Please provide complete and well-written solutions to the following exercises.
Due on May 15th before noon.

Homework 5

Exercise 1. Recall that a positive random variable X is said to have the memoryless property if

$$\mathbb{P}[X \geq t+s | X \geq t] = \mathbb{P}[X \geq s], \qquad \text{for all } s,t>0.$$

(i) Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a function with the property that $g(x+y) = g(x)+g(y)$ for all $x, y \geq 0$, and assume that g is continuous at 0. Show that $g(x) = \alpha x$ for some constant α. For that purpose, proceed in the following steps:

— Show that $g(0) = 0$.

— Show by induction that $g(n) = ng(1)$ for all integers $n \geq 1$.

— Show that $g(x) = xg(1)$ for all rational numbers $x \geq 0$.

— Show that g is continuous on $\mathbb{R}^+$.

— Deduce by continuity that $g(x) = xg(1)$ for all $x \in \mathbb{R}^+$.

(ii) Using (i), prove that a continuous positive random variable X has the memoryless property if and only if it is an exponential random variable.

Exercise 2. Let $T_1 \sim \mathrm{Exp}(\lambda_1)$ and $T_2 \sim \mathrm{Exp}(\lambda_2)$ be independent.

(i) Show that the minimum is also exponential: $T_1 \wedge T_2 \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$.

(ii) Compute the expectation and the variance of $T_1 \wedge T_2$, $T_1 \vee T_2$, and $T_1 \vee T_2 - T_1 \wedge T_2$. (Hint: Use (i) and note that $T_1 \vee T_2 = T_1 + T_2 - T_1 \wedge T_2$.)

(iii) Show that

$$\mathbb{P}[T_1 < T_2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

(iv) Let I be the random variable defined by

$$I = \begin{cases} 1, & \text{if } T_1 \leq T_2; \\ 2, & \text{if } T_1 > T_2; \end{cases}$$

and show that I and $T_1 \wedge T_2$ are independent.

Exercise 3. Alice and Bob enter a beauty parlor simultaneously: Alice to get a haircut and Bob to get a manicure. Suppose that the haircut (resp. the manicure) is exponentially distributed with mean 30 minutes (resp. 20 minutes). What is the probability Alice gets done first? What is the expected amount of time until Alice and Bob are both done?

Exercise 4. The number of hours between successive trains is T, which is uniformly distributed between 1 hour and 2 hours. Passengers arrive at the station according to a Poisson process with rate 24 per hour. Compte the expectation and variance of the number of people who get on each train.

Exercise 5. A machine has two critically important parts and is subject to three different types of shocks. Shocks of type i occur at times of a Poisson process with rate λ_i. Shocks of types 1 break part 1, those of type 2 break part 2, while those of type 3 break both parts. Let T_1 and T_2 be the failure times of the two parts.

(i) What is the averaged time before part of the machine gets broken?

(ii) Compute $\mathbb{P}\left[T_1 > s, T_2 > t\right]$ and find the distribution of T_1 and T_2.

(iii) Are T_1 and T_2 independent?

Exercise 6. Consider a Poisson process with rate λ and let L_t be the time of the last arrival in the interval $[0, t]$, with the convention that $L_t = 0$ if there was no arrival. Compute $\mathbb{E}\left[t - L_t\right]$. What happens when we let $t \uparrow \infty$?

Exercise 7. Suppose that traffic on a road follows a Poisson process with rate λ cars per minute. A giraffe needs a gap of length at least c minutes in the traffic to cross the road. To compute the time the giraffe will have to wait to cross the road, let $t_1, t_2, t_3, \ldots$ be the interarrival times for the cars and let $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \ldots + t_n$, then the giraffe will start to cross the road at time T_{J-1} and complete his journey at time $T_{J-1} + c$. Compute $\mathbb{E}\left[T_{J-1} + c\right]$.