}}+\varepsilon^{1/3}\|\phi\|_{W^{1,\infty}}.$$

It remains to replace $\tilde\mu_\varepsilon$ by μ_ε in this estimate. By definition (8.104), with $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}\lesssim 1$ and $|\nabla\phi|\le \mathbb{1}_{B_R(z)}\|\phi\|_{W^{1,\infty}}$, and using the result of Step 1 in the form $\varepsilon^{2/3}RN_\varepsilon(\mathcal{E}^*_{\varepsilon,R})^{1/2}\lesssim 1$, we find

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\phi(\tilde\mu_\varepsilon-\mu_\varepsilon)\Big|&\le N_\varepsilon\int_{B_R(z)}|\nabla\phi||\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2|\\&\lesssim RN_\varepsilon\|\phi\|_{W^{1,\infty}}\Big(\int_{B_R(z)}(1-|u_\varepsilon|^2)^2\Big)^{1/2}\\&\lesssim \varepsilon RN_\varepsilon(\mathcal{E}^*_{\varepsilon,R})^{1/2}\|\phi\|_{W^{1,\infty}}\lesssim\varepsilon^{1/3}\|\phi\|_{W^{1,\infty}},\end{aligned}\tag{8.142}$$

and the result (8.141) follows.

Step 3. Energy and number of vortices.

In this step, we show that (8.138) is essentially an equality, in the sense that for all $\varepsilon^{1/2}<r\ll 1$,

$$\sup_z\int_{\mathbb{R}^2}\chi^z_R|\nu^r_{\varepsilon,R}|\lesssim N_\varepsilon+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\lesssim N_\varepsilon+\sup_z\int_{\mathbb{R}^2}\chi^z_R|\nu^r_{\varepsilon,R}|.\tag{8.143}$$

The lower bound already follows from (8.138). We now turn to the upper bound. Since the energy excess satisfies $\mathcal{D}^z_{\varepsilon,R}\lesssim N_\varepsilon^2$, we deduce from (8.141),

$$\mathcal{E}^z_{\varepsilon,R}\le\mathcal{D}^z_{\varepsilon,R}+\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi^z_R\mu_\varepsilon\le\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi^z_R\nu^r_{\varepsilon,R}+O\bigg(N_\varepsilon^2+r|\log\varepsilon|\Big(N_\varepsilon+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg).\tag{8.144}$$

Taking the supremum in z, and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side with $r\ll1$, the upper bound in (8.143) follows.

Step 4. Estimate on the total variation of the vorticity.

In this step, we prove that for all $e^{-o(|\log\varepsilon|)}<r\ll 1$,

$$\sup_z\int_{\mathbb{R}^2}\chi^z_R|\nu^r_{\varepsilon,R}|\le(1+o(1))\sup_z\int_{\mathbb{R}^2}\chi^z_R\nu^r_{\varepsilon,R}+O(N_\varepsilon).\tag{8.145}$$

This result is used in Step 5 below in order to replace $\int a\chi^z_R\nu^r_{\varepsilon,R}$ (resp. $\int a\chi^z_R\mu_\varepsilon$) by $\int\chi^z_R\nu^r_{\varepsilon,R}$ (resp. $\int\chi^z_R\mu_\varepsilon$), which happens to be crucial if we want to avoid integrability assumptions on ∇h, as we do here.

The lower bound (8.139) of Step 2 with $\phi=a\chi^y_R$ yields for all $y\in\mathbb{R}^2$, using the upper bound in (8.143) to replace the energy $\mathcal{E}^*_{\varepsilon,R}$ in the error terms,

$$\begin{aligned}\mathcal{E}^y_{\varepsilon,R}\ge\frac12\int_{\mathcal{B}^r_{\varepsilon,R}}a\chi^y_R\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ge\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi^y_R|\nu^r_{\varepsilon,R}|\\-O\bigg((|\log r|+r|\log\varepsilon|)\Big(N_\varepsilon+\sup_z\int_{\mathbb{R}^2}\chi^z_R|\nu^r_{\varepsilon,R}|\Big)+\Big(N_\varepsilon+\sup_z\int_{\mathbb{R}^2}\chi^z_R|\nu^r_{\varepsilon,R}|\Big)\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg).\end{aligned}$$

For $e^{-o(|\log\varepsilon|)}<r\ll1$, using the result of Step 1 in the form $\log\mathcal{E}^*_{\varepsilon,R}\ll|\log\varepsilon|$, we obtain for all $y\in\mathbb{R}^2$,

$$\mathcal{E}^y_{\varepsilon,R}\ge\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi^y_R|\nu^r_{\varepsilon,R}|-o(|\log\varepsilon|)\sup_z\int_{\mathbb{R}^2}\chi^z_R|\nu^r_{\varepsilon,R}|-o(N_\varepsilon|\log\varepsilon|).\tag{8.146}$$

On the other hand, the upper bound (8.144) yields

$$\mathcal{E}^y_{\varepsilon,R} \le \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi^y_R\nu^r_{\varepsilon,R} + O(N_\varepsilon|\log\varepsilon|) + o(1)\mathcal{E}^*_{\varepsilon,R}, \tag{8.147}$$

and thus, taking the supremum over y and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side,

$$\mathcal{E}^*_{\varepsilon,R} \le \frac{|\log\varepsilon|}{2}(1+o(1))\sup_z\int_{\mathbb{R}^2} a\chi^z_R|\nu^r_{\varepsilon,R}| + O(N_\varepsilon|\log\varepsilon|),$$

so that (8.147) takes the form, for all $y\in\mathbb{R}^2$,

$$\mathcal{E}^y_{\varepsilon,R} \le \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi^y_R\nu^r_{\varepsilon,R} + O(N_\varepsilon|\log\varepsilon|) + o(|\log\varepsilon|)\sup_z\int_{\mathbb{R}^2}\chi^z_R|\nu^r_{\varepsilon,R}|.$$

Combining this with (8.146), dividing both sides by $\frac12|\log\varepsilon|$, and taking the supremum over y, we find

$$\sup_z\int_{\mathbb{R}^2}\chi^z_R(\nu^r_{\varepsilon,R})^- \lesssim \sup_z\int_{\mathbb{R}^2} a\chi^z_R(|\nu^r_{\varepsilon,R}|-\nu^r_{\varepsilon,R}) \le O(N_\varepsilon) + o(1)\sup_z\int\chi^z_R|\nu^r_{\varepsilon,R}|.$$

This implies

$$\sup_z\int_{\mathbb{R}^2}\chi^z_R|\nu^r_{\varepsilon,R}| = \sup_z\int_{\mathbb{R}^2}\chi^z_R(\nu^r_{\varepsilon,R}+2(\nu^r_{\varepsilon,R})^-) \le \sup_z\int_{\mathbb{R}^2}\chi^z_R\nu^r_{\varepsilon,R} + O(N_\varepsilon) + o(1)\sup_z\int_{\mathbb{R}^2} a\chi^z_R|\nu^r_{\varepsilon,R}|,$$

and the result (8.145) follows after absorbing the last term in the left-hand side.

Step 5. Refined bound on the energy.

In this step, we prove $\mathcal{E}^*_{\varepsilon,R} \lesssim (N_\varepsilon + \log|\log\varepsilon|)|\log\varepsilon|$. By (8.138) this implies in particular $\sup_z\int_{\mathbb{R}^2}\chi^z_R|\nu^r_{\varepsilon,R}| \lesssim N_\varepsilon + \log|\log\varepsilon|$. In the regime $N_\varepsilon \gtrsim \log|\log\varepsilon|$, these bounds are already the optimal ones. The regime with a "small" number of vortices $1 \ll N_\varepsilon \ll \log|\log\varepsilon|$ is treated in Steps 6–7 below.

Let $e^{-o(|\log\varepsilon|)} < r \ll 1$ to be suitably chosen later. Using (8.141), the bound on the energy excess $\mathcal{D}^*_{\varepsilon,R} \lesssim N^2_\varepsilon$ yields for all $z\in R\mathbb{Z}^2$,

$$\mathcal{E}^z_{\varepsilon,R} \le \mathcal{D}^z_{\varepsilon,R} + \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi^z_R\mu_\varepsilon \lesssim N^2_\varepsilon + |\log\varepsilon|\int_{\mathbb{R}^2}\chi^z_R|\nu^r_{\varepsilon,R}| + r(N_\varepsilon|\log\varepsilon| + \mathcal{E}^*_{\varepsilon,R}),$$

and hence, using the result (8.145) of Step 4,

$$\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon| + |\log\varepsilon|\sup_z\int_{\mathbb{R}^2}\chi^z_R\nu^r_{\varepsilon,R} + r\mathcal{E}^*_{\varepsilon,R}.$$

Using (8.141) again, and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side with $r \ll 1$, this takes the form

$$\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon| + |\log\varepsilon|\sup_z\int_{\mathbb{R}^2}\chi^z_R\mu_\varepsilon. \tag{8.148}$$

It remains to estimate $\int_{\mathbb{R}^2}\chi^z_R\mu_\varepsilon$. Decomposing $\mu_\varepsilon = N_\varepsilon\mathrm{curl}\,\mathrm{v}_\varepsilon + \mathrm{curl}\,(j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon)$, using the pointwise estimate of Lemma 8.4.2 for $j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon$, using $|\nabla\chi^z_R| \lesssim R^{-1}\mathbb{1}_{B_{2R}(z)}$, $\|\nabla\chi^z_R\|_{\mathrm{L}^2} \lesssim 1$, $\|\mathrm{curl}\,\mathrm{v}_\varepsilon\|_{\mathrm{L}^1} \lesssim 1$, $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty} \lesssim 1$, and using the result of Step 1 in the form $\varepsilon\mathcal{E}^*_{\varepsilon,R} \lesssim 1$, we find

$$\int_{\mathbb{R}^2}\chi^z_R\mu_\varepsilon = N_\varepsilon\int_{\mathbb{R}^2}\chi^z_R\mathrm{curl}\,\mathrm{v}_\varepsilon - \int_{\mathbb{R}^2}\nabla^\perp\chi^z_R\cdot(j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon) \lesssim N_\varepsilon + \int_{\mathbb{R}^2}|\nabla\chi^z_R||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|.$$

Regarding the last integral, we distinguish between the contributions inside and outside the balls $\mathcal{B}^r_{\varepsilon,R}$, with $|\nabla\chi^z_R| \lesssim R^{-1}\mathbb{1}_{B_{2R}(z)} \le R^{-1}\chi^z_{2R}$, $\|\nabla\chi^z_R\|_{\mathrm{L}^2} \lesssim 1$, and $|B_{2R}(z)\cap\mathcal{B}^r_{\varepsilon,R}| \lesssim r^2$,

$$\begin{aligned}\int_{\mathbb{R}^2}\chi^z_R\mu_\varepsilon &\lesssim N_\varepsilon + \int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}}|\nabla\chi^z_R||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + R^{-1}\int_{B_{2R}(z)\cap\mathcal{B}^r_{\varepsilon,R}}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|\\ &\lesssim N_\varepsilon + \Big(\int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}}\chi^z_{2R}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2\Big)^{1/2} + rR^{-1}\Big(\int_{B_{2R}(z)}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2\Big)^{1/2}. \quad (8.149)\end{aligned}$$

Estimating the last right-hand side term by $rR^{-1}(\mathcal{E}^*_{\varepsilon,R})^{1/2}$, using (8.140) to estimate the first, using the bound on the energy excess $\mathcal{D}^*_{\varepsilon,R}\lesssim N_\varepsilon^2$, and noting that $k^{1/2}\log^{1/2}(2+k)\ll k$ holds for $k\gg1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\chi^z_R\mu_\varepsilon &\lesssim N_\varepsilon + (\mathcal{D}^*_{\varepsilon,R})^{1/2} + rR^{-1}(\mathcal{E}^*_{\varepsilon,R})^{1/2} + r^{1/2}(N_\varepsilon|\log\varepsilon| + \mathcal{E}^*_{\varepsilon,R})^{1/2}\\ &\qquad + \Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^{1/2}\Big(|\log r| + \log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\Big)^{1/2}\\ &\lesssim N_\varepsilon + r^{1/2}(N_\varepsilon|\log\varepsilon|)^{1/2} + r^{1/2}(\mathcal{E}^*_{\varepsilon,R})^{1/2} + o(1)\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} + |\log r|^{1/2}\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^{1/2}.\end{aligned}$$

Combining this with (8.148) yields

$$\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} \lesssim N_\varepsilon + r^{1/2}(N_\varepsilon|\log\varepsilon|)^{1/2} + r^{1/2}(\mathcal{E}^*_{\varepsilon,R})^{1/2} + o(1)\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} + |\log r|^{1/2}\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^{1/2},$$

and hence,

$$\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} \lesssim N_\varepsilon + |\log r| + r^{1/2}|\log\varepsilon|.$$

The result then follows from the choice $r = |\log\varepsilon|^{-2}$.

Step 6. Refined lower bound in the regime with a "small" number of vortices.

In this step, we study the regime $1\ll N_\varepsilon \lesssim \log|\log\varepsilon|$, for which the result of Step 5 is not optimal. More precisely, we consider the whole regime $1\ll N_\varepsilon\lesssim|\log\varepsilon|$ and we show the following: for all $r_0\in(\varepsilon^{1/2},\bar r)$ and $r\ge r_0$, there exists a locally finite union of disjoint closed balls $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$, monotone in r, covering the set $\{x : ||u_\varepsilon(x)|-1|\ge|\log\varepsilon|^{-1}\}$, such that for all z the sum of the radii of the balls intersecting $B_R(z)$ is bounded by Cr, and such that for all $\varepsilon>0$ small enough, and all $r_0\le r$ satisfying

$$\varepsilon^{1/2} < r_0 \ll N_\varepsilon^2|\log\varepsilon|^{-1}(N_\varepsilon + \log|\log\varepsilon|)^{-1}, \qquad e^{-o(N_\varepsilon)}\le r\ll 1, \quad (8.150)$$

we have for all $z\in\mathbb{R}^2$,

$$\begin{aligned}&\frac12\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\ &\qquad\qquad \ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi^z_R\nu^{r_0}_{\varepsilon,R} - o(1)\Big(\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2 - o(N_\varepsilon^2). \quad (8.151)\end{aligned}$$

We split the proof into three further substeps.

Substep 6.1. Enlarged balls: in this step, given some fixed $r_0\in(\varepsilon^{1/2},\bar r)$, we construct the enlarged collections of balls $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ for $r\ge r_0$.

According to [382, Proposition 4.8], and using the energy estimate of Step 5, we have

$$\mathcal{H}^1(\{x \in B_R(z), ||u_\varepsilon(x)|-1| \ge |\log\varepsilon|^{-1}\}) \le C\varepsilon|\log\varepsilon|^2\mathcal{E}^*_{\varepsilon,R} \le C\varepsilon|\log\varepsilon|^4,$$

where $\mathcal{H}^1$ denotes the 1-dimensional Haussdorff measure. From [382, Section 4.4.1] and [383, Section 2.2], it follows that we may cover the set $\{x : ||u_\varepsilon(x)|-1| \ge |\log\varepsilon|^{-1}\}$ by a locally finite union of disjoint closed balls such that for all z the sum of the radii of the balls intersecting $B_R(z)$ is bounded by $C\varepsilon|\log\varepsilon|^4$. We then combine this collection of balls with the collection $\mathcal{B}^{r_0}_{\varepsilon,R}$. Inductively merging as in [382, Lemma 4.1] any two such balls that intersect into a ball with the same total radius, we obtain a new collection $\tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}$ of disjoint closed balls that cover the set $\{x : ||u_\varepsilon(x)|-1| \ge |\log\varepsilon|^{-1}\}$, and such that for all z the sum of the radii of the balls intersecting $B_R(z)$ is bounded by $r_0 + C\varepsilon|\log\varepsilon|^6 \le Cr_0$.

Let us now grow the balls of this new collection $\tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}$ following Sandier's ball construction, as described e.g. in [382, Theorem 4.2]. This consists in growing simultaneously all the balls keeping their centers fixed and multiplying their radius by the same factor t. If some balls touch at some point during the growth, the corresponding balls are merged into one larger ball containing the previous ones and of same total radius. This construction ensures that the balls always remain disjoint. Stopping the growth process at some value of the factor t, and setting $r = tr_0$, we denote by $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ the corresponding locally finite collection of disjoint closed balls. By construction, for all z, the sum of the radii of the balls that intersect $B_R(z)$ is bounded by $Ct(r_0 + C\varepsilon|\log\varepsilon|^6) \le Cr$. Note that by construction $\mathcal{B}^{r_0}_{\varepsilon,R} \subset \tilde{\mathcal{B}}^{r_0}_{\varepsilon,R} = \tilde{\mathcal{B}}^{r_0,r_0}_{\varepsilon,R}$, but for $r > r_0$ the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ has a priori no clear relation with the collection $\mathcal{B}^{r}_{\varepsilon,R}$.

Substep 6.2. Preliminary estimate.

According to [396, Lemma 3.2] (applied with $c = d$ and $\lambda = 1$), we have, for any $\mathbb{S}^1$-valued map v with degree d on a generic ball B of radius r, and for any vector field $A : \partial B \to \mathbb{R}^2$,

$$\frac{1}{2}\int_{\partial B}|\nabla v - ivA|^2 + \frac{1}{2}\int_B |\operatorname{curl} A|^2 \ge \frac{\pi d^2}{r} - \frac{\pi d^2}{2} + \frac{1}{2}\int_{\partial B}\Big|\nabla v - ivA - ivd\frac{\tau}{r}\Big|^2,$$

where τ denotes the unit tangent to the circle ∂B. Applying it to $v = \frac{u_\varepsilon}{|u_\varepsilon|}$ and $A = N_\varepsilon \mathrm{v}_\varepsilon$, and noting that $|\nabla u_\varepsilon - iu_\varepsilon F|^2 = |u_\varepsilon|^2|\nabla\frac{u_\varepsilon}{|u_\varepsilon|} - i\frac{u_\varepsilon}{|u_\varepsilon|}F|^2 + |\nabla|u_\varepsilon||^2$ holds for any real-valued vector field F, we obtain the following improved lower bound on annuli: if $||u_\varepsilon|-1| \le |\log\varepsilon|^{-1}$ holds on ∂B, then we have

$$\begin{aligned}(1+O(|\log\varepsilon|^{-1}))\frac{1}{2}\int_{\partial B}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{1}{2}N_\varepsilon^2\int_B|\operatorname{curl}\mathrm{v}_\varepsilon|^2 \\ \ge \frac{\pi d^2}{r} - \frac{\pi d^2}{2} + \frac{1}{2}(1-O(|\log\varepsilon|^{-1}))\int_{\partial B}\Big|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon - iu_\varepsilon d\frac{\tau}{r}\Big|^2. \quad (8.152)\end{aligned}$$

Substep 6.3. Proof of (8.151).

Let $r_0 > 0$ be chosen as in (8.150). We start from Lemma 8.5.1(i) with $\phi = a\chi_R^z$, combined with the refined energy estimate of Step 5 and the choice of r_0, which yields

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{B}^{r_0}_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \\ \ge \frac{\log(r_0/\varepsilon)}{2}\int a\chi_R^z|\nu^{r_0}_{\varepsilon,R}| - o(N_\varepsilon^2) - C\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big). \quad (8.153)\end{aligned}$$

We next need to show that this lower bound for the energy is essentially maintained during the ball growth and merging process, hence holds as well for the collections $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ with $r > r_0$.

Assume that some ball $B=\bar B(y,s)$ gets grown into $B'=\bar B(y,ts)$ without merging, for some $t\ge 1$, and assume that $B'\setminus B$ does not intersect $\tilde{\mathcal B}_{\varepsilon,R}^{r_0}$, so that $||u_\varepsilon|-1|\le|\log\varepsilon|^{-1}$ holds on $B'\setminus B$. Let d denote the degree of B (hence of B'). Since by assumption we have

$$\begin{aligned}|a(x)\chi_R^z(x)-a(y)\chi_R^z(y)|&\le \chi_R^z(y)|a(x)-a(y)|+a(x)|\chi_R^z(x)-\chi_R^z(y)|\\&\le C\big(R^{-1}+\chi_R^z(y)\big)|x-y|,\end{aligned}\tag{8.154}$$

we may write

$$\begin{aligned}\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)&\ge\frac{a(y)\chi_R^z(y)}{2}\int_{B'\setminus B}|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\\&-CR^{-1}\int_{B'\setminus B}|\cdot-y||\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2-C\chi_R^z(y)\int_{B'\setminus B}|\cdot-y||\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2.\end{aligned}$$

Using that $|u_\varepsilon|\le 1+|\log\varepsilon|^{-1}$ holds on $B'\setminus B$, the last right-hand side term above is estimated as follows,

$$\begin{aligned}&\int_{B'\setminus B}|\cdot-y||\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\\&\le 2\int_{B'\setminus B}|\cdot-y|\,|u_\varepsilon|^2\Big|\frac{\tau d}{|\cdot-y|}\Big|^2+2\int_{B'\setminus B}|\cdot-y|\,\Big|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon-iu_\varepsilon\frac{\tau d}{|\cdot-y|}\Big|^2\\&\le Cd^2ts+2ts\int_{B'\setminus B}\Big|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon-iu_\varepsilon\frac{\tau d}{|\cdot-y|}\Big|^2,\end{aligned}\tag{8.155}$$

where $\tau(x)=(x-y)^\perp/|x-y|$ is the unit tangent to the circle centered at y, and we may then deduce

$$\begin{aligned}\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)&\ge\frac{a(y)\chi_R^z(y)}{2}\int_{B'\setminus B}|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\\&-CtsR^{-1}\int_{B'\setminus B}|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2-Cd^2ts\chi_R^z(y)\\&-Cts\chi_R^z(y)\int_{B'\setminus B}\Big|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon-iu_\varepsilon\frac{\tau d}{|\cdot-y|}\Big|^2.\end{aligned}\tag{8.156}$$

Again using that $||u_\varepsilon|-1|\le|\log\varepsilon|^{-1}$ holds on $B'\setminus B$, the estimate (8.152) on the ball $B(y,\rho)$ for ρ integrated between s and ts takes the form

$$\begin{aligned}(1+C|\log\varepsilon|^{-1})\frac12\int_{B'\setminus B}|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2&\ge\pi d^2\log t-\frac{\pi}{2}d^2ts-\frac12N_\varepsilon^2ts\int_{B'}|\mathrm{curl}\,\mathrm{v}_\varepsilon|^2\\&+(1-C|\log\varepsilon|^{-1})\frac12\int_{B'\setminus B}\Big|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon-iu_\varepsilon\frac{\tau d}{|\cdot-y|}\Big|^2.\end{aligned}$$

Combining this with (8.156), we are then led to

$$\begin{aligned}&(1+C|\log\varepsilon|^{-1})\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\&\ge a(y)\chi_R^z(y)\pi d^2\log t-Cd^2ts-\frac12N_\varepsilon^2ts\int_{B'}|\mathrm{curl}\,\mathrm{v}_\varepsilon|^2-CtsR^{-1}\int_{B'\setminus B}|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\\&+\Big(\frac{a(y)}{2}(1-C|\log\varepsilon|^{-1})-Cts\Big)\chi_R^z(y)\int_{B'\setminus B}\Big|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon-iu_\varepsilon\frac{\tau d}{|\cdot-y|}\Big|^2.\end{aligned}\tag{8.157}$$

For ε small enough and $ts \le \min\{1, \frac{1}{4C}\inf a\} =: \tilde r$ (note that by assumption $\tilde r \simeq 1$), the last right-hand side term is nonnegative, so that we conclude

$$\begin{aligned}(1+C|\log\varepsilon|^{-1})\frac12\int_{B'\setminus B} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\ \ge a(y)\chi_R^z(y)\pi d^2\log t - Cd^2ts - \frac12 N_\varepsilon^2 ts\int_{B'}|\mathrm{curl}\,\mathrm{v}_\varepsilon|^2 - CtsR^{-1}\int_{B'\setminus B}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\\ \ge a(y)\chi_R^z(y)\pi d^2\log t - Cts(d^2+N_\varepsilon^2) - CtsR^{-1}\mathcal{E}^*_{\varepsilon,R}. \quad (8.158)\end{aligned}$$

If the ball $B=\bar B(y,s)$ belongs to the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ for some $r\ge r_0$, only a finite number of balls of the collection $\mathcal{B}^{r_0}_{\varepsilon,R}$ are included in the ball B. Denote them by $B^j=\bar B(y_j,s_j)$, $j=1,\ldots,k$. By definition, the degree d of B is then equal to $d=\sum_j d_j$, where d_j denotes the degree of B^j. We may then write

$$\begin{aligned}a(y)\chi_R^z(y)d^2 \ge a(y)\chi_R^z(y)\sum_j d_j \ge \sum_j a(y_j)\chi_R^z(y_j)d_j - C\sum_j |d_j||y-y_j|\mathbb{1}_{B_{2R}(z)}(y_j)\\ \ge \sum_j a(y_j)\chi_R^z(y_j)d_j - Cs\sum_j |d_j|\mathbb{1}_{B_{2R}(z)}(y_j),\end{aligned}$$

and hence, in terms of the point-vortex measure $\nu^{r_0}_{\varepsilon,R}$,

$$a(y)\chi_R^z(y)d^2 \ge \frac{1}{2\pi}\int_B a\chi_R^z\nu^{r_0}_{\varepsilon,R} - Cs\int_{B_{2R}(z)}|\nu^{r_0}_{\varepsilon,R}|. \quad (8.159)$$

Therefore, if the ball $B=\bar B(y,s)$ belongs to the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ for some $r\ge r_0$ and gets grown without merging into a ball $B'=\bar B(y,ts)$ for some $t\ge 1$ with $ts\le\tilde r$, then combining (8.158) and (8.159) yields

$$\begin{aligned}(1+C|\log\varepsilon|^{-1})\frac12\int_{B'\setminus B} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\ \ge \frac{\log t}{2}\int_B a\chi_R^z\nu^{r_0}_{\varepsilon,R} - Cs\log t\int_{B_{2R}(z)}|\nu^{r_0}_{\varepsilon,R}| - Cts\Big(N_\varepsilon + \int_{B_{2R}(z)}|\nu^{r_0}_{\varepsilon,R}|\Big)^2 - CtsR^{-1}\mathcal{E}^*_{\varepsilon,R},\end{aligned}$$

and hence, using Lemma 8.5.1(ii), the inequality $|\log t|\le t$ for $t\ge1$, and the choice $R\gtrsim|\log\varepsilon|$,

$$\frac12\int_{B'\setminus B} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \ge \frac{\log t}{2}\int_B a\chi_R^z\nu^{r_0}_{\varepsilon,R} - Cts\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2.$$

By construction of the ball growth and merging process, this easily implies the following: if a ball $B=\bar B(y_B,s_B)$ belongs to the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ for some $r_0\le r\le\tilde r$, then we have

$$\frac12\int_{B\setminus\tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \ge \frac{\log(r/r_0)}{2}\int_B a\chi_R^z\nu^{r_0}_{\varepsilon,R} - Cs_B\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2.$$

Summing this estimate over all the balls B of the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ that intersect $B_{2R}(z)$, and recalling that the sum of the radii of these balls is by construction bounded by Cr, we deduce for any $r_0\le r\le\tilde r$,

$$\frac12\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}\setminus\tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \ge \frac{\log(r/r_0)}{2}\int_{\mathbb{R}^2} a\chi_R^z\nu^{r_0}_{\varepsilon,R} - Cr\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2.$$

Combining this with (8.153), and recalling that by definition $\mathcal{B}^{r_0}_{\varepsilon,R} \subset \tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}$, we deduce

$$\frac{1}{2}\int_{\tilde{B}^{r_0,r}_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)$$
$$\geq \frac{\log(r/\varepsilon)}{2}\int_{\mathbb{R}^2} a\chi^z_R \nu^{r_0}_{\varepsilon,R} - Cr\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2 - o(N_\varepsilon^2) - C\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big), \quad (8.160)$$

and hence, using Lemma 8.5.1(ii) and the choice (8.150) of r,

$$\frac{1}{2}\int_{\tilde{B}^{r_0,r}_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)$$
$$\geq \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi^z_R \nu^{r_0}_{\varepsilon,R} - C|\log r|\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big) - Cr\Big(\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2 - o(N_\varepsilon^2) - C\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)$$
$$\geq \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi^z_R \nu^{r_0}_{\varepsilon,R} - o(1)\Big(\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2 - o(N_\varepsilon^2),$$

that is, (8.151).

Step 7. Optimal bound on the energy.

In this step, we prove $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon|$, thus completing the result of Step 5 in all regimes $1 \ll N_\varepsilon \lesssim |\log\varepsilon|$. Note that by Step 3 this also implies $\sup_z \int_{\mathbb{R}^2} \chi^z_R |\nu^r_{\varepsilon,R}| \lesssim N_\varepsilon$.

By Step 5, it only remains to consider the regime with a "small" number of vortices $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$. Let $r_0 \leq r \ll 1$ be fixed as in (8.150). On the one hand, using the estimate (8.141), we deduce from the result (8.151) of Step 6,

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)$$
$$\leq \mathcal{D}^z_{\varepsilon,R} + O\bigg(r_0|\log\varepsilon|\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg) + o(1)\Big(\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2 + o(N_\varepsilon^2)$$

and hence, using the assumption $\mathcal{D}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$, the suboptimal energy bound of Step 5, and the choice (8.150) of r_0,

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \lesssim N_\varepsilon^2 + o(1)\Big(\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2. \quad (8.161)$$

On the other hand, combining the estimates (8.148) and (8.149) (with $\mathcal{B}^r_{\varepsilon,R}$ replaced by $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$) of Step 5, we find

$$\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon| + |\log\varepsilon|\Big(\sup_z \int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} \chi^z_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2} + r|\log\varepsilon|R^{-1}(\mathcal{E}^*_{\varepsilon,R})^{1/2}.$$

Now inserting (8.161) yields

$$\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon| + o(1)\mathcal{E}^*_{\varepsilon,R} + |\log\varepsilon|R^{-1}(\mathcal{E}^*_{\varepsilon,R})^{1/2},$$

and thus, recalling the choice $R \gtrsim |\log\varepsilon|$, and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side, the result $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon|$ follows.

Step 8. Conclusion.

The optimal energy bound $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon|$ is now proved. In the present step, we check that the rest of the statements follow from this bound. We split the proof into seven further substeps.

Substep 8.1. Proof of (i).

The result (8.126) follows from (8.139) in Step 2 with $\phi = a\chi_R^z$, combined with the optimal energy bound. Repeating the argument of Step 6 with the optimal energy bound rather than with the suboptimal bound of Step 5, the choice (8.150) can be replaced by $\varepsilon^{1/2} < r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$. For such a choice of r_0, and for $r \ge r_0$ as in (8.150), the result (8.151) together with the optimal energy bound directly implies the result (8.127) for a “small” number of vortices $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$.

Substep 8.2. Proof of (ii).

The bound (8.128) on the number of vortices follows from the result (8.143) of Step 3 together with the optimal energy bound. It remains to prove that in the regime $1 \ll N_\varepsilon \ll |\log\varepsilon|^{1/2}$ for $e^{-o(1)N_\varepsilon^{-1}|\log\varepsilon|} \le r < \bar r$ each ball of the collection $\mathcal{B}_{\varepsilon,R}^r$ has a nonnegative degree. This is a refinement of the result of Step 4. The lower bound (8.139) of Step 2 with $\phi = a\chi_R^z$ can be rewritten as follows, using the optimal energy bound, for all $z \in \mathbb{R}^2$,

$$\begin{aligned}|\log\varepsilon|\int_{\mathbb{R}^2} a\chi_R^z(\nu_{\varepsilon,R}^r)^- &= \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z(|\nu_{\varepsilon,R}^r| - \nu_{\varepsilon,R}^r)\\ &\le \mathcal{E}_{\varepsilon,R}^z - \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\nu_{\varepsilon,R}^r + O\Big(rN_\varepsilon|\log\varepsilon| + r^2N_\varepsilon^2 + N_\varepsilon|\log r|\Big) + o(N_\varepsilon^2),\end{aligned}$$

and hence, using (8.141) to replace $\nu_{\varepsilon,R}^r$ by μ_ε in the right-hand side, and using the assumption $\mathcal{D}_{\varepsilon,R}^z \lesssim N_\varepsilon^2$, we find

$$|\log\varepsilon|\int_{\mathbb{R}^2} \chi_R^z(\nu_{\varepsilon,R}^r)^- \lesssim N_\varepsilon^2 + rN_\varepsilon|\log\varepsilon| + N_\varepsilon|\log r|. \tag{8.162}$$

Dividing both sides by $|\log\varepsilon|$, we deduce in the regime $N_\varepsilon \ll |\log\varepsilon|^{1/2}$ with $e^{-o(1)N_\varepsilon^{-1}|\log\varepsilon|} \le r \ll N_\varepsilon^{-1}$,

$$\sup_z \int_{\mathbb{R}^2} \chi_R^z(\nu_{\varepsilon,R}^r)^- \ll 1,$$

which means that for ε small enough there exists no single ball $B^j \in \mathcal{B}_{\varepsilon,R}^r$ with negative degree $d_j < 0$. This proves the result for $r \ll N_\varepsilon^{-1}$. Now for $N_\varepsilon^{-1} \lesssim r < \bar r$ the same property must hold, since, by monotonicity of the collection $\mathcal{B}_{\varepsilon,R}^r$ with respect to r, for any $r > r'$ the degree of a ball $B \in \mathcal{B}_{\varepsilon,R}^r$ equals the sum of the degrees of all the balls $B' \in \mathcal{B}_\varepsilon(r')$ with $B' \subset B$.

Substep 8.3. Proof of (v).

In the regime $N_\varepsilon \gg \log|\log\varepsilon|$, for $e^{-o(N_\varepsilon)} \le r \ll N_\varepsilon|\log\varepsilon|^{-1}$, the result (8.133) follows from (8.140) together with the optimal energy bound. Monotonicity of $\mathcal{B}_{\varepsilon,R}^r$ with respect to r then implies (8.133) for all $r \ge e^{-o(N_\varepsilon)}$ in the regime $N_\varepsilon \gg \log|\log\varepsilon|$. In the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$, it suffices to argue as for (8.140) in Step 2, but with the lower bound (8.139) replaced by its refined version (8.151): for $r_0 \le r$ with $\varepsilon^{1/2} < r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$ and $e^{-o(N_\varepsilon)} \le r \ll 1$, the estimate (8.151) together with (8.141) indeed yields

$$\begin{aligned}&\frac12\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0,r}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\ &\qquad\le \frac12\int_{\mathbb{R}^2} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\nu_{\varepsilon,R}^{r_0} + o(N_\varepsilon^2)\\ &\qquad\qquad\le \mathcal{D}_{\varepsilon,R}^z + r_0N_\varepsilon|\log\varepsilon| + o(N_\varepsilon^2) = \mathcal{D}_{\varepsilon,R}^z + o(N_\varepsilon^2),\end{aligned}$$

and the result (8.134) follows by monotonicity of $\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0,r}$ with respect to r.

Substep 8.4. Proof of (iii).

The Jacobian estimate (8.129) follows from Lemma 8.5.1(iii) together with the optimal energy bound, and the estimate (8.130) with $\gamma = 1$ similarly follows from (8.142). The result (8.130) for all $\gamma \in [0,1]$ is then obtained by interpolation (as e.g. in [262]) provided we also manage to prove, for all $\phi \in \mathrm{L}^\infty(\mathbb{R}^2)$ supported in a ball $B_R(z)$,

$$\Big|\int_{\mathbb{R}^2} \phi(\tilde\mu_\varepsilon - \mu_\varepsilon)\Big| \lesssim RN_\varepsilon|\log\varepsilon|\|\phi\|_{\mathrm{L}^\infty}. \tag{8.163}$$

Let $\phi \in \mathrm{L}^\infty(\mathbb{R}^2)$ be supported in $B_R(z)$, for some $z \in \mathbb{R}^2$. By definition (8.104), we find

$$\begin{aligned}
&\int_{\mathbb{R}^2} \phi(\tilde\mu_\varepsilon - \mu_\varepsilon) = N_\varepsilon \int_{\mathbb{R}^2} \phi\left((1-|u_\varepsilon|^2)\operatorname{curl}\mathrm{v}_\varepsilon + 2\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon, u_\varepsilon\rangle\cdot \mathrm{v}_\varepsilon^\perp\right)\\
&\le N_\varepsilon\|\phi\|_{\mathrm{L}^\infty}\int_{B_R(z)}\left(|1-|u_\varepsilon|^2||\operatorname{curl}\mathrm{v}_\varepsilon| + 2|\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon| + 2|\mathrm{v}_\varepsilon||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|\right),
\end{aligned}$$

and hence we obtain with the optimal energy bound, with $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}, \|\operatorname{curl}\mathrm{v}_\varepsilon\|_{\mathrm{L}^2} \lesssim 1$,

$$\int_{\mathbb{R}^2}\phi(\tilde\mu_\varepsilon - \mu_\varepsilon) \lesssim \left(\varepsilon N_\varepsilon^2|\log\varepsilon| + RN_\varepsilon|\log\varepsilon|\right)\|\phi\|_{\mathrm{L}^\infty},$$

that is, (8.163).

Substep 8.5. Proof of (iv) in the regime $N_\varepsilon \gg \log|\log\varepsilon|$.

We focus on the regime $N_\varepsilon \gg \log|\log\varepsilon|$. Let $\varepsilon^{1/2} < r \ll 1$ to be later optimized as a function of ε. We write as before $\mathcal{B}^r_{\varepsilon,R} = \biguplus_j B^j$, $B^j = \bar B(y_j, r_j)$, we denote by d_j the degree of B^j, and we set $\nu^r_{\varepsilon,R} = 2\pi\sum_j d_j\delta_{y_j}$. Given $\phi \in W^{1,\infty}(\mathbb{R}^2)$ supported in the ball $B_R(z)$, we decompose

$$\begin{aligned}
&\int_{\mathbb{R}^2}\phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\nu^r_{\varepsilon,R}\Big)\\
&\le \int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}}\phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\
&\qquad + \sum_j\Big|\int_{B^j}\phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi\phi(y_j)d_j|\log\varepsilon|\Big|\\
&\le \|a^{-1}\phi\|_{\mathrm{L}^\infty}\int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\
&\qquad + \|a^{-1}\phi\|_{\mathrm{L}^\infty}\sum_j\chi^z_R(y_j)\Big|\int_{B^j} a\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi a(y_j)d_j|\log\varepsilon|\Big|\\
&\qquad + r\|a^{-1}\phi\|_{W^{1,\infty}}\int_{B_{2R}(z)\cap\mathcal{B}^r_{\varepsilon,R}}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).
\end{aligned}\tag{8.164}$$

Combined with the optimal energy bound, the localized lower bound (8.119) in Lemma 8.5.1(i) with $\phi = a$ yields for all j,

$$\begin{aligned}
&\frac12\int_{B^j} a\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\
&\qquad\ge \pi a(y_j)|d_j||\log\varepsilon| - O\big(r_jN_\varepsilon|\log\varepsilon| + |d_j||\log r| + |d_j|\log N_\varepsilon\big),
\end{aligned}$$

and hence

$$\Big|\int_{B^j} a\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi a(y_j)|d_j||\log\varepsilon|\Big| \\ \le \int_{B^j} a\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi a(y_j)|d_j||\log\varepsilon| \\ + O\big(r_j N_\varepsilon|\log\varepsilon| + |d_j||\log r| + |d_j|\log N_\varepsilon\big).$$

Noting that $\chi_R^z(y_j) \le \chi_R^z(y) + O(R^{-1}r_j)\chi_{2R}^z(y_j)$ holds for $y \in B_j$, we obtain

$$\chi_R^z(y_j)\Big|\int_{B^j} a\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi a(y_j)|d_j||\log\varepsilon|\Big| \\ \le \int_{B^j} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi a(y_j)\chi_R^z(y_j)|d_j||\log\varepsilon| \\ + \chi_{2R}^z(y_j)O\big(r_j N_\varepsilon|\log\varepsilon| + |d_j||\log r| + |d_j|\log N_\varepsilon\big).$$

Inserting this into (8.164), and using the bound of item (ii) on the number of vortices, we find

$$\int_{\mathbb{R}^2} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\nu_{\varepsilon,R}^r\Big) \\ \le \|a^{-1}\phi\|_{\mathrm{L}^\infty}\int_{\mathbb{R}^2} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\nu_{\varepsilon,R}^r\Big) \\ + r\|a^{-1}\phi\|_{W^{1,\infty}}\int_{B_{2R}(z)\cap\mathcal{B}_{\varepsilon,R}^r}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \\ + O\big(rN_\varepsilon|\log\varepsilon| + N_\varepsilon|\log r| + N_\varepsilon\log N_\varepsilon\big)\|\phi\|_{\mathrm{L}^\infty},$$

where the second right-hand side term is estimated by $r\mathcal{E}_{\varepsilon,R}^*\|a^{-1}\phi\|_{W^{1,\infty}} \lesssim rN_\varepsilon|\log\varepsilon|\|a^{-1}\phi\|_{W^{1,\infty}}$, and where (8.141) can be used to replace $\nu_{\varepsilon,R}^r$ by μ_ε in both sides up to an error of order $(rN_\varepsilon|\log\varepsilon| + 1)\|\phi\|_{\mathrm{L}^\infty}$. In the present regime $N_\varepsilon \gg \log|\log\varepsilon|$, we may choose $e^{-o(N_\varepsilon)} \le r \ll N_\varepsilon|\log\varepsilon|^{-1}$, and the conclusion (8.132) follows for that choice.

Substep 8.6. Proof of (iv) in the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$.

We turn to the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$, in which case the proof of (iv) needs to be adapted in the spirit of the computations in Step 6. Let $\phi \in W^{1,\infty}(\mathbb{R}^2)$ be supported in the ball $B_R(z)$, and let $e^{-o(1)|\log\varepsilon|/N_\varepsilon} \le r_0 \ll N_\varepsilon/|\log\varepsilon|$. First arguing as in Substep 8.5 with this choice of r_0, we obtain

$$\int_{\mathcal{B}_{\varepsilon,R}^{r_0}} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - \log(r_0/\varepsilon)\nu_{\varepsilon,R}^{r_0}\Big) \\ \le \|a^{-1}\phi\|_{\mathrm{L}^\infty}\int_{\mathcal{B}_{\varepsilon,R}^{r_0}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - \log(r_0/\varepsilon)\nu_{\varepsilon,R}^{r_0}\Big) + o(N_\varepsilon^2)\|\phi\|_{W^{1,\infty}}. \tag{8.165}$$

Now we consider the modified ball collection $\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0;r}$ with $r \ge r_0$, as constructed in Step 6.1. Assume that some ball $B = \bar{B}(y,s)$ gets grown into $B' = \bar{B}(y,ts)$ without merging, for some $t \ge 1$, and assume that $B' \setminus B$ does not intersect $\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0}$, so that by construction $||u_\varepsilon| - 1| \le |\log\varepsilon|^{-1}$ holds on $B' \setminus B$. Let

d denote the degree of B (hence of B'). We may then decompose

$$\begin{aligned}
&\bigg|\frac12\int_{B'\setminus B}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi\phi(y)d\log t\bigg|\\
&\qquad\le\|a^{-1}\phi\|_{\mathrm{L}^\infty}\bigg|\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi a(y)\chi_R^z(y)d\log t\bigg|\\
&\qquad\qquad+\|a^{-1}\phi\|_{W^{1,\infty}}\frac12\int_{B'\setminus B}\chi_R^z|\cdot-y|\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big),
\end{aligned}$$

and hence, for $ts\le 1$, decomposing $\chi_R^z(x)\le\chi_R^z(y)+O(R^{-1})$ for all $x\in B'\setminus B$, using the optimal energy bound and the choice $R\gtrsim|\log\varepsilon|$,

$$\begin{aligned}
&\bigg|\frac12\int_{B'\setminus B}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi\phi(y)d\log t\bigg|\\
&\qquad\le\|a^{-1}\phi\|_{\mathrm{L}^\infty}\bigg|\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi a(y)\chi_R^z(y)d\log t\bigg|\\
&\qquad+\|a^{-1}\phi\|_{W^{1,\infty}}\frac{\chi_R^z(y)}{2}\int_{B'\setminus B}|\cdot-y|\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)+CtsN_\varepsilon\|\phi\|_{W^{1,\infty}}.
\end{aligned}$$

Arguing as in (8.155) yields

$$\begin{aligned}
&\bigg|\frac12\int_{B'\setminus B}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi\phi(y)d\log t\bigg|\\
&\le\|a^{-1}\phi\|_{\mathrm{L}^\infty}\bigg|\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi a(y)\chi_R^z(y)d\log t\bigg|+CtsN_\varepsilon\|\phi\|_{W^{1,\infty}}\\
&\qquad+ts\chi_R^z(y)\|a^{-1}\phi\|_{W^{1,\infty}}\bigg(Cd^2+\int_{B'\setminus B}\Big|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon-iu_\varepsilon\frac{d\tau}{|\cdot-y|}\Big|^2+\int_{B'\setminus B}\frac{a}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\bigg).
\end{aligned}\tag{8.166}$$

Let us estimate the last right-hand side term of (8.166). Applying the lower bound (8.151) with ε replaced by 2ε (with $\varepsilon<1/2$), together with the optimal energy bound, we obtain, for $r\ge r_0$ with $e^{-o(N_\varepsilon)}\le r\ll 1$,

$$\begin{aligned}
\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi_R^z|\nu_{\varepsilon,R}^{r_0}|-\frac{\log 2}{2}\int_{\mathbb{R}^2}a\chi_R^z|\nu_{\varepsilon,R}^{r_0}|-o(N_\varepsilon^2)&=\frac{|\log(2\varepsilon)|}{2}\int_{\mathbb{R}^2}a\chi_R^z|\nu_{\varepsilon,R}^{r_0}|-o(N_\varepsilon^2)\\
&\le\frac12\int_{\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0,r}}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2(2\varepsilon)^2}(1-|u_\varepsilon|^2)^2\Big)\\
&\le\mathcal{D}_{\varepsilon,R}^*+\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi_R^z\mu_\varepsilon-\frac{3}{16\varepsilon^2}\int_{\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0,r}}a^2\chi_R^z(1-|u_\varepsilon|^2)^2.
\end{aligned}$$

Using (8.141), the bound of item (ii) on the number of vortices, and the choice of r_0, we then find

$$\begin{aligned}
\frac{3}{16\varepsilon^2}\int_{\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0,r}}a^2\chi_R^z(1-|u_\varepsilon|^2)^2&\le\mathcal{D}_{\varepsilon,R}^*+\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi_R^z(\mu_\varepsilon-\nu_{\varepsilon,R}^{r_0})+\frac{\log 2}{2}\int_{\mathbb{R}^2}a\chi_R^z|\nu_{\varepsilon,R}^{r_0}|+o(N_\varepsilon^2)\\
&\le\mathcal{D}_{\varepsilon,R}^*+o(N_\varepsilon^2)\lesssim N_\varepsilon^2.
\end{aligned}$$

Combining this with the result (8.134) of item (v), we deduce the (suboptimal) estimate

$$\sup_z\int_{\mathbb{R}^2}\frac{\chi_R^z}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\lesssim N_\varepsilon^2.\tag{8.167}$$

Injecting this result into (8.166), together with the bound of item (ii) on the number of vortices, we find

$$\begin{aligned}\Big|\frac12\int_{B'\setminus B}&\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi\phi(y)d\log t\Big|\\&\le\|a^{-1}\phi\|_{\mathrm{L}^\infty}\Big|\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi a(y)\chi_R^z(y)d\log t\Big|\\&\qquad+CtsN_\varepsilon^2\|\phi\|_{W^{1,\infty}}+ts\chi_R^z(y)\|a^{-1}\phi\|_{W^{1,\infty}}\int_{B'\setminus B}\Big|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon-iu_\varepsilon\frac{d\tau}{|\cdot-y|}\Big|^2.\end{aligned}\tag{8.168}$$

Recalling the improved lower bound (8.157), and combining it with the bound of item (ii) on the number of vortices, with the assumption $\|\mathrm{curl}\,\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}\lesssim 1$, with the optimal energy bound, and with the choice $R\gtrsim|\log\varepsilon|$, we find for $ts\le 1$,

$$\begin{aligned}(1+O(|\log\varepsilon|^{-1}))\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ge\pi a(y)\chi_R^z(y)d\log t-CtsN_\varepsilon^2\\+(1-O(|\log\varepsilon|^{-1}+ts))\frac{a(y)\chi_R^z(y)}{2}\int_{B'\setminus B}\Big|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon-iu_\varepsilon\frac{d\tau}{|\cdot-y|}\Big|^2.\end{aligned}$$

Injecting this estimate into (8.168) yields for $ts\ll 1$,

$$\begin{aligned}\Big|\frac12\int_{B'\setminus B}&\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi\phi(y)d\log t\Big|\\&\le C\|\phi\|_{W^{1,\infty}}\Big(\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi a(y)\chi_R^z(y)d\log t\Big)\\&\qquad+CtsN_\varepsilon^2\|\phi\|_{W^{1,\infty}}+C|\log\varepsilon|^{-1}\|\phi\|_{W^{1,\infty}}\int_{B'\setminus B}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).\end{aligned}$$

Using the bound of item (ii) on the number of vortices, we find

$$\Big|2\pi\phi(y)d\log t-\log t\int_B\phi\nu_{\varepsilon,R}^{r_0}\Big|\le\|\nabla\phi\|_{\mathrm{L}^\infty}s\log t\int_B|\nu_{\varepsilon,R}^{r_0}|\le C\|\nabla\phi\|_{\mathrm{L}^\infty}tsN_\varepsilon,$$

so that the above becomes

$$\begin{aligned}\Big|\frac12\int_{B'\setminus B}&\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\frac{\log t}{2}\int_{B'}\phi\nu_{\varepsilon,R}^{r_0}\Big|\\&\le C\|\phi\|_{W^{1,\infty}}\Big(\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\frac{\log t}{2}\int_{B'}a\chi_R^z\nu_{\varepsilon,R}^{r_0}\Big)\\&\qquad+CtsN_\varepsilon^2\|\phi\|_{W^{1,\infty}}+C|\log\varepsilon|^{-1}\|\phi\|_{W^{1,\infty}}\int_{B'\setminus B}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).\end{aligned}$$

By construction of the ball growth and merging process, this easily implies the following: if a ball $B=\bar B(y_B,s_B)$ belongs to the collection $\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0,r}$ for some $r_0\le r\ll 1$, then we have

$$\begin{aligned}\Big|\frac12\int_{B\setminus\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0}}&\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\frac{\log(r/r_0)}{2}\int_B\phi\nu_{\varepsilon,R}^{r_0}\Big|\\&\le C\|\phi\|_{W^{1,\infty}}\Big(\int_{B\setminus\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0}}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\log(r/r_0)\int_Ba\chi_R^z\nu_{\varepsilon,R}^{r_0}\Big)\\&\qquad+Cs_BN_\varepsilon^2\|\phi\|_{W^{1,\infty}}+C|\log\varepsilon|^{-1}\|\phi\|_{W^{1,\infty}}\int_B\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).\end{aligned}$$

Summing this estimate over all balls B of the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ that intersect $B_R(z)$, recalling that the sum of the radii of these balls is by construction bounded by Cr, and using the optimal energy bound and the bound of item (ii) on the number of vortices, we deduce for $r_0 \le r \ll 1$,

$$\begin{aligned}
&\Big|\frac{1}{2}\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}\setminus\tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \frac{\log(r/r_0)}{2}\int_{\mathbb{R}^2}\phi\nu^{r_0}_{\varepsilon,R}\Big| \\
&\le C\|\phi\|_{W^{1,\infty}}\bigg(\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}\setminus\tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \log(r/r_0)\int_{\mathbb{R}^2} a\chi^z_R\nu^{r_0}_{\varepsilon,R}\bigg) \\
&\qquad + CrN_\varepsilon^2\|\phi\|_{W^{1,\infty}} + C|\log\varepsilon|^{-1}\mathcal{E}^z_{\varepsilon,2R}\|\phi\|_{W^{1,\infty}} \\
&\le C\|\phi\|_{W^{1,\infty}}\bigg(\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}\setminus\tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \\
&\qquad\qquad\qquad - \log(r/r_0)\int_{\mathbb{R}^2} a\chi^z_R\nu^{r_0}_{\varepsilon,R} + o(N_\varepsilon^2)\bigg).
\end{aligned}$$

Combining this with (8.165), and recalling that by definition $\mathcal{B}^{r_0}_{\varepsilon,R} \subset \tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}$, we deduce

$$\begin{aligned}
&\Big|\frac{1}{2}\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \frac{\log(r/\varepsilon)}{2}\int_{\mathbb{R}^2}\phi\nu^{r_0}_{\varepsilon,R}\Big| \\
&\le C\|\phi\|_{W^{1,\infty}}\bigg(\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \log(r/\varepsilon)\int_{\mathbb{R}^2} a\chi^z_R\nu^{r_0}_{\varepsilon,R} + o(N_\varepsilon^2)\bigg).
\end{aligned}$$

Using (8.141) to replace $\nu^{r_0}_{\varepsilon,R}$ by μ_ε in both sides up to an error of order $(r_0N_\varepsilon|\log\varepsilon|+1)\|\phi\|_{W^{1,\infty}} \ll N_\varepsilon^2\|\phi\|_{W^{1,\infty}}$, the result (8.132) follows.

Substep 8.7. Proof of (vi).

We adapt an argument by Struwe [403] (see also [384, Proof of Lemma 4.7]). Recalling that $|B_{2R}(z)\cap\mathcal{B}^r_{\varepsilon,R}| \lesssim r^2$, a direct application of the Hölder inequality yields

$$\int_{\mathcal{B}^r_{\varepsilon,R}} \chi^z_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim r^{2-p}\Big(\int_{\mathcal{B}^r_{\varepsilon,R}} \chi^z_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{p/2} \lesssim r^{2-p}(N_\varepsilon|\log\varepsilon|)^{p/2},$$

which only implies the result if we are allowed to choose the total radius r small enough. Otherwise, it is useful to rather work on dyadic "annuli". For each integer $0\le k\le K_\varepsilon := \lfloor\log_2(r/\varepsilon^{1/2})\rfloor$, define the "annulus" $E_k := \mathcal{B}^{r2^{-k}}_{\varepsilon,R}\setminus\mathcal{B}^{r2^{-k-1}}_{\varepsilon,R}$. We set for simplicity $s_k := r2^{-k}$. Applying the Hölder inequality separately on each annulus yields

$$\begin{aligned}
\int_{\mathcal{B}^r_{\varepsilon,R}} \chi^z_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p &\le \Big(\int_{\mathcal{B}^{\sqrt{\varepsilon}}_{\varepsilon,R}} \chi^z_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{p/2}|B_{2R}(z)\cap\mathcal{B}^{\sqrt{\varepsilon}}_{\varepsilon,R}|^{1-p/2} \\
&\qquad + \sum_{k=0}^{K_\varepsilon}\Big(\int_{E_k}\chi^z_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{p/2}|B_{2R}(z)\cap E_k|^{1-p/2}.
\end{aligned}$$

Using that $|B_{2R}(z)\cap\mathcal{B}^{\sqrt{\varepsilon}}_{\varepsilon,R}| \lesssim \varepsilon$, that $|B_{2R}(z)\cap E_k| \lesssim s_k^2$, and that the integral over $\mathcal{B}^{\sqrt{\varepsilon}}_{\varepsilon,R}$ in the right-hand side is bounded by $\mathcal{E}^z_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon|$, we deduce

$$\int_{\mathcal{B}^r_{\varepsilon,R}} \chi^z_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim \varepsilon^{1-p/2}(N_\varepsilon|\log\varepsilon|)^{p/2} + \sum_{k=0}^{K_\varepsilon} s_k^{2-p}\Big(\int_{\mathbb{R}^2\setminus\mathcal{B}^{s_{k+1}}_{\varepsilon,R}} \chi^z_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{p/2}. \tag{8.169}$$

It remains to estimate the last integrals. Using Lemma 8.5.1(i)–(ii) in the forms (8.120) and (8.121), together with the optimal energy bound, we obtain

$$\frac{1}{2}\int_{\mathcal{B}^{s_{k+1}}_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \\ \ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\nu^{s_{k+1}}_{\varepsilon,R} - O\big(N_\varepsilon|\log s_{k+1}| + s_{k+1}N_\varepsilon|\log\varepsilon|\big) - o(N_\varepsilon^2),$$

and hence, using (8.141) to replace $\nu^{s_{k+1}}_{\varepsilon,R}$ by μ_ε,

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus\mathcal{B}^{s_{k+1}}_{\varepsilon,R}} a\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \le \mathcal{D}^z_{\varepsilon,R} + O(N_\varepsilon|\log s_{k+1}| + s_{k+1}N_\varepsilon|\log\varepsilon|) + o(N_\varepsilon^2).$$

If $r \ll N_\varepsilon|\log\varepsilon|^{-1}$, then $s_k \le r \ll N_\varepsilon|\log\varepsilon|^{-1}$ for all k, so that we find

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus\mathcal{B}^{s_{k+1}}_{\varepsilon,R}} \chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \lesssim N_\varepsilon^2 + N_\varepsilon(|\log r| + k). \tag{8.170}$$

Inserting this into (8.169) yields for all $p<2$, with $r \ll N_\varepsilon|\log\varepsilon|^{-1}$,

$$\int_{\mathcal{B}^r_{\varepsilon,R}} \chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim \varepsilon^{1-p/2}(N_\varepsilon|\log\varepsilon|)^{p/2} + \sum_{k=0}^{K_\varepsilon}(r2^{-k})^{2-p}\Big(N_\varepsilon^p + N_\varepsilon^{p/2}|\log r|^{p/2} + N_\varepsilon^{p/2}k^{p/2}\Big) \\ \lesssim_p \varepsilon^{1-p/2}(N_\varepsilon|\log\varepsilon|)^{p/2} + r^{2-p}N_\varepsilon^p + r^{2-p}N_\varepsilon^{p/2}|\log r|^{p/2}.$$

In the regime $N_\varepsilon \gg \log|\log\varepsilon|$, we may choose $e^{-o(N_\varepsilon)} \le r \ll N_\varepsilon|\log\varepsilon|^{-1}$, and the above yields for that choice

$$\int_{\mathcal{B}^r_{\varepsilon,R}} \chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \ll_p N_\varepsilon^p, \tag{8.171}$$

that is, (8.135).

We now consider the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$. In that case, we need to prove (8.171) for larger values of the radius $r \ge e^{-o(N_\varepsilon)}$, and the above argument no longer holds. Given $\varepsilon^{1/2} < r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$, we replace the initial total radius $\varepsilon^{1/2}$ by r_0, and for $r_0 \le r \ll 1$ we consider the modified dyadic "annuli" $\tilde E_k := \tilde{\mathcal{B}}^{r_0,r2^{-k}}_{\varepsilon,R} \setminus \tilde{\mathcal{B}}^{r_0,r2^{-k-1}\vee r_0}_{\varepsilon,R}$, with $0 \le k \le K := \lfloor \log_2(r/r_0)\rfloor$. We set for simplicity $\tilde s_k := (r2^{-k}) \vee r_0$. The decomposition (8.169) is then replaced by

$$\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} \chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim r_0^{2-p}(N_\varepsilon|\log\varepsilon|)^{p/2} + \sum_{k=0}^{K} s_k^{2-p}\Big(\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,\tilde s_{k+1}}_{\varepsilon,R}} \chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{p/2}, \tag{8.172}$$

where it remains to adapt the estimate (8.170) for the last integrals. The lower bound (8.160) of Step 6 together with the optimal energy bound and with the bound of item (ii) on the number of vortices yields

$$\frac{1}{2}\int_{\tilde{\mathcal{B}}^{r_0,\tilde s_{k+1}}_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \ge \frac{\log(\tilde s_{k+1}/\varepsilon)}{2}\int_{\mathbb{R}^2} a\chi_R^z\nu^{r_0}_{\varepsilon,R} - o(N_\varepsilon^2) \\ \ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\nu^{r_0}_{\varepsilon,R} - O(N_\varepsilon|\log s_{k+1}|) - o(N_\varepsilon^2),$$

and hence, using (8.141) to replace $\nu^{r_0}_{\varepsilon,R}$ by μ_ε,

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,\tilde{s}_{k+1}}_{\varepsilon,R}} a\chi^z_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \le \mathcal{D}^z_{\varepsilon,R} + O(N_\varepsilon|\log s_{k+1}| + r_0 N_\varepsilon|\log\varepsilon|) + o(N_\varepsilon^2).$$

The choice $r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$ then yields

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,\tilde{s}_{k+1}}_{\varepsilon,R}} a\chi^z_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \lesssim N_\varepsilon^2 + N_\varepsilon(|\log r| + k).$$

Inserting this into (8.172), the result (8.136) follows. $\square$

Based on the above vortex-balls construction, we have the following approximation result, which is obtained as in [382, Proposition 9.6] (see also Step 2 of the proof of Proposition 6.2.11 in Chapter 6).

Lemma 8.5.3. *Let $\varepsilon^{1/2} < r_0 \le r < \bar{r}$, and let $\mathcal{B}^r_{\varepsilon,R}$ and $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ denote the collections of the balls constructed in Proposition 8.5.2. Then, given $\Gamma_\varepsilon \in W^{2,\infty}(\mathbb{R}^2)^2$, there exist approximate vector fields $\bar{\Gamma}_\varepsilon, \tilde{\Gamma}_\varepsilon \in W^{2,\infty}(\mathbb{R}^2)^2$ such that $\bar{\Gamma}_\varepsilon$ is constant in each ball of the collection $\mathcal{B}^r_{\varepsilon,R}$ and $\tilde{\Gamma}_\varepsilon$ is constant in each ball of the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$, such that $\|\bar{\Gamma}_\varepsilon\|_{\mathrm{L}^\infty} \le \|\Gamma_\varepsilon\|_{\mathrm{L}^\infty}$ and $\|\tilde{\Gamma}_\varepsilon\|_{\mathrm{L}^\infty} \le \|\Gamma_\varepsilon\|_{\mathrm{L}^\infty}$, such that for all $0 \le \gamma \le 1$,*

$$\|\bar{\Gamma}_\varepsilon - \Gamma_\varepsilon\|_{C^\gamma} + \|\tilde{\Gamma}_\varepsilon - \Gamma_\varepsilon\|_{C^\gamma} \lesssim r^{1-\gamma}\|\nabla\Gamma_\varepsilon\|_{\mathrm{L}^\infty},$$

and such that for all $R \ge 1$,

$$\sup_z \|\nabla(\bar{\Gamma}_\varepsilon - \Gamma_\varepsilon)\|_{\mathrm{L}^1(B_R(z))} + \sup_z \|\nabla(\tilde{\Gamma}_\varepsilon - \Gamma_\varepsilon)\|_{\mathrm{L}^1(B_R(z))} \lesssim rR^2\|\nabla\Gamma_\varepsilon\|_{W^{1,\infty}}. \qquad \Diamond$$

8.5.2 Additional results

In order to control the velocity of the vortices, the following quantitative version of the "product estimate" of [381] is needed; the proof is omitted, as it is a direct adaptation of [395, Appendix A] (further deforming the metric in a non-constant way in the time direction; see also [381, Section III]).

Lemma 8.5.4 (Product estimate)**.** *Denote by M_ε any quantity such that for all $q > 0$,*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^q M_\varepsilon = \lim_{\varepsilon\downarrow 0}|\log\varepsilon|M_\varepsilon^{-q} = \lim_{\varepsilon\downarrow 0}|\log\varepsilon|^{-1}\log M_\varepsilon = 0.$$

Let $u_\varepsilon : [0,T]\times\mathbb{R}^2 \to \mathbb{C}$, $\mathrm{v}_\varepsilon : [0,T]\times\mathbb{R}^2 \to \mathbb{R}^2$, and $\mathrm{p}_\varepsilon : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$. Assume that $\mathcal{E}^{*,t}_{\varepsilon,R} \lesssim |\log\varepsilon|^2$ for all t, and that $\bar{\mathcal{E}}^{*,T}_{\varepsilon,R} \le M_\varepsilon$, where we have set

$$\bar{\mathcal{E}}^{*,T}_{\varepsilon,R} := \sup_z \int_0^T \bigg(\mathcal{E}^{z,t}_{\varepsilon,R} + \int_{\mathbb{R}^2}\chi^z_R|\partial_t u^t_\varepsilon - iu^t_\varepsilon N_\varepsilon \mathrm{p}^t_\varepsilon|^2\bigg)dt.$$

Then, for all $X \in W^{1,\infty}([0,T]\times\mathbb{R}^2)^2$ and $Y \in W^{1,\infty}([0,T]\times\mathbb{R}^2)$, we have for all $z \in \mathbb{R}^2$,

$$\begin{aligned}&\bigg|\int_0^T\int_{\mathbb{R}^2}\chi^z_R\tilde{V}_\varepsilon\cdot XY\bigg| \\ &\quad\le \frac{1 + C\frac{\log M_\varepsilon}{|\log\varepsilon|}}{|\log\varepsilon|}\bigg(\int_0^T\int_{\mathbb{R}^2}\chi^z_R|(\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon)Y|^2 + \int_0^T\int_{\mathbb{R}^2}\chi^z_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot X|^2\bigg) \\ &\qquad + C\big(1 + \|(X,Y)\|^5_{W^{1,\infty}([0,T]\times\mathbb{R}^2)}\big)\big(M_\varepsilon^{-1/8} + \varepsilon N_\varepsilon\big)\Big(\bar{\mathcal{E}}^{*,T}_{\varepsilon,R} + \sup_{0\le\tau\le T}\mathcal{E}^{*,\tau}_{\varepsilon,R} + N_\varepsilon^2\Big). \qquad \Diamond\end{aligned}$$

We now turn to some useful a priori estimates on the solution u_ε of equation (8.6). We start with the following (very suboptimal) a priori bound on the velocity of the vortices, adapted from [395, Lemma 4.1].

Lemma 8.5.5 (A priori bound on velocity)**.** *Let $\alpha \ge 0$, $\beta \in \mathbb{R}$, and let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy* (8.43)*. Let $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}$ and $\mathrm{v}_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{R}^2$ be the solutions of* (8.6) *and* (8.51) *as in Proposition 8.2.2(i) and in Assumption 8.3.1(a), respectively, for some $T > 0$. Let $0 < \varepsilon \ll 1$, $1 \le N_\varepsilon \lesssim \varepsilon^{-1}$, and $R \ge 1$ with $\varepsilon R^\theta \ll 1$ for some $\theta > 0$, and assume that $\mathcal{E}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon|\log\varepsilon|$ for all t. Then, in each of the considered regimes* (GL_1), (GL_2), (GL_3), (GL_1'), *and* (GL_2'), *we have for all $\theta > 0$ and all $t \in [0,T)$,*

$$\alpha^2 \sup_z \int_0^t \int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 \lesssim_{t,\theta} N_\varepsilon|\log\varepsilon|^3 + R^\theta N_\varepsilon^2|\log\varepsilon|^2 \lesssim R^\theta N_\varepsilon(N_\varepsilon + |\log\varepsilon|)|\log\varepsilon|^2. \qquad \Diamond$$

Proof. Integrating identity (8.112) in time, reorganizing the terms, and setting $D^{z,t}_{\varepsilon,R} := \int_0^t \int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2$, we obtain

$$\begin{aligned}\lambda_\varepsilon\alpha D^{z,t}_{\varepsilon,R} = \hat{\mathcal{E}}^{z,\circ}_{\varepsilon,R} - \hat{\mathcal{E}}^{z,t}_{\varepsilon,R} - \int_0^t\int_{\mathbb{R}^2} a\nabla\chi_R^z \cdot \langle \partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle + \int_0^t\int_{\mathbb{R}^2} N_\varepsilon\chi_R^z\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle \operatorname{div}(a\mathrm{v}_\varepsilon) \\ + \int_{\mathbb{R}^2}\frac{aN_\varepsilon^2}{2}(1-|u_\varepsilon^t|^2)(\psi^{z,t}_{\varepsilon,R} - \chi_R^z|\mathrm{v}_\varepsilon^t|^2) - \int_{\mathbb{R}^2}\frac{aN_\varepsilon^2}{2}(1-|u_\varepsilon^\circ|^2)(\psi^{z,\circ}_{\varepsilon,R} - \chi_R^z|\mathrm{v}_\varepsilon^\circ|^2) \\ + \int_0^t\int_{\mathbb{R}^2} a\chi_R^z\Big(N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon\cdot V_\varepsilon - \frac{|\log\varepsilon|}{2}F^\perp\cdot V_\varepsilon\Big). \quad (8.173)\end{aligned}$$

Noting that $|\nabla\chi_R^z| \lesssim R^{-1}(\chi_R^z)^{1/2}$, using the pointwise estimates of Lemma 8.4.2 for V_ε and $j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon$, and using assumptions (8.43), the properties of v_ε in Assumption 8.3.1(a), the bound (8.98) on $\psi^z_{\varepsilon,R}$, and Lemma 8.4.1 in the form $\hat{\mathcal{E}}^{z,t}_{\varepsilon,R} \lesssim \mathcal{E}^{*,t}_{\varepsilon,R} + o(N_\varepsilon^2) \lesssim_t N_\varepsilon|\log\varepsilon|$, we find for $\theta > 0$ small enough, in the considered regimes,

$$\begin{aligned}\lambda_\varepsilon\alpha D^{z,t}_{\varepsilon,R} \quad &\lesssim_{t,\theta} \quad N_\varepsilon|\log\varepsilon| + R^{-1}(N_\varepsilon|\log\varepsilon|)^{1/2}(D^{z,t}_{\varepsilon,R})^{1/2} + N_\varepsilon(1+\varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2})(D^{z,t}_{\varepsilon,R})^{1/2} \\ &\qquad + \varepsilon N_\varepsilon^2(N_\varepsilon|\log\varepsilon|)^{1/2}\Big(1 + \frac{|\log\varepsilon|}{N_\varepsilon}(\lambda_\varepsilon R^\theta + 1\wedge\lambda_\varepsilon^{1/2} + R^{-1+\theta})\Big) \\ &\qquad + N_\varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2}(1+\varepsilon N_\varepsilon) + \varepsilon\lambda_\varepsilon^{-1/2}N_\varepsilon^2|\log\varepsilon| \\ &\qquad + (N_\varepsilon + \lambda_\varepsilon|\log\varepsilon|)\big((1+\varepsilon N_\varepsilon)(N_\varepsilon|\log\varepsilon|)^{1/2} + N_\varepsilon R^\theta\big)(D^{z,t}_{\varepsilon,R})^{1/2} \\ &\lesssim_\theta \quad |\log\varepsilon|(N_\varepsilon + |\log\varepsilon|) + \big(N_\varepsilon|\log\varepsilon|R^\theta + |\log\varepsilon|(N_\varepsilon|\log\varepsilon|)^{1/2}\big)(D^{z,t}_{\varepsilon,R})^{1/2} + o(1).\end{aligned}$$

Absorbing $(D^{z,t}_{\varepsilon,R})^{1/2}$ in the left-hand side, the result follows. ☐

The following optimal a priori estimate is also crucially needed in our analysis in the presence of pinning, due to the absence of a factor $\frac{1}{2}$ in front of the quantity $\frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2$ as it appears in the term $I^H_{\varepsilon,\varrho,R}$ in Lemma 8.4.4. A simple computation based on the energy lower bound of Proposition 8.5.2 yields a similar bound with N_ε replaced by N_ε^2 (see indeed (8.167)), but the optimal result below is much more subtle. It is proved as a combination of the Pohozaev vortex-balls construction of [382, Section 5], together with some careful cut-off techniques inspired by [382, Proof of Proposition 13.4].

Lemma 8.5.6. *Let $\alpha \ge 0$, $\beta \in \mathbb{R}$, and let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy* (8.43)*. Let $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}$ and $\mathrm{v}_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{R}^2$ be the solutions of* (8.6) *and* (8.51) *as in Proposition 8.2.2(i) and in Assumption 8.3.1(a), respectively, for some $T > 0$. Let $0 < \varepsilon \ll 1$, $1 \le N_\varepsilon \lesssim |\log\varepsilon|$, and $R \ge 1$ with $\varepsilon R|\log\varepsilon|^3 \lesssim 1$, and assume that $\mathcal{E}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon|\log\varepsilon|$ for all t. Then, in* *the nondegenerate dissipative case, in each of the considered regimes* (GL_1), (GL_2), (GL_1'), *and* (GL_2'), *we have for all* $t \in [0,T)$,

$$\alpha^2 \sup_z \int_0^t \int_{\mathbb{R}^2} \frac{\chi_R^z}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 \lesssim_t N_\varepsilon. \tag{8.174}$$

$\Diamond$

Proof. To simplify notation, we focus on the case $z=0$, but the result of course holds uniformly with respect to the translation $z \in R\mathbb{Z}^2$. We split the proof into three steps.

Step 1. Pohozaev estimate on balls.

In this step, we prove the following Pohozaev type estimate, adapted from [382, Theorem 5.1]: for any ball $B_r(x_0)$ with $r \le 1$, we have

$$\begin{aligned}\alpha^2 \int_0^t \int_{B_r(x_0)} \frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 &\lesssim_t r\lambda_\varepsilon N_\varepsilon |\log\varepsilon|^3 \\ &+ r\int_0^t \int_{\partial B_r(x_0)} \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+|f|)\Big).\end{aligned} \tag{8.175}$$

For any smooth vector field X and any bounded open set $U \subset \mathbb{R}^2$, we have by integration by parts

$$-\int_U \chi_R \nabla X : \tilde S_\varepsilon = \int_U \chi_R \operatorname{div} \tilde S_\varepsilon \cdot X + \int_U X\cdot \tilde S_\varepsilon \cdot \nabla\chi_R - \int_{\partial U} \chi_R X\cdot \tilde S_\varepsilon \cdot n,$$

and hence, for $U = B_r(x_0)$, $r>0$, and $X = x - x_0$,

$$-\int_{B_r(x_0)} \chi_R \operatorname{Tr} \tilde S_\varepsilon = \int_{B_r(x_0)} \chi_R \operatorname{div} \tilde S_\varepsilon \cdot (x-x_0) + \int_{B_r(x_0)} (x-x_0)\cdot \tilde S_\varepsilon \cdot \nabla \chi_R - r\int_{\partial B_r(x_0)} \chi_R\, \tilde S_\varepsilon : n\otimes n.$$

By definition (8.107) of the modulated stress-energy tensor $\tilde S_\varepsilon$, this means

$$\begin{aligned}\int_{B_r(x_0)} a\chi_R\Big(\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big) &= \int_{B_r(x_0)} \chi_R \operatorname{div} \tilde S_\varepsilon\cdot(x-x_0) + \int_{B_r(x_0)} (x-x_0)\cdot \tilde S_\varepsilon \cdot \nabla\chi_R \\ &+ r\int_{\partial B_r(x_0)} \frac{a\chi_R}{2}\Big(|n^\perp\cdot(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)|^2 - |n\cdot(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)|^2 \\ &\qquad + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)\big(N_\varepsilon^2(|n^\perp\cdot \mathrm{v}_\varepsilon|^2 - |n\cdot \mathrm{v}_\varepsilon|^2) + f\big)\Big),\end{aligned}$$

so that we may simply estimate

$$\begin{aligned}\int_{B_r(x_0)} \frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 &\le r\int_{B_r(x_0)} |\operatorname{div} \tilde S_\varepsilon| + r\int_{B_r(x_0)} |\nabla\chi_R||\tilde S_\varepsilon| + \int_{B_r(x_0)} a|1-|u_\varepsilon|^2||f| \\ &+ r\int_{\partial B_r(x_0)} \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+|f|)\Big).\end{aligned} \tag{8.176}$$

It remains to estimate the first three right-hand side terms. Using the pointwise estimates of Lemma 8.4.2, and using assumption (8.43) and the boundedness properties of $\mathrm{v}_\varepsilon, \mathrm{p}_\varepsilon$ in Assumption 8.3.1(a), Lemma 8.4.3 directly yields in the considered regimes,

$$\begin{aligned}|\operatorname{div} \tilde S_\varepsilon| &\lesssim \lambda_\varepsilon|\log\varepsilon||\partial_t u_\varepsilon||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + N_\varepsilon(1+\lambda_\varepsilon^{1/2}|\log\varepsilon|)(1+|1-|u_\varepsilon|^2|)|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| \\ &+ \lambda_\varepsilon N_\varepsilon|\log\varepsilon||\partial_t u_\varepsilon|(1+|1-|u_\varepsilon|^2|) + (N_\varepsilon + \lambda_\varepsilon|\log\varepsilon|)|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \varepsilon^{-2}(1-|u_\varepsilon|^2)^2 \\ &+ |1-|u_\varepsilon|^2|\big(N_\varepsilon^2(N_\varepsilon+\lambda_\varepsilon|\log\varepsilon|) + \lambda_\varepsilon^2|\log\varepsilon|^2\big) + N_\varepsilon^2(N_\varepsilon + \lambda_\varepsilon|\log\varepsilon|),\end{aligned}$$

which gives for $N_\varepsilon \lesssim |\log\varepsilon|$,

$$|\mathrm{div}\ \tilde S_\varepsilon| \lesssim \lambda_\varepsilon|\partial_t u_\varepsilon|^2 + \lambda_\varepsilon|\log\varepsilon|^2|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \lambda_\varepsilon N_\varepsilon^2|\log\varepsilon|^2(1+(1-|u_\varepsilon|^2)^2) + \varepsilon^{-2}(1-|u_\varepsilon|^2)^2.$$

By Lemma 8.5.5 with $R=1$, we deduce for all $r\le 1$,

$$\alpha^2\int_0^t\int_{B_r(x_0)}|\mathrm{div}\ \tilde S_\varepsilon| \lesssim_t \lambda_\varepsilon N_\varepsilon|\log\varepsilon|^3 + \lambda_\varepsilon N_\varepsilon^2|\log\varepsilon|^2(1+\varepsilon^2N_\varepsilon|\log\varepsilon|) \lesssim \lambda_\varepsilon N_\varepsilon|\log\varepsilon|^3.$$

Inserting this into (8.176), and noting that (8.43) in the form $\|f\|_{\mathrm{L}^\infty} \lesssim |\log\varepsilon|^2$ yields

$$\int_{B_r(x_0)} a|1-|u_\varepsilon|^2||f| \lesssim_t \varepsilon r(N_\varepsilon|\log\varepsilon|)^{1/2}\|f\|_{\mathrm{L}^\infty} \lesssim \varepsilon r|\log\varepsilon|^3,$$

and

$$\begin{aligned}\int_{B_r(x_0)}|\nabla\chi_R||\tilde S_\varepsilon| &\lesssim R^{-1}\int_{B_r(x_0)}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + \varepsilon^2(N_\varepsilon^4|\mathrm{v}_\varepsilon|^4+|f|^2)\Big)\\ &\lesssim R^{-1}\big(N_\varepsilon|\log\varepsilon| + \varepsilon^2(N_\varepsilon^4+\|f\|^2_{\mathrm{L}^\infty})\big) \lesssim N_\varepsilon|\log\varepsilon|,\end{aligned}$$

the result (8.175) follows.

Step 2. Estimate inside small balls.

In this step, we prove the desired estimate (8.174) for the integral restricted to suitable small balls centered at the vortex locations. More precisely, since we have by assumption $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon| \lesssim |\log\varepsilon|^2$, we may apply [382, Proposition 4.8] with $M=\varepsilon^{\kappa-1}$ and $\delta=\varepsilon^{\kappa/4}$ for any $\kappa\in(0,1)$. This yields a finite union $\hat{\mathcal{B}}_{\varepsilon,0}$ of disjoint closed balls with total radius $r(\hat{\mathcal{B}}_{\varepsilon,0})=\varepsilon^{\kappa/2}$, covering the set $\{x\in B_{2R} : ||u_\varepsilon(x)|-1|\ge\varepsilon^{\kappa/4}\}$. We then prove that

$$\alpha^2\int_0^t\int_{\hat{\mathcal{B}}_{\varepsilon,0}}\frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \lesssim_t N_\varepsilon. \tag{8.177}$$

For that purpose, we let the initial collection of balls $\hat{\mathcal{B}}_{\varepsilon,0}$ grow, and we use the Pohozaev estimate of Step 1 as in [382, Proof of Theorem 5.1]. By [382, Theorem 4.2], there exists a monotone family $(\hat{\mathcal{B}}^s_\varepsilon)_{s\ge0}$ of unions of disjoint closed balls, such that $\hat{\mathcal{B}}^0_\varepsilon=\hat{\mathcal{B}}_{\varepsilon,0}$, $\hat{\mathcal{B}}^s_\varepsilon$ has total radius $r(\hat{\mathcal{B}}^s_\varepsilon)=e^s r(\hat{\mathcal{B}}_{\varepsilon,0})$ for all $s\ge0$, and $\hat{\mathcal{B}}^s_\varepsilon = e^{s-r}\hat{\mathcal{B}}^r_\varepsilon$ for all $0\le r\le s$ with $[r,s]\subset\mathbb{R}^+\setminus\mathcal{T}_\varepsilon$, for some finite set $\mathcal{T}_\varepsilon\subset\mathbb{R}^+$ (corresponding to the merging times in the growth process). For all $s\ge0$ with $r(\hat{\mathcal{B}}^s_\varepsilon)\le1$, the result (8.175) of Step 1 gives the following estimate, for all $\theta>0$,

$$\begin{aligned}\alpha^2\int_0^t\int_{\hat{\mathcal{B}}^s_\varepsilon}\frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 &\lesssim_t r(\hat{\mathcal{B}}^s_\varepsilon)N_\varepsilon|\log\varepsilon|^3\\ &+\sum_{B_r(x)\in\hat{\mathcal{B}}^s_\varepsilon} r\int_0^t\int_{\partial B_r(x)}\frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2+|1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+f)\Big).\end{aligned}$$

Integrating this estimate over s and applying [382, Proposition 4.1], we find, for all $s\ge0$ with $r(\hat{\mathcal{B}}_\varepsilon(s))\le1$,

$$\begin{aligned}s\alpha^2\int_0^t\int_{\hat{\mathcal{B}}_{\varepsilon,0}}\frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 &\le \alpha^2\int_0^s dv\int_0^t\int_{\hat{\mathcal{B}}^v_\varepsilon}\frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\\ \lesssim_t s\,r(\hat{\mathcal{B}}^s_\varepsilon)N_\varepsilon|\log\varepsilon|^3 &+\int_0^t\int_{\hat{\mathcal{B}}^s_\varepsilon\setminus\hat{\mathcal{B}}_{\varepsilon,0}}\frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2+|1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+f)\Big),\end{aligned}$$

and hence, using assumption (8.43), the boundedness of v_ε in Assumption 8.3.1(a), and the assumed energy bound,

$$s\alpha^2\int_0^t\int_{\hat{\mathcal{B}}_{\varepsilon,0}}\frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\lesssim_t s\,r(\hat{\mathcal{B}}^s_\varepsilon)N_\varepsilon|\log\varepsilon|^3+N_\varepsilon|\log\varepsilon|.$$

Recalling that $r(\hat{\mathcal{B}}^s_\varepsilon)=e^s\varepsilon^{\kappa/2}$, this yields for all $s\geq 1$ with $r(\hat{\mathcal{B}}^s_\varepsilon)\leq 1$,

$$\alpha^2\int_0^t\int_{\hat{\mathcal{B}}_{\varepsilon,0}}\frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\lesssim_t e^s\varepsilon^{\kappa/2}N_\varepsilon|\log\varepsilon|^3+\frac{N_\varepsilon|\log\varepsilon|}{s},$$

and the result (8.177) now follows for the choice $s=|\log\varepsilon^{\kappa/4}|$.

Step 3. Estimate outside small balls.

It remains to show that the desired estimate (8.174) also holds for the integral restricted to the complement of the small balls $\hat{\mathcal{B}}_{\varepsilon,0}$. More precisely, we prove in this step for all $\theta>0$,

$$\alpha\int_0^t\int_{||u_\varepsilon|-1|\leq\varepsilon^{\kappa/4}}\chi_R\Big(|\nabla|u_\varepsilon||^2+\frac{a(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}\Big)\lesssim_{t,\theta}\varepsilon^{\kappa/4}R^\theta|\log\varepsilon|^2+\varepsilon R|\log\varepsilon|^3 \tag{8.178}$$

The conclusion (8.174) of course follows from this together with (8.177), choosing $\theta>0$ small enough.

In order to prove (8.178), we adapt the argument of [382, Proof of Proposition 13.4]. For $0<\varepsilon\leq 2^{-4/\kappa}$, we define a cut-off function ζ_ε as follows,

$$\zeta_\varepsilon(y):=\begin{cases}y, & \text{if } 0\leq y\leq 1/2;\\ \frac12+\frac{y-1/2}{1-2\varepsilon^{\kappa/4}}, & \text{if } 1/2\leq y\leq 1-\varepsilon^{\kappa/4};\\ 1, & \text{if } 1-\varepsilon^{\kappa/4}\leq y\leq 1+\varepsilon^{\kappa/4};\\ 1+\frac{y-1-\varepsilon^{\kappa/4}}{1-2\varepsilon^{\kappa/4}}, & \text{if } 1+\varepsilon^{\kappa/4}\leq y\leq 3/2;\\ y, & \text{if } y\geq 3/2.\end{cases}$$

Writing $u_\varepsilon:=\rho_\varepsilon e^{i\varphi_\varepsilon}$ locally, the equation (8.6) for u_ε yields in particular

$$\begin{aligned}\alpha\lambda_\varepsilon\partial_t\rho_\varepsilon-\beta\lambda_\varepsilon|\log\varepsilon|\rho_\varepsilon\partial_t\varphi_\varepsilon\\ =\triangle\rho_\varepsilon-\rho_\varepsilon|\nabla\varphi_\varepsilon|^2+\frac{a\rho_\varepsilon}{\varepsilon^2}(1-\rho_\varepsilon^2)+\nabla h\cdot\nabla\rho_\varepsilon-\rho_\varepsilon|\log\varepsilon|F^\perp\cdot\nabla\varphi_\varepsilon+f\rho_\varepsilon.\end{aligned}\tag{8.179}$$

Testing this equation against $\chi_R(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)$, and rearranging the terms, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\chi_R(1-\zeta_\varepsilon'(\rho_\varepsilon))|\nabla\rho_\varepsilon|^2+\int_{\mathbb{R}^2}\frac{a\chi_R}{\varepsilon^2}\rho_\varepsilon(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)(1-\rho_\varepsilon^2)=\alpha\lambda_\varepsilon\int_{\mathbb{R}^2}\chi_R(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)\partial_t\rho_\varepsilon\\ -\beta\lambda_\varepsilon|\log\varepsilon|\int_{\mathbb{R}^2}\chi_R\rho_\varepsilon(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)\partial_t\varphi_\varepsilon+\int_{\mathbb{R}^2}(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)\nabla\chi_R\cdot\nabla\rho_\varepsilon+\int_{\mathbb{R}^2}\chi_R(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)\rho_\varepsilon|\nabla\varphi_\varepsilon|^2\\ -\int_{\mathbb{R}^2}\chi_R(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)\nabla h\cdot\nabla\rho_\varepsilon+|\log\varepsilon|\int_{\mathbb{R}^2}\chi_R\rho_\varepsilon(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)F^\perp\cdot\nabla\varphi_\varepsilon-\int_{\mathbb{R}^2}\chi_R(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)f\rho_\varepsilon.\end{aligned}\tag{8.180}$$

Using that the cut-off function ζ_ε satisfies for all $y\geq 0$

$$|\zeta_\varepsilon(y)-y|\lesssim\varepsilon^{\kappa/4}\mathbb{1}_{|y-1|\leq 1/2},\qquad |\zeta_\varepsilon(y)-y|\leq|1-y|\leq|1-y^2|,\tag{8.181}$$

$$|\zeta_\varepsilon'(y)-1|\lesssim\mathbb{1}_{|y-1|\leq\varepsilon^{\kappa/4}}+\varepsilon^{\kappa/4}\mathbb{1}_{|y-1|\leq 1/2},\qquad(\zeta_\varepsilon(y)-y)(1-y)\geq 0,\tag{8.182}$$

and noting that

$$\int_{|\rho_\varepsilon-1|\le\varepsilon^{\kappa/4}}\frac{a\chi_R}{5\varepsilon^2}(1-\rho_\varepsilon^2)^2\le\int_{|\rho_\varepsilon-1|\le\varepsilon^{\kappa/4}}\frac{a\chi_R}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon)(1-\rho_\varepsilon^2)\le\int_{\mathbb{R}^2}\frac{a\chi_R}{\varepsilon^2}\rho_\varepsilon(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)(1-\rho_\varepsilon^2),$$

we obtain from (8.43), (8.180) and (8.181),

$$\begin{aligned}\int_{|\rho_\varepsilon-1|\le\varepsilon^{\kappa/4}}\chi_R\Big(|\nabla\rho_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2\Big)&\lesssim\varepsilon^{\kappa/4}\int_{|\rho_\varepsilon-1|\le1/2}\chi_R(|\nabla\rho_\varepsilon|^2+\rho_\varepsilon^2|\nabla\varphi_\varepsilon|^2)\\&\quad+\lambda_\varepsilon|\log\varepsilon|\int_{|\rho_\varepsilon-1|\le1/2}\chi_R|1-\rho_\varepsilon^2|(|\partial_t\rho_\varepsilon|+\rho_\varepsilon|\partial_t\varphi_\varepsilon|)\\&\quad+(1+\lambda_\varepsilon|\log\varepsilon|)\int_{|\rho_\varepsilon-1|\le1/2}\chi_R|1-\rho_\varepsilon^2|(|\nabla\rho_\varepsilon|+\rho_\varepsilon|\nabla\varphi_\varepsilon|)\\&\quad+\int_{|\rho_\varepsilon-1|\le1/2}\chi_R|f||1-\rho_\varepsilon^2|+\int_{|\rho_\varepsilon-1|\le1/2}|\nabla\chi_R||1-\rho_\varepsilon^2||\nabla\rho_\varepsilon|.\end{aligned}$$

Since $|\nabla u_\varepsilon|^2=|\nabla\rho_\varepsilon|^2+\rho_\varepsilon^2|\nabla\varphi_\varepsilon|^2$, and $|\partial_t u_\varepsilon|^2=|\partial_t\rho_\varepsilon|^2+\rho_\varepsilon^2|\partial_t\varphi_\varepsilon|^2$, we obtain with assumption (8.43),

$$\begin{aligned}&\int_{|\rho_\varepsilon-1|\le\varepsilon^{\kappa/4}}\chi_R\Big(|\nabla|u_\varepsilon||^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\&\qquad\lesssim\varepsilon^{\kappa/4}\|\nabla u_\varepsilon\|^2_{\mathrm{L}^2(B_{2R})}+\lambda_\varepsilon|\log\varepsilon|\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}\|\partial_t u_\varepsilon\|_{\mathrm{L}^2(B_{2R})}\\&\qquad+(1+\lambda_\varepsilon|\log\varepsilon|)\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}\|\nabla u_\varepsilon\|_{\mathrm{L}^2(B_{2R})}+R(1+\lambda_\varepsilon^2|\log\varepsilon|^2)\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}.\end{aligned}$$

By the integrability properties of v_ε in Assumption 8.3.1(a), we have for all $\theta>0$,

$$\|\nabla u_\varepsilon\|_{\mathrm{L}^2(B_{2R})}\lesssim_\theta\|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_{2R})}+N_\varepsilon(R^\theta+\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}),$$

hence, by Lemma 8.5.5,

$$\alpha\int_0^t\int_{|\rho_\varepsilon-1|\le\varepsilon^{\kappa/4}}\chi_R\Big(|\nabla|u_\varepsilon||^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\lesssim_{t,\theta}\varepsilon^{\kappa/4}R^\theta|\log\varepsilon|^2+\varepsilon R|\log\varepsilon|^3,$$

and the result (8.178) follows. ☐

8.6 Mean-field limit in the dissipative case

In this section we prove Theorem 8.1.2, that is, the mean-field limit result in the dissipative case ($\alpha>0$) in both critical regimes (GL_1) and (GL_2) as well as in the subcritical regimes (GL_1') and (GL_2'). More precisely, the following result states that the rescaled supercurrent density $N_\varepsilon^{-1}j_\varepsilon$ remains close to the solution v_ε of equation (8.51). Combining this with the results of Section 8.3.1 (in particular, with Lemma 8.3.3), the result of Theorem 8.1.2 follows. The proof consists in making use of the various estimates and technical tools for vortex analysis developed in Section 8.5 in order to estimate the terms in the decomposition of the time derivative of the modulated energy excess given by Lemma 8.4.4, and then deduce the smallness of the modulated energy excess by a Grönwall argument. (In this section, as we assume $\alpha>0$, all multiplicative constants are implicitly allowed to additionally depend on an upper bound on α^{-1}.)

Proposition 8.6.1. *Let $\alpha>0$, $\beta\in\mathbb{R}$, $\alpha^2+\beta^2=1$, and let $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, $F:\mathbb{R}^2\to\mathbb{R}^2$, $f:\mathbb{R}^2\to\mathbb{R}$ satisfy* (8.43). *Let $u_\varepsilon:[0,T)\times\mathbb{R}^2\to\mathbb{C}$ and $\mathrm{v}_\varepsilon:[0,T)\times\mathbb{R}^2\to\mathbb{R}^2$ be solutions of* (8.6) *and* (8.51) *as in Proposition 8.2.2(i) and in Proposition 8.3.2, respectively, for some $T>0$. Let $0<\varepsilon\ll1$, $1\ll N_\varepsilon\lesssim|\log\varepsilon|$, $R\ge1$, $|\log\varepsilon|/N_\varepsilon\ll R\lesssim|\log\varepsilon|^n$, for some $n\ge1$, and assume that the initial modulated energy excess satisfies $\mathcal{D}^{*,\circ}_{\varepsilon,R}\ll N_\varepsilon^2$. Then,* *(i) if* $\log|\log\varepsilon| \ll N_\varepsilon \lesssim |\log\varepsilon|$*, in each of the regimes* (GL$_1$), (GL$_2$), (GL$_1'$)*, and* (GL$_2'$)*, we have* $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ *for all* $t \in [0,T)$*;*

(ii) if $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$*, in the parabolic case* $\alpha = 1$*,* $\beta = 0$*, either in the regime* (GL$_1$)*, or in the regime* (GL$_2'$) *with* $\lambda_\varepsilon \lesssim e^{o(N_\varepsilon)}/|\log\varepsilon|$*, the same conclusion* $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ *holds for all* $t \in [0,T)$*.*

In particular, in both cases, we deduce $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ *in* $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ *as* $\varepsilon \downarrow 0$*. If we further assume* $\mathcal{D}^{*,\circ}_{\varepsilon,\infty} \ll N_\varepsilon^2$*, then for any* $\ell \ge 1$ *we obtain more precisely for all* $t \in [0,T)$ *and* $L \ge 1$*,*

$$\sup_z \|N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon\|_{(\mathrm{L}^1+\mathrm{L}^2)(B_L(z))} \ll_{t,\ell} \Big(1 + \frac{L}{|\log\varepsilon|^\ell}\Big)^2. \tag{8.183}$$

$\Diamond$

Remark 8.6.2. If we further assume $\|u_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1$ for all t, then the proof shows that the convergence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ actually holds in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^p_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ for all $p < 2$. In the parabolic case $\beta = 0$ without forcing $F = f = 0$, a maximum principle type argument gives that $\|u_\varepsilon^\circ\|_{\mathrm{L}^\infty} \le 1$ implies $\|u_\varepsilon^t\|_{\mathrm{L}^\infty} \le 1$ for all $t \ge 0$ (see e.g. [117, Proposition 4.4]). [2] However, the same argument fails in the presence of forcing $F, f \neq 0$. Moreover, such a uniform L^∞-bound on u_ε is expected to fail in the Gross-Pitaevskii case $\alpha = 0$ due to the time reversibility of the equation in that case, and similarly it is expected to fail as well in the mixed-flow case $\alpha > 0$, $\beta \neq 0$. We therefore systematically avoid the use of such L^∞-estimates here. $\Diamond$

Proof. We choose $R \gg |\log\varepsilon|/N_\varepsilon$ with $R^{\theta_0} \lesssim |\log\varepsilon|$ for some $\theta_0 > 0$. Given the assumption $\mathcal{D}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$ on the initial data, for all $\varepsilon > 0$ we define $T_\varepsilon > 0$ as the maximum time $\le T$ such that $\mathcal{D}^{*,t}_{\varepsilon,R} \le N_\varepsilon^2$ holds for all $t \le T_\varepsilon$. By Lemma 8.4.1 and Proposition 8.5.2, we deduce $\hat{\mathcal{D}}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$ and for all $t \le T_\varepsilon$,

$$\mathcal{E}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon|\log\varepsilon|, \qquad \hat{\mathcal{E}}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon|\log\varepsilon|, \qquad \hat{\mathcal{D}}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon^2, \qquad \mathcal{D}^{*,t}_{\varepsilon,R} \lesssim \hat{\mathcal{D}}^{*,t}_{\varepsilon,R} + o_t(N_\varepsilon^2). \tag{8.184}$$

The strategy of the proof consists in showing that for all $t \le T_\varepsilon$,

$$\hat{\mathcal{D}}^{*,t}_{\varepsilon,R} \lesssim_t o(N_\varepsilon^2) + \int_0^t \hat{\mathcal{D}}^*_{\varepsilon,R}. \tag{8.185}$$

By the Grönwall inequality, this implies $\hat{\mathcal{D}}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$, hence $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \le T_\varepsilon$. This gives in particular $T_\varepsilon = T$ for all $\varepsilon > 0$ small enough, and the main conclusion follows.

To simplify notation, we focus on (8.185) with the left-hand side $\hat{\mathcal{D}}^t_{\varepsilon,R}$ centered at $z = 0$, but the result of course holds uniformly with respect to the translation. We start with the general mixed-flow case in the regime $\log|\log\varepsilon| \ll N_\varepsilon \lesssim |\log\varepsilon|$. The proof of (8.185) in that case is split into three

2. The usual maximum principle type argument is indeed as follows: Denoting by $\xi^z(x) := e^{-|x-z|}$ the exponential cut-off centered at $z \in \mathbb{Z}^2$, writing $u_\varepsilon := \rho_\varepsilon e^{i\varphi_\varepsilon}$ locally, and testing equation (8.179) for ρ_ε (in the parabolic case $\beta = 0$) with the positive part $\xi^z(\rho_\varepsilon - 1)_+$, we find

$$\begin{aligned}\frac{\lambda_\varepsilon}{2}\partial_t \int_{\mathbb{R}^2} \xi^z(\rho_\varepsilon - 1)_+^2 = &-\int_{\rho_\varepsilon>1} \xi^z|\nabla\rho_\varepsilon|^2 - \int_{\mathbb{R}^2}(\rho_\varepsilon - 1)_+\nabla\xi^z\cdot\nabla\rho_\varepsilon - \int_{\mathbb{R}^2}\xi^z\rho_\varepsilon(\rho_\varepsilon-1)_+|\nabla\varphi_\varepsilon|^2 + \int_{\mathbb{R}^2}\frac{a\xi^z}{2}\rho_\varepsilon(\rho_\varepsilon-1)_+(1-\rho_\varepsilon^2)\\ &+\frac{1}{2}\int_{\mathbb{R}^2}\xi^z\nabla h\cdot\nabla(\rho_\varepsilon-1)_+^2 - |\log\varepsilon|\int_{\mathbb{R}^2}\xi^z\rho_\varepsilon(\rho_\varepsilon-1)_+F^\perp\cdot\nabla\varphi_\varepsilon + \int_{\mathbb{R}^2} f\xi^z\rho_\varepsilon(\rho_\varepsilon-1)_+,\end{aligned}$$

and hence, using the estimate $|\nabla\xi^z| \lesssim \xi^z$, and the inequality $2xy \le x^2 + y^2$,

$$\frac{\lambda_\varepsilon}{2}\partial_t\int_{\mathbb{R}^2}\xi^z(\rho_\varepsilon-1)_+^2 \lesssim (1 + \|\nabla h\|_{W^{1,\infty}})\int_{\mathbb{R}^2}\xi^z(\rho_\varepsilon-1)_+^2 + (|\log\varepsilon|^2\|F\|^2_{\mathrm{L}^\infty} + \|f\|_{\mathrm{L}^\infty})\int_{\mathbb{R}^2}\xi^z\rho_\varepsilon(\rho_\varepsilon-1)_+.$$

Therefore, in the case $F = f = 0$ (which implies $\nabla h = 0$ by the choice (8.7)), if the initial data satisfies $|u_\varepsilon^\circ| \le 1$, the Grönwall inequality implies $|u_\varepsilon^t| \le 1$ for all $t \ge 0$.

steps, while the additional stated consequences are deduced in Step 4. Finally, Step 5 describes the modifications needed to treat the parabolic case in the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$.

Let us first introduce some notation. In the regime $\log|\log\varepsilon| \ll N_\varepsilon \lesssim |\log\varepsilon|$, for all $t \le T_\varepsilon$, as we are in the framework of Proposition 8.5.2 with $u_\varepsilon^t, \mathrm{v}_\varepsilon^t$, we let $\mathcal{B}_\varepsilon^t := \mathcal{B}_{\varepsilon,R}^t$ denote the constructed collection of disjoint closed balls $\mathcal{B}_{\varepsilon,R}^{r_\varepsilon}(u_\varepsilon^t, \mathrm{v}_\varepsilon^t)$ with total radius $r_\varepsilon := |\log\varepsilon|^{-4}e^{-\sqrt{N_\varepsilon}}$, hence $e^{-o(N_\varepsilon)} \le r_\varepsilon \ll N_\varepsilon|\log\varepsilon|^{-1}$. Let then $\bar\Gamma_\varepsilon^t$ denote the corresponding approximation of Γ_ε^t given by Lemma 8.5.3. We decompose $\Gamma_\varepsilon := \alpha\Gamma_{\varepsilon,0} - \beta\Gamma_{\varepsilon,0}^\perp$ with

$$\Gamma_{\varepsilon,0} := \lambda_\varepsilon^{-1}\Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big).$$

Step 1. Time derivative of the modulated energy excess.

Lemma 8.4.4 yields the following decomposition,

$$\partial_t\hat{\mathcal{D}}_{\varepsilon,R} = I_{\varepsilon,R}^S + I_{\varepsilon,R}^V + I_{\varepsilon,R}^E + I_{\varepsilon,R}^D + I_{\varepsilon,R}^H + I_{\varepsilon,R}^d + I_{\varepsilon,R}^g + I_{\varepsilon,R}^n + I_{\varepsilon,R}', \tag{8.186}$$

where the eight first terms are as in the statement of Lemma 8.4.4, and where the error $I'_{\varepsilon,R}$ is estimated as follows (cf. (8.110)) in the considered regimes,

$$\int_0^t |I'_{\varepsilon,R}| \lesssim_t \varepsilon R(N_\varepsilon|\log\varepsilon|)^{1/2}|\log\varepsilon|^2 = o(N_\varepsilon^2).$$

Step 2. Estimating the error terms.

In this step, we consider the regime $\log|\log\varepsilon| \ll N_\varepsilon \lesssim |\log\varepsilon|$, we study the three error terms $I_{\varepsilon,R}^d$, $I_{\varepsilon,R}^g$, and $I_{\varepsilon,R}^n$, and we prove for all $t \le T_\varepsilon$,

$$\int_0^t (I_{\varepsilon,R}^d + I_{\varepsilon,R}^g + I_{\varepsilon,R}^n) \lesssim_t o(N_\varepsilon^2) + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int_0^t\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2. \tag{8.187}$$

We start with the estimation of $I_{\varepsilon,R}^n$. Using (8.184), Lemma 8.5.5, and the boundedness properties of p_ε (cf. Proposition 8.3.2), the quantity $\bar{\mathcal{E}}_{\varepsilon,R}^*$ defined in Lemma 8.5.4 is estimated as follows in the considered regimes, for all $\theta > 0$,

$$\begin{aligned}\bar{\mathcal{E}}_{\varepsilon,R}^{*,t} \lesssim \sup_z \int_0^t \mathcal{E}_{\varepsilon,R}^z + \sup_z \int_0^t\int_{\mathbb{R}^2}\chi_R^z\big(|\partial_t u_\varepsilon|^2 + N_\varepsilon^2|\mathrm{p}_\varepsilon|^2 + N_\varepsilon^2|1-|u_\varepsilon|^2||\mathrm{p}_\varepsilon|^2\big)&\\ \lesssim_{t,\theta} R^\theta N_\varepsilon|\log\varepsilon|^3 + \lambda_\varepsilon^{-1}N_\varepsilon^2 \lesssim R^\theta|\log\varepsilon|^4&,\end{aligned}$$

hence, for $\theta > 0$ small enough, $\bar{\mathcal{E}}_{\varepsilon,R}^{*,t} \lesssim_t |\log\varepsilon|^5$. Using the obvious estimate $|\nabla\chi_R| \lesssim R^{-1}\chi_R^{1/2}$, Lemma 8.5.4 then yields

$$\begin{aligned}\Big|\int_0^t\int_{\mathbb{R}^2} a\tilde V_\varepsilon\cdot\nabla^\perp\chi_R\Big| &\lesssim_t |\log\varepsilon|^{-1}\\ &+ R^{-1}|\log\varepsilon|^{-1}\Big(\int_0^t\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon|^2 + \int_0^t\int_{B_{2R}}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2\Big),\end{aligned}$$

and hence,

$$\begin{aligned}\Big|\int_0^t I_{\varepsilon,R}^n\Big| \lesssim_t 1 + R^{-1}\int_0^t\int_{B_{2R}}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + |f|)\Big)&\\ + R^{-1}\int_0^t\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon|^2&.\end{aligned}$$

Using (8.184), (8.43), and the integrability properties of v_ε (cf. Proposition 8.3.2), with the choice $R \gg |\log\varepsilon|/N_\varepsilon$, we conclude

$$\Big|\int_0^t I^n_{\varepsilon,R}\Big| \lesssim_t 1 + R^{-1}N_\varepsilon|\log\varepsilon| + R^{-1}\int_0^t\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon|^2$$
$$\lesssim o(N_\varepsilon^2) + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int_0^t\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon|^2. \quad (8.188)$$

We now turn to the estimation of $I^g_{\varepsilon,R}$. Using (8.43) and the pointwise estimates of Lemma 8.4.2, we find

$$\begin{aligned}|I^g_{\varepsilon,R}| \lesssim \|\Gamma_\varepsilon - \bar\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\bigg(&N_\varepsilon\int_{B_{2R}}(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + N_\varepsilon|1-|u_\varepsilon|^2|)|\operatorname{curl}\mathrm{v}_\varepsilon|\\
&+ N_\varepsilon\int_{B_{2R}}|1-|u_\varepsilon|^2||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon| + \lambda_\varepsilon\int_{\mathbb{R}^2}\chi_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\
&+ \lambda_\varepsilon|\log\varepsilon|\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|\\
&+ (N_\varepsilon + \lambda_\varepsilon|\log\varepsilon|)\int_{\mathbb{R}^2}\chi_R(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + N_\varepsilon^2|1-|u_\varepsilon|^2||\mathrm{v}_\varepsilon|^2)\\
&+ N_\varepsilon^2\int_{\mathbb{R}^2}\chi_R|\mathrm{v}_\varepsilon|^2(N_\varepsilon|\mathrm{v}_\varepsilon| + |\log\varepsilon||F|) + \lambda_\varepsilon N_\varepsilon|\log\varepsilon||\beta|\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|(|\mathrm{v}_\varepsilon| + |1-|u_\varepsilon|^2|)\bigg).\end{aligned}$$

By (8.184), by Lemma 8.5.3 in the form $\|\Gamma_\varepsilon - \bar\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \lesssim r_\varepsilon = |\log\varepsilon|^{-4}e^{-\sqrt{N_\varepsilon}}$, and by the integrability properties of v_ε (cf. Proposition 8.3.2), we deduce in the considered regimes for all $\theta > 0$,

$$|I^g_{\varepsilon,R}| \lesssim_{t,\theta} \frac{e^{-\sqrt{N_\varepsilon}}}{|\log\varepsilon|^4}R^\theta N_\varepsilon|\log\varepsilon|^2\Big(1 + \int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2\Big)^{1/2}, \quad (8.189)$$

and hence, for $\theta > 0$ small enough such that $R^\theta \lesssim |\log\varepsilon|$, we conclude

$$|I^g_{\varepsilon,R}| \lesssim_t o(N_\varepsilon^2) + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2. \quad (8.190)$$

Regarding the last term $I^d_{\varepsilon,R}$, the definition of the pressure p_ε in (8.51) simply yields $I^d_{\varepsilon,R} = 0$, and the conclusion (8.187) follows.

Step 3. Estimating the dominant terms.

In this step, we consider the regime $\log|\log\varepsilon| \ll N_\varepsilon \lesssim |\log\varepsilon|$ and we turn to the estimation of the five first terms in (8.186), showing more precisely that for all $t \le T_\varepsilon$,

$$\hat{\mathcal{D}}^t_{\varepsilon,R} \lesssim_t o(N_\varepsilon^2) + \int_0^t \hat{\mathcal{D}}_{\varepsilon,R}. \quad (8.191)$$

As this result obviously holds uniformly with respect to translations of the cut-off functions, the conclusion (8.185) follows.

We start with the estimation of the first term $I^S_{\varepsilon,R}$. Since for all t the field $\bar\Gamma^t_\varepsilon$ is by definition constant in each ball of the collection $\mathcal{B}^t_\varepsilon$ and satisfies $\|\nabla\bar\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty} \lesssim \|\nabla\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty}$, we obtain

$$|I^S_{\varepsilon,R}| \lesssim \int_{\mathbb{R}^2\setminus\mathcal{B}_\varepsilon}\chi_R|\tilde S_\varepsilon| \lesssim \int_{\mathbb{R}^2\setminus\mathcal{B}_\varepsilon}a\chi_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)$$
$$+ \int_{\mathbb{R}^2}\chi_R|1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + |f|).$$

Since $\mathcal{B}_\varepsilon$ has total radius $r_\varepsilon := |\log\varepsilon|^{-4}e^{-\sqrt{N_\varepsilon}}$, and since the choice $N_\varepsilon \gg \log|\log\varepsilon|$ ensures $r_\varepsilon \geq e^{-o(N_\varepsilon)}$, we may apply Proposition 8.5.2(v), which shows that the first integral in the above right-hand side is bounded by $\mathcal{D}^*_{\varepsilon,R} + o(N_\varepsilon^2)$. Further using (8.184), (8.43), and the integrability properties of v_ε (cf. Proposition 8.3.2), we obtain in the considered regimes,

$$|I^S_{\varepsilon,R}| \lesssim \mathcal{D}_{\varepsilon,R} + o(N_\varepsilon^2) + \varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2}(N_\varepsilon^2 + R\lambda_\varepsilon^2|\log\varepsilon|^2) \lesssim \hat{\mathcal{D}}_{\varepsilon,R} + o(N_\varepsilon^2). \tag{8.192}$$

We turn to $I^H_{\varepsilon,R}$. Since $\|(\Gamma_\varepsilon, \nabla h)\|_{\mathrm{L}^\infty} \lesssim_t 1$, Lemma 8.5.6 yields

$$\int_0^t I^H_{\varepsilon,R} = O_t(N_\varepsilon) + \int_0^t \int_{\mathbb{R}^2} \frac{a\chi_R}{2}\Gamma_\varepsilon^\perp \cdot \nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big),$$

and hence by Proposition 8.5.2(iv) and by (8.184),

$$\int_0^t I^H_{\varepsilon,R} \lesssim_t o(N_\varepsilon^2) + \int_0^t \mathcal{D}_{\varepsilon,R} \lesssim_t o(N_\varepsilon^2) + \int_0^t \hat{\mathcal{D}}_{\varepsilon,R}. \tag{8.193}$$

The term $I^D_{\varepsilon,R}$ is simply estimated by

$$I^D_{\varepsilon,R} \leq -\frac{\lambda_\varepsilon\alpha}{2}\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac{\lambda_\varepsilon\alpha}{2}\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon^\perp|^2. \tag{8.194}$$

We finally turn to $I^V_{\varepsilon,R}$. Using $\alpha^2+\beta^2=1$, we have by definition

$$\Gamma_{\varepsilon,0} - \beta\Gamma_\varepsilon^\perp = \Gamma_{\varepsilon,0} - \beta(\alpha\Gamma_{\varepsilon,0}^\perp + \beta\Gamma_{\varepsilon,0}) = \alpha^2\Gamma_{\varepsilon,0} - \alpha\beta\Gamma_{\varepsilon,0}^\perp = \alpha\Gamma_\varepsilon,$$

and hence $I^V_{\varepsilon,R}$ takes on the following guise,

$$I^V_{\varepsilon,R} = \lambda_\varepsilon|\log\varepsilon|\int_{\mathbb{R}^2}\frac{a\chi_R}{2}\tilde{V}_\varepsilon\cdot(\Gamma_{\varepsilon,0} - \beta\Gamma_\varepsilon^\perp) = \lambda_\varepsilon\alpha|\log\varepsilon|\int_{\mathbb{R}^2}\frac{a\chi_R}{2}\tilde{V}_\varepsilon\cdot\Gamma_\varepsilon. \tag{8.195}$$

As shown in Step 2, the quantity $\bar{\mathcal{E}}^*_{\varepsilon,R}$ defined in Lemma 8.5.4 satisfies $\bar{\mathcal{E}}^{*,t}_{\varepsilon,R} \lesssim_t |\log\varepsilon|^5$. In the regime $\log|\log\varepsilon| \ll N_\varepsilon \lesssim |\log\varepsilon|$, choosing e.g. $M_\varepsilon := \exp((N_\varepsilon\log|\log\varepsilon|)^{1/2})$, Lemma 8.5.4 yields for any $\Lambda \simeq 1$,

$$\begin{aligned}\Big|\int_0^t I^V_{\varepsilon,R}\Big| \leq o_t(1) + \lambda_\varepsilon\alpha\Big(1 + \frac{C_t(N_\varepsilon\log|\log\varepsilon|)^{1/2}}{|\log\varepsilon|}\Big)\\ \times\bigg(\frac{1}{\Lambda}\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2 + \frac{\Lambda}{4}\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2\bigg),\end{aligned}$$

and thus, using the optimal energy bound (8.184), we obtain in the considered regimes,

$$\begin{aligned}\Big|\int_0^t I^V_{\varepsilon,R}\Big| \leq o_t(N_\varepsilon^2) + \Big(\lambda_\varepsilon + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\Big)\frac{\alpha}{\Lambda}\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2\\ + \frac{\lambda_\varepsilon\alpha\Lambda}{4}\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2.\end{aligned} \tag{8.196}$$

We now distinguish between two cases:

$$\text{(Case 1)}\qquad \int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2 \leq 5\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2, \tag{8.197}$$

$$\text{(Case 2)}\qquad \int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2 > 5\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2. \tag{8.198}$$

In Case 1, choosing $\Lambda = 2$ in (8.196) yields

$$\Big|\int_0^t I^V_{\varepsilon,R}\Big| \le o_t(N_\varepsilon^2) + \Big(\lambda_\varepsilon + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\Big)\frac{\alpha}{2}\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2.$$

In Case 2, the condition (8.198) can be rewritten as

$$\frac14\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2 \le \Big(\frac14+\frac{1}{10}\Big)\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac12\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2,$$

and choosing $\Lambda = 4$ in (8.196) then yields, with $N_\varepsilon/|\log\varepsilon| \lesssim \lambda_\varepsilon$ in the considered regimes,

$$\Big|\int_0^t I^V_{\varepsilon,R}\Big| \le o_t(N_\varepsilon^2) + \lambda_\varepsilon\alpha\bigg(\Big(\frac14+\frac{1}{10}+o(1)\Big)\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac12\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2\bigg).$$

Further noting that in Case 1 the condition (8.197) together with the energy bound (8.184) yields

$$o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 \le o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int_0^t\int_{\mathbb{R}^2} a\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \lesssim_t o(N_\varepsilon^2),$$

and combining this with (8.187) and (8.194), we observe an exact recombination of the terms, and obtain in Case 1,

$$\int_0^t (I^V_{\varepsilon,R} + I^D_{\varepsilon,R} + I^d_{\varepsilon,R} + I^g_{\varepsilon,R} + I^n_{\varepsilon,R} + I'_{\varepsilon,R}) \le \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int_{\mathbb{R}^2} a\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2|\Gamma_\varepsilon|^2 + o_t(N_\varepsilon^2), \tag{8.199}$$

and in Case 2,

$$\int_0^t (I^V_{\varepsilon,R} + I^D_{\varepsilon,R} + I^d_{\varepsilon,R} + I^g_{\varepsilon,R} + I^n_{\varepsilon,R} + I'_{\varepsilon,R}) \le -\frac{\lambda_\varepsilon\alpha}{2}\Big(\frac12-\frac15-o(1)\Big)\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}|^2 + \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int_{\mathbb{R}^2} a\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2|\Gamma_\varepsilon|^2 + o_t(N_\varepsilon^2),$$

so that (8.199) holds in both cases for $\varepsilon > 0$ small enough. Using $\alpha^2+\beta^2 = 1$, we have by definition $\Gamma_\varepsilon\cdot\Gamma_{\varepsilon,0} = \alpha|\Gamma_{\varepsilon,0}|^2 = \alpha|\Gamma_\varepsilon|^2$, and hence the term $I^E_{\varepsilon,R}$ takes on the following guise, in terms of Γ_ε, $\Gamma_{\varepsilon,0}$,

$$I^E_{\varepsilon,R} = -\frac{\lambda_\varepsilon}{2}|\log\varepsilon|\int_{\mathbb{R}^2} a\chi_R\Gamma_\varepsilon\cdot\Gamma_{\varepsilon,0}\,\mu_\varepsilon = -\frac{\lambda_\varepsilon\alpha}{2}|\log\varepsilon|\int_{\mathbb{R}^2} a\chi_R|\Gamma_\varepsilon|^2\mu_\varepsilon.$$

Together with (8.199), this leads to

$$\int_0^t (I^V_{\varepsilon,R} + I^E_{\varepsilon,R} + I^D_{\varepsilon,R} + I^d_{\varepsilon,R} + I^g_{\varepsilon,R} + I^n_{\varepsilon,R} + I'_{\varepsilon,R}) \le \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int_{\mathbb{R}^2} a\chi_R\big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 - |\log\varepsilon|\mu_\varepsilon\big)|\Gamma_\varepsilon|^2 + o_t(N_\varepsilon^2).$$

Combining this with (8.186), (8.192), (8.193), and with $\hat{\mathcal{D}}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$, we conclude

$$\hat{\mathcal{D}}^t_{\varepsilon,R} \le o_t(N_\varepsilon^2) + C_t \int_0^t \hat{\mathcal{D}}_{\varepsilon,R} + \frac{\lambda_\varepsilon \alpha}{2} \int_0^t \int_{\mathbb{R}^2} a\chi_R \big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 - |\log \varepsilon|\mu_\varepsilon\big)|\Gamma_\varepsilon|^2, \tag{8.200}$$

and the result (8.191) now follows from Proposition 8.5.2(iv).

Step 4. Consequences.

In the previous steps, the results $T_\varepsilon = T$ and $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \in [0,T)$ are established in the case (i) of the statement (that is, in the regime $\log|\log\varepsilon| \ll N_\varepsilon \lesssim |\log\varepsilon|$). We now show that it implies the stated convergence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$.

For all $t \in [0,T)$, since there holds $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$, Proposition 8.5.2(v)–(vi) implies

$$\sup_z \int_{\mathbb{R}^2 \setminus \mathcal{B}_\varepsilon} \chi_R^z |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \ll_t N_\varepsilon^2,$$

and for all $1 \le p < 2$,

$$\sup_z \int_{\mathcal{B}_\varepsilon} \chi_R^z |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \ll_t N_\varepsilon^p.$$

Using the pointwise estimates of Lemma 8.4.2, we deduce

$$\begin{aligned}\sup_z \int_{B(z)} |j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon| &\lesssim_t \sup_z \int_{B(z)} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + \varepsilon N_\varepsilon |\log\varepsilon| \\ &\lesssim_t \sup_z \int_{\mathcal{B}_\varepsilon} \chi_R^z |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + \sup_z \Big(\int_{B(z)\setminus\mathcal{B}_\varepsilon} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2} + o(N_\varepsilon) \ll_t N_\varepsilon,\end{aligned}$$

hence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$. More precisely, we may decompose for all $L \ge 1$,

$$\begin{aligned}\sup_z \|j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon\|_{(\mathrm{L}^1+\mathrm{L}^2)(B_L(z))} &\lesssim_t \sup_z \int_{\mathcal{B}_\varepsilon} \chi_R^z |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + \sup_z \|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_L(z)\setminus\mathcal{B}_\varepsilon)} \\ &\quad + N_\varepsilon \sup_z \|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_L(z))} + \sup_z \|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_L(z))} \|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_L(z))},\end{aligned}$$

hence

$$\begin{aligned}&\sup_z \|j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon\|_{(\mathrm{L}^1+\mathrm{L}^2)(B_L(z))} \\ &\qquad \lesssim_t o(N_\varepsilon)(1+L/R) + \varepsilon N_\varepsilon (N_\varepsilon |\log\varepsilon|)^{1/2}(1+L/R) + \varepsilon N_\varepsilon |\log\varepsilon|(1+L/R)^2,\end{aligned}$$

and the result (8.183) follows. As mentioned in Remark 8.6.2, under the additional assumption that $\|u_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1$, the convergence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ also holds in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^2)^2)$ for all $1 \le p < 2$; this follows from a similar argument as above, replacing the pointwise estimate of Lemma 8.4.2 for $j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon$ by

$$|j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon| \le |u_\varepsilon||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + N_\varepsilon |1-|u_\varepsilon|^2||\mathrm{v}_\varepsilon|.$$

Step 5. Refinement in the purely parabolic case.

In this step, we consider the parabolic case ($\alpha = 1$, $\beta = 0$) both in the critical regime (GL_1) and in the subcritical regime (GL_2') with $\lambda_\varepsilon \le e^{o(N_\varepsilon)}/|\log\varepsilon|$, and we show that the additional assumption $N_\varepsilon \gg \log|\log\varepsilon|$ can then be dropped. In the proof in Steps 1–4 above, the main limitation comes from the fact that we need to use balls $\mathcal{B}_\varepsilon$ with a particularly small total radius r_ε in order to obtain smallness of the error term $I^g_{\varepsilon,\varrho,R}$ in (8.189), while on the other hand the term $I^S_{\varepsilon,\varrho,R}$ corresponds to the energy outside the small balls $\mathcal{B}_\varepsilon$ so that we need to choose $r_\varepsilon \ge e^{-o(N_\varepsilon)}$ in order to apply Proposition 8.5.2(v). As we now show, the worst terms in $I^g_{\varepsilon,\varrho,R}$ vanish in the parabolic case, and the total radius r_ε may then be chosen much larger.

Let us thus consider the parabolic case ($\alpha = 1$, $\beta = 0$) in the regime (GL$_1$) and in the subcritical regime (GL$'_2$) with $\lambda_\varepsilon \lesssim e^{o(N_\varepsilon)}/|\log\varepsilon|$, and with a "small" number of vortices $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$. Choose $\varepsilon^{1/2} < \tilde r^0_\varepsilon \ll N_\varepsilon|\log\varepsilon|^{-1}$ and let $\tilde r_\varepsilon := (\lambda_\varepsilon|\log\varepsilon|)^{-2} \ge e^{-o(N_\varepsilon)}$. For all $t \le T_\varepsilon$, as we are in the framework of Proposition 8.5.2 with $u^t_\varepsilon, \mathrm{v}^t_\varepsilon$, we let $\tilde{\mathcal{B}}^t_\varepsilon := \tilde{\mathcal{B}}^t_{\varepsilon,R}$ denote the corresponding collection of disjoint closed balls $\tilde{\mathcal{B}}^{\tilde r^0_\varepsilon,\tilde r_\varepsilon}_{\varepsilon,R}(u^t_\varepsilon, \mathrm{v}^t_\varepsilon)$. Let then $\tilde\Gamma^t_\varepsilon$ denote the associated approximation of Γ^t_ε given by Lemma 8.5.3. As in Step 1, Lemma 8.4.4 yields the following decomposition, with the approximate vector field $\bar\Gamma_\varepsilon$ replaced by $\tilde\Gamma_\varepsilon$,

$$\partial_t\hat{\mathcal{D}}_{\varepsilon,R} = I^S_{\varepsilon,R} + I^V_{\varepsilon,R} + I^E_{\varepsilon,R} + I^D_{\varepsilon,R} + I^H_{\varepsilon,R} + I^d_{\varepsilon,R} + I^g_{\varepsilon,R} + I^n_{\varepsilon,R} + I'_{\varepsilon,R},$$

where all the terms are estimated just as above, except $I^S_{\varepsilon,R}$, $I^V_{\varepsilon,R}$, and $I^g_{\varepsilon,R}$. We start with the discussion of $I^g_{\varepsilon,R}$. For $\alpha = 1$, $\beta = 0$, this term takes on the following simpler form,

$$\begin{aligned} I^g_{\varepsilon,R} = \int_{\mathbb{R}^2} a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot(\Gamma_\varepsilon - \tilde\Gamma_\varepsilon)\operatorname{curl}\mathrm{v}_\varepsilon \\ + \int_{\mathbb{R}^2} \lambda_\varepsilon a\chi_R(\Gamma_\varepsilon - \tilde\Gamma_\varepsilon)^\perp\cdot\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle \\ + \int_{\mathbb{R}^2} \frac{a\chi_R}{2}(\tilde\Gamma_\varepsilon - \Gamma_\varepsilon)^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \\ + \int_{\mathbb{R}^2} a\chi_R(\tilde\Gamma_\varepsilon - \Gamma_\varepsilon)\cdot(N_\varepsilon \mathrm{v}_\varepsilon + |\log\varepsilon|F^\perp/2)\mu_\varepsilon. \end{aligned} \tag{8.201}$$

We estimate each of the four right-hand side terms separately. We start with the first term. Using the pointwise estimates of Lemma 8.4.2 and the integrability properties of v_ε (cf. Proposition 8.3.2), we find

$$\begin{aligned} &\int_{\mathbb{R}^2} a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot(\Gamma_\varepsilon - \tilde\Gamma_\varepsilon)\operatorname{curl}\mathrm{v}_\varepsilon \\ &\quad\lesssim N_\varepsilon\|\Gamma_\varepsilon - \tilde\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\Big(\int_{\tilde{\mathcal{B}}^t_\varepsilon}\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + \Big(\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^t_\varepsilon}\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2}\Big) \\ &\qquad + N_\varepsilon\|\Gamma_\varepsilon - \tilde\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\Big(\int_{\mathbb{R}^2}\chi_R|1-|u_\varepsilon|^2||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + N_\varepsilon\int_{\mathbb{R}^2}\chi_R|1-|u_\varepsilon|^2||\operatorname{curl}\mathrm{v}_\varepsilon|\Big), \end{aligned}$$

and hence, using (8.184) and Proposition 8.5.2(v)–(vi) with $p = 1$ to estimate the first two integrals in the right-hand side, and using Lemma 8.5.3 in the form $\|\Gamma^t_\varepsilon - \tilde\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty} \lesssim_t \tilde r_\varepsilon \ll 1$,

$$\int_{\mathbb{R}^2} a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot(\Gamma_\varepsilon - \tilde\Gamma_\varepsilon)\operatorname{curl}\mathrm{v}_\varepsilon \lesssim N_\varepsilon^2\|\Gamma_\varepsilon - \tilde\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \ll_t N_\varepsilon^2.$$

For the second right-hand side term in (8.201), using (8.184) and again Lemma 8.5.3, with $\tilde r_\varepsilon\lambda_\varepsilon \ll N_\varepsilon|\log\varepsilon|^{-1}$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} \lambda_\varepsilon a\chi_R(\Gamma_\varepsilon - \tilde\Gamma_\varepsilon)^\perp\cdot\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle \\ &\quad\lesssim \lambda_\varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2}\|\Gamma_\varepsilon - \tilde\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\Big(\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2\Big)^{1/2} \\ &\quad\lesssim o(N_\varepsilon^2) + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2. \end{aligned}$$

For the third term in (8.201), using (8.184), (8.43), and Lemma 8.5.3 in the form $\|(\tilde\Gamma_\varepsilon-\Gamma_\varepsilon)^\perp\cdot\nabla h\|_{\mathrm{L}^\infty}\lesssim_t \tilde r_\varepsilon\lambda_\varepsilon\ll N_\varepsilon|\log\varepsilon|^{-1}$, we find

$$\int_{\mathbb{R}^2}\frac{a\chi_R}{2}(\tilde\Gamma_\varepsilon-\Gamma_\varepsilon)^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ll_t N_\varepsilon^2.$$

It remains to estimate the fourth term in (8.201). Using (8.184), Proposition 8.5.2(iii) in the form (8.131) with $\gamma=1/2$, the regularity properties of v_ε (cf. Proposition 8.3.2), (8.43) in the form $\|F\|_{C^{1/2}}\lesssim\lambda_\varepsilon$, and Lemma 8.5.3 in the form $\|\tilde\Gamma_\varepsilon-\Gamma_\varepsilon\|_{C^{1/2}}\lesssim_t\tilde r_\varepsilon^{1/2}=(\lambda_\varepsilon|\log\varepsilon|)^{-1}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}a\chi_R(\tilde\Gamma_\varepsilon-\Gamma_\varepsilon)\cdot(N_\varepsilon\mathrm{v}_\varepsilon+|\log\varepsilon|F^\perp/2)\mu_\varepsilon&\lesssim N_\varepsilon\|a\chi_R(\tilde\Gamma_\varepsilon-\Gamma_\varepsilon)\cdot(N_\varepsilon\mathrm{v}_\varepsilon+|\log\varepsilon|F^\perp/2)\|_{C^{1/2}}\\&\lesssim N_\varepsilon(N_\varepsilon+\lambda_\varepsilon|\log\varepsilon|)\|\tilde\Gamma_\varepsilon-\Gamma_\varepsilon\|_{C^{1/2}}\ll_t N_\varepsilon^2.\end{aligned}$$

Inserting these various estimates into (8.201) leads to

$$I^g_{\varepsilon,R}\lesssim_t o(N_\varepsilon^2)+o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2,$$

proving that (8.190) again holds in the present setting. We turn to the discussion of $I^S_{\varepsilon,R}$. Since the total radius satisfies $\tilde r_\varepsilon\ge e^{-o(N_\varepsilon)}$, we may apply Proposition 8.5.2(v), so that the same argument as in Step 3 leads to the estimate (8.192) for $I^S_{\varepsilon,R}$. It remains to discuss the estimation of the term $I^V_{\varepsilon,R}$. In the regime $1\ll N_\varepsilon\lesssim\log|\log\varepsilon|$, the assumption on λ_ε leads to $\lambda_\varepsilon\lesssim e^{o(N_\varepsilon)}/|\log\varepsilon|\ll N_\varepsilon/\log|\log\varepsilon|$, that is, $N_\varepsilon/(\lambda_\varepsilon\log|\log\varepsilon|)\gg1$. Writing $I^V_{\varepsilon,R}$ as in (8.195), we may thus apply Lemma 8.5.4 with the choice

$$M_\varepsilon:=\exp\bigg(\Big(\frac{N_\varepsilon}{\lambda_\varepsilon\log|\log\varepsilon|}\Big)^{1/2}\log|\log\varepsilon|\bigg),$$

and hence, for any $\Lambda\simeq1$, noting that $\lambda_\varepsilon\frac{\log M_\varepsilon}{|\log\varepsilon|}=\frac{1}{|\log\varepsilon|}(N_\varepsilon\lambda_\varepsilon\log|\log\varepsilon|)^{1/2}=o(\frac{N_\varepsilon}{|\log\varepsilon|})$,

$$\begin{aligned}&\Big|\int_0^t I^V_{\varepsilon,R}\Big|=\lambda_\varepsilon\alpha|\log\varepsilon|\Big|\int_0^t\int_{\mathbb{R}^2}\frac{a\chi_R}{2}\tilde V_\varepsilon\cdot\Gamma_\varepsilon\Big|\\&\le o_t(1)+\alpha\Big(\lambda_\varepsilon+o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\Big)\Big(\frac{1}{\Lambda}\int_0^t\int_{\mathbb{R}^2}a\chi_R|\partial_tu_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2+\frac{\Lambda}{4}\int_0^t\int_{\mathbb{R}^2}a\chi_R|(\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2\Big).\end{aligned}$$

Further using the optimal energy bound (8.184), the estimate (8.196) follows. With these ingredients at hand, we may now repeat the argument in Steps 2–3 and again conclude with (8.185). Finally, the convergence $N_\varepsilon^{-1}j_\varepsilon-\mathrm{v}_\varepsilon\to0$ follows as in Step 4, with $\mathcal{B}_\varepsilon$ replaced by $\tilde{\mathcal{B}}_\varepsilon$. ☐

8.7 Mean-field limit in the Gross-Pitaevskii case

In this section, we prove Theorem 8.1.4, that is, the mean-field limit result in the Gross-Pitaevskii case ($\alpha=0$, $\beta=1$) in the regime (GP). More precisely, the rescaled supercurrent density $N_\varepsilon^{-1}j_\varepsilon$ is shown to remain close to the solution v_ε of equation (8.53). Combining this with the results of Section 8.3.2 (in particular, with Lemma 8.3.5), the result of Theorem 8.1.4 follows.

8.7.1 Preliminary: vorticity estimate

In the present context, it is not required to adapt the vortex-balls construction and the localized lower bound of Section 8.5 to the regime $N_\varepsilon\gg|\log\varepsilon|$: we only need the following elementary estimate on the number of vortices based on a bound on the modulated energy excess. Since the vector field ∇h

is assumed here to decay at infinity, the proof is considerably reduced. We deduce in particular that in the considered regime $N_\varepsilon \gg |\log \varepsilon|$ the modulated energy $\mathcal{E}_{\varepsilon,R}$ and the excess $\mathcal{D}_{\varepsilon,R}$ are interchangeable up to an error of order $o(N_\varepsilon^2)$. (Note that in the absence of pinning and forcing no cut-off is needed and the corresponding property is completely trivial: the excess is then indeed simply defined by $\mathcal{D}_\varepsilon = \mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|$, cf. [395].)

Lemma 8.7.1. *Let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, with $a \le 1$ and $\|\nabla h\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim 1$, let $u_\varepsilon : \mathbb{R}^2 \to \mathbb{C}$, $\mathrm{v}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$, with $\|\mathrm{curl}\,\mathrm{v}_\varepsilon\|_{\mathrm{L}^1 \cap \mathrm{L}^\infty} \lesssim 1$ and $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty} \lesssim 1$. Assume that $0 < \varepsilon \ll 1$, $|\log \varepsilon| \ll N_\varepsilon \lesssim \varepsilon^{-1}$, $R \ge 1$, and assume that the modulated energy excess satisfies $\mathcal{D}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$. Then,*

$$\sup_z \|\mu_\varepsilon\|_{(\dot H^1 \cap W^{1,\infty}(B_R(z)))^*} \lesssim N_\varepsilon,$$

hence in particular

$$\sup_z |\mathcal{E}^z_{\varepsilon,R} - \mathcal{D}^z_{\varepsilon,R}| \lesssim N_\varepsilon |\log \varepsilon| \ll N_\varepsilon^2. \qquad \lozenge$$

Proof. Let $\phi \in \dot H^1 \cap W^{1,\infty}(\mathbb{R}^2)$ be supported in a ball of radius R. We decompose

$$\int_{\mathbb{R}^2} \phi \mu_\varepsilon = \int_{\mathbb{R}^2} \phi\big(N_\varepsilon \mathrm{curl}\,\mathrm{v}_\varepsilon + \mathrm{curl}\,(j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon)\big) = N_\varepsilon \int_{\mathbb{R}^2} \phi \,\mathrm{curl}\,\mathrm{v}_\varepsilon - \int_{\mathbb{R}^2} \nabla^\perp \phi \cdot (j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon),$$

hence, using the pointwise estimates of Lemma 8.4.2,

$$\int_{\mathbb{R}^2} \phi \mu_\varepsilon \lesssim N_\varepsilon \|\phi\|_{\mathrm{L}^\infty} + (\mathcal{E}^*_{\varepsilon,R})^{1/2} \|\nabla \phi\|_{\mathrm{L}^2} + \varepsilon \mathcal{E}^*_{\varepsilon,R} \|\nabla \phi\|_{\mathrm{L}^\infty}. \tag{8.202}$$

In particular, using the assumptions $\mathcal{D}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$ and $\|\nabla h\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim 1$, we obtain

$$\mathcal{E}^z_{\varepsilon,R} = \mathcal{D}^z_{\varepsilon,R} + |\log \varepsilon| \int_{\mathbb{R}^2} a \chi_R^z \mu_\varepsilon \lesssim N_\varepsilon^2 + |\log \varepsilon| (\mathcal{E}^*_{\varepsilon,R})^{1/2} + \varepsilon \mathcal{E}^*_{\varepsilon,R},$$

which implies, taking the supremum in z and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side, for $\varepsilon > 0$ small enough,

$$\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon^2 + (1 + \varepsilon N_\varepsilon)^2 |\log \varepsilon|^2 \lesssim N_\varepsilon^2.$$

Inserting this into (8.202) yields $\int_{\mathbb{R}^2} \phi \mu_\varepsilon \lesssim N_\varepsilon \|\phi\|_{\dot H^1 \cap W^{1,\infty}}$, and the result follows. ☐

8.7.2 Modulated energy argument

We now show that the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon$ remains close to the solution v_ε of equation (8.53). The proof consists in estimating the terms in the decomposition of the time derivative of the modulated energy excess given by Lemma 8.4.4, and then deducing the smallness of the modulated energy by a Grönwall argument. Note that in the present regime $N_\varepsilon \gg |\log \varepsilon|$ the situation is greatly simplified with respect to Section 8.6, since Lemma 8.7.1 above ensures that the modulated energy $\mathcal{E}_{\varepsilon,R}$ and the excess $\mathcal{D}_{\varepsilon,R}$ are now interchangeable up to an error $o(N_\varepsilon^2)$: the different terms appearing in Lemma 8.4.4 thus only need to be estimated by means of the modulated energy $\mathcal{E}_{\varepsilon,R}$ without having to take care to substract the correct vortex self-interaction energy. In particular, the vector field Γ_ε does no longer need to be truncated on small balls around the vortex locations, and we simply set $\bar\Gamma_\varepsilon = \Gamma_\varepsilon$. For this choice, all the terms involving the vortex velocity $\tilde V_{\varepsilon,\varrho}$ in Lemma 8.4.4 are easily seen to vanish. This simplification is crucial since in the present conservative case no good a priori control on the vortex velocity is available (apart from rough a priori estimates of the form $\|\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\|_{\mathrm{L}^2} \lesssim \varepsilon^{-2}$), which indeed prevents us from adapting this modulated energy argument to the case $N_\varepsilon \lesssim |\log \varepsilon|$.

Proposition 8.7.2. *Let $\alpha = 0$, $\beta = 1$, and let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy* (8.44). *Let $u_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{C}$ and $\mathrm{v}_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{R}^2$ be solutions of* (8.6) *and* (8.53) *as in Proposition 8.2.2(ii) and in Proposition 8.3.4, respectively, for some $T > 0$. Let $0 < \varepsilon \ll 1$, $|\log \varepsilon| \ll N_\varepsilon \ll \varepsilon^{-1}$, $R \gtrsim \|\partial_t u_\varepsilon\|_{\mathrm{L}^\infty_T \mathrm{L}^2} + |\log \varepsilon|^2$, and assume that the initial modulated energy satisfies $\mathcal{E}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$. Then, in the regime* (GP), *we have $\mathcal{E}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \in [0,T)$, and in particular $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ holds in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$. Under the stronger assumption $\mathcal{E}^{*,\circ}_\varepsilon \ll N_\varepsilon^2$, the same convergence holds in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); (\mathrm{L}^1 + \mathrm{L}^2)(\mathbb{R}^2)^2)$.* $\Diamond$

Proof. In the sequel, we choose $1 \ll \varrho \le R$ with $\varrho^{\theta_0} \ll (\varepsilon N_\varepsilon)^{-1}$ for some $\theta_0 > 0$. Regarding the global truncation at the scale R, it is not really needed in the present context (as a consequence of the decay assumption on the fields $\nabla h, F, f$), and can be sent to infinity arbitrarily fast; here it suffices to choose $R \ge \sup_{t \in [0,T)} \|\partial_t u_\varepsilon\|_{\mathrm{L}^2} + |\log \varepsilon|^2$ (where the right-hand side is indeed finite by Proposition 8.2.2(ii)). Given the assumption $\mathcal{E}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$ on the initial data, for all $\varepsilon > 0$, we define $T_\varepsilon > 0$ as the maximum time $\le T$ such that $\mathcal{E}^{*,t}_{\varepsilon,R} \le N_\varepsilon^2$ holds for all $t \le T_\varepsilon$. By Lemmas 8.4.1 and 8.7.1, we deduce $\hat{\mathcal{D}}^{*,\circ}_{\varepsilon,\varrho,R} \ll N_\varepsilon^2$ and for all $t \le T_\varepsilon$,

$$\mathcal{D}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon^2, \quad \hat{\mathcal{E}}^{*,t}_{\varepsilon,\varrho,R} \lesssim_t N_\varepsilon^2, \quad \hat{\mathcal{D}}^{*,t}_{\varepsilon,\varrho,R} \lesssim_t N_\varepsilon^2, \quad \mathcal{E}^{*,t}_{\varepsilon,R} \lesssim \hat{\mathcal{E}}^{*,t}_{\varepsilon,\varrho,R} + o_t(N_\varepsilon^2), \quad \hat{\mathcal{E}}^{*,t}_{\varepsilon,\varrho,R} \lesssim \hat{\mathcal{D}}^{*,t}_{\varepsilon,\varrho,R} + o_t(N_\varepsilon^2). \tag{8.203}$$

The strategy of the proof now consists in showing that for all $t \le T_\varepsilon$,

$$\hat{\mathcal{E}}^{*,t}_{\varepsilon,\varrho,R} \lesssim_t o(N_\varepsilon^2) + \int_0^t \hat{\mathcal{E}}^*_{\varepsilon,\varrho,R}. \tag{8.204}$$

This estimate is proved in Step 1 below. To simplify notation, we focus on (8.204) with the left-hand side $\hat{\mathcal{E}}^t_{\varepsilon,\varrho,R}$ centered at $z = 0$, but the result of course holds uniformly with respect to the translation. By the Grönwall inequality, it implies $\hat{\mathcal{E}}^{*,t}_{\varepsilon,\varrho,R} \ll_t N_\varepsilon^2$, hence $\mathcal{E}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \le T_\varepsilon$. This yields in particular $T_\varepsilon = T$ for all $\varepsilon > 0$ small enough, and the main conclusion follows, while the additional stated consequences are deduced in Step 2.

Step 1. Proof of (8.204).

Using the constraint $0 = a^{-1} \operatorname{div}(a \mathrm{v}_\varepsilon) = \operatorname{div} \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon \cdot \nabla h$, and choosing $\bar{\Gamma}_\varepsilon := \Gamma_\varepsilon$, the result of Lemma 8.4.4 takes the following simpler form,

$$\partial_t \hat{\mathcal{D}}_{\varepsilon,\varrho,R} = I^S_{\varepsilon,\varrho,R} + I^V_{\varepsilon,\varrho,R} + I^E_{\varepsilon,\varrho,R} + I^H_{\varepsilon,\varrho,R} + I^n_{\varepsilon,\varrho,R} + I'_{\varepsilon,\varrho,R}, \tag{8.205}$$

where we have set

$$\begin{aligned}
I^S_{\varepsilon,\varrho,R} &:= -\int_{\mathbb{R}^2} \chi_R \nabla \Gamma_\varepsilon^\perp : \tilde{S}_\varepsilon, \\
I^V_{\varepsilon,\varrho,R} &:= \int_{\mathbb{R}^2} \frac{a \chi_R |\log \varepsilon|}{2} \tilde{V}_{\varepsilon,\varrho} \cdot \Big(-\lambda_\varepsilon \Gamma_\varepsilon^\perp + \nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log \varepsilon|} \mathrm{v}_\varepsilon \Big), \\
I^E_{\varepsilon,\varrho,R} &:= -\int_{\mathbb{R}^2} \frac{a \chi_R |\log \varepsilon|}{2} \Gamma_\varepsilon \cdot \Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log \varepsilon|} \mathrm{v}_\varepsilon \Big) \mu_\varepsilon, \\
I^H_{\varepsilon,\varrho,R} &:= \int_{\mathbb{R}^2} \frac{a \chi_R}{2} \Gamma_\varepsilon^\perp \cdot \nabla h \Big(|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2} (1 - |u_\varepsilon|^2)^2 - |\log \varepsilon| \mu_\varepsilon \Big), \\
I^n_{\varepsilon,\varrho,R} &:= -\int_{\mathbb{R}^2} \nabla \chi_R \cdot \tilde{S}_\varepsilon \cdot \Gamma_\varepsilon^\perp - \int_{\mathbb{R}^2} a \nabla \chi_R \cdot \Big(\langle \partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon, \nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon \rangle + \frac{|\log \varepsilon|}{2} \tilde{V}_{\varepsilon,\varrho}^\perp \Big),
\end{aligned}$$

and where the error $I'_{\varepsilon,\varrho,R}$ is estimated as follows (cf. (8.111)),

$$|I'_{\varepsilon,\varrho,R}| \lesssim_{t,\theta} \varepsilon N_\varepsilon \mathcal{E}^*_{\varepsilon,R} + N_\varepsilon (\mathcal{E}^*_{\varepsilon,R})^{1/2} \|\nabla (\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho})\|_{\mathrm{L}^2} + \varepsilon N_\varepsilon^2 \varrho^\theta (\mathcal{E}^*_{\varepsilon,R})^{1/2}.$$

Choosing $\theta > 0$ small enough, and using Proposition 8.3.4 in the form $\|\nabla(\mathrm{p}_\varepsilon^t - \mathrm{p}_{\varepsilon,\varrho}^t)\|_{\mathrm{L}^2} \ll_t 1$ (cf. (8.72)), we obtain

$$|I'_{\varepsilon,\varrho,R}| \lesssim_{t,\theta} \mathcal{E}^*_{\varepsilon,R} + o(N_\varepsilon)(\mathcal{E}^*_{\varepsilon,R})^{1/2}. \tag{8.206}$$

The choice (8.53) for Γ_ε gives $I^V_{\varepsilon,\varrho,R} = I^E_{\varepsilon,\varrho,R} = 0$, hence

$$\partial_t \hat{\mathcal{D}}_{\varepsilon,\varrho,R} = I^S_{\varepsilon,\varrho,R} + I^H_{\varepsilon,\varrho,R} + I^n_{\varepsilon,\varrho,R} + I'_{\varepsilon,\varrho,R}. \tag{8.207}$$

It remains to estimate the first three right-hand side terms. By (8.44) in the form $\|f\|_{\mathrm{L}^2} \lesssim N_\varepsilon^2$, and by the integrability properties of v_ε (cf. Proposition 8.3.4), the first right-hand side term $I^S_{\varepsilon,\varrho,R}$ is estimated as follows, for all $t \le T_\varepsilon$,

$$\begin{aligned} I^S_{\varepsilon,\varrho,R} &\lesssim \|\nabla\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^2} \chi_R \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + |f|)\Big) \\ &\lesssim_t \mathcal{E}_{\varepsilon,R} + \varepsilon N_\varepsilon^2 (\mathcal{E}_{\varepsilon,R})^{1/2} \lesssim \mathcal{E}_{\varepsilon,R} + o(N_\varepsilon^2). \end{aligned} \tag{8.208}$$

We turn to the second right-hand side term in (8.207). Lemma 8.7.1 yields

$$\begin{aligned} I^H_{\varepsilon,\varrho,R} &\le \|\Gamma_\varepsilon^\perp \cdot \nabla h\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^2} \chi_R \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 \Big) + |\log\varepsilon| \bigg| \int_{\mathbb{R}^2} a\chi_R \Gamma_\varepsilon^\perp \cdot \nabla h \, \mu_\varepsilon \bigg| \\ &\lesssim \mathcal{E}_{\varepsilon,R} \|\Gamma_\varepsilon^\perp \cdot \nabla h\|_{\mathrm{L}^\infty} + N_\varepsilon |\log\varepsilon| \|a\chi_R \Gamma_\varepsilon^\perp \cdot \nabla h\|_{\dot{H}^1 \cap W^{1,\infty}}, \end{aligned}$$

and hence, using (8.44) and the properties of v_ε (cf. Proposition 8.3.4),

$$I^H_{\varepsilon,\varrho,R} \lesssim_t \mathcal{E}_{\varepsilon,R} + N_\varepsilon |\log\varepsilon| \le \mathcal{E}_{\varepsilon,R} + o(N_\varepsilon^2). \tag{8.209}$$

It remains to estimate the third right-hand side term in (8.207). Arguing as above, we find

$$\begin{aligned} I^n_{\varepsilon,\varrho,R} &\lesssim R^{-1}\|\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \int_{B_{2R}} \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + |f|)\Big) \\ &\qquad + R^{-1}|\log\varepsilon| \int_{B_{2R}} |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| \\ &\lesssim_t \mathcal{E}^*_{\varepsilon,R} + o(N_\varepsilon^2) + R^{-1}|\log\varepsilon|(\mathcal{E}^*_{\varepsilon,R})^{1/2} \|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\|_{\mathrm{L}^2(B_{2R})}. \end{aligned}$$

The properties of p_ε (cf. Proposition 8.3.4) yield for all $\theta > 0$,

$$\begin{aligned} \|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\|_{\mathrm{L}^2(B_{2R})} &\lesssim \|\partial_t u_\varepsilon\|_{\mathrm{L}^2(B_{2R})} + N_\varepsilon \|\mathrm{p}_{\varepsilon,\varrho}\|_{\mathrm{L}^2(B_{2R})} + N_\varepsilon \|\mathrm{p}_{\varepsilon,\varrho}\|_{\mathrm{L}^\infty(B_{2R})} \|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})} \\ &\lesssim_{t,\theta} \|\partial_t u_\varepsilon\|_{\mathrm{L}^2(B_{2R})} + N_\varepsilon \varrho^\theta + \varepsilon N_\varepsilon (\mathcal{E}^*_{\varepsilon,R})^{1/2}, \end{aligned}$$

so that the above takes the form

$$I^n_{\varepsilon,\varrho,R} \lesssim_{t,\theta} \mathcal{E}^*_{\varepsilon,R} + R^{-2}|\log\varepsilon|^2 \|\partial_t u_\varepsilon\|^2_{\mathrm{L}^2(B_{2R})} + R^{-2(1-\theta)} N_\varepsilon^2 |\log\varepsilon|^2 + o(N_\varepsilon^2).$$

Using the choice $R \gtrsim \|\partial_t u_\varepsilon\|_{\mathrm{L}^2} + |\log\varepsilon|^2$, and choosing $\theta > 0$ small enough, we deduce $I^n_{\varepsilon,\varrho,R} \lesssim_t \mathcal{E}^*_{\varepsilon,R} + o(N_\varepsilon^2)$. Combining this with (8.206), (8.207), (8.208), and (8.209), we conclude

$$\partial_t \hat{\mathcal{D}}_{\varepsilon,\varrho,R} \lesssim_t \mathcal{E}^*_{\varepsilon,R} + o(N_\varepsilon^2).$$

Integrating this in time with $\hat{\mathcal{D}}^{*,\circ}_{\varepsilon,\varrho,R} \ll N_\varepsilon^2$, using (8.203), and noting that the same result holds uniformly with respect to translations of the cut-off functions, the conclusion (8.204) follows.

Step 2. Conclusion.

As explained, the result of Step 1 implies $T_\varepsilon = T$ and $\mathcal{E}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \in [0, T)$. We now show that it implies the convergence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$. Using the pointwise estimates of Lemma 8.4.2, we obtain

$$\|j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon\|_{(\mathrm{L}^1+\mathrm{L}^2)(B_R(z))} \lesssim \|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_R(z))}\big(1+\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_R(z))}\big) + N_\varepsilon \|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_R(z))}$$
$$\ll_t N_\varepsilon(1+\varepsilon N_\varepsilon) \lesssim N_\varepsilon,$$

and the conclusion follows, letting $R \uparrow \infty$. □

8.8 Mean-field limit in the superdense parabolic case

In this section, we turn to the dissipative case ($\alpha > 0$) in the superdense regime (GL$_3$), and we prove Theorem 8.1.3. More precisely, we make use of the modulated energy strategy and show that the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon$ remains close to the solution v_ε of equation (8.52). Combining this with the convergence results of Section 8.3.3, the statement of Theorem 8.1.3 follows.

Regarding the modulated energy argument, note that the proof of Proposition 8.6.1 indicates that we expect to find in this superdense regime, with the corresponding notation,

$$\hat{\mathcal{D}}^t_{\varepsilon,R} \le o_t(N_\varepsilon^2) + C_t(1+\alpha\lambda_\varepsilon)\int_0^t \hat{\mathcal{D}}_{\varepsilon,R}. \tag{8.210}$$

As $\lambda_\varepsilon \gg 1$, the Grönwall inequality does of course not allow us to conclude that $\hat{\mathcal{D}}^t_{\varepsilon,R} \ll_t N_\varepsilon^2$ for any $t > 0$. (Note that in the Gross-Pitaevskii case $\alpha = 0$ the prefactor λ_ε formally disappears in (8.210), and this is indeed the situation successfully treated in Section 8.7.) The strategy in the sequel consists in refining as much as possible the magnitude of the error $o(N_\varepsilon^2)$ in (8.210), and showing that it can be reduced to $O(N_\varepsilon^{2-\delta})$ for some $\delta > 0$. With $\lambda_\varepsilon = N_\varepsilon/|\log\varepsilon| \gg 1$, the Grönwall inequality then indeed leads to $\hat{\mathcal{D}}^t_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \ge 0$ in the regime $|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon| \log|\log\varepsilon|$. Since in Chapter 7 (see also Section 8.3.3) the well-posedness of the degenerate limiting equation (8.52) could only be established in the parabolic case ($\alpha = 1$, $\beta = 0$), we have to restrict to that case here.

8.8.1 Preliminary results: vortex analysis

We adapt crucial vortex analysis estimates of Section 8.5 to the present situation with a large number of vortices $N_\varepsilon \gg |\log\varepsilon|$. We start with establishing the following version of Proposition 8.5.2.

Proposition 8.8.1 (Refined lower bound)**.** *Let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, with $1 \lesssim a \le 1$ and $\|\nabla h\|_{\mathrm{L}^\infty} \lesssim 1$, let $u_\varepsilon : \mathbb{R}^2 \to \mathbb{C}$, $\mathrm{v}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$, with $\|\mathrm{curl}\, \mathrm{v}_\varepsilon\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}, \|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty} \lesssim 1$. Let $0 < \varepsilon \ll 1$, $N_\varepsilon \gtrsim |\log\varepsilon|$, and $R \ge 1$ with $\log N_\varepsilon \ll |\log\varepsilon|$ and $|\log\varepsilon| \lesssim R \lesssim |\log\varepsilon|^n$ for some $n \ge 1$, and assume that $\mathcal{D}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$. Then $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$ holds for all $\varepsilon > 0$ small enough. Moreover, for some $\bar r \simeq 1$, for all $\varepsilon > 0$ small enough and all $r \in (\varepsilon^{1/2}, \bar r)$, letting $\mathcal{B}^r_{\varepsilon,R}$ and $\nu^r_{\varepsilon,R}$ denote the locally finite union of disjoint closed balls and the point-vortex measure constructed in Lemma 8.5.1, the following properties hold,*

(i) Lower bound: *in the regime $N_\varepsilon \gg \log|\log\varepsilon|$, we have for all $\varepsilon^{1/2} < r < \bar r$ and all $z \in \mathbb{R}^2$,*

$$\frac12 \int_{\mathcal{B}^r_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)$$
$$\ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi^z_R|\nu^r_{\varepsilon,R}| - O\Big(rN_\varepsilon^2 + \frac{N_\varepsilon^2}{|\log\varepsilon|}(|\log r| + \log N_\varepsilon)\Big); \tag{8.211}$$

(ii) Number of vortices: *for* $\varepsilon^{1/2} < r \ll 1$,

$$\sup_z \int_{B_R(z)} |\nu^r_{\varepsilon,R}| \lesssim \frac{N_\varepsilon^2}{|\log \varepsilon|}; \tag{8.212}$$

(iii) Jacobian estimate: *for* $\varepsilon^{1/2} < r \ll 1$, *for all* $\gamma \in [0,1]$,

$$\sup_z \|\nu^r_{\varepsilon,R} - \tilde\mu_\varepsilon\|_{(C_c^\gamma(B_R(z)))^*} \lesssim r^\gamma \frac{N_\varepsilon^2}{|\log \varepsilon|} + \varepsilon^{\gamma/2} N_\varepsilon^2, \tag{8.213}$$

$$\sup_z \|\mu_\varepsilon - \tilde\mu_\varepsilon\|_{(C_c^\gamma(B_R(z)))^*} \lesssim \varepsilon^\gamma N_\varepsilon^2 |\log \varepsilon|^n; \tag{8.214}$$

(iv) Excess energy estimate: *for all* $\phi \in W^{1,\infty}(\mathbb{R}^2)$ *supported in a ball of radius* R,

$$\int_{\mathbb{R}^2} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log \varepsilon| \mu_\varepsilon\Big) \lesssim \Big(\mathcal{D}^*_{\varepsilon,R} + \frac{N_\varepsilon^2}{|\log \varepsilon|} \log N_\varepsilon\Big) \|\phi\|_{W^{1,\infty}}; \tag{8.215}$$

(v) Energy outside small balls: *for all* $\gamma \ge 1$, *all* $N_\varepsilon^{-\gamma} \le r < \bar r$, *and all* $z \in \mathbb{R}^2$,

$$\int_{\mathbb{R}^2 \setminus \mathcal{B}^r_{\varepsilon,R}} a\chi_R^z \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \le \mathcal{D}^z_{\varepsilon,R} + O_\gamma\Big(\frac{N_\varepsilon^2}{|\log \varepsilon|} \log N_\varepsilon\Big). \tag{8.216}$$

$\Diamond$

Proof. We split the proof into six steps. The main work consists in checking that the assumptions imply the optimal bound on the energy $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$. This main conclusion is obtained in Step 5, while the various other statements are deduced in Step 6.

Step 1. Rough a priori estimate on the energy.

In this step, we prove $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon^2 + R^2 |\log \varepsilon|^2$, and hence by the choice of R we deduce $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon^2 + |\log \varepsilon|^m$ for some $m \ge 4$.

Decomposing $\mu_\varepsilon = N_\varepsilon \mathrm{curl}\, \mathrm{v}_\varepsilon + \mathrm{curl}\,(j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon)$, the assumption $\mathcal{D}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$ yields for all $z \in \mathbb{R}^2$,

$$\begin{aligned} \mathcal{E}^z_{\varepsilon,R} &\le \mathcal{D}^*_{\varepsilon,R} + \frac{|\log \varepsilon|}{2} \int_{\mathbb{R}^2} a\chi_R^z \mu_\varepsilon \\ &\lesssim N_\varepsilon^2 + N_\varepsilon |\log \varepsilon| \int_{\mathbb{R}^2} a\chi_R^z |\mathrm{curl}\, \mathrm{v}_\varepsilon| + |\log \varepsilon| \int_{\mathbb{R}^2} |\nabla(a\chi_R^z)| |j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon|. \end{aligned} \tag{8.217}$$

Using the pointwise estimate of Lemma 8.4.2 for $j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon$, using $|\nabla(a\chi_R^z)| \lesssim \mathbb{1}_{B_{2R}(z)}$, $\|\mathrm{curl}\, \mathrm{v}_\varepsilon\|_{\mathrm{L}^1} \lesssim 1$, and $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty} \lesssim 1$, we obtain

$$\begin{aligned} \mathcal{E}^z_{\varepsilon,R} &\lesssim N_\varepsilon^2 + |\log \varepsilon| \Big(\int_{B_{2R}(z)} (1-|u_\varepsilon|^2)^2\Big)^{1/2} \Big(\int_{B_{2R}(z)} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2} \\ &\qquad + R|\log \varepsilon| \Big(\int_{B_{2R}(z)} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2} + RN_\varepsilon |\log \varepsilon| \Big(\int_{B_{2R}(z)} (1-|u_\varepsilon|^2)^2\Big)^{1/2} \\ &\lesssim N_\varepsilon^2 + \varepsilon |\log \varepsilon| \mathcal{E}^*_{\varepsilon,R} + R|\log \varepsilon| (\mathcal{E}^*_{\varepsilon,R})^{1/2}. \end{aligned}$$

Taking the supremum over z, and absorbing $\mathcal{E}^*_{\varepsilon,R}$ into the left-hand side, the result follows.

Step 2. Applying Lemma 8.5.1.

By assumption $\log N_\varepsilon \ll |\log \varepsilon|$, the result of Step 1 yields in particular $\log \mathcal{E}^*_{\varepsilon,R} \ll |\log \varepsilon|$, which allows to apply Lemma 8.5.1. For fixed $r \in (\varepsilon^{1/2}, \bar r)$, let $\mathcal{B}^r_{\varepsilon,R} = \biguplus_j B^j$ denote the union of disjoint closed balls given by Lemma 8.5.1, and let $\nu^r_{\varepsilon,R}$ denote the associated point-vortex measure. Using Lemma 8.5.1(ii) in the form

$$\int_{B_R(z)} |\nu^r_{\varepsilon,R}| = \sum_{j:y_j\in B_R(z)} |d_j| \lesssim \frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}, \tag{8.218}$$

Lemma 8.5.1(i) gives, for all $\phi\in W^{1,\infty}(\mathbb{R}^2)$ with $\phi\ge 0$, if ϕ is supported in a ball of radius R,

$$\begin{aligned}\frac12\int_{\mathcal{B}^r_{\varepsilon,R}} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}\phi|\nu^r_{\varepsilon,R}| - O(r\mathcal{E}^*_{\varepsilon,R})\|\nabla\phi\|_{\mathrm{L}^\infty}\\ - O\bigg(r^2N_\varepsilon^2+|\log r|\frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}+\frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg)\|\phi\|_{\mathrm{L}^\infty}. \end{aligned}\tag{8.219}$$

Arguing as in Step 2 of the proof of Proposition 8.5.2, we then find for all z,

$$\begin{aligned}\int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\ \le \mathcal{D}^z_{\varepsilon,R}+O\bigg(1+(|\log r|+r|\log\varepsilon|)\frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}+\frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg),\end{aligned}\tag{8.220}$$

and in addition,

$$\Big|\int_{\mathbb{R}^2}\phi(\mu_\varepsilon-\nu^r_{\varepsilon,R})\Big| \lesssim \Big(r\frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}+\varepsilon^{1/3}\Big)\|\phi\|_{W^{1,\infty}}, \tag{8.221}$$

$$\Big|\int_{\mathbb{R}^2}\phi(\tilde\mu_\varepsilon-\mu_\varepsilon)\Big| \lesssim \varepsilon RN_\varepsilon(\mathcal{E}^*_{\varepsilon,R})^{1/2}\|\phi\|_{W^{1,\infty}} \lesssim \varepsilon^{1/3}\|\phi\|_{W^{1,\infty}}. \tag{8.222}$$

Step 3. Energy and number of vortices.

In this step, we show that (8.218) is essentially an equality, in the sense that for all $\varepsilon^{1/2}<r\ll 1$,

$$\sup_z\int_{\mathbb{R}^2}\chi_R^z|\nu^r_{\varepsilon,R}| \lesssim \frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} \lesssim \frac{N_\varepsilon^2}{|\log\varepsilon|}+\sup_z\int_{\mathbb{R}^2}\chi_R^z|\nu^r_{\varepsilon,R}|. \tag{8.223}$$

The lower bound already follows from (8.218). We now turn to the upper bound. Since the energy excess satisfies $\mathcal{D}^z_{\varepsilon,R}\lesssim N_\varepsilon^2$, we deduce from (8.221),

$$\mathcal{E}^z_{\varepsilon,R} \le \mathcal{D}^z_{\varepsilon,R}+\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi_R^z\mu_\varepsilon \le \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi_R^z\nu^r_{\varepsilon,R}+O\big(N_\varepsilon^2+r\mathcal{E}^*_{\varepsilon,R}\big). \tag{8.224}$$

Taking the supremum in z, and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side with $r\ll 1$, the upper bound in (8.223) follows.

Step 4. Estimate on the total variation of the vorticity.

In this step, we prove that for all $e^{-o(|\log\varepsilon|)}<r\ll 1$,

$$\sup_z\int_{\mathbb{R}^2}\chi_R^z|\nu^r_{\varepsilon,R}| \le (1+o(1))\sup_z\int_{\mathbb{R}^2}\chi_R^z\nu^r_{\varepsilon,R}+O\Big(\frac{N_\varepsilon^2}{|\log\varepsilon|}\Big). \tag{8.225}$$

The lower bound (8.219) of Step 2 with $\phi=a\chi_R^y$ yields for all $y\in\mathbb{R}^2$, using the upper bound in (8.223) to replace the energy $\mathcal{E}^*_{\varepsilon,R}$ in the error terms,

$$\begin{aligned}\mathcal{E}^y_{\varepsilon,R}\ge\frac12\int_{\mathcal{B}^r_{\varepsilon,R}}a\chi_R^y\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ge\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi_R^y|\nu^r_{\varepsilon,R}|\\ -O\Big(\frac{N_\varepsilon^2}{|\log\varepsilon|}+\sup_z\int_{\mathbb{R}^2}\chi_R^z|\nu^r_{\varepsilon,R}|\Big)\bigg(|\log r|+r|\log\varepsilon|+\log\Big(2+\frac{N_\varepsilon^2+\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg).\end{aligned}$$

For $e^{-o(|\log\varepsilon|)} < r \ll 1$, using the result of Step 1 in the form $\log(N_\varepsilon^2 + \mathcal{E}_{\varepsilon,R}^*) \ll |\log\varepsilon|$, we obtain for all $y \in \mathbb{R}^2$,

$$\mathcal{E}_{\varepsilon,R}^y \geq \frac{|\log\varepsilon|}{2} \int_{\mathbb{R}^2} a\chi_R^y |\nu_{\varepsilon,R}^r| - o(|\log\varepsilon|) \sup_z \int_{\mathbb{R}^2} \chi_R^z |\nu_{\varepsilon,R}^r| - o(N_\varepsilon^2). \tag{8.226}$$

On the other hand, the upper bound (8.224) yields

$$\mathcal{E}_{\varepsilon,R}^y \leq \frac{|\log\varepsilon|}{2} \int_{\mathbb{R}^2} a\chi_R^y \nu_{\varepsilon,R}^r + O(N_\varepsilon^2) + o(1)\mathcal{E}_{\varepsilon,R}^*, \tag{8.227}$$

and thus, taking the supremum over y and absorbing $\mathcal{E}_{\varepsilon,R}^*$ in the left-hand side,

$$\mathcal{E}_{\varepsilon,R}^* \leq \frac{|\log\varepsilon|}{2} \sup_z \int_{\mathbb{R}^2} a\chi_R^z |\nu_{\varepsilon,R}^r| + O(N_\varepsilon^2),$$

so that (8.227) takes the form, for all $y \in \mathbb{R}^2$,

$$\mathcal{E}_{\varepsilon,R}^y \leq \frac{|\log\varepsilon|}{2} \int_{\mathbb{R}^2} a\chi_R^y \nu_{\varepsilon,R}^r + O(N_\varepsilon^2) + o(|\log\varepsilon|) \sup_z \int_{\mathbb{R}^2} \chi_R^z |\nu_{\varepsilon,R}^r|.$$

Combining this with (8.226), dividing both sides by $\frac{1}{2}|\log\varepsilon|$, and taking the supremum over y, we find

$$\sup_z \int_{\mathbb{R}^2} \chi_R^z (\nu_{\varepsilon,R}^r)^- \lesssim \sup_z \int_{\mathbb{R}^2} a\chi_R^z (|\nu_{\varepsilon,R}^r| - \nu_{\varepsilon,R}^r) \leq O\Big(\frac{N_\varepsilon^2}{|\log\varepsilon|}\Big) + o(1) \sup_z \int_{\mathbb{R}^2} \chi_R^z |\nu_{\varepsilon,R}^r|,$$

hence

$$\begin{aligned} \sup_z \int_{\mathbb{R}^2} \chi_R^z |\nu_{\varepsilon,R}^r| &= \sup_z \int_{\mathbb{R}^2} \chi_R^z (\nu_{\varepsilon,R}^r + 2(\nu_{\varepsilon,R}^r)^-) \\ &\leq \sup_z \int_{\mathbb{R}^2} \chi_R^z \nu_{\varepsilon,R}^r + O\Big(\frac{N_\varepsilon^2}{|\log\varepsilon|}\Big) + o(1) \sup_z \int_{\mathbb{R}^2} \chi_R^z |\nu_{\varepsilon,R}^r|, \end{aligned}$$

and the result (8.146) follows after absorbing in the left-hand side the last right-hand side term.

Step 5. Refined bound on the energy.

In this step, we prove the optimal energy bound $\mathcal{E}_{\varepsilon,R}^* \lesssim N_\varepsilon^2$. By (8.218) this yields in particular $\sup_z \int_{\mathbb{R}^2} \chi_R^z |\nu_{\varepsilon,R}^r| \lesssim N_\varepsilon^2/|\log\varepsilon|$.

Let $e^{-o(|\log\varepsilon|)} < r \ll 1$ be suitably chosen later. Using (8.221), the bound on the energy excess $\mathcal{D}_{\varepsilon,R}^* \lesssim N_\varepsilon^2$ yields for all $z \in R\mathbb{Z}^2$,

$$\mathcal{E}_{\varepsilon,R}^z \leq \mathcal{D}_{\varepsilon,R}^z + \frac{|\log\varepsilon|}{2} \int_{\mathbb{R}^2} a\chi_R^z \mu_\varepsilon \lesssim N_\varepsilon^2 + r\mathcal{E}_{\varepsilon,R}^* + |\log\varepsilon| \int_{\mathbb{R}^2} \chi_R^z |\nu_{\varepsilon,R}^r|,$$

and hence, using the result (8.225) of Step 4, and absorbing $\mathcal{E}_{\varepsilon,R}^*$ in the left-hand side with $r \ll 1$,

$$\mathcal{E}_{\varepsilon,R}^* \lesssim N_\varepsilon^2 + |\log\varepsilon| \sup_z \int_{\mathbb{R}^2} \chi_R^z \nu_{\varepsilon,R}^r \lesssim N_\varepsilon^2 + |\log\varepsilon| \sup_z \int_{\mathbb{R}^2} \chi_R^z \mu_\varepsilon. \tag{8.228}$$

It remains to estimate $\int_{\mathbb{R}^2} \chi_R^z \mu_\varepsilon$. Arguing as in Step 5 of the proof of Proposition 8.5.2, we find

$$\int_{\mathbb{R}^2} \chi_R^z \mu_\varepsilon \lesssim N_\varepsilon + \Big(\int_{\mathbb{R}^2 \setminus \mathcal{B}_{\varepsilon,R}^r} \chi_{2R}^z |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2} + rR^{-1}\Big(\int_{B_{2R}(z)} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2}, \tag{8.229}$$

and then using (8.220) to estimate the first right-hand side term,

$$\int_{\mathbb{R}^2} \chi_R^z \mu_\varepsilon \lesssim N_\varepsilon + (\mathcal{D}^*_{\varepsilon,R})^{1/2} + rR^{-1}(\mathcal{E}^*_{\varepsilon,R})^{1/2} + r^{1/2}(N_\varepsilon^2 + \mathcal{E}^*_{\varepsilon,R})^{1/2} + \Big(\frac{N_\varepsilon^2 + \mathcal{E}^*_{\varepsilon,R}}{|\log \varepsilon|}\Big)^{1/2}\Big(|\log r| + \log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\Big)^{1/2}$$
$$\lesssim N_\varepsilon + r^{1/2}(\mathcal{E}^*_{\varepsilon,R})^{1/2} + o(1)\frac{N_\varepsilon^2 + \mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} + |\log r|^{1/2}\Big(\frac{N_\varepsilon^2 + \mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^{1/2}.$$

Combining this with (8.228) leads to

$$\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} \lesssim \frac{N_\varepsilon^2}{|\log\varepsilon|} + r^{1/2}(\mathcal{E}^*_{\varepsilon,R})^{1/2} + o(1)\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} + |\log r|^{1/2}\Big(\frac{N_\varepsilon^2 + \mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^{1/2},$$

and hence,

$$\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} \lesssim \frac{N_\varepsilon^2}{|\log\varepsilon|} + |\log r|.$$

The result then follows e.g. from the choice $r = |\log\varepsilon|^{-1}$.

Step 6. Conclusion.

The optimal energy bound $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$ is now proved. In the present step, we check that the remaining statements follow from this estimate. The result (8.211) follows from (8.219) in Step 2 with $\phi = a\chi_R^z$, combined with the optimal energy bound. The bound (8.212) on the number of vortices follows from the result (8.223) of Step 3 together with the optimal energy bound. For $r = N_\varepsilon^{-\gamma}$, $\gamma \ge 1$, the result (8.216) follows from (8.220) together with the optimal energy bound. Monotonicity of $\mathcal{B}^r_{\varepsilon,R}$ with respect to r then implies (8.216) for all $r \ge N_\varepsilon^{-\gamma}$. It remains to establish items (iii) and (iv). We split the proof into two further substeps.

Substep 6.1. Proof of (iii).

The Jacobian estimate (8.213) follows from Lemma 8.5.1(iii) together with the optimal energy bound, and the estimate (8.214) with $\gamma = 1$ similarly follows from (8.222) and from the bound $R \lesssim |\log\varepsilon|^n$. As in Substep 8.4 of the proof of Proposition 8.5.2, we further find for all $\phi \in \mathrm{L}^\infty(\mathbb{R}^2)$ supported in a ball $B_R(z)$, $z \in \mathbb{R}^2$,

$$\Big|\int_{\mathbb{R}^2} \phi(\tilde\mu_\varepsilon - \mu_\varepsilon)\Big|$$
$$\lesssim N_\varepsilon \|\phi\|_{\mathrm{L}^\infty} \int_{B_R(z)} \big(|1-|u_\varepsilon|^2||\mathrm{curl}\,\mathrm{v}_\varepsilon| + 2|\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + 2|\mathrm{v}_\varepsilon||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|\big)$$
$$\lesssim RN_\varepsilon^2 \|\phi\|_{\mathrm{L}^\infty}, \quad (8.230)$$

and the result (8.214) then follows by interpolation for all $\gamma \in [0,1]$.

Substep 6.2. Proof of (iv).

Let $\varepsilon^{1/2} < r \ll 1$ to be later optimized as a function of ε. Arguing as in Substep 8.5 of the proof of Proposition 8.5.2, we find for all $\phi \in W^{1,\infty}(\mathbb{R}^2)$ supported in the ball $B_R(z)$,

$$\int_{\mathbb{R}^2} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\nu^r_{\varepsilon,R}\Big)$$
$$\le \|a^{-1}\phi\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^2} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\nu^r_{\varepsilon,R}\Big)$$
$$+ O\Big(\frac{N_\varepsilon^2}{|\log\varepsilon|}(|\log r| + \log N_\varepsilon)\Big)\|a^{-1}\phi\|_{\mathrm{L}^\infty} + O(rN_\varepsilon^2)\|a^{-1}\phi\|_{W^{1,\infty}}.$$

Using (8.221) to replace $\nu^r_{\varepsilon,R}$ by μ_ε in both sides up to an error of order $(1+rN_\varepsilon^2)\|\phi\|_{W^{1,\infty}}$, and choosing e.g. $r=N_\varepsilon^{-1}$, the conclusion (8.215) follows. $\square$

We now establish the following version of the (suboptimal) a priori estimate of Lemma 8.5.5 on the velocity of the vortices in the case with a large number of vortices $N_\varepsilon \gg |\log\varepsilon|$.

Lemma 8.8.2 (A priori bound on velocity)**.** *Let $\alpha\ge 0$, $\beta\in\mathbb{R}$, and let $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, $F:\mathbb{R}^2\to\mathbb{R}^2$, $f:\mathbb{R}^2\to\mathbb{R}$ satisfy* (8.43)*. Let $u_\varepsilon:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{C}$ and $\mathrm{v}_\varepsilon:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{R}^2$ be the solutions of* (8.6) *and* (8.52) *as in Propositions 8.2.2(i) and 8.3.6, respectively. Let $0<\varepsilon\ll 1$, $|\log\varepsilon|\ll N_\varepsilon\lesssim\varepsilon^{-1}$, and $R\ge 1$ with $\varepsilon R\lesssim 1$, and assume that $\mathcal{E}^{*,t}_{\varepsilon,R}\lesssim_t N_\varepsilon^2$ for all $t\ge 0$. Then, in the regime* (GL$_3$)*, we have for all $\theta>0$ and all $t\ge 0$,*

$$\alpha^2\sup_z\int_0^t\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2\lesssim_{t,\theta}(1+\varepsilon RN_\varepsilon)N_\varepsilon|\log\varepsilon|+R^\theta N_\varepsilon^2|\log\varepsilon|^2\lesssim R^\theta N_\varepsilon^2|\log\varepsilon|^2. \qquad\Diamond$$

Proof. Set $D^{z,t}_{\varepsilon,R}:=\int_0^t\int_{\mathbb{R}^2}a\chi_R^z|\partial_t u_\varepsilon|^2$. From identity (8.173), noting that $|\nabla\chi_R^z|\lesssim R^{-1}(\chi_R^z)^{1/2}$, using the pointwise estimates of Lemma 8.4.2 for V_ε and $j_\varepsilon-N_\varepsilon\mathrm{v}_\varepsilon$, and using assumption (8.43), the bound (8.98) on $\psi^z_{\varepsilon,R}$, and the definition of $\hat{\mathcal{E}}^{z,t}_{\varepsilon,R}$, we find in the considered regime,

$$\begin{aligned}\lambda_\varepsilon\alpha D^{z,t}_{\varepsilon,R}\lesssim_{t,\theta} N_\varepsilon^2\big(1+\|\mathrm{v}_\varepsilon\|^2_{\mathrm{L}^\infty_t\mathrm{L}^4}\big)\big(1+\|\partial_t\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t(\mathrm{L}^2\cap\mathrm{L}^\infty(B_R))}\big)\\ +\varepsilon RN_\varepsilon^3\big(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}\big)\big(1+\|\Gamma_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}\big)+\varepsilon N_\varepsilon^2|\log\varepsilon|\|\mathrm{div}\,(a\mathrm{v}_\varepsilon)\|_{\mathrm{L}^\infty_t\mathrm{L}^2}\\ +N_\varepsilon^2\big(1+\|\mathrm{v}_\varepsilon\|^2_{\mathrm{L}^\infty_t(\mathrm{L}^2\cap\mathrm{L}^\infty(B_{2R}))}+\|\,\mathrm{div}\,(a\mathrm{v}_\varepsilon)\|_{\mathrm{L}^\infty_t(\mathrm{L}^2\cap\mathrm{L}^\infty)}\big)(D^{z,t}_{\varepsilon,R})^{1/2}+R^{-1}N_\varepsilon(D^{z,t}_{\varepsilon,R})^{1/2},\end{aligned}$$

and hence, using the properties of v_ε in (8.80), for any $\theta>0$,

$$\lambda_\varepsilon\alpha D^{z,t}_{\varepsilon,R}\lesssim_{t,\theta}N_\varepsilon^2+\varepsilon RN_\varepsilon^3+N_\varepsilon^2R^\theta(D^{z,t}_{\varepsilon,R})^{1/2}.$$

Absorbing $(D^{z,t}_{\varepsilon,R})^{1/2}$ in the left-hand side, and choosing $\theta>0$ small enough, the result follows. $\square$

We finally turn to the adaptation of the crucial a priori estimate of Lemma 8.5.6 to the case with a large number of vortices $N_\varepsilon\gg|\log\varepsilon|$.

Lemma 8.8.3. *Let $\alpha\ge 0$, $\beta\in\mathbb{R}$, and let $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, $F:\mathbb{R}^2\to\mathbb{R}^2$, $f:\mathbb{R}^2\to\mathbb{R}$ satisfy* (8.43)*. Let $u_\varepsilon:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{C}$ and $\mathrm{v}_\varepsilon:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{R}^2$ be the solutions of* (8.6) *and* (8.52) *as in Propositions 8.2.2(i) and 8.3.6, respectively. Let $0<\varepsilon\ll1$, $|\log\varepsilon|\ll N_\varepsilon\lesssim\varepsilon^{-1}$, and $R\ge1$ with $\varepsilon RN_\varepsilon^3\lesssim 1$, and assume that $\mathcal{E}^{*,t}_{\varepsilon,R}\lesssim_t N_\varepsilon^2$ for all $t\ge0$. Then, in the regime* (GL$_3$)*, we have for all $t\ge 0$,*

$$\alpha^2\sup_z\int_0^t\int_{\mathbb{R}^2}\frac{\chi_R^z}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\lesssim_t\frac{N_\varepsilon^2}{|\log\varepsilon|}. \qquad\Diamond$$

Proof. Using the pointwise estimates of Lemma 8.4.2, assumption (8.43), and the properties of v_ε in (8.80), Lemma 8.4.3 directly yields

$$\begin{aligned}|\,\mathrm{div}\,\tilde S_\varepsilon|\lesssim\big((\lambda_\varepsilon+\beta N_\varepsilon)|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|+\beta N_\varepsilon^2+\beta N_\varepsilon^2|1-|u_\varepsilon|^2|\big)\big(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}\big)|\partial_t u_\varepsilon|\\ +\big((\lambda_\varepsilon+\beta N_\varepsilon)N_\varepsilon\|\mathrm{p}_\varepsilon\|_{\mathrm{L}^\infty}+N_\varepsilon\|\mathrm{curl}\,\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}+N_\varepsilon^2\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}\big)(1+|1-|u_\varepsilon|^2|)|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|\\ +N_\varepsilon(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty})^3(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2+(1-|u_\varepsilon|^2)^2+N_\varepsilon^2)\\ +N_\varepsilon^2|1-|u_\varepsilon|^2|\big(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}\big)\big(N_\varepsilon(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty})^3+\lambda_\varepsilon\|\mathrm{p}_\varepsilon\|_{\mathrm{L}^\infty}+\|\mathrm{curl}\,\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}\big).\end{aligned}$$

Using the assumption $\mathcal{E}^{*,t}_{\varepsilon,R}\lesssim_t N_\varepsilon^2$, Lemma 8.8.2 with $R=1$, and the properties of v_ε in (8.80), we find for $r\le 1$,

$$\int_0^t\int_{B_r(x_0)}|\mathrm{div}\,\tilde S_\varepsilon|\lesssim_t N_\varepsilon^4|\log\varepsilon|(1+\beta|\log\varepsilon|)\lesssim N_\varepsilon^4|\log\varepsilon|^2.$$

Further noting that assumption (8.43) yields

$$\int_{B_r(x_0)} a|1-|u_\varepsilon|^2||f| \lesssim_t \varepsilon r N_\varepsilon \|f\|_{\mathrm{L}^\infty} \lesssim \varepsilon r N_\varepsilon^3,$$

and also

$$\begin{aligned}\int_{B_r(x_0)} |\nabla\chi_R||\tilde S_\varepsilon| &\lesssim R^{-1}\int_{B_r(x_0)} \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + \varepsilon^2(N_\varepsilon^4|\mathrm{v}_\varepsilon|^4 + |f|^2)\Big)\\ &\lesssim R^{-1}\big(N_\varepsilon^2 + \varepsilon^2(N_\varepsilon^4\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}^4 + \|f\|_{\mathrm{L}^\infty}^2)\big) \lesssim_t R^{-1}N_\varepsilon^2,\end{aligned}$$

and arguing as in Step 1 of the proof of Lemma 8.5.6, we deduce the following Pohozaev type estimate, adapted from [382, Theorem 5.1]: for any ball $B_r(x_0)$ with $r \le 1$, we have

$$\begin{aligned}\int_0^t \int_{B_r(x_0)} \frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 &\lesssim_t rN_\varepsilon^4|\log\varepsilon|^2\\ &+ r\int_0^t\int_{\partial B_r(x_0)} \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+|f|)\Big).\end{aligned}$$

The conclusion then follows from a direct adaptation of Steps 2–3 of the proof of Lemma 8.5.6. □

8.8.2 Modulated energy argument

With the above vortex analysis estimates at hand, we consider the superdense regime (GL$_3$) with $|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|\log|\log\varepsilon|$, and we adapt the modulated energy argument of Section 8.6 to show that the rescaled supercurrent density $N_\varepsilon^{-1}j_\varepsilon$ remains close to the solution v_ε of equation (8.52). Although the well-posedness result of Section 8.3.3 for equation (8.52) (hence the final statement of Theorem 8.1.3) is reduced to the parabolic case ($\alpha = 1$, $\beta = 0$), we show below that the modulated energy argument formally works in the mixed-flow case as well. (As we assume $\alpha > 0$, all multiplicative constants are implicitly allowed to additionally depend on an upper bound on α^{-1}.)

Proposition 8.8.4. *Let $\alpha > 0$, $\beta \in \mathbb{R}$, $\alpha^2+\beta^2 = 1$, and let $h:\mathbb{R}^2\to\mathbb{R}$, $a := e^h$, $F:\mathbb{R}^2\to\mathbb{R}^2$, $f:\mathbb{R}^2\to\mathbb{R}$ satisfy* (8.43)*. Let $u_\varepsilon:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{C}$ be the solution of* (8.6) *as in Proposition 8.2.2(i). Assume that for some $T>0$ for all $\varepsilon>0$ there exists a solution $\mathrm{v}_\varepsilon:[0,T)\times\mathbb{R}^2\to\mathbb{R}^2$ of the following mixed-flow version of* (8.52),

$$\begin{aligned}\partial_t \mathrm{v}_\varepsilon &= \nabla \mathrm{p}_\varepsilon + \Gamma_\varepsilon \mathrm{curl}\,\mathrm{v}_\varepsilon, \qquad \mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}^\circ, \\ \Gamma_\varepsilon &:= \lambda_\varepsilon^{-1}(\alpha - \mathbb{J}\beta)\Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big), \qquad \mathrm{p}_\varepsilon := (\lambda_\varepsilon\alpha a)^{-1}\,\mathrm{div}\,(a\mathrm{v}_\varepsilon),\end{aligned} \tag{8.231}$$

and assume that v_ε satisfies the estimates (8.80) *on $[0,T)$. Let $0<\varepsilon\ll 1$, $|\log\varepsilon| \lesssim N_\varepsilon \ll |\log\varepsilon|\log|\log\varepsilon|$, and $|\log\varepsilon| \lesssim R \lesssim |\log\varepsilon|^n$ for some $n\ge 1$. Assume that the initial modulated energy excess satisfies $\mathcal{D}_{\varepsilon,R}^{*,\circ} \lesssim N_\varepsilon^{2-\delta}$ for some $\delta>0$. Then we have $\mathcal{D}_{\varepsilon,R}^{*,t} \ll_t N_\varepsilon^2$ for all $t\in[0,T)$, hence in particular $N_\varepsilon^{-1}j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon\downarrow 0$.* ◊

Proof. Let $|\log\varepsilon| \lesssim N_\varepsilon \lesssim |\log\varepsilon|^n$ and $|\log\varepsilon| \lesssim R \lesssim |\log\varepsilon|^n$ for some $n\ge1$. Given the assumption $\mathcal{D}_{\varepsilon,R}^{*,\circ} \ll N_\varepsilon^2$ on the initial data, for all $\varepsilon>0$ we define $T_\varepsilon>0$ as the maximum time $\le T$ such that $\mathcal{D}_{\varepsilon,R}^{*,t} \le N_\varepsilon^2$ holds for all $t \le T_\varepsilon$. By the proof of Lemma 8.4.1 and by Proposition 8.8.1, we deduce for all $t\le T_\varepsilon$,

$$\mathcal{E}_{\varepsilon,R}^{*,t} \lesssim_t N_\varepsilon^2, \qquad \hat{\mathcal{E}}_{\varepsilon,R}^{*,t} \lesssim_t N_\varepsilon^2, \qquad \hat{\mathcal{D}}_{\varepsilon,R}^{*,t} \lesssim_t N_\varepsilon^2, \qquad \mathcal{D}_{\varepsilon,R}^{*,t} \lesssim \hat{\mathcal{D}}_{\varepsilon,R}^{*,t} + o_t(\varepsilon^{1/2}). \tag{8.232}$$

The strategy of the proof consists in showing that for all $t \le T_\varepsilon$,

$$\hat{\mathcal{D}}^{*,t}_{\varepsilon,R} \lesssim_t \hat{\mathcal{D}}^{*,\circ}_{\varepsilon,R} + N_\varepsilon \lambda_\varepsilon^3 \log|\log\varepsilon| + \lambda_\varepsilon N_\varepsilon \log N_\varepsilon + \lambda_\varepsilon \int_0^t \hat{\mathcal{D}}^*_{\varepsilon,R}. \tag{8.233}$$

Combined with (8.232) and with the Grönwall inequality, this implies

$$\mathcal{D}^{*,t}_{\varepsilon,R} \lesssim_t e^{C_t\lambda_\varepsilon}\big(\mathcal{D}^{*,\circ}_{\varepsilon,R} + N_\varepsilon \lambda_\varepsilon^3 \log|\log\varepsilon| + \lambda_\varepsilon N_\varepsilon \log N_\varepsilon\big).$$

Then choosing $|\log\varepsilon| \lesssim N_\varepsilon \ll |\log\varepsilon| \log|\log\varepsilon|$ and $\mathcal{D}^\circ_{\varepsilon,R} \lesssim N_\varepsilon^{2-\delta}$ for some $\delta > 0$, we deduce $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \le T_\varepsilon$. This gives in particular $T_\varepsilon = T$ for $\varepsilon > 0$ small enough, and the conclusion follows.

To simplify notation, we focus on (8.233) with the left-hand side $\hat{\mathcal{D}}^t_{\varepsilon,R}$ centered at $z = 0$, but the result of course holds uniformly with respect to the translation. Using the definition of the pressure in (8.51), the result of Lemma 8.4.4 yields

$$\partial_t \hat{\mathcal{D}}_{\varepsilon,R} = I^S_{\varepsilon,R} + I^V_{\varepsilon,R} + I^E_{\varepsilon,R} + I^D_{\varepsilon,R} + I^H_{\varepsilon,R} + I^d_{\varepsilon,R} + I^g_{\varepsilon,R} + I^n_{\varepsilon,R} + I'_{\varepsilon,R}, \tag{8.234}$$

where the eight first terms are as in the statement of Lemma 8.4.4 while the error $I'_{\varepsilon,R}$ is estimated as follows (cf. (8.110)),

$$\int_0^t |I'_{\varepsilon,\varrho,R}| \lesssim_t \varepsilon R(N_\varepsilon^2 + |\log\varepsilon|^2)(\mathcal{E}^*_{\varepsilon,R})^{1/2} \lesssim_t \varepsilon^{1/2}.$$

Let us first introduce some notation. For all $t \le T_\varepsilon$, as we are in the framework of Proposition 8.8.1 with $u^t_\varepsilon, \mathrm{v}^t_\varepsilon$, we let $\mathcal{B}^t_\varepsilon := \mathcal{B}^t_{\varepsilon,R}$ denote the constructed collection of disjoint closed balls $\mathcal{B}^{r_\varepsilon}_{\varepsilon,R}(u^t_\varepsilon, \mathrm{v}^t_\varepsilon)$ with total radius $r_\varepsilon := N_\varepsilon^{-4}$. Let then $\bar\Gamma^t_\varepsilon$ denote the corresponding approximation of Γ^t_ε given by Lemma 8.5.3. We decompose $\Gamma_\varepsilon := \alpha\Gamma_{\varepsilon,0} - \beta\Gamma^\perp_{\varepsilon,0}$ with

$$\Gamma_{\varepsilon,0} := \lambda_\varepsilon^{-1}\Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big).$$

Step 1. Estimating the error terms.

In this step, we prove for all $t \le T_\varepsilon$,

$$\int_0^t (I^d_{\varepsilon,R} + I^g_{\varepsilon,R} + I^n_{\varepsilon,R}) \lesssim_t 1 + R^{-1}N_\varepsilon^2 + (R^{-1} + N_\varepsilon^{-2}) \int_0^t \int_{\mathbb{R}^2} \chi_R |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \,\mathrm{p}_\varepsilon|^2. \tag{8.235}$$

We start with the estimation of $I^g_{\varepsilon,R}$. Using assumption (8.43) and the pointwise estimates of Lemma 8.4.2, we find

$$\begin{aligned}
|I^g_{\varepsilon,R}| \lesssim \|\Gamma_\varepsilon - \bar\Gamma_\varepsilon\|_{\mathrm{L}^\infty}(1 + \|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty})\bigg(& N_\varepsilon \int_{\mathbb{R}^2} \chi_R\big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + N_\varepsilon|1 - |u_\varepsilon|^2|\big)|\mathrm{curl}\,\mathrm{v}_\varepsilon| \\
&+ N_\varepsilon \int_{\mathbb{R}^2} \chi_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1 - |u_\varepsilon|^2)^2\Big) \\
&+ N_\varepsilon^3 \int_{\mathbb{R}^2} \chi_R(1 + |1 - |u_\varepsilon|^2|)|\mathrm{v}_\varepsilon|^2 + \lambda_\varepsilon \int_{\mathbb{R}^2} \chi_R |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| \\
&+ \beta N_\varepsilon \int_{\mathbb{R}^2} \chi_R |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|\big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + N_\varepsilon|1 - |u_\varepsilon|^2| + N_\varepsilon|\mathrm{v}_\varepsilon|\big)\bigg).
\end{aligned}$$

By Lemma 8.5.3 in the form $\|\Gamma_\varepsilon - \bar\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \lesssim r_\varepsilon = N_\varepsilon^{-4}$, and by the properties of v_ε in (8.80), we deduce for $\theta > 0$ small enough such that $R^\theta \lesssim |\log\varepsilon| \lesssim N_\varepsilon$,

$$\begin{aligned}
|I^g_{\varepsilon,R}| &\lesssim r_\varepsilon N_\varepsilon^3 R^\theta + r_\varepsilon(\lambda_\varepsilon N_\varepsilon + R^\theta N_\varepsilon^2)\Big(\int_{\mathbb{R}^2} \chi_R |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2\Big)^{1/2} \\
&\lesssim 1 + N_\varepsilon^{-1}\Big(\int_{\mathbb{R}^2} \chi_R |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2\Big)^{1/2} \lesssim 1 + N_\varepsilon^{-2} \int_{\mathbb{R}^2} \chi_R |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2.
\end{aligned} \tag{8.236}$$

We now turn to the estimation of $I^n_{\varepsilon,R}$. Using Lemma 8.8.2 and the properties of v_ε in (8.80), the quantity $\bar{\mathcal{E}}^*_{\varepsilon,R}$ defined in Lemma 8.5.4 is estimated as follows, for $\theta > 0$ small enough,

$$\begin{aligned}\bar{\mathcal{E}}^{*,t}_{\varepsilon,R} &\lesssim \sup_z \int_0^t \mathcal{E}^z_{\varepsilon,R} + \sup_z \int_0^t \int_{\mathbb{R}^2} \chi^z_R\big(|\partial_t u_\varepsilon|^2 + N_\varepsilon^2|\mathrm{p}_\varepsilon|^2 + N_\varepsilon^2|1-|u_\varepsilon|^2||\mathrm{p}_\varepsilon|^2\big)\\ &\lesssim_{t,\theta} N_\varepsilon^2 + (1+\varepsilon R N_\varepsilon)N_\varepsilon|\log\varepsilon| + R^\theta N_\varepsilon^2|\log\varepsilon|^2 + N_\varepsilon|\log\varepsilon|\|\mathrm{div}\,(a\mathrm{v}_\varepsilon)\|^2_{\mathrm{L}^\infty_t(\mathrm{L}^2\cap\mathrm{L}^\infty)}\\ &\lesssim_{t,\theta} \varepsilon R N_\varepsilon^2|\log\varepsilon| + R^\theta N_\varepsilon^2|\log\varepsilon|^2 \lesssim N_\varepsilon^2|\log\varepsilon|^3 \lesssim |\log\varepsilon|^{n+3}.\end{aligned}$$

Using the obvious estimate $|\nabla\chi_R| \lesssim R^{-1}\chi_R^{1/2}$, and using Lemma 8.5.3 in the form $\|\bar\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \lesssim \|\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \lesssim 1$, Lemma 8.5.4 then yields

$$\begin{aligned}\Big|\int_0^t\int_{\mathbb{R}^2} a\tilde V_\varepsilon\cdot\nabla^\perp\chi_R\Big| &\lesssim_t |\log\varepsilon|^{-1}\\ &\quad + R^{-1}|\log\varepsilon|^{-1}\bigg(\int_0^t\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon|^2 + \int_0^t\int_{B_{2R}}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\bigg),\end{aligned}$$

and hence,

$$\begin{aligned}\Big|\int_0^t I^n_{\varepsilon,R}\Big| &\lesssim_t 1 + R^{-1}\int_0^t\int_{B_{2R}}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+|f|)\Big)\\ &\qquad + R^{-1}\int_0^t\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon|^2.\end{aligned}$$

Using (8.232), assumption (8.43), and the properties of v_ε in (8.80), we conclude

$$\begin{aligned}\Big|\int_0^t I^n_{\varepsilon,R}\Big| &\lesssim_t 1 + R^{-1}N_\varepsilon^2 + \varepsilon N_\varepsilon^3\big(1+\|\mathrm{v}_\varepsilon\|^2_{\mathrm{L}^\infty_t\mathrm{L}^4}\big) + R^{-1}\int_0^t\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon|^2\\ &\lesssim_t 1 + R^{-1}N_\varepsilon^2 + R^{-1}\int_0^t\int_{\mathbb{R}^2}\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon|^2.\end{aligned}$$

Regarding the last term $I^d_{\varepsilon,R}$, the definition of the pressure in (8.231) simply yields $I^d_{\varepsilon,R}=0$, and the conclusion (8.235) follows.

Step 2. Estimating the dominant terms.

In this step, we turn to the estimation of the five first terms in (8.234), showing more precisely that for all $t \le T_\varepsilon$,

$$\hat{\mathcal{D}}^t_{\varepsilon,R} \lesssim_t \hat{\mathcal{D}}^\circ_{\varepsilon,R} + N_\varepsilon\lambda_\varepsilon^3\log|\log\varepsilon| + \lambda_\varepsilon N_\varepsilon\log N_\varepsilon + \lambda_\varepsilon\int_0^t \hat D_{\varepsilon,R}. \tag{8.237}$$

As this result obviously holds uniformly with respect to translations of the cut-off functions, the conclusion (8.233) follows.

We start with the estimation of the first term $I^S_{\varepsilon,R}$. Since for all t the field $\bar\Gamma^t_\varepsilon$ is by definition constant in each ball of the collection $\mathcal{B}^t_\varepsilon$ and satisfies $\|\nabla\bar\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty} \lesssim \|\nabla\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty} \lesssim 1$, we obtain

$$\begin{aligned}|I^S_{\varepsilon,R}| &\lesssim \int_{\mathbb{R}^2\setminus\mathcal{B}_\varepsilon}\chi_R|\tilde S_\varepsilon|\\ &\lesssim \int_{\mathbb{R}^2\setminus\mathcal{B}_\varepsilon} a\chi_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) + \int_{\mathbb{R}^2}\chi_R|1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+|f|).\end{aligned}$$

Since $\mathcal{B}_\varepsilon$ has total radius $r_\varepsilon = N_\varepsilon^{-4}$, Proposition 8.8.1(v) yields

$$|I^S_{\varepsilon,R}| \lesssim \mathcal{D}_{\varepsilon,R} + \lambda_\varepsilon N_\varepsilon \log N_\varepsilon + \int_{\mathbb{R}^2} \chi_R |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + |f|).$$

Further using (8.232), assumption (8.43), and the properties of v_ε in (8.80), we conclude

$$|I^S_{\varepsilon,R}| \lesssim \hat{\mathcal{D}}_{\varepsilon,R} + \lambda_\varepsilon N_\varepsilon \log N_\varepsilon. \tag{8.238}$$

We turn to $I^H_{\varepsilon,R}$. Using the assumption (8.43) and the properties of v_ε in (8.80), Lemma 8.8.3 yields

$$\int_0^t I^H_{\varepsilon,R} = O_t(\lambda_\varepsilon N_\varepsilon) + \int_0^t \int_{\mathbb{R}^2} \frac{a\chi_R}{2} \Gamma_\varepsilon^\perp \cdot \nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big),$$

and hence by Proposition 8.8.1(iv) and by (8.232),

$$\int_0^t I^H_{\varepsilon,R} \lesssim_t \lambda_\varepsilon N_\varepsilon \log N_\varepsilon + \int_0^t \mathcal{D}_{\varepsilon,R} \lesssim_t \lambda_\varepsilon N_\varepsilon \log N_\varepsilon + \int_0^t \hat{\mathcal{D}}_{\varepsilon,R}. \tag{8.239}$$

The term $I^D_{\varepsilon,R}$ is simply estimated by

$$I^D_{\varepsilon,R} \le -\frac{\lambda_\varepsilon \alpha}{2} \int_{\mathbb{R}^2} a\chi_R |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac{\lambda_\varepsilon\alpha}{2} \int_{\mathbb{R}^2} a\chi_R |(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot \Gamma_\varepsilon^\perp|^2. \tag{8.240}$$

We finally turn to $I^V_{\varepsilon,R}$. Using $\alpha^2+\beta^2=1$, we have by definition $\Gamma_{\varepsilon,0} - \beta\Gamma_\varepsilon^\perp = \alpha\Gamma_\varepsilon$, and hence $I^V_{\varepsilon,R}$ takes on the following guise,

$$I^V_{\varepsilon,R} = N_\varepsilon \int_{\mathbb{R}^2} \frac{a\chi_R}{2} \tilde{V}_\varepsilon \cdot (\Gamma_{\varepsilon,0} - \beta\Gamma_\varepsilon^\perp) = \alpha N_\varepsilon \int_{\mathbb{R}^2} \frac{a\chi_R}{2} \tilde{V}_\varepsilon \cdot \Gamma_\varepsilon.$$

As shown in Step 1, the quantity $\bar{\mathcal{E}}^*_{\varepsilon,R}$ defined in Lemma 8.5.4 satisfies $\bar{\mathcal{E}}^{*,t}_{\varepsilon,R} \lesssim_t |\log\varepsilon|^{n+3}$. Choosing e.g. $M_\varepsilon := \exp((\lambda_\varepsilon \log|\log\varepsilon|) \wedge |\log\varepsilon|^{1/2})$, Lemma 8.5.4 then yields for any $\Lambda \simeq 1$,

$$\begin{aligned}\Big|\int_0^t I^V_{\varepsilon,R}\Big| \le o_t(1) + \lambda_\varepsilon\alpha\Big(1 + O_t\Big(|\log\varepsilon|^{-1/2} \wedge \frac{\lambda_\varepsilon \log|\log\varepsilon|}{|\log\varepsilon|}\Big)\Big)& \\ \times\Big(\frac{1}{\Lambda}\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac{\Lambda}{4}\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2\Big)&,\end{aligned}$$

and thus, using the optimal energy bound (8.232),

$$\begin{aligned}\Big|\int_0^t I^V_{\varepsilon,R}\Big| \le\ & O_t\big(N_\varepsilon\lambda_\varepsilon^3 \log|\log\varepsilon|\big) \\ &+ \Big(1 + O_t\Big(|\log\varepsilon|^{-1/2} \wedge \frac{\lambda_\varepsilon\log|\log\varepsilon|}{|\log\varepsilon|}\Big)\Big)\frac{\lambda_\varepsilon\alpha}{\Lambda}\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2 \\ &+ \frac{\lambda_\varepsilon\alpha\Lambda}{4}\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2.\end{aligned} \tag{8.241}$$

We now distinguish between two cases:

$$\text{(Case 1)} \qquad \int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2 \le 5\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2, \tag{8.242}$$

$$\text{(Case 2)} \qquad \int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2 > 5\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2. \tag{8.243}$$

In Case 1, choosing $\Lambda = 2$ in (8.241) yields

$$\Big|\int_0^t I_{\varepsilon,R}^V\Big| \le O_t\big(N_\varepsilon \lambda_\varepsilon^3 \log|\log\varepsilon|\big) + \frac{\lambda_\varepsilon \alpha}{2}\Big(1 + O_t\Big(\frac{\lambda_\varepsilon \log|\log\varepsilon|}{|\log\varepsilon|}\Big)\Big)\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 \\ + \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2.$$

In Case 2, the condition (8.243) can be rewritten as

$$\frac{1}{4}\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2 \\ \le \Big(\frac14 + \frac1{10}\Big)\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac12\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2,$$

and choosing $\Lambda = 4$ in (8.241) then yields

$$\Big|\int_0^t I_{\varepsilon,R}^V\Big| \le O_t\big(N_\varepsilon \lambda_\varepsilon^3 \log|\log\varepsilon|\big) + \lambda_\varepsilon\alpha\Big(\frac14 + \frac1{10} + o_t(1)\Big)\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 \\ + \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int_{\mathbb{R}^2} a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2.$$

Further noting that in Case 1 the condition (8.242) together with the energy bound (8.232) yields

$$\Big(R^{-1} + N_\varepsilon^{-2} + \frac{\lambda_\varepsilon^2\log|\log\varepsilon|}{|\log\varepsilon|}\Big)\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 \\ \lesssim \Big(R^{-1} + N_\varepsilon^{-2} + \frac{\lambda_\varepsilon^2\log|\log\varepsilon|}{|\log\varepsilon|}\Big)\int_0^t\int_{\mathbb{R}^2} a\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \lesssim_t N_\varepsilon\lambda_\varepsilon^3\log|\log\varepsilon|,$$

and combining this with (8.235) and (8.240), we observe an exact recombination of the terms, and obtain in Case 1,

$$\int_0^t (I_{\varepsilon,R}^V + I_{\varepsilon,R}^D + I_{\varepsilon,R}^g + I_{\varepsilon,R}^n + I_{\varepsilon,R}') \\ \le \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int_{\mathbb{R}^2} a\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2|\Gamma_\varepsilon|^2 + O_t(N_\varepsilon\lambda_\varepsilon^3\log|\log\varepsilon|), \quad (8.244)$$

and in Case 2,

$$\int_0^t (I_{\varepsilon,R}^V + I_{\varepsilon,R}^D + I_{\varepsilon,R}^g + I_{\varepsilon,R}^n + I_{\varepsilon,R}') \le \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int_{\mathbb{R}^2} a\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2|\Gamma_\varepsilon|^2 + O_t(N_\varepsilon\lambda_\varepsilon^3\log|\log\varepsilon|) \\ - \frac{\lambda_\varepsilon\alpha}{2}\Big(\frac12 - \frac15 - o_t(1)\Big)\int_0^t\int_{\mathbb{R}^2} a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2,$$

so that (8.244) holds in both cases for $\varepsilon > 0$ small enough. Using $\alpha^2 + \beta^2 = 1$, we have by definition $\Gamma_\varepsilon \cdot \Gamma_{\varepsilon,0} = \alpha|\Gamma_{\varepsilon,0}|^2 = \alpha|\Gamma_\varepsilon|^2$, and hence the term $I_{\varepsilon,R}^E$ takes on the following guise, in terms of Γ_ε, $\Gamma_{\varepsilon,0}$,

$$I_{\varepsilon,R}^E = -\frac{\lambda_\varepsilon}{2}|\log\varepsilon|\int_{\mathbb{R}^2} a\chi_R\Gamma_\varepsilon\cdot\Gamma_{\varepsilon,0}\,\mu_\varepsilon = -\frac{\lambda_\varepsilon\alpha}{2}|\log\varepsilon|\int_{\mathbb{R}^2} a\chi_R|\Gamma_\varepsilon|^2\mu_\varepsilon.$$

Together with (8.244), this yields

$$\int_0^t (I^V_{\varepsilon,R} + I^E_{\varepsilon,R} + I^D_{\varepsilon,R} + I^g_{\varepsilon,R} + I^n_{\varepsilon,R} + I'_{\varepsilon,R}) \\ \le \frac{\lambda_\varepsilon \alpha}{2} \int_0^t \int_{\mathbb{R}^2} a\chi_R \big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 - |\log\varepsilon|\mu_\varepsilon\big)|\Gamma_\varepsilon|^2 + O_t(N_\varepsilon \lambda_\varepsilon^3 \log|\log\varepsilon|).$$

Combining this with (8.234), (8.238), and (8.239), we obtain

$$\hat{\mathcal{D}}^t_{\varepsilon,R} - \hat{\mathcal{D}}^\circ_{\varepsilon,R} \lesssim_t \int_0^t \hat{\mathcal{D}}_{\varepsilon,R} + \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int_{\mathbb{R}^2} a\chi_R\big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 - |\log\varepsilon|\mu_\varepsilon\big)|\Gamma_\varepsilon|^2 \\ + N_\varepsilon\lambda_\varepsilon^3 \log|\log\varepsilon| + \lambda_\varepsilon N_\varepsilon \log N_\varepsilon,$$

and the result (8.237) now follows from Proposition 8.8.1(iv).

Step 3. Conclusion.

As explained at the beginning of the proof, in the regime $|\log\varepsilon| \lesssim N_\varepsilon \ll |\log\varepsilon|\log|\log\varepsilon|$ with $\mathcal{D}^\circ_{\varepsilon,R} \lesssim N_\varepsilon^{2-\delta}$ for some $\delta > 0$, the estimate (8.233) implies $T_\varepsilon = T$ and $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \in [0,T)$. We now show that it implies the convergence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$. For all $t\in[0,T)$, Proposition 8.8.1(v) gives

$$\sup_z \int_{\mathbb{R}^2\setminus\mathcal{B}_\varepsilon} \chi_R^z |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \ll_t N_\varepsilon^2,$$

and for all $1\le p<2$,

$$\sup_z \int_{\mathcal{B}_\varepsilon} \chi_R^z |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim |\mathcal{B}_\varepsilon|^{1-p/2}(\mathcal{E}^*_{\varepsilon,R})^{p/2} \lesssim_t r_\varepsilon^{2-p} N_\varepsilon^p \ll_p N_\varepsilon^p.$$

Using the pointwise estimates of Lemma 8.4.2, we deduce

$$\sup_z \int_{B(z)} |j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon| \lesssim_t \sup_z \int_{B(z)} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + \varepsilon N_\varepsilon^2 \\ \lesssim_t \sup_z \int_{\mathcal{B}_\varepsilon} \chi_R^z |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + \sup_z \Big(\int_{B(z)\setminus\mathcal{B}_\varepsilon} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2} + \varepsilon N_\varepsilon^2 \ll_t N_\varepsilon,$$

hence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$. □

8.9 Small pin separation limit

In this section, we aim to combine the mean-field limit with the homogenization limit of a small pin separation $\eta_\varepsilon \downarrow 0$. Only partial results are obtained here for this double limit. We focus on the dissipative regimes (GL_1), (GL_2), (GL'_1), and (GL'_2), and for simplicity we restrict to the periodic setting, that is, $\hat{h}(x) = \eta_\varepsilon \hat{h}^0(x, x/\eta_\varepsilon)$ with $\hat{h}^0$ periodic in its second variable. (As we assume $\alpha > 0$, all multiplicative constants in this section are implicitly allowed to additionally depend on an upper bound on α^{-1}.)

8.9.1 Modulated energy argument

In this section, we adapt the result of Proposition 8.6.1 to the case with fast oscillating pinning. Since for simplicity we have not been looking for precise rates of convergence in Proposition 8.6.1 (that is, refinements of the $o(N_\varepsilon^2)$ error in (8.185)), we are only in position to treat inexplicit diagonal regimes $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$, for some suitable $\eta_{\varepsilon,0}$ depending on the data of the problem. Further refinements are left to the interested reader.

Proposition 8.9.1. *We consider the regimes* (GL_1), (GL_2), (GL_1'), *and* (GL_2') *with fast oscillating pinning potential* (8.27). *The solution* v_ε *of the corresponding limiting equation* (8.51) *exists up to time* $\eta_\varepsilon T$, *where* $T > 0$ *is as in Proposition 8.3.2. In particular, in the regimes* (GL_1) *and* (GL_2) *with* $\beta = 0$ *the time* T *can be chosen infinite, while in the regimes* (GL_1), (GL_1'), *and* (GL_2') *there exists some sequence* $\eta_{\varepsilon,0} \ll 1$ *(depending only on* ε *and* N_ε*) such that for all* $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ *the time* $\eta_\varepsilon T$ *can be chosen arbitrarily large for* $\varepsilon > 0$ *small enough.*
Moreover, there exists some exponent $\sigma > 0$ *and some increasing bijection* $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ *such that, if the initial modulated energy satisfies* $\mathcal{D}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$, *we have in the considered regimes, with the same restrictions as in Proposition 8.6.1, for all* $0 \le t < \eta_\varepsilon T$,

$$\sup_{0\le s\le t} \mathcal{D}^s_{\varepsilon,R} \le N_\varepsilon^2 \qquad \Longrightarrow \qquad \hat{\mathcal{D}}^t_{\varepsilon,R} \le \theta(t/\eta_\varepsilon)\Big(\eta_\varepsilon^{-\sigma} o(N_\varepsilon^2) + \eta_\varepsilon^{-1}\int_0^t \hat{\mathcal{D}}_{\varepsilon,R}\Big). \tag{8.245}$$

$\Diamond$

Proof. We adapt the proof of Proposition 8.6.1 to the present case with fast oscillating pinning. For that purpose we first need to check how the solution v_ε of the limiting equations (8.51) depends on the small parameter η_ε, thus adapting the result of Proposition 8.3.2. A scaling argument shows that the solution v_ε exists up to time $\eta_\varepsilon T$, where T is as in Proposition 8.3.2. Moreover, an inspection of the proofs in Chapter 7 together with a scaling argument shows that all the estimates in Proposition 8.3.2 still hold up to multiplicative constants of the form $\eta_\varepsilon^{-\sigma}\theta(t/\eta_\varepsilon)$, for all $0 \le t < \eta_\varepsilon T$, for some exponent $\sigma \ge 0$ and some increasing bijection $\theta : \mathbb{R}^+ \to \mathbb{R}^+$. (Of course this is but a rough estimate, but it is enough for our purposes here.) Note that a scaling argument yields more precisely for all $0 \le t < \eta_\varepsilon T$,

$$\|\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty} \le \theta(t/\eta_\varepsilon), \qquad \|\nabla\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty} \le \eta_\varepsilon^{-1}\theta(t/\eta_\varepsilon),$$

for some increasing bijection $\theta : \mathbb{R}^+ \to \mathbb{R}^+$. Repeating the proof of Proposition 8.6.1, but now taking into account this η_ε-dependence, the conclusion follows. $\square$

8.9.2 Local relaxation for slowed-down dynamics

The result of Proposition 8.9.1 a priori prevents us from applying a Grönwall argument beyond times of order η_ε. As the following shows, in this short timescale, in each (mesoscopic) periodicity cell, the vorticity gets projected onto the invariant measure for the cell dynamics associated with the initial vector field Γ°_ε (where Γ_ε is the vector field driving the limiting equation (8.51)). This initial-boundary layer is captured in the framework of 2-scale convergence. The proof of this short-time result is very easy since the non-linearity does not play any role in this timescale. In contrast, in the next sections, we give formal arguments that on larger timescales the effective vector field is given by the cell vector field projected onto the corresponding invariant measure (which is indeed in agreement with the present short-time result), but on such large timescales the nonlinearity truly enters into play and a rigorous justification is still missing.

Proposition 8.9.2. *Let Assumption 8.1.1(a) hold, with the initial data* $(u^\circ_\varepsilon, \mathrm{v}^\circ_\varepsilon, \mathrm{v}^\circ)$ *satisfying the well-preparedness condition* (8.16). *We consider the regimes* (GL_1), (GL_2), (GL_1'), *and* (GL_2') *with fast oscillating pinning potential* (8.27). *Let* $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}$ *be the solution of* (8.6) *as in Proposition 8.2.2(i). Let* $T > 0$ *denote the finite existence time given by Proposition 8.3.2 in the regime* (GL_2) *in the mixed-flow case* $\beta \neq 0$, *and set* $T := \infty$ *otherwise. Let also* $\hat{\mathrm{m}}_0$ *denote the unique solution of the following transport equation on* $\mathbb{R}^+ \times Q$, *for all* $x \in \mathbb{R}^2$,

$$\partial_t \hat{\mathrm{m}}_0(x,\cdot) = -\operatorname{div}_y\big(\Gamma^\circ(x,\cdot)^\perp \hat{\mathrm{m}}_0(x,\cdot)\big), \qquad \hat{\mathrm{m}}_0(x,\cdot)|_{t=0} = \operatorname{curl}\mathrm{v}^\circ(x), \tag{8.246}$$
$$\Gamma^\circ(x,y) := (\alpha - \mathbb{J}\beta)\big(\nabla_2^\perp \hat{h}^0(x,y) - \hat{F}(x)^\perp - 2\kappa\mathrm{v}^\circ(x)\big),$$

where $\kappa := 1$ in the regime (GL_1)*, $\kappa := \lambda$ in the regime* (GL_2)*, and $\kappa := 0$ in the regimes* (GL_1') *and* (GL_2')*. Then there exists $\eta_{\varepsilon,0} \ll 1$ (depending on all the data of the problem) such that for all $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ the rescaled vorticity $N_\varepsilon^{-1}\mu_\varepsilon^{\eta_\varepsilon t}$ 2-scale converges to $\hat{\mathrm{m}}_0^t$, that is, for all $\phi \in C_c^\infty([0,T) \times \mathbb{R}^2; C_{\mathrm{per}}^\infty(Q))$,*

$$\lim_{\varepsilon\downarrow 0} \iint_{\mathbb{R}^+\times\mathbb{R}^2} \phi(t,x,x/\eta_\varepsilon) N_\varepsilon^{-1}\mu_\varepsilon^{\eta_\varepsilon t}(x)dxdt = \iiint_{\mathbb{R}^+\times\mathbb{R}^2\times Q} \phi(t,x,y)\hat{\mathrm{m}}_0^t(x,y)dydxdt. \qquad \Diamond$$

Proof. Let $\mathrm{v}_\varepsilon : [0,\eta_\varepsilon T)\times\mathbb{R}^2 \to \mathbb{R}^2$ denote the solution of the limiting equations (8.51) with oscillating pinning (8.27), as given by Proposition 8.9.1. Now applying Proposition 8.9.1 in the form (8.245), and choosing a sequence $\eta_{\varepsilon,0} \ll 1$ going sufficiently slowly to 0 so that the error $\eta_{\varepsilon,0}^{-\sigma} o(N_\varepsilon^2)$ in (8.245) remains of order $o(N_\varepsilon^2)$, the Grönwall inequality implies for all $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ that $\mathcal{D}_{\varepsilon,R}^{*,\eta_\varepsilon t} \lesssim_t o(N_\varepsilon^2)$ holds for all $0 \le t < T$. Hence, arguing as in Step 5 of the proof of Proposition 8.6.1, we deduce $N_\varepsilon^{-1} j_\varepsilon^{\eta_\varepsilon t}(x) - \mathrm{v}_\varepsilon^{\eta_\varepsilon t}(x) \to 0$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$. It remains to determine the asymptotic behaviour of $\mathrm{v}_\varepsilon^{\eta_\varepsilon t}$. We split the proof into two steps.

Step 1. 2-scale convergence of $\operatorname{curl} \mathrm{v}_\varepsilon^{\eta_\varepsilon t}$.

Let $\hat{\mathrm{v}}_\varepsilon^t := \mathrm{v}_\varepsilon^{\eta_\varepsilon t}$ and $\hat{\mathrm{m}}_\varepsilon := \operatorname{curl} \hat{\mathrm{v}}_\varepsilon$. Taking the curl in both sides of (8.51), we deduce

$$\partial_t \hat{\mathrm{m}}_\varepsilon = -\eta_\varepsilon \operatorname{div}(\hat{\Gamma}_\varepsilon^\perp \hat{\mathrm{m}}_\varepsilon), \qquad \hat{\Gamma}_\varepsilon := \lambda_\varepsilon^{-1}(\alpha - \mathbb{J}\beta)\Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|}\hat{\mathrm{v}}_\varepsilon\Big), \qquad \hat{\mathrm{m}}_\varepsilon|_{t=0} = \operatorname{curl} \mathrm{v}_\varepsilon^\circ. \tag{8.247}$$

By Lemma 7.4.1(iii) in the dissipative case $\alpha > 0$, with $\|h\|_{W^{1,\infty}}$, $\|\lambda_\varepsilon^{-1}(\nabla^\perp h - F^\perp)\|_{\mathrm{L}^\infty}$, $\|\mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^\infty}$, $\|\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2} \lesssim 1$, we deduce that $\int_{\mathbb{R}^2} |\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ|^2 \lesssim t$ for all $t \in [0,\eta_\varepsilon T)$. On the other hand, Lemma 7.4.2 implies by scaling $\|\operatorname{curl} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_{t/\eta_\varepsilon} 1$. After time rescaling, this implies for all $t \in [0,T)$,

$$\int_{\mathbb{R}^2} |\hat{\mathrm{v}}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ|^2 \lesssim_t \eta_\varepsilon, \qquad \text{and} \qquad \|\hat{\mathrm{m}}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1. \tag{8.248}$$

Nguetseng's 2-scale compactness theorem [344, 11] (e.g. in the form of [172, Theorem 3.2]) then states that there exists $\tilde{\mathrm{m}}_0 \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2 \times Q))$ such that up to a subsequence $\hat{\mathrm{m}}_\varepsilon$ 2-scale converges to $\tilde{\mathrm{m}}_0$, in the sense that for all $\phi \in C_c^\infty([0,T) \times \mathbb{R}^2; C_{\mathrm{per}}^\infty(Q))$ we have

$$\lim_{\varepsilon\downarrow 0} \int_0^T \int_{\mathbb{R}^2} \phi(t,x,x/\eta_\varepsilon)\hat{\mathrm{m}}_\varepsilon^t(x)dxdt = \int_0^T \iint_{\mathbb{R}^2\times Q} \phi(t,x,y)\tilde{\mathrm{m}}_0^t(x,y)dydxdt.$$

Testing equation (8.247) with $\phi(t,x,x/\eta_\varepsilon)$, we find

$$\begin{aligned} -\int_{\mathbb{R}^2} \phi(0,x,x/\eta_\varepsilon)\operatorname{curl} \mathrm{v}^\circ(x)dx - \int_0^T \int_{\mathbb{R}^2} \partial_t\phi(t,x,x/\eta_\varepsilon)\hat{\mathrm{m}}_\varepsilon^t(x)dxdt \\ = \int_0^T \int_{\mathbb{R}^2} \hat{\mathrm{m}}_\varepsilon^t(x)(\eta_\varepsilon\nabla_1\phi(t,x,x/\eta_\varepsilon) + \nabla_2\phi(t,x,x/\eta_\varepsilon)) \cdot \hat{\Gamma}_\varepsilon^t(x)^\perp dxdt, \end{aligned}$$

and hence, passing to the limit $\varepsilon \downarrow 0$ along the subsequence, and noting that $\hat{\mathrm{v}}_\varepsilon \to \mathrm{v}^\circ$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2))$ (cf. (8.248)), we obtain in the considered regimes,

$$\begin{aligned} -\iint_{\mathbb{R}^2\times Q} \phi(0,x,y)\operatorname{curl} \mathrm{v}^\circ(x)dydx - \int_0^T \iint_{\mathbb{R}^2\times Q} \partial_t\phi(t,x,y)\tilde{\mathrm{m}}_0^t(x,y)dydxdt \\ = \int_0^T \iint_{\mathbb{R}^2\times Q} \tilde{\mathrm{m}}_0^t(x,y)\nabla_2\phi(t,x,y) \cdot \Gamma^\circ(x,y)^\perp dydxdt. \end{aligned}$$

This proves that $\tilde{\mathrm{m}}_0$ satisfies the weak formulation of the linear transport equation (8.246), and therefore coincides with its unique solution, $\tilde{\mathrm{m}}_0 = \hat{\mathrm{m}}_0$.

Step 2. Conclusion.

Let $\phi \in C_c^\infty([0,T) \times \mathbb{R}^2; C^\infty_{\mathrm{per}}(Q))$, with $\phi(t,x,y) = 0$ for $|x| > R_0$. Integration by parts yields

$$\begin{aligned}\Big| \int_0^T \int_{\mathbb{R}^2} \phi(t,x,x/\eta_\varepsilon) \operatorname{curl}(N_\varepsilon^{-1} j_\varepsilon^{\eta_\varepsilon t})(x) dx dt - \int_0^T \iint_{\mathbb{R}^2\times Q} \phi(t,x,y) \hat{\mathrm{m}}_0^t(x,y) dy dx dt \Big| \\ \le \eta_\varepsilon^{-1} \|\nabla_{1,2}\phi\|_{\mathrm{L}^\infty} \int_0^T \int_{B_{R_0}} |N_\varepsilon^{-1} j_\varepsilon^{\eta_\varepsilon t} - \hat{\mathrm{v}}_\varepsilon^t| \\ + \Big| \int_0^T \int_{\mathbb{R}^2} \phi(t,x,x/\eta_\varepsilon) \operatorname{curl} \hat{\mathrm{v}}_\varepsilon^t(x) dx dt - \int_0^T \iint_{\mathbb{R}^2\times Q} \phi(t,x,y) \hat{\mathrm{m}}_0^t(x,y) dy dx dt \Big|. \quad (8.249)\end{aligned}$$

By Step 1, the second right-hand side term goes to 0. It remains to estimate the first term. At the beginning of the proof, we have shown that $\int_0^T \int_{B_{R_0}} |N_\varepsilon^{-1} j_\varepsilon^{\eta_\varepsilon t} - \hat{\mathrm{v}}_\varepsilon^t| \to 0$ holds uniformly with respect to the choice of $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$. Now choosing $\eta_{\varepsilon,0} \ll 1$ going sufficiently slowly to 0, we conclude that for all $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ the first right-hand side term in (8.249) also goes to 0. $\square$

8.9.3 Homogenization diagonal result

Although the result of Proposition 8.9.1 a priori prevents us from applying a Grönwall argument beyond times of order η_ε, it is possible to find some perturbative diagonal regime where the conclusion holds for all times. (While this regime is still denoted below by $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ for some sequence $\eta_{\varepsilon,0} \ll 1$ going sufficiently slowly to 0, it should be emphasized that the sequence $\eta_{\varepsilon,0}$ needs here to be taken incomparably much larger than in Propositions 8.9.1 and 8.9.2.) In such a diagonal regime, the homogenization limit may simply be performed *after* the mean-field limit.

Corollary 8.9.3. *We consider the regimes* (GL_1), (GL_2), (GL_1'), *and* (GL_2') *with fast oscillating pinning potential* (8.27), *and in the regime* (GL_2) *we restrict to the parabolic case* $\beta = 0$. *Then there exists* $\eta_{\varepsilon,0} \ll 1$ *(depending on all the data of the problem) such that for all* $\eta_{\varepsilon,0} \ll \eta_\varepsilon \ll 1$ *the statement of Proposition 8.6.1 holds in each of the corresponding regimes.* $\Diamond$

Proof. Since the regime (GL_2) is excluded here in the mixed-flow case $\beta \ne 0$, Proposition 8.9.1 asserts that the solution v_ε of (8.51) with oscillating pinning exists up to time $\eta_\varepsilon T$, and that in addition for $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ with $\eta_{\varepsilon,0}$ going sufficiently slowly to 0 the time $\eta_\varepsilon T$ can be chosen arbitrarily large for $\varepsilon > 0$ small enough. Now given the assumption $\mathcal{D}_{\varepsilon,R}^{*,\circ} \ll N_\varepsilon^2$ on the initial data, for all $\varepsilon > 0$ we define $T_\varepsilon > 0$ as the maximum time such that $\mathcal{D}_{\varepsilon,R}^t \le N_\varepsilon^2$ holds for all $t \le T_\varepsilon$, so that Proposition 8.9.1 yields for all $0 \le t \le T_\varepsilon$,

$$\hat{\mathcal{D}}_{\varepsilon,R}^t \le \theta(t/\eta_\varepsilon)\Big(\eta_\varepsilon^{-\sigma} o(N_\varepsilon^2) + \eta_\varepsilon^{-1} \int_0^t \hat{\mathcal{D}}_{\varepsilon,R}\Big),$$

for some exponent $\sigma \ge 0$ and some increasing bijection $\theta : \mathbb{R}^+ \to \mathbb{R}^+$. Hence we find by the Grönwall inequality for all $0 \le t \le T_\varepsilon$,

$$\hat{\mathcal{D}}_{\varepsilon,R}^t \lesssim \theta\Big(\frac{t+1}{\eta_\varepsilon^\sigma}\Big) o(N_\varepsilon^2),$$

for some other exponent $\sigma \ge 1$ and some other increasing bijection $\theta : \mathbb{R}^+ \to \mathbb{R}^+$. Let the sequence $\eta_{\varepsilon,0}$ go sufficiently slowly to 0 so that

$$\theta^{-1}\Big(\frac{N_\varepsilon}{\sqrt{o(N_\varepsilon^2)}}\Big)^{-1/\sigma} \ll \eta_{\varepsilon,0} \ll 1.$$

Then for all $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ we deduce $\hat{\mathcal{D}}^t_{\varepsilon,R} \ll N_\varepsilon^2$ for all $t \ge 0$, and the conclusion follows as in Step 4 of the proof of Proposition 8.6.1. □

In this diagonal regime, the problem is thus reduced to the determination of the asymptotic behavior as $\varepsilon \downarrow 0$ of the solution v_ε of the limiting equation (8.51) with fast oscillating pinning potential (8.27). As the following shows, we may further replace v_ε by the solution $\bar{\mathrm{v}}_\varepsilon$ of the simpler corresponding equations in Lemma 8.3.3 with fast oscillating pinning potential. Determining the asymptotic behavior of $\bar{\mathrm{v}}_\varepsilon$ is then a homogenization problem; this is precisely the content of Corollary 8.1.5 as stated in the introduction. (Note that the correct choice of the diagonal regime $\eta_{\varepsilon,0} \ll 1$ could be made completely explicit here in terms of the rate of convergence of $N_\varepsilon/|\log \varepsilon|$ to its limit; this is however not made precise since we are anyway limited to some unclear diagonal regime when combining this result with Corollary 8.9.3.)

Corollary 8.9.4. *We consider the regimes* (GL_1), (GL_2), (GL_1'), *and* (GL_2') *with fast oscillating pinning potential* (8.27)*, and in the regime* (GL_2) *we restrict to the parabolic case* $\beta = 0$*. Let* v_ε *be the solution of* (8.51) *with fast oscillating pinning as in Proposition 8.9.1, and let* $\bar{\mathrm{v}}_\varepsilon$ *be the solution of the corresponding equation* (8.63)–(8.66) *in Lemma 8.3.3 with* $\nabla\hat{h}(x)$ *replaced by* $\nabla_2\hat{h}^0(x, x/\eta_\varepsilon)$*. Then there exists* $\eta_{\varepsilon,0} \ll 1$ *(depending on all the data of the problem) such that for all* $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ *the solutions* v_ε *and* $\bar{\mathrm{v}}_\varepsilon$ *exist on arbitrarily large time intervals as* $\varepsilon \downarrow 0$*, and the same convergence results hold as in Lemma 8.3.3 in the form* $\mathrm{v}_\varepsilon - \bar{\mathrm{v}}_\varepsilon \to 0$. ◊

Proof. This convergence result directly follows from the computations in the proof of Lemma 8.3.3, now taking into account the η_ε-dependence of v_ε and $\bar{\mathrm{v}}_\varepsilon$, and applying the Grönwall inequality in a diagonal regime as in the proof of Corollary 8.9.3. □

In the next sections 8.9.4–8.9.5, we examine the homogenization problems arising in the above result. Although the justification of the homogenization of the nonlinear equation arising in the critical regimes seems to be out of reach (in the dissipative case), the situation is much simpler in the subcritical regimes.

8.9.4 Critical regimes: formal asymptotics

In this section, we investigate the asymptotic behavior of the mean-field equations in the critical regimes (GL_1) and (GL_2) with fast oscillating pinning (8.27). In order to extract the effective equations that should rule the system in the limit $\eta_\varepsilon \downarrow 0$, we use a formal 2-scale expansion (see e.g. [50] for a general presentation) and justify Heuristics 8.1.7 as stated in the introduction. However, as emphasized in Remark 8.9.5 below, due to both the nonlinear nonlocal character of the mean-field equations and their instability as $\eta_\varepsilon \downarrow 0$, the rigorous justification of this homogenization limit seems to be a very difficult task, and is not pursued here. Regarding the interpretation of the formal limiting equations as a stick-slip model, we refer to the introduction (see Section 8.1.3).

Formal justification of Heuristics 8.1.7. We focus on the regime (GL_1), while the formal justification is easily adapted to the regime (GL_2). The only difference is that in the regime (GL_2) it is further needed to restrict to the parabolic case $\beta = 0$ in order to get global existence for the solution $\bar{\mathrm{v}}_\varepsilon$ of (8.64) with fast oscillating pinning, since otherwise the finite existence time would a priori shrink to 0 as $\eta_\varepsilon \downarrow 0$ (cf. Proposition 8.9.1). Let $\bar{\mathrm{v}}_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ denote the unique (global) smooth solution of (8.63) with $\nabla\hat{h}(x)$ replaced by $\nabla_2\hat{h}^0(x, x/\eta_\varepsilon)$,

$$\partial_t \bar{\mathrm{v}}_\varepsilon = \nabla \bar{\mathrm{p}}_\varepsilon + \bar{\Gamma}_\varepsilon^\perp \operatorname{curl} \bar{\mathrm{v}}_\varepsilon, \qquad \operatorname{div} \bar{\mathrm{v}}_\varepsilon = 0, \qquad \bar{\mathrm{v}}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ,$$
$$\bar{\Gamma}_\varepsilon := (\alpha - \mathbb{J}\beta)\big(\nabla_2\hat{h}^0(\cdot, \cdot/\eta_\varepsilon) - \hat{F} + 2\bar{\mathrm{v}}_\varepsilon^\perp\big),$$

with $\hat{h}^0$ and $\hat{F}$ independent of ε. Let us recall the more convenient vorticity formulation of this equation: the vorticity $\bar{\mathrm{m}}_\varepsilon := \operatorname{curl}\bar{\mathrm{v}}_\varepsilon$ satisfies

$$\partial_t \bar{\mathrm{m}}_\varepsilon = \operatorname{div}(\bar{\Gamma}_\varepsilon \bar{\mathrm{m}}_\varepsilon), \qquad \bar{\mathrm{v}}_\varepsilon = \nabla^\perp \bar{g}_\varepsilon, \qquad \triangle \bar{g}_\varepsilon = \bar{\mathrm{m}}_\varepsilon. \tag{8.250}$$

As a consequence of Lemmas 7.4.1(iii) and 7.4.2, we find $\|\bar{\mathrm{v}}_\varepsilon^t - \mathrm{v}^\circ\|_{\mathrm{L}^2} \lesssim_t 1$ and by scaling $\|\bar{\mathrm{m}}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_{t/\eta_\varepsilon} 1$. In order to obtain the effective equations satisfied by v_ε in the limit $\eta_\varepsilon \downarrow 0$, we use a formal 2-scale expansion: we assume that v_ε satisfies the following natural 2-scale Ansatz,

$$\begin{aligned} \bar{\mathrm{v}}_\varepsilon^t(x) &= \bar{\mathrm{v}}_0(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + \eta_\varepsilon \bar{\mathrm{v}}_1(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + O(\eta_\varepsilon^2), \\ \bar{\mathrm{m}}_\varepsilon^t(x) &= \bar{\mathrm{m}}_0(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + \eta_\varepsilon \bar{\mathrm{m}}_1(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + O(\eta_\varepsilon^2), \\ \bar{g}_\varepsilon^t(x) &= \bar{g}_0(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + \eta_\varepsilon \bar{g}_1(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + \eta_\varepsilon^2 \bar{g}_2(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + O(\eta_\varepsilon^3). \end{aligned} \tag{8.251}$$

We denote by (t, τ, x, y) the coordinates corresponding with $(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon)$. Injecting the above ansatz into equation (8.250), and formally identifying the powers of η_ε, we derive the following equations,

$$\begin{gathered} \partial_\tau \bar{\mathrm{m}}_0 = \operatorname{div}_y(\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0), \\ \partial_t \bar{\mathrm{m}}_0 + \partial_\tau \bar{\mathrm{m}}_1 = \operatorname{div}_x(\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0) + \operatorname{div}_y(\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_1) + \operatorname{div}_y(\Gamma^1[\bar{\mathrm{v}}_1]\bar{\mathrm{m}}_0), \\ \bar{\mathrm{v}}_0 = \nabla_x^\perp \bar{g}_0 + \nabla_y^\perp \bar{g}_1, \\ \nabla_y \bar{g}_0 = 0, \quad \triangle_y \bar{g}_1 = 0, \quad \triangle_x \bar{g}_0 + 2\nabla_x \cdot \nabla_y \bar{g}_1 + \triangle_y \bar{g}_2 = \bar{\mathrm{m}}_0, \end{gathered} \tag{8.252}$$

where for any vector field w we have defined for simplicity the following vector fields,

$$\begin{aligned} \Gamma^0[w] &:= (\alpha - \mathbb{J}\beta)(\nabla_2 \hat{h}^0 - \hat{F} + 2w^\perp), \\ \Gamma^1[w] &:= 2(\alpha - \mathbb{J}\beta) w^\perp. \end{aligned}$$

The first two equations in the last line of (8.252) imply that both $\bar{g}_0$ and $\bar{g}_1$ are independent of the y-variable. The third equation in (8.252) then ensures that $\bar{\mathrm{v}}_0 = \nabla_x^\perp \bar{g}_0$ is also independent of the y-variable. Averaging both the first and the last equations on the periodicity cell Q, and denoting for simplicity $\langle\cdot\rangle := \int_Q dy$ the averaging operator, we find

$$\partial_\tau \langle \bar{\mathrm{m}}_0 \rangle = 0, \qquad \bar{\mathrm{v}}_0 = \nabla_x^\perp \bar{g}_0, \qquad \triangle_x \bar{g}_0 = \langle \bar{\mathrm{m}}_0 \rangle,$$

which implies that $\langle \bar{\mathrm{m}}_0 \rangle$ is independent of the τ-variable, hence the same holds for $\bar{g}_0$ and $\bar{\mathrm{v}}_0$. The 2-scale Ansatz (8.251) then takes on the following more precise form,

$$\begin{aligned} \bar{\mathrm{v}}_\varepsilon^t(x) &= \bar{\mathrm{v}}_0^t(x) + \eta_\varepsilon \bar{\mathrm{v}}_1(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + O(\eta_\varepsilon^2), \\ \bar{\mathrm{m}}_\varepsilon^t(x) &= \bar{\mathrm{m}}_0(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + \eta_\varepsilon \bar{\mathrm{m}}_1(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + O(\eta_\varepsilon^2). \end{aligned}$$

Further averaging the second equation in (8.252) on the periodicity cell Q, we obtain

$$\begin{gathered} \partial_\tau \bar{\mathrm{m}}_0 = \operatorname{div}_y(\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0), \\ \partial_t \langle \bar{\mathrm{m}}_0 \rangle + \partial_\tau \langle \bar{\mathrm{m}}_1 \rangle = \operatorname{div}_x(\langle \Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0 \rangle), \\ \bar{\mathrm{v}}_0 = \nabla_x^\perp \triangle_x^{-1} \langle \bar{\mathrm{m}}_0 \rangle. \end{gathered} \tag{8.253}$$

Let us now take a closer look at these equations (8.253). For all $x \in \mathbb{R}^2$ and $t \in \mathbb{R}^+$, consider the periodic flow $\phi_{x,t} : \mathbb{R}^+ \times Q \to Q$ associated with the periodic vector field $-\Gamma^0[\bar{\mathrm{v}}_0^t](x, \cdot) : Q \to \mathbb{R}^2$,

$$\partial_\tau \phi_{x,t}^\tau(y) = -\Gamma^0[\bar{\mathrm{v}}_0^t](x, \phi_{x,t}^\tau(y)), \qquad \phi_{x,t}^\tau(y)|_{\tau=0} = y.$$

The first equation in (8.253) then yields

$$\bar{\mathrm{m}}_0(t,\tau,x,y) = \big((\phi^\tau_{x,t})_*\bar{\mathrm{m}}_0(t,0,x,\cdot)\big)(y).$$

Now applying $s^{-1}\int_0^s d\tau$ to both sides of the second equation in (8.253), passing to the limit $s\uparrow\infty$, and recalling that $\langle\bar{\mathrm{m}}_0\rangle$ is independent of the τ-variable, we formally deduce

$$\partial_t\langle\bar{\mathrm{m}}_0\rangle(t,x) = \operatorname{div}_x\int_Q\Big(\lim_{s\uparrow\infty}s^{-1}\int_0^s\Gamma^0[\bar{\mathrm{v}}_0^t](x,\phi^\tau_{x,t}(y))d\tau\Big)\bar{\mathrm{m}}_0(t,0,x,y)dy. \tag{8.254}$$

By assumption, the periodic vector field $-\Gamma^0[\bar{\mathrm{v}}_0^t](x,\cdot)$ admits a unique stable (normalized) invariant measure $\mu_x[\bar{\mathrm{v}}_0^t]\in\mathcal{P}(Q)$. By the ergodic theorem, for any $\psi\in C_{\mathrm{per}}(Q)$, we deduce for $\mu_x[\bar{\mathrm{v}}_0^t]$-almost all $y\in Q$,

$$\lim_{s\uparrow\infty}s^{-1}\int_0^s\psi(\phi^\tau_{x,t}(y))d\tau = \langle\psi\,\mu_x[\bar{\mathrm{v}}_0^t]\rangle.$$

In view of the unique stability assumption, it is most natural to admit that the above also holds for $\bar{\mathrm{m}}_0(t,0,x,\cdot)$-almost all $y\in Q$, in which case we find

$$\begin{aligned}\lim_{s\uparrow\infty}\int_Q\psi(y)\Big(s^{-1}\int_0^s\bar{\mathrm{m}}_0(t,\tau,x,y)d\tau\Big)dy &= \lim_{s\uparrow\infty}\int_Q\Big(s^{-1}\int_0^s\psi(\phi^\tau_{x,t}(y))d\tau\Big)\bar{\mathrm{m}}_0(t,0,x,y)dy\\ &= \langle\bar{\mathrm{m}}_0\rangle(t,x)\langle\psi\,\mu_x[\bar{\mathrm{v}}_0^t]\rangle,\end{aligned}$$

that is,

$$\lim_{s\uparrow\infty}s^{-1}\int_0^s\bar{\mathrm{m}}_0(t,\tau,x,y)d\tau = \langle\bar{\mathrm{m}}_0\rangle(t,x)\mu_x[\bar{\mathrm{v}}_0^t],$$

in the weak-* sense of measures. In particular, the limit in the right-hand side of (8.254) is explicitly computed,

$$\partial_t\langle\bar{\mathrm{m}}_0\rangle(t,x) = \operatorname{div}_x\big(\langle\Gamma^0[\bar{\mathrm{v}}_0^t](x,\cdot)\mu_x[\bar{\mathrm{v}}_0^t]\rangle\langle\bar{\mathrm{m}}_0\rangle(t,x)\big). \tag{8.255}$$

Combining this with the first and the last equations in (8.253), the heuristics follows. ☐

Remark 8.9.5 (Obstacles to a rigorous justification)**.** As described below, there are essentially three distinct weaknesses in the above formal justification of Heuristics 8.1.7.

(a) The first part of the justification consists in formally deriving the relations (8.253) for the 2-scale expansion of $\bar{\mathrm{v}}_\varepsilon$. This derivation is based on formally inserting the 2-scale Ansatz in the equation for $\bar{\mathrm{v}}_\varepsilon$ and identifying the powers of η_ε. However, due to both the nonlinear nonlocal character of the equation for $\bar{\mathrm{v}}_\varepsilon$ and its instability as $\eta_\varepsilon\downarrow 0$, a rigorous justification seems difficult to obtain, as we explain here.

In order to justify formal 2-scale expansions, a powerful tool is given by Nguetseng's 2-scale weak compactness theorem [344, 11]. Since the equation for v_ε is nonlinear, this technique is not well suited for the present situation, and since the nonlinearity is in addition nonlocal, E's technique of 2-scale Young measures [172] is also useless here. (If we try to argue by 2-scale weak compactness, we would deduce that $\bar{\mathrm{m}}_\varepsilon$, $\bar{\mathrm{v}}_\varepsilon$, and the product $\bar{\mathrm{m}}_\varepsilon\bar{\mathrm{v}}_\varepsilon$ 2-scale converge weakly-* to some $\bar{\mathrm{m}}_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{M}(\mathbb{R}^2;\mathcal{M}_{\mathrm{per}}(Q)))$, some $\bar{\mathrm{v}}_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2))$, and some $\bar{Q}_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{M}(\mathbb{R}^2;\mathcal{M}_{\mathrm{per}}(Q))^2)$, respectively; cf. Lemma 8.9.10. Compensated compactness in the form of Delort's weak continuity theorem [143] actually ensures that $\langle\bar{Q}_0\rangle=\langle\bar{\mathrm{m}}_0\rangle\bar{\mathrm{v}}_0$, but the stronger microscopic identification $\bar{Q}_0=\bar{\mathrm{m}}_0\bar{\mathrm{v}}_0$ would further be needed, which seems very unclear by such a weak compactness approach.)

Another way to proceed (see e.g. [135, Section 3.1]) consists in approximating the solution $\bar{\mathrm{v}}_\varepsilon$ with the first terms of its formal 2-scale expansion (8.251): by definition this approximation satisfies the very same equation as $\bar{\mathrm{v}}_\varepsilon$ up to a small error, and this could be combined with a quantitative uniqueness principle to ensure that $\bar{\mathrm{v}}_\varepsilon$ remains close to its expansion. However, the linear part of the equation with fast oscillating forcing and the nonlinear interaction part are difficult to conciliate, and we do not know of any stability estimate which does not blow up in the homogenization limit. On the one hand, the L^1-contraction principle for the vorticity holds in the linear case but interacts badly with the nonlinearity. On the other hand, the nonlinear interaction part calls for energy type estimates (that is, estimates on the L^2-distance between supercurrent densities), but the evolution of such metrics (as well as of the 2-Wasserstein distance) is sensitive to the blowing Lipschitz norm of the oscillating forcing vector field. This issue is linked with the particularly strong instability of the equation upon perturbations as $\eta_\varepsilon \downarrow 0$.

(b) The last part of the justification consists in checking that the relations (8.253) imply the closed equation (8.255) for the averaged vorticity $\langle \mathrm{m}_0\rangle$. If the (normalized) invariant measure $\mu_x[\bar{\mathrm{v}}^t]$ was truly unique for all x, t, then the given justification would be perfectly rigorous. Unfortunately, in the periodic setting, due to the gradient structure, this uniqueness (or *unique ergodicity*) is impossible, while the uniqueness assumption for a *stable* invariant measure seems more reasonable. The flaw in the above justification then lies in the assumption that unstable invariant measures do not play any role in the limit in (8.254), which is however not obvious and would require some argument.

(c) Finally, the well-posedness of the limiting equation (8.29) or (8.30) is unclear. The main difficulty is that the map $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d : (x, Z) \mapsto \Gamma_{\mathrm{hom}}[Z](x)$ is not even expected to be Lipschitz-continuous in Z: indeed, as explained in Remark 8.9.9 and Proposition 8.9.11, for fixed x, this map typically vanishes for Z in some bounded domain (pinning phenomenon), and is expected to have a power-law behavior with some power < 1 at the boundary of this domain (fractional depinning rate). $\lozenge$

Remark 8.9.6 (Vanishing viscosity)**.** For simplicity, we may consider the corresponding homogenization problems with a vanishing viscosity, that is, adding in the right-hand side of equation (8.63) or (8.64) for $\bar{\mathrm{v}}_\varepsilon$ a term $+D\eta_\varepsilon \triangle \bar{\mathrm{v}}_\varepsilon$ for some fixed constant $D > 0$. A similar formal 2-scale expansion as above then yields the following modification of the relations (8.253), in the case of the regime (GL_1),

$$\begin{aligned} \partial_\tau \bar{\mathrm{m}}_0 &= D\triangle_y \bar{\mathrm{m}}_0 + \mathrm{div}_y(\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0), \\ \partial_t\langle \bar{\mathrm{m}}_0\rangle + \partial_\tau\langle \bar{\mathrm{m}}_1\rangle &= \mathrm{div}_x(\langle \Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0\rangle), \\ \bar{\mathrm{v}}_0 &= \nabla_x^\perp \triangle_x^{-1}\langle \bar{\mathrm{m}}_0\rangle. \end{aligned} \tag{8.256}$$

From these relations the interpretation is now much easier: the first equation implies the (exponential) convergence of $\bar{\mathrm{m}}_0(t, \tau, x, \cdot)$ towards $\langle \bar{\mathrm{m}}_0\rangle(t, x)\tilde{\mu}_x^D[\bar{\mathrm{v}}_0^t]$ as $\tau \uparrow \infty$, where the viscous invariant measure $\tilde{\mu}_x^D[\bar{\mathrm{v}}_0^t] \in \mathcal{P}_{\mathrm{per}}(Q)$ is the unique (smooth) solution of the following equation on the periodicity cell Q,

$$D\triangle_y \tilde{\mu}_x^D[\bar{\mathrm{v}}_0^t] + \mathrm{div}_y(\Gamma^0[\bar{\mathrm{v}}_0^t]\tilde{\mu}_x^D[\bar{\mathrm{v}}_0^t]) = 0.$$

The formal limiting equation then takes exactly the same form as in Heuristics 8.1.7, but with $\Gamma_{\mathrm{hom}}[w]$ replaced by its better-behaved viscous analogue,

$$\tilde{\Gamma}^D_{\mathrm{hom}}[w](x) := \int_Q \Gamma_x[w](y)\, d\tilde{\mu}_x^D[w](y).$$

In this case, the last two difficulties (b) and (c) pointed out in Remark 8.9.5 above disappear: the viscous invariant measure is easily checked to be always uniquely defined, and the corresponding limiting equation for $\bar{\mathrm{m}}$ is well-posed. Nevertheless, the difficulty (a) remains unchanged (that is, the rigorous derivation of the relations (8.256) for the 2-scale limit), and finding a rigorous proof remains very challenging. $\lozenge$

Remark 8.9.7 (Conservative case)**.** In this remark, resulting from a discussion with Anne-Laure Dalibard, we briefly explain that the homogenization problem is much simpler for the corresponding equation (8.63) in the conservative case ($\alpha = 0$, $\beta = 1$), that is,

$$\partial_t \bar{\mathrm{v}}_\varepsilon = \nabla \bar{\mathrm{p}}_\varepsilon + \bar{\Gamma}_\varepsilon^\perp \mathrm{curl}\, \bar{\mathrm{v}}_\varepsilon + D\eta_\varepsilon \triangle \bar{\mathrm{v}}_\varepsilon, \qquad \mathrm{div}\ \bar{\mathrm{v}}_\varepsilon = 0, \qquad \bar{\mathrm{v}}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ,$$
$$\bar{\Gamma}_\varepsilon := -\big(\nabla_2^\perp \hat{h}^0(\cdot, \cdot/\eta_\varepsilon) - \hat{F}^\perp - 2\bar{\mathrm{v}}_\varepsilon\big),$$

where we have included for simplicity a vanishing viscosity as in Remark 8.9.6 above, and where $D > 0$, $\hat{h}^0$, and $\hat{F}$ are independent of ε. In vorticity form, in terms of $\bar{\mathrm{m}}_\varepsilon := \mathrm{curl}\, \bar{\mathrm{v}}_\varepsilon$, this equation becomes

$$\partial_t \bar{\mathrm{m}}_\varepsilon = D\eta_\varepsilon \triangle \bar{\mathrm{m}}_\varepsilon + \mathrm{div}\,(\bar{\Gamma}_\varepsilon \bar{\mathrm{m}}_\varepsilon), \qquad \bar{\Gamma}_\varepsilon := -\big(\nabla_2^\perp \hat{h}^0(\cdot, \cdot/\eta_\varepsilon) - \hat{F}^\perp - 2\nabla^\perp \triangle^{-1} \bar{\mathrm{m}}_\varepsilon\big), \qquad \bar{\mathrm{m}}_\varepsilon|_{t=0} = \mathrm{m}_\varepsilon^\circ .$$

Since the divergence of the vector field $\bar{\Gamma}_\varepsilon$ is given by $\mathrm{div}\ \bar{\Gamma}_\varepsilon^\perp = -(\nabla_1 \cdot \nabla_2^\perp \hat{h}^0)(\cdot, \cdot/\eta_\varepsilon) - \mathrm{curl}\, \hat{F}$ and is bounded in $\mathrm{L}^\infty(\mathbb{R}^2)$, assuming that $\mathrm{m}_\varepsilon^\circ$ is bounded in $\mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^2)$, we deduce that the vorticity $\bar{\mathrm{m}}_\varepsilon$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^2))$. Therefore, $\bar{\mathrm{v}}_\varepsilon = \nabla^\perp \triangle^{-1} \bar{\mathrm{m}}_\varepsilon$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty \cap H^1_{\mathrm{loc}}(\mathbb{R}^2)^2)$. Up to an extraction, and using the Aubin-Simon lemma, we conclude that $\bar{\mathrm{m}}_\varepsilon$ 2-scale converges weakly to some $\bar{\mathrm{m}}_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2 \times Q))$, and that $\bar{\mathrm{v}}_\varepsilon$ converges strongly to some $\bar{\mathrm{v}}_0$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)^2$. This strong compactness for $\bar{\mathrm{v}}_\varepsilon$ allows to pass to the limit in the product $\bar{\mathrm{m}}_\varepsilon \bar{\mathrm{v}}_\varepsilon$, and we easily deduce that the limiting vorticity $\bar{\mathrm{m}}_0$ satisfies the expected equation. $\lozenge$

8.9.5 Subcritical regimes

In the subcritical regimes (GL_1') and (GL_2'), the interaction of the vortices vanishes in the limit, and we are left with a much simpler *linear* transport equation for the vorticity $\bar{\mathrm{m}}_\varepsilon := \mathrm{curl}\, \bar{\mathrm{v}}_\varepsilon$ with fast oscillating pinning force (as is obtained by taking the curl of equation (8.65) or (8.66) in the form of Corollary 8.9.4),

$$\partial_t \bar{\mathrm{m}}_\varepsilon = \mathrm{div}\ \big(\bar{\Gamma}_\varepsilon \bar{\mathrm{m}}_\varepsilon\big), \qquad \bar{\mathrm{m}}_\varepsilon|_{t=0} = \mathrm{curl}\, \mathrm{v}_\varepsilon^\circ,$$
$$\bar{\Gamma}_\varepsilon(x) := \bar{\Gamma}(x, x/\eta_\varepsilon), \qquad \bar{\Gamma}(x, y) := (\alpha - \mathbb{J}\beta)(\nabla_2 \hat{h}^0(x, y) - \hat{F}(x)).$$

With its fast oscillating gradient part, this linear transport equation is referred to as a *washboard* or *wiggly system*. Obviously the macroscopic dynamics strongly depends on microstructural events, for instance if some mass gets stuck in local minima: the typical mental picture is that of a particle sliding down a rough slope (like a washboard), thus taking a jerky path downwards, sometimes getting stuck along the way. Due to its gradient part, the corresponding vertical flow $\bar{\Gamma}(x, \cdot)$ on the periodicity cell Q cannot be uniquely ergodic, so that the problem of determining the asymptotic behavior of the solution $\bar{\mathrm{m}}_\varepsilon$ lies outside the classical theory of averaging. This problem was first studied in dimension 1 by [1, 60], and later investigated in dimension 2 by Menon [319].

Menon's results [319] show that the space $\mathbb{R}^2$ splits into three regions associated with different dynamical properties: (1) an open set where the mass gets stuck (pinning region), (2) a transition region with a combination of sticking and slipping, and (3) the rest of the plane with only slipping. The slipping region is actually further split into countably many resonance zones where the limiting vector field has a constant direction given by the (rational) rotation number of the underlying microscopic cell flow, and the direction of the vector field varies continuously but not smoothly across the boundary of the resonance zones: given an initial position far from the pinning region, its path downwards is typically rough like a Cantor function. The dynamics is indeed particularly rich in dimensions $d \geq 2$: through the forcing $\hat{F}$, the macroscopic variable x acts as a bifurcation parameter for the topology of the underlying microscopic cell flow, and the bifurcations in the topology generate changes in the macroscopic motion between stick and slip, as well as between (rational) slipping directions. Note that Menon's results [319] are only partially justified, and are restricted to dimension $d = 2$ (due to some key topological arguments).

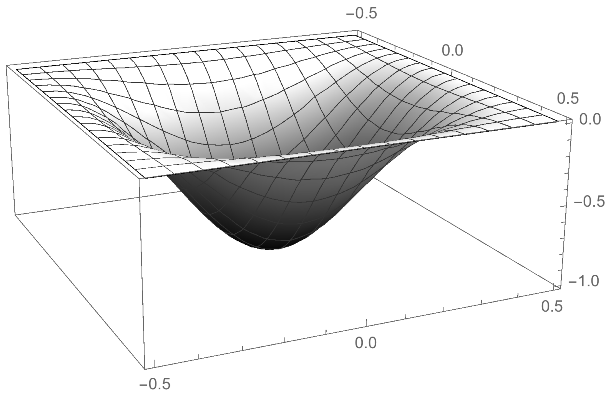


Figure 8.4 – In dimension 2, a typical choice for the pinning potential is e.g. $\tilde{h}^0(x) := -\cos(\pi x_1)^2 \cos(\pi x_2)^2$ for $x \in Q = [-\frac{1}{2}, \frac{1}{2})^2$.

Simplified model

In order to exemplify the complexity of the structure of the limiting motion described above, let us consider (in general dimension d, say) the easier case of a constant forcing $\hat{F} := F_0 \in \mathbb{R}^d$ together with a wiggly potential $\tilde{h}^0$ that only depends on the microscopic variable; we thus consider the following linear transport equation,

$$\begin{aligned} \partial_t \tilde{\mathrm{m}}_\varepsilon = \operatorname{div}(\Gamma_\varepsilon^{F_0} \tilde{\mathrm{m}}_\varepsilon), \qquad & \tilde{\mathrm{m}}_\varepsilon|_{t=0} = \tilde{\mathrm{m}}_\varepsilon^\circ, \\ \Gamma_\varepsilon^{F_0}(x) = \Gamma^{F_0}(x/\eta_\varepsilon), \qquad & \Gamma^{F_0}(y) = (\alpha - \mathbb{J}\beta)(\nabla \tilde{h}^0(y) - F_0). \end{aligned} \tag{8.257}$$

In this context, there is a true separation of scales in the limit $\eta_\varepsilon \downarrow 0$, and we may simply study the bifurcation of the limiting motion with respect to the *constant* forcing F_0. This system is a very particular case of the general nonlinear systems studied in [137] under additional well-preparedness conditions, but a more precise result is obtained here (see also [85, 247, 172, 254] for the easier incompressible case, and [188, 136] for the corresponding Hamiltonian setting).

We first introduce some notation and make some regularity assumptions. The periodic vector field $-\Gamma^{F_0}$ on the unit cell $Q \subset \mathbb{R}^d$ defines a dynamical system on the d-torus Q. Assume that $\tilde{h}^0$ is smooth and non-degenerate, in the sense that for $F_0 \neq 0$ this dynamical system admits a finite number of (normalized) ergodic invariant measures $(\mu_k^{F_0})_{k=1}^{L_{F_0}} \subset \mathcal{P}(Q)$, $1 \leq L_{F_0} < \infty$. For $F_0 = 0$ we only assume that the dynamical system admits a finite number of (normalized) ergodic invariant measures supported on $\operatorname{int} Q$, while the boundary ∂Q is assumed to be made of unstable fixed points of the dynamics, thus yielding an infinite family $(\delta_p)_{p \in \partial Q}$ of ergodic measures on this boundary. (This assumption is motivated by the typical choice $\tilde{h}_0 \leq 0$, $(\tilde{h}_0)^{-1}(\{0\}) = \partial Q$; cf. the explicit example in Figure 8.4.) For all $1 \leq k \leq L_{F_0}$ we define the minimal invariant sets $A_k^{F_0} := \operatorname{supp} \mu_k^{F_0}$, and we let $B_k^{F_0}$ denote the set of $\mu_k^{F_0}$-generic points. We order the ergodic measures in such a way that $|B_k^{F_0}| > 0$ holds for all $1 \leq k \leq K_{F_0}$, and $|B_k^{F_0}| = 0$ for all $K_{F_0} + 1 \leq k \leq L_{F_0}$, with $1 \leq K_{F_0} \leq L_{F_0}$. By construction we have

$$\Big| Q \setminus \biguplus_{k=1}^{K_{F_0}} B_k^{F_0} \Big| = 0.$$

Note that in dimension $d = 2$ the dynamical picture is particularly simple, as Denjoy's version of the Poincaré-Bendixson theorem on the 2-torus [144] (see also [392]) asserts that minimal invariant sets are either fixed points, periodic orbits, or the whole torus.

The limiting behavior of the solution $\tilde{\mathrm{m}}_\varepsilon$ of (8.257) is then characterized as follows; note that the result is much simpler in the case $K_F = 1$, that is, if there exists a unique stable (normalized) invariant measure.

Theorem 8.9.8. *Let the above notation and assumptions hold, and let $\tilde{\mathrm{m}}_\varepsilon^\circ \in \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^d)$ satisfy*

$$\tilde{\mathrm{m}}_\varepsilon^\circ(x) - \omega^\circ(x, x/\eta_\varepsilon) \to 0, \tag{8.258}$$

strongly in $\mathrm{L}^1(\mathbb{R}^d)$ as $\varepsilon \downarrow 0$, for some $\omega^\circ \in \mathrm{L}^1(\mathbb{R}^d; C_{\mathrm{per}}(Q))$. Let $F_0 \in \mathbb{R}^d$, and denote by $\tilde{\mathrm{m}}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^d))$ the unique solution to the transport equation (8.257) with initial data $\tilde{\mathrm{m}}_\varepsilon^\circ$. Then we have for all $t \geq 0$,

$$\tilde{\mathrm{m}}_\varepsilon^t \ \overset{*}{\rightharpoonup}\ \tilde{\mathrm{m}}^t := \sum_{k=1}^{K_{F_0}} \tilde{\mathrm{m}}_k^t,$$

where for all k we denote by $\tilde{\mathrm{m}}_k \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^d))$ the unique solution of the (constant-coefficient) transport equation

$$\partial_t \tilde{\mathrm{m}}_k = \operatorname{div}(\Gamma_k^{F_0} \tilde{\mathrm{m}}_k), \qquad \Gamma_k^{F_0} := \int_Q \Gamma^{F_0}(y) d\mu_k^{F_0}(y), \qquad \tilde{\mathrm{m}}_k|_{t=0} = \tilde{\mathrm{m}}_k^\circ := \int_{B_k^{F_0}} \omega^\circ(\cdot, y) dy.$$

In particular, if the stable invariant sets of the dynamical system generated by the periodic vector field $-\Gamma^{F_0}$ are all reduced to a point (that is, if $A_k^{F_0}$ is a point for all $1 \leq k \leq K_{F_0}$), then we have for all $t \geq 0$,

$$\tilde{\mathrm{m}}_\varepsilon^t \ \overset{*}{\rightharpoonup}\ \tilde{\mathrm{m}}^\circ := \int_Q \omega^\circ(\cdot, y) dy = \sum_{k=1}^{K_{F_0}} \tilde{\mathrm{m}}_k^\circ. \qquad \Diamond$$

Remarks 8.9.9.

(a) *Stick-slip motion.* In this remark, we consider the behavior of the limiting vorticity $\tilde{\mathrm{m}}$ as a function of the forcing F_0, and we argue that the space $\mathbb{R}^d$ of values of F_0 splits into three regions: (1) an open bounded set around 0 for which the limiting solution is stuck $\tilde{\mathrm{m}} = \tilde{\mathrm{m}}^\circ$ (pinning phenomenon), (2) a transition region for which a part of the mass is stuck and another part is transported, and (3) the rest of $\mathbb{R}^d$ for which there is only transport (with possibly a superposition of different effective velocities). The link with Menon's results [319] is thus clear. A natural question consists in studying the precise behavior of the effective velocity as a function of F_0 beyond the pinning region. The behavior at the depinning threshold, that is, for forcing F_0 just across the boundary of the pinning region, is shortly addressed in the sequel of this section (see Proposition 8.9.11 below). On the other hand, for very large $|F_0| \gg 1$, the deviation of the effective velocity due to the wiggly potential $\tilde{h}^0$ naturally tends to 0,

$$-\Gamma_k^{F_0} = (\alpha - \mathbb{J}\beta)F_0 - (\alpha - \mathbb{J}\beta) \int_Q \nabla \tilde{h}^0 d\mu_k^{F_0} = (1 + o(1))(\alpha - \mathbb{J}\beta)F_0.$$

We first consider the case $F_0 = 0$, hence $-\Gamma^0 = -\alpha \nabla \tilde{h}^0 + \beta \nabla^\perp \tilde{h}^0$. For energy reasons, we note that the only invariant sets are then necessarily made of unions of fixed points of the dynamics. The last part of Theorem 8.9.8 then allows to conclude that the limiting solution $\tilde{\mathrm{m}}$ is constant in time. Next, for F_0 close enough to 0, the stable invariant sets of $-\Gamma^{F_0}$ are still made of stable fixed points, which are simply deformations of the stable fixed points of $-\Gamma^0$, and we conclude that the limiting solution $\tilde{\mathrm{m}}$ still remains constant. In contrast, for larger values of F_0, the topological nature of the stable invariant sets may change, yielding a possible combination of both stable fixed points and other types of stable sets, hence by Theorem 8.9.8 a combination of pinning and transport. Finally, for $|F_0| > \|\nabla \tilde{h}^0\|_{\mathrm{L}^\infty}$, we note that the map $-\Gamma^{F_0}$ no longer has any fixed point (since the condition on F_0 implies $|\Gamma^{F_0}|^2 = (\alpha^2 + \beta^2)|\nabla \tilde{h}^0 - F_0|^2 > 0$), so that Theorem 8.9.8 yields pure transport in that case.

(b) *Initial-boundary layer.* While the initial data $\tilde{\mathrm{m}}_\varepsilon^\circ$ may have some microscopic heterogeneities, which are assumed to be given by $\omega^\circ(\cdot, \cdot/\eta_\varepsilon)$, it is instantaneously relaxed to an invariant measure $\sum_{k=1}^{K_F} \mu_k(\cdot/\eta_\varepsilon)\tilde{\mathrm{m}}_k^\circ$ in a timescale of order $O(\eta_\varepsilon)$. This initial-boundary layer at the microscopic scale could be described in similar terms as in Proposition 8.9.2. $\Diamond$

We now turn to the proof of Theorem 8.9.8. It is obtained by 2-scale convergence methods. More precisely, we use the following L^1-version of Nguetseng's 2-scale compactness theorem [344, 11]; as it is not standard in this form, we include a short proof (see also [137, end of Section 2.1]).

Lemma 8.9.10 (à la Nguetseng). *Let $(g_\eta)_\eta$ be a bounded sequence in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1(\mathbb{R}^d))$. Further assume that it is tight, in the sense that for all $T > 0$,*

$$\lim_{L\uparrow\infty} \limsup_{\eta\downarrow 0} \sup_{t\in[0,T]} \int_{|x|>L} |g_\eta^t| = 0. \tag{8.259}$$

Then, there exists a subsequence, still denoted by $(g_\eta)_\eta$, and an element $g_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{M}(\mathbb{R}^d; \mathcal{M}_{\mathrm{per}}(Q)))$ (where $\mathcal{M}$ (resp. $\mathcal{M}_{\mathrm{per}}$) denotes the space of Radon measures (resp. periodic Radon measures)), such that we have for all $T > 0$ and all $\psi \in \mathrm{L}^1([0, T]; C_b(\mathbb{R}^d; C_{\mathrm{per}}(Q)))$,

$$\lim_{\eta\downarrow 0} \int_0^T \int_{\mathbb{R}^d} \psi^t(x, x/\eta) g_\eta^t(x) dx dt = \int_0^T \iint_{\mathbb{R}^d\times Q} \psi^t(x, y) dg_0^t(x, y) dt. \tag{8.260}$$

We say that g_η two-scale converges weakly- to g_0. Moreover, if there holds $\psi_\eta \to \psi$ strongly in $\mathrm{L}^1([0, T]; C_b(\mathbb{R}^d; C_{\mathrm{per}}(Q)))$, then we find*

$$\lim_{\eta\downarrow 0} \int_0^T \int_{\mathbb{R}^d} \psi_\eta^t(x, x/\eta) g_\eta^t(x) dx dt = \int_0^T \iint_{\mathbb{R}^d\times Q} \psi^t(x, y) dg_0^t(x, y) dt. \qquad \Diamond$$

Proof. Let $T > 0$ be fixed. The boundedness assumption on g_η gives $\sup_\eta \|g_\eta\|_{\mathrm{L}^\infty([0,T];\mathrm{L}^1(\mathbb{R}^d))} \le C_T$, so that we find for all $\psi \in \mathrm{L}^1([0, T]; C_0(\mathbb{R}^d; C_{\mathrm{per}}(Q)))$,

$$\left| \int_0^T \int_{\mathbb{R}^d} \psi^t(x, x/\eta) g_\eta^t(x) dx dt \right| \le C_T \|\psi\|_{\mathrm{L}^1([0,T];C_0(\mathbb{R}^d;C_{\mathrm{per}}(Q)))}.$$

The sequence $(g_\eta)_\eta$ may thus be seen as a bounded sequence of elements in the dual of the Banach space $\mathrm{L}^1([0, T]; C_0(\mathbb{R}^d; C_{\mathrm{per}}(Q)))$, that is, a bounded sequence in $\mathrm{L}^\infty([0, T]; \mathcal{M}(\mathbb{R}^d; \mathcal{M}_{\mathrm{per}}(Q)))$ (for the operator norm). Let this dual space be endowed with the corresponding weak-* topology (also called vague topology in this context). By the Banach-Alaoglu theorem, we deduce that there is a subsequence, still denoted by $(g_\eta)_\eta$, and an element $g_0 \in \mathrm{L}^\infty([0, T]; \mathcal{M}(\mathbb{R}^d; \mathcal{M}_{\mathrm{per}}(Q)))$ such that g_η converges to g_0 in this weak-* (vague) topology, which precisely means that (8.260) holds for all $\psi \in \mathrm{L}^1([0, T]; C_0(\mathbb{R}^d; C_{\mathrm{per}}(Q)))$. Combining this with the additional tightness assumption (8.259) allows to extend this to all test functions $\psi \in \mathrm{L}^1([0, T]; C_b(\mathbb{R}^d; C_{\mathrm{per}}(Q)))$. $\square$

With this weak compactness result at hand, we now give a proof of Theorem 8.9.8.

Proof of Theorem 8.9.8. Let F_0 be fixed, and write for simplicity $A_k := A_k^{F_0}$, $B_k := B_k^{F_0}$, and $\mu_k := \mu_k^{F_0}$. We split the proof into four steps.

Step 1. 2-scale compactness argument.

In this step, we show that up to a subsequence the solution $\tilde{\mathrm{m}}_\varepsilon$ of (8.257) 2-scale converges weakly-* (in the sense of Lemma 8.9.10) to some limit $\tilde{\mathrm{m}}_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{M}^+(\mathbb{R}^d; \mathcal{M}^+_{\mathrm{per}}(Q)))$. Moreover, denoting for simplicity by $\langle\cdot\rangle := \int_Q dy$ the averaging operator, the limit $\tilde{\mathrm{m}}_0$ satisfies the following equations,

$$-\operatorname{div}_y(\Gamma^F \tilde{\mathrm{m}}_0) = 0, \tag{8.261}$$

$$\partial_t\langle\tilde{\mathrm{m}}_0\rangle = \operatorname{div}_x\langle\Gamma^F\tilde{\mathrm{m}}_0\rangle, \qquad \langle\tilde{\mathrm{m}}_0\rangle|_{t=0} = \langle\omega^\circ\rangle = \tilde{\mathrm{m}}^\circ. \tag{8.262}$$

Equation (8.261) means that $\tilde{\mathrm{m}}_0^t(x,\cdot)$ is an invariant measure for the vector field $-\Gamma^F$ on Q for almost all t, x. For $F \neq 0$, by assumption, we may then decompose $\tilde{\mathrm{m}}_0$ as a linear combination

$$\tilde{\mathrm{m}}_0^t(x,y) = \sum_{k=1}^{L_{F_0}} \xi_k^t(x)\mu_k(y). \tag{8.263}$$

For $F_0 = 0$, by assumption, a similar decomposition holds in $\operatorname{int} Q$: there exists some $\hat{\mathrm{m}}_0$ in the space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{M}(\mathbb{R}^d; \mathcal{M}_{\mathrm{per}}(Q)))$ such that for all t, x the measure $\hat{\mathrm{m}}_0^t(x,\cdot)$ is supported in ∂Q, and such that

$$\tilde{\mathrm{m}}_0^t(x,y) = \hat{\mathrm{m}}_0^t(x,y) + \sum_{k=1}^{L_0} \xi_k^t(x)\mu_k(y).$$

Since $\tilde{\mathrm{m}}_\varepsilon$ is nonnegative and has constant mass 1, it is bounded in $\mathrm{L}^\infty(\mathbb{R}^+; \mathrm{L}^1(\mathbb{R}^d))$. Moreover, as the velocity field Γ^F_ε is bounded in $\mathrm{L}^\infty(\mathbb{R}^d)^d$, the tightness of the initial data $(\tilde{\mathrm{m}}^\circ_\varepsilon)_\varepsilon$ easily implies the tightness of the solutions $(\tilde{\mathrm{m}}_\varepsilon)_\varepsilon$ in the sense of (8.259). Therefore, by Lemma 8.9.10, up to a subsequence, $\tilde{\mathrm{m}}_\varepsilon$ 2-scale converges weakly-* to some $\tilde{\mathrm{m}}_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{M}^+(\mathbb{R}^d; \mathcal{M}^+_{\mathrm{per}}(Q)))$. We now prove that this limit satisfies equations (8.261) and (8.262). Testing the equation for $\tilde{\mathrm{m}}_\varepsilon$ against a test function $\psi^t(x, x/\eta_\varepsilon)$ with $\psi \in C^1_c(\mathbb{R}^+ \times \mathbb{R}^d; C^1_{\mathrm{per}}(Q))$, we find

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \partial_t\psi^t(x, x/\eta_\varepsilon)d\tilde{\mathrm{m}}^t_\varepsilon(x)dt + \int_{\mathbb{R}^d}\psi^0(x, x/\eta_\varepsilon)d\tilde{\mathrm{m}}^\circ_\varepsilon(x)&\\ = \int_{\mathbb{R}^+}\int_{\mathbb{R}^d}(\eta_\varepsilon^{-1}\nabla_y\psi^t(x,x/\eta_\varepsilon) + \nabla_x\psi^t(x,x/\eta_\varepsilon))\cdot\Gamma^F(x/\eta_\varepsilon)d\tilde{\mathrm{m}}^t_\varepsilon(x)dt.&\end{aligned}$$

Choosing $\psi^t(x,y) := \eta_\varepsilon\phi^t(x,y)$ with $\phi \in C^1_c(\mathbb{R}^+ \times \mathbb{R}^d; C^1_{\mathrm{per}}(Q))$, and letting $\varepsilon \downarrow 0$ (along the subsequence), we find

$$\int_{\mathbb{R}^+}\iint_{\mathbb{R}^d\times Q}\nabla_y\phi^t(x,y)\cdot\Gamma^F(y)d\tilde{\mathrm{m}}^t_0(x,y)dt = 0,$$

that is (8.261). Now choosing $\psi^t(x,y) := \phi^t(x)$ with $\phi \in C^1_c(\mathbb{R}^+ \times \mathbb{R}^d)$, letting $\varepsilon \downarrow 0$ (along the subsequence), and using assumption (8.258), we obtain

$$\int_{\mathbb{R}^+}\iint_{\mathbb{R}^d\times Q}\partial_t\phi^t(x)d\tilde{\mathrm{m}}^t_0(x,y)dt + \iint_{\mathbb{R}^d\times Q}\phi^0(x)d\omega^\circ(x,y) = \int_{\mathbb{R}^+}\iint_{\mathbb{R}^d\times Q}\nabla\phi(t,x)\cdot\Gamma^F(y)d\tilde{\mathrm{m}}^t_0(x,y)dt,$$

that is (8.262).

Step 2. Localization.

Let $1 \le k \le K_{F_0}$ be fixed. Denote by B'_k the 1-periodic extension of $B_k \subset Q$ on $\mathbb{R}^d$. In this step, we show that, if $\tilde{\mathrm{m}}^\circ_\varepsilon(\mathbb{R}^d \setminus \eta_\varepsilon B'_k) = 0$ for all ε, then $\xi^t_j(x) = 0$ holds for all $j \neq k$ for almost all t, x. In particular, this implies $\tilde{\mathrm{m}}^t_0(x,y) = \xi^t_k(x)\mu_k(y)$ almost everywhere.

Given the smoothness assumptions, viewing B_k as the attraction basin associated with A_k, it follows that we must have $n\cdot\Gamma^F = 0$ on the boundary ∂B_k. Note that the method of propagation along characteristics together with the Liouville-Ostrogradski formula yields the following estimate for the solution $\tilde{\mathrm{m}}_\varepsilon$ of (8.257),

$$\|\tilde{\mathrm{m}}^t_\varepsilon\|_{\mathrm{L}^\infty} \le \|\tilde{\mathrm{m}}^\circ_\varepsilon\|_{\mathrm{L}^\infty}\exp(t\|\operatorname{div}\Gamma^{F_0}_\varepsilon\|_{\mathrm{L}^\infty}) \le \|\tilde{\mathrm{m}}^\circ_\varepsilon\|_{\mathrm{L}^\infty}\exp(\alpha\eta_\varepsilon^{-1}t\|\triangle\tilde{h}^0\|_{\mathrm{L}^\infty}),$$

hence $\tilde{\mathrm{m}}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^d))$ (although of course no uniform bound holds in that space). We may then deduce by integration by parts, for all $t \geq 0$,

$$\partial_t \int_{\eta_\varepsilon B'_k} d\tilde{\mathrm{m}}^t_\varepsilon = \int_{\eta_\varepsilon \partial B'_k} n \cdot \Gamma^F_\varepsilon(x)\tilde{\mathrm{m}}^t_\varepsilon(x) d\sigma(x) = 0,$$

that is, $\tilde{\mathrm{m}}^t_\varepsilon(\eta_\varepsilon B'_k) = \tilde{\mathrm{m}}^\circ_\varepsilon(\eta_\varepsilon B'_k) = 1$, and the conclusion follows from the decomposition (8.263).

Step 3. Convergence of partitioned initial data.

Decompose $\tilde{\mathrm{m}}^\circ_\varepsilon = \sum_{k=1}^{K_{F_0}} \tilde{\mathrm{m}}^\circ_{\varepsilon,k}$ with $\tilde{\mathrm{m}}^\circ_{\varepsilon,k} := \tilde{\mathrm{m}}^\circ_\varepsilon \mathbb{1}_{\eta_\varepsilon B'_k}$. In this step, for all k, we show that $\tilde{\mathrm{m}}^\circ_{\varepsilon,k}$ converges weakly in $\mathrm{L}^1(\mathbb{R}^2)$ to $\tilde{\mathrm{m}}^\circ_k := \int_{B_k} \omega^\circ(\cdot, y)dy$.

For any test function $\phi \in \mathrm{L}^\infty(\mathbb{R}^2)$, assumption (8.258) yields

$$\limsup_{k\uparrow\infty} \Big| \int_{\mathbb{R}^d} \phi d\tilde{\mathrm{m}}^\circ_{\varepsilon,k} - \int_{\eta_\varepsilon B'_k} \phi(x)\omega^\circ(x, x/\eta_\varepsilon)dx \Big| \leq \limsup_{k\uparrow\infty} \int_{\mathbb{R}^d} |\phi(x)|\, |\tilde{\mathrm{m}}^\circ_\varepsilon(x) - \omega^\circ(x, x/\eta_\varepsilon)|dx = 0,$$

while by periodicity we may compute (see e.g. [11, proof of Lemma 5.2])

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^d} \phi(x)\omega^\circ(x, x/\eta_\varepsilon)\mathbb{1}_{x/\eta_\varepsilon \in B'_k} dx = \iint_{\mathbb{R}^d \times B_k} \phi(x)\omega^\circ(x, y)dxdy = \int_{\mathbb{R}^d} \phi d\tilde{\mathrm{m}}^\circ_k,$$

and the result follows.

Step 4. Conclusion.

By linearity, with the choice of the $\tilde{\mathrm{m}}^\circ_{\varepsilon,k}$'s in Step 3, we may decompose $\tilde{\mathrm{m}}_\varepsilon = \sum_{k=1}^{K_{F_0}} \tilde{\mathrm{m}}_{\varepsilon,k}$, where for all k the function $\tilde{\mathrm{m}}_{\varepsilon,k} \in \mathrm{L}^\infty(\mathbb{R}^+;\mathcal{M}^+(\mathbb{R}^d))$ is the unique solution of the following equation,

$$\partial_t \tilde{\mathrm{m}}_{\varepsilon,k} = \mathrm{div}\,(\Gamma^{F_0}_\varepsilon \tilde{\mathrm{m}}_{\varepsilon,k}), \qquad \tilde{\mathrm{m}}_{\varepsilon,k}|_{t=0} = \tilde{\mathrm{m}}^\circ_{\varepsilon,k}.$$

Up to a subsequence, for all k, we know by Step 1 that $\tilde{\mathrm{m}}_{\varepsilon,k}$ 2-scale converges weakly-* to some $\tilde{\mathrm{m}}_{0,k} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{M}^+(\mathbb{R}^2;\mathcal{M}^+_{\mathrm{per}}(Q)))$, which satisfies

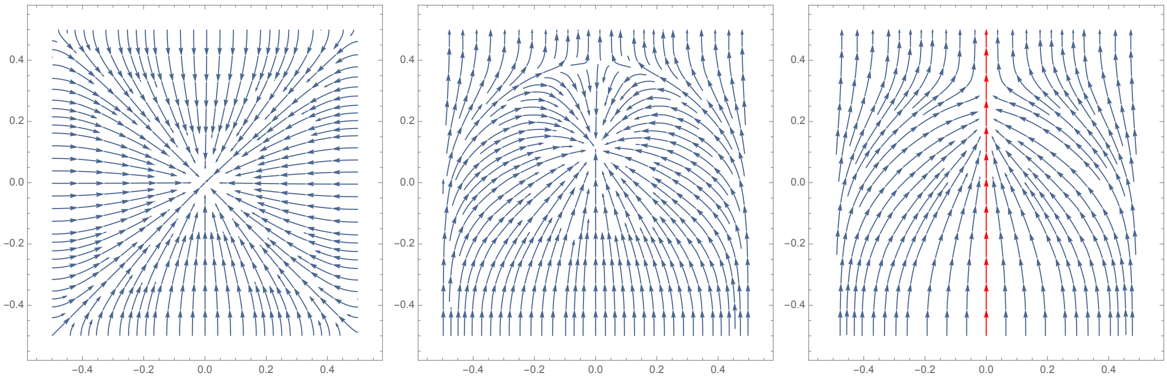
$$\begin{gathered} -\,\mathrm{div}_y(\Gamma^{F_0}\tilde{\mathrm{m}}_{0,k}) = 0, \\ \partial_t\langle\tilde{\mathrm{m}}_{0,k}\rangle = \mathrm{div}_x\langle\Gamma^{F_0}\tilde{\mathrm{m}}_{0,k}\rangle, \qquad \langle\tilde{\mathrm{m}}_{0,k}\rangle|_{t=0} = \tilde{\mathrm{m}}^\circ_k, \end{gathered}$$

where the first equation implies for $\tilde{\mathrm{m}}_{0,k}$ a similar decomposition (8.263) as in Step 1. By Step 2, since we have by construction $\tilde{\mathrm{m}}^\circ_{\varepsilon,k}(\mathbb{R}^2 \setminus \eta_\varepsilon B'_k) = 0$ for all ε, we deduce $\tilde{\mathrm{m}}^t_{0,k}(x, y) = \langle\tilde{\mathrm{m}}^t_{0,k}(x, \cdot)\rangle\mu_k(y)$. Inserting this form into the above equations, we find

$$\partial_t\langle\tilde{\mathrm{m}}_{0,k}\rangle = \mathrm{div}\,(\Gamma^{F_0}_k\langle\tilde{\mathrm{m}}_{0,k}\rangle), \qquad \Gamma^{F_0}_k := \langle\Gamma^{F_0}\mu_k\rangle, \qquad \langle\tilde{\mathrm{m}}_{0,k}\rangle|_{t=0} = \tilde{\mathrm{m}}^\circ_k.$$

This is now a linear transport equation for $\langle\tilde{\mathrm{m}}_{0,k}\rangle$. Uniqueness allows us to get rid of all extractions of subsequences, and the conclusion follows, since by linearity we necessarily have $\tilde{\mathrm{m}}_0 = \sum_{k=1}^{K_{F_0}} \tilde{\mathrm{m}}_{0,k}$, where $\tilde{\mathrm{m}}_0$ is the weak limit extracted in Step 1. □

As noticed in Remark 8.9.9(a), the question of determining the depinning rate at the depinning threshold is of particular interest. While obtaining a complete answer seems difficult due to the variety of possible dynamical behaviors, we consider the simplest situation when the depinning is due to the bifurcation of a unique stable fixed point into a stable periodic orbit. A square-root power law is then obtained under some non-degeneracy condition. An additional assumption is made for simplicity, which reduces the computation to a 1D setting (being then comparable to some explicit computations in [1, 60, 250]; see also [147, 149]). This assumption is typically satisfied for $\beta = 0$ and for a forcing F_0 that is parallel to a coordinate axis when the pinning potential $\tilde{h}^0$ has similar symmetries as in the example of Figure 8.4 (see indeed Figure 8.5). Yet, we believe that the same result holds in more general situations.



(a) $\kappa = 0$ (b) $0 < \kappa < \kappa_c = \pi$ (c) $\kappa > \kappa_c = \pi$

Figure 8.5 – In dimension $d = 2$, for the typical example of pinning potential $\tilde{h}^0$ given in Figure 8.4, with $\alpha = 1$, $\beta = 0$, we plot the stream lines of the vector field $-\Gamma^{(0,\kappa)}$ for growing values of κ. The assumptions of Proposition 8.9.11 are clearly seen to be satisfied: for $\kappa < \kappa_c = \pi$ there is a unique stable fixed point, while for $\kappa > \kappa_c = \pi$ the stable fixed point gives way to a periodic orbit with image $\mathcal{O} = \{0\} \times [-1/2, 1/2)$.

Proposition 8.9.11. *Let $e \in \mathbb{S}^{d-1}$ be some direction, and consider equation (8.257) with $F_0 = \kappa e$. Assume that the vector field $-\Gamma^{\kappa e}$ has a unique stable invariant set for all $\kappa \geq 0$, and assume that there exists a critical value $\kappa_c > 0$ such that this invariant set is a fixed point for $0 \leq \kappa < \kappa_c$, and is a periodic orbit for $\kappa > \kappa_c$. Further assume that the image of the periodic orbit $\mathcal{O} \subset Q$ remains the same for all $\kappa > \kappa_c$. Assume that $\tilde{h}^0$ is smooth, and is non-degenerate in the following sense: for all x and all $|v| = 1$, if $v \cdot \nabla(\alpha\nabla - \beta\nabla^{\perp})\tilde{h}^0(x) = 0$ holds, then $(v \cdot \nabla)^2(\alpha\nabla - \beta\nabla^{\perp})\tilde{h}^0(x) \neq 0$. Then the effective velocity $\Gamma_1^{\kappa e}$ defined in Theorem 8.9.8 satisfies as $\kappa \downarrow \kappa_c$,*

$$\Gamma_1^{\kappa e} = C(1 + o(1))(\kappa - \kappa_c)^{1/2} e,$$

for some constant $C > 0$ depending on the shape of the pinning potential $\tilde{h}^0$. ◊

Remarks 8.9.12.

(a) While Proposition 8.9.11 above is proved in the particularly simple situation of the bifurcation of a fixed point into a periodic orbit, it would be interesting to determine the best general lower bound on the Hölder regularity of the multivalued map $F_0 \mapsto \{\Gamma_1^{F_0}, \ldots, \Gamma_{K_{F_0}}^{F_0}\}$ at the depinning threshold, for smooth $\tilde{h}^0$. We do not pursue this question here, but note that at least the continuity of this map essentially follows from the argument in [319, Section 7.2] together with the result on circle maps in [366, Theorem I.1].

(b) Without the non-degeneracy assumption for the pinning potential $\tilde{h}^0$, the behavior can be very different: if $\tilde{h}^0$ is degenerate at order k for some $0 \leq k \leq \infty$, in the sense that the power 2 in the expansion (8.265) near the critical point is replaced by a power $k + 2$, then we indeed rather obtain $\Gamma_1^{\kappa e} \sim C(\kappa - \kappa_c)^{1-1/(k+2)} e$ as $\kappa \downarrow \kappa_c$. (Although in this case the effective velocity $\Gamma_1^{\kappa e}$ is still a Hölder function of κ, and is at least of class $C^{1/2}$, examples of non-smooth pinning potentials $\tilde{h}^0 \in C^{0,1}(\mathbb{R}^d)$ can be constructed for which the Hölder property fails at $\kappa = \kappa_c$; see e.g. [250, Example 1.3].) ◊

Proof of Proposition 8.9.11. Choose an arc-length parametrization $(\phi^t)_{0 \leq t \leq T}$ of the periodic orbit $\mathcal{O}$, where $|\partial_t \phi^t| = 1$ for all $t \geq 0$, and where the period $T \in \mathbb{R}^+$ is the total length of the orbit. Since $\mathcal{O}$ is the image of the (unique stable) periodic orbit of $-\Gamma^{\kappa e}$ for all $\kappa > \kappa_c$, we find $\partial_t \phi^t =$

$-\Gamma^{\kappa e}(\phi^t)/|\Gamma^{\kappa e}(\phi^t)|$ for all $t \geq 0$. We then deduce that for all $\kappa > \kappa_c$ the unique stable ergodic invariant measure $\mu_\kappa \in \mathcal{P}_{\mathrm{per}}(Q)$ has the following form, for all test function $f \in C^\infty_{\mathrm{per}}(Q)$,

$$\int_Q f d\mu_\kappa = \Big(\int_0^T f(\phi^t)|\Gamma^{\kappa e}(\phi^t)|^{-1}dt\Big)\Big(\int_0^T |\Gamma^{\kappa e}(\phi^t)|^{-1}dt\Big)^{-1},$$

so that according to Theorem 8.9.8 the effective velocity is given by

$$\Gamma_1^{\kappa e} = \Big(\int_0^T \Gamma^{\kappa e}(\phi^t)|\Gamma^{\kappa e}(\phi^t)|^{-1}dt\Big)\Big(\int_0^T |\Gamma^{\kappa e}(\phi^t)|^{-1}dt\Big)^{-1} = (\phi^0 - \phi^T)\Big(\int_0^T |\Gamma^{\kappa e}(\phi^t)|^{-1}dt\Big)^{-1}.$$

Now setting $\tilde{e} := \phi^T - \phi^0$, we obtain

$$-\Gamma_1^{\kappa e} = \Big(\int_0^T |\Gamma^{\kappa e}(\phi^t)|^{-1}dt\Big)^{-1}\tilde{e}.$$

Consider the finite collection $(t_j)_{j=1}^J$ of all points $t \in [0,T]$ such that $\Gamma^{\kappa_c e}(\phi^t) = 0$. By smoothness of $\tilde{h}^0$ and by the minimality assumption defining κ_c, the function $f(t) := |\Gamma^{\kappa_c e}(\phi^t)|$ is smooth, hence satisfies for all j,

$$f'(t_j) = 0, \qquad f''(t_j) \geq 0, \tag{8.264}$$

and also

$$0 = \partial_t \Gamma^{\kappa_c e}(\phi^t)|_{t=t_j} = \partial_t \phi^{t_j} \cdot \nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}).$$

A direct computation then yields

$$\begin{aligned} f''(t_j) &= |\Gamma^{\kappa_c e}(\phi^{t_j})|^{-1}|\partial_t\phi^{t_j} \cdot \nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j})|^2 \\ &\qquad - 2|\Gamma^{\kappa_c e}(\phi^{t_j})|^{-1}|\partial_t\phi^{t_j} \cdot \nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}) \cdot \partial_t\phi^{t_j}|^2 \\ &\qquad + |\Gamma^{\kappa_c e}(\phi^{t_j})|^{-1}\partial_t\phi^{t_j} \cdot \nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}) \cdot \nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}) \cdot \partial_t\phi^{t_j} \\ &\qquad + (\partial_t\phi^{t_j})^{\otimes 3} \odot \nabla^2(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}) \\ &= (\partial_t\phi^{t_j})^{\otimes 3} \odot \nabla^2(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}), \end{aligned}$$

where $\odot$ denotes the complete contraction of 3-tensors. The non-degeneracy assumption now implies $f''(t_j) \neq 0$. Combined with (8.264), this yields

$$2C_j := f''(t_j) = (\partial_t\phi^{t_j})^{\otimes 3} \odot \nabla^2(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}) > 0.$$

A Taylor expansion around t_j allows to write for $|t - t_j| \ll 1$,

$$|\Gamma^{\kappa_c e}(\phi^t)| = C_j(t-t_j)^2 + O((t-t_j)^3). \tag{8.265}$$

Let $\delta > 0$ be small enough such that $|t_j - t_{j+1}| > 2\delta$ is satisfied for all j, and define

$$I_\delta := \biguplus_{j=1}^J [t_j - \delta, t_j + \delta], \qquad c_\delta := \inf_{t\in[0,T]\setminus I_\delta} |\Gamma^{\kappa_c e}(\phi^t)| > 0.$$

For $\kappa > \kappa_c$ sufficiently close to κ_c, we may then compute

$$\begin{aligned} \int_{[0,T]\setminus I_\delta} |\Gamma^{\kappa e}(\phi^t)|^{-1}dt = (\alpha^2+\beta^2)^{-1/2} &\int_{[0,T]\setminus I_\delta} |\nabla\tilde{h}^0(\phi^t) - \kappa e|^{-1}dt \\ &\leq T(\alpha^2+\beta^2)^{-1/2}(c_\delta - |\kappa - \kappa_c|)^{-1}, \end{aligned}$$

and hence, setting for simplicity $e_{\alpha,\beta}:=\alpha e-\beta e^\perp$,

$$\begin{aligned}\int_0^T|\Gamma^{\kappa e}(\phi^t)|^{-1}dt&\le\int_{I_\delta}|\Gamma^{\kappa_c e}(\phi^t)-(\kappa-\kappa_c)e_{\alpha,\beta}|^{-1}dt+T(\alpha^2+\beta^2)^{-1/2}(c_\delta-|\kappa-\kappa_c|)^{-1}\\
&\le\sum_{j=1}^J\int_{t_j-\delta}^{t_j+\delta}\big(|C_j(t-t_j)^2\partial_t\phi^{t_j}-(\kappa-\kappa_c)e_{\alpha,\beta}|-C|t-t_j|^3\big)^{-1}dt+C(c_\delta-|\kappa-\kappa_c|)^{-1}\\
&=2\sum_{j=1}^J\int_0^\delta\Big(\big(C_j^2t^4+(\kappa-\kappa_c)^2-2C_jt^2(\kappa-\kappa_c)\partial_t\phi^{t_j}\cdot e_{\alpha,\beta}\big)^{1/2}-Ct^3\Big)^{-1}dt+C(c_\delta-|\kappa-\kappa_c|)^{-1}\\
&=\frac{2}{C_j^{1/2}(\kappa-\kappa_c)^{1/2}}\sum_{j=1}^J\int_0^{\delta\left(\frac{C_j}{\kappa-\kappa_c}\right)^{1/2}}\Big(\big(t^4-2t^2\partial_t\phi^{t_j}\cdot e_{\alpha,\beta}+1\big)^{1/2}-CC_j^{-1/2}(\kappa-\kappa_c)^{1/2}t^3\Big)^{-1}dt\\
&\qquad+C(c_\delta-|\kappa-\kappa_c|)^{-1}.\end{aligned}$$

Multiplying by $(\kappa-\kappa_c)^{1/2}$ and letting $\kappa\downarrow\kappa_c$, this yields

$$\limsup_{\kappa\downarrow\kappa_c}(\kappa-\kappa_c)^{1/2}|\Gamma_1^{\kappa e}|^{-1}\le\frac{2}{|\tilde e|C_j^{1/2}}\sum_{j=1}^J\int_0^\infty(t^4-2t^2\partial_t\phi^{t_j}\cdot e_{\alpha,\beta}+1)^{-1/2}dt. \tag{8.266}$$

Symmetrically, we have a similar lower bound

$$\begin{aligned}&\int_0^T|\Gamma^{\kappa e}(\phi^t)|^{-1}dt\\
&\ge\sum_{j=1}^J\int_{t_j-\delta}^{t_j+\delta}\big(|C_j(t-t_j)^2\partial_t\phi^{t_j}-(\kappa-\kappa_c)e_{\alpha,\beta}|+C|t-t_j|^3\big)^{-1}dt-C(c_\delta-|\kappa-\kappa_c|)^{-1}\\
&=\frac{2}{C_j^{1/2}(\kappa-\kappa_c)^{1/2}}\sum_{j=1}^J\int_0^{\delta\left(\frac{C_j}{\kappa-\kappa_c}\right)^{1/2}}\Big(\big(t^4-2t^2\partial_t\phi^{t_j}\cdot e_{\alpha,\beta}+1\big)^{1/2}+CC_j^{-1/2}(\kappa-\kappa_c)^{1/2}t^3\Big)^{-1}dt\\
&\qquad-C(c_\delta-|\kappa-\kappa_c|)^{-1},\end{aligned}$$

so that equality actually holds in (8.266),

$$\lim_{\kappa\downarrow\kappa_c}(\kappa-\kappa_c)^{1/2}|\Gamma_1^{\kappa e}|^{-1}=\frac{2}{|\tilde e|C_j^{1/2}}\sum_{j=1}^J\int_0^\infty(t^4-2t^2\partial_t\phi^{t_j}\cdot e_{\alpha,\beta}+1)^{-1/2}dt,$$

and the result follows. ☐

8.9.6 Small applied force implies macroscopic frozenness

Beyond diagonal regimes, we may at least prove the following intuitive result: in the presence of a small applied force $\|F\|_{\mathrm{L}^\infty}\ll\|\nabla h\|_{\mathrm{L}^\infty}$, but with fast oscillating pinning potential, the vortices are pinned in the limit. Based on energy methods, the proof below is limited to the subcritical Ginzburg-Landau regimes (GL_1') and (GL_2').

Proposition 8.9.13. *Let $\alpha>0$, $\beta\in\mathbb{R}$, $\alpha^2+\beta^2=1$, let Assumption 8.1.1(a) hold with the initial data $(u_\varepsilon^\circ,\mathrm{v}_\varepsilon^\circ,\mathrm{v}^\circ)$ satisfying the well-preparedness condition* (8.16)*, and assume that*

$$1\ll N_\varepsilon\ll|\log\varepsilon|,\qquad\frac{N_\varepsilon}{|\log\varepsilon|}\ll\lambda_\varepsilon\lesssim1,\qquad\frac{\varepsilon}{\lambda_\varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2}}\ll\eta_\varepsilon\ll1,$$
$$h(x):=\lambda_\varepsilon\eta_\varepsilon\hat h^0(x,x/\eta_\varepsilon),\qquad\|F\|_{W^{1,\infty}}\ll\lambda_\varepsilon,$$

with $\hat h^0$ independent of ε. Let $u_\varepsilon:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{C}$ be the solution of (8.6) *as in Proposition 8.2.2(i). We consider the regime* (GL_1') *with* $\mathrm{v}_\varepsilon^\circ=\mathrm{v}^\circ$, *and the regime* (GL_2') *with* $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)=0$. *Then for all $\gamma\in(0,1)$ there holds $N_\varepsilon^{-1}\mu_\varepsilon\overset{*}{\rightharpoonup}\operatorname{curl}\mathrm{v}^\circ$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;(C_c^{0,\gamma}(\mathbb{R}^2))^*)$.* ◊

Proof. We choose $\mathrm{v}_\varepsilon:=\mathrm{v}_\varepsilon^\circ$ in the definition of the modulated energy (8.12), thus setting for all $z\in\mathbb{R}^2$,

$$\mathcal{E}_{\varepsilon,R}^z:=\int_{\mathbb{R}^2}\frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big),\qquad \mathcal{D}_{\varepsilon,R}^z:=\mathcal{E}_{\varepsilon,R}^z-\frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2}a\chi_R^z\mu_\varepsilon,$$

and $\mathcal{E}_{\varepsilon,R}^*:=\sup_z\mathcal{E}_{\varepsilon,R}^z$, $\mathcal{D}_{\varepsilon,R}^*:=\sup_z\mathcal{D}_{\varepsilon,R}^z$ (where the suprema implicitly run over $z\in R\mathbb{Z}^2$). We further consider the following modification of this modulated energy, including suitable lower-order terms,

$$\hat{\mathcal{E}}_{\varepsilon,R}^z:=\int_{\mathbb{R}^2}\frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2+(1-|u_\varepsilon|^2)(f-N_\varepsilon^2|\mathrm{v}_\varepsilon^\circ|^2-N_\varepsilon|\log\varepsilon|\,\mathrm{v}_\varepsilon^\circ\cdot F^\perp)\Big),$$

and $\hat{\mathcal{E}}_{\varepsilon,R}^*:=\sup_z\hat{\mathcal{E}}_{\varepsilon,R}^z$. The lower bound assumption on the pin separation η_ε allows to choose the cut-off length $R\ge1$ in such a way that

$$\lambda_\varepsilon^{-1}\ll R\ll\varepsilon^{-1}\frac{(N_\varepsilon|\log\varepsilon|)^{1/2}}{\lambda_\varepsilon|\log\varepsilon|^2},\qquad R\ll\eta_\varepsilon\varepsilon^{-1}(N_\varepsilon|\log\varepsilon|)^{1/2}.$$

By Proposition 8.5.2, the assumption on the initial data implies $\mathcal{E}_{\varepsilon,R}^{*,\circ}\le C_0N_\varepsilon|\log\varepsilon|$ for some $C_0\simeq1$. Let $T>0$ be fixed, and define $T_\varepsilon>0$ as the maximum time $\le T$ such that the bound $\mathcal{E}_{\varepsilon,R}^{*,t}\le(C_0+1)N_\varepsilon|\log\varepsilon|$ holds for all $t\le T_\varepsilon$. Note that, using the bound $\|f\|_{\mathrm{L}^\infty}\lesssim\lambda_\varepsilon\eta_\varepsilon^{-1}+\lambda_\varepsilon^2|\log\varepsilon|^2$ (cf. (8.7)), the choice of η_ε and R, and the assumption $\|\mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^2\cap\mathrm{L}^\infty(B_{2R})}\lesssim_\theta R^\theta$ for all $\theta>0$, we deduce for all $t\le T_\varepsilon$,

$$\begin{aligned}|\hat{\mathcal{E}}_{\varepsilon,R}^{z,t}-\mathcal{E}_{\varepsilon,R}^{z,t}|&\lesssim\int_{\mathbb{R}^2}\chi_R^z|1-|u_\varepsilon^t|^2|(|f|+N_\varepsilon^2|\mathrm{v}_\varepsilon^\circ|^2+N_\varepsilon|\log\varepsilon||\mathrm{v}_\varepsilon^\circ||F|)\\&\lesssim\varepsilon R(\lambda_\varepsilon\eta_\varepsilon^{-1}+\lambda_\varepsilon^2|\log\varepsilon|^2)(\mathcal{E}_{\varepsilon,R}^{z,t})^{1/2}+\varepsilon R^\theta o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|)(\mathcal{E}_{\varepsilon,R}^{z,t})^{1/2}\ll\lambda_\varepsilon N_\varepsilon|\log\varepsilon|,\end{aligned}\tag{8.267}$$

hence in particular $\hat{\mathcal{E}}_{\varepsilon,R}^{*,t}\lesssim N_\varepsilon|\log\varepsilon|$ for all $t\le T_\varepsilon$. We split the proof into three steps.

Step 1. Evolution of the modulated energy.

In this step, for all $\varepsilon>0$ small enough, we show that $T_\varepsilon=T$, and that for all $t\le T$,

$$\frac{\lambda_\varepsilon\alpha}{4}\int_0^t\int_{\mathbb{R}^2}a\chi_R^z|\partial_tu_\varepsilon|^2\le\hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ}-\hat{\mathcal{E}}_{\varepsilon,R}^{z,t}+o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|)\lesssim_t N_\varepsilon|\log\varepsilon|.\tag{8.268}$$

The time derivative of the modulated energy $\hat{\mathcal{E}}_{\varepsilon,R}^z$ is computed as follows, by integration by parts,

$$\begin{aligned}\partial_t\hat{\mathcal{E}}_{\varepsilon,R}^z&=\int_{\mathbb{R}^2}a\chi_R^z\Big(\langle\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ,\nabla\partial_tu_\varepsilon\rangle-N_\varepsilon\mathrm{v}_\varepsilon^\circ\cdot\langle\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ,i\partial_tu_\varepsilon\rangle\\&\qquad-\frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)\langle u_\varepsilon,\partial_tu_\varepsilon\rangle-(f-N_\varepsilon^2|\mathrm{v}_\varepsilon^\circ|^2-N_\varepsilon|\log\varepsilon|\,\mathrm{v}_\varepsilon^\circ\cdot F^\perp)\langle u_\varepsilon,\partial_tu_\varepsilon\rangle\Big)\\&=-\int_{\mathbb{R}^2}a\chi_R^z\Big\langle\triangle u_\varepsilon+\frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)+\nabla h\cdot\nabla u_\varepsilon+i|\log\varepsilon|F^\perp\cdot\nabla u_\varepsilon+fu_\varepsilon,\partial_tu_\varepsilon\Big\rangle\\&\qquad+N_\varepsilon\int_{\mathbb{R}^2}a\chi_R^z(\mathrm{v}_\varepsilon^\circ\cdot\nabla h+\operatorname{div}\mathrm{v}_\varepsilon^\circ)\langle\partial_tu_\varepsilon,iu_\varepsilon\rangle-\int_{\mathbb{R}^2}a\nabla\chi_R^z\cdot\langle\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ,\partial_tu_\varepsilon\rangle\\&\qquad-\int_{\mathbb{R}^2}a\chi_R^z(|\log\varepsilon|F^\perp+2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ,i\partial_tu_\varepsilon\rangle,\end{aligned}$$

hence, inserting equation (8.6) in the first right-hand side term,

$$\partial_t\hat{\mathcal{E}}^z_{\varepsilon,R} = -\lambda_\varepsilon\alpha\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 - \int_{\mathbb{R}^2} a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon \mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle$$
$$+ N_\varepsilon\int_{\mathbb{R}^2}\chi_R^z\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle - \int_{\mathbb{R}^2} a\nabla\chi_R^z\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, \partial_t u_\varepsilon\rangle.$$

In particular, using the energy bound $\mathcal{E}^{*,t}_{\varepsilon,R}\lesssim N_\varepsilon|\log\varepsilon|$, we find for all $t\le T_\varepsilon$,

$$\begin{aligned}\partial_t\hat{\mathcal{E}}^z_{\varepsilon,R} &\le -\frac{\lambda_\varepsilon\alpha}{2}\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 - \int_{\mathbb{R}^2} a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle\\
&\qquad + C_t\lambda_\varepsilon^{-1}N_\varepsilon^2\int_{\mathbb{R}^2}\chi_R^z|\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)|^2(1+|1-|u_\varepsilon|^2|) + C_t\lambda_\varepsilon^{-1}R^{-2}\int_{B_{2R}(z)}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2\\
&\le -\frac{\lambda_\varepsilon\alpha}{2}\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 - \int_{\mathbb{R}^2} a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle\\
&\qquad + C_t\lambda_\varepsilon^{-1}N_\varepsilon^2\|\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)\|^2_{\mathrm{L}^2\cap\mathrm{L}^\infty(B_{2R})} + C_t\lambda_\varepsilon^{-1}R^{-2}N_\varepsilon|\log\varepsilon|,\end{aligned}$$

so that the assumptions on $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)$ and the choice of the cut-off length R yield

$$\partial_t\hat{\mathcal{E}}^z_{\varepsilon,R} \le -\frac{\lambda_\varepsilon\alpha}{2}\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 - \int_{\mathbb{R}^2} a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle$$
$$+ o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|). \quad (8.269)$$

Again using the Cauchy-Schwarz inequality to estimate the second right-hand side term, with $\|F\|_{\mathrm{L}^\infty}\lesssim\lambda_\varepsilon$ and $\|\mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^\infty}\lesssim 1$, we find the following rough a priori estimate,

$$\begin{aligned}\partial_t\hat{\mathcal{E}}^z_{\varepsilon,R} &\le -\frac{\lambda_\varepsilon\alpha}{4}\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 + C\lambda_\varepsilon|\log\varepsilon|^2\int_{\mathbb{R}^2} a\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2 + o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|)\\
&\le -\frac{\lambda_\varepsilon\alpha}{4}\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 + O_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|^3),\end{aligned}$$

and thus, integrating in time with $\lambda_\varepsilon\lesssim 1$, we find for all $t\le T_\varepsilon$,

$$\frac{\lambda_\varepsilon\alpha}{4}\int_0^t\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 \le \hat{\mathcal{E}}^{z,\circ}_{\varepsilon,R} - \hat{\mathcal{E}}^{z,t}_{\varepsilon,R} + o_t(|\log\varepsilon|^4)\lesssim_t |\log\varepsilon|^4.$$

This rough estimate now allows us to apply the product estimate in Lemma 8.5.4 (with $\mathrm{v}_\varepsilon = \mathrm{v}_\varepsilon^\circ$ and $\mathrm{p}_\varepsilon = 0$), using $|\log\varepsilon|\|F\|_{\mathrm{L}^\infty} + N_\varepsilon \ll \lambda_\varepsilon|\log\varepsilon|$, to the effect of

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{R}^2} a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle\Big|\\
&\lesssim\ \frac{|\log\varepsilon|\|F\|_{\mathrm{L}^\infty} + N_\varepsilon}{|\log\varepsilon|}\Big(\int_0^t\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 + \int_0^t\int_{\mathbb{R}^2} a\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2\Big) + o_t(1)\\
&\lesssim\ o(\lambda_\varepsilon)\int_0^t\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 + o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|).\end{aligned}$$

Inserting this into (8.269) and integrating in time, we find for all $t\le T_\varepsilon$,

$$\hat{\mathcal{E}}^{z,t}_{\varepsilon,R} - \hat{\mathcal{E}}^{z,\circ}_{\varepsilon,R} \le -\Big(\frac{\lambda_\varepsilon\alpha}{2} - o(\lambda_\varepsilon)\Big)\int_0^t\int_{\mathbb{R}^2} a\chi_R^z|\partial_t u_\varepsilon|^2 + o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|),$$

and the result (8.268) follows for all $t \le T_\varepsilon$. In particular, combined with (8.267), this yields for all $t \le T_\varepsilon$,

$$\mathcal{E}_{\varepsilon,R}^{z,t} \le \hat{\mathcal{E}}_{\varepsilon,R}^{z,t} + o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \le \hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ} + o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \le \mathcal{E}_{\varepsilon,R}^{z,\circ} + o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \le (C_0 + o_t(1))N_\varepsilon|\log\varepsilon|,$$

and thus, taking the supremum in z, the conclusion $T_\varepsilon = T$ follows for $\varepsilon > 0$ small enough.

Step 2. Lower bound on the modulated energy.

In this step, we prove that for all $t \le T$,

$$\mathcal{E}_{\varepsilon,R}^{z,t} \ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\mu_\varepsilon^t - o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|),$$

and hence, combined with the well-preparedness assumption $\mathcal{D}_{\varepsilon,R}^{z,\circ} = \mathcal{E}_{\varepsilon,R}^{z,\circ} - \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\mu_\varepsilon^\circ \le o(N_\varepsilon^2)$, and with (8.267),

$$\hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ} - \hat{\mathcal{E}}_{\varepsilon,R}^{z,t} \le \mathcal{E}_{\varepsilon,R}^{z,\circ} - \mathcal{E}_{\varepsilon,R}^{z,t} + o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \le \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z(\mu_\varepsilon^\circ - \mu_\varepsilon^t) + o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|).$$

As we show, this is a simple consequence of Lemma 8.5.1. (However note that we may not directly apply Proposition 8.5.2(i)–(iii), since in the present situation the assumption $R \gtrsim |\log\varepsilon|$ does not hold.) Noting that $\|\nabla(a\chi_R^z)\|_{\mathrm{L}^\infty} \lesssim \lambda_\varepsilon + R^{-1} \lesssim \lambda_\varepsilon$, we deduce from Lemma 8.5.1(i) with $\phi = a\chi_R^z$, $\mathcal{E}_{\varepsilon,R}^* \lesssim_t N_\varepsilon|\log\varepsilon|$, and $e^{-N_\varepsilon} \lesssim r \ll 1$,

$$\begin{aligned}
\mathcal{E}_{\varepsilon,R}^z &\ge \frac{\log(r/\varepsilon)}{2}\int_{\mathbb{R}^2} a\chi_R^z|\nu_{\varepsilon,R}^r| - O_t(\lambda_\varepsilon rN_\varepsilon|\log\varepsilon|) - O_t(r^2N_\varepsilon^2) - O_t(N_\varepsilon\log N_\varepsilon)\\
&\ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z|\nu_{\varepsilon,R}^r| - O(|\log r|)\int_{\mathbb{R}^2}\chi_R^z|\nu_{\varepsilon,R}^r| - o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|),
\end{aligned}$$

hence by Lemma 8.5.1(ii), with the choice of the radius $r \gtrsim e^{-N_\varepsilon}$,

$$\mathcal{E}_{\varepsilon,R}^z \ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z|\nu_{\varepsilon,R}^r| - O_t(N_\varepsilon|\log r|) - o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\nu_{\varepsilon,R}^r - o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|).$$

By Lemma 8.5.1(iii) in the form (8.125) with $\gamma = 1$, and by (8.142), using again $\|\nabla(a\chi_R^z)\|_{\mathrm{L}^\infty} \lesssim \lambda_\varepsilon$, we may now replace $\nu_{\varepsilon,R}^r$ by μ_ε in the right-hand side,

$$\begin{aligned}
\mathcal{E}_{\varepsilon,R}^z \ge \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\chi_R^z\mu_\varepsilon &- \lambda_\varepsilon|\log\varepsilon|O_t\big(\varepsilon RN_\varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2} + rN_\varepsilon\big)\\
&- |\log\varepsilon|O_t\big(\varepsilon^{1/2}N_\varepsilon|\log\varepsilon|\big) - o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|),
\end{aligned}$$

and the result follows from the choice of $R \ll \varepsilon^{-1}(N_\varepsilon|\log\varepsilon|)^{-1/2}$.

Step 3. Estimate on the total vorticity.

In this step we show for all $t \le T$,

$$\Big|\int_{\mathbb{R}^2} a\chi_R^z(\mu_\varepsilon^t - \mu_\varepsilon^\circ)\Big| \ll_t \lambda_\varepsilon N_\varepsilon.$$

We first prove (a weaker version of) the result with the weight a replaced by 1, and the conclusion then follows by noting that $a = \exp(\lambda_\varepsilon\eta_\varepsilon\hat{h}^0)$ indeed quickly converges to 1 as $\varepsilon \downarrow 0$. Using identity (8.102), we write

$$\begin{aligned}
\int_{\mathbb{R}^2}\chi_R^z(\mu_\varepsilon^t - \mu_\varepsilon^\circ) &= \int_0^t\int_{\mathbb{R}^2}\chi_R^z\partial_t\mu_\varepsilon^t = \int_0^t\int_{\mathbb{R}^2}\chi_R^z\operatorname{curl}V_\varepsilon^t = -\int_0^t\int_{\mathbb{R}^2}\nabla^\perp\chi_R^z\cdot V_\varepsilon^t\\
&= -2\int_0^t\int_{\mathbb{R}^2}\nabla^\perp\chi_R^z\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle + N_\varepsilon\int_0^t\int_{\mathbb{R}^2}\nabla^\perp\chi_R^z\cdot\mathrm{v}_\varepsilon^\circ\,\partial_t(1-|u_\varepsilon|^2).
\end{aligned}$$

Applying the product estimate of Lemma 8.5.4 as in Step 1, with $|\nabla\chi_R|\lesssim R^{-1}\chi_R^{1/2}$, we find for all $|\log\varepsilon|^{-2}\lesssim K\lesssim|\log\varepsilon|^2$ and for all $t\le T$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\chi_R^z(\mu_\varepsilon^t-\mu_\varepsilon^\circ)\Big|&\lesssim\frac{1}{|\log\varepsilon|}\Big(K^{-2}\int_0^t\int_{\mathbb{R}^2}\chi_R^z|\partial_t u_\varepsilon|^2+K^2R^{-2}\int_0^t\int_{B_{2R}}|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon^\circ|^2\Big)\\&\qquad+o_t(|\log\varepsilon|^{-1})+N_\varepsilon\int_{\mathbb{R}^2}|1-|u_\varepsilon^t|^2||\nabla^\perp\chi_R^z|+N_\varepsilon\int_{\mathbb{R}^2}|1-|u_\varepsilon^\circ|^2||\nabla^\perp\chi_R^z|\\&\lesssim_t\frac{K^{-2}}{|\log\varepsilon|}\int_0^t\int_{\mathbb{R}^2}\chi_R^z|\partial_t u_\varepsilon|^2+K^2R^{-2}N_\varepsilon+\varepsilon N_\varepsilon|\log\varepsilon|+o(|\log\varepsilon|^{-1}).\end{aligned}$$

Using (8.268) to estimate the first right-hand side term, and choosing $\lambda_\varepsilon^{-1}\ll K^2\ll\lambda_\varepsilon R^2$, we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\chi_R^z(\mu_\varepsilon^t-\mu_\varepsilon^\circ)\Big|&\lesssim_t\frac{K^{-2}}{\lambda_\varepsilon|\log\varepsilon|}(\hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ}-\hat{\mathcal{E}}_{\varepsilon,R}^{z,t})_++o(K^{-2}N_\varepsilon)+K^2R^{-2}N_\varepsilon+o(|\log\varepsilon|^{-1})\\&\lesssim_t o(|\log\varepsilon|^{-1})(\hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ}-\hat{\mathcal{E}}_{\varepsilon,R}^{z,t})_++o(\lambda_\varepsilon N_\varepsilon).\end{aligned}\tag{8.270}$$

It remains to smuggle the weight a into the left-hand side. For all $t\le T$, applying Lemma 8.5.1(iii) in the form (8.125) with $\gamma=1$, as well as (8.142), and using the choice of $R\ll\varepsilon^{-1}(N_\varepsilon|\log\varepsilon|)^{-1/2}$, we find for any total radius $\varepsilon^{1/2}<r\ll1$,

$$\Big|\int_{\mathbb{R}^2}(1-a)\chi_R^z(\mu_\varepsilon^t-\nu_{\varepsilon,R}^{r,t})\Big|\lesssim_t\lambda_\varepsilon rN_\varepsilon+\varepsilon^{1/2}N_\varepsilon|\log\varepsilon|+\lambda_\varepsilon\varepsilon RN_\varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2}\ll\lambda_\varepsilon N_\varepsilon,$$

and hence, by Lemma 8.5.1(ii) with $\|1-a\|_{\mathrm{L}^\infty}\lesssim\lambda_\varepsilon\eta_\varepsilon\ll\lambda_\varepsilon$,

$$\Big|\int_{\mathbb{R}^2}(1-a)\chi_R^z\mu_\varepsilon^t\Big|\lesssim\|1-a\|_{\mathrm{L}^\infty}\int_{\mathbb{R}^2}\chi_R^z|\nu_{\varepsilon,R}^{r,t}|+o(\lambda_\varepsilon N_\varepsilon)\ll\lambda_\varepsilon N_\varepsilon.$$

Combining this with (8.270) and with the result of Step 2, we deduce

$$\Big|\int_{\mathbb{R}^2}a\chi_R^z(\mu_\varepsilon^t-\mu_\varepsilon^\circ)\Big|\lesssim_t o(|\log\varepsilon|^{-1})(\hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ}-\hat{\mathcal{E}}_{\varepsilon,R}^{z,t})_++o(\lambda_\varepsilon N_\varepsilon)\lesssim_t o(1)\Big|\int_{\mathbb{R}^2}a\chi_R^z(\mu_\varepsilon^t-\mu_\varepsilon^\circ)\Big|+o(\lambda_\varepsilon N_\varepsilon),$$

and the result follows.

Step 4. Conclusion.

Combining the results of Steps 1–2 with the well-preparedness assumption $\hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ}\le\frac12|\log\varepsilon|\int_{\mathbb{R}^2}a\chi_R^z\mu_\varepsilon^\circ+o(N_\varepsilon^2)$, we find

$$\frac{\lambda_\varepsilon\alpha}{2}\int_0^T\int_{\mathbb{R}^2}a\chi_R^z|\partial_t u_\varepsilon|^2\le|\log\varepsilon|\int_{\mathbb{R}^2}a\chi_R^z(\mu_\varepsilon^\circ-\mu_\varepsilon^T)+o_T(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|),$$

and hence by the result of Step 3,

$$\int_0^T\int_{\mathbb{R}^2}a\chi_R^z|\partial_t u_\varepsilon|^2\ll_T N_\varepsilon|\log\varepsilon|.$$

The product estimate of [395, Appendix A] (see also Lemma 8.5.4) then yields for all $X\in W^{1,\infty}([0,T]\times\mathbb{R}^2)^2$ and all $|\log\varepsilon|^{-1}\lesssim K\lesssim|\log\varepsilon|$,

$$\begin{aligned}\Big|\int_0^T\int_{\mathbb{R}^2}\chi_R^zX\cdot V_\varepsilon\Big|&\lesssim\frac{1}{|\log\varepsilon|}\Big(\frac1K\int_0^T\int_{\mathbb{R}^2}\chi_R^z|\partial_t u_\varepsilon|^2+K\int_0^T\int_{\mathbb{R}^2}\chi_R^z|X\cdot(\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ)|^2\Big)\\&\qquad+o(1)\big(1+\|X\|_{W^{1,\infty}([0,T]\times\mathbb{R}^2)}^5\big)\\&\lesssim_T\big(o(K^{-1}N_\varepsilon)+KN_\varepsilon+o(1)\big)\big(1+\|X\|_{W^{1,\infty}([0,T]\times\mathbb{R}^2)}^5\big),\end{aligned}$$

hence, for a suitable choice of K,

$$\sup_z \Big| \int_0^T \int_{\mathbb{R}^2} \chi_R^z X \cdot V_\varepsilon \Big| \ll_T N_\varepsilon \big(1 + \|X\|^5_{W^{1,\infty}([0,T]\times\mathbb{R}^2)}\big).$$

This proves $N_\varepsilon^{-1} V_\varepsilon \xrightharpoonup{*} 0$ in $(C_c^1([0,T]\times\mathbb{R}^2))^*$, so that identity (8.102) yields $\partial_t(N_\varepsilon^{-1}\mu_\varepsilon) = N_\varepsilon^{-1}\mathrm{curl}\, V_\varepsilon \xrightharpoonup{*} 0$ in $(C^1([0,T]; C_c^2(\mathbb{R}^2)))^*$. Arguing as in Step 5 of the proof of Proposition 8.6.1, the well-preparedness assumption on the initial data implies $N_\varepsilon^{-1} j_\varepsilon^\circ \to \mathrm{v}^\circ$ in $\mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2$, hence in particular $N_\varepsilon^{-1}\mu_\varepsilon^\circ \xrightharpoonup{*} \mathrm{curl}\,\mathrm{v}^\circ$ in $(C_c^1(\mathbb{R}^2))^*$. We easily conclude $N_\varepsilon^{-1}\mu_\varepsilon \xrightharpoonup{*} \mathrm{curl}\,\mathrm{v}^\circ$ in $(C([0,T]; C_c^2(\mathbb{R}^2)))^*$. The conclusion then follows, noting that by Lemma 8.5.1(iii) and by (8.130) the sequence $(N_\varepsilon^{-1}\mu_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty([0,T]; (C_c^{0,\gamma}(\mathbb{R}^2))^*)$ for all $\gamma > 0$, and using interpolation (as e.g. in [262]). $\square$

8.A Appendix: Well-posedness for the modified Ginzburg-Landau equation

In this appendix, we address global well-posedness for equation (8.6), proving Proposition 8.2.2 as well as additional regularity. We start with the decaying setting, that is, the case when $\nabla h, F, f$ are assumed to have some decay at infinity. Note that in this setting no transport is expected to occur at infinity. As is classical since the work of Bethuel and Smets [59] (see also [323]), we consider the existence of solutions u_ε of (8.6) in the affine space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$ for some "reference map" U, which is typically chosen smooth and equal (in polar coordinates) to $e^{iD_\varepsilon\theta}$ outside a ball at the origin, for some given $D_\varepsilon \in \mathbb{Z}$. Such a choice $U = U_{D_\varepsilon}$ imposes a total degree D_ε at infinity. More generally, we consider here the following spaces of "admissible" reference maps, for all $k \geq 0$,

$$\begin{aligned} E_k(\mathbb{R}^2) := \big\{ U \in \mathrm{L}^\infty(\mathbb{R}^2;\mathbb{C}) : \nabla^2 U \in H^k(\mathbb{R}^2;\mathbb{C}), \nabla|U| \in \mathrm{L}^2(\mathbb{R}^2), \\ 1 - |U|^2 \in \mathrm{L}^2(\mathbb{R}^2), \nabla U \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C})\ \forall p > 2 \big\}. \end{aligned}$$

(Note that this definition slightly differs from the usual one in [59], but it is more suitable in this form in the presence of pinning and forcing.) The map U_{D_ε} above clearly belongs to the space $E_\infty(\mathbb{R}^2)$. Global well-posedness and regularity in this framework are provided by the following proposition. Note that a stronger decay of the coefficients $\nabla h, F, f$ is required in the Gross-Pitaevskii case, although we do not know whether it is necessary.

Proposition 8.A.1 (Well-posedness for (8.6) — decaying setting). *Set $a := e^h$ with $h : \mathbb{R}^2 \to \mathbb{R}$.*

(i) Dissipative case $\alpha > 0$, $\beta \in \mathbb{R}$:
Given $h \in W^{1,\infty}(\mathbb{R}^2)$, $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, $f \in \mathrm{L}^2 \cap \mathrm{L}^\infty(\mathbb{R}^2)$, with $\nabla h, F \in \mathrm{L}^p(\mathbb{R}^2)^2$ for some $p < \infty$, and $u_\varepsilon^\circ \in U + H^1(\mathbb{R}^2;\mathbb{C})$ for some $U \in E_0(\mathbb{R}^2)$, there exists a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$ of (8.6) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u_ε°.
Moreover, if for some $k \geq 0$ we have $h \in W^{k+1,\infty}(\mathbb{R}^2)$, $F \in W^{k,\infty}(\mathbb{R}^2)^2$, $f \in H^k \cap W^{k,\infty}(\mathbb{R}^2)$, with $\nabla h, F \in W^{k,p}(\mathbb{R}^2)^2$ for some $p < \infty$, and $U \in E_k(\mathbb{R}^2)$, then $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([\delta,\infty); U + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ for all $\delta > 0$, and if in addition $u_\varepsilon^\circ \in U + H^{k+1}(\mathbb{R}^2;\mathbb{C})$, then $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$.

(ii) Gross-Pitaevskii case $\alpha = 0$, $\beta \in \mathbb{R}$:
Given $h \in W^{2,\infty}(\mathbb{R}^2)$, $\nabla h \in H^1(\mathbb{R}^2)^2$, $F \in H^2 \cap W^{2,\infty}(\mathbb{R}^2)^2$ with $\mathrm{div}\, F = 0$, $f \in \mathrm{L}^2 \cap \mathrm{L}^\infty(\mathbb{R}^2)$, and $u_\varepsilon^\circ \in U + H^1(\mathbb{R}^2;\mathbb{C})$ for some $U \in E_0(\mathbb{R}^2)$, there exists a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$ of (8.6) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u_ε°.
Moreover, if for some $k \geq 0$ we have $h \in W^{k+2,\infty}(\mathbb{R}^2)$, $\nabla h \in H^{k+1}(\mathbb{R}^2)^2$, $F \in H^{k+2} \cap W^{k+2,\infty}(\mathbb{R}^2)^2$ with $\mathrm{div}\, F = 0$, $f \in H^{k+1} \cap W^{k+1,\infty}(\mathbb{R}^2)$, and $u_\varepsilon^\circ \in U + H^{k+1}(\mathbb{R}^2;\mathbb{C})$ for some $U \in E_{k+1}(\mathbb{R}^2)$, then $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$. $\Diamond$

The proof below is based on arguments by [59, 323], which need to be adapted in the present context with both pinning and forcing. The conservative case $\alpha = 0$ is however more delicate, and we then use the structure of the equation to make a crucial change of variables that transforms the first-order terms into zeroth-order ones. As shown in the proof, in the dissipative regime, the decay assumption $\nabla h, F \in \mathrm{L}^p(\mathbb{R}^2)^2$ (for some $p < \infty$) can be replaced by $(|\nabla h| + |F|)\nabla U \in \mathrm{L}^2(\mathbb{R}^2; \mathbb{C})^2$.

Proof of Proposition 8.A.1. We split the proof into seven steps. We start with the (easiest) case $\alpha > 0$, and then turn to the conservative case $\alpha = 0$ in Steps 4–7.

Step 1. Local existence in $U + H^{k+1}(\mathbb{R}^2; \mathbb{C})$ for $\alpha > 0$.

In this step, given $k \geq 0$, we assume $h \in W^{k+1,\infty}(\mathbb{R}^2)$, $F \in W^{k,\infty}(\mathbb{R}^2)^2$, $f \in H^k \cap W^{k,\infty}(\mathbb{R}^2)$, $\nabla h, F \in W^{k,p}(\mathbb{R}^2)$ for some $p < \infty$, and $u_\varepsilon^\circ \in U + H^{k+1}(\mathbb{R}^2; \mathbb{C})$ for some $U \in E_k(\mathbb{R}^2)$, and we prove that there exists some $T > 0$ and a unique solution $u_\varepsilon \in \mathrm{L}^\infty([0,T); U + H^{k+1}(\mathbb{R}^2; \mathbb{C}))$ of (8.6) on $[0,T) \times \mathbb{R}^2$. To simplify notation, we replace equation (8.6) by its rescaled version

$$(\alpha + i\beta)\partial_t u = \triangle u + au(1 - |u|^2) + \nabla h \cdot \nabla u + iF^\perp \cdot \nabla u + fu, \qquad u|_{t=0} = u^\circ. \tag{8.271}$$

We start with the case $k = 0$, and briefly comment afterwards on the adaptations needed for $k \geq 1$. We argue by a fixed-point argument in the set $E_{U,u^\circ}(C_0, T) := \{u : \|u - U\|_{\mathrm{L}^\infty_T H^1} \leq C_0,\ u|_{t=0} = u^\circ\}$, for some $C_0, T > 0$ to be suitably chosen. We denote by $C \geq 1$ any constant that only depends on an upper bound on α, α^{-1}, $|\beta|$, $\|h\|_{W^{1,\infty}}$, $\|(F, f, U)\|_{\mathrm{L}^\infty}$, $\|1 - |U|^2\|_{\mathrm{L}^2}$, $\|\triangle U\|_{\mathrm{L}^2}$, $\|f\|_{\mathrm{L}^2}$, and $\|(|F| + |\nabla h|)\nabla U\|_{\mathrm{L}^2}$, and we add a subscript to indicate dependence on further parameters.

The kernel of the semigroup operator $e^{(\alpha+i\beta)^{-1}t\triangle}$ is given explicitly by

$$S^t(x) := (\alpha + i\beta)(4\pi t)^{-1} e^{-(\alpha+i\beta)|x|^2/(4t)}.$$

Since $\alpha > 0$, this kernel decays just like the standard heat kernel,

$$|S^t(x)| \leq Ct^{-1} e^{-\alpha|x|^2/(4t)}, \tag{8.272}$$

and we have the following obvious estimates, for all $1 \leq r \leq \infty$, $k \geq 1$,

$$\|S^t\|_{\mathrm{L}^r} \leq Ct^{\frac{1}{r}-1}, \qquad \|\nabla^k S^t\|_{\mathrm{L}^r} \leq C_k t^{\frac{1}{r}-1-\frac{k}{2}}. \tag{8.273}$$

Setting $\hat u := u - U$, we may rewrite equation (8.271) as follows:

$$\begin{aligned}(\alpha + i\beta)\partial_t \hat u = \triangle \hat u + \triangle U + a(\hat u + U)(1 - |U|^2) - 2a(\hat u + U)\langle U, \hat u\rangle - a(\hat u + U)|\hat u|^2 \\ + \nabla h \cdot \nabla \hat u + \nabla h \cdot \nabla U + iF^\perp \cdot \nabla \hat u + iF^\perp \cdot \nabla U + f\hat u + fU,\end{aligned} \tag{8.274}$$

with initial data $\hat u|_{t=0} = \hat u^\circ := u^\circ - U$. Any solution $\hat u \in \mathrm{L}^\infty([0,T); H^1(\mathbb{R}^2; \mathbb{C}))$ satisfies the Duhamel formula $\hat u = \Xi_{U,\hat u^\circ}(\hat u)$, where we have set

$$\begin{aligned}\Xi_{U,\hat u^\circ}(\hat u)^t &:= S^t * \hat u^\circ + (\alpha + i\beta)^{-1}\int_0^t S^{t-s} * Z_{U,\hat u^\circ}(\hat u^s)ds, \\ Z_{U,\hat u^\circ}(\hat u^s) &:= \triangle U + a(\hat u^s + U)(1 - |U|^2) - 2a(\hat u^s + U)\langle U, \hat u^s\rangle - a(\hat u^s + U)|\hat u^s|^2 \\ &\qquad + \nabla h \cdot \nabla \hat u^s + \nabla h \cdot \nabla U + iF^\perp \cdot \nabla \hat u^s + iF^\perp \cdot \nabla U + f\hat u^s + fU.\end{aligned}$$

Let us examine the map $\Xi_{U,\hat u^\circ}$ more closely. Using (8.273) in the forms $\|S^t\|_{\mathrm{L}^1} \leq C$ and $\|\nabla S^t\|_{\mathrm{L}^1} \leq Ct^{-1/2}$, we obtain by the triangle inequality

$$\|\Xi_{U,\hat u^\circ}(\hat u)^t\|_{H^1} \leq \|S^t\|_{\mathrm{L}^1}\|\hat u^\circ\|_{H^1} + C\int_0^t (1 + (t-s)^{-1/2})\Big(1 + \|\hat u^s\|_{\mathrm{L}^2} + \|\hat u^s\|_{\mathrm{L}^6}^3 + \|\nabla \hat u^s\|_{\mathrm{L}^2}\Big)ds,$$

and hence, by the Sobolev embedding in the form $\|\hat u^s\|_{\mathrm{L}^6}\le C\|\hat u^s\|_{H^1}$, for all $\hat u\in -U+E_{U,u^\circ}(C_0,T)$,

$$\|\Xi_{U,\hat u^\circ}(\hat u)\|_{\mathrm{L}^\infty_T H^1}\le C\|\hat u^\circ\|_{H^1}+C(T+T^{1/2})(1+C_0^3).$$

Similarly, again using the Sobolev embedding, we easily find for all $\hat u,\hat v\in -U+E_{U,u^\circ}(C_0,T)$

$$\begin{aligned}\|\Xi_{U,\hat u^\circ}(\hat u)-\Xi_{U,\hat u^\circ}(\hat v)\|_{\mathrm{L}^\infty_T H^1}&\le C\int_0^t(1+(t-s)^{-1/2})(1+\|\hat u^s\|_{H^1}^2+\|\hat v^s\|_{H^1}^2)\|\hat u^s-\hat v^s\|_{H^1}ds\\&\le C(T+T^{1/2})(1+C_0^2)\|\hat u-\hat v\|_{\mathrm{L}^\infty_T H^1}.\end{aligned}$$

Choosing $C_0:=1+C\|\hat u^\circ\|_{H^1}$ and $T:=1\wedge(4C(1+C_0^3))^{-2}$, we deduce that $\Xi_{U,\hat u^\circ}$ maps the set $-U+E_{U,u^\circ}(C_0,T)$ into itself, and is contracting on that set. The conclusion follows from a fixed-point argument.

Let us now briefly comment on the case $k\ge1$ and explain how to adapt the argument above. We again proceed by a fixed point argument, but estimating this time $\Xi_{U,\hat u^\circ}(w)$ in $H^{k+1}(\mathbb{R}^2;\mathbb{C})$ as follows

$$\|\Xi_{U,\hat u^\circ}(\hat u)^t\|_{H^{k+1}}\le\|S^t\|_{\mathrm{L}^1}\|\hat u^\circ\|_{H^{k+1}}+C\int_0^t(\|S^{t-s}\|_{\mathrm{L}^1}+\|\nabla S^{t-s}\|_{\mathrm{L}^1})\|Z_{U,\hat u^\circ}(\hat u^s)\|_{H^k},$$

where we easily check with the Sobolev embedding that

$$\|Z_{U,\hat u^\circ}(\hat u^s)\|_{H^k}\le C_k(1+\|\hat u^s\|_{H^{k+1}}^3),\tag{8.275}$$

for some constant $C_k\ge1$ that only depends on an upper bound on α, α^{-1}, $|\beta|$, k, $\|h\|_{W^{k+1,\infty}}$, $\|F\|_{W^{k,\infty}}$, $\|f\|_{H^k\cap W^{k,\infty}}$, $\|U\|_{\mathrm{L}^\infty}$, $\|\nabla|U|\|_{\mathrm{L}^2}$, $\|\nabla^2U\|_{H^k}$, $\|1-|U|^2\|_{\mathrm{L}^2}$, and $\sum_{j\le k}\|(|\nabla^jF|+|\nabla^j\nabla h|)\nabla U\|_{\mathrm{L}^2}$. Similarly estimating the H^{k+1}-norm of the difference $\Xi_{U,\hat u^\circ}(\hat u)-\Xi_{U,\hat u^\circ}(\hat v)$, the result follows.

Step 2. Regularizing effect for $\alpha>0$.

In this step, given $k\ge0$, we assume $h\in W^{k+1,\infty}(\mathbb{R}^2)$, $F\in W^{k,\infty}(\mathbb{R}^2)^2$, $f\in H^k\cap W^{k,\infty}(\mathbb{R}^2)$, $\nabla h,F\in W^{k,p}(\mathbb{R}^2)^2$ for some $p<\infty$, and $U\in E_k(\mathbb{R}^2)$, and we prove that any solution $u\in\mathrm{L}^\infty([0,T);U+H^1(\mathbb{R}^2;\mathbb{C}))$ of (8.271) satisfies $u\in\mathrm{L}^\infty([\delta,T);U+H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ for all $\delta>0$. We denote by $C_k\ge1$ any constant that only depends on an upper bound on α, α^{-1}, $|\beta|$, k, $\|h\|_{W^{k+1,\infty}}$, $\|F\|_{W^{k,\infty}}$, $\|f\|_{H^k\cap W^{k,\infty}}$, $\|U\|_{\mathrm{L}^\infty}$, $\|1-|U|^2\|_{\mathrm{L}^2}$, $\|\nabla|U|\|_{\mathrm{L}^2}$, $\|\nabla^2U\|_{H^k}$, $\sum_{j\le k}\|(|\nabla^jF|+|\nabla^j\nabla h|)\nabla U\|_{\mathrm{L}^2}$, and $\|u^\circ-U\|_{H^1}$. We write C for such a constant in the case $k=1$. We denote by $C_{k,t}\ge1$ any constant that additionally depends on an upper bound on t, t^{-1}, and $\|u-U\|_{\mathrm{L}^\infty_t H^1}$. We add a subscript to indicate dependence on further parameters.

Let $u\in\mathrm{L}^\infty([0,T);U+H^1(\mathbb{R}^2;\mathbb{C}))$ be a solution of (8.271), and let $\hat u:=u-U$. We prove by induction that $\|\hat u^t\|_{H^{k+1}}\le C_{k,t}$ for all $t\in(0,T)$ and $k\ge0$. As it is obvious for $k=0$, we assume that it holds for some $k\ge0$ and we then deduce that it also holds for k replaced by $k+1$. Using the Duhamel formula $\hat u=\Xi_{U,\hat u^\circ}(\hat u)$ as in Step 1, we find

$$\begin{aligned}\|\nabla^{k+1}\hat u^t\|_{\mathrm{L}^2}&\le\|\nabla^kS^t\|_{\mathrm{L}^1}\|\nabla\hat u^\circ\|_{\mathrm{L}^2}\\&\quad+C\int_{t/2}^t\|\nabla S^{t-s}*\nabla^kZ_{U,\hat u^\circ}(\hat u^s)\|_{\mathrm{L}^2}ds+C\int_0^{t/2}\|\nabla^{k+1}S^{t-s}*Z_{U,\hat u^\circ}(\hat u^s)\|_{\mathrm{L}^2}ds.\end{aligned}\tag{8.276}$$

A finer estimate than (8.275) is now needed. Arguing as in [59, Lemma 2] by means of various Sobolev embeddings, we have for all $1<r<2$,

$$\|\nabla Z_{U,\hat u^\circ}(\hat u^t)\|_{\mathrm{L}^2+\mathrm{L}^r}\le C_r(1+\|\hat u^t\|_{H^1}^3+\|\hat u^t\|_{H^2}).\tag{8.277}$$

(Note that we cannot choose $r=2$ above because of terms of the form $\||\hat u^s|^2\nabla\hat u^s\|_{\mathrm{L}^r}$, and that the term $\|\hat u^t\|_{H^2}$ in the right-hand side simply comes from the forcing terms $(\nabla h+iF^\perp)\cdot\nabla\hat u^t$ in the expression for $Z_{U,\hat{u}^\circ}(\hat{u}^t)$.) By a similar argument (see e.g. [323, Step 1 of the proof of Proposition A.8]), we find for all $k \geq 0$ and $1 < r < 2$,

$$\|\nabla^k Z_{U,\hat{u}^\circ}(\hat{u}^t)\|_{\mathrm{L}^2+\mathrm{L}^r} \leq C_{k,r}(1 + \|\hat{u}^t\|_{H^k}^3 + \|\hat{u}^t\|_{H^{k+1}}). \tag{8.278}$$

We may then deduce from (8.276) together with Young's convolution inequality and with (8.273), for all $1 < r < 2$,

$$\begin{aligned}
\|\nabla^{k+1}\hat{u}^t\|_{\mathrm{L}^2} &\leq \|\nabla^k S^t\|_{\mathrm{L}^1}\|\nabla\hat{u}^\circ\|_{\mathrm{L}^2} + C\int_{t/2}^t \|\nabla S^{t-s}\|_{\mathrm{L}^1\cap\mathrm{L}^{2r/(3r-2)}}\|\nabla^k Z_{U,\hat{u}^\circ}(\hat{u}^s)\|_{\mathrm{L}^2+\mathrm{L}^r}ds \\
&\qquad + C\int_0^{t/2} \|\nabla^{k+1}S^{t-s}\|_{\mathrm{L}^1}\|Z_{U,\hat{u}^\circ}(\hat{u}^s)\|_{\mathrm{L}^2}ds \\
&\leq Ct^{-k/2} + C_{k,r}\int_{t/2}^t ((t-s)^{-1/2} + (t-s)^{-1/r})(1 + \|\hat{u}^s\|_{H^k}^3 + \|\hat{u}^s\|_{H^{k+1}})ds \\
&\qquad + C\int_0^{t/2} (t-s)^{-(k+1)/2}(1 + \|\hat{u}^s\|_{H^1}^3)ds \\
&\leq C_{k,t} + C_{k,t}\sup_{t/2\leq s\leq t}\|\hat{u}^s\|_{H^k}^3 + C_{k,t}\bigg(\int_0^t \|\nabla^{k+1}\hat{u}^s\|_{\mathrm{L}^2}^3 ds\bigg)^{1/3}.
\end{aligned}$$

By induction hypothesis, this yields $\|\nabla^{k+1}\hat{u}^t\|_{\mathrm{L}^2}^3 \leq C_{k,t} + C_{k,t}\int_0^t \|\nabla^{k+1}\hat{u}^s\|_{\mathrm{L}^2}^3 ds$, and the result follows from the Grönwall inequality.

Step 3. Global existence for $\alpha > 0$.

In this step, we assume $h \in \mathrm{L}^\infty(\mathbb{R}^2)$, $f \in \mathrm{L}^2 \cap \mathrm{L}^\infty(\mathbb{R}^2)$, $\nabla h, F \in \mathrm{L}^p \cap \mathrm{L}^\infty(\mathbb{R}^2)$ for some $p < \infty$, $u^\circ \in U + H^1(\mathbb{R}^2;\mathbb{C})$, and $U \in E_0(\mathbb{R}^2)$, and we prove that (8.271) admits a unique global solution $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$. We denote by $C > 0$ any constant that only depends on an upper bound on α, α^{-1}, $|\beta|$, $\|h\|_{W^{1,\infty}}$, $\|(F,U)\|_{\mathrm{L}^\infty}$, $\|1 - |U|^2\|_{\mathrm{L}^2}$, $\|\triangle U\|_{\mathrm{L}^2}$, $\|f\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}$, and $\|(|F| + |\nabla h|)\nabla U\|_{\mathrm{L}^2}$.

Given a solution $u \in \mathrm{L}^\infty([0,T); U + H^1(\mathbb{R}^2;\mathbb{C}))$ of (8.271), we claim that the following a priori estimate holds for all $t \in [0,T)$,

$$\frac{\alpha}{2}\int_0^t\int_{\mathbb{R}^2}|\partial_t u|^2 + \frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla(u^t - U)|^2 + \frac{a}{2}(1 - |u^t|^2)^2 + |u^t - U|^2\Big) \leq Ce^{Ct}(1 + \|u^\circ - U\|_{H^1}^2). \tag{8.279}$$

Combining this with the local existence result of Step 1 in the space $U + H^1(\mathbb{R}^2;\mathbb{C})$, we deduce that local solutions can be extended globally in that space, and the result follows. It thus remains to prove the claim (8.279). For simplicity, we assume in the computations below that $u \in \mathrm{L}^\infty([0,T); U + H^2(\mathbb{R}^2;\mathbb{C}))$, which in particular implies $\partial_t u \in \mathrm{L}^\infty([0,T);\mathrm{L}^2(\mathbb{R}^2;\mathbb{C}))$ by (8.271). The general result then follows from a simple approximation argument based on the local existence result of Step 1 in the space $U + H^2(\mathbb{R}^2;\mathbb{C})$.

We set for simplicity $(\alpha + i\beta)^{-1} = \alpha' + i\beta'$, $\alpha' > 0$. Using equation (8.271), we compute the following time derivative, suitably regrouping the terms and integrating by parts,

$$\begin{aligned}
\frac{1}{2}\partial_t\int_{\mathbb{R}^2}|u - U|^2 &= \int_{\mathbb{R}^2}\langle u - U, (\alpha' + i\beta')(\triangle u + au(1 - |u|^2) + \nabla h\cdot\nabla u + iF^\perp\cdot\nabla u + fu)\rangle \\
&= -\alpha'\int_{\mathbb{R}^2}|\nabla(u - U)|^2 + \alpha'\int_{\mathbb{R}^2}a|u - U|^2(1 - |u|^2) \\
&\qquad + \int_{\mathbb{R}^2}\langle u - U, (\alpha' + i\beta')(\nabla h\cdot\nabla(u - U) + iF^\perp\cdot\nabla(u - U) + f(u - U))\rangle \\
&\qquad + \int_{\mathbb{R}^2}\langle u - U, (\alpha' + i\beta')(\triangle U + aU(1 - |u|^2) + \nabla h\cdot\nabla U + iF^\perp\cdot\nabla U + fU)\rangle,
\end{aligned}$$

which we may now estimate as follows

$$\begin{aligned}\frac{1}{2}\partial_t\int_{\mathbb{R}^2}|u-U|^2 &\le -\alpha'\int_{\mathbb{R}^2}|\nabla(u-U)|^2+C\int_{\mathbb{R}^2}|u-U|^2+C\int_{\mathbb{R}^2}|u-U||\nabla(u-U)|\\&\qquad+\int_{\mathbb{R}^2}|u-U|(|\triangle U|+|1-|u|^2|+(|\nabla h|+|F|)|\nabla U|+|f|)\\&\le -\frac{\alpha'}{2}\int_{\mathbb{R}^2}|\nabla(u-U)|^2+C+C\int_{\mathbb{R}^2}|u-U|^2+C\int_{\mathbb{R}^2}(1-|u|^2)^2.\end{aligned}$$

On the other hand, again using the equation, and integrating by parts, we compute

$$\begin{aligned}\frac{1}{2}\partial_t\int_{\mathbb{R}^2}|\nabla(u-U)|^2 &= \int_{\mathbb{R}^2}\langle\nabla(u-U),\nabla\partial_t u\rangle=-\int_{\mathbb{R}^2}\langle\triangle(u-U),\partial_t u\rangle\\&= -\int_{\mathbb{R}^2}\langle(\alpha+i\beta)\partial_t u-\triangle U-au(1-|u|^2)-\nabla h\cdot\nabla u-iF^\perp\cdot\nabla u-fu,\partial_t u\rangle\\&= -\alpha\int_{\mathbb{R}^2}|\partial_t u|^2-\frac{1}{4}\partial_t\int_{\mathbb{R}^2}a(1-|u|^2)^2+\int_{\mathbb{R}^2}\langle\nabla h\cdot\nabla(u-U)+iF^\perp\cdot\nabla(u-U)+f(u-U),\partial_t u\rangle\\&\qquad+\int_{\mathbb{R}^2}\langle\triangle U+\nabla h\cdot\nabla U+iF^\perp\cdot\nabla U+fU,\partial_t u\rangle\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{2}\partial_t\int_{\mathbb{R}^2}|\nabla(u-U)|^2+\frac{1}{4}\partial_t\int_{\mathbb{R}^2}a(1-|u|^2)^2\\&\le -\alpha\int_{\mathbb{R}^2}|\partial_t u|^2+C\int_{\mathbb{R}^2}|\partial_t u|(|u-U|+|\nabla(u-U)|)\\&\qquad\qquad+C\int_{\mathbb{R}^2}|\partial_t u|(|\triangle U|+(|\nabla h|+|F|)|\nabla U|+|f|)\\&\le -\frac{\alpha}{2}\int_{\mathbb{R}^2}|\partial_t u|^2+C+C\int_{\mathbb{R}^2}|u-U|^2+C\int_{\mathbb{R}^2}|\nabla(u-U)|^2.\end{aligned}$$

We may thus conclude

$$\begin{aligned}\frac{\alpha}{2}\int_{\mathbb{R}^2}|\partial_t u|^2+\partial_t\int_{\mathbb{R}^2}\Big(\frac{1}{2}|\nabla(u-U)|^2+\frac{a}{4}(1-|u|^2)^2+\frac{1}{2}|u-U|^2\Big)\\\le C+C\int_{\mathbb{R}^2}\Big(\frac{1}{2}|\nabla(u-U)|^2+\frac{a}{4}(1-|u|^2)^2+\frac{1}{2}|u-U|^2\Big),\end{aligned}$$

and the claim (8.279) follows from the Grönwall inequality.

Step 4. A useful change of variables.

We now turn to the conservative case $\alpha=0$. The first-order terms (that are, forcing terms) in the right-hand side of (8.6) can then no longer be treated as errors, since the lost derivative is not retrieved by the Schrödinger operator. The proof of local existence in Step 1 can thus not be adapted to this case. The global estimates in Step 3 similarly fail, as no dissipation is available to absorb the first-order terms. To remedy this, we start by performing a useful change of variables transforming first-order terms into zeroth-order ones, which are much easier to deal with. Since by assumption $\operatorname{div} F=0$ with $F\in \mathrm{L}^\infty(\mathbb{R}^2)^2$, we deduce from a Hodge decomposition that there exists $\psi\in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that $F=-2\nabla^\perp\psi$. Using the relation $a=e^h$, and setting $w_\varepsilon:=\sqrt{a}u_\varepsilon e^{i|\log\varepsilon|\psi}$, a straightforward computation shows that the equation (8.6) for u_ε is equivalent to

$$\begin{cases}\lambda_\varepsilon(\alpha+i|\log\varepsilon|\beta)\partial_t w_\varepsilon=\triangle w_\varepsilon+\frac{w_\varepsilon}{\varepsilon^2}(a-|w_\varepsilon|^2)+(f_0+ig_0)w_\varepsilon, & \text{in } \mathbb{R}^+\times\mathbb{R}^2,\\ w_\varepsilon|_{t=0}=w_\varepsilon^\circ:=\sqrt{a}e^{i|\log\varepsilon|\psi}u_\varepsilon^\circ.\end{cases}\tag{8.280}$$

where we have set

$$f_0 := f - \frac{\triangle\sqrt{a}}{\sqrt{a}} + \frac{1}{4}|\log\varepsilon|^2|F|^2, \qquad g_0 := \frac{1}{2}|\log\varepsilon|a^{-1}\mathrm{curl}\,(aF).$$

We look for solutions w_ε of the above in the class $W + H^1(\mathbb{R}^2;\mathbb{C})$, for a "weighted reference map" W, that is an element of

$$E_k^a(\mathbb{R}^2) := \{W \in \mathrm{L}^\infty(\mathbb{R}^2;\mathbb{C}) : \nabla^2 W \in H^k(\mathbb{R}^2;\mathbb{C}), \nabla|W| \in \mathrm{L}^2(\mathbb{R}^2),\\ a - |W|^2 \in \mathrm{L}^2(\mathbb{R}^2), \nabla W \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C})\ \forall p > 2\}.$$

For $k \geq 0$, and $\nabla h, \nabla\psi \in H^{k+1}(\mathbb{R}^2)^2$, we indeed observe that w_ε is a solution of (8.280) in $\mathrm{L}^\infty([0,T); W + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ for some $W \in E_k^a$ if and only if u_ε is a solution of (8.6) in $\mathrm{L}^\infty([0,T); U + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ for some $U \in E_k$.

Step 5. Local existence for $\alpha = 0$.

In this step, given $k \geq 0$, we assume $h \in W^{k+1,\infty}(\mathbb{R}^2)$, $\nabla h \in H^k(\mathbb{R}^2)^2$, $f_0, g_0 \in H^{k+1} \cap W^{k+1,\infty}(\mathbb{R}^2)$, and $w^\circ \in W + H^{k+1}(\mathbb{R}^2;\mathbb{C})$ for some $W \in E_{k+1}^a(\mathbb{R}^2)$, and we prove that there exists some $T > 0$ and a unique solution $w_\varepsilon \in \mathrm{L}^\infty([0,T); W + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ of (8.280) on $[0,T) \times \mathbb{R}^2$. To simplify notation, we replace equation (8.280) (with $\alpha = 0$) by its rescaled version

$$i\partial_t w = \triangle w + w(a - |w|^2) + (f_0 + ig_0)w, \qquad w|_{t=0} = w^\circ. \tag{8.281}$$

We start with the case $k = 0$, and comment afterwards on the adaptations needed for $k \geq 1$. We argue by a fixed-point argument in the set $E_{W,w^\circ}(C_0, T) := \{w : \|w - W\|_{\mathrm{L}_T^\infty H^1} \leq C_0, w|_{t=0} = w^\circ\}$, for some $C_0, T > 0$ to be suitably chosen. We denote by $C \geq 1$ any constant that only depends on an upper bound on $\|\nabla h\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}$, $\|(f_0, g_0)\|_{H^1 \cap W^{1,\infty}}$, $\|(h, W)\|_{\mathrm{L}^\infty}$, $\|a - |W|^2\|_{\mathrm{L}^2}$, $\|\nabla|W|\|_{\mathrm{L}^2}$, and $\|\triangle W\|_{H^1}$, and we add a subscript to indicate dependence on further parameters.

Let S^t denote the kernel of the semigroup operator $e^{-it\triangle}$. Setting $\hat{w} := w - W$, we may rewrite equation (8.281) as follows:

$$i\partial_t \hat{w} = \triangle\hat{w} + \triangle W + (\hat{w} + W)(a - |W|^2) - 2(\hat{w} + W)\langle W, \hat{w}\rangle - (\hat{w} + W)|\hat{w}|^2\\ + (f_0 + ig_0)\hat{w} + (f_0 + ig_0)W,$$

with initial data $\hat{w}|_{t=0} = \hat{w}^\circ := w^\circ - W$. Any solution $\hat{w} \in \mathrm{L}^\infty([0,T); H^1(\mathbb{R}^2;\mathbb{C}))$ satisfies the Duhamel formula $\hat{w} = \Xi_{W,\hat{w}^\circ}(\hat{w})$, where we have set

$$\Xi_{W,\hat{w}^\circ}(\hat{w})^t := S^t * \hat{w}^\circ - i\int_0^t S^{t-s} * Z_{W,\hat{w}^\circ}(w^s)ds,\\ Z_{W,\hat{w}^\circ}(\hat{w}^s) := \triangle W + (\hat{w}^s + W)(a - |W|^2) - 2(\hat{w}^s + W)\langle W, \hat{w}^s\rangle - (\hat{w}^s + W)|\hat{w}^s|^2\\ + (f_0 + ig_0)\hat{w}^s + (f_0 + ig_0)W.$$

Similarly as in Step 1, we find $\|Z_{W,\hat{w}^\circ}(\hat{w}^s)\|_{\mathrm{L}^2} \leq C(1 + \|\hat{w}^s\|_{H^1}^3)$. On the other hand, arguing as in [59, Lemma 2] by means of various Sobolev embeddings, we have the following version of (8.277) without forcing: we may decompose $\nabla Z_{W,\hat{w}^\circ}(\hat{w}^s) = Z^1_{W,\hat{w}^\circ}(\hat{w}^s) + Z^2_{W,\hat{w}^\circ}(w^s)$, such that for all $1 < r < 2$,

$$\|Z^1_{W,\hat{w}^\circ}(\hat{w}^s)\|_{\mathrm{L}^2} \leq C(1 + \|\hat{w}^s\|_{H^1}^3), \qquad \|Z^2_{W,\hat{w}^\circ}(\hat{w}^s)\|_{\mathrm{L}^r} \leq C_r(1 + \|\hat{w}^s\|_{H^1}^3). \tag{8.282}$$

(Recall that we cannot choose $r = 2$ above because of terms of the form $\||\hat{w}^s|^2\nabla\hat{w}^s\|_{\mathrm{L}^r}$.) Let us now examine the map $\Xi_{W,\hat{w}^\circ}$ more closely. We have

$$\|\Xi_{W,\hat{w}^\circ}(\hat{w})^t\|_{H^1} \leq \|S^t * (\hat{w}^\circ, \nabla\hat{w}^\circ)\|_{\mathrm{L}^2} + \left\|\int_0^t e^{-i(t-s)\triangle}(Z_{W,\hat{w}^\circ}(\hat{w}^s), Z^1_{W,\hat{w}^\circ}(\hat{w}^s), Z^2_{W,\hat{w}^\circ}(\hat{w}^s))ds\right\|_{\mathrm{L}^2},$$

and hence by the Strichartz estimates for the Schrödinger operator [270], for all $1 < r \le 2$,

$$\|\Xi_{W,\hat w^\circ}(\hat w)\|_{\mathrm{L}^\infty_T H^1} \le C\|\hat w^\circ\|_{H^1} + C\|(Z_{W,\hat w^\circ}(\hat w), Z^1_{W,\hat w^\circ}(\hat w))\|_{\mathrm{L}^1_T \mathrm{L}^2} + C_r\|Z^2_{W,\hat w^\circ}(\hat w)\|_{\mathrm{L}^{2r/(3r-2)}_T \mathrm{L}^r}.$$

The above estimates for $Z_{W,\hat w^\circ}$ then yield for all $1 < r < 2$,

$$\|\Xi_{W,\hat w^\circ}(\hat w)\|_{\mathrm{L}^\infty_T H^1} \le C\|\hat w^\circ\|_{H^1} + (CT + C_r T^{\frac32 - \frac1r})(1 + \|\hat w\|^3_{\mathrm{L}^\infty_T H^1}).$$

Choosing $r = 4/3$, this yields in particular, for all $\hat w \in -W + E_{W,\hat w^\circ}(C_0, T)$,

$$\|\Xi_{W,\hat w^\circ}(\hat w)\|_{\mathrm{L}^\infty_T H^1} \le C\|\hat w^\circ\|_{H^1} + C(T + T^{3/4})(1 + C_0^3).$$

Similarly, again using Sobolev embeddings and Strichartz estimates, we easily find for all $\hat v, \hat w \in -W + E_{W,\hat w^\circ}(C_0, T)$

$$\|\Xi_{W,\hat w^\circ}(\hat v) - \Xi_{W,\hat w^\circ}(\hat w)\|_{\mathrm{L}^\infty_T H^1} \le C(T + T^{3/4})(1 + C_0^2)\|\hat v - \hat w\|_{\mathrm{L}^\infty_T H^1}.$$

Choosing $C_0 := 1 + C\|\hat w^\circ\|_{H^1}$ and $T := 1 \wedge (4C(1 + C_0^3))^{-4/3}$, we may then deduce that $\Xi_{W,\hat w^\circ}$ maps the set $-W + E_{W,\hat w^\circ}(C_0, T)$ into itself, and is contracting on that set. The conclusion follows from a fixed-point argument.

Let us now briefly comment on the case $k \ge 1$ and explain how to adapt the above argument. We again proceed by a fixed point argument, estimating this time $\Xi_{W,\hat w^\circ}(\hat w)$ hence $Z_{W,\hat w^\circ}(\hat w)$ in $H^{k+1}(\mathbb{R}^2; \mathbb{C})$. Arguing similarly as e.g. in [323, Step 1 of the proof of Proposition A.8] by means of various Sobolev embeddings, we have the following version of (8.278) without forcing: for all $k \ge 1$,

$$\|\nabla^{k+1} Z_{W,\hat w^\circ}(\hat w)\|_{\mathrm{L}^\infty_t(\mathrm{L}^2 + \mathrm{L}^r)} \le C_k(1 + \|\hat w\|^3_{\mathrm{L}^\infty_t H^{k+1}}), \tag{8.283}$$

for some constant $C_k \ge 1$ that only depends on an upper bound on k, $\|\nabla h\|_{H^k \cap W^{k,\infty}}$, $\|(h, W)\|_{\mathrm{L}^\infty}$, $\|(f_0, g_0)\|_{H^{k+1} \cap W^{k+1,\infty}}$, $\|a - |W|^2\|_{\mathrm{L}^2}$, $\|\nabla |W|\|_{\mathrm{L}^2}$, and $\|\nabla^2 W\|_{H^{k+1}}$. The result then easily follows as above.

Step 6. Global existence for $\alpha = 0$.

In this step, we assume $h \in \mathrm{L}^\infty(\mathbb{R}^2)$, $f_0 \in \mathrm{L}^2 \cap \mathrm{L}^\infty(\mathbb{R}^2)$, $g_0 \in H^1 \cap W^{1,\infty}(\mathbb{R}^2)$, and $w^\circ \in W + H^1(\mathbb{R}^2; \mathbb{C})$ for some $W \in E_0^a(\mathbb{R}^2)$, and we prove that (8.281) admits a unique global solution $w \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W + H^1(\mathbb{R}^2; \mathbb{C}))$. We denote by $C > 0$ any constant that only depends on an upper bound on $\|h\|_{\mathrm{L}^\infty}$, $\|f_0\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}$, $\|g_0\|_{H^1 \cap W^{1,\infty}}$, $\|W\|_{\mathrm{L}^\infty}$, $\|1 - |W|^2\|_{\mathrm{L}^2}$, and $\|\triangle W\|_{\mathrm{L}^2}$.

Given a solution $w \in \mathrm{L}^\infty([0, T); W + H^1(\mathbb{R}^2; \mathbb{C}))$ of (8.281), we claim that the following a priori estimate holds for all $t \in [0, T)$,

$$\int_{\mathbb{R}^2} \Big(|\nabla(w^t - W)|^2 + \frac12 (a - |w^t|^2)^2 + |w^t - W|^2 \Big) \le Ce^{Ct}(1 + \|w^\circ - W\|^2_{H^1}). \tag{8.284}$$

Combining this with the local existence result of Step 5 in the space $W + H^1(\mathbb{R}^2; \mathbb{C})$, we deduce that local solutions can be extended globally in that space, and the result follows. So it remains to prove the claim (8.284). For simplicity, we assume in the computations below that $w \in \mathrm{L}^\infty([0, T); W + H^2(\mathbb{R}^2; \mathbb{C}))$, which in particular implies $\partial_t w \in \mathrm{L}^\infty([0, T); \mathrm{L}^2(\mathbb{R}^2; \mathbb{C}))$ by (8.281). The general result then follows from a simple approximation argument based on the local existence result of Step 5 in the space $W + H^2(\mathbb{R}^2; \mathbb{C})$.

Using equation (8.281), we compute the following time derivative, suitably regrouping the terms and integrating by parts,

$$\begin{aligned}
\frac12 \partial_t \int_{\mathbb{R}^2} |w - W|^2 &= \int_{\mathbb{R}^2} \langle i(w - W), \triangle w + w(a - |w|^2) + f_0 w + i g_0 w \rangle \\
&= \int_{\mathbb{R}^2} \langle i(w - W), \triangle W + W(a - |w|^2) + f_0 W + i g_0 W \rangle + \int_{\mathbb{R}^2} g_0 |w - W|^2 \\
&\le C + C \int_{\mathbb{R}^2} |w - W|^2 + C \int_{\mathbb{R}^2} (a - |w|^2)^2.
\end{aligned} \tag{8.285}$$

Likewise, we compute

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^2} |\nabla(w-W)|^2 &= 2\int_{\mathbb{R}^2} \langle \nabla(w-W), \nabla\partial_t w\rangle \\
&= -2\int_{\mathbb{R}^2} \langle \triangle(w-W), \partial_t w - g_0 w\rangle \\
&\qquad +2\int_{\mathbb{R}^2} \langle \nabla(w-W), g_0\nabla(w-W) + g_0\nabla W + (w-W)\nabla g_0 + W\nabla g_0\rangle \\
&\le -2\int_{\mathbb{R}^2} \langle \triangle(w-W), \partial_t w - g_0 w\rangle + C + C\int_{\mathbb{R}^2} |\nabla(w-W)|^2 + C\int_{\mathbb{R}^2} |w-W|^2, \qquad (8.286)
\end{aligned}$$

where we have

$$\begin{aligned}
&-2\int_{\mathbb{R}^2} \langle \triangle(w-W), \partial_t w - g_0 w\rangle \\
&= -2\int_{\mathbb{R}^2} \langle i(\partial_t w - g_0 w) - w(a-|w|^2) - f_0 w - \triangle W, \partial_t w - g_0 w\rangle \\
&= 2\int_{\mathbb{R}^2} \langle w(a-|w|^2) + f_0 w + \triangle W, \partial_t w - g_0 w\rangle \\
&= -\partial_t \int_{\mathbb{R}^2} \Big(\frac{1}{2}(a-|w|^2)^2 - f_0|w|^2 - 2\langle \triangle W, w\rangle\Big) \\
&\qquad +2\int_{\mathbb{R}^2} g_0(a-|w|^2)^2 - 2\int_{\mathbb{R}^2} a g_0(a-|w|^2) - 2\int_{\mathbb{R}^2} f_0 g_0|w|^2 - 2\int_{\mathbb{R}^2} g_0\langle \triangle W, w\rangle \\
&\le -\partial_t \int_{\mathbb{R}^2} \Big(\frac{1}{2}(a-|w|^2)^2 - f_0|w-W|^2 - 2\langle w, \triangle W + f_0 W\rangle\Big) \\
&\qquad +C + C\int_{\mathbb{R}^2} (a-|w|^2)^2 + C\int_{\mathbb{R}^2} |w-W|^2.
\end{aligned}$$

Combining this with (8.285) and (8.286), we obtain

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^2} \Big((C-f_0)|w-W|^2 + |\nabla(w-W)|^2 + \frac{1}{2}(a-|w|^2)^2 - 2\langle w, \triangle W + f_0 W\rangle\Big)& \\
\le C + C\int_{\mathbb{R}^2} \big(|w-W|^2 + |\nabla(w-W)|^2 + (a-|w|^2)^2\big)&,
\end{aligned}$$

and the result easily follows from the Grönwall inequality, choosing a large enough constant C in the left-hand side.

Step 7. Propagation of regularity for $\alpha = 0$.

In this step, given $k \ge 0$, we assume $h \in W^{k+1,\infty}(\mathbb{R}^2)$, $\nabla h \in H^k(\mathbb{R}^2)^2$, $f_0, g_0 \in H^{k+1} \cap W^{k+1,\infty}(\mathbb{R}^2)$, and $w^\circ \in W + H^{k+1}(\mathbb{R}^2;\mathbb{C})$ for some $W \in E^a_{k+1}(\mathbb{R}^2)$, and we prove that the global solution w of Step 6 belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$. We denote by $C_k \ge 1$ any constant that only depends on an upper bound on k, $\|\nabla h\|_{H^k\cap W^{k,\infty}}$, $\|(f_0,g_0)\|_{H^{k+1}\cap W^{k+1,\infty}}$, $\|(h,W)\|_{\mathrm{L}^\infty}$, $\|a-|W|^2\|_{\mathrm{L}^2}$, $\|\nabla|W|\|_{\mathrm{L}^2}$, and $\|\nabla^2 W\|_{H^{k+1}}$. We add a subscript to indicate dependence on further parameters.

Let $w \in \mathrm{L}^\infty([0,T); W + H^1(\mathbb{R}^2;\mathbb{C}))$ be a solution of (8.271), and let $\hat w := w - W$. We argue by induction: as the result is obvious for $k=0$, we assume that it holds for some $k \ge 0$ and we deduce that it then also holds for k replaced by $k+1$. By a similar argument as e.g. in [59, Lemma 4] or in [323, Step 1 of the proof of Proposition A.8], we have the following version of (8.278) without forcing (which generalizes (8.282) to higher derivatives): for all $k \ge 0$ we may decompose $\nabla^{k+1} Z_{W,\hat w^\circ}(\hat w^t) = \nabla^{k+1} Z^1_{W,\hat w^\circ}(\hat w^t) + \nabla^{k+1} Z^2_{W,\hat w^\circ}(w^t)$ such that for all $1 < r < 2$

$$\|\nabla^{k+1} Z^1_{W,\hat w^\circ}(\hat w^t)\|_{\mathrm{L}^2} + \|\nabla^{k+1} Z^2_{W,\hat w^\circ}(\hat w^t)\|_{\mathrm{L}^r} \le C_{k,r}(1 + \|\hat w^t\|^3_{H^{k+1}}),$$

or even more precisely,

$$\|\nabla^{k+1} Z^1_{W,\hat{w}^\circ}(\hat{w}^t)\|_{\mathrm{L}^2} + \|\nabla^{k+1} Z^2_{W,\hat{w}^\circ}(\hat{w}^t)\|_{\mathrm{L}^r} \le C_{k,r}(1+\|\hat{w}^t\|^2_{H^k})(1+\|\hat{w}^t\|_{H^{k+1}}). \tag{8.287}$$

Using Duhamel's formula $\hat{w} = \Xi_{W,\hat{w}^\circ}(\hat{w})$ and applying the Strichartz estimates for the Schrödinger operator [270] as in Step 5, we find for all $k \ge 0$ and $1 < r \le 2$

$$\begin{aligned}\|\nabla^{k+1}\hat{w}^t\|_{\mathrm{L}^2} &\le \|S^t * \nabla^{k+1}\hat{w}^\circ\|_{\mathrm{L}^2} + \Big\|\int_0^t S^{t-s} * \nabla^{k+1} Z_{W,\hat{w}^\circ}(\hat{w}^s)ds\Big\|_{\mathrm{L}^2} \\ &\le C\|\nabla^{k+1}\hat{w}^\circ\|_{\mathrm{L}^2} + C\|\nabla^{k+1} Z^1_{W,\hat{w}^\circ}(\hat{w})\|_{\mathrm{L}^1_t\mathrm{L}^2} + C_r\|\nabla^{k+1} Z^2_{W,\hat{w}^\circ}(\hat{w})\|_{\mathrm{L}^{2r/(3r-2)}_t\mathrm{L}^r},\end{aligned}$$

and hence, by (8.287), for all $k \ge 0$,

$$\|\hat{w}^t\|_{H^{k+1}} \le C_k\|\hat{w}^\circ\|_{H^{k+1}} + C_{k,r}(1+t)(1+\|\hat{w}\|^2_{\mathrm{L}^\infty_t H^k})(1+\|\hat{w}\|_{\mathrm{L}^{2r/(3r-2)}_t H^{k+1}}).$$

The result then follows from the induction hypothesis and the Grönwall inequality. □

In the dissipative case, we now prove a well-posedness result for equation (8.6) in the general non-decaying setting, that is, without decay assumption on the coefficients $\nabla h, F, f$. Since the forcing does not decay, subtle advection forces may occur at infinity, preventing the solution u_ε from staying in the same affine space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$ for any stationary reference map U. The well-posedness result below is therefore simply obtained in the space $\mathrm{L}^\infty(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$, which yields no information at all on the behavior of the constructed solution at infinity. It is in particular completely unclear whether the total degree of the solution remains well-defined for positive times. In the proof below, the key observation is that the Grönwall argument for the energy in Step 3 of the proof of Proposition 8.A.1 above can be localized by means of an exponential cut-off. Note that the same argument does not seem applicable to the Gross-Pitaevskii case.

Proposition 8.A.2 (Well-posedness for (8.6) — non-decaying setting)**.** *Set $a := e^h$, with $h : \mathbb{R}^2 \to \mathbb{R}$. In the dissipative case $\alpha > 0$, $\beta \in \mathbb{R}$, given $h \in W^{1,\infty}(\mathbb{R}^2)$, $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, $f \in \mathrm{L}^\infty(\mathbb{R}^2)$, and $u^\circ_\varepsilon \in H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, there exists a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ of (8.6) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u°_ε, and this solution satisfies $\partial_t u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$. Moreover, if for some $k \ge 0$ we have $h \in W^{k+1,\infty}(\mathbb{R}^2)$, $F \in W^{k,\infty}(\mathbb{R}^2)^2$, $f \in W^{k,\infty}(\mathbb{R}^2)$, and $u^\circ_\varepsilon \in H^{k+1}_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, then $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{k+1}_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ and $\partial_t u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^k_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$.* ◊

Proof. We split the proof into four steps. We denote by $\xi^z(x) := e^{-|x-z|}$ the exponential cut-off centered at $z \in \mathbb{Z}^2$, and $\xi(x) := \xi^0(x) = e^{-|x|}$. To simplify notation, we replace equation (8.6) by its rescaled version

$$(\alpha + i\beta)\partial_t u = \triangle u + au(1-|u|^2) + \nabla h \cdot \nabla u + iF^\perp \cdot \nabla u + fu, \qquad u|_{t=0} = u^\circ. \tag{8.288}$$

Step 1. Global existence with $k = 0$.

In this step, we assume $h \in W^{1,\infty}(\mathbb{R}^2)$, $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, $f \in \mathrm{L}^\infty(\mathbb{R}^2)$, and $u^\circ \in H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, and we prove that there exists a global solution $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ of (8.288) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u°. We denote by $C \ge 1$ any constant that only depends on an upper bound on α, α^{-1}, $|\beta|$, $\|(h, \nabla h, F, f)\|_{\mathrm{L}^\infty}$, and $\|u^\circ\|_{H^1_{\mathrm{uloc}}}$.

We argue by approximation: for all $n \ge 1$, we let $\chi_n := \chi(\cdot/n)$ for some fixed cut-off function χ with $\chi|_{B_1} \equiv 1$ and $\chi|_{\mathbb{R}^2\setminus B_2} \equiv 0$, and we set $h_n := \chi_n h$, $a_n := e^{h_n}$, $F_n := \chi_n F$, and $f_n := \chi_n f$. Note that by construction $\|(h_n, \nabla h_n, F_n, f_n)\|_{\mathrm{L}^\infty} \le C$. We also need to approximate the initial data $u^\circ \in H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$: for all $n \ge 1$, we let $\rho_n := n^2\rho(nx)$ for some $\rho \in C^\infty_c(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2}\rho = 1$, and we set $u^\circ_n := \chi_n(u^\circ * \rho_n) + 1 - \chi_n$. By definition, we have $u^\circ_n \in E_0$, the sequence $(u^\circ_n)_n$ is bounded in

$H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, and as $n\uparrow\infty$ we obtain $u_n^\circ\to u^\circ$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{C})$, and $a_n\to a$, $\nabla h_n\to\nabla h$, and $F_n\to F$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)^2$. By Proposition 8.A.1, there exists a unique global solution $u_n\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;U+H^1(\mathbb{R}^2;\mathbb{C}))$ of the following truncated equation on $\mathbb{R}^+\times\mathbb{R}^2$,

$$(\alpha+i\beta)\partial_t u_n=\triangle u_n+a_nu_n(1-|u_n|^2)+\nabla h_n\cdot\nabla u_n+iF_n^\perp\cdot\nabla u_n+f_nu_n,\quad u_n|_{t=0}=u_n^\circ.\quad (8.289)$$

In order to pass to the limit $n\uparrow\infty$ in (the weak formulation of) this equation, we prove the boundedness of the sequence $(u_n)_n$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$, that is, we claim that the following a priori estimate holds for all $t\ge0$,

$$\|u_n^t\|_{H^1_{\mathrm{uloc}}}\le\sup_z\|u_n^t\|_{H^1(B(z))}+\alpha^{1/2}\sup_z\|\partial_tu_n\|_{\mathrm{L}^2_t\mathrm{L}^2(B(z))}\le Ce^{Ct}.\quad (8.290)$$

Before proving this estimate, we show how to conclude from this. Up to a subsequence, the sequence u_n converges weakly-* to some u in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$. Since moreover ∂_tu_n is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(B(z);\mathbb{C}))$, uniformly in z, and as $H^1(B(z);\mathbb{C})$ is compactly embedded into $\mathrm{L}^2(B(z);\mathbb{C})$, we deduce from the Aubin-Simon lemma that $u_n\to u$ strongly in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$. This allows to pass to the limit in the weak formulation of equation (8.289), and deduce that the limit u is a global solution of (8.288) on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data u°.

It remains to prove (8.290). We set for simplicity $(\alpha+i\beta)^{-1}=\alpha'+i\beta'$, $\alpha'>0$. Using equation (8.289), integrating by parts, and using $|\nabla\xi^z|\le\xi^z$, we compute the following time derivative, for all $z\in R\mathbb{Z}^2$,

$$\begin{aligned}\frac12\partial_t\int_{\mathbb{R}^2}\xi^z|u_n|^2&=\int_{\mathbb{R}^2}\xi^z\langle u_n,(\alpha'+i\beta')(\triangle u_n+a_nu_n(1-|u_n|^2)+\nabla h_n\cdot\nabla u_n+iF_n^\perp\cdot\nabla u_n+f_nu_n)\rangle\\&\le\int_{\mathbb{R}^2}\xi^z\langle u_n,(\alpha'+i\beta')\triangle u_n\rangle+\alpha'\int_{\mathbb{R}^2}a_n\xi^z|u_n|^2(1-|u_n|^2)+C\int_{\mathbb{R}^2}\xi^z|u_n||\nabla u_n|+C\int_{\mathbb{R}^2}\xi^z|u_n|^2\\&\le-\alpha'\int_{\mathbb{R}^2}\xi^z|\nabla u_n|^2+C\int_{\mathbb{R}^2}\xi^z|u_n||\nabla u_n|+C\int_{\mathbb{R}^2}\xi^z|u_n|^2,\end{aligned}$$

and hence

$$\frac12\partial_t\int_{\mathbb{R}^2}\xi^z|u_n|^2\le-\frac{\alpha'}{2}\int_{\mathbb{R}^2}\xi^z|\nabla u_n|^2+C\int_{\mathbb{R}^2}\xi^z|u_n|^2.$$

On the other hand, integration by parts yields

$$\frac12\partial_t\int_{\mathbb{R}^2}\xi^z|\nabla u_n|^2=\int_{\mathbb{R}^2}\xi^z\langle\nabla u_n,\nabla\partial_tu_n\rangle=-\int_{\mathbb{R}^2}\xi^z\langle\triangle u_n,\partial_tu_n\rangle-\int_{\mathbb{R}^2}\nabla\xi^z\cdot\langle\nabla u_n,\partial_tu_n\rangle,$$

hence, inserting equation (8.289) in the first right-hand side term,

$$\begin{aligned}&\frac12\partial_t\int_{\mathbb{R}^2}\xi^z|\nabla u_n|^2\\&\quad=-\int_{\mathbb{R}^2}\xi^z\langle(\alpha+i\beta)\partial_tu_n-a_nu_n(1-|u_n|^2)-\nabla h_n\cdot\nabla u_n-iF_n^\perp\cdot\nabla u_n-f_nu_n,\partial_tu_n\rangle\\&\qquad\qquad-\int_{\mathbb{R}^2}\nabla\xi^z\cdot\langle\nabla u_n,\partial_tu_n\rangle\\&\quad\le-\alpha\int_{\mathbb{R}^2}\xi^z|\partial_tu_n|^2-\frac14\partial_t\int_{\mathbb{R}^2}a_n\xi^z(1-|u_n|^2)^2+C\int_{\mathbb{R}^2}\xi^z(|u_n|+|\nabla u_n|)|\partial_tu_n|,\end{aligned}$$

and thus

$$\frac12\partial_t\int_{\mathbb{R}^2}\xi^z|\nabla u_n|^2+\frac14\partial_t\int_{\mathbb{R}^2}a_n\xi^z(1-|u_n|^2)^2\le-\frac{\alpha}{2}\int_{\mathbb{R}^2}\xi^z|\partial_tu_n|^2+C\int_{\mathbb{R}^2}\xi^z(|u_n|^2+|\nabla u_n|^2).$$

We may then conclude

$$\frac{1}{2}\partial_t\int_{\mathbb{R}^2}\xi^z(|u_n|^2+|\nabla u_n|^2)+\frac{1}{4}\partial_t\int_{\mathbb{R}^2}a_n\xi^z(1-|u_n|^2)^2+\frac{\alpha}{2}\int_{\mathbb{R}^2}\xi^z|\partial_t u_n|^2\le C\int_{\mathbb{R}^2}\xi^z(|u_n|^2+|\nabla u_n|^2).$$

By the Grönwall inequality, this yields for all $t\ge 0$ and $z\in R\mathbb{Z}^2$,

$$\begin{aligned}\int_{\mathbb{R}^2}\xi^z(|u_n^t|^2+|\nabla u_n^t|^2)+\frac{1}{2}\int_{\mathbb{R}^2}a_n\xi^z(1-|u_n^t|^2)^2+\alpha\int_0^t\int_{\mathbb{R}^2}\xi^z|\partial_t u_n|^2&\\ \le e^{Ct}\Big(\int_{\mathbb{R}^2}\xi^z(|u_n^\circ|^2+|\nabla u_n^\circ|^2)+\frac{1}{2}\int_{\mathbb{R}^2}a_n\xi^z(1-|u_n^\circ|^2)^2\Big),&\end{aligned}$$

and hence, using the Sobolev embedding of $H^1_{\mathrm{uloc}}(\mathbb{R}^2)$ into $\mathrm{L}^4_{\mathrm{uloc}}(\mathbb{R}^2)$ (see e.g. (8.293) below),

$$\begin{aligned}\int_{\mathbb{R}^2}\xi^z(|u_n^t|^2+|\nabla u_n^t|^2)+\frac{1}{2}\int_{\mathbb{R}^2}a_n\xi^z(1-|u_n^t|^2)^2+\alpha\int_0^t\int_{\mathbb{R}^2}\xi^z|\partial_t u_n|^2&\\ \le e^{Ct}\Big(1+\int_{\mathbb{R}^2}\xi^z(|u_n^\circ|^2+|\nabla u_n^\circ|^2)\Big)^2.&\end{aligned}$$

The claim (8.290) then follows from the boundedness of u_n° in $H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, noting that

$$\|\zeta\|^2_{\mathrm{L}^2_{\mathrm{uloc}}}\simeq\sup_{z\in\mathbb{R}^2}\int_{\mathbb{R}^2}\xi^z|\zeta|^2. \tag{8.291}$$

Step 2. Global existence with $k\ge 0$.

In this step, given $k\ge 0$, we assume $h\in W^{k+1,\infty}(\mathbb{R}^2)$, $F\in W^{k,\infty}(\mathbb{R}^2)^2$, $f\in W^{k,\infty}(\mathbb{R}^2)$, and $u^\circ\in H^{k+1}_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, and we prove that the global solution u constructed in Step 1 then belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{k+1}_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$. We denote by $C_k\ge 1$ any constant that only depends on an upper bound on k, α, α^{-1}, $|\beta|$, $\|(h,\nabla h,F,f)\|_{W^{k,\infty}}$, and $\|u^\circ\|_{H^{k+1}_{\mathrm{uloc}}}$, and we write $C_{k,t}$ if it additionally depends on an upper bound on t.

We argue again by approximation. We consider the truncations $h_n,a_n,F_n,f_n,u_n^\circ$ defined in Step 1, as well as the solution u_n to the corresponding equation (8.289). We claim that for all $k\ge 0$, for all $t\ge 0$,

$$\|u_n^t\|_{H^{k+1}_{\mathrm{uloc}}}+\|\partial_t u_n^t\|_{H^k_{\mathrm{uloc}}}\le C_{k,t}. \tag{8.292}$$

The desired result then follows by passing to the limit $n\uparrow\infty$. This result is proved by induction on k. As for $k=0$ the result already follows from Step 1, we assume that $\|u_n^t\|_{H^k_{\mathrm{uloc}}}\le C_{k,t}$ holds for some $k\ge 1$, and we deduce that (8.292) also holds for this k. Integrating by parts, we find

$$\begin{aligned}\frac{1}{2}\partial_t\int_{\mathbb{R}^2}\xi^z|\nabla^{k+1}u_n|^2&=\int_{\mathbb{R}^2}\xi^z\langle\nabla^{k+1}u_n,\nabla^{k+1}\partial_t u_n\rangle\\ &\le C\int_{\mathbb{R}^2}\xi^z|\nabla^{k+1}u_n||\nabla^k\partial_t u_n|-\int_{\mathbb{R}^2}\xi^z\langle\nabla^k\triangle u_n,\nabla^k\partial_t u_n\rangle,\end{aligned}$$

hence, inserting equation (8.289) in the first right-hand side term, and developing the terms,

$$\begin{aligned}
&\frac{1}{2}\partial_t \int_{\mathbb{R}^2} \xi^z |\nabla^{k+1} u_n|^2 \\
&\leq -\alpha \int_{\mathbb{R}^2} \xi^z |\nabla^k \partial_t u_n|^2 + C \int_{\mathbb{R}^2} \xi^z |\nabla^{k+1} u_n||\nabla^k \partial_t u_n| \\
&\qquad + \int_{\mathbb{R}^2} \xi^z \big\langle \nabla^k \big(a_n u_n (1-|u_n|^2) + \nabla h_n \cdot \nabla u_n + iF_n^\perp \cdot \nabla u_n + f_n u_n\big), \nabla^k \partial_t u_n \big\rangle \\
&\leq -\alpha \int_{\mathbb{R}^2} \xi^z |\nabla^k \partial_t u_n|^2 + C_k \sum_{j=0}^{k+1} \int_{\mathbb{R}^2} \xi^z |\nabla^j u_n||\nabla^k \partial_t u_n| + C_k \sum_{j=0}^{k-1} \int_{\mathbb{R}^2} \xi^z |\nabla^j u_n|^3 |\nabla^k \partial_t u_n| \\
&\qquad + C \int_{\mathbb{R}^2} \xi^z |u_n|^2 |\nabla^k u_n||\nabla^k \partial_t u_n| \\
&\leq -\frac{\alpha}{2} \int_{\mathbb{R}^2} \xi^z |\nabla^k \partial_t u_n|^2 + C_k \sum_{j=0}^{k+1} \int_{\mathbb{R}^2} \xi^z |\nabla^j u_n|^2 + C_k \sum_{j=0}^{k-1} \int_{\mathbb{R}^2} \xi^z |\nabla^j u_n|^6 \\
&\qquad + C \int_{\mathbb{R}^2} \xi^z |u_n|^4 |\nabla^k u_n|^2.
\end{aligned}$$

Note that the Sobolev embedding in the balls $B_2(x)$ yields

$$\begin{aligned}
\int_{\mathbb{R}^2} \xi^z |\nabla^j u_n|^6 &\lesssim \sum_{x \in \mathbb{Z}^2} \xi^z(x) \int_{B_2(x)} |\nabla^j u_n|^6 \\
&\lesssim \sum_{x \in \mathbb{Z}^2} \xi^z(x) \Big(\int_{B_2(x)} (|\nabla^j u_n|^2 + |\nabla^{j+1} u_n|^2) \Big)^3 \\
&\lesssim \Big(\sum_{x \in \mathbb{Z}^2} \xi^z(x) \int_{B_2(x)} (|\nabla^j u_n|^2 + |\nabla^{j+1} u_n|^2) \Big)^3 \\
&\lesssim \Big(\int_{\mathbb{R}^2} \xi^z (|\nabla^j u_n|^2 + |\nabla^{j+1} u_n|^2) \Big)^3, \qquad (8.293)
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_{\mathbb{R}^2} \xi^z |u_n|^4 |\nabla^k u_n|^2 &\leq \Big(\int_{\mathbb{R}^2} \xi^z |u_n|^8 \Big)^{1/2} \Big(\int_{\mathbb{R}^2} \xi^z |\nabla^k u_n|^4 \Big)^{1/2} \\
&\lesssim \Big(\int_{\mathbb{R}^2} \xi^z |\nabla u_n|^2 \Big)^2 \Big(\int_{\mathbb{R}^2} \xi^z (|\nabla^k u_n|^2 + |\nabla^{k+1} u_n|^2) \Big).
\end{aligned}$$

Inserting these estimates in the above, and using (8.291), we obtain

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^2} \xi^z |\nabla^{k+1} u_n|^2 + \alpha \int_{\mathbb{R}^2} \xi^z |\nabla^k \partial_t u_n|^2 \\
&\leq C_k \sum_{j=0}^{k} \Big(1 + \int_{\mathbb{R}^2} \xi^z |\nabla^j u_n|^2 \Big)^3 + C_k \Big(1 + \int_{\mathbb{R}^2} \xi^z |\nabla u_n|^2 \Big)^2 \int_{\mathbb{R}^2} \xi^z |\nabla^{k+1} u_n|^2 \\
&\leq C_k \big(1 + \|u_n\|^6_{H^k_{\mathrm{uloc}}} \big) + C_k \big(1 + \|u_n\|^4_{H^1_{\mathrm{uloc}}} \big) \int_{\mathbb{R}^2} \xi^z |\nabla^{k+1} u_n|^2.
\end{aligned}$$

By the induction hypothesis, we deduce for all $t \geq 0$

$$\partial_t \int_{\mathbb{R}^2} \xi^z |\nabla^{k+1} u_n^t|^2 + \alpha \int_{\mathbb{R}^2} \xi^z |\nabla^k \partial_t u_n^t|^2 \leq C_{k,t} + C_{k,t} \int_{\mathbb{R}^2} \xi^z |\nabla^{k+1} u_n^t|^2,$$

and the result (8.292) follows from the Grönwall inequality, taking the supremum over $z \in \mathbb{Z}^2$.

Step 3. Uniqueness.

In this step, we assume $h \in W^{1,\infty}(\mathbb{R}^2)$, $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, and $f \in \mathrm{L}^\infty(\mathbb{R}^2)$, and we prove that there exists at most one global solution $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ of (8.288) on $\mathbb{R}^+ \times \mathbb{R}^2$ with given initial data u°. We denote by $C \geq 1$ any constant that only depends on an upper bound on α, α^{-1}, $|\beta|$, and $\|(h, \nabla h, F, f)\|_{\mathrm{L}^\infty}$.

Let $u_1, u_2 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ denote two solutions as above. We set for simplicity $(\alpha + i\beta)^{-1} = \alpha' + i\beta'$, $\alpha' > 0$. Using equation (8.288) and integrating by parts, we find

$$\begin{aligned}\frac{1}{2}\partial_t \int_{\mathbb{R}^2} \xi^z |u_1 - u_2|^2 &\leq -\alpha' \int_{\mathbb{R}^2} \xi^z |\nabla(u_1 - u_2)|^2 + C \int_{\mathbb{R}^2} \xi^z |u_1 - u_2||\nabla(u_1 - u_2)| + C \int_{\mathbb{R}^2} \xi^z |u_1 - u_2|^2 \\ &\qquad + \int_{\mathbb{R}^2} a\xi^z \big\langle u_1 - u_2, (\alpha' + i\beta')\big(u_1(1 - |u_1|^2) - u_2(1 - |u_2|^2)\big)\big\rangle \\ &\leq -\frac{\alpha'}{2} \int_{\mathbb{R}^2} \xi^z |\nabla(u_1 - u_2)|^2 + C \int_{\mathbb{R}^2} \xi^z |u_1 - u_2|^2 (1 + |u_1| + |u_2|)^2. \end{aligned} \tag{8.294}$$

It remains to estimate the last integral. For that purpose, we decompose

$$\begin{aligned}\int_{\mathbb{R}^2} \xi^z |u_1 - u_2|^2 (|u_1| + |u_2|)^2 &\lesssim \sum_{x \in \mathbb{Z}^2} \xi^z(x) \int_{B_2(x)} |u_1 - u_2|^2 (|u_1| + |u_2|)^2 \\ &\lesssim \sum_{x \in \mathbb{Z}^2} \xi^z(x) \Big(\int_{B_2(x)} |u_1 - u_2|^4 \Big)^{1/2} \Big(\int_{B_2(x)} (|u_1| + |u_2|)^4 \Big)^{1/2},\end{aligned}$$

hence, using the Sobolev embedding of $H^{3/4}(B_2(x))$ (and of $H^1(B_2(x))$) into $\mathrm{L}^4(B_2(x))$,

$$\int_{\mathbb{R}^2} \xi^z |u_1 - u_2|^2 (|u_1| + |u_2|)^2 \lesssim \|(u_1, u_2)\|^2_{H^1_{\mathrm{uloc}}} \sum_{x \in \mathbb{Z}^2} \xi^z(x) \|u_1 - u_2\|^2_{H^{3/4}(B_2(x))}.$$

Using interpolation and Young's inequality then yields for all $K \geq 1$,

$$\begin{aligned}&\int_{\mathbb{R}^2} \xi^z |u_1 - u_2|^2 (|u_1| + |u_2|)^2 \lesssim \|(u_1, u_2)\|^2_{H^1_{\mathrm{uloc}}} \sum_{x \in \mathbb{Z}^2} \xi^z(x) \|u_1 - u_2\|^{3/2}_{H^1(B_2(x))} \|u_1 - u_2\|^{1/2}_{\mathrm{L}^2(B_2(x))} \\ &\lesssim K^{-1} \sum_{x \in \mathbb{Z}^2} \xi^z(x) \int_{B_2(x)} |\nabla(u_1 - u_2)|^2 + K^3 (1 + \|(u_1, u_2)\|^8_{H^1_{\mathrm{uloc}}}) \sum_{x \in \mathbb{Z}^2} \xi^z(x) \int_{B_2(x)} |u_1 - u_2|^2 \\ &\lesssim K^{-1} \int_{\mathbb{R}^2} \xi^z |\nabla(u_1 - u_2)|^2 + K^3 (1 + \|(u_1, u_2)\|^8_{H^1_{\mathrm{uloc}}}) \int_{\mathbb{R}^2} \xi^z |u_1 - u_2|^2.\end{aligned}$$

Inserting this into (8.294) with $K \simeq 1$ large enough, we find

$$\frac{1}{2}\partial_t \int_{\mathbb{R}^2} \xi^z |u_1 - u_2|^2 \leq C\big(1 + \|(u_1, u_2)\|^8_{H^1_{\mathrm{uloc}}}\big) \int_{\mathbb{R}^2} \xi^z |u_1 - u_2|^2,$$

and the conclusion $u_1 = u_2$ follows from the Grönwall inequality. $\square$

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Abstract

This thesis is devoted to the mathematical study of effects of disorder in various physical systems. We start with three stochastic homogenization problems in connection with static classical physics questions. First, motivated by the rigorous derivation of nonlinear elasticity from the statistical physics of polymer-chain networks, we establish the existence of effective properties for randomly heterogeneous hyperelastic materials under general growth assumptions. Second, in the simplest linearized setting, we investigate the so-called Clausius-Mossotti formulas for the effective properties of dilute two-phase dispersed media: we provide the first general and rigorous proof of these formulas, as well as an extension to higher orders. Third, again for linearized models, we propose to study deviations with respect to effective properties and we establish the first general theory of fluctuations in stochastic homogenization. In the second part of this thesis, the focus is on the interplay between disorder and interactions, and more precisely we study the dynamics of Ginzburg-Landau vortices in 2D type-II superconductors in the presence of several impurities. Although a complete mathematical understanding of the complex glassy properties of such systems seems out of reach, we rigorously establish the mean-field dynamics of a large number of vortices, and we investigate the homogenization of the fluid-like mean-field equations and their stick-slip properties.

Keywords: stochastic homogenization, fluctuations, Clausius-Mossotti formula, Ginzburg-Landau, vortex liquid, mean-field limit.

Quelques résultats en mathématique des milieux désordonnés

Résumé

Cette thèse est consacrée à l'étude mathématique des effets de désordre dans divers systèmes physiques. On commence par trois problèmes d'homogénéisation stochastique en lien avec des questions statiques de physique classique. Premièrement, en vue de la déduction rigoureuse de l'élasticité non linéaire à partir de la physique statistique de réseaux de chaînes de polymères, on établit l'existence de propriétés effectives pour des matériaux hyperélastiques hétérogènes aléatoires sous des hypothèses générales de croissance. Deuxièmement, dans un cadre linéarisé simplifié, on étudie les formules de Clausius-Mossotti pour les propriétés effectives d'alliages binaires dilués : on donne la première preuve générale et rigoureuse de ces formules, ainsi qu'une extension aux ordres supérieurs. Troisièmement, encore pour des systèmes linéarisés, on propose d'étudier les déviations par rapport aux propriétés effectives et on établit la première théorie générale des fluctuations en homogénéisation stochastique. Dans la seconde partie de cette thèse, on se focalise sur la compétition entre désordre et interactions, et on étudie plus particulièrement la dynamique des vortex de Ginzburg-Landau dans des supraconducteurs 2D de type II en présence d'impuretés. Bien que la compréhension mathématique des propriétés vitreuses complexes de ces systèmes semble hors de portée, on établit rigoureusement la limite de champ moyen pour la dynamique d'un grand nombre de vortex, et on étudie l'homogénéisation de ces équations limites et leurs propriétés.

Mots clés : homogénéisation stochastique, fluctuations, formule de Clausius-Mossotti, Ginzburg-Landau, liquide de vortex, limite de champ moyen.