Please provide complete and well-written solutions to the following exercises.
Due on April 17th before noon.

Homework 2

Exercise 1. Let P be the transition matrix of a Markov chain with finite state space Ω. A function $f : \Omega \to \mathbb{R}$ is said to be P-harmonic if

$$f(x) = \sum_{y \in \Omega} P_{xy} f(y), \qquad \text{for all } x \in \Omega.$$

Show that if P is irreducible, then any P-harmonic function is a constant function.

Exercise 2. Consider the following transition matrices. Determine their stationary distributions and whether they satisfy the detailed balance condition. Justify your answers.

$$\text{(i)} \quad \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \qquad\qquad \text{(ii)} \quad \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.4 & 0 & 0.6 \end{pmatrix}$$

Exercise 3. Let $(X_n)_n$ be a Markov chain with finite state space $\Omega = \{1, \ldots, m\}$ and with transition matrix P given by

$$P_{i,i+1} = p_i \;\; \forall 1 \le i < m, \qquad P_{i,i-1} = q_i \;\; \forall 1 < i \le m, \qquad P_{ij} = 0 \;\; \forall |i-j| > 1,$$

for some constants $p_i, q_i > 0$ with $p_i + q_i \le 1$ for $1 < i < m$, and some $0 < p_1, q_m \le 1$. Such chains are known as "birth and death chains". Compute the stationary distribution, show that it is unique and satisfies the detailed balance condition.

Exercise 4. A graph is represented as a pair $\Gamma = (V, A)$ where

- V is the set of vertices.
- $A \in \{0,1\}^{|V| \times |V|}$ is the adjacency matrix, that is, A_{uv} is equal to 1 if there is an edge connecting u and v and is equal to 0 otherwise. By convention, set $A_{vv} = 0$ for all v.

A simple random walk on the graph Γ is a Markov chain $(X_n)_n$ on V described as follows: given $X_n = v$, the chain jumps at time $n+1$ to a randomly chosen neighbour of v in Γ.

(i) Compute the transition probability for the chain $(X_n)_n$. Recall that the degree d_v of a vertex v is defined as the number of neighbors it has, that is, $d_v := \sum_u A_{uv}$.

(ii) Show that the chain is irreducible iff the graph is connected.

(iii) Determine the unique stationary distribution for the chain and check that it satisfies the detailed balance condition.

(iv) As a concrete example, consider the simple random walk of a queen on a chessboard. A chessboard is an 8×8 grid of squares. A queen can move any number or squares horizontally, vertically, or diagonally. The simple random walk of the queen is the sequence of the queen's positions on the board if one picks queen's legal moves at random. Viewing this as a walk on a graph, compute the degrees of the vertices in the corresponding graph, and deduce the stationary distribution for the walk of the queen. What is the expected number of moves to return to a given corner if we start there?

Exercise 5. Let $(X_n)_n$ be an irreducible Markov chain with finite state space Ω. Given a state $y \in \Omega$, consider the number of visits to y at times $\leq n$,

$$N_n(y) := \sharp\{0 \leq m \leq n : X_m = y\}.$$

(i) Show that the times between returns to y, $\{T_y^{k+1} - T_y^k\}_{k \geq 1}$, are iid random variables with expectation $\mathbb{E}_y T_y < \infty$.

(ii) Using the strong law of large numbers, deduce that the time of the kth return to y,

$$R(k) := \min\{n \geq 1 : N_n(y) = k\},$$

satisfies $\frac{1}{k} R(k) \to \mathbb{E}_y T_y < \infty$ almost surely as $k \uparrow \infty$.

(iii) Use the above to conclude that $\frac{1}{n} N_n(y) \to (\mathbb{E}_y T_y)^{-1}$ almost surely as $n \uparrow \infty$. In other words, in the long run, the proportion of time the chain spends at site y is given by $(\mathbb{E}_y T_y)^{-1}$.

Exercise 6. A professor has two light bulbs in his garage. When both are burned out, they are replaced, and the next day starts with two working light bulbs. Suppose that when both are working, one of the two will go out with probability 0.02 (each has probability 0.01 and we ignore the possibility of losing two on the same day). However, when only one is there, it will burn out with probability 0.05.

(i) What is the expected time between light bulb replacements?

(ii) What is the long-run fraction of time that there is exactly one bulb working?