Please provide complete and well-written solutions to the following exercises.
Due on May 8th before noon.

Homework 4

Exercise 1. In a coin-flipping game, consider the double-your-bet strategy: start by betting \$1, double your bet until you win, and stop playing once you have won. More precisely, letting $p > 0$ be the probability to win a bet, and letting $X_1, X_2, \ldots$ be iid random variables with $\mathbb{P}[X_1 = 1] = p$ and $\mathbb{P}[X_1 = -1] = 1 - p$, your net fortune can be written as

$$M_n := \sum_{m=1}^{n \wedge T} 2^{m-1} X_m, \qquad T := \min\{m \geq 1 : X_m = 1\}.$$

Show that $M_n \to M_T = 1$ a.s. as $n \uparrow \infty$, which shows that in principle this betting strategy makes you win eventually, whatever the value of $p > 0$. However note that $\mathbb{E}[M_n] = 2p - 1$ for all n (which is < 1 if $p < 1$): where is the catch?

Exercise 2. Consider an election with 2 candidates and c voters. Assume that candidate 1 gets a votes and candidate 2 gets b votes, with $a > b$, $a + b = c$, so that candidate 1 eventually wins the election. The votes are counted one by one in a uniformly random ordering, and we would like to keep a running tally of who is currently winning.

(i) Denote by S_n the number of votes for candidate 1 minus the number of votes for candidate 2 after n votes have been counted. Define $M_n := S_{c-n}/(c-n)$ and show that $(M_n)_n$ is a martingale.

(ii) Define $T := \min\{0 \leq n \leq c : M_n = 0\}$, and set $T = c - 1$ if there is no n with $M_n = 0$. Show that T is a stopping time.

(iii) With the above ingredients, show that that the probability that candidate 1 is always ahead throughout the running tally is equal to $\frac{a-b}{a+b}$.

Exercise 3. Let (S_n) be the simple random walk, that is, $S_n = S_0 + X_1 + \ldots + X_n$ where $X_1, X_2, \ldots$ are iid random variables with $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = -1] = \frac{1}{2}$.

(i) Let T be the first time that the walk hits 0 or m. Using that $(S_n)_n$, $(S_n^2 - n)_n$, and $(S_n^3 - 3nS_n)_n$ are martingales, show that for all $0 < k < m$,

$$\mathbb{P}_k[S_T = m] = \frac{k}{m}, \qquad \mathbb{E}_k T = k(m-k), \qquad \mathbb{E}_k[T | S_T = m] = \frac{1}{3}(m^2 - k^2).$$

(ii) Using a martingale involving S_n^4, further compute $\mathrm{Var}_k[T]$.

Exercise 4. Consider the random walk $S_n = S_0 + X_1 + \ldots + X_n$, where $X_1, X_2, \ldots$ are iid integer-valued random variables with $\mathbb{E}X_i > 0$, $\mathbb{P}[X_i \geq -1] = 1$, and $\mathbb{P}[X_i = -1] > 0$. Let $\phi(\theta) = \mathbb{E}[\exp(\theta X_1)]$ be the moment generating function and let $V_a = \min\{n \geq 0 : S_n = a\}$ be the first visit time to $a \in \mathbb{Z}$.

(i) Show that there exists a unique $\alpha < 0$ with $\phi(\alpha) = 1$.

(ii) Deduce that $(\exp(\alpha S_n))_n$ is a martingale.

(iii) Prove that $\mathbb{P}_x[V_a < \infty] = e^{\alpha(x-a)}$ for all $a < x$.

Exercise 5. Consider a Markov chain with finite state space Ω. We use martingale theory to provide an alternative proof of the characterization of exit probabilities and expected exit times.

(i) Given $a, b \in \Omega$, let $\tau := V_a \wedge V_b$ the first visit time to a or b. Assume that a function $h : \Omega \to \mathbb{R}$ satisfies $h(a) = 1$, $h(b) = 0$, and

$$h(x) = \sum_y P_{xy} h(y) \qquad \text{for all } x \neq a, b.$$

Show that $(h(X_{n\wedge\tau}))_n$ is a martingale. Provided $\mathbb{P}_x[\tau < \infty] > 0$ for all $x \neq a, b$, deduce that $h(x) = \mathbb{P}_x[V_a < V_b]$.

(ii) Given $A \subset \Omega$, let $V_A := \min\{n \geq 0 : X_n \in A\}$ be the first visit time to A. Assume that a function $g : \Omega \to \mathbb{R}$ satisfies $g(x) = 0$ for all $x \in A$ and

$$g(x) = 1 + \sum_y P_{xy} g(y) \qquad \text{for all } x \notin A.$$

Show that $(g(X_{n\wedge V_A}) + n \wedge V_A)_n$ is a martingale. Provided $\mathbb{P}_x[V_A < \infty] > 0$ for all $x \notin A$, deduce that $g(x) = \mathbb{E}_x V_A$.

Exercise 6. Let $(X_n)_n$ be an irreducible Markov chain with state space $\{0, 1, 2, \ldots\}$ and assume that there exists a nonnegative function ϕ such that $\lim_{x\uparrow\infty} \phi(x) = \infty$ and $\mathbb{E}_x \phi(X_1) \leq \phi(x)$ for all $x \geq K$. Then show that $(X_n)_n$ is recurrent.