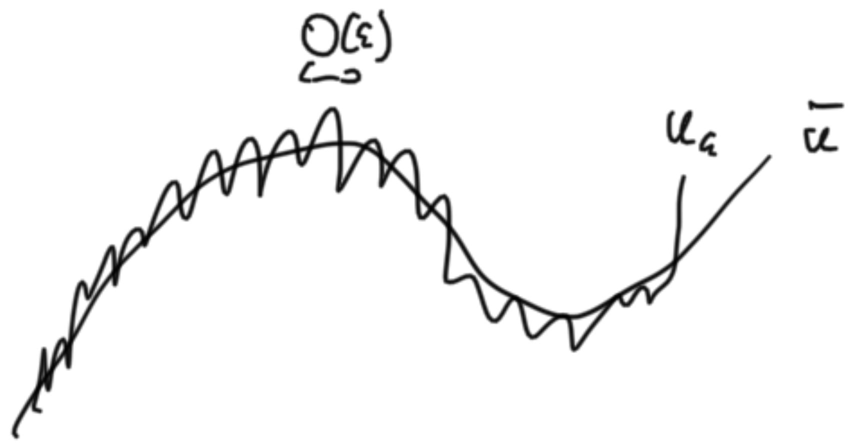
M285K - course #3

II. QUALITATIVE THEORY.

II.1 Formal 2-scale expansion

$$\left[\, -\nabla\cdot a\left(\tfrac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon = \nabla\cdot f \quad \text{in } \mathbb{R}^d, \quad f\in L^2(\mathbb{R}^d). \right.$$



2 scales : $\begin{cases} \text{micro} & O(\varepsilon) \\ \text{macro} & O(1) \end{cases}$

Ansatz : $u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\, u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots$

where $u_0, u_1, \ldots$ are $\begin{cases} \text{smooth in } x \text{ variable} \\ \text{stationary in } \frac{x}{\varepsilon} \equiv y, \end{cases}$ $\left(\begin{array}{l} u_n(x,y;a) \\ = u_n(x,0;a(\cdot+y)) \end{array} \right)$

Equation becomes: $\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$.

$$\hookrightarrow \; -\left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right)\cdot a(y)\left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right)\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \dots\right) = \nabla_x \cdot f(x).$$

$$= -\frac{1}{\varepsilon^2}\nabla_y \cdot a(y)\nabla_y u_0$$

$$- \frac{1}{\varepsilon}\left(\nabla_y \cdot a(y)\nabla_x u_0 + \cancel{\nabla_x \cdot a(y)\nabla_y u_0} + \nabla_y \cdot a(y)\nabla_y u_1\right)\Big]$$

$$- \left(\nabla_x \cdot a(y)\nabla_x u_0 + \nabla_y \cdot a(y)\nabla_x u_1 + \nabla_x \cdot a(y)\nabla_y u_1 + \nabla_y \cdot a(y)\nabla_y u_2\right)$$

$$- \dots$$

Separate the orders of magnitude:

* $O(\frac{1}{\varepsilon^2})$: $-\nabla_y \cdot a(y)\nabla_y u_0 = 0$

$\Rightarrow \nabla_y u_0 = 0 \rightarrow u_0 = u_0(x)$

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$* \; O\left(\frac{1}{\varepsilon}\right): \; -\nabla_y \cdot a(y) \nabla_x u_0(x) - \nabla_y \cdot a(y) \nabla_y u_1 = 0$

$$\left[-\nabla_y \cdot a(y) \left(\nabla_x u_0(x) + \nabla_y u_1(x,y) \right) = 0 \right]$$

Auxiliary problem: $-\nabla \cdot a \left(e_j + \nabla \varphi_j\right) = 0$ in $\mathbb{R}^d$.

& [assume for the moment that $\exists !$ solution φ_j stationary.

$\hookrightarrow u_1(x,y) \equiv \sum_{j=1}^{d} \varphi_j(y) \, \nabla_j u_0(x).$

$= \varphi_j(y) \, \nabla_j u_0(x)$ (Einstein's convention)

$* \; O(1): \; -\nabla_y \cdot a(y) \nabla_x u_0 - \nabla_x \cdot a(y) \nabla_y u_1 \; [-\nabla_y \cdot a(y) \nabla_x u_1]$

$$\left[- \nabla_y \cdot a(y) \nabla_y u_2 \right] = \nabla \cdot f$$

Take the expectation $\mathbb{E}$ & use $\mathbb{E}\, e_j \cdot \nabla_y \overset{\text{stat}}{v}(y)$

$$= \lim_{h \to 0} \mathbb{E} \; \frac{v(y + e_j h) - v(y)}{h} = 0 .$$

$$C_0 \quad - \nabla_x \cdot \mathbb{E}\left[a(y) \left(\nabla_x u_0 + \underbrace{\nabla_y u_1}_{\nabla \varphi_j(y) \nabla_j u_0(x)} \right) \right] = \nabla \cdot f .$$

$$- \nabla_x \cdot \underbrace{\mathbb{E}\left[a \left(e_j + \nabla \varphi_j \right) \right]}_{\bar{a} e_j} \nabla_j u_0(x) = \nabla \cdot f$$

$$-\nabla\cdot \underbrace{a}\,\nabla u_0 = \nabla\cdot f\,.$$

<u>Conclusion.</u>

Expect $\nabla u_\varepsilon \rightharpoonup \nabla \bar u$ in $L^2(\mathbb{R}^d)$ (qual.) homog result

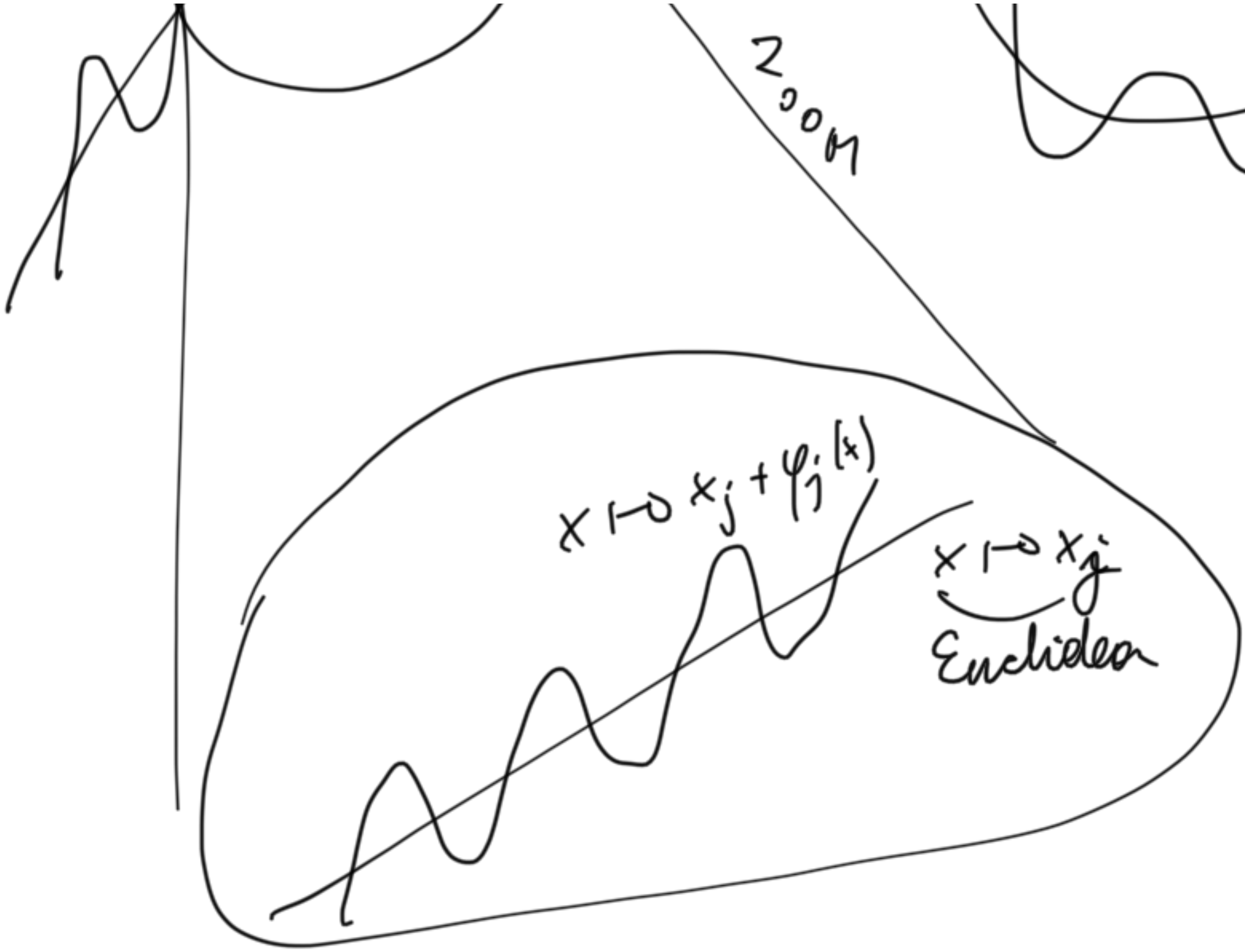
& $-\nabla\cdot \bar a \nabla \bar u = \nabla\cdot f$ in $\mathbb{R}^d$ homogenized eqn

$\bar a e_j = \mathbb{E}\left[a(\nabla\varphi_j + e_j)\right]$ cell formula

$\boxed{-\nabla\cdot a(\nabla\varphi_j + e_j) = 0 \text{ in } \mathbb{R}^d.}$ corrector equation

$\varphi = \begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_d\end{pmatrix}.$

<u>Geometric interpretation.</u>



$-\Delta x_j = 0$

$-\nabla \cdot a \nabla (x_j + \varphi_j) = 0$

$\Rightarrow$ corrector φ_j

= correction to Eucl. coord to make them a. harm.

$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon \varphi_j\left(\frac{x}{\varepsilon}\right) \underbrace{\nabla_j \bar{u}(x)}_{\text{slope of limit profile.}} + O(\varepsilon^2)$$

2-scale expansion for u_ε

$$\nabla u_\varepsilon(x) = \left(e_j + \nabla \varphi_j\left(\frac{x}{\varepsilon}\right)\right) \nabla_j \bar{u}(x)$$

expected accurate in $L^2(\mathbb{R}^d)$.

$\to 0$

<u>Justification in 1D</u>: $\quad \bar{a} = \mathbb{E}\left[\frac{1}{a}\right]^{-1}.$

Corrector eqn $\quad -\left(a(\varphi'+1)\right)' = 0$

$$a(\varphi'+1) = cst$$

$$\varphi' = \frac{cst}{a} - 1$$

$$\mathbb{E}\varphi' = 0 \Rightarrow \varphi' = \mathbb{E}\left[\frac{1}{a}\right]^{-1}\left(\frac{1}{a} - \mathbb{E}\left[\frac{1}{a}\right]\right).$$

$$\hookrightarrow \bar{a} = \mathbb{E}\left[a(\varphi'+1)\right] = \mathbb{E}\left[\frac{1}{a}\right]^{-1}. \quad \underline{\text{ok}}.$$

<u>Rem</u>. Homog provides reduction in complexity:

(1) solve corrector eqn: φ, $\bar{a}$

↳ off-line computation

(2) $\left[-\nabla\cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon = \nabla\cdot g\right]$

$\Rightarrow$ deduce approximation for free: $\begin{cases} u_\varepsilon = \bar{u} + \varepsilon\,\varphi_j(\tfrac{\cdot}{\varepsilon})\,\partial_j\bar{u} \\ \left[-\nabla\cdot \bar{a}\nabla\bar{u} = \nabla\cdot g\right] \end{cases}$

II.2 Stationary differential calculus.

Action $\tau = (\tau_x)_{x\in\mathbb{R}^d}$ of $(\mathbb{R}^d, +)$ on $(\Omega, \mathbb{P})$, $\tau_x a = a(\cdot + x)$

induces action $T = (T_x)_{x\in\mathbb{R}^d}$ on $L^p(\Omega)$, $(T_x F)(a)$

$= F(\tau_x a)$.

(Koopman operator)

Observation. Stationarity yields canonical isomorphism

$L^p(\Omega) \cong L^p(\mathbb{R}^d \times \Omega) = \{ \psi \in L^p_{\mathrm{loc}}(\mathbb{R}^d, L^p(\Omega)) random variables

τ-stat random fields

s.t. $v(x,\omega) = v(0,\tau_x\omega)$ }

1) $L^p(\Omega) \xrightarrow{\sim} L^p_\tau(\mathbb{R}^d\times\Omega)$: $F \mapsto F^\#$

with $F^\#(x,\omega) = F(\tau_x\omega) = T_x F(\omega)$.

2) $L^p_\tau(\mathbb{R}^d\times\Omega) \xrightarrow{\sim} L^p(\Omega)$: $v \mapsto v^b$

with $v^b(\omega) = v(0,\omega)$.

$$\left(\text{e.g. } \mathbb{E}\int_Q |F^\#(x,\cdot)|^p\,dx = \int_Q \underbrace{\mathbb{E}\left[|F^\#(x,\cdot)|^p\right]}_{\substack{\mathbb{E}\left[|F^\#(0,\cdot)|^p\right] \\ = \mathbb{E}\left[|F|^p\right]}}\,dx\,. \right)$$

Lemma. T is a (d-parameter) C_0-group of isometries on $L^p(\Omega)$ $(p<\infty)$.

Proof. • $T_0 = Id$, $T_x T_y = T_{x+y}$.

• $\|T_x F\|^p_{L^p(\Omega)} = \mathbb{E}\left[|F(\tau_x \omega)|^p\right] = \mathbb{E}\left[|F|^p\right] \to$ isometries.

• $\|T_x F - F\|_{L^p(\Omega)} \to 0$ as $|x| \to 0$ $\forall F \in L^p(\Omega)$.

Indeed: $\|T_x F - F\|^p_{L^p(\Omega)} = \mathbb{E}\,|F(\tau_x\omega) - F(\omega)|^p$

$$= \mathbb{E} \int_Q |F(\tau_{x+y}\omega) - F(\tau_y \omega)|^p dy$$

$$= \mathbb{E}\Big[\underbrace{\underbrace{\int_Q |F^\#(x+y,\omega) - F^\#(y,\omega)|^p dy}_{\xrightarrow{|x|\to 0} 0 \text{ because } y \mapsto F^\#(y,\omega) \text{ measurable } \forall' \omega.}\Big]}_{\to 0 \text{ as } |x| \to 0}$$

$\to 0$ as $|x| \to \infty$
by dominated convergence $\square$

<u>Corollary</u>. The generator ∇^{st} "stationary gradient" of T is a densely-defined skew-adjoint operator on $L^p(\Omega)$ $(p < \infty)$ with domain $\underbrace{W^{1,p}(\Omega)} = \{ F \in L^p(\Omega) :$
$\nabla_j^{st} F = \lim_{h \to 0} \frac{1}{h}(T_{he_j} F - F)$ exists in $L^p(\Omega)$ $\forall j = 1 \ldots d \}$.

More generally, define stationary Sobolev spaces $W^{s,p}(\Omega)$ as domain of $(\underbrace{-\nabla^{st} \cdot \nabla^{st}}_{\Delta^{st}})^{s/2}$

& $H^s(\Omega) = W^{s,2}(\Omega)$.

Lemma.
$$\begin{cases} W^{1,p}(\Omega) \cong W^{1,p}_{loc}(\mathbb{R}^d, L^p(\Omega)) \cap L^p_\tau(\mathbb{R}^d \times \Omega) \\ \quad \text{via canonical isomorphism.} \\ \& \quad (\nabla^{st} F)^\# = \nabla F^\# . \end{cases}$$

Proof. If $F \in W^{1,p}(\Omega)$,

Then $\quad D_j^{st} F = \lim_{h \to 0} \dfrac{T_{he_j} F - F}{h}$ in $L^p(\Omega)$

$\rightsquigarrow \forall \chi \in C_c^\infty(\mathbb{R}^d)$:

$$\int \chi \, (D_j^{st} F)^\# = \lim_{h \to 0} \int \chi \left(\frac{T_{he_j} F - F}{h} \right)^\# \quad \text{in } L^p(\Omega)$$

$\mathbb{R}^d \quad L^p_{loc}(\mathbb{R}^d; L^p(\Omega)) \quad h \to 0 \quad \int_{\mathbb{R}^d}$

$$= \lim_h \int_{\mathbb{R}^d} \chi \, \frac{F^\#(\cdot + he_j) - F^\#}{h}$$

$$= \lim_h \int_{\mathbb{R}^d} \frac{\chi(\cdot - he_j) - \chi}{h} \, F^\#$$

$$= -\int_{\mathbb{R}^d} (D_j \chi) \, F^\# \quad \text{in } L^p(\Omega).$$

$$\Rightarrow F^\# \in W^{1,p}_{loc}(\mathbb{R}^d; L^p(\Omega))$$

with $DF^\# = (D^{st}F)^\#$. $\square$

Malliavin derivative = derivative wrt ω.]

Stat der = der wrt τ]

$$F^{\#}(\underset{\varnothing}{x}, \underset{X}{a}) \, .$$