M285K - course #12

III.2 Large-scale regularity.

d) Perturbative version.

Theorem. (perturbative annealed L^p estimate)

$\exists\, C = C(d,\alpha,\beta) > 0$

$\forall\, |p-2|, |q-2| \leq \frac{1}{C}$; $\forall\, h \in C_c^\infty(\mathbb{R}^d, L^q(\Omega)^d)$

consider unique sol. $\nabla w \in L^2(\mathbb{R}^d, L^2(\Omega))$

of $-\nabla \cdot a \nabla w = \nabla \cdot h$ in $\mathbb{R}^d$.

We have: $\| [\nabla w] \|_{L^p(\mathbb{R}^d; L^q(\Omega))} \lesssim \| [h] \|_{L^p(\mathbb{R}^d; L^q(\Omega))}$.

(recall $[h](x) = \left(\fint_{B(x)} |h|^2 \right)^{\frac{1}{2}}$.)

Lemma (dual CZ - Shen '07).

Given $1 \leq p_0 < p_1 \leq \infty$, $F, G \in L^{p_0} \cap L^{p_1}(\mathbb{R}^d)$, $F, G \geq 0$.

Assume $\forall D$ ball $\exists F_D^0, F_D^1$ meas with $\begin{cases} F \leq F_D^0 + F_D^1 \\ F_D^1 \leq F + F_D^0 \end{cases}$

such that
$$\begin{cases} \left(\fint_D |F_D^0|^{p_0} \right)^{\frac{1}{p_0}} \leq C_0 \left(\fint_{C_0 D} |G|^{p_0} \right)^{\frac{1}{p_0}} & \text{(local dependence)} \\ \left(\fint_{\frac{1}{C_0} D} |F_D^1|^{p_1} \right)^{\frac{1}{p_1}} \leq C_0 \left(\fint_D |F_D^1|^{p_0} \right)^{\frac{1}{p_0}} & \text{(reverse Jensen)} \end{cases}$$

Then $\forall\, p_0 < p < p_1$:
$$\int_{\mathbb{R}^d} |F|^p \lesssim_{C_0, p, p_0, p_1} \int_{\mathbb{R}^d} |G|^p.$$

Proof of theorem.

$$-\nabla\cdot a\nabla w = \nabla\cdot h \quad \text{in } \mathbb{R}^d.$$

$\forall$ ball D : decompose $\nabla w = \nabla w_D^0 + \nabla w_D^1$

with
$$\begin{cases} -\nabla\cdot a\nabla w_D^0 = \nabla\cdot(h \mathbb{1}_D) & \text{in } \mathbb{R}^d \\ -\nabla\cdot a\nabla w_D^1 = \underbrace{\nabla\cdot(h\mathbb{1}_{\mathbb{R}^d\setminus D})}_{=0 \text{ in } D} & \text{in } \mathbb{R}^d \end{cases}$$

Step 1: Meyers' estimate: $-\nabla\cdot a\nabla v = \nabla\cdot g$, $g\in C_c^\infty(\mathbb{R}^d, L^\infty(\Omega))$

$$\exists\, C_0 = C_0(d,\alpha,\beta) > 0 : \forall\, |p-2| \le \frac{1}{C_0}, \quad \int_{\mathbb{R}^d} [\nabla v]^p \lesssim \int_{\mathbb{R}^d} [g]^p. \tag{2.5}$$

already proven!

Step 2: Reverse Jensen: $-\nabla\cdot a\nabla v = 0$ in B_r

$\exists\, C_0 = C_0(d,\alpha,\beta) > 0 : \forall\, p_0 < 2 \underbrace{\leq p_1}$, $|p_0 - 2|, |p_1 - 2| \leq \frac{1}{C_0}$

$$\left(\fint_{B_{r/2}} [\nabla v]^{p_1}\right)^{1/p_1} \lesssim \left(\fint_{B_r} [\nabla v]^{p_0}\right)^{1/p_0} \quad \underline{a.s}$$

* Caccioppoli inequality: $\displaystyle\fint_{B_{r/2}} |\nabla v|^2 \lesssim \frac{1}{r^2}\fint_{B_r} |v - k|^2, \quad \forall k \in \mathbb{R}.$

($\sim$ reverse Poincaré)

$-\nabla\cdot a\nabla v = 0$ in B_r

Choose cut-off $\chi_r = \begin{cases} 1 & \text{in } B_{r/2} \\ 0 & \text{outside } B_r \end{cases}$ & $|\nabla \chi_r| \lesssim \frac{1}{r}$.

Test equation with $\chi_r^2 (v - k)$

$$\Rightarrow \int \chi_r^2\, \nabla v \cdot a\nabla v + \int 2\chi_r (v-k)\, \nabla\chi_r \cdot a\nabla v = 0$$

$$\int \chi_r \underbrace{\nabla v \cdot a \nabla v}_{\substack{\vee\!\!/ \\ \alpha |\nabla v|^2}} \quad \int 2 \chi_r \quad \chi_r$$

$$\to \int \chi_r^2 |\nabla v|^2 \lesssim \int |\nabla \chi_r|^2 |v-k|^2$$

$$\int_{B_{r/2}} |\nabla v|^2 \leq \int \chi_r^2 |\nabla v|^2, \qquad \int |\nabla \chi_r|^2 |v-k|^2 \leq \frac{1}{r^2} \int_{B_r} |v-k|^2 \quad !$$

* Apply Poincaré-Sobolev, $k = \fint_{B_r} v$.

$$\to \left(\fint_{B_{r/2}} |\nabla v|^2 \right)^{\frac{1}{2}} \lesssim \left(\fint_{B_r} |\nabla v|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{2d}}.$$

Or if we like:

$$\left(\fint [\nabla v]^{2} \right)^{\frac{1}{2}} \lesssim \left(\fint [\nabla v]^{\frac{4}{d+2}} \right)^{\frac{d+2}{2d}}$$

$$\left(\fint_{B_{r/4}} [\nabla v]^{2}\right)^{\frac{1}{2}} \lesssim \left(\fint_{B_r} [\nabla v] \right).$$

* Use Gehring's lemma: If D_0 ball $\subseteq \mathbb{R}^d$, if $G \geq 0$ is measurable,
& if $\exists p_0 < 2$: $\forall D \subset D_0$ ball
$$\left(\fint_{\frac{1}{2}D} G^2\right)^{\frac{1}{2}} \lesssim \left(\fint_D G^{p_0}\right)^{\frac{1}{p_0}}.$$
Then $\exists p_1 > 2$: $\left(\fint_{\frac{1}{2}D_0} G^{p_1}\right)^{\frac{1}{p_1}} \lesssim \left(\fint_{D_0} G^{p_0}\right)^{\frac{1}{p_0}}.$

$\to$ conclude!

<u>Step 3</u>: $\forall\, 2 \leq q < p < 2 + \frac{1}{C_0}$, $\qquad -\nabla\cdot a\nabla w = \nabla\cdot h$

$$\| [\nabla v] \|_{L^p(\mathbb{R}^d, L^q(\Omega))} \lesssim \| [h] \|_{L^p(\mathbb{R}^d; L^q(\Omega))}.$$

Let $2 \leq p_0 < p_1 \leq 2 + \frac{1}{C_0}$ fixed.

$\forall D$ ball: $\nabla w = \nabla w_D^0 + \nabla w_D^1$ as above.

① $-\nabla \cdot a \nabla w_D^0 = \nabla \cdot (h \mathbb{1}_D)$ $\quad D = B_r(x)$, $r \geq 1$

Step 1 $\Rightarrow \int_{\mathbb{R}^d} [\nabla w_D^0]^{p_0} \lesssim \int_{\mathbb{R}^d} \underbrace{[h \mathbb{1}_D]^{p_0}} \lesssim \int_{2D} [h]^{p_0}$

$\overset{\mathbb{E}[\cdot]}{\Rightarrow} \int_{\mathbb{R}^d} \| [\nabla w_D^0] \|_{L^{p_0}(\Omega)}^{p_0} \lesssim \int_{2D} \| [h] \|_{L^{p_0}(\Omega)}^{p_0}$ $\quad \sigma^{p_0}$

$\quad \geq$

$\int_D \| [\nabla w_D^0] \|_{L^{p_0}(\Omega)}^{p_0}$ $\quad$ by energy (L^2) estimate; also true for $r < 1$.

$|F_D^0|^{p_0}$

② $-\nabla \cdot a \nabla w_D^1 = \nabla \cdot (h \mathbb{1}_{\mathbb{R}^d \setminus D})$ $\quad (= 0 \text{ in } D)$.

$$\left(\fint_{\frac{1}{2}D} \underbrace{\| [\nabla w_D^1] \|_{L^{p_0}(\Omega)}^{p_1}}_{|F_D^1|^{p_1}} \right)^{\frac{1}{p_1}} \underset{(p_1 \geq p_0)}{\leq} \left\| \left(\fint_{\frac{1}{2}D} [\nabla w_D^1]^{p_1} \right)^{\frac{1}{p_1}} \right\|_{L^{p_0}(\Omega)}$$

$$\underset{\text{Step 2}}{\lesssim} \left\| \left(\fint_D [\nabla w_D^1]^{p_0} \right)^{\frac{1}{p_0}} \right\|_{L^{p_0}(\Omega)}$$

$$= \left(\fint_D \underbrace{\| [\nabla w_D^1] \|_{L^{p_0}(\Omega)}^{p_0}}_{|F_D^1|^{p_0}} \right)^{1/p_0}.$$

Use dual CZ lemma: $\left\{ \begin{array}{l} F = \| [\nabla w] \|_{L^{p_0}(\Omega)} \\ F_D^0 = \| [\nabla w_D^0] \|_{L^{p_0}(\Omega)} \\ F_D^1 = \quad w_D^1 \\ G = \| [h] \|_{L^{p_0}(\Omega)}. \end{array} \right.$ $\to$ conclude!

Step 4: dualization + interpolation.

* $\underbrace{2 \le q < p < 2 + \frac{1}{C_0}}$:

$$\|[\nabla w]\|_{L^{p'}(\mathbb{R}^d;\, L^{q'}(\Omega))} = \sup\left\{ \int_{\mathbb{R}^d} h' \cdot \nabla w \;:\; \|[h']\|_{L^{p}(\mathbb{R}^d;\, L^{q}(\Omega))} = 1 \right\}$$

Consider the dual equation: $-\nabla \cdot a^* \nabla w' = \nabla \cdot h'$.

$$\begin{aligned} \to \int h' \cdot \nabla w &= -\int \nabla w' \cdot a \nabla w \\ &= \int \nabla w' \cdot h \\ &\le \underbrace{\|[\nabla w']\|_{L^p(\mathbb{R}^d;\, L^q(\Omega))}}_{\text{by Step 3:}} \|[h]\|_{L^{p'}(\mathbb{R}^d;\, L^{q'}(\Omega))}. \end{aligned}$$

$$\lesssim \| [h] \|_{L^p(\mathbb{R}^d; L^q(\Omega))}$$

$$\to \| [\nabla w] \|_{L^{p'}(\mathbb{R}^d; L^{q'}(\Omega))} \lesssim \| [h] \|_{L^{p'}(\mathbb{R}^d; L^{q'}(\Omega))} \, .$$

Conclusion: the result is proven for $2 \leq q < p \leq 2 + \frac{1}{C_0}$

& $\left(2 + \frac{1}{C_0}\right)' \leq p < q \leq 2$

$\to$ interpolate and get all exponents. $\square$

III.3 Stochastic corrector estimates.

Theorem.

(i) $\mathbb{E}\left[[\![\nabla \varphi, \nabla \sigma]\!]^{2p} \right]^{\frac{1}{2p}} \lesssim (Cp)^C$

(stretched exponential moments: $\mathbb{E} \exp[|\nabla \varphi|^{\frac{1}{4C}}] < \infty$.)

(ii) $\forall g \in C_c^\infty(\mathbb{R}^d)$

(ii) $\forall g \in C_c^\infty \ldots$

$$\mathbb{E}\left[\left|\int g(\nabla\varphi, \nabla\sigma)\right|^{2p}\right]^{\frac{1}{p}} \lesssim (Cp)^c \int |g|^2.$$

(iii) $$\mathbb{E}\left[\left[(\varphi,\sigma) - \fint_B (\varphi,\sigma)\right]^{2p}(x)\right]^{\frac{1}{2p}} \lesssim (Cp)^C \times \underbrace{\begin{cases} 1 & d>2 \\ \log(2+|x|)^{1/2} & d=2 \\ \langle x\rangle^{1/2} & d=1 \end{cases}}_{\text{same growth as for GFF}}$$

same growth as for GFF. $= \Delta^{-1}\nabla\cdot\xi$

$\varphi = (\nabla\cdot a\nabla)^{-1}\nabla\cdot(ae)$

Proof:
- luckily (i) & (ii)
- (ii) $\Rightarrow$ (iii) by integration.