# M285K - course #16

## IV.1 Pathwise structure of fluctuations.

$$-\nabla\cdot a\left(\tfrac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon = \nabla\cdot f$$

**Proposition** (Principle #1 – reduction to commutator).

$\forall f,g \in C_c^\infty(\mathbb{R}^d)^d$,

$$\bar{\mathcal{H}}(h) = \nabla(-\nabla\cdot\bar{a}\nabla)^{-1}\nabla\cdot$$

$$\int_{\mathbb{R}^d} g\cdot(\nabla u_\varepsilon - \nabla\bar{u}) = \int_{\mathbb{R}^d} \bar{\mathcal{H}}^*(g)\cdot \overbrace{\Xi_\varepsilon[\nabla u_\varepsilon]}^{a\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon - \bar{a}\nabla u_\varepsilon}$$

& similarly for flux.

**Proposition** (CLT scaling).

$\forall f,g \in C_c^\infty(\mathbb{R}^d)^d, \quad \forall q<\infty$

$$\mathbb{E}\left[\left|\varepsilon^{-\frac{d}{2}}\int g\cdot(\nabla u_\varepsilon - \mathbb{E}\nabla u_\varepsilon)\right|^q\right]^{\frac{1}{q}} \lesssim \|g\|_{L^4(\mathbb{R}^d)}\|f\|_{L^4(\mathbb{R}^d)}.$$

Proof. By scaling: choose $\varepsilon = 1$.

Malliavin: $\qquad \left\| \int g \cdot (\nabla u - \mathbb{E} \nabla u) \right\|_{L^q(\Omega)} \lesssim_q \left\| D \int g \cdot \nabla u \right\|_{L^q(\Omega, h)}$ )

Duality: consider auxiliary problem $-\nabla \cdot a^* \nabla v = \nabla \cdot g$

$$D \int g \cdot \nabla u = \int g \cdot \nabla D u = - \int \nabla v \cdot a \nabla D u$$

$$= \int \nabla v \cdot D a \, \nabla u$$

$$\begin{cases} -\nabla \cdot a \nabla u = \nabla \cdot f \\ -\nabla \cdot a \nabla D u = \nabla \cdot (D a \nabla u) \end{cases}$$

Recall $D_z a(x) = \delta(x - z) \, a_0'(G(z))$.

$a(x) = a_0(G(x))$

$\|\pi\|_h \lesssim \left\| \left( \int_{B(\cdot)} |\pi| \right) \right\|_{L^2}$

$\Rightarrow \left\| \int g \cdot (\nabla u - \mathbb{E} \nabla u) \right\|_{L^q(\Omega)} \lesssim \left\| \nabla v \otimes \nabla u \right\|_{L^q(\Omega, h)}$

$$\lesssim \left\| [\nabla v]\,[\nabla u] \right\|_{L^q(\Omega, L^2(\mathbb{R}^d))}$$

$$\lesssim \left\| [\nabla v] \right\|_{L^4(\mathbb{R}^d, L^{2q}(\Omega))} \left\| [\nabla u] \right\|_{L^4(\mathbb{R}^d; L^{2q}(\Omega))} .$$

$L^p$ annealed regularity

$$\lesssim_q \left\| [g] \right\|_{L^4(\mathbb{R}^d)} \left\| [f] \right\|_{L^4(\mathbb{R}^d)} .$$

$$\leq \| g \|_{L^4} \| f \|_{L^4} .$$

Rescale: $\displaystyle \varepsilon^{-d/2} \int g \cdot \nabla u_\varepsilon[f] = \int \varepsilon^{d/2} g(\varepsilon \cdot) \cdot \underbrace{\nabla u_\varepsilon[f](\varepsilon \cdot)}_{\nabla u[f(\varepsilon \cdot)]}$

$$\| \cdot \|_{L^q(\Omega)} \lesssim \varepsilon^{d/2} \| g(\varepsilon \cdot) \|_{L^4} \| f(\varepsilon \cdot) \|_{L^4} = \| g \|_{L^4} \| f \|_{L^4} . \qquad \square .$$

Theorem (Principle #2 — 2-scale expansion for commutators).

$\forall f, g \in C_c^\infty(\mathbb{R}^d)^d, \quad q < \infty$

$$\Big\| \varepsilon^{-d/2} \int g \cdot \Big( \Xi_\varepsilon [\nabla u_\varepsilon] - \mathbb{E}\big[\Xi_\varepsilon[\nabla u_\varepsilon]\big]\Big)$$

$$- \varepsilon^{-d/2} \int g \cdot \underbrace{\Xi_\varepsilon^{\circ} [\nabla \bar{u}]}_{\left(a(\nabla\varphi_i + e_i) - \bar{a}(\nabla\varphi_i + e_i)\right)\left(\frac{\cdot}{\varepsilon}\right) \nabla_i \bar{u}} \Big\|_{L^q(\Omega)}$$

$$\lesssim_q \varepsilon \mu_d\left(\tfrac{1}{\varepsilon}\right) \left( \|g\|_{L^4(\mathbb{R}^d)} \|\mu_d(|\cdot|) \nabla f\|_{L^4} + \|\mu_d(|\cdot|) \nabla g\|_{L^4} \|f\|_{L^4} \right).$$

where $\mu_d(r) := \begin{cases} 1, & d > 2 \\ \log^{1/2}(2+r), & d = 2 \\ (1+r)^{1/2}, & d = 1 \end{cases}$

Proof. By scaling, set $\varepsilon = 1$.

Step 1:

$$\begin{cases} D\,\Xi[\nabla u]_j = (\nabla\varphi_j^{*} + e_j)\cdot Da\,\nabla u \\ \qquad - \underbrace{\nabla}_{\hookrightarrow O(\varepsilon) \text{ after rescaling}}\cdot\left[(a\varphi_j^{*} + \sigma_j^{*})\nabla Du\right] - \nabla\cdot\left[\varphi_j^{*}\, Da\,\nabla u\right]. \\ D\,\Xi^{0}[\nabla \bar u]_j = (\nabla\varphi_j^{*} + e_j)\cdot Da\,(\nabla\varphi_i + e_i)\nabla_i \bar u \\ \qquad - (\nabla_i \bar u)\,\nabla\cdot\left[(a\varphi_j^{*} + \sigma_j^{*})\nabla D\varphi_i\right] - (\nabla_i\bar u)\,\nabla\cdot\left[\varphi_j^{*}\,Da(\nabla\varphi_i + e_i)\right] \end{cases}$$

Second formula = exercise

First formula:

$$D\,\Xi[\nabla u]_j = e_j\cdot Da\,\nabla u + \underbrace{e_j\cdot(a-\bar a)\nabla Du}_{= (a^{*}-\bar a^{*})e_j\cdot\nabla Du}$$

$$\big(a^* - \sigma\big) \ldots \nabla\cdot\sigma_j \ldots$$
$$= -a^* \nabla\varphi_j + \nabla\cdot\sigma_j$$

$$= -\nabla\varphi_j^* \cdot a \nabla D u + (\nabla\cdot\sigma_j^*)\cdot\nabla D u$$
$$= -\nabla\cdot\big((a\varphi_j^* + \sigma_j^*)\nabla D u\big) + \varphi_j^* \underbrace{\nabla\cdot a\nabla D u}_{-\nabla\cdot D a \nabla u}$$
$$\left.\begin{array}{l} -\nabla\cdot\big(\varphi_j^* D a \nabla u\big) \\ + \nabla\varphi_j^* \cdot D a \nabla u \end{array}\right.$$

## Step 2:

Mollify:
$$\Big\| \int\rho \cdot \big(\Xi[\nabla u] - \mathbb{E}[\Xi[\nabla u]]\big) - \int\rho\cdot \Xi^0[\nabla\bar u] \Big\|_{L^q(\Omega)}$$
$$\lesssim_q \Big\| \int\rho \cdot D\big(\Xi[\nabla u] - \Xi^0[\nabla\bar u]\big)\Big\|_{L^q(\Omega, h)}$$
$$\le \Big\| \int\rho_j\, (\nabla\varphi_j^* + e_j)\cdot D a\, \big(\nabla u - (\nabla\varphi_i + e_i)\partial_i \bar u\big)\Big\|_{L^q(\Omega,h)} \Big\}\ (I)$$

$$+ \left\| \int \nabla \rho_j \cdot (\sigma \varphi_j^* - \sigma_j^*) \nabla D u \right\|_{L^q(\Omega, h)} \Big\} (\mathrm{II}).$$

$$+ \text{etc. (exercise)}$$

$$\left( [\rho]_\infty(x) = \sup_{B(x)} |\rho| \right)$$

$$(\mathrm{I}) \lesssim \left\| \rho \circledast (\nabla \varphi^* + e) \circledast (\nabla u - (\nabla \varphi_i + e_i) \nabla_i \bar{u}) \right\|_{L^q(\Omega, h)}$$

$$\lesssim \left\| [\rho]_\infty [\nabla \varphi^* + e] [\nabla u - (\nabla \varphi_i + e_i) \nabla_i \bar{u}] \right\|_{L^q(\Omega, L^2(\mathbb{R}^d))}$$

$$\underset{\lesssim_q}{\text{corrector estimates}} \left\| [\rho]_\infty [\nabla u - (\nabla \varphi_i + e_i) \nabla_i \bar{u}] \right\|_{L^2(\mathbb{R}^d, L^{2q}(\Omega))}$$

$$\leq \left\| [\rho]_\infty \right\|_{L^4(\mathbb{R}^d)} \underbrace{\left\| [\nabla u - (\nabla \varphi_i + e_i) \nabla_i \bar{u}] \right\|_{L^4(\mathbb{R}^d; L^{2q}(\Omega))}}_{\text{theory of oscillations (Chapter III)}}$$

$$\hookrightarrow \lesssim \| \mu_1(|\cdot|) [\nabla^2 \bar{u}] \|_{L^4(\mathbb{R}^d)} .$$

$\lesssim_q \| \mu_d(\cdot) [D g] \|_{L^q(\mathbb{R}^d)}$

$\lesssim \| \mu_d(|\cdot|) [D g] \|_{L^q}$ by standard constant-coeff $L^p$ regularity.

$$(II) = \left\| \int \nabla \rho_j \cdot (a \varphi_j^\sigma - \sigma_j^\sigma) \nabla D u \right\|_{L^q(\Omega, h)}$$

[use auxiliary problem:
$$-\nabla \cdot a^* \nabla z = \nabla \cdot ((a \varphi_j^\sigma + \sigma_j^\sigma) \nabla \rho_j).$$

$$= \left\| \int \nabla z \cdot a \nabla D u \right\|_{L^q(\Omega, h)}$$

$$= \left\| \int \nabla z \cdot D a \nabla u \right\|_{L^q(\Omega, h)}$$

$$\lesssim \| \nabla z \otimes \nabla u \|_{L^q(\Omega; h)}$$

$$\lesssim \left\| [Dz]\,[Du] \right\|_{L^q(\Omega,\, L^2(\mathbb{R}^d))}$$

$$\lesssim \underbrace{\left\| [Dz] \right\|_{L^4(\mathbb{R}^d,\, L^{2q}(\Omega))}}_{\substack{\text{annealed reg:} \\ \lesssim \left\| [(\varphi^\circ_i, \sigma^\circ_i)\, Dg] \right\|_{L^4(\mathbb{R}^d;\, L^{4q}(\Omega))} \\ \lesssim \left\| \mu_d\, [Dg]_\circ \right\|_{L^4(\mathbb{R}^d)} \\ \text{by corrector estimate!}}} \underbrace{\left\| [Du] \right\|_{L^4(\mathbb{R}^d,\, L^{2q}(\Omega))}}_{\substack{\lesssim \left\| [f] \right\|_{L^4(\mathbb{R}^d)}. \\ \text{by annealed} \\ L^p \text{ reg.}}}$$

$\square$

<u>Corollary</u>. $\forall f, g \in C^\infty_c(\mathbb{R}^d)^d, \quad \forall q < \infty$

$$\varepsilon^{-\frac{d}{2}} \int g \cdot \left( Du_\varepsilon - \mathbb{E}[Du_\varepsilon] \right) \overset{\text{Principle 1}}{=} \varepsilon^{-\frac{d}{2}} \int \mathcal{H}^*(g) \cdot \left( \Xi_\varepsilon [Du_\varepsilon] - \mathbb{E}\left[ \Xi_\varepsilon [Du_\varepsilon] \right] \right)$$

$$= \varepsilon^{\frac{d}{2}} \int \mathcal{H}(g) \cdot \Xi_\varepsilon[\nabla u] + O_{L^q(\Omega)}(\varepsilon \mu_d(\varepsilon))$$

$$\mathrm{Cov}\left[\nabla u_\varepsilon - \mathbb{E}[\nabla u_\varepsilon]\right] \approx \nabla(-\nabla \cdot \bar{a} \nabla)^{-1} \nabla \cdot \left[\left[(a - \bar{a})(\nabla \varphi_i + e_i)\right]\left(\tfrac{\cdot}{\varepsilon}\right) \nabla_i \bar{u}\right].$$

nonlocal 2-scale expansion

"pathwise structure of fluctuations.

## IV.2 Scaling limit of $\Xi^\circ$.

**Goal**: $\varepsilon^{-\frac{d}{2}} \int h : \left(a\left(\tfrac{\cdot}{\varepsilon}\right) - \mathbb{E}[a]\right) \xrightarrow{\text{law}} \int h : K_\circ \xi$ (white noise)

with $K_\circ : K_\circ = \int_{\mathbb{R}^d} \mathrm{Cov}(a(x), a(0))\, dx$.

$\Rightarrow$ expect same result for $\Xi^\circ$:

$$\varepsilon^{-d/2}\int g\cdot \boxed{\mathcal{H}}^{0}_{\varepsilon}[\nabla \bar u] \xrightarrow{\text{law}} \int \nabla \bar u \otimes g : \bar{K}_0\, \xi$$

with $\bar K_0 : \bar K_0 = \int_{\mathbb{R}^d} \underbrace{\mathrm{Cov}\big((a-\bar a)(\nabla\varphi + e)(x), (a-\bar a)(\nabla\varphi+e)(0)\big)}_{\sim |x|^{-d}}\, dx$

... makes sense using that $\boxed{D[(a-\bar a)(\nabla\varphi + e)] = Da + \text{error}}$.

2 parts:
* convergence of covariance : Helffer-Sjöstrand identity : $\mathrm{Var}\, X = \mathbb{E}\langle DX, (1+\mathcal{L})^{-1} DX\rangle_{\mathfrak{H}}$
* (asymptotic normality : Malliavin-Stein method.

**Theorem** ("second-order Poincaré inequality"
— Chatterjee '08, Nourdin, Peccati, Reinert '08, Ledoux, Nourdin, Peccati '15).

$[\forall X \in L^2(\Omega),\ \mathbb{E}X = 0,\ \mathrm{Var}\, X = 1$

$\forall X \in L^2(\Omega),\ \mathbb{E}X = 0,\ \mathrm{Var}\,X = 1,$

$$d_{TV}(\underset{\text{standard Gaussian}}{X, N}) + W_2(X, N) \lesssim \left(\mathrm{Var}\left[\langle DX, (1+\mathcal{L})^{-1} DX\rangle_h\right]\right)^{\frac{1}{2}}$$

$$\underset{\text{Poincaré inequality.}}{\lesssim} \left(\mathbb{E}\left[\|D^2X\|^4_{h\to h}\right]\right)^{\frac{1}{4}} \left(\mathbb{E}\left[\|DX\|_h^4\right]\right)^{\frac{1}{4}}.$$

1st order Poincaré: $\underbrace{\mathrm{Var}\,X}_{\text{distance to constants}} \lesssim \|DX\|_{L^2(\Omega, h)}$

2nd order Poincaré: distance to normality $\lesssim \|D^2X\| \, \|DX\|.$

if $D^2X = 0 \to X = \int S \cdot G + ct$ is gaussian.

<u>Proof</u>. We focus of $d_{TV}$ (argument for $W_2$ is different)

Stein's method: $\big[$ $X$ is Gaussian $\iff \mathbb{E}\, X h(X) = \mathbb{E}\, h'(X) \quad \forall h \in C_c^\infty(\mathbb{R}).$

$\mathbb{E}X=0$
$\mathrm{Var}X=1$

& expect: if $\mathbb{E}\, X h(X) \approx \mathbb{E}\, h'(X)$ $\forall h$
then $X$ should be approx Gaussian.

Argue as follows: $\forall h \in L^\infty(\mathbb{R})$, define its Stein's transform $S_h$

$$S_h'(x) - x S_h(x) = h(x) - \underbrace{\mathbb{E}[h(N)]}_{\int h(y) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy}.$$

Then: $$d_{TV}(X,N) = \sup\{\mathbb{E}\, h(X) - \mathbb{E}\, h(N) : \underset{\|h\|_{L^\infty}=1}{h \in L^\infty(\mathbb{R})}\}.$$

$$= \sup\{\mathbb{E}\, S_h'(X) - \mathbb{E}\, X S_h(X) : \underset{\|h\|_{L^\infty}=1}{h \in L^\infty(\mathbb{R})}\}$$

Because $\mathbb{E}X=0$, can write $X = \mathcal{L}Z$, for some $Z \in \mathrm{dom}\,\mathcal{L}$.

$\leadsto$ $\mathbb{E}\, X S_h(X) = \mathbb{E}\, S_h(X) \mathcal{L}Z = \mathbb{E}\langle DS_h(X), DZ\rangle_{\mathfrak{H}}$

$\mathbb{E}\, S_h'(X) \langle \ldots$

$$= \mathbb{E}\, S_h'(X) \langle DX, DZ \rangle_n$$

$$\left[ DZ = D\mathcal{L}^{-1} X = (\mathcal{L}+1)^{-1} DX, \right.$$

$$= \mathbb{E}\, S_h'(X) \langle DX, (1+\mathcal{L})^{-1} DX \rangle_n .$$

$$\rightsquigarrow d_{TV}(X, N) \leq \sup \left\{ \mathbb{E}\left[ |S_h'(X)|^2 \right]^{\frac{1}{2}} \mathbb{E}\left[ |1 - \langle DX, (1+\mathcal{L})^{-1} DX \rangle_n |^2 \right]^{\frac{1}{2}} : \begin{array}{l} h \in L^\infty \\ \|h\|_{L^\infty} \leq 1 \end{array} \right\}$$

Exercise : $|S_h'| \leq 2 \|h\|_{L^\infty}$ o.e.

$$\leadsto d_{TV}(X, N) \leq \sqrt{2}\, \underbrace{\mathbb{E}\left[ |1 - \langle DX, (1+\mathcal{L})^{-1} DX \rangle_n |^2 \right]^{\frac{1}{2}}}_{\mathrm{Var}\left[ \langle DX, (1+\mathcal{L})^{-1} DX \rangle_n \right]^{\frac{1}{2}}} .$$

$\square$