Measure theory : exercises

Bachelor 3

Academic year 2017-2018

Chapter 3 : Product measures

Around Fubini's theorem

1. Let $(c_{n,m})_{n,m\geq 1} \subset \mathbb{R}^+$, and prove that

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} c_{n,m}\right) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} c_{n,m}\right).$$

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f : \Omega \to [0, \infty]$. Show that f is $\mathcal{F}$-measurable if and only if the set $A(f) := \{(t, \omega) : 0 < t < f(\omega)\} \subset \mathbb{R}^+ \times \Omega$ belongs to $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$.
3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f_n : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable functions. Assuming that $\sum_n \int |f_n|\, d\mu < \infty$ and that μ is σ-finite, prove that

$$\sum_n \int f_n \, d\mu \;=\; \int \sum_n f_n \, d\mu.$$

   Is this also true without the σ-finiteness assumption ?
4. Consider the measure space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mu)$, with $\mu = \lambda \times \sharp$, where λ is the Lebesgue measure on $\mathbb{R}$ and where $\sharp$ is the counting measure on $\mathbb{R}$. Compare the iterated integrals

$$\int_{\mathbb{R}}\left(\int_{\{a\}} d\lambda\right) d\sharp \qquad \text{and} \qquad \int_{\mathbb{R}}\left(\int_{\{a\}} d\sharp\right) d\lambda.$$

   Does this contradict Fubini's theorem ?
5. Consider the measure space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mu)$, with $\mu = \lambda \times \sharp$ as in the previous exercise. Consider the diagonal $D = \{(x, x) : x \in \mathbb{R}\}$. Compare the iterated integrals

$$\int_{[0,1]}\left(\int_{[0,1]} 1_D \, d\lambda\right) d\sharp \qquad \text{and} \qquad \int_{[0,1]}\left(\int_{[0,1]} 1_D \, d\sharp\right) d\lambda.$$

   Does this contradict Fubini's theorem ?
6. Let $f(x, y) = e^{-xy} - 2e^{-2xy}$. Check that f is Lebesgue-measurable on $\mathbb{R}^2$ and compare the iterated Lebesgue integrals

$$\int_{[0,1]}\left(\int_{[1,\infty)} f(x,y)dy\right) dx \quad \text{and} \quad \int_{[1,\infty)}\left(\int_{[0,1]} f(x,y)dx\right) dy.$$

   Does this contradict Fubini's theorem ?
7. Let $f : (-1, 1) \times (-1, 1) \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{otherwise.} \end{cases}$$

   Check that f is Lebesgue-measurable on $(-1, 1) \times (-1, 1)$ and compare the iterated integrals

$$\int_{(-1.1)}\left(\int_{(-1.1)} f(x,y)\, dx\right) dy \qquad \text{and} \qquad \int_{(-1.1)}\left(\int_{(-1.1)} f(x,y)\, dy\right) dx.$$

   Is f integrable on $(-1, 1) \times (-1, 1)$?

8. Let $[a,b] \subset [0,1)$ and $g\colon (0,1) \to \mathbb{R}^+$ a continuous function, supported inside (a,b), such that $\int_0^1 g(y)\,dy = 1$.
   (a) Given $\epsilon > 0$, construct a continuous function $G : (0,1) \times (0,1) \to \mathbb{R}^+$, supported inside $(0,1) \times (a,b)$, such that

   $$G(\tfrac{1}{2}, y) = g(y) \qquad \text{and} \qquad \int_0^1 \int_0^1 G(x,y)\,dx\,dy = \epsilon.$$

   (b) Let $g_n\colon (0,1) \to \mathbb{R}^+$ be continuous functions such that

   $$\operatorname{supp} g_n \subset (\delta_n, \delta_{n+1}) \qquad \text{and} \qquad \int_0^1 g_n(y)\,dy = 1,$$

   with $0 = \delta_1 < \delta_2 < \cdots < \delta_n \to 1$ as $n \to \infty$. For all $n \geq 1$, let G_n be the function associated with g_n by the construction of item (a) with $\epsilon = 2^{-n}$. Show that

   $$H(x,y) := \sum_{n=1}^{\infty} G_n(x,y)$$

   is a nonnegative continuous function on $(0,1) \times (0,1)$ with

   $$\iint_{(0,1)\times(0,1)} H(x,y)\,dx\,dy = 1 \qquad \text{and} \qquad \int_0^1 H(\tfrac{1}{2}, y)\,dy = +\infty.$$

   (c) Does this contradict Fubini's theorem ?

Around Cavalieri's principle

9. Let $f : \mathbb{R} \to \mathbb{R}$ be nonnegative and Lebesgue-measurable.
   (a) Set

   $$A(f) = \{(x,y) \in \mathbb{R}^2 : 0 < y < f(x)\}.$$

   Show that $A(f)$ is Lebesgue-measurable in $\mathbb{R}^2$ and that the integral of f on $\mathbb{R}$ coincides with the measure of the set $A(f)$ in $\mathbb{R}^2$.
   (b) Show that the graph of f,

   $$G(f) = \{(x, f(x));\, x \in \mathbb{R}\},$$

   has zero Lebesgue measure in $\mathbb{R}^2$.
   (*Hint* : for $i, j \in \mathbb{Z}$, set

   $$I_{ij} = f^{-1}([i, i+1)) \cap [j, j+1),$$
   $$G_{ij}(f) = \{(x, f(x));\, x \in I_{ij}\},$$

   ans show that $G_{ij}(f)$ has zero measure.)
   (c) Let $A(f)$ be defined as above. Check that

   $$\mathbb{1}_{A(f)}(x,y) = \mathbb{1}_{(y,\infty)}(f(x)) \quad \text{for all } (x,y) \in \mathbb{R} \times \mathbb{R}^+.$$

   Deduce Cavalieri's principle : for all $f : \mathbb{R} \to \mathbb{R}$ nonnegative and Lebesgue-measurable,

   $$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}^+} m(\{x \in \mathbb{R} : f(x) > y\})\,dy.$$

10. Let X be a nonnegative random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[X^p] < \infty$, for some $p > 0$. Prove that

   $$\mathbb{E}[X^p] = \int_0^\infty p\,x^{p-1}\,\mathbb{P}[X > x]\,dx.$$

Miscellaneous

11. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space, and let $h : \Omega \times \mathbb{R}^d \to \mathbb{C}$ such that $h(\cdot, x)$ is measurable for any fixed $x \in \mathbb{R}^d$ (meaning that h is a random field). Show that the following two properties are equivalent :

    (a) h is measurable on $\Omega \times \mathbb{R}^d$;

    (b) h is almost stochastically continuous, that is, for all $x \in \mathbb{R}^d$,

    $$\mathbb{P}\{\omega \in \Omega : |h(\omega, x+y) - h(\omega, x)| > \delta\} \xrightarrow{y \to 0} 0.$$

    Show that, if h is stationary (that is, $\mathbb{P}\{\omega : h(\omega, x) \in B\} = \mathbb{P}\{\omega : h(\omega, 0) \in B\}$ for all $x \in \mathbb{R}^d$ and all Borel set $B \subset \mathbb{C}$), then these properties are further equivalent to the following :

    (c) h is stochastically continuous, that is, for all $x \in \mathbb{R}^d$,

    $$\mathbb{P}\{\omega \in \Omega : |h(\omega, x+y) - h(\omega, x)| > \delta\} \xrightarrow{y \to 0} 0.$$

    (This can be viewed as a stochastic version of Lusin's theorem.) Show that in general without the stationarity assumption property (c) is strictly stronger than (a) and (b).