Please provide complete and well-written solutions to the following exercises.
Due on April 10th before noon.

Homework 1

Exercise 1. A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability $\frac{3}{4}$ and goes to the other hotel with probability $\frac{1}{4}$.

(i) Find the transition matrix for the chain.

(ii) If the driver starts at the airport at time 0, find the probability for each of his three possible locations at time 2, and the probability that he is at hotel B at time 3.

Exercise 2. Suppose that the probability it rains today is 0.4 if neither of the last two days was rainy, but 0.5 if at least one of the last two days was rainy. Let $\Omega = \{S, R\}$, where S=sunny and R=rainy. Let W_t be the weather of day t.

(i) Show that $(W_t)_{t\geq 0}$ is not a Markov chain.

(ii) Expand the state space into the set of pairs $\Sigma := \Omega^2$. For each $t \geq 0$, define $X_t := (W_{t-1}, W_t) \in \Sigma$. Show that $(X_t)_{t\geq 0}$ is a Markov chain on Σ. Identify its transition matrix.

(iii) What is the two-step transition matrix?

(iv) What is the probability that it will rain on Wednesday if it didn't rain on Sunday and Monday?

Exercise 3. Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains. Give reasons for your answers.

$$\text{(i)} \quad \begin{pmatrix} 0.4 & 0.3 & 0.3 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0.4 \end{pmatrix} \qquad \text{(ii)} \quad \begin{pmatrix} 0.1 & 0 & 0 & 0.4 & 0.5 & 0 \\ 0.1 & 0.2 & 0.2 & 0 & 0.5 & 0 \\ 0 & 0.1 & 0.3 & 0 & 0 & 0.6 \\ 0.1 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

Exercise 4. Let $(X_t)_{t\geq 0}$ be a Markov chain with finite state space Ω. Given $y \in \Omega$, define $L_y := \max\{n \geq 0 : X_n = y\}$ and $M_y := \min\{n \geq 0 : X_n = X_{n+1} = X_{n+2} = y\}$. Are L_y and M_y stopping times? Prove your assertions.

Exercise 5. Let $(X_t)_{t\geq 0}$ be a Markov chain with finite state space Ω and with transition matrix P. Given $x, y \in \Omega$, consider $T_y := \min\{n \geq 1 : X_n = y\}$ the first return time to y, and show that

$$\mathbb{P}_x[T_y \leq j] \;\geq\; (P^j)_{xy}.$$

Exercise 6. Let $(X_t)_{t\geq 0}$ be a Markov chain with finite state space Ω. Given $y \in \Omega$, assume that $\mathbb{P}_x[T_y \leq k] \geq \alpha > 0$ holds for all x. Then show that $\mathbb{P}_x[T_y > nk] \leq (1-\alpha)^n$ for all x.