Midterm — May 3rd

Please provide complete and well-written solutions to the following exercises. All answers must be properly explained and justified.

Exercise 1. (/3) Given a parameter $0 \le p \le 1$, consider the Markov chain with state space $\Omega = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & p & 1-p & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Depending on the parameter p, identify the transient states and the classes of recurrent states. Also determine the period of each state.

(ii) Depending on p, for each recurrent class, find the stationary distribution for the chain restricted to that class. Determine all stationary distributions for the full chain.

(iii) Depending on p, compute all entries of the limit $\lim_{n\uparrow\infty} P^n$ if it exists.

Exercise 2. (/2) At the beginning of each day, a machine is inspected to determine its working condition, which is classified as state 1 = new, 2, 3, or 4 = broken. We assume that a machine in state 1 remains in state 1 with probability 0.95 and otherwise is degraded to state 2, that a machine in state 2 remains in state 2 with probability 0.9 and otherwise is degraded to state 3, and that a machine in state 3 remains in state 3 with probability 0.875 and otherwise is degraded to state 4.

(i) Suppose that a broken machine requires three days to fix it. To incorporate that into the Markov chain, we add states 5 and 6 and say that $P_{45} = P_{56} = P_{61} = 1$. What is the long-run time fraction that the machine is working?

(ii) Suppose now that we have the option of performing preventative maintenance when the machine is in state 3 and that this maintenance takes one day and returns to state 1. Find the time fraction that the machine is working under this new policy.

Exercise 3. (/2) A company gives each of its employees the title of either 'programmer' or 'project manager'. In any given year, 70% of programmers remain in that position, 20% are promoted to project manager, and 10% are fired. On the other hand, 95% project managers remain in that position and only 5% are fired. How long on average does a programmer work before they are fired?

Exercise 4. (/1) Folk wisdom holds that in the summer it rains 1/3 of the time, and that a rainy day is followed by a second one with probability 1/2. Supposing that the weather can be modeled by a two-state Markov chain, what is its transition matrix?

Exercise 5. (/2) Let $(S_n)_n$ be the simple random walk, that is, $S_n = S_0 + X_1 + \ldots + X_n$ where $X_1, X_2, \ldots$ are iid random variables with $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = -1] = \frac{1}{2}$.

(i) Show that $(S_n)_n$, $(S_n^2 - n)_n$, $(S_n^3 - 3nS_n)_n$ are martingales with respect to $(X_n)_n$.

(ii) Construct a martingale involving S_n^4.