# M285K - course #19

## V.2 $C^{1,\alpha}$ large-scale regularity.

$$\mathrm{Exc}(\nabla u, D) := \inf_{e \in \mathbb{R}^d} \fint_D |\nabla u - (\nabla \varphi_e + e)|^2$$

**Theorem** ($C^{1,\alpha}$ regularity for $a$-harmonic fcts).

$\forall \alpha \in (0,1)$ $\exists C_0 < \infty$ such that the following holds:

Let $r_* = r_*[C_0] := \inf\{ r > 0 : \forall R \geq r, \ \frac{1}{R^2} \fint_{B_R} |(\varphi, \sigma) - \fint_{B_R} (\varphi, \sigma)|^2 \leq \frac{1}{C_0} \}$

Given $R \geq r_*$, let $u \in H^1(B_R)$ be $a$-harmonic

$$-\nabla \cdot a \nabla u = 0 \quad \text{in } B_R$$

Then, (i) Large-scale Lipschitz regularity: $\forall \underline{r_*} \le r \le R$,

$$\fint_{B_r} |\nabla u|^2 \lesssim \fint_{B_R} |\nabla u|^2 .$$

(ii) Large-scale $C^{1,\alpha}$ regularity: $\forall r_* \le r \le R$,

$$\mathrm{Exc}(\nabla u, B_r) \lesssim \left(\frac{r}{R}\right)^{2\alpha} \mathrm{Exc}(\nabla u, B_R) .$$

... compare to: $\begin{cases} v \in H^1(B_R), \ -\Delta v = 0 \text{ in } B_R \\ \fint_{B_r} \left| \nabla v - \fint_{B_r} \nabla v \right|^2 \lesssim \left(\frac{r}{R}\right)^2 \fint_{B_R} \left| \nabla v - \fint_{B_R} \nabla v \right|^2 \ \left(\underline{C^{1,1} \text{ regularity}}\right) \end{cases}$

Proof is by iterating the following:

Proposition. $\exists \varepsilon > 0$ such that the following holds:

Let $R>0$, let $u \in H^1(B_R)$ be a-harmonic

$$-\nabla\cdot a\nabla u = 0 \text{ on } B_R$$

Then $\exists e \in \mathbb{R}^d$ $\forall 0 < r \in R$,

$$\fint_{B_r} |\nabla u - (\nabla\varphi_e + e)|^2 \lesssim \Big[\underbrace{\left(\frac{r}{R}\right)^2}_{\substack{\text{like}\\ \text{for harmonic}\\ \text{fcts.}}} + \underbrace{\delta_R^{2\varepsilon}\left(\frac{R}{r}\right)^{d+2}}_{\substack{\text{control}\\ \text{of homog}\\ \text{error.}}}\Big] \fint_{B_R} |\nabla u|^2$$

where $\delta_R := \frac{1}{R}\left(\fint_{B_R} \left|(\varphi,\sigma) - \fint_{B_R}(\varphi,\sigma)\right|^2\right)^{\frac{1}{2}}$.

Moreover, we have non-degeneracy property: $\forall e \in \mathbb{R}^d$,

[illegible]

$$|e|^2(1-C\delta_R) \lesssim \fint_{B_{R/2}} |\nabla\varphi_e + e|^2 \lesssim |e|^2(1+\delta_R).$$

Idea: * approximate $u$ by $\bar{u} + \underbrace{\eta}_{\text{cut-off}}\varphi_i \nabla_i \bar{u}$

$$\begin{cases} -\nabla\cdot\bar{a}\nabla\bar{u} = 0 & \text{in } B_R \\ \bar{u} = u & \text{on } \partial B_R \end{cases}$$

* estimate homog error in terms of sublinearity of $(\varphi,\sigma)$.

* use $C^2$ regularity for $\bar{u}$ away from boundary.

Proof. By scaling, assume $R=1$

& wlog: assume $\fint_B (\varphi,\sigma) = 0$.

$\boxed{\text{Step 1}}$: PDE ingredients: (a) Weighted energy estimate:

$\forall\, 0 \leq \varepsilon << 1$: we have
$$\left\{\begin{array}{l} -\nabla\cdot a\nabla w = \nabla\cdot g \text{ in } B \\ w = 0 \quad \text{on } \partial B \end{array}\right.$$
$$\Longrightarrow \int_B (1-|x|)^\varepsilon |\nabla w|^2 \lesssim \int_B (1-|x|)^\varepsilon |g|^2.$$

(b) Inner $C^2$ regularity for homog eqn:

$$-\nabla\cdot \bar{a}\nabla \bar{u} = 0 \text{ in } B$$
$$\Longrightarrow \sup_{B_{1-\rho}} \left(\rho\, |\nabla^2 \bar{u}| + |\nabla \bar{u}|\right) \lesssim \left(\frac{1}{\rho^d}\int_B |\nabla \bar{u}|^2\right)^{\frac{1}{2}}.$$
$\forall \rho \in (0,1)$

(a) Let $\chi^2 = (1-|x|)^\varepsilon$.

Test eqn $\begin{cases} -\nabla\cdot a\nabla w = \nabla\cdot g & \text{in } B \\ w \in H^1_0(B) \end{cases}$

with $\chi^2 w$:

Get: $\displaystyle\int \chi^2 \overbrace{Dw\cdot aDw}^{\gtrsim |Dw|^2} + 2\int \chi w\, D\chi\cdot aDw$

$\displaystyle = -\int \chi^2 Dw\cdot g - 2\int \chi w\, D\chi\cdot g.$

$\displaystyle \hookrightarrow \int \chi^2 |Dw|^2 \lesssim \underbrace{\int |D\chi|^2 |w|^2}_{\substack{\text{want to absorb} \\ \text{for } \chi = (1-|x|)^\varepsilon \\ \varepsilon \ll 1.}} + \int |\chi|^2 |g|^2$ (Caccioppoli estimate)

Use some version of Hardy ineqn: $\displaystyle\int_B (1-|x|)^{\varepsilon-2} |w|^2 \lesssim \int_B (1-|x|)^\varepsilon |Dw|^2.$

Proof: Con form on radial setting: $\forall v:[0,1]\to\mathbb{R}$, $v(1)=0$

$$\int_0^1 r^{d-1}(1-r)^{\varepsilon-2} v(r)^2\,dr \lesssim \int_0^1 r^{d-1}(1-r)^{\varepsilon} v'(r)^2\,dr.$$

Notice $\frac{d}{dr}\left(r^{d-1}\frac{(1-r)^{\varepsilon-1}}{1-\varepsilon}\right) \geq r^{d-1}(1-r)^{\varepsilon-2}$

$$\Rightarrow \int_0^1 r^{d-1}(1-r)^{\varepsilon-2} v(r)^2\,dr \leq \overbrace{\int_0^1 \frac{d}{dr}\left(r^{d-1}\frac{(1-r)^{\varepsilon-1}}{1-\varepsilon}v(r)^2\right)dr}^{\leq 0}$$

$$-2\int_0^1 r^{d-1}\frac{(1-r)^{\varepsilon-1}}{1-\varepsilon}v(r)v'(r)\,dr$$

$$\overset{C\text{-}S}{\leq} \frac{2}{1-\varepsilon}\left(\int_0^1 r^{d-1}(1-r)^{\varepsilon-2}v(r)^2\,dr\right)^{\frac{1}{2}} \leftarrow$$

$$\times\left(\int_0^1 r^{d-1}(1-r)^{\varepsilon}v'(r)^2\,dr\right)^{\frac{1}{2}}.$$

Let's go back to Caccioppoli:

$$\int \chi^2|\nabla v|^2 \leq \int |\nabla\chi|^2|v|^2 + \int \chi^2|\rho|^2.$$

$\int \chi \ |Dw| \ \lesssim \int |D\chi| \ |w| \ + \int \chi \rho$

$\chi^2 = (1-|x|)^\varepsilon, \quad |D\chi|^2 \lesssim \varepsilon^2 (1-|x|)^{\varepsilon-2}.$

$$c_0 \underbrace{\int (1-|x|)^\varepsilon |Dw|^2} \lesssim \varepsilon^2 \underbrace{\int (1-|x|)^{\varepsilon-2} |w|^2}_{\overset{\text{Hardy}}{\lesssim} \int (1-|x|)^\varepsilon |Dw|^2.} + \underbrace{\int (1-|x|)^\varepsilon \rho^2}.$$

absorbed for $\varepsilon << 1$ small.

(b) $-\nabla \cdot \bar{a} \nabla \bar{u} = 0$ in $B$

Want to show: $\sup_{B_{1-\rho}} \left(\rho |D^2 \bar{u}| + |D\bar{u}|\right) \lesssim \left(\frac{1}{\rho^d} \int_B |D\bar{u}|^2\right)^{\frac{1}{2}}.$

$\gtrsim$

Suffices to show: $\forall z \in B_{1-\rho}$: $\rho |D^2 \bar{u}(z)| + |D\bar{u}(z)| \lesssim \left(\fint_{B_\rho(z)} |D\bar{u}|^2\right)^{\frac{1}{2}}.$

By scaling, suffices to show: $\sup_B |\nabla^2 \bar{u}| + |\nabla \bar{u}| \lesssim \left( \int_{B_2} |\nabla \bar{u}|^2 \right)^{1/2}$.

By Sobolev: $\int_B \left( |\nabla^{k+1} \bar{u}|^2 + \dots + |\nabla^2 \bar{u}|^2 \right) \lesssim \int_{B_2} |\nabla \bar{u}|^2$ for $k > \frac{d}{2} + 1$.

in fact consequence of Caccioppoli.

$$\left( \begin{array}{l} \int_B |\nabla^{\ell+1} \bar{u}|^2 \lesssim \int_{B_2} |\nabla^\ell \bar{u}|^2 \quad \forall \ell . \\ \text{for } -\nabla \cdot \bar{a} \nabla \partial^\ell \bar{u} = 0 \text{ in domain.} \end{array} \right.$$

$\boxed{\text{Step 2}}$ Harmonic approximation

$$\begin{cases} -\nabla \cdot \bar{a} \nabla \bar{u} = 0 & \text{in } B_R \\ \bar{u} = u & \text{in } \partial B_R . \end{cases}$$

Note that $\left[ \int_B |D\bar{u}|^2 \lesssim \int_B |Du|^2. \right.$

Proof: $\begin{cases} -\nabla \cdot \bar{a} \nabla(\bar{u} - u) = \nabla \cdot \bar{a} \nabla u \\ \bar{u} - u \in H_0^1(B) \end{cases}$

$\hookrightarrow$ energy: $\int_B |D(\bar{u} - u)|^2 \lesssim \int_B |Du|^2$ $\eta$.

## $\boxed{\text{Step 3}}$ Equation for error.

Error : $w = u - \bar{u} - \eta \varphi_i D_i \bar{u}$

with cut-off $\eta$ : $\begin{cases} \eta = 1 \text{ in } B_{1-2\rho} \\ \eta = 0 \text{ outside } B_{1-\rho} \end{cases}$ for some thickness $\rho \in (0,1)$.

$$|D\eta| \lesssim \tfrac{1}{\rho}.$$

Equation for $w$: $\left[\begin{aligned} &-\nabla\cdot a\nabla w \\ &= \nabla\cdot\left[\underbrace{(1-\eta)(a-\bar{a})\nabla\bar{u}} + \underbrace{(\varphi_i a - \sigma_i)}\,\underbrace{\nabla(\eta\,\partial_i\bar{u})}\right]\end{aligned}\right.$

Proof: Compute $\nabla w = \underline{\nabla u} - \nabla\bar{u} - (\eta\,\partial_i\bar{u})\nabla\varphi_i - \varphi_i\nabla(\eta\,\partial_i\bar{u})$.

$$\leadsto -\nabla\cdot a\nabla w = \nabla\cdot\Big(\underline{a\nabla\bar{u}} + (\eta\,\partial_i\bar{u})\,\underbrace{a\nabla\varphi_i} + a\varphi_i\nabla(\eta\,\partial_i\bar{u})\Big)$$

[in $B$]

$$\left[\begin{aligned} &= a(\nabla\varphi_i + e_i) - \bar{a}e_i \\ &\quad - (a-\bar{a})e_i \\ &= \nabla\cdot\sigma_i - \underbrace{(a-\bar{a})}e_i \end{aligned}\right.$$

$$= \nabla\cdot\Big((1-\eta)a\nabla\bar{u} + \underline{\eta\bar{a}\nabla\bar{u}}$$

$$+ (\eta\,\partial_i\bar{u})\,\underline{(\nabla\cdot\sigma_i)} + \underline{a\varphi_i}\,\nabla(\eta\,\partial_i\bar{u})\Big)$$

1) $$\nabla \cdot \left( \eta \, \bar{a} \, \nabla \bar{u} \right) = - \nabla \cdot \left( (1-\eta) \, \bar{a} \, \nabla \bar{u} \right)$$

2) $$\nabla \cdot \left( (\eta \nabla_i \bar{u}) (\nabla \cdot \sigma_i) \right) = \nabla (\eta \nabla_i \bar{u}) \cdot (\nabla \cdot \sigma_i)$$

$$= - \nabla \cdot \left( \sigma_i \, \nabla (\eta \nabla_i \bar{u}) \right) .$$