M285K - course #4

No course this Friday 15/1. & Monday holiday

Next one: Wednesday 20/1.

∇^{st} on $L^p(\Omega)$ generator of $x \mapsto T_x F = F(a(\cdot + x))$.

∇ on $L^p_{\tau}(\mathbb{R}^d \times \Omega)$ gen of $x \mapsto F^\#(\cdot + x, a)$

$\Big) \#$ pb

<u>Lemma</u> (corrector eqn.).

$$\left\{\begin{array}{l} \left\{\begin{array}{l} -\nabla \cdot a(\nabla\varphi_j + e_j) = 0 \text{ in } \mathbb{R}^d \text{ (weak) a.s.} \\ \varphi_j \in H^1_{loc}(\mathbb{R}^d; L^2(\Omega)) \\ \nabla\varphi_j \text{ stat.} \end{array}\right. \\ \Leftrightarrow \nabla\varphi_j = G_j^\# : \quad -\nabla^{st} \cdot a^b(G_j + e_j) = 0 \text{ in } \Omega \text{ (weak)} \\ \left(\text{i.e. } \left(\mathbb{E}\left[\nabla^{st} H \cdot a^b(G_j + e_j)\right] = 0 \quad \forall H \in H^1(\Omega).\right)\right) \end{array}\right.$$

Proof. $\boxed{\Rightarrow}$ $\forall h \in C_c^\infty(\mathbb{R}^d)$, $\forall H \in L^2(\Omega)$

$$\mathbb{E}\Big[H \int Dh \cdot a(\overset{G_j^\#}{D\varphi_j} + e_j)\Big] = 0$$

$$= \mathbb{E}\Big[\int_{\mathbb{R}^d} dy \; \underbrace{H^\#(0,a) \;\; Dh(y) \cdot a(y) \,\big(G_j^\#(y,a) + e_j\big)}_{\text{by stat: } H^\#(-y,\, a(\cdot+y)) \; Dh(y)\cdot a(y) \,(G_j^\#(0, a(\cdot+y)) + e_j)}\Big]$$

$$= \mathbb{E}\Big[\underbrace{\int_{\mathbb{R}^d} dy \;\; H^\#(-y, a) \;\; Dh(y)}\cdot a(0) \;\big(G_j^\#(0,a) + e_j\big)\Big]$$

$$= \mathbb{E}\Big[\underbrace{\Big(\int_{\mathbb{R}^d} Dh(y)\, H^\#(-y)\, dy\Big)}_{\int_{\mathbb{R}^d} h(y)\, \underset{(D^{st}H)^\#}{DH^\#}(-y)} \cdot \; a^b \;(G_j + e_j)\Big].$$

Take $h \to \delta_0$:

$$\mathbb{E}\left[\nabla^{st} H \cdot a^b (G_j + e_j) \right] = 0 .$$

$$\boxed{\Leftarrow} \quad \left[- \nabla^{st} \cdot a^b (G_j + e_j) = 0 . \right]$$

$\forall\, h \in C_c^\infty(\mathbb{R}^d)$, $H \in L^2(\Omega)$,

define $\vartheta_{h,H}(x) := \int_{\mathbb{R}^d} dy\ h(x-y)\, H^\#(y)$.

$$\vartheta_{h,H} \in \cancel{H^1_{loc}}\, C_b^\infty (\mathbb{R}^d ; L^2(\Omega)) \rightarrow \vartheta^b_{hH} \in H^1(\Omega)$$

$$\nabla^{st} \vartheta^b_{hH} = \int_{\mathbb{R}^d} dy\ Dh(-y)\, H^\#(y).$$

$$\mathbb{E}\left[\nabla^{st} \vartheta^b \cdot a^b (G_j + e_j) \right] = 0 .$$

$$= \dots = \mathbb{E}\left[H \int_{\mathbb{R}^d} Dh \cdot a (G_j^\# + e_j) \right]$$

a.s $\int \; Dh \quad (G_j^\# \quad) = ?$

$\to \int_{\mathbb{R}^d} \nabla h \cdot a\,(e_j + e_j) = 0.$ $\square$

<u>Careful</u>: PDEs on Ω are not an easy matter.

<u>Lemma</u> (lack of fctal tools on Ω).

Assume $\exists\, G_0 \in L^2(\Omega)$ <u>s.t.</u> $\begin{cases} G_0^\# \text{ is Gaussian field} \\ \mathbb{E}\, G_0^\# = 0 \\ \mathbb{E}\, G_0^\#(x)\, G_0^\#(y) = C_0(x-y), \; C_0 \in C^1. \end{cases}$

(i) Poincaré's inequality $\|F - \mathbb{E}F\|_{L^2(\Omega)} \leq C \|D^{st} F\|_{L^2(\Omega)}$ on $H^1(\Omega)$ is false $\forall C > 0$.

(ii) Rellich's theorem $H^m(\Omega) \subset\subset L^2(\Omega)$ is false $\forall m > 0$

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(iii) Sobolev's inequality $\|F\|_{L^q(\Omega)} \leq C \|F\|_{H^m(\Omega)}$ is false $\forall m > 0,\ q > 2,\ C > 0$.

Proof. (i) $F_n = n^{-d/2} \int_{\mathbb{R}^d} h\left(\frac{y}{n}\right) \sigma_0^{\#}(y)\, dy, \quad h \in C_c^\infty(\mathbb{R}^d)$.

$$\|F_n\|^2_{L^2(\Omega)} = n^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h\left(\frac{y}{n}\right) h\left(\frac{z}{n}\right) C_0(y-z)\, dy\, dz$$

$$\xrightarrow{n \to \infty} \left(\int_{\mathbb{R}^d} |h|^2\right) \left(\int_{\mathbb{R}^d} C_0\right).$$

$$\underbrace{n^2 \|\nabla^{st} F_n\|^2_{L^2(\Omega)}} \to \left(\int_{\mathbb{R}^d} |Dh|^2\right) \left(\int C_0\right).$$

(ii) F_n bnd in $H^m(\Omega)$ $\forall m$

but F_n has no limit in $sL^2(\Omega)$.

(iii) <u>exercise</u> : $(F_n)^p$, $n \uparrow \infty$, $p \uparrow \infty$ Δ

II.3 Corrector equation.

<u>Theorem</u> (Papanicolaou - Varadhan '79, Kozlov '79).

Assume a is stat. and ergodic.

Then $\exists!\ \varphi = (\varphi_1 \dots \varphi_d) \in H^1_{loc}(\mathbb{R}^d ; L^2(\Omega))^d$.

<u>s.t.</u>

$$\begin{cases} -\nabla \cdot a(\nabla \varphi_j + e_j) = 0 \quad \text{in } \mathbb{R}^d \text{ (weak) } a.s. \\ \nabla \varphi_j \text{ is stationary } (\underline{ie}\ \nabla \varphi_j(\cdot + x, a) = \nabla \varphi_j(\cdot, a(\cdot + x))) \\ \mathbb{E}\, \nabla \varphi_j = 0 \end{cases}$$

(omitted top portion) $\int_B \varphi_j = 0$.

Moreover, $\mathbb{E}[|\nabla \varphi_j|^2] \leq \left(\frac{\beta}{\alpha}\right)^2 - 1$. $\quad (\alpha \mathrm{Id} \leq a \leq \beta \mathrm{Id})$

<u>Remark</u>: In general φ is not stationary itself.

a) In 1D: $\varphi(x) = \frac{1}{\mathbb{E}[\frac{1}{a}]} \int_0^x \left(\frac{1}{a} - \mathbb{E}[\frac{1}{a}] \right)$ $\quad (\varphi(0) = 0)$.

$\hookrightarrow$ CLT: $\mathbb{E}[|\varphi(x)|^2] \sim |x|$ not bounded

$\rightarrow \varphi$ not stationary.

b) In perturbative small-disorder regime:

$\boxed{a_\delta = \mathrm{Id} + \delta b}$

$\delta << 1$

b random coeff field, $\mathbb{E}\, b = 0$

$\mathbb{E}[b(x)\, b(y)] = C(x-y)$, $C \in L^1$

Formally: $\varphi_j = \delta \psi_j + O(\delta^2)$

where $-\Delta \psi_j = \nabla \cdot (b)$ "linearized" wrt randomness

$\psi_j = (-\Delta)^{-1} \nabla \cdot b$

$$\mathbb{E}\left[|\psi_j(x)|^2 \right] = \mathbb{E}\left[|\Delta^{-1} \nabla \cdot b|^2 (x) \right]$$

$$= \mathbb{E}\left[\left| \int \nabla G(x-y)\, b(y)\, dy \right|^2 \right]$$

(Green for $-\Delta$)

$$= \mathbb{E}\left[\iint dy\, dz\; \nabla G(x-y)\, \nabla G(x-z)\; b(y)\, b(z) \right]$$

$$= \iint dy\, dz\; \nabla G(x-y)\, \nabla G(x-z)\; \underbrace{C(y-z)}_{\in L^1}$$

$$\sim \int |\nabla G|^2 \qquad G \sim |x|^{2-d}$$

$< \infty$ (at ∞) $\quad \nabla G \sim |x|^{1-d}$

iff $2(1-d) < -d$

$d > 2$

$\Rightarrow$ expect stationary corrector only for $d > 2$

... require mixing.

Remark. In periodic setting: $-\nabla \cdot a_{per} (\nabla\varphi + e) = 0$

$\Rightarrow \exists! \, \varphi \in H^1_{per}$ by Poincaré.

$\mathbb{E}[\nabla^{st} F] = \cancel{\mathbb{E}[(\nabla^{st} 1) F]} = 0$

Proof (of existence of corrector).

Difficulty: $Y = \{ \nabla^{st} F : F \in H^1(\Omega) \}$ not closed in $L^2(\Omega)^d$.

Exercise: weak closure of $Y \equiv X := \{ G \in L^2(\Omega)^d :$

$\nabla_j^{st} G_k = \nabla_k^{st} G_j$ (weak) $\forall j,k$

& $\underline{\mathbb{E}[G] = 0}$ $\}$

Remark: $G \in X \iff G^\# \in L^2_\tau(\mathbb{R}^d \times \Omega)$

with $(\nabla_j G_k^\# = \nabla_k G_j^\#)$ (weak) $\forall j,k$.

& $\mathbb{E}\ G^\# = 0$.

Poincaré lemma $\implies$ $G^\# = \nabla v$, $v \in H^1_{loc}(\mathbb{R}^d, L^2(\Omega))$

s.t. ∇v stat, $\mathbb{E}\, \nabla v = 0$.

Consequence: $\begin{cases} -\nabla \cdot a(\nabla \varphi_j + e_j) = 0 \text{ in } \mathbb{R}^d \\ (\nabla \varphi_j) \text{ stat.}, \ \mathbb{E}\, \nabla \varphi_j = 0 \end{cases}$

$$\Longleftrightarrow (D\varphi_j)^b = G_j \; : \; \begin{cases} -\nabla^{st} \cdot a^b (G_j + e_j) = 0 \text{ in } \Omega \\ G_j \in X \end{cases}$$

Notice that $\mathbb{E}[G \cdot a^b G] \geq \alpha \, \mathbb{E}[|G|^2] \quad \forall G \in X$.

By Lax-Milgram: $\exists! \, G_j \in X$ st $\mathbb{E}[G \cdot a^b G_j] = -\mathbb{E}[G \cdot a^b e_j]$
$\forall G \in X$.

In particular: $\forall H \in H^1(\Omega)$, $\nabla^{st} H \in X$,

$$\mathbb{E}[\nabla^{st} H \cdot a^b (G_j + e_j)] = 0$$

$\to$ as above: $G_j^{\#} = D\varphi_j$, $\varphi_j \in H^1_{loc}(\mathbb{R}^d, L^2(\Omega))$.

Remark: construction of φ.

$$\varphi_j(x) := \Big(\underbrace{e^{x\cdot\nabla^{st}}}_{T_x} - \fint_B e^{y\cdot\nabla^{st}}\,dy \Big) (\Delta^{st})^{-1} \nabla^{st}\cdot G_j$$

- well-defined: by fctal calculus.

$$\mathbb{E}\,|\varphi_j(x)|^2 \le |x|^2\, \mathbb{E}\big[|\nabla^{st}(\Delta^{st})^{-1}\nabla^{st} G_j|^2 \big]$$

$$\le |x|^2\, \mathbb{E}\big[|G_j|^2\big].$$

- $$\nabla_k \varphi_j(x) = \underbrace{e^{x\cdot\nabla^{st}}}_{T_x} \big[\nabla_k^{st} (\Delta^{st})^{-1} \nabla_l^{st} (G_j)_l \big]$$

$$\nabla_l^{st}\ \underbrace{\nabla_k^{st} (G_j)_l}$$

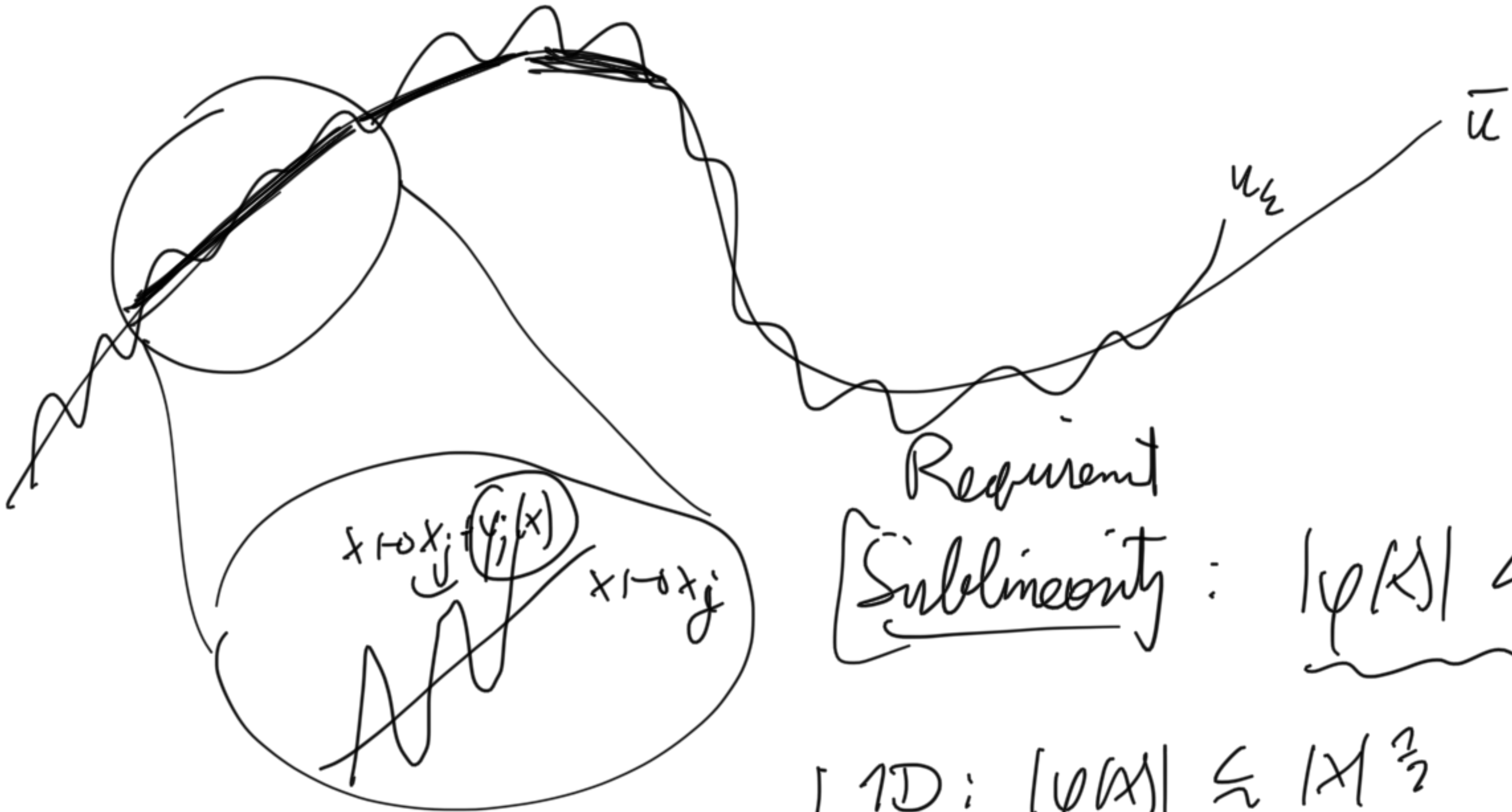
$$\underbrace{\nabla_l^{st} (G_j)_k}$$

$$\Delta^{st} (G_j)_k$$

$$\underbrace{\qquad (G_j)_h \qquad}$$

$$(G_j^{\#})_h(x).$$

$\square$



Requirement

$\boxed{\underline{\text{Sublinearity}}: \ |\varphi(x)| \ll |x|.}$

$$\begin{cases} 1D: & |\varphi(x)| \lesssim |x|^{\frac{1}{2}} \\ d>2: & \text{expect } |\varphi(x)| \lesssim 1. \ \text{ if mixing.} \end{cases}$$

Corollary (qualitative sublinearity).

If $v \in H^1_{loc}(\mathbb{R}^d, L^2(\Omega))$ s.t. $\begin{cases} \nabla v \text{ stat.} \\ \mathbb{E}\nabla v = 0 \end{cases}$

Then: $\varepsilon v\left(\frac{\cdot}{\varepsilon}\right) \longrightarrow 0$ in $L^2(\mathbb{R}^d)$ a.s.

(& in fact in $L^q(\mathbb{R}^d)$ $\forall q < \frac{2d}{d-2}$).