M285K - course #11

Proof of dual CZ lemma

Lemma (Caffarelli-Peral '98, Shen '07).

Let $1 \leq p_0 < p_1 < \infty$, let $C_0 > 0$, let $F, G \overset{\geq 0}{\in} L^{p_0} \cap L^{p_1}(\mathbb{R}^d)$.

Assume $\forall$ ball $D \subseteq \mathbb{R}^d$ $\exists$ meas F_D^0, F_D^1 with $\begin{cases} F \leq F_D^0 + F_D^1 \\ F_D^1 \leq F + F_D^0 \text{ in } D \end{cases}$

such that: $$\begin{cases} \left(\fint_D |F_D^0|^{p_0} \right)^{1/p_0} \leq C_0 \left(\fint_{C_0 D} |G|^{p_0} \right)^{1/p_0} & \text{(local dependence)} \\ \left(\fint_{\frac{1}{C_0} D} |F_D^1|^{p_1} \right)^{1/p_1} \leq C_0 \left(\fint_D |F_D^1|^{p_0} \right)^{1/p_0} & \text{(reverse Jensen).} \end{cases}$$

Then $\forall p_0 < p < p_1$: $\displaystyle\int_{\mathbb{R}^d} |F|^p \lesssim_{C_0, p_0, p_1, p} \int |G|^p$.

$\mathbb{R}$ $\mathbb{R}^d$

Proof.

Recall maximal fct $M(h)(x) = \sup\left\{ ⨍_D |h| \ : \ D \text{ ball} \ni x \right\}$.

& properties:

* Lebesgue diff thm: $M(h) \geq |h|$ a.e.
* weak L^1 estimate: $h \in L^1 \Rightarrow \forall \alpha > 0: \ \left|\{ |Mh| > \alpha \}\right| \lesssim \frac{1}{\alpha} \int |h|$.
* L^p estimate: $\forall 1 < p \leq \infty$: $\|Mh\|_{L^p} \lesssim_p \|h\|_{L^p}$.

Step 1: Enough to prove for some $T \gg 1 \geq \theta > 0$:

$$\left|\{ |M(|F|^{p_1})| > T \}\right| \lesssim \left(\left(\frac{1}{T}\right)^{p_1/p_0} + \frac{\theta}{T} \right) \left|\{ M(|F|^{p_1}) > 1 \}\right| + \left|\{ M(|G|^{p_0}) > \theta \}\right|$$

By homogeneity: $\forall t \geq 0$,

$$\left|\left\{ |M|F|^{p_0}| > T t \right\}\right| \lesssim \left(\left(\frac{1}{T}\right)^{p_1/p_0} + \frac{\theta}{T} \right) \left|\left\{ M|F|^{p_0} > t \right\}\right|$$
$$+ \left|\left\{ M|G|^{p_0} > \theta t \right\}\right|.$$

Apply: $\int_0^\infty (-)\, t^{\frac{p}{p_0}-1}\, dt$

$$\Rightarrow \left(\frac{1}{T}\right)^{p/p_0} \int |M|F|^{p_0}|^{p/p_0} \lesssim \left(\left(\frac{1}{T}\right)^{p_1/p_0} + \frac{\theta}{T} \right) \int |M|F|^{p_0}|^{p/p_0}$$
$$+ \left(\frac{1}{\theta}\right)^{p/p_0} \int |M|G|^{p_0}|^{p/p_0}.$$

Choose $\theta = T \left(\frac{1}{T}\right)^{p_1/p_0}$, & $T \gg 1$: can absorb the first rhs term

$$\Rightarrow \int |M|F|^{p_0}|^{p/p_0} \lesssim T^{\frac{p p_1}{p_0^2}} \int |M|G|^{p_0}|^{p/p_0}$$

$$\int \underbrace{M|F|^{p}}_{\substack{\geq \\ |F|^{p}\ \text{a.e.} \\ \text{by Lebesgue.}}} \lesssim \int \underbrace{M|G|^{p}}_{\substack{(p>p_0)\ \lesssim_{p,p_0} \int |G|^{p} \\ L^{p/p_0}\ \text{estimate}}} .$$

<u>Step 2</u>:

$D = B_r(x)$
$RD = B_{Rr}(x)$

[Enough to prove for some $T \gg 1 \geq \theta$, $\forall$ ball $D \subseteq \mathbb{R}^d$:

$$\left.\begin{array}{l} |D \cap \{M|G|^{p_0} \leq \theta\}| > 0 \\ \displaystyle\fint_{RD} |F|^{p_0} \leq \left(\tfrac{6}{5}\right)^d \ \forall R \geq 5 \end{array}\right\} \overset{(*)}{\Longrightarrow} \begin{array}{l} |5D \cap \{M|F|^{p_0} > T\}| \\ \lesssim \left(\left(\frac{1}{T}\right)^{\frac{p_1}{p_0}} + \frac{\theta}{T} \right) |D| \end{array}$$
]

Consider the set $E = \{ M|F|^{p_0} \geq \frac{6}{5} \}$

As $\displaystyle\int_{\mathbb{R}^d} |F|^{p_0} < \infty \Rightarrow \lim_{R \to \infty} \fint_{B_R} |F|^{p_0} \to 0$

$$\forall x \in E \quad \exists\, r(x) \in (0,\infty)$$

$$\text{s.t.:}\quad \begin{cases} \displaystyle\fint_{B_R(x)} |F|^{p_0} = \left(\frac{\delta}{5}\right)^d & \text{for } R = 5r(x) \\[2ex] \displaystyle\fint_{B_R(x)} |F|^{p_0} \le \left(\frac{\delta}{5}\right)^d & \text{for } R \geq 5r(x). \end{cases}$$

$\{B_{r(x)}(x)\}_{x\in E}$ covers the set E.

Vitali's covering thm: $\exists$ countable subset $\mathcal{F} \subseteq \{B_{r(x)}(x)\}_{x\in E}$

$$\text{s.t.}\quad \begin{cases} D \cap D' = \emptyset \quad \forall\, D \neq D' \in \mathcal{F}. \\[1ex] E \subset \displaystyle\bigcup_{D\in\mathcal{F}} 5D. \end{cases}$$

We get: $|\{M|F|^{p_0} > T\}| = |E \cap \{M|F|^{p_0} > T\}| \quad (\forall T \geq \frac{\delta}{5}$

We get: [illegible]

$$\le \sum_{D\in\mathcal{F}} \underbrace{|5D \cap \{M|F|^{p_0} > T\}|}$$ by covering property.

2 cases:

① Case 1: $|D \cap \{M|G|^{p_0} \le \theta\}| = 0$

Then: $|5D \cap \{M|F|^{p_0} > T\}| \le |5D| \simeq |D| = |D \cap \{M|G|^{p_0} > \theta\}|$.

ok.

② Case 2: $|D \cap \{M|G|^{p_0} > \theta\}| > 0$, $D = B_{r(x)}(x)$, $x \in E$.

By definition of $r(x)$: $\displaystyle\fint_{\substack{RD \\ = B_{Rr(x)}(x)}} |F|^{p_0} \le \left(\frac{\delta}{5}\right)^{d}\theta \quad \forall R \ge 5$.

Can apply (*) & deduce: $|5D \cap \{M|F|^{p_0} > T\}| \lesssim \left(\left(\frac{1}{T}\right)^{\frac{p_1}{p_0}} + \frac{\theta}{T} \right) |D|$.

Note that $M|F|^{p_0} \geq 1$ on D.

Indeed: $\forall y \in D = B_{r(x)}(x)$,

$$M|F|^{p_0}(y) \geq \fint_{\substack{B_{6r(x)}(y) \\ \cup \\ B_{5r(x)}(x)}} |F|^{p_0} \geq \left(\frac{5}{6}\right)^d \underbrace{\fint_{B_{5r(x)}(x)} |F|^{p_0}}_{= \left(\frac{6}{5}\right)^d \text{ by choice of } r(x).} = 1.$$

$$\Rightarrow |5D \cap \{M|F|^{p_0} > T\}| \lesssim \left(\left(\frac{1}{T}\right)^{\frac{p_1}{p_0}} + \frac{\theta}{T} \right) |D \cap \{M|F|^{p_0} \geq 1\}|.$$

<u>Case 1+2</u>: in all cases:

$$|5D \cap \{M|F|^{p_0} > T\}| \lesssim \left(\left(\frac{1}{T}\right)^{\frac{p_1}{p_0}} + \frac{\theta}{T} \right) |D \cap \{M|F|^{p_0} > 1\}|$$

$$+ |D \cap \{M|G|^{p_0} > \theta\}|.$$

$\Rightarrow$ by disjointness of $\mathcal{F}$:

$$|\{M|F|^{p_0} > T\}| \lesssim \left(\left(\frac{1}{T}\right)^{\frac{p_1}{p_0}} + \frac{\theta}{T} \right) |\{M|F|^{p_0} > 1\}|$$

$$+ |\{M|G|^{p_0} > \theta\}|.$$

<u>Step 3</u>: Conclusion = proof of $(*)$

$\exists T \gg 1 \geq \theta$ s.t. $\forall$ ball $D \subseteq \mathbb{R}^d$

$|D \cap \{M|G|^{p_0} \leq \theta\}| > 0)$ $\Rightarrow$ $(|5D \cap \{M|F|^{p_0} > T\}|)$

$$\left[\ \fint_{RD} |F|^{p_0} \le \left(\tfrac{6}{5}\right)^d \quad \forall R \geq 5 \right]$$

$$\lesssim \left(\left(\tfrac{1}{T}\right)^{\frac{p_1}{p_0}} + \frac{\theta}{T} \right) |D|.$$

* First note that $\fint_{LD} |G|^{p_0} \lesssim \theta \quad \forall L \geq 1.$)

Indeed, by assumption $\exists y \in D: \ M|G|^{p_0}(y) \le \theta$

$$\forall R>0: \fint_{B_R(y)} |G|^{p_0} \le \theta.$$)

For $D = B_r(x)$, get $LD = B_{Lr}(x) \subset B_{(L+1)r}(y)$

$$\Rightarrow \fint_{LD} |G|^{p_0} \le \left(\frac{L+1}{L}\right)^d \fint_{B_{(L+1)r}(y)} |G|^{p_0} \le \left(\frac{L+1}{L}\right)^d \theta.$$

* By assumption of the lemma, we decompose

$$\begin{cases} F \leq F_D^0 + F_D^1 & \text{in } 10C_0 D \qquad (C_0 \geq 1) \\ F_D^1 \leq F + F_D^0 \end{cases}$$

&

$$\begin{cases} \left(\fint_{10C_0 D} |F_D^0|^{p_0} \right)^{\frac{1}{p_0}} \leq C_0 \left(\fint_{10C_0^2 D} |F|^{p_0} \right)^{\frac{1}{p_0}} \\ \left(\fint_{10D} |F_D^1|^{p_1} \right)^{\frac{1}{p_1}} \leq C_0 \left(\fint_{10C_0 D} |F_D^1|^{p_0} \right)^{\frac{1}{p_0}} \end{cases}$$

Hence $|F|^{p_0} \leq 2^{p_0-1} |F_D^0|^{p_0} + 2^{p_0-1} |F_D^1|^{p_0}$

$$M(\mathbb{1}_{10D} |F|^{p_0}) \leq 2^{p_0-1} M(\mathbb{1}_{10D} |F_D^0|^{p_0}) + 2^{p_0-1} M(\mathbb{1}_{10D} |F_D^1|^{p_0}).$$

Therefore:

$$|5D \cap \{ |M(\mathbb{1}_{10D} |F|^{p_0})| > T \}$$

$$\leq \underbrace{\left|\cancel{5D} \cap \left\{ M\left(\mathbb{1}_{10D}|F_D^0|^{p_0}\right) > \frac{T}{2} \right\}\right|}_{\lesssim \frac{1}{T}\int_{10D}|F_D^0|^{p_0} \text{ weak } L^1 \text{ estimate}} + \underbrace{\left|\cancel{5D} \cap \left\{ M\left(\mathbb{1}_{10D}|F_D^1|^{p_0}\right)^{\frac{p_1}{p_0}} > \left(\frac{T}{2}\right)^{p_1/p_0} \right\}\right|}_{\lesssim \left(\frac{1}{T}\right)^{\frac{p_1}{p_0}} \int \left|M\left(\mathbb{1}_{10D}|F_D^1|^{p_0}\right)\right|^{\frac{p_1}{p_0}}} .$$

(Markov inequ.)

$$\lesssim \left(\frac{1}{T}\right)^{\frac{p_1}{p_0}} \int_{10D} |F_D^1|^{p_1}$$ (by $L^{\frac{p_1}{p_0}}$ estimate).

$$\Rightarrow \frac{1}{|D|}\left|5D \cap \left\{ M\left(\mathbb{1}_{10D}|F|^{p_0}\right) > T \right\}\right|$$

$$\lesssim \frac{1}{T} \underbrace{\fint_{10D} |F_D^0|^{p_0}}_{\lesssim \fint_{10C_0D}|F_D^0|^{p_0}} + \left(\frac{1}{T}\right)^{\frac{p_1}{p_0}} \underbrace{\fint_{10D} |F_D^1|^{p_1}}_{\lesssim \left(\fint_{10C_0D}|F_D^1|^{p_0}\right)^{\frac{p_1}{p_0}}}$$

by assumption.

$$\lesssim \fint_{10C_0^2 D} |G|^{p_0} \quad \text{by assumptio}$$

$$\lesssim \theta$$

$$\lesssim \left(\fint_{10C_0 D} |F_D^0|^{p_0}\right)^{\frac{p_1}{p_0}} + \left(\fint_{10C_0 D} |F|^{p_0}\right)^{\frac{p_1}{p_0}}$$

$$\lesssim \theta^{\frac{p_1}{p_0}} + 1.$$

$$\lesssim 1.$$

$$\lesssim \frac{\theta}{T} + \left(\frac{1}{T}\right)^{p_1/p_0}.$$

- Remains to show $|5D \cap \{M|F|^{p_0} > T\}|$

$$\leq |5D \cap \{M(\mathbb{1}_{10D}|F|^{p_0}) > T\}|, \quad \forall T \geq \left(\frac{12}{5}\right)^d.$$

In fact, we show $M|F|^{p_0} \leq \max\left\{\left(\frac{12}{5}\right)^d, M(\mathbb{1}_{10D}|F|^{p_0})\right\}$ on $5D$

Let $y \in 5D$, say $D = B_r(x)$.

- for $R \leq 5r$: $B_R(y) \subset B_{R+5r}(x) \subset B_{10r}(x) = 10D$.

$$\hookrightarrow \fint_{B_R(y)} |F|^{p_0} \leq \fint_{B_R(y)} \mathbb{1}_{10D} |F|^{p_0} \leq \underline{M(\mathbb{1}_{10D} |F|^{p_0})(y)}.$$

- for $R > 5r$: $B_R(y) \subseteq B_{R+5r}(x) \subset B_{2R}(x) = \frac{2R}{r} D$.

$$\hookrightarrow \fint_{B_R(y)} |F|^{p_0} \leq 2^d \fint_{\underbrace{\frac{2R}{r}}_{>10} D} |F|^{p_0} \leq 2^d \left(\frac{6}{5}\right)^d.$$

$\square$