Please provide complete and well-written solutions to the following exercises.
Due on April 24th before noon.

Homework 3

Exercise 1. The Markov chain associated with a manufacturing process may be described as follows: A part to be manufactured will begin the process by entering step 1. After step 1, 20% of the parts must be reworked (i.e., returned to step 1), 10% of the parts are thrown away, and 70% proceed to step 2. After step 2, 5% of the parts must be returned to the step 1, 10% to step 2, 5% are scrapped, and 80% emerge to be sold for a profit.

(i) Formulate a Markov chain with state space $\{1, 2, 3, 4\}$, where 3 corresponds to a part being scrapped, and 4 corresponds to a part being sold for a profit.

(ii) Compute the probability that a part is scrapped in the manufacturing process. Compute the expected number of steps for a part to be fully processed (that is, either scrapped or sold).

Exercise 2. Consider the following gambling game. Player 1 picks a three-coin pattern (for example HTH), and then player 2 picks another (say THH). A coin is flipped repeatedly and outcomes are recorded until one of the two patterns appears. Somewhat surprisingly, player 2 has a considerable advantage in this game. No matter what player 1 picks, player 2 can win with probability $\geq \frac{2}{3}$. Indeed, show that:

- if player 1 picks HHH, then player 2 wins with probability $\frac{7}{8}$ by picking THH;
- if player 1 picks HHT, then player 2 wins with probability $\frac{3}{4}$ by picking THH;
- if player 1 picks HTH, then player 2 wins with probability $\frac{2}{3}$ by picking HHT;
- if player 1 picks HTT, then player 2 wins with probability $\frac{2}{3}$ by picking HHT.

Exercise 3. Roll a fair dice repeatedly and let $Y_1, Y_2, \ldots$ be the resulting numbers. Let $X_n = |\{Y_1, Y_2, \ldots, Y_n\}|$ be the number of values we have seen in the first n rolls for $n \geq 1$, and set $X_0 = 0$.

(i) Show that $(X_n)_n$ is a Markov chain and find its transition matrix.

(ii) Let $\tau := \min\{n : X_n = 6\}$ be the number of trials before having seen all 6 numbers at least once. Compute $\mathbb{E}\tau$.

Exercise 4. Consider a population model where each individual in the nth generation gives birth to an iid number of children. Let p_k be the probability that an individual has k children, and let $\mu := \sum_k kp_k$ be the expected number of children per individual. Let X_n be the number of individuals in the nth generation.

(i) Show that $(X_n)_n$ is a Markov chain on an infinite state space. This is known as a "branching process".

We aim to investigate the probability $R_1 := \mathbb{P}_1[T_0 < \infty]$ that the population gets extinct when starting from one individual.

(ii) Show by induction that $\mathbb{E}X_n = \mu^n \mathbb{E}X_0$. If $\mu < 1$, deduce $R_1 = 1$, that is, extinction occurs almost surely.

(iii) In terms of the generating function $\phi(t) := \sum_{k=0}^{\infty} t^k p_k$, show that the probability $\rho_n = \mathbb{P}_1[X_n = 0]$ satisfies

$$\rho_n = \phi(\rho_{n-1}), \qquad \text{for all } n \geq 1.$$

Deduce that the extinction probability R_1 is the smallest solution of the equation $x = \phi(x)$ with $0 \leq x \leq 1$.

(iv) If $\mu > 1$, show that $R_1 < 1$, that is, there is a positive probability to avoid extinction. (*Hint:* Note that $\phi(0) = p_0$, $\phi(1) = 1$, and $\phi'(1^-) = \mu$. Split the cases $p_0 = 0$ and $p_0 > 0$ to deduce that there exists a solution $0 \leq x < 1$ to the equation $x = \phi(x)$.)

(v) If $\mu = 1$ and $p_1 < 1$, show that $R_1 = 1$, that is, extinction occurs almost surely. (*Hint:* Show that $\phi'(y) < 1$ for $y < 1$ and deduce that the equation $x = \phi(x)$ has no solution $x < 1$.)

(vi) Show that $\mathbb{P}_n[T_0 < \infty] = (R_1)^n$. In particular, the condition $R_1 < 1$ means that the extinction probability is arbitrarily small when starting from a large enough population: $\lim_{n\uparrow\infty} \mathbb{P}_n[T_0 < \infty] = 0$.

(vii) Consider the branching process where the number of children is chosen to follow a shifted geometric distribution, that is, $p_k = p(1-p)^k$ for all $k \geq 0$. Compute the extinction probability R_1.

Exercise 5. Examples of martingales for Markov processes.

(i) Let $(X_n)_n$ be a Markov chain with transition matrix P on a finite state space Ω, let $f : \Omega \times \mathbb{N} \to \mathbb{R}$ be a function such that

$$f(x,n) = \sum_{y \in \Omega} P_{xy} f(y, n+1), \qquad \text{for all } x, n,$$

and set $M_n := f(X_n, n)$. Show that $(M_n)_n$ is a martingale with respect to $(X_n)_n$.

(ii) Let $(X_n)_n$ be a Markov chain with transition matrix P on a finite state space Ω, with two absorbing states $a, b \in \Omega$, and set $h(x) := \mathbb{P}_x[V_a < V_b]$. Show that $(h(X_n))_n$ is a martingale with respect to $(X_n)_n$.

(iii) If $(X_n)_n$ is the branching process defined in Exercise 4, show that the sequences $((R_1)^{X_n})_n$ and $(X_n/\mu^n)_n$ are both martingales.