M285K - course #14

Theorem.

(i) $\mathbb{E}\left[\left[\nabla(\varphi,\sigma)\right]^{2p}\right]^{\frac{1}{2p}} \leq (Cp)^C$

(ii) $\forall g \in C_c^\infty(\mathbb{R}^d)$,

$$\mathbb{E}\left[\left|\int g\cdot\nabla(\varphi,\sigma)\right|^{2p}\right]^{\frac{1}{2p}} \leq (Cp)^C \, \|g\|_{L^2(\mathbb{R}^d)}$$

(iii) $$\mathbb{E}\left[\left[(\varphi,\sigma)-\fint_B(\varphi,\sigma)\right]^{2p}(x)\right]^{\frac{1}{2p}} \leq (Cp)^C \times \begin{cases} 1, & d>2 \\ \log(2+|x|)^{\frac{1}{2}}, & d=2 \\ \langle x\rangle^{\frac{1}{2}}, & d=1. \end{cases}$$

Proof.

Step 1. $\forall g \in C_c^\infty(\mathbb{R}^d)^d$, $\forall p \gg 1$, $R \geq 1$, $q \geq 1$:

$$\left\|\int g\cdot\nabla\varphi\right\|^2_{L^{2p}(\Omega)} \leq p\,\|g\|^2_{L^2(\mathbb{R}^d)}\,\left\|\left(\int_{B_R}\left[\nabla\varphi+e\right]^{2q}\right)^{\frac{1}{q}}\right\|_{L^p(\Omega)}.$$

Step 2.
$$\begin{cases} \forall\, 1<p<\infty,\ |q-1|\ll 1,\ 1\le r\ll R, \\ \left\| \left(\fint_{B_R} [\nabla\varphi+e]^{2q} \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)} \lesssim \left\| \fint_{B_r} \nabla\varphi \right\|^2_{L^{2p}(\Omega)} + 1 \end{cases}$$

Step 3: Proof (i) & (ii) for $\nabla\varphi$ by buckling.

* Get: $$\left\| \left(\fint_{B_R} [\nabla\varphi+e]^{2q} \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)} \lesssim 1 + \Bigg\| \underbrace{\fint_{B_r} \nabla\varphi}_{\int_{\mathbb{R}^d} r^{-d} 1_{B_r} \nabla\varphi} \Bigg\|^2_{L^{2p}(\Omega)} \qquad \left(\begin{array}{l} \forall\, 1<p<\infty \\ |q-1|\ll 1 \\ 1\le r\ll R \end{array} \right)$$

$$\lesssim 1 + \dots \| r^{-d} 1_{B_r} \|^2 \dots \left\| \left(\int [\nabla\varphi+e]^{q} \right)^{\frac{1}{q}} \right\|$$

$$\sim 1 + p \underbrace{\| \chi \, d B_r \|_{L^2(\mathbb{R}^d)}}_{\lesssim_r r^{-d}} \left\| \left(\int_{B_R} [\nabla \varphi + e] \right) \right\|_{L^p(\Omega)}$$

(CLT decay!) $\forall p >> 1$.

$$\lesssim_r 1 + p \underbrace{r^{-d} R^{d/q}}_{\left(\frac{R}{r}\right)^d R^{-d/q'} \,=\, pC^d R^{-d/q'}} \left\| \left(\fint_{B_R} [\nabla \varphi + e]^{2q} \right)^{1/q} \right\|_{L^p(\Omega)}$$

choose $q > 1$, $|q-1| << 1$.

& $r = \frac{1}{C} R$, for some large enough C.

& R s.t. $pC^d R^{-d/q'} << 1$

$$R >> (pC^d)^{q'/d}.$$

$\Rightarrow$ can absorb.

$$\to \left\| \left(\fint_{B_R} [\nabla\varphi + e]^{2q} \right)^{\frac{1}{q}} \right\|_{L^{\rho}(\Omega)} \lesssim 1 \quad \forall R \gg (\rho C^{d})^{q'/d}$$

$$\begin{cases} p \gg 1 \\ |q-1| \ll 1, \ \rho > 1 . \end{cases}$$

* Note that $\| [\nabla\varphi + e] \|_{L^{2p}(\Omega)} \lesssim R^{d/p'} \Big\| \underbrace{\fint_{B_R} [\nabla\varphi + e]^2}_{} \Big\|_{L^{p}(\Omega)}$

$$\le \left(\fint_{B_R} [\nabla\varphi + e]^{2q} \right)^{\frac{1}{q}} \text{ by Jensen.}$$

Indeed $\left(\fint_{B_R} [\nabla\varphi + e]^{2p} \right)^{\frac{1}{p}} \lesssim R^{-\frac{d}{p}} \left(\sum_{z \in \frac{1}{C}\mathbb{Z}^d \cap B_R} [\nabla\varphi + e]^{2p} \right)^{\frac{1}{p}}$

$(R \geq 1)$

$$\lesssim R^{-d/p} \sum_{z \in \frac{1}{C}\mathbb{Z}^d \cap B_R} [\nabla\varphi + e] \quad \text{by } \ell^p\text{-}\ell^1 \text{ estimate.}$$

$$\lesssim R^{d/p'} \fint_{B_{2R}} [\nabla\varphi + e]^2$$

& take $\|\cdot\|_{L^p(\Omega)}$

- Conclude $\|[\nabla\varphi + e]\|_{L^{2p}(\Omega)} \lesssim \underbrace{R^{d/p'} \quad \forall p \gg 1 \quad [R \gg (C^d p)^{q'/d}]}_{(Cp)^C \ \forall p \gg 1.}$

$\Rightarrow$ proof (i) $\underset{\text{Step 1}}{\Rightarrow}$ (ii)

Step 4. Proof of (iii) by integration

Decompose

$$\left\| \left[\varphi - \fint_B \varphi\right](x) \right\|_{L^{2p}(\Omega)} = \left\| \left(\fint_{B(x)} \left|\varphi - \fint_B \varphi\right|^2 \right)^{\frac{1}{2}} \right\|_{L^{2p}(\Omega)}$$

$$\leq \underbrace{\left\| \left(\fint_{B(x)} \underbrace{\left|\varphi - \fint_{B(x)} \varphi\right|^2}_{\text{Poincaré} \quad \lesssim \fint_{B(x)} |D\varphi|^2 = [D\varphi]^2} \right)^{\frac{1}{2}} \right\|_{L^{2p}(\Omega)}}_{\text{by (i)}: \ \lesssim (Cp)^C} + \left\| \fint_B \varphi - \fint_{B(x)} \varphi \right\|_{L^{2p}(\Omega)}$$

For the last term: consider $-\Delta h_x = \dfrac{\mathbb{1}_B}{|B|} - \dfrac{\mathbb{1}_{B(x)}}{|B|}$

$$C_0 \left\| \fint_B \varphi - \fint_{B(x)} \varphi \right\|_{L^{2p}(\Omega)} = \left\| \int_{\mathbb{R}^d} \nabla h_x \cdot \nabla \varphi \right\|_{L^{2p}(\Omega)}$$

$$\overset{\text{by item (ii)}}{\lesssim} (Cp)^C \, \| \nabla h_x \|_{L^2(\mathbb{R}^d)} .$$

& exercise: $\| \nabla h_x \|_{L^2(\mathbb{R}^d)} \lesssim \begin{cases} 1 & d > 2 \\ \log(2+|x|)^{\frac{1}{2}}, & d = 2 \\ \langle x \rangle^{1/2} & d = 1 \end{cases} .$

$$d > 2: \ \| \nabla h_x \|_{L^2} \leq 2 \left\| \nabla \Delta^{-1} \frac{\mathbb{1}_B}{|B|} \right\|_{L^2}$$

$$\lesssim \left\| \frac{\mathbb{1}_B}{|B|} \right\|_{L^{\frac{2d}{d+2}>1}} \text{ by HLS}$$

$$\lesssim 1 .$$

$\Rightarrow$ proof of (iii) for φ.

Step 5 : some results for σ ,
$$\begin{cases} \nabla\cdot\sigma = q = a(\nabla\varphi+e) - \bar{a}e \\ -\Delta\sigma = \nabla\times q . \end{cases}$$

• [Proof that $\| \int_{\mathbb{R}^d} g \cdot q \|_{L^{2p}(\Omega)} \lesssim \|g\|_{L^2(\mathbb{R}^d)} (Cp)^C$.

Use Malliavin calculus :
$$\| \int g \cdot q \|_{L^{2p}(\Omega)} \lesssim \| D \int g \cdot q \|_{L^{2p}(\Omega, h)} .$$

& compute $D \int g \cdot q = \int g \cdot Da(\nabla\varphi + e) + \int g \cdot a \nabla D\varphi$

Consider auxiliary problem $-\nabla\cdot a^* \nabla v = \nabla\cdot(\tilde{a} g)$

$$\text{So} \int g \cdot a \nabla D\varphi = -\int \nabla v \cdot a \nabla D\varphi \qquad \left(-\nabla \cdot a \nabla D\varphi = \nabla \cdot Da(\nabla\varphi + e)\right)$$

$$= \int \nabla v \cdot Da(\nabla \varphi + e).$$

$$\Rightarrow D\int g \cdot q = \int (g + \nabla v) \cdot Da(\nabla\varphi + e)$$

& repeat argument of Step 1. /

Proof of (ii) for σ: $\left\| \int g \cdot \nabla\sigma \right\|_{L^{2p}(\langle\rangle)} \lesssim (Cp)^C \|g\|_{L^2(\mathbb{R}^d)}$.

$$-\Delta \sigma_{ijk} = (\nabla \times q_i)_{jk} \qquad \text{consider eqn } \left[-\Delta z = \nabla \cdot g\right.$$

$$\int g \cdot \nabla \sigma_{ijk} = -\int \nabla z \cdot \nabla \sigma_{ijk}$$

$$= \int \nabla_j z \cdot (q_i)_k - \int \nabla_k z \cdot (q_i)_j$$

$$\Rightarrow \left\| \int g \cdot \nabla\sigma \right\|_{L^{2p}(\Omega)} \lesssim (Cp)^p \underbrace{\|\nabla z\|_{L^2(\mathbb{R}^d)}}_{\lesssim \|g\|_{L^2(\mathbb{R}^d)}.}$$ by result for q!

* [Proof of (i) for σ: $\|[\nabla\sigma]\|_{L^{2p}(\Omega)} \lesssim (Cp)^c$.

$$-\Delta\sigma = \nabla \times q.$$

By localized CZ: $\forall R > 0$, $\forall 1 < p < \infty$

$$\left(\fint_{B_R} [\nabla\sigma]^{2p} \right)^{\frac{1}{2p}} \lesssim_p \left(\fint_{B_{2R}} [q]^{2p} \right)^{\frac{1}{2p}} + \left(\fint_{B_{2R}} [\nabla\sigma]^2 \right)^{\frac{1}{2}}.$$

Take $\|\cdot\|_{L^{2p}(\Omega)}$ & use stationarity:

$$\|[\nabla\sigma]\|_{L^{2p}(\Omega)} \lesssim_p \underbrace{\|[q]\|_{L^{2p}(\Omega)}}_{\substack{\lesssim 1+\|[\nabla\varphi]\|_{L^{2p}(\Omega)} \\ \lesssim (C_p)^C}} + \Big\| \underbrace{\fint_{B_{2R}} [\nabla\sigma]^2}_{\substack{\text{for } R\uparrow\infty:\ \text{by ergodic thm,} \\ \text{get } \mathbb{E}\left[[\nabla\sigma]^2\right]<\infty.}} \Big\|_{L^p(\Omega)}$$

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* Proof of (iii) for σ: deduced from (ii)+(i) for σ. $\square$

III.4 Quantitative corrector result.

Recall $\begin{cases} -\nabla\cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon = \nabla\cdot g \text{ in } \mathbb{R}^d \\ -\nabla\cdot \bar{a}\nabla \bar{u} = \nabla\cdot g \end{cases}$

2-scale expansion: $\begin{cases} u_\varepsilon \sim \bar{u} + \varepsilon\varphi_i(\tfrac{\cdot}{\varepsilon})\partial_i\bar{u} \\ \nabla u_\varepsilon \sim \left(\nabla\varphi_i(\tfrac{\cdot}{\varepsilon}) + e_i\right)\partial_i\bar{u} \end{cases}$

We have seen that $w_\varepsilon = u_\varepsilon - \bar{u} - \varepsilon\varphi_i(\tfrac{\cdot}{\varepsilon})\partial_i\bar{u}$

satisfies $-\nabla\cdot a(\tfrac{\cdot}{\varepsilon})\nabla w_\varepsilon = \nabla\cdot\left[\varepsilon(a\varphi_i - \sigma_i)(\tfrac{\cdot}{\varepsilon})\nabla\partial_i\bar{u}\right].$

In Chapter III: $\begin{cases} \text{"}(\varphi,\sigma)(x) \ll |x|\text{"} \\ \varepsilon(\varphi,\sigma)(\tfrac{x}{\varepsilon}) \to 0 \end{cases}$ qualitative sublinearity. $\Rightarrow$ only qualitative corrector result.

$\forall\, 1 < p, q < \infty$

Corollary :

$$\|\nabla w_\varepsilon\|_{L^p(\mathbb{R}^d, L^q(\Omega))} \lesssim_{p,q,\delta} \|\varepsilon\,(\varphi,\sigma)(\tfrac{\cdot}{\varepsilon})\,\nabla^2 \bar{u}\|_{L^p(\mathbb{R}^d, L^{q+\delta}(\Omega))}$$

by L^p annealed reg.

$$\leq \Big\| \underbrace{\|\varepsilon[(\varphi,\sigma)](\tfrac{\cdot}{\varepsilon})\|_{L^{q+\delta}(\Omega)}}_{\text{by corrector estimates: } \lesssim_{q,\delta} \begin{cases} \varepsilon & d>2 \\ \varepsilon \log(2+\frac{|\cdot|}{\varepsilon})^{1/2} & d=2 \\ \varepsilon \langle \frac{\cdot}{\varepsilon}\rangle^{1/2} & d=1 \end{cases}} \nabla^2 \bar{u} \Big\|_{L^p(\mathbb{R}^d)}$$

Set

$$\mu_d(s) = \begin{cases} 1, & d>2 \\ \log(2+s)^{1/2}, & d=2 \\ (1+s)^{1/2}, & d=1 \end{cases}$$

$$\leq \varepsilon\,\mu_d(\tfrac{1}{\varepsilon})\,\|\mu_d(|\cdot|)\,\nabla^2 \bar{u}\|_{L^p(\mathbb{R}^d)}$$

$$\leq \varepsilon\,\mu_d(\tfrac{1}{\varepsilon})\,\|\mu_d(|\cdot|)\,\nabla f\|_{L^p(\mathbb{R}^d)}$$

by standard weighted CZ thry. for $-\nabla\cdot\bar{a}\nabla\bar{u} = \nabla\cdot f$.