Please provide complete and well-written solutions to the following exercises.
Due on May 29th before noon.

Homework 7

Exercise 1. A light bulb burns for an amount of time having distribution F with mean μ_F then burns out. A janitor comes at times of a rate λ Poisson process to check the bulb and will replace the bulb if it is burnt out.

(i) At what rate are bulbs replaced?

(ii) What is the long-run fraction of time that the right bulb works?

(iii) What is the long-run fraction of visits on which the bulb is working?

Exercise 2. In front of an airport, there is an area where hotel shuttle vans park. Customers arrive at times of a Poisson process with rate 10 per hour looking for transportation to hotel A nearby. When 7 people are in the van, it leaves for the 36-minute round trip to the hotel. Travelers who arrive while the van is gone go to some other hotel instead.

(i) What is the long-run fraction of the customers who actually go to hotel A?

(ii) What is the long-run average amount of time that a person who actually goes to hotel A ends up waiting in the van?

Exercise 3. People arrive at a college admissions office at rate 1 per minute. When k people have arrived a tour starts. Student tour guides are paid \$20 for each tour they conduct. The college estimates that it loses \$0.1 in good will for each minute a person waits. What is the optimal tour group size k?

Exercise 4. Let $(X_k)_{k\geq 1}$ be a sequence of iid random variables with finite variance. Let $(w_k)_{k\geq 1}$ be a sequence of real numbers such that the following limits exist,

$$\bar{w} := \lim_{n\uparrow\infty} \frac{1}{n}\sum_{k=1}^{n} w_k < \infty, \qquad \lim_{n\uparrow\infty} \frac{1}{n}\sum_{k=1}^{n} w_k^2 < \infty.$$

The goal of the exercise is to show the following strong law of large numbers for the weighted sums $S_n = \sum_{k=1}^{n} w_k X_k$:

$$\frac{S_n}{n} \to \bar{w}\mathbb{E}\left[X_1\right], \qquad \text{almost surely,} \qquad \text{as } n\uparrow\infty. \tag{0.1}$$

(i) Write

$$\frac{S_n}{n} = \frac{1}{n}\sum_{k=1}^{n} w_k \mathbb{E}\left[X_k\right] + \frac{1}{n}\sum_{k=1}^{n} w_k (X_k - \mathbb{E}\left[X_k\right]),$$

and deduce that it suffices to prove the assertion (0.1) assuming $\mathbb{E}\left[X_1\right] = 0$. We may therefore assume $\mathbb{E}\left[X_1\right] = 0$ in the following steps.

(ii) Use Chebychev's inequality to show that

$$\mathbb{P}\left[\frac{S_n}{n} \geq \frac{1}{\log n}\right] \leq \frac{(\log n)^2}{n}\mathbb{E}\left[X_1^2\right]\frac{1}{n}\sum_{k=1}^{n} w_k^2.$$

(iii) Invoking the Borel–Cantelli lemma, deduce

$$\mathbb{P}\left[\lim_{n\uparrow\infty}\frac{1}{n^2}S_{n^2} = 0\right] = 1.$$

More generally, show that for any increasing sequence of integers $(n_k)_k$ we can choose a subsequence $(n_{k_r})_r$ such that

$$\mathbb{P}\left[\lim_{r\uparrow\infty}\frac{1}{n_{k_r}}S_{n_{k_r}} = 0\right] = 1.$$

Conclude that $\lim_{n\uparrow\infty}\frac{1}{n}S_n = 0$ almost surely, that is, (0.1).

Exercise 5. Consider a GI/G/1 queue, where arrivals are given by a renewal process of rate λ and where service times are iid copies $(\sigma_k)_k$ of a random variable σ with finite mean and variance. Letting $W_{Q,k}$ denote the time that the kth customer spends in the queue, assume that the following limit exists almost surely and is finite,

$$\lim_{n\uparrow\infty}\frac{1}{n}\sum_{k=1}^{n} W_{Q,k} < \infty.$$

Then, letting $R(t)$ denote the remaining service time of the current customer in the server at time t, prove that the following limit exists almost surely and is given by the following formula, representing the long-run average load in the server,

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T R(t)\,dt = \tfrac{1}{2}\lambda\mathbb{E}\left[\sigma^2\right]. \tag{0.2}$$

(*Hint:* Using the following further notation,

$$\begin{aligned} T_k &:= k\text{th arrival time},\\ N(t) &:= \text{number of arrivals up to time } t,\\ N_d(t) &:= \text{number of departures up to time } t, \end{aligned}$$

show that

$$R(t) = \sum_{k=1}^{\infty}(T_k + W_{Q,k} + \sigma_k - t)\mathbb{1}_{T_k+W_{Q,k}\leq t<T_k+W_{Q,k}+\sigma_k}.$$

and deduce

$$\sum_{k=1}^{N_d(T)}\frac{\sigma_k^2}{2} \leq \int_0^T R(t)\,dt \leq \sum_{k=1}^{N(T)}\frac{\sigma_k^2}{2},$$

from which the conclusion (0.2) can be inferred.)

Exercise 6. Consider a M/G/1 queue, where arrivals are given by a Poisson process of rate λ and service times are iid copies $(\sigma_k)_k$ of a random variable σ with finite mean and variance. We use the same notation as in Exercise 5 and we assume that the following limits exist almost surely,

$$W_Q := \lim_{n\uparrow\infty} \frac{1}{n}\sum_{k=1}^{n} W_{Q,k} < \infty, \qquad \lim_{n\uparrow\infty} \frac{1}{n}\sum_{k=1}^{n} W_{Q,k}^2 < \infty.$$

The goal of this exercise is to show the following Pollaczek–Khinchine formula for the long-run average time W_Q that a customer spends in the queue:

$$W_Q = \frac{\lambda \mathbb{E}\left[\sigma^2\right]}{2(1-\lambda\mathbb{E}\left[\sigma\right])}. \tag{0.3}$$

(i) Let $S_Q(t)$ denote the sum of service times of all customers in the queue at time t. Show that

$$S_Q(t) = \sum_{k=1}^{\infty} \sigma_k \mathbb{1}_{T_k \le t < T_k + W_{Q,k}},$$

and deduce

$$\sum_{k=1}^{N_d(T)} \sigma_k W_{Q,k} \le \int_0^T S_Q(t)\, dt \le \sum_{k=1}^{N(T)} \sigma_k W_{Q,k}.$$

(ii) Combining (i) with Exercise 4, show that almost surely,

$$\lim_{T\uparrow\infty} \frac{1}{T}\int_0^T S_Q(t)\, dt = \lambda W_Q \mathbb{E}\left[\sigma\right].$$

(iii) Letting $R(t)$ denote the remaining service time of the current customer in the server, use the PASTA property to infer

$$W_Q = \lim_{T\uparrow\infty} \frac{1}{T}\int_0^T \big(S_Q(t) + R(t)\big)\, dt.$$

Deduce the conclusion (0.3) from (ii) and Exercise 5.