Measure theory : exercises

Bachelor 3

Academic year 2017-2018

Chapter 4 : Convergence

Reminder on different types of convergence

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A sequence $(f_n)_n$ of measurable functions converge

— *in measure* to f if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x \in \Omega \,:\, |f_n(x) - f(x)| > \epsilon\}) = 0;$$

— *in* L^p to f, with $1 \le p \le \infty$, if $f \in L^p$, $f_n \in L^p$ for all n, and

$$\lim_{n\to\infty} \int_\Omega |f_n - f|^p \, d\mu = 0\,;$$

— *almost uniformly* to f if for all $\epsilon > 0$, there exists $\Omega_\epsilon \subset \Omega$ such that

$$\mu(\Omega \setminus \Omega_\epsilon) < \epsilon \text{ and } f_n \to f \text{ uniformly on } \Omega_\epsilon\,;$$

— *almost everywhere* to f if there exists a set $N \subset \Omega$ of zero measure such that $f_n(x) \to f(x)$ for all $x \in \Omega \setminus N$.

1. Convergence almost everywhere does not imply L^p convergence, except if the sequence $(f_n)_n$ is bounded by a function $g \in L^p$.
2. L^p convergence implies convergence in measure.
3. Convergence almost everywhere does not imply convergence in measure, except if $\mu(\Omega) < \infty$.
4. Convergence in measure does not imply convergence almost everywhere, but only convergence almost everywhere of a subsequence.
5. Convergence in measure does not imply L^p convergence, except if $(f_n)_n$ is bounded by a function $g \in L^p$.
6. Convergence almost everywhere does not imply almost uniform convergence, except if $\mu(\Omega) < \infty$ (Egorov's theorem).

Convergence of random variables

7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space given by $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, $\mathbb{P} = \lambda|_{\mathcal{F}}$.
   (a) Consider the sequence $(X_n)_n$ defined by $X_n : [0,1] \to \mathbb{R} : \omega \mapsto \sqrt{n}(-\omega)^n$. Does $(X_n)_n$ converge in L^1, L^2, L^3, in probability, and almost everywhere ? Is it uniformly integrable ?
   (b) Consider the sequence $(Y_n)_n$ defined by $Y_1 = \mathbb{1}_{[0,1]}$ and $Y_{2^n+j} = \sqrt{2^n}\mathbb{1}_{[j2^{-n},(j+1)2^{-n}]}$ for all $0 \le j \le 2^n - 1$ and $n \ge 1$. Does $(Y_n)_n$ converge in L^1, L^2, L^3, in probability, and almost everywhere ? Is it uniformly integrable ?
   (c) Consider the sequence $(Z_n)_n$ defined by $Z_n = n\mathbb{1}_{[0,\frac{1}{n}]} - n\mathbb{1}_{[1-\frac{1}{n},1]}$. Show that there exists a random variable Z such that $Z_n \to Z$ in probability and $\mathbb{E}[Z_n] \to \mathbb{E}[Z]$, but that Z_n does not converge in L^1. Which assumption of which theorem is not satisfied ?
8. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_n)_n$ be a sequence of random variables. Show that

$$X_n \to X \text{ in probability} \qquad \Leftrightarrow \qquad \mathbb{E}[|X_n - X| \wedge 1] \to 0.$$

9. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some $p \in [1,\infty)$, consider $B_p := \{X \in L^p(\Omega) : \mathbb{E}[|X|^p]^{\frac{1}{p}} \le 1\}$. Show that B_p is uniformly integrable for $p \in (1,\infty)$ but not for $p = 1$.
10. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_n)_n$ be a sequence of Gaussian random variables such that $X_n \to X$ in probability. Show that X is also Gaussian and that $\mathbb{E}[X_n^p] \to \mathbb{E}[X^p]$ holds for all $p \ge 1$.
    Hint : Show that $(X_n)_n$ is uniformly integrable.

L^p spaces

11. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $1 \le q < p < \infty$.
    (a) If $\mu(\Omega) < \infty$, show that for all $u \in L^p(\Omega)$ we have

$$\frac{\|u\|_{L^q}}{\mu(\Omega)^{1/q}} \le \frac{\|u\|_{L^p}}{\mu(\Omega)^{1/p}},$$

    hence $L^p(\Omega) \subset L^q(\Omega)$ (but the converse is in general false).

    (b) Let A be a countable set. For all $1 \le r < \infty$, we define the space $\ell^r(A)$ as

$$\ell^r(A) = \left\{ (x_k)_{k\in A} \;:\; \|(x_k)_k\|_{\ell^r(A)} := \left(\sum_{k\in A} |x_k|^r \right)^{1/r} < \infty \right\}.$$

    Show that for all $(x_k)_k \in \ell^q(A)$ we have

$$\|(x_k)_k\|_{\ell^p(A)} \le \|(x_k)_k\|_{\ell^q(A)},$$

    hence $\ell^q(A) \subset \ell^p(A)$ (but the converse is in general false).

    (c) Let $u : \mathbb{R} \to \mathbb{R}$ be defined by

$$u(x) = \begin{cases} (1/x)^{1/q} & \text{if } x \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

    Show that $u \in L^p(\mathbb{R})$ but $u \notin L^q(\mathbb{R})$, hence $L^p(\mathbb{R}) \not\subset L^q(\mathbb{R})$.

    (d) Let $v : \mathbb{R} \to \mathbb{R}$ be defined by

$$v(x) = \begin{cases} (1/x)^{1/p} & \text{if } 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

    Show that $v \in L^q(\mathbb{R})$ but $v \notin L^p(\mathbb{R})$, hence $L^p(\mathbb{R}) \not\supset L^q(\mathbb{R})$.

    (e) Is it true that $L^q(\mathbb{R}) \supset L^\infty(\mathbb{R})$? And that $L^q(\mathbb{R}) \subset L^\infty(\mathbb{R})$?

    (f) Is it true that $\cap_{1\le r<\infty} L^r(\mathbb{R}) \subset L^\infty(\mathbb{R})$? And what if $\mathbb{R}$ is replaced by a compact subset ?

12. Let $1 \le p < \infty$. Construct a measurable function f on $\mathbb{R}$ such that $f \in L^p(\mathbb{R})$ but $f \notin L^q(\mathbb{R})$ for all $q \ne p$.

13. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that the map $p \mapsto \|\cdot\|_p^p$ is log-convex, that is, for all $1 \le p, q < \infty$ and all measurable functions u on Ω, we have for all $0 \le \theta \le 1$,

$$\|u\|_{L^{\theta p+(1-\theta)q}}^{\theta p+(1-\theta)q} \le \|u\|_{L^p}^{\theta p} \|u\|_{L^q}^{(1-\theta)q}.$$

    Deduce that, for all $p \le q$, we have $\cap_{r\in[p,q]} L^r(\Omega) = L^p(\Omega) \cap L^q(\Omega)$. In particular, if u is measurable on Ω, the set $\{p \in [1,\infty] : u \in L^p\}$ is convex (hence an interval). Examining the previous exercises, deduce that any convex subset of $[1,\infty]$ can be obtained in this form.