M285K - course #21

V.2 $C^{1,\alpha}$ large-scale regularity

Main result. $\forall \alpha \in (0,1)$ $\exists C_\alpha < \infty$ such that the following holds:

Set $r_* = r_*[C_\alpha]$. Given $R \geq r_*$, let $u \in H^1(B_R)$ be a-harm.

$$\left\{ -\nabla \cdot a \nabla u = 0 \text{ in } B_R. \right.$$

Then: (i) $C^{1,\alpha}$ regularity: $\forall r_* \leq r \leq R$

$$\mathrm{Exc}(\nabla u; B_r) \leq \left(\frac{r}{R}\right)^{2\alpha} \mathrm{Exc}(\nabla u; B_R)$$

(remember $\mathrm{Exc}(\nabla u; B_R) = \inf_{e \in \mathbb{R}^d} \fint_{B_R} |\nabla u - (\nabla \varphi_e + e)|^2$.)

(ii) Non-degeneracy: $\forall r \geq r_*$ $\forall e$,

$$|e|^2 \leq \fint_{B_r} |\nabla \varphi_e + e|^2 \lesssim |e|^2.$$

already proven.

(iii) Lipschitz regularity: $\forall r_* \leq r \leq R$,

$$\fint_{B_r} |\nabla u|^2 \lesssim \fint_{B_R} |\nabla u|^2 .$$

<u>Proof</u>: Remains to prove (iii).

① Effective gradient: $\forall r_* \leq r \leq R \quad \exists!\, e_r \in \mathbb{R}^d$
s.t. $\mathcal{E}_\alpha(\nabla u, B_r) = \fint_{B_r} |\nabla u - (\nabla\varphi_{e_r} + e_r)|^2$.

Uniqueness: assume that e_1 & e_2 both make it.

$$\text{Then } \mathcal{E}_\alpha(\nabla u; B_r) \leq \fint_{B_r} \underbrace{\Big|\nabla u - \Big(\overbrace{\nabla\varphi_{\frac{e_1+e_2}{2}} + \frac{e_1+e_2}{2}}^{\frac{1}{2}(\nabla\varphi_{e_1}+e_1) + \frac{1}{2}(\nabla\varphi_{e_2}+e_2)}\Big)}_{\frac{1}{2}(\nabla u - (\nabla\varphi_{e_1}+e_1)) + \frac{1}{2}(\nabla u - (\nabla\varphi_{e_2}+e_2))}\Big|^2$$

$$\left[\; \left|\frac{1}{2}(a+b)\right|^2 = \frac{1}{2}(a^2+b^2) - \frac{1}{4}(a-b)^2 \right.$$

$$= \frac{1}{2} \overbrace{⨍_{B_r} |\nabla u - (\nabla\varphi_{e_1}+e_1)|^2}^{\mathrm{Exc}(\nabla u, B_r)} + \frac{1}{2} \overbrace{⨍_{B_r} |\nabla u - (\nabla\varphi_{e_2}+e_2)|^2}^{\mathrm{Exc}(\nabla u, B_r)}$$

$$- \frac{1}{4} \underbrace{⨍_{B_r} |\nabla\varphi_{e_1-e_2} + e_1 - e_2|^2}.$$

$$\Rightarrow ⨍_{B_r} |\nabla\varphi_{e_1-e_2} + e_1 - e_2|^2 = 0 \quad \Rightarrow \text{by non-deg: } e_1 = e_2!$$

② Dependence on scales: $\left[\; |e_r - e_R|^2 \lesssim \mathrm{Exc}(\nabla u, B_R). \right.$

Suffices to show: $\left[\; \forall\, 2^{-N-1}R \le r \le 2^{-N}R : \; |e_r - e_{2^{-N}R}|^2 \lesssim \mathrm{Exc}(\nabla u; B_{2^{-N}R}). \right]$ $(✡)$

Indeed: $|e_r - e_R|$ let us choose N: $2^{-N-1}R \le r \le 2^{-N}R$.

$$\le \underbrace{|e_r - e_{2^{-N}R}|}_{\substack{\lesssim \mathcal{E}xc(\nabla u, 2^{-N}R)^{\frac12} \\ \lesssim 2^{-N\alpha}\mathcal{E}xc(\nabla u, R)^{\frac12}}} + \sum_{n=0}^{N-1} \underbrace{|e_{2^{-n-1}R} - e_{2^{-n}R}|}_{\substack{\lesssim \mathcal{E}xc(\nabla u, 2^{-n}R)^{\frac12} \\ \lesssim 2^{-n\alpha}\mathcal{E}xc(\nabla u, B_R)^{\frac12}}}$$

$$\lesssim \mathcal{E}xc(\nabla u, B_R)^{\frac12}.$$

Remains to prove (✡). $\forall\, 2^{-N-1}R \le r \le 2^{-N}R$,

$$|e_r - e_{2^{-N}R}|^2 \lesssim \fint_{B_r} \left|(\nabla\varphi_{e_r} + e_r) - (\nabla\varphi_{e_{2^{-N}R}} + e_{2^{-N}R})\right|^2 \quad \text{by non-deg.}$$

$$\lesssim \underbrace{\fint_{B_r} |\nabla u - (\nabla\varphi_{e_r} + e_r)|^2}_{= \mathcal{E}xc(\nabla u, B_r)} + \underbrace{\fint_{B_r} |\nabla u - (\nabla\varphi_{e_{2^{-N}R}} + e_{2^{-N}R})|^2}_{\lesssim \fint_{B_{\dots}} |\nabla u - \dots|^2 \ (r \sim 2^{-N}R)}$$

$$\lesssim \mathcal{E}xc(\nabla u, B_{2^{-N}R}) \qquad = \mathcal{E}xc(\nabla u, B_{2^{-N}R}).$$
$$(r \sim 2^{-N}R)$$

③ Conclusion:

$$\fint_{B_r} |\nabla u|^2 \lesssim \underbrace{\fint_{B_r} |\nabla u - (\nabla \varphi_{e_r} + e_r)|^2}_{} + \underbrace{\fint_{B_r} |\nabla \varphi_{e_r} + e_r|^2}_{}$$

$\forall r_* \leq r \leq R.$

$$= \mathcal{E}xc(\nabla u, B_r)$$
$$\lesssim \left(\frac{r}{R}\right)^{2\alpha} \mathcal{E}xc(\nabla u, B_R)$$
by $C^{1,\alpha}$ reg.
$$\lesssim \left(\frac{r}{R}\right)^{2\alpha} \fint_{B_R} |\nabla u|^2$$

$$\lesssim |e_r|^2 \quad \text{non-deg.}$$
$$\lesssim |e_R|^2 + |e_r - e_R|^2$$
$$\lesssim |e_R|^2 + \mathcal{E}xc(\nabla u, B_R)$$
$$\lesssim \fint_{B_R} |\nabla \varphi_{e_R} + e_R|^2 + \mathcal{E}xc(\nabla u, B_R)$$
non-deg
$$\lesssim \fint_{B_R} |\nabla u|^2 + \underbrace{\mathcal{E}xc(\nabla u, B_R)}_{\lesssim \fint_{B_R} |\nabla u|^2}$$

$$\lesssim \fint_{B_R} |\nabla u|^2 .$$

$\square$

<u>Corollary</u> (Liouville principle).

Let a be stat. ergodic.
Almost surely, the following Liouville principle holds:

If u is a-harmonic on $\mathbb{R}^d$, <u>ie</u> $-\nabla\cdot a\nabla u = 0$ on $\mathbb{R}^d$

& is sub-quadratic, <u>ie</u> $\left(\exists \beta < 1 : R^{-2(1+\beta)} \fint_{B_R} |u|^2 \xrightarrow{R\uparrow\infty} 0.\right)$

Then u is a-linear, <u>ie</u> $u(x) = c + \varphi_e(x) + e\cdot x$
for some $c \in \mathbb{R}$,
$e \in \mathbb{R}^d$.

$$R^{-2\beta} \fint |\nabla u|^2 \quad \lesssim \quad R^{-2(1+\beta)} \int |u|^2$$

Proof. u sub-quadr $\Rightarrow \exists \beta<1$: $\quad R^{-\beta}\int_{B_R} |\nabla u| \lesssim \underbrace{R^{\cdots}\int_{B_R} |u|}_{\substack{\text{by Caccioppoli} \\ \xrightarrow{R\uparrow\infty} 0}}$.

$$\Rightarrow \nabla u \text{ is sublinear.}$$

$$\Rightarrow R^{-2\beta}\,\mathcal{E}_{\mathcal{L}}(\nabla u, B_R) \leq R^{-2\beta}\fint_{B_R} |\nabla u|^2 \xrightarrow{R\uparrow\infty} 0.$$

$$\Rightarrow \forall r \geq r_*: \quad \mathcal{E}_{\mathcal{L}}(\nabla u, B_r) \lesssim \underbrace{\left(\frac{r}{R}\right)^{2\beta}\mathcal{E}_{\mathcal{L}}(\nabla u, B_R)}_{\xrightarrow{R\uparrow\infty} 0}$$

$$\Rightarrow \mathcal{E}_{\mathcal{L}}(\nabla u, B_r) = 0$$

$$\Rightarrow \inf_e \fint_{B_r} |\nabla u - (\nabla\varphi_e + e)|^2 = 0$$

$\square$

Corollary. (large-scale $C^{1,\alpha}$ Schauder theory).

$\forall \alpha \in (0,1)$ $\exists C_\alpha < \infty$ such that the following holds:

Let $r_* = r_*[C_\alpha]$. Given $R \geq r_*$, let $h \in L^2(B_R)^d$, $u \in H^1(B_R)$ satisfy:

$$-\nabla \cdot a \nabla u = \nabla \cdot h \quad \text{in } B_R.$$

Then: (i) $C^{1,\alpha}$ regularity:

$$\sup_{r_* \leq r \leq R} \frac{1}{r^{2\alpha}} \operatorname{Exc}(\nabla u; B_r) \lesssim \frac{1}{R^{2\alpha}} \underbrace{\operatorname{Exc}(\nabla u; B_R)}^{\leq \fint_{B_R} |\nabla u|^2}$$

$$+ \sup_{r_* \leq r \leq R} \frac{1}{r^{2\alpha}} \fint_{B_r} \Big| h - \fint_{B_r} h \Big|^2.$$

In particular, if $R = \infty$, $h, \nabla u \in L^2(\mathbb{R}^d)^d$ satisfy:

$$\sup_{r \geq r_*} \frac{1}{r^{2\alpha}} \operatorname{Exc}(\nabla u; B_r) \lesssim \sup_{r \geq r_*} \frac{1}{r^{2\alpha}} \fint_{B_r} \Big| h - \fint_{B_r} h \Big|^2.$$

(ii) Lipschitz reg: $\forall \alpha>0$,

$$\sup_{r_* \le r \le R} \fint_{B_r} |Du|^2 \lesssim \fint_{B_R} |Du|^2 + \sup_{r_* \le r \le R} \frac{1}{r^{2\alpha}} \fint_{B_r} \Big|h - \fint_{B_r} h\Big|^2.$$

<u>Proof</u>. Let $\alpha' \in (\alpha, 1)$, choose $r_0 = r_*[C_{\alpha'}]$.

<u>Step 1</u> : <u>$C^{1,\alpha}$ regularity</u>.

① $\begin{cases} \forall r_* \le r \le \rho \le R \\ \mathcal{E}x(Du, B_r) \lesssim \left(\frac{r}{\rho}\right)^{2\alpha'} \mathcal{E}x(Du, B_\rho) + \left(\frac{\rho}{r}\right)^d \fint_{B_\rho} \Big|h - \fint_{B_\rho} h\Big|^2. \end{cases}$

Consider $\begin{cases} -\nabla \cdot a \nabla w = \nabla \cdot h & \text{in } B_\rho \\ w \in H^1_0(B_\rho) \end{cases}$ $\rightarrow$ $\displaystyle\int_{B_\rho} |\nabla w|^2 \lesssim \int_{B_\rho} \Big|h - \fint_{B_\rho} h\Big|^2$

energy estimate.

By def $u-w$ is a-harm in B_R

$\Rightarrow$ $C^{1,\alpha'}$ reg for a-harm fcts gives:

$$\mathcal{E}xc(\nabla u-\nabla w\,;B_r) \lesssim \left(\frac{r}{\rho}\right)^{2\alpha'} \mathcal{E}xc(\nabla u-\nabla w\,;B_\rho).$$

Hence: $$\mathcal{E}xc(\nabla u, B_r) = \inf_e \fint_{B_r} |\nabla u - (\nabla\varphi_e + e)|^2$$

$$\lesssim \mathcal{E}xc(\nabla u-\nabla w, B_r) + \fint_{B_r} |\nabla w|^2$$

$$\mathcal{E}xc(\nabla u-\nabla w, B_r) \lesssim \left(\frac{r}{\rho}\right)^{2\alpha'} \mathcal{E}xc(\nabla u-\nabla w, B_\rho)$$

$$\lesssim \left[\left(\frac{r}{\rho}\right)^{2\alpha'} \mathcal{E}xc(\nabla u, B_\rho)\right] + \left(\frac{r}{\rho}\right)^{2\alpha'} \fint_{B_\rho} |\nabla w|^2$$

$$\left(\frac{r}{\rho}\right)^{2\alpha'} \fint_{B_\rho} |\nabla w|^2 \lesssim \fint_{B_\rho} \Big|h - \fint_{B_\rho} h\Big|^2.$$

$$\fint_{B_r} |\nabla w|^2 \lesssim \left(\frac{r}{\rho}\right)^{-d} \fint_{B_\rho} |\nabla w|^2$$

$$\lesssim \left(\frac{r}{\rho}\right)^{-d} \fint_{B_\rho} \Big|h - \fint_{B_\rho} h\Big|^2$$

$B_\rho \qquad B_\rho$

② Conclusion.

Set $\theta = \frac{r}{\rho}$: $\forall \frac{1}{\theta} r_* \leq \rho \leq R$,

$$\mathrm{Exc}(\nabla u, B_{\theta\rho}) \lesssim \theta^{2\alpha'} \mathrm{Exc}(\nabla u, B_\rho) + \theta^{-d} ⨍_{B_\rho} \Big|h - ⨍_{B_\rho} h\Big|^2 .$$

Divide by $(\theta\rho)^{2\alpha}$ & take supremum over $\frac{1}{\theta} r_* \leq \rho \leq R$:

$$\sup_{r_* \leq r \leq \theta R} r^{-2\alpha} \mathrm{Exc}(\nabla u, B_r) \lesssim \theta^{2(\alpha' - \alpha)} \sup_{r_* \leq r \leq R} r^{-2\alpha} \mathrm{Exc}(\nabla u, B_r)$$

$$+ \theta^{-d-2\alpha} \sup_{r_* \leq r \leq R} r^{-2\alpha} ⨍_{B_r} \Big|h - ⨍_{B_r} h\Big|^2 .$$

Choose θ small enough:

$$\sup_{r_\delta \le r \le R} r^{-2\alpha} \mathcal{E}_{cc}(\nabla u, B_r) \lesssim_\theta \overbrace{\sup_{\theta R \le r \le R} r^{-2\alpha} \mathcal{E}_{cc}(\nabla u, B_r)}^{\lesssim_\theta R^{-2\alpha} \fint_{B_R} |\nabla u|^2}$$

$$+ \sup_{r_\delta \le r \le R} r^{-2\alpha} \fint_{B_r} \left| h - \fint_{B_r} h \right|^2$$

<u>Step 2</u>: Lipschitz $\to$ <u>exercise</u> $\square$