M285K - course #20

V.2 Large-scale $C^{1,\alpha}$ reg.

Compare to:

$$\fint_{B_r} |\nabla\bar{u} - \nabla\bar{u}(0)|^2 \lesssim \left(\frac{r}{R}\right)^2 \fint_{B_R} |\nabla\bar{u}|^2.$$

Proposition. $\exists\, \varepsilon > 0$ such that the following holds:

Let $R > 0$, let $u \in H^1(B_R)$ be a-harm.

$$-\nabla \cdot a\nabla u = 0 \text{ in } B_R.$$

Then $\exists\, e \in \mathbb{R}^d$ $\forall r \leq R$

$$\fint_{B_r} |\nabla u - (\nabla\varphi_e + e)|^2 \leq \left[\left(\frac{r}{R}\right)^2 + \delta_R^{2\varepsilon} \left(\frac{R}{r}\right)^{d+2} \right] \fint_{B_R} |\nabla u|^2.$$

where $\delta_R := \frac{1}{R} \left(\fint_{B_R} \left| (\varphi, \sigma) - \fint_{B_R} (\varphi, \sigma) \right|^2 \right)^{\frac{1}{2}}.$

Moreover $|e|^2(1 - C\delta_R) \leq \fint |\nabla\varphi_e + e|^2 \leq |e|^2(1 + \delta_R)$

Moreover, $M(1+\sigma_R) \sim \int_{B_{R/2}} |\nabla\varphi_e + e| \lesssim R^d (1+\sigma_R)$.

Proof. By scaling: $R=1$. WLOG: $\fint_B (\varphi,\sigma) = 0$.

Idea: approximate $u \sim \bar{u} + \varphi_i \partial_i \bar{u}$

$$\begin{cases} -\nabla\cdot \bar{a}\nabla\bar{u} = 0 & \text{in } B_R \\ \bar{u} = u & \text{on } \partial B_R \end{cases}$$

Recall: $\int_B |\nabla\bar{u}|^2 \lesssim \int_B |\nabla u|^2$.

Step 1. PDE ingredient: (a) Weighted energy estimate:

$\forall\, 0 < \varepsilon \ll 1$: $\begin{cases} -\nabla\cdot a\nabla w = \nabla\cdot g & \text{in } B \\ w \in H^1_0(B) \end{cases}$

$\Rightarrow \int (1-|x|)^{\varepsilon} |\nabla w|^2 \lesssim \int (1-|x|)^{\varepsilon} |g|^2.$

(a') Hardy ineqn:

$$w \in H^1_0(B) \Rightarrow \int_B (1-|x|)^{\varepsilon-2}|w|^2 \lesssim \int_B (1-|x|)^{\varepsilon}|\nabla w|^2.$$

(b) Inner regularity for homog eqn:

$$-\nabla\cdot\bar{a}\nabla\bar{u} = 0 \text{ in } B$$

$$\Rightarrow \sup_{B_{1-\rho}} \left(|\nabla\bar{u}| + \rho|\nabla^2\bar{u}|\right) \lesssim \left(\rho^{-d}\int_B |\nabla\bar{u}|^2\right)^{1/2}.$$

<u>Step 3</u>: Eqn for 2-scale expansion error

$$w = u - \bar{u} - \eta\,\varphi_i \nabla_i \bar{u}$$

where cut-off η : $\begin{cases} \eta|_{B_{1-2\rho}} = 1 \\ \eta|_{\mathbb{R}^d \setminus B_{1-\rho}} = 0 \\ |\nabla\eta| \lesssim 1/\rho . \end{cases}$ ($\rho \overset{\leq \frac{1}{4}}{>} 0$ thickness to be optimized).

$$
C_v - \nabla \cdot a \nabla w = \nabla \cdot \Big[(1-\eta)(a-\bar{a})\nabla\bar{u} + (\varphi_i a - \sigma_i)\nabla(\eta \nabla_i \bar{u}) \Big].
$$

Step 4. $\Big[\int_B (1-|x|)^\varepsilon |\nabla w|^2 \lesssim (\rho^\varepsilon + \delta_1 \rho^{-d-2}) \int |\nabla u|^2 \Big]$

where $\delta_1 := \fint_B |(\varphi,\sigma) - \cancel{\fint_B (\varphi,\sigma)}|^2 = \fint_B |\varphi,\sigma|^2$.

Weighted energy est:

$$\int_B (1-|x|)^\varepsilon |\nabla w|^2 \lesssim \int_B (1-|x|)^\varepsilon \Big(\underbrace{(1-\eta)}(a-\bar a)\nabla \bar u + (\varphi_i a - \sigma_i)\nabla(\eta \nabla_i \bar u)\Big)^2 .$$

$$\lesssim \underbrace{\int_{B\setminus B_{1-2\rho}} \underbrace{(1-|x|)^\varepsilon}_{\lesssim \rho^\varepsilon} |\nabla \bar u|^2}_{\substack{\lesssim \rho^\varepsilon \int_B |\nabla \bar u|^2 \\ \lesssim \rho^\varepsilon \int_B |\nabla u|^2 .}} + \underbrace{\underbrace{\int_{B_{1-\rho}} |(\varphi,\sigma)|^2}_{\lesssim \delta_1} \underbrace{|\nabla(\eta \nabla_i \bar u)|^2}_{\substack{\le |\nabla^2 \bar u|^2 + \rho^{-2}|\nabla \bar u|^2 \\ \lesssim \rho^{-2-d}\int_B |\nabla \bar u|^2 \\ \text{by inner reg.} \\ \lesssim \rho^{-2-d}\int_B |\nabla u|^2}}}_{\lesssim \delta_1 \rho^{-d-2} \int_B |\nabla u|^2 .}$$

Step 5. **Conclusion.**

Consider: $u - \bar{u}(0) - (x_i + \varphi_i) \nabla_i \bar{u}(0)$

a-harm!

by Caccioppoli

$$\fint_{B_r} \left| \nabla u - (\nabla \varphi_i + e_i) \nabla_i \bar{u}(0) \right|^2 \lesssim r^{-2} \fint_{B_{2r}} \Big| \underbrace{u - \bar{u}(0) - (x_i + \varphi_i) \nabla_i \bar{u}(0)}_{} \Big|^2 .$$

$$= \underbrace{(u - \bar{u} - \varphi_i \nabla_i \bar{u})}_{= w \text{ on } B_{1-2\rho}}$$

$$+ \bar{u} - \bar{u}(0) - x_i \nabla_i \bar{u}(0)$$

$$+ \varphi_i (\nabla_i \bar{u} - \nabla_i \bar{u}(0))$$

$(2r \leq 1 - 2\rho)$

$\boxed{r, \rho \leq \frac{1}{4}}$

$$\lesssim r^{-2} \fint_{B_{2r}} |w|^2 + r^{-2} \underbrace{\sup_{B_{2r}} |\bar{u} - \bar{u}(0) - x_i \nabla_i \bar{u}(0)|^2}_{\substack{\lesssim r^4 \sup_{B_{2r}} |\nabla^2 \bar{u}|^2 \\ \lesssim r^4 \int_B |\nabla \bar{u}|^2 . \\ \text{inner reg.}}} + r^{-2} \overbrace{\sup_{B_{2r}} |\nabla_i \bar{u} - \nabla_i \bar{u}(0)|^2}^{\lesssim r^2 \int_B |\nabla \bar{u}|^2 .} \times \underbrace{\fint_{B_{2r}} |\varphi|^2}_{\substack{\lesssim r^{-d} \int_B |\varphi|^2 \\ \lesssim r^{-d} \delta_1 .}}$$

$$\lesssim \underbrace{r^{-2} \fint_{B_{2r}} |w|^2}_{\substack{\lesssim r^{-2-d} \int_B |w|^2 (1-|x|)^{\varepsilon-2} \\ \lesssim r^{-2-d} \int_B (1-|x|)^{\varepsilon} |\nabla v|^2 \text{ by Hardy!} \\ \lesssim r^{-d-2} \left(\rho^{\varepsilon} + \delta_1 \rho^{-d-2}\right) \int_B |\nabla u|^2}} + \left(r^2 + r^{-d}\delta_2\right) \underbrace{\int_B |\nabla \bar{u}|^2}_{\lesssim \int_B |\nabla u|^2}.$$

$$\lesssim \left[r^2 + r^{-d-2}\left(\rho^{\varepsilon} + \delta_1 \rho^{-d-2}\right)\right] \int_B |\nabla u|^2.$$

$$\forall\, r, \rho \leq \tfrac{1}{4}.$$

Optimize in ρ: if $\delta_1 \ll 1$, $\rho = \delta_1^{\frac{1}{d+2+\varepsilon}}$

$$C_0 \lesssim \left[r^2 + r^{-d-2}\, \delta_1^{\frac{\varepsilon}{d+2+\varepsilon}} \right] \int_B |\nabla u|^2 .$$

$2\varepsilon'$

Step 6. Non-degeneracy. $|e|^2(1 - C\delta_1) \lesssim \int_{B_{1/2}} |\nabla \varphi_e + \underset{\sim}{e}|^2 \lesssim |e|^2 (1 + \underset{\sim}{\delta_1})$

Upper bound: $\displaystyle\int_{B_{1/2}} |\nabla\varphi_e + e|^2 \lesssim \int_B |\varphi_e + \underset{\sim}{e}x|^2$ Caccioppoli.

$$\lesssim |e|^2 + |e|^2 \underbrace{\int_B |\varphi|^2}_{\delta_2} !$$

Lower bound: $\displaystyle\int_{B_{1/2}} |\nabla\varphi_e + e|^2 \gtrsim \int_{B_{1/2}} \Big| \varphi_e + \underset{\sim}{e}x - \fint_{B_{1/2}} \varphi_e \Big|^2$ Poincaré.

$$\geq |e|^2 - \underbrace{C \int_{B_{1/2}} |\varphi_e - ⨍ \varphi_e|^2}_{\leq C \int_B |\varphi_e|^2 = C\delta_1 .}$$

$\sim$

$\square$

<u>Main theorem</u> (large-scale $C^{1,\alpha}$ reg for a-harm. fcts).

$\forall \alpha \in (0,1)$ $\exists C_0 < \infty$ large such that the following holds:

Let $r_* = \inf \left\{ r \geq 0 : \forall R \geq r , \underbrace{\frac{1}{R}\left(⨍_{B_R} |(\varphi,\sigma) - ⨍_{B_R} (\varphi,\sigma)|^2 \right)^{1/2}}_{\equiv \delta_R} \leq \frac{1}{C_0} \right\}$

Given $R \geq r_*$, let $u \in H^1(B_R)$ be a-harm.

$$-\nabla\cdot a\nabla u = 0 \quad \text{in } B_R$$

Thm: (i) Lipschitz reg: $\forall\, r_* \le r \le R$,

$$\fint_{B_r} |\nabla u|^2 \lesssim \fint_{B_R} |\nabla u|^2$$

(ii) $C^{1,\alpha}$ reg: $\forall\, r_* \le r \le R$,

$$\mathrm{Exc}(\nabla u, B_r) \lesssim \left(\frac{r}{R}\right)^{2\alpha} \mathrm{Exc}(\nabla u, B_R)$$

where
$$\mathrm{Exc}(\nabla u, B_r) := \inf_{e \in \mathbb{R}^d} \fint_{B_r} \Big|\nabla u - \underbrace{(\nabla\phi_e + e)}_{\text{"effective grad of } \nabla u\text{"}}\Big|^2 .$$

Moreover: non-degeneracy on $r \ge r_*$

$$|e|^2 \lesssim \fint_{B_r} |e + \nabla\phi_e|^2 \lesssim |e|^2 .$$

Proof. $-\nabla\cdot a\nabla u = 0$ on B_R, $R \geq r_*$.

Step 1: Proof of $C^{1,\alpha}$ reg. by Campanato iteration.

By Proposition: $\quad \forall\, r \leq R' \leq R$

$$\mathcal{E}_x(\nabla u, B_r) \leq C\left(\left(\frac{r}{R'}\right)^2 + \delta_{R'}^{2\varepsilon}\left(\frac{R'}{r}\right)^{d+2}\right) \fint_{B_{R'}} |\nabla u|^2 .$$

Apply it to u replaced by $u - (\varphi_e + e\cdot x)$:

$$\mathcal{E}_x(\nabla u, B_r) \leq C\left(\left(\frac{r}{R'}\right)^2 + \delta_{R'}^{2\varepsilon}\left(\frac{R'}{r}\right)^{d+2}\right) \mathcal{E}_x(\nabla u, B_{R'}).$$

Size of one iteration step $\theta = \frac{\ }{R'}$.

Given $\alpha \in (0,1)$, choose θ small s.t. $C\theta^2 \leq \frac{1}{2}\theta^{2\alpha}$

choose C_0 large s.t. $C C_0^{-2\varepsilon} \theta^{-d-2} \leq \frac{1}{2}\theta^{2\alpha}$.

$\Rightarrow \forall\ R' \overset{\leq R}{\geq} r_*$: $\mathrm{Exc}(\nabla u, B_{\theta R'}) \leq \theta^{2\alpha}\, \mathrm{Exc}(\nabla u, B_{R'})$.

$\Rightarrow \forall\ R' \leq R,\ \theta^{n-1}R' \geq r_*$:

$$\mathrm{Exc}(\nabla u, B_{\theta^n R'}) \leq (\theta^n)^{2\alpha}\, \mathrm{Exc}(\nabla u, B_{R'}).$$

For $r_* \leq r \leq R$: choose n s.t. $\theta^{n+1}R \leq r \leq \theta^n R$

$$C_d\, \mathrm{Exc}(\nabla u, B_r) \leq \theta^{-d}\, \mathrm{Exc}(\nabla u, B_{\theta^n R}).$$

$$\le \theta^{-d} \underbrace{\left(\theta^{m}\right)^{2\alpha}}_{\le \left(\frac{r}{R}\right)^{2\alpha} \theta^{-2\alpha}} \mathcal{E}_{2\alpha}(\nabla u, B_R)$$

$$\theta^{-d-2\alpha} \left(\frac{r}{R}\right)^{2\alpha} \mathcal{E}_{2\alpha}(\nabla u, B_R) .$$

Next time Wednesday March 3rd.