M285K - course #23

V.3 Large-scale & annealed L^p regularity.

Theorem (Annealed L^p reg.).

$\forall h \in C_c^\infty(\mathbb{R}^d; L^\infty(\Omega))^d$, if $\nabla u \in L^\infty(\Omega; L^2(\mathbb{R}^d))$ satisfies

$$-\nabla \cdot a \nabla u = \nabla \cdot h,$$

Then $\forall\, 1 < p, q < \infty$, $\forall \delta > 0$:

$$\| [\nabla u] \|_{L^p(\mathbb{R}^d; L^q(\Omega))} \lesssim_{p,q,\delta} \| [h] \|_{L^p(\mathbb{R}^d, L^{q+\delta}(\Omega))}.$$

$$[\nabla u](x) = \Big(\fint_{B(x)} |\nabla u|^2 \Big)^{\frac{1}{2}}.$$

Proof.

Step 1. $\forall D$ ball $\subset \mathbb{R}^d$, decompose $\nabla u = \nabla u_D^0 + \nabla u_D^1$

$\ldots \nabla u_D^0 = \nabla \cdot (h 1_D)$

$$\begin{cases} -\nabla\cdot a\nabla u_D = \nabla\cdot(h 1_D) \\ -\nabla\cdot a\nabla u_D^1 = \nabla\cdot(h 1_{\mathbb{R}^d\setminus D}). \end{cases}$$

&

$$\begin{cases} ① \left(\fint_D \mathbb{E}\left[\left(\fint_{B_\delta(x)} |\nabla u_D^\circ|^2\right)^{q/2}\right]\right)^{1/q} \lesssim_q \left(\fint_{6D} \mathbb{E}\left[\left(\fint_{B_\delta(x)} |h|^2\right)^{q/2}\right]\right)^{1/q} \quad \forall 1<q<\infty \\ ② \left(\fint_{\frac{1}{24}D} \mathbb{E}\left[\left(\fint_{B_\delta(x)} |\nabla u_D^1|^2\right)^{q/2}\right]^{p/q}\right)^{1/p} \lesssim_q \left(\fint_D \mathbb{E}\left[\left(\fint_{B_\delta(x)} |\nabla u_D^1|^2\right)^{q/2}\right]\right)^{1/q} \quad \forall 1<q<\infty,\ \forall 1\le p\le\infty \end{cases}$$

$\hookrightarrow$ use dual CZ lemma: $\forall 1<p,q<\infty$,

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_\delta(x)} |\nabla u|^2\right)^{q/2}\right]^{p/q} \lesssim_{p,q} \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_\delta(x)} |h|^{2}\right)^{q/2}\right]^{p/q}.$$

① Suffices to argue: $\fint \left(\fint |\nabla u^\circ|^2\right)^{q/2} \lesssim \fint \left(\fint |h|^2\right)^{q/2}$

① Suffices to prove: $\int_D \left(\int_{B_*(x)} |\nabla u_D|^2 \right)^{q/2} \lesssim \int_{6D} \left(\int_{B_*(x)} |h|^2 \right)^{q/2}$.

ⓐ $D = B(x_D, r_D)$, $r_D \geq \frac{1}{4} r_*(x_D)$.

$\hookrightarrow$ $\forall 1 < q < \infty$: by quenched result,

$$\int_{\mathbb{R}^d} \left(f_{B_*(x)} |\nabla u_D^\circ|^2 \right)^{q/2} dx \lesssim_q \int_{\mathbb{R}^d} \left(f_{B_*(x)} |h \mathbb{1}_D|^2 \right)^{q/2} dx$$

$$\lesssim \int_{6D} \left(f_{B_*(x)} |h|^2 \right)^{q/2} dx$$

by Lip. prop. & $r_*(x_D) \leq 4 r_D$.

ⓑ $r_D \leq \frac{1}{4} r_*(x_D)$:

$$f_{B_*(x_D)} |\nabla u_D^\circ|^2 \lesssim |B_*(x_D)|^{-1} \int_D |h|^2 \quad \text{by energy est.}$$

$$\lesssim f \left(f |h|^2 \right) dx \quad \rightarrow \text{conclude}$$

$\frac{1}{2}D \quad B_\delta(x_0)$

② $\forall x_0 \in \frac{1}{12}D$: Lip reg gives $\fint_{B_\delta(x_0)} |Du^1_D|^2 \lesssim \fint_D \left(\fint_{B_\delta(x)} |Du^1_D|^2\right) dx$]

$\Rightarrow \forall\, 2 \le q < \infty$, $1 \le p \le \infty$: $\fint_{\frac{7}{12}D} \mathbb{E}\left[\left(\fint_{B_\delta(x)} |Du^1_D|^2\right)^{\frac{q}{2}}\right]^{p/q}$

$$\lesssim \mathbb{E}\left[\left(\fint_D \left(\fint_{B_\delta(x)} |Du^1_D|^2\right)\right)^{\frac{q}{2}}\right]^{p/q}$$

$$\lesssim \mathbb{E}\left[\left(\fint_D \left(\fint_{B_\delta(x)} |Du^1_D|^2\right)^{\frac{q}{2}}\right)\right]^{p/q}$$ by Jensen $q \ge 2$.

$\to$ need another argument for $q < 2$:

Caccioppoli type reg.: $\left[\forall x_0 \in \frac{1}{2}D\right.$: $\left(\fint |Du^1_D|^2\right)^{1/2} \lesssim \fint \left(\fint |Du^1_D|^2\right)^{\frac{1}{2}}$.

Suffice to prove : $\left[\ \ldots \frac{1}{24} \ldots \left(\fint_{B_\delta(x_0)} \ldots\right) \ldots \fint_D \left(\fint_{B_\delta(x)} \ldots\right)\right.$

a) if $r_D \leq 6 r_\delta(x_0)$: clear –

b) if $r_D > 6 r_\delta(x_0)$: $\fint_{B_\delta(x_0)} |\nabla u_D^1|^2 \overset{\text{hyp. on } \delta\text{'s}}{\lesssim} \fint_{\substack{B(x_0, r_\delta(x_0) + \frac{1}{24} r_D) \\ \subseteq \frac{1}{4}D}} |\nabla u_D^1|^2 \lesssim \fint_{\frac{1}{4}D} |\nabla u_D^1|^2$

Moreover, since u_D^1 is a-harm in $\frac{1}{2}D$: get $\fint_{\frac{1}{4}D} |\nabla u_D^1|^2 \lesssim \left(\fint_{\frac{1}{2}D} |\nabla u_D^1|\right)^2$. ($\star$)

So $\fint_{B_\delta(x)} |\nabla u_D^1|^2 \lesssim \left(\fint_{\frac{1}{2}D} |\nabla u_D^1|\right)^2 \lesssim \left[\fint_D \left(\fint_{B_\delta(x)} |\nabla u_D^1|^2\right)^{\frac{1}{2}} dx\right]^2$.

Proof of ($\star$) : $\left[-\nabla \cdot a \nabla w = 0 \text{ in } B \right.$

then $\displaystyle\fint_{\frac{1}{2}B} |Dw|^2 \lesssim \left(\fint_B |Dw|\right)^2.$

Assume $\displaystyle\fint_B w = 0$, Let χ cut-off $\begin{cases} \chi = 1 \text{ on } \frac{1}{2}B \\ \chi = 0 \text{ on } \mathbb{R}^d \setminus B. \end{cases}$

* $$\fint_{\frac{1}{2}B} |Dw|^2 \lesssim \fint_B |D(\chi w)|^2$$
$$\lesssim \fint_B |D\chi|^2 |w|^2 + \fint_B \chi^2 |Dw|^2$$
$$\lesssim \fint_B |D\chi|^2 |w|^2. \quad \text{by Caccioppoli.}$$

* $$\left(\int |\chi w|^p\right)^{\frac{1}{p}} \lesssim \left(\int |D(\chi w)|^2 + |\chi w|^2\right)^{\frac{1}{2}}$$ by Sobolev for some $p > 2$ large.

$$\lesssim \left(\int_B |D(\chi w)|^2 \right)^{\frac{1}{2}} \qquad \text{by Poincaré}$$

$$\lesssim \left(\int_B |D\chi|^2 |w|^2 \right)^{\frac{1}{2}} \qquad \text{by Caccioppoli.}$$

$$* \left(\int_B |D\chi|^2 |w|^2 \right)^{\frac{1}{2}} , \quad \begin{cases} \chi = \xi^q , \quad q = 2\,\frac{p-1}{p-2} \\ |D\chi| \lesssim \xi^{q-1}. \end{cases}$$

$$\lesssim \left(\int_B \xi^{2(q-1)} |w|^2 \right)^{\frac{1}{2}} = \left(\int_B |\xi^q w|^{\frac{2}{q'}} |w|^{\frac{2}{q}} \right)^{\frac{1}{2}}$$

$$\overset{\text{Hölder}}{\lesssim} \left(\int_B |\chi w|^p \right)^{\frac{1}{2(p-1)}} \left(\int_B |w| \right)^{\frac{1}{2}}.$$

$$\left(\int |D \ldots|^2 |w|^2 \right)^{\frac{q-1}{2q}} \left(\int |w| \right)^{\frac{1}{2}}$$

$$\lesssim \left(\int_B |\nabla \chi| \, |w| \right) \left(\int_B |w| \right) .$$

$$\leadsto \int |\nabla \chi|^2 |w|^2 \lesssim \left(\int |w| \right)^2 \lesssim \left(\int_B |Dw| \right)^2 .$$

<u>Conclusion up to here</u> : $\forall \, 1 < p, q < \infty$,

$$\left[\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_*(x)} |Du|^2 \right)^{\frac{q}{2}} \right]^{p/q} \lesssim \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_*(x)} |h|^2 \right)^{\frac{q}{2}} \right]^{p/q} .\right.$$

<u>Step 2.</u> $\forall \, 1 \leq p \leq q \leq \infty$

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_*(x)} |h|^2 \right)^{\frac{q}{2}} \right]^{p/q} \begin{cases} \lesssim_{p,q,r} \int_{\mathbb{R}^d} \mathbb{E}\left[[h]^r \right]^{p/r} & \forall r > q \\ \int \mathbb{E}\left[[h]^r \right]^{p/r} & \forall \end{cases}$$

$$\lesssim_{p,q,r} \int_{\mathbb{R}^d} \mathbb{E}\left[|h|^r \right]^{q} \quad \forall r < q.$$

Upper bound:

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_{r_*}(x)} |h|^2 \right)^{q/2} \right]^{p/q}$$

condition on r_*

$$\lesssim \int_{\mathbb{R}^d} \mathbb{E}\left[\underbrace{\sum_{n=0}^{\infty}}_{\text{}} \mathbb{1}_{2^{n-1} \le r_* \le 2^n - 1} \left(\fint_{B_{2^n}(x)} |h|^2 \right)^{q/2} \right]^{p/q} \geq 1$$

$r_* \sim 2^n$

$$\left(\sum_{n=0}^{\infty} \alpha_n \right)^{p/q} = \left(\sum_{n=0}^{\infty} \underbrace{2^{-n}} 2^n \alpha_n \right)^{p/q}$$

$$\overset{\text{Jensen}}{\le} \sum_{n=0}^{\infty} 2^{-n} 2^{n\frac{p}{q}} \alpha_n^{p/q}.$$

$$\lesssim \sum_n 2^{n\left(\frac{p}{q}-1\right)} \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\left(\fint_{B_{2^n}(x)} |h|^2 \mathbb{1}_{r_* \sim 2^n}\right)^{q/2}\right]^{p/q}.$$

$$\left(\fint_{B_R(x)} |h|^2\right)^{q/2} \lesssim \begin{cases} q>2: \text{ Jensen} \to \fint_{B_R(x)} [h]^q \\ q<2: \ \ell^1\text{-}\ell^2 \text{ est} \to R^{dq\left(\frac{1}{q}-\frac{1}{2}\right)} \fint_{B_{2R}(x)} [h]^q. \end{cases}$$

$$\lesssim \sum_n 2^{\beta n} \underbrace{\int_{\mathbb{R}^d} \mathbb{E}\left[[h]^q \mathbb{1}_{r_* \sim 2^n}\right]^{p/q}}_{\lesssim \int_{\mathbb{R}^d} \mathbb{E}\left[[h]^r\right]^{p/r} \underbrace{\mathbb{P}\left[r_* \sim 2^n\right]^{\frac{p}{qr}(r-q)}}_{\lesssim C_\alpha 2^{-n\alpha} \ \forall \alpha>0} \ \forall r>q}$$

because $\mathbb{E}\, r_*^s < \infty \ \forall s$.

$\square$

RECAP :

Chapter II : qual. theory

↳ general methods, systematic theory based on "2-scale" weak conv.

Chapter III & IV : quant theory

↳ fine description of osc & fluct. + optimal rates.

Ingredients : – Malliavin calculus : unravel dependence on a "directly".

– annealed L^p regularity : estimate borderline decay of dependence on a.

Simplified setting : Gaussian + ~~integrable~~ correlation

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Chapter V : notion of large-scale regularity.