Measure theory : exercises

Bachelor 3

Academic year 2017-2018

Chapter 4 : Independence

1. Let $\mathcal{R}^{\mathbb{N}_0}$ denote the smallest σ-algebra on $\mathbb{R}^{\mathbb{N}_0}$ such that all projections $\pi_j((x_n)_n) = x_j$, $j \geq 1$, are $\mathcal{R}^{\mathbb{N}_0}$-measurable.
   (a) Show that $\mathcal{R}^{\mathbb{N}_0} = \sigma(\mathcal{P})$, where
   $$\mathcal{P} := \{A_1 \times \ldots \times A_k \times \mathbb{R}^{\mathbb{N}_0} : A_1, \ldots, A_k \in \mathcal{R},\, k \geq 1\}$$
   is a π-system.
   (b) Given a family $(f_n)_n$ of maps $f_n : \Omega \to \mathbb{R}$ on a measurable space $(\Omega, \mathcal{F})$, show that $(f_n)_n : \Omega \to \mathbb{R}^{\mathbb{N}_0}$ is $(\mathcal{F}, \mathcal{R}^{\mathbb{N}_0})$-measurable if and only if $f_n : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable for all n.
   (c) Let $(P_n)_n$ be a sequence of probability measures on $\mathcal{R}$. Show that there exists a unique probability measure $\mathbb{P}$ on $\mathcal{R}^{\mathbb{N}_0}$ such that
   $$\mathbb{P}(A_1 \times \ldots \times A_k \times \mathbb{R}^{\mathbb{N}_0}) = \prod_{n=1}^{k} P_n(A_n),$$
   for all $A_1, \ldots, A_k \in \mathcal{R}$ and all $k \geq 1$. Notation : $\mathbb{P} := \otimes_{n=1}^{\infty} P_n$.
   Hint : Consider a probability space $(\Omega_0, \mathcal{F}_0, P_0)$ and a sequence of independent random variables $(Y_n)_n$ such that $(P_0)_{Y_n} = P_n$ for all n, and consider the image measure $(P_0)_{(Y_n)_n}$.
2. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(A_n)_n$ be a sequence of independent events such that $\mathbb{P}[\cup_n A_n] = 1$ and $\mathbb{P}[A_n] < 1$ for all n. Show that $\mathbb{P}[\limsup_n A_n] = 1$.
3. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(A_n)_n$ be a sequence of independent events with $\mathbb{P}[A_n] = p \in (0,1)$ for all n. Show that the probability space cannot have any atom (that is, there exists no $B \in F$ with $\mathbb{P}[B] > 0$ such that for all $C \in \mathcal{F}$ with $C \subset B$ there holds either $\mathbb{P}[C] = 0$ or $\mathbb{P}[B \setminus C] = 0$). In particular, the probability space cannot be discrete.
4. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(A_n)_n$ be a sequence of events. The goal of this exercise is to establish the following generalized Borel-Cantelli lemma :
   $$\sum_n \mathbb{P}[A_n] = \infty \quad \text{and} \quad \liminf_{N\uparrow\infty} \frac{\sum_{j,k=1}^{n} \mathbb{P}[A_j \cap A_k]}{(\sum_{k=1}^{n} \mathbb{P}[A_k])^2} \leq 1 \quad \Longrightarrow \quad \mathbb{P}\left[\limsup_{n\uparrow\infty} A_n\right] = 1.$$
   What does this statement become in the case of independent events ?
   Hint : Let $N_n := \sum_{k=1}^{n} \mathbb{1}_{A_k}$ and examine $\mathbb{P}[N_n \leq x]$ for any given $x \leq \mathbb{E}[N_n] \uparrow \infty$.
5. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_n)_n$ be a sequence of independent random variables with $\mathbb{E}[X_n] = 0$ and $\sup_n \mathbb{E}\left[X_n^4\right] < \infty$. Show that $\frac{1}{n}\sum_{k=1}^{n} X_k \to 0$ a.s., even though the random variables X_n are not identically distributed.
6. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_n)_n$ be a sequence of independent and identically distributed random variables. Prove that
   $$\mathbb{P}\left[\limsup_{n\uparrow\infty} \frac{|X_n|}{\sqrt{n}} < \infty\right] = 1 \qquad \Longrightarrow \qquad \mathbb{E}\left[X_1^2\right] < \infty.$$
   Hint : First show that for some $K > 0$ there holds $\mathbb{P}[\limsup_{n\uparrow\infty} \frac{1}{\sqrt{n}}|X_n| \leq K\,] = 1$ and use the Borel-Cantelli lemma.

7. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_n)_n$ be a sequence of independent and identically distributed random variables with $\mathbb{P}[X_n = 0] = 1 - \mathbb{P}[X_n = 1] = p$ for all n. Show that

$$p \neq \frac{1}{2} \quad \Longrightarrow \quad \mathbb{P}\left[\limsup_n \left\{\sum_{k=1}^n X_k = 0\right\}\right] = 0.$$

8. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X : \Omega \to [0, 1)$ be a random variable such that for all $k = 0, 1, \ldots, 2^n - 1$ and all $n \geq 1$ there holds

$$\mathbb{P}\left[\frac{k}{2^n} \leq X < \frac{k+1}{2^n}\right] = \frac{1}{2^n}.$$

Show that $\mathbb{E}\left[X^2\right] = \frac{1}{3}$.