M285K - course #2

I.3 Setting.

$$-\nabla\cdot a^\omega\left(\tfrac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon^\omega = \nabla\cdot f \quad \text{in } \mathbb{R}^d$$ (avoid boundary issues)

with $a^\omega : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ measurable

& unif elliptic ie ($\exists \alpha, \beta > 0$) $\begin{cases} e\cdot a^\omega(x) e \geq \alpha |e|^2 \\ |a^\omega(x) e| \leq \beta |e| \end{cases}$ $\forall x, e \in \mathbb{R}^d$, $\omega \in \Omega$

Lemma (Lax-Milgram). $\forall \omega$ $\forall f \in L^2(\Omega)$
$\exists!$ solution u_ε^ω in $\dot H^1(\mathbb{R}^d) = \{v \in H^1_{loc}(\mathbb{R}^d)$ st $\nabla v \in L^2\}$

In addition: $\|\nabla u_\varepsilon^\omega\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{\alpha}\|f\|_{L^2(\mathbb{R}^d)}$.

<u>Micro model</u>: * periodic setting: a periodic

* random setting: statistical ensemble of coeff fields.

Natural requirement: stationarity = "law of a is invariant under shifts"

$$a(\cdot) \overset{\text{law}}{=} a(\cdot + y) \quad \forall y.$$

<u>Definition</u>. Let $(\Omega, \mathcal{F}, \mathbb{P})$ pr space.

- A <u>random field</u> on $\mathbb{R}^d$ is a map $\varphi : \mathbb{R}^d \times \Omega \to \mathbb{R}$ <u>st</u>. $\forall x \in \mathbb{R}^d$: $\varphi(x, \cdot) : \Omega \to \mathbb{R}$ meas.

- It is <u>stationary</u> if the finite-dim law is shift-invariant
i.e. $\forall$ finite $F \subset \mathbb{R}^d$:

ie $\forall$ given ...
the law of $\{ v(x+y,\cdot) \}_{x\in E}$ is independent of the shift.

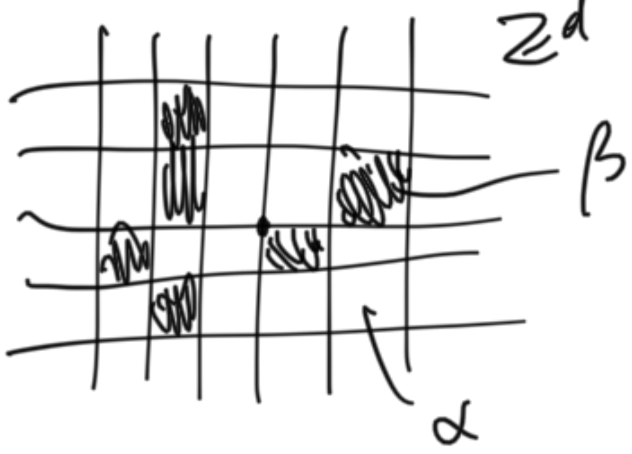
- It is measurable if $v : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is jointly measurable.

(In part: $\forall \omega$, realization $v(\cdot,\omega) : \mathbb{R}^d \to \mathbb{R}$ meas.)

Lemma / Exercise (von Neumann '32) – stoch version of Lusin's thm.

A stationary random field is measurable
$\Longleftrightarrow$ it is stochastically continuous
ie $\forall x \in \mathbb{R}^d, \forall \delta > 0 : \mathbb{P}\left[|v(x+y,\cdot) - v(x,\cdot)| > \delta \right] \xrightarrow{|y|\to 0} 0$.

Examples: • random checkerboard



$$\begin{cases} \text{iid rv } \{V_z\}_{z\in\mathbb{Z}^d},\ V_z \in \{\alpha,\beta\} \\ \perp\!\!\!\perp\ U \sim \text{uniform on } Q = [-\tfrac{1}{2},\tfrac{1}{2})^d. \end{cases}$$

Co $$a^\omega(x) = \sum_{z\in\mathbb{Z}^d} V_z^\omega \, \mathbb{1}_{x\in U^\omega + z + Q}$$

- inclusion model

- <u>Gaussian model</u>: G Gaussian random field

$$\begin{cases} \text{centered} & \mathbb{E}[G(x)] = 0 \quad \forall x \\ \text{covariance} & \mathbb{E}[G(x)\,G(y)] = K(x,y) \equiv C(x-y) \end{cases}$$

covariance fct

$$\mathbb{E}[G(x)\,G(y)] = \mathbb{E}[G(x+z)\,G(y+z)]$$

$\Rightarrow$ stationary!

$a_0 : \mathbb{R} \xrightarrow{\text{smooth}} \mathbb{R}^{d\times d}$ with $\overset{\text{unif}}{\text{elliptic}}$ values $[\alpha,\beta]$ | + assume that C continuous

Set $a(x,\omega) = a_0(G(x,\omega))$

Rem: $\mathbb{E}\,|G(x)-G(y)|^2 = \mathbb{E}\, G(x)^2 + G(y)^2 - 2G(x)G(y)$

$= 2\left(C(0) - C(x-y)\right) \underset{|x-y|\to 0}{\longrightarrow} 0$

$\Rightarrow G$ stoch cts $\Rightarrow G$ meas stat r.f.

In the sequel: $(\Omega, \mathcal{F}, \mathbb{P})$ generated by a (coeff field)

More precisely, concrete choice:

- $\Omega = \{$ coeff fields $\mathbf{a} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ meas & unif ell $[\alpha,\beta]\}$
- $\mathcal{F}$ = σ-algebra generated by $\{a \mapsto \int_{\mathbb{R}^d} a\varphi : \varphi \in C^\infty_c(\mathbb{R}^d)\}$
- group $(\mathbb{R}^d,+)$ acts on Ω : $(\tau_z a)(x) = a(x+z)$

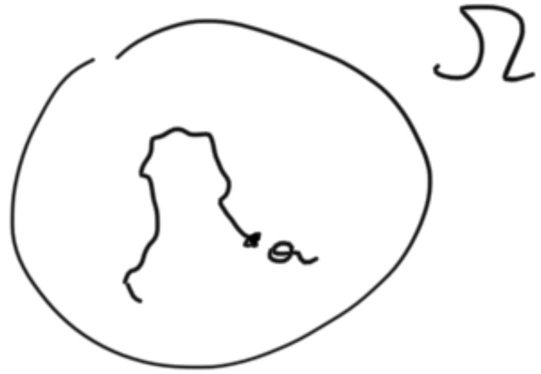
• stationarity: $\underbrace{\mathbb{P}[\tau_z A] = \mathbb{P}[A]}$ $\forall z \in \mathbb{R}^d$ $\forall A \in \mathcal{F}$. $\quad \widehat{\mathbb{R}^d}$

Rem. τ measure-preserving action of $(\mathbb{R}^d, +)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ | dynamical system

↳ $\tau_0 = id$, $\tau_x \tau_y = \tau_{x+y}$.

$\mathbb{R}^d \times \Omega \to \Omega$

$(z, \omega) \mapsto \tau_z \omega$ meas.



• $v : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is τ-stationary

if $v(x, \omega) = v(0, \underbrace{\omega(\cdot + x)}_{\tau_x \omega})$ ($\Rightarrow$ v is meas. stat random field in the previous sense)

• ergodicity: if $A \in \mathcal{F}$ satisfies $\tau_z A = A$ $\forall z$ then $\mathbb{P}A$ is 0 or 1.

(connecting condition)

<u>Ergodic theorem</u> (pointwise version - cts setting) (Birkhoff, Khinchin).

If τ ergodic on $(\Omega, \mathcal{F}, \mathbb{P})$,

Then $\forall v \in L^1(\Omega)$ we have $\forall' \omega \in \Omega$:

$$\frac{1}{R^d}\int_{RQ} v(\tau_z \omega)\, dz \xrightarrow{R \to \infty} \mathbb{E}[v].$$

~ generalization of LLN for iid random variables.

(Proof: difficult - SLLN, much easier: L^2 conv. result, Von Neumann mean erg thm)

Interpretation $(d=1)$: $v = \mathbb{1}_E$

$$\frac{1}{R}\int^R \mathbb{1}_{\tau_z\omega \in E}\, dz \to \mathbb{P}[E]$$

$$\underbrace{\int_0 \mathbb{1}_{\tau_z a \in E}}_{\substack{\text{averaged amount of} \\ \text{time spent in } E}}$$

Time average $\longrightarrow$ statistical average

Corollary. $\begin{cases} \forall v : \mathbb{R}^d \times \Omega \to \mathbb{R} \quad \tau\text{-stat.}, \ \mathbb{E}|v| < \infty \\ \text{Then } \forall' a : \quad v\left(\frac{\cdot}{\varepsilon}, a\right) \xrightarrow{*L^\infty} \mathbb{E}[v] . \end{cases}$

Proof. Show that $\forall g \in L^1(\mathbb{R}^d) : \int g \, v\left(\frac{\cdot}{\varepsilon}, a\right) \to \mathbb{E}[v] \int g$.

① $g = \mathbb{1}_Q$: $\displaystyle\int g \, v\left(\frac{\cdot}{\varepsilon}, a\right) = \int_Q v\left(\frac{\cdot}{\varepsilon}, a\right) = \varepsilon^d \int_{\frac{1}{\varepsilon}Q} \underbrace{v(z, a)}_{v(0, \tau_z a)} \, dz$

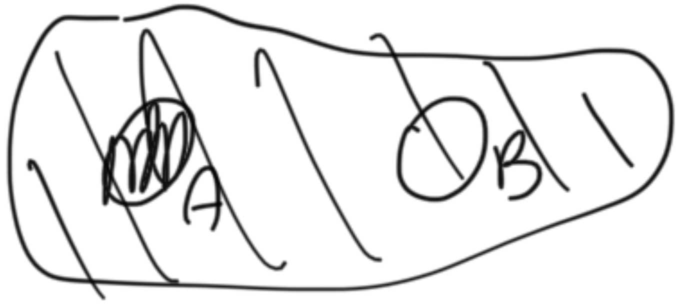
$$(\forall' a) \quad \xrightarrow{\text{erg thm}} \mathbb{E}[v] \int \mathbb{1}_Q .$$

②

(2) $g = \sum \alpha_k 1\!\!1_{Q_k}$

then approx!

Stronger assumption: τ mixing if $\mathbb{P}[A \cap \tau_x B] \xrightarrow{|x|\to\infty} \mathbb{P}A\,\mathbb{P}B$.



decorrelation.

Quantitative: e.g. Rosenblatt α-mixing

- $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{F}$ sub-σ-alg.

$$\alpha_0(\mathcal{G}_1, \mathcal{G}_2) = \sup\Big\{ \big|\mathbb{P}[G_1 \cap G_2] - \mathbb{P}G_1\,\mathbb{P}G_2\big| : \begin{array}{l} G_1 \in \mathcal{G}_1, \\ G_2 \in \mathcal{G}_2 \end{array} \Big\}$$

- $\alpha(R, D) = \sup\big\{ \alpha_0\big(\sigma(a|_{S_1}), \sigma(a|_{S_2})\big) : S_1, S_2 \in \mathbb{R}^d,$

assume diameter $\qquad \mathrm{dist}(S_1, S_2) \geq R, \ \mathrm{diam}(S_i) \leq D \}$.

- α-mixing if $\alpha(R, D) \xrightarrow{R \to \infty} 0 \quad (\forall D)$

<u>Example</u>: finite range of dependence $\Leftrightarrow$ $\alpha(R, D) = 0 \ \forall R \geq R_0$
$\Leftrightarrow$ $a|_{S_1} \perp\!\!\!\perp a|_{S_2} \ \forall \, \mathrm{dist}(S_1, S_2) \geq R_0$.

I.4 Explicit 1D case. $(d = 1)$

$$-\left(a\left(\tfrac{\cdot}{\varepsilon}\right) u_\varepsilon'\right)' = f' \text{ in } \mathbb{R}, \qquad u_\varepsilon' \in L^2(\mathbb{R})$$
$$(f \in L^2(\mathbb{R}))$$

$$-a\left(\tfrac{\cdot}{\varepsilon}\right) u_\varepsilon' = f \ \cancel{+C}$$

$$u_\varepsilon' = -\frac{1}{a\left(\frac{\cdot}{\varepsilon}\right)} f.$$

trivial in 1D: sol. operator = mult by $\frac{1}{a\left(\frac{\cdot}{\varepsilon}\right)}$

(cf in 1D: $\nabla \Delta^{-1} \nabla \cdot = Id$)

<u>Lemma</u> (1D qual. homog.).

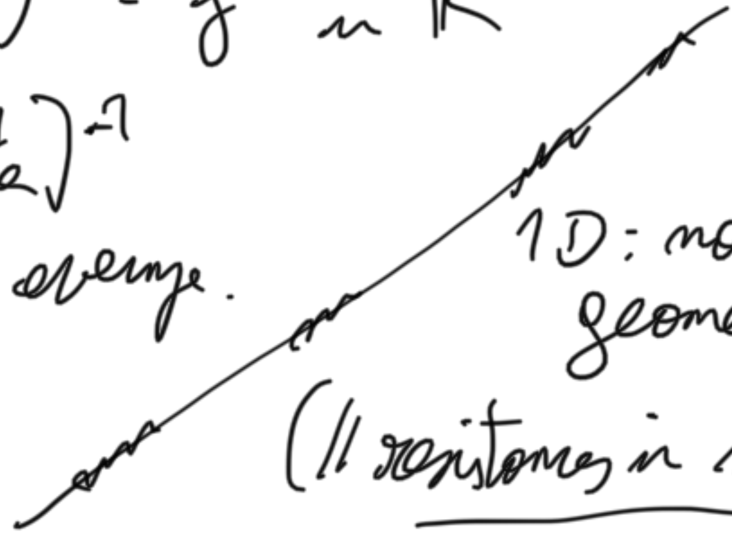
If a stat. ergodic,

Then $u'_\varepsilon \xrightarrow{L^2(\mathbb{R})} -\mathbb{E}\left[\frac{1}{a}\right] f = \bar{u}'$

where $-(\bar{a}\,\bar{u}')' = f'$ in $\mathbb{R}$

with $\bar{a} = \mathbb{E}\left[\frac{1}{a}\right]^{-1}$

geom average.



1D: no geometry

(// resistances in series)

<u>Proof</u>: erg. thm. $\triangle$

<u>Lemma</u> (corrector result).

= description of oscillation

If a stat. erg.,

Then $u'_\varepsilon - \bar{u}'\left(1 + \underbrace{\left(\frac{\bar{a}}{a(\frac{\cdot}{\varepsilon})} - 1\right)}_{\to 0}\right) = 0$

Rem. Less trivial for Dirichlet BC on $[0,1]$: $\begin{cases} -\left(a(\tfrac{\cdot}{\varepsilon})\, v_\varepsilon'\right)' = f', & [0,1] \\ v_\varepsilon(0) = v_\varepsilon(1) = 0. \end{cases}$

Again $v_\varepsilon' \to \bar v'$: $\begin{cases} -(\bar a \bar v')' = f' \\ \bar v(0) = \bar v(1) = 0. \end{cases}$

$$v_\varepsilon' - \bar v'\Big(1 + \underbrace{\Big(\frac{\bar a}{a(\frac{\cdot}{\varepsilon})} - 1\Big)}_{\text{relevant oscillation}}\Big) \underset{\text{(exercise)}}{=} \underbrace{\frac{1}{a(\frac{\cdot}{\varepsilon})}\left(-\int_0^{\cdot} f + \frac{\int_0^1 f \frac{1}{a(\frac{\cdot}{\varepsilon})}}{\int_0^1 \frac{1}{a(\frac{\cdot}{\varepsilon})}}\right)}_{\to 0 \ \forall a \text{ by erg. thm}}. \quad \triangle$$

Convergence rate: need quant error estimate for $\int_0^1 f \frac{1}{a(\frac{\cdot}{\varepsilon})} \xrightarrow{\varepsilon \downarrow 0} \mathbb{E}\left[\frac{1}{a}\right] \int_0^1 f$

$\to$ need quant. erg., e.g. mixing.

Here explicit: $\mathbb{E}\left|\int_0^1 f \frac{1}{a(\frac{\cdot}{\varepsilon})} - \underbrace{\mathbb{E}\left[\frac{1}{a}\right]\int_0^1 f}\right|^2 = \mathrm{Var}\left[\int_0^1 f \frac{1}{a(\frac{\cdot}{\varepsilon})}\right]$

$$= \int_0^1 dx \int_0^1 dy \; f(x) f(y) \underbrace{\mathrm{Cov}\left[\frac{1}{a(\frac{x}{\varepsilon})}, \frac{1}{a(\frac{y}{\varepsilon})}\right]}_{\text{stat: } \mathrm{Cov}\left[\frac{1}{a(\frac{x-y}{\varepsilon})}, \frac{1}{a(0)}\right]}$$

$x \to \varepsilon x + y$
$y \to y$

$$= \varepsilon \int_0^{\frac{1}{\varepsilon}} dx \int_0^1 dy \; \underbrace{f(y+\varepsilon x) f(y)}_{\leq \frac{1}{2}\left(|f(y+\varepsilon x)|^2 + |f(y)|^2\right)} \mathrm{Cov}\left[\frac{1}{a(x)}, \frac{1}{a(0)}\right].$$

$$\lesssim \varepsilon \left(\int_0^1 |f|^2\right) \cdot \underbrace{\int_{-\infty}^{\infty} dx \left|\mathrm{Cov}\left[\frac{1}{a(x)}, \frac{1}{a(0)}\right]\right|}.$$

$\to$ rate $O(\sqrt{\varepsilon})$ in $L^2(\Omega)$ if decay of correl of $\frac{1}{a}$

e.g. a α-mixing with integrable α-mix coeff

$$\Rightarrow \left| \mathrm{Cov}\left(\frac{1}{a(x)}, \frac{1}{a(0)} \right) \right| \leq \alpha(|x|, D=0).$$

Exercise:
$$\begin{cases} \mathbb{E}\left[|v_\varepsilon - \bar{v}|^2(x) \right]^{1/2} \lesssim \sqrt{\varepsilon} \\ \mathbb{E}\left[\left| v_\varepsilon' - \bar{v}'\left(1 + \left(\frac{\bar{a}}{a(\frac{\cdot}{\varepsilon})} - 1\right)\right) \right|^2(x) \right]^{1/2} \lesssim \sqrt{\varepsilon}. \end{cases}$$

Fluctuations: macro observables $\int g\, u_\varepsilon'$, $g \in C_c^\infty(\mathbb{R}^d)$.

$$= -\int g f \frac{1}{a}(\tfrac{\cdot}{\varepsilon})$$

ⓐ convergence of variance: $\mathrm{Var}\left[\varepsilon^{-1/2} \int g\, u_\varepsilon' \right] \to \int |g f|^2 \left(\int \mathrm{Cov}\left(\frac{1}{a(x)}, \frac{1}{a(0)} \right) dx \right)$

CLT

scaling

b) asymptotic normality: stronger mixing, use favorite proof ...

$$d_{TV}\left(\frac{\varepsilon^{-\frac{1}{2}} \int g\, u_\varepsilon' - \mathbb{E}[\cdot]}{\mathrm{Var}[\cdot]^{\frac{1}{2}}}, \mathcal{N} \right) \lesssim \sqrt{\varepsilon}.$$

standard Gaussian

No lecture on Friday 8/1.

Next lecture: Monday 11/1!