M285K - course #6

Theorem.

$$-\nabla\cdot a\left(\tfrac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon = \nabla\cdot f \quad \text{on } \mathbb{R}^d, \quad f\in L^2(\mathbb{R}^d)$$

$$-\nabla\cdot \bar a\,\nabla \bar u = \nabla\cdot f \quad \text{on } \mathbb{R}^d$$

Then, $\begin{cases} \nabla u_\varepsilon \rightharpoonup \nabla \bar u \\ a\left(\tfrac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon \rightharpoonup \bar a\,\nabla \bar u \end{cases}$ in $L^2(\mathbb{R}^d)^d$ a.s.

Proof. Step 1: compactness.

A priori estimate

$$\underbrace{\int_{\mathbb{R}^d}\nabla u_\varepsilon\cdot a\left(\tfrac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon}_{\geq\ \alpha\|\nabla u_\varepsilon\|_{L^2}^2} = -\underbrace{\int_{\mathbb{R}^d}\nabla u_\varepsilon\cdot f}_{\leq\ \beta\|f\|_{L^2}\|\nabla u_\varepsilon\|_{L^2}}$$

$$\rightarrow \|\nabla u_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq \frac{\beta}{\alpha}\|f\|_{L^2}.$$

Hence: a.s. UTPE $\nabla u_\varepsilon \rightharpoonup \nabla u_0$ ... $L^2(\mathbb{R}^d)^d$

$$\begin{cases} \dots & \text{in } L^2(\mathbb{R}^d) \\ a\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon \rightharpoonup p_0 \end{cases}$$

$$\begin{cases} u_0 \in \dot{H}^1(\mathbb{R}^d) \\ p_0 \in L^2(\mathbb{R}^d)^d \end{cases}$$

Step 2 : Tartar's oscillating test fct method (Tartar '77, Murat '78)

Idea: recall $u_\varepsilon \sim \bar{u} + \varepsilon\varphi_i\left(\frac{\cdot}{\varepsilon}\right)\nabla_i \bar{u}$

$\hookrightarrow$ given $\chi \in C_c^\infty(\mathbb{R}^d)$, supported in domain $D \subseteq \mathbb{R}^d$.

then consider test fct: $\chi + \varepsilon\varphi_i'\left(\frac{\cdot}{\varepsilon}\right)\nabla_i\chi$

Get: $\left[\int \nabla\left[\chi + \varepsilon\varphi_i'\left(\frac{\cdot}{\varepsilon}\right)\nabla_i\chi\right] \cdot a\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon = -\int \nabla\left[\chi + \varepsilon\varphi_i'\left(\frac{\cdot}{\varepsilon}\right)\nabla_i\chi\right] \cdot f\right.$

Expand: $\int_D (\nabla_i\chi)\left(e_i + \nabla\varphi_i'\left(\frac{\cdot}{\varepsilon}\right)\right) \cdot a\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon + \int_D \varepsilon\varphi_i\left(\frac{\cdot}{\varepsilon}\right)\nabla\nabla_i\chi \cdot a\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon$

$$= -\underbrace{\int_D (D_i\chi)\,(e_i + \underbrace{D\varphi_i(\tfrac{\cdot}{\varepsilon})}_{\to 0 \text{ in } L^2_{loc} \text{ a.s}})\cdot f}\; \underbrace{-\int_D \varepsilon\, \underbrace{\varphi_i(\tfrac{\cdot}{\varepsilon})}_{\to 0 \text{ in } L^2_{loc}}\, D D_i\chi \cdot f}_{\to 0}$$

$$\Rightarrow \text{a.s, UTE}: \quad \underbrace{\int_D (D_i\chi)\,(e_i + \underbrace{D\varphi_i(\tfrac{\cdot}{\varepsilon})}_{\to e_i})\cdot \underbrace{a(\tfrac{\cdot}{\varepsilon})\, Du_\varepsilon}_{\to p_0}} \xrightarrow{\varepsilon \to 0} \underbrace{-\int_D D\chi \cdot f}$$

Tartar's compensated compactness: "use IBP to transform W-W into S-W"

$$\int_D (D_i\chi)\,(e_i + D\varphi_i(\tfrac{\cdot}{\varepsilon}))\cdot a(\tfrac{\cdot}{\varepsilon})\, Du_\varepsilon \qquad u_\varepsilon|_D \in H^1(D)$$

$$= \int_D \underbrace{(D_i\chi)}\; \underbrace{\overbrace{a^t(\tfrac{\cdot}{\varepsilon})(e_i + D\varphi_i(\tfrac{\cdot}{\varepsilon}))}^{\operatorname{div}=0}} \cdot \overbrace{D\Big(u_\varepsilon - \fint_D u_\varepsilon\Big)}^{\text{grad}}$$

$$= - \int_D \nabla \nabla_i \chi \cdot \underbrace{a'\left(\tfrac{\cdot}{\varepsilon}\right)\left(e_i + \nabla \varphi_i'\left(\tfrac{\cdot}{\varepsilon}\right)\right)}_{\rightharpoonup \mathbb{E}\left[a'(e_i+\nabla\varphi_i')\right] = \bar{a}' e_i \text{ in } L^2_{loc}} \underbrace{\left(u_\varepsilon - \fint_D u_\varepsilon\right)}_{\to u_0 - \fint_D u_0}$$

$$- 0 \quad (\text{corrector eqn})$$

$\nabla u_\varepsilon \rightharpoonup \nabla u_0$ in $L^2(D)$

Poincaré: $u_\varepsilon - \fint_D u_\varepsilon \rightharpoonup u_0 - \fint_D u_0$ in $H^1(D)$

Rellich: $\quad \ldots \to \ldots \quad$ in $L^2(D)$

Conclusion: $$\underbrace{- \int_D \nabla \nabla_i \chi \cdot \bar{a}' e_i \left(u_0 - \fint_D u_0\right)}_{= \int_D \nabla \chi \cdot \bar{a} \nabla u_0} = - \int_D \nabla \chi \cdot f$$

$$\underline{\text{ie}} \quad - \nabla \cdot \bar{a} \nabla u_0 = \nabla \cdot f \text{ in } \mathbb{R}^d$$

Conclude: $\nabla u_\varepsilon \rightharpoonup \nabla u_0 = \nabla \bar{u}$ in $L^2(\mathbb{R}^d)^d$ a.s. $\square$

⊛ Reformulation of the proof: compensated compactness

Lemma (Div-curl, Tartar, Murat ...)

Let $(v_\varepsilon)_\varepsilon \subseteq H^1_{loc}(\mathbb{R}^d)$, $(q_\varepsilon)_\varepsilon \subseteq L^2(\mathbb{R}^d)^d$

such that
$$\begin{cases} \nabla v_\varepsilon \rightharpoonup \nabla v_0 & \text{in } L^2(\mathbb{R}^d) \\ q_\varepsilon \rightharpoonup q_0 & \text{in } L^2(\mathbb{R}^d) \\ \nabla\cdot q_\varepsilon \to \nabla\cdot q_0 & \text{in } H^{-1}(\mathbb{R}^d) \end{cases}$$

Thm: $\underbrace{q_\varepsilon}\cdot\underbrace{\nabla v_\varepsilon} \to q_0 \cdot \nabla v_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$

Proof. Let $D \subseteq \mathbb{R}^d$ domain, $\chi \in C_c^\infty(\mathbb{R}^d)$ supported in D.

$v_\varepsilon|_D \in H^1(D)$

Poincaré: $v_\varepsilon - \fint_D v_\varepsilon \rightharpoonup v_0 - \fint_D v_0 \quad \text{in } H^1(D)$

$\to \qquad L^2(D)$

$$\int \chi \, q_\varepsilon \cdot \nabla v_\varepsilon = \int \chi \, q_\varepsilon \cdot \nabla \left(v_\varepsilon - \fint v_\varepsilon \right)$$

$$= - \int \nabla \chi \cdot \underbrace{q_\varepsilon}_{\rightharpoonup_{\text{in } L^2} q_0} \underbrace{\left(v_\varepsilon - \fint v_\varepsilon \right)}_{\to v_0 - \fint v_0 \text{ in } L^2}$$

$$- \int \chi \, (\nabla \cdot q_\varepsilon) \, \left(v_\varepsilon - \fint v_\varepsilon \right)$$

$$\underbrace{\qquad}_{\to \nabla\cdot q_0 \text{ in } H^{-1}} \qquad \underbrace{\qquad\qquad}_{\to v_0 - f v_0 \text{ in } H^{1}} . \qquad \square$$

Proof of qual homog :

Remember a.s UTE
$$\begin{cases} \nabla u_\varepsilon \rightharpoonup \nabla u_0 \\ a(\tfrac{\cdot}{\varepsilon}) \nabla u_\varepsilon \rightharpoonup p_0 \end{cases} \text{ in } L^2$$

In particular : $-\nabla\cdot\left[a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon\right] = \nabla\cdot f$

$$\Rightarrow \boxed{-\nabla\cdot[p_0] = \nabla\cdot f .}$$

Remains to get a relation between p_0 & ∇u_0.

Ergodic thm a.s
$$\begin{cases} \nabla\varphi_i^{1}(\tfrac{\cdot}{\varepsilon}) + e_i \rightharpoonup e_i & \text{in } L^2_{loc} \\ a^{1}(\tfrac{\cdot}{\varepsilon})\left[\nabla\varphi_i^{1}(\tfrac{\cdot}{\varepsilon}) + e_i\right] \rightharpoonup \bar{a}^{1} e_i \end{cases}$$

Get: $\underbrace{\overbrace{\left(\nabla\varphi_i^1\left(\tfrac{\cdot}{\varepsilon}\right)+e_i\right)}^{\text{grad}}}_{\to e_i} \cdot \underbrace{\overbrace{a\left(\tfrac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon}^{\nabla\cdot = -\nabla\cdot f}}_{\to p_0} = \underbrace{\overbrace{a'\left(\tfrac{\cdot}{\varepsilon}\right)\left(\nabla\varphi_i'\left(\tfrac{\cdot}{\varepsilon}\right)+e_i\right)}^{\nabla\cdot = 0}}_{\to \bar{a}' e_i} \cdot \underbrace{\nabla u_\varepsilon}_{\to \nabla u_0}$

$$\Rightarrow p_0 = \bar{a}\nabla u_0 \qquad \square$$

(*) Another reform: 2-scale convergence.

(Nguetseng '89, Allaire '91: periodic
& random: Bourgeat-Mikelic-Wright '94)

Definition. Given domain $D\subseteq\mathbb{R}^d$, $(v_\varepsilon)_\varepsilon \subseteq L^2(D\times\Omega)$.

Then $v_\varepsilon \xrightarrow{2\text{-sc}} v_0$, i.e. v_ε "2-scale stochastically converges in the mean" to v_0

if $\displaystyle \mathbb{E}\int_D v_\varepsilon(x,\cdot)\, \psi\left(x, \tau_{\frac{x}{\varepsilon}}\cdot\right) dx$

$$\longrightarrow \mathbb{E}\int_D v_0(x,\cdot)\, \psi(x,\cdot)\, dx .$$

$\forall\, \psi \in L^2(D\times\Omega)$ "admissible", ie $(x,\omega)\mapsto \psi(x,\tau_x\omega)$ measurable on $D\times\Omega$.

<u>Lemma</u>.

(i) if $F\in L^2(\Omega)$, $\psi\in L^2(D)$, then $(x,\omega)\mapsto F(\omega)\psi(x)$ is admissible in $L^2(D\times\Omega)$.

(ii) elements of $L^2(D, \mathcal{B}(\Omega))$, $\mathcal{B}(\Omega) = \{F\in \mathrm{Mes}(\Omega) : \sup_\omega |F(\omega)| < \infty\}$ are admissible

(iii) $C(D, L^\infty(\Omega))$ are admissible.

(exercise)

<u>Theorem</u> (2-scale compactness).

If $(v_\varepsilon)_\varepsilon$ is bounded in $L^2(D\times\Omega)$

Then $\exists$ subsequence s.t. $v_\varepsilon \xrightarrow{2\text{-sc}} v_0$

for some $v_0 \in L^2(D\times\Omega)$.

Proof.

$$\left| \mathbb{E} \int_D \psi(x, T_{x/\varepsilon}\,\cdot)\, v_\varepsilon(x,\cdot)\, dx \right| \leq \underbrace{\|v_\varepsilon\|_{L^2(D\times\Omega)}}_{\leq C} \|\psi\|_{L^2(D\times\Omega)}$$

By construction, $(\Omega, \mathbb{P})$ is separable $\Rightarrow$ $L^2(\Omega)$ separable.

$\Rightarrow$ $\exists$ countable $S \subseteq L^2(D\times\Omega)$ of admissible elements that is dense in $L^2(D\times\Omega)$

$\forall \psi \in S$: $\exists$ subsequence st $\mathbb{E} \int_D \psi(x, \tau_{\frac{x}{\varepsilon}} \cdot) v_\varepsilon(x, \cdot)\, dx \xrightarrow[\varepsilon \to 0]{} \mathcal{L}(\psi)$.

Diagonal: $\exists$ subsequence st. $\mathbb{E} \int_D \psi(x, \tau_{\frac{x}{\varepsilon}} \cdot) v_\varepsilon(x, \cdot)\, dx \to \mathcal{L}(\psi)$
$\forall \psi \in S$.

Note that $\begin{cases} \mathcal{L} \text{ linear on } S \\ \& \text{ bounded on } S: \quad |\mathcal{L}(\psi)| \leq C \|\psi\|_{L^2(D \times \Omega)} \end{cases}$

$\to \mathcal{L}$ extends uniquely as a bnd linear fctal on $L^2(D \times \Omega)$.

Riesz thm: $\mathcal{L}(\psi) = \mathbb{E} \int_D v_0(x, \cdot)\, \psi(x, \cdot)$ for some $v_0 \in L^2(D \times \Omega)$.

$\square$

<u>Proof of homog thm</u>: $-\nabla \cdot \underset{\smile}{a}\left(\frac{\cdot}{\varepsilon}\right) \underset{\smile}{\nabla u_\varepsilon} = \nabla \cdot f$ in $\mathbb{R}^d$.

$D \subseteq \mathbb{R}^d$ domain, $u_\varepsilon|_D \in H^1(D)$, $\int_D u_\varepsilon = 0$.

Comportmen : $\left\{\begin{array}{l} u_\varepsilon|_D \xrightarrow{2sc} u_0 \\ \nabla u_\varepsilon|_D \xrightarrow{2sc} v_0 \end{array}\right]$

① $\chi \in C_c^\infty(D)$:

$$\mathbb{E}\int \chi \nabla u_\varepsilon = -\mathbb{E}\int \nabla\chi \, u_\varepsilon$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\mathbb{E}\int \chi v_0 \qquad\qquad -\mathbb{E}\int \nabla\chi \, u_0$$

$$\to \mathbb{E}\, v_0 = \mathbb{E}\,\nabla u_0$$

$$\to v_0 = \nabla u_0 + \xi_0 , \quad \mathbb{E}\,\xi_0 = 0.$$

② $\chi \in C_c^\infty(D)$, $F \in W_\tau^{1,\infty}(\Omega)$:

$$\mathbb{E}\int \chi(x)\, \varepsilon F(\tau_{\frac{x}{\varepsilon}}\cdot)\, \nabla u_\varepsilon(x)\, dx = \overbrace{-\mathbb{E}\int \nabla\chi(x)\, \varepsilon F(\tau_{\frac{x}{\varepsilon}}\cdot)\, u_\varepsilon(x)\, dx}^{\to 0}$$

$\to 0$

$$- \mathbb{E} \underbrace{\int \chi(x) (\nabla^{st} F)(\tau_{\frac{x}{\varepsilon}} \cdot)\, u_\varepsilon(x)\, dx}$$

$$\to \mathbb{E} \int \chi\, (\nabla^{st} F)\, u_0 = 0$$

$$\nabla^{st} u_0 = 0 \quad \text{a.e.}$$

$$\Rightarrow \begin{cases} u_0 = u_0(x) \\ v_0 = \nabla u_0 + \xi_0, \quad \mathbb{E}\xi_0 = 0. \end{cases}$$

ergodicity: $u_0 = u_0(x)$!

③ $\chi \in C_c^\infty(D)$, $F \in W^{1,\infty}_\tau(\Omega)^d$, $\nabla^{st} \cdot F = 0$, $\mathbb{E} F = 0$:

$$\underbrace{\mathbb{E} \int \chi(x)\, F(\tau_{\frac{x}{\varepsilon}} \cdot) \cdot \nabla u_\varepsilon(x)\, dx} = -\underbrace{\mathbb{E} \int \nabla\chi(x)\, F(\tau_{\frac{x}{\varepsilon}} \cdot)\, u_\varepsilon(x)\, dx}$$

$$\to \mathbb{E}\left[F \cdot \int \chi(x)\, v_0(x)\, dx\right] \qquad \to \mathbb{E}\left[F \int \nabla\chi \cdot u_0\right]$$

$$\Rightarrow \mathbb{E}[F \cdot v_0(x, \cdot)] = \cancel{\underbrace{\mathbb{E}[F]}_{=0} \cdot \nabla u_0(x)}$$

$$= 0$$

$$v_0(x,\cdot) \perp \{F \in H^1(\Omega)^d : \nabla^{st} \cdot F = 0,\ \mathbb{E} F = 0\}.$$

$$\Rightarrow v_0(x,\cdot) = \nabla_{u_0}(x) + \xi_0(x,\cdot), \quad \mathbb{E}[\xi_0] = 0$$

$$\Rightarrow \xi_0 \in \left\{ F \in L^2(\Omega)^d : \begin{array}{l} \nabla^{st}_i F_j = \nabla^{st}_j F_i \\ \& \ \mathbb{E} F = 0 \end{array} \right\}$$

④ $\chi \in C_c^\infty(D)$, $F \in W^{1,\infty}_\tau(\Omega)$:

$$\mathbb{E} \int \underbrace{\nabla\left[\chi(x)_\varepsilon F(\tau_{\frac{x}{\varepsilon}} \cdot)\right]}_{\begin{array}{c} \nabla\chi(x) \otimes_\varepsilon F(\tau_{\frac{x}{\varepsilon}} \cdot) \ ] \to 0 \\ + \boxed{\chi(x)(\nabla^{st} F)(\tau_{\frac{x}{\varepsilon}} \cdot)} \end{array}} \cdot a(\tfrac{\cdot}{\varepsilon}) \nabla u_\varepsilon = - \mathbb{E} \int \nabla\left[\chi(x)_\varepsilon F(\tau_{\frac{x}{\varepsilon}} \cdot)\right] \cdot f$$

$$\hookrightarrow \ \mathbb{E} \int (\nabla^{st} F) \ldots a^b \ldots v_0(x,\cdot) \ldots = - \mathbb{E} \int [\nabla^{st} F] f(x) = 0$$

$$\mathbb{E}\left[(\nabla^{st} \cdot a \; v_0(x)\right] = \ldots$$

$$\Rightarrow \; -\nabla^{st} \cdot a^b \, v_0(x,\cdot) = 0$$

$$-\nabla^{st} \cdot a^b \left(\nabla u_0(x) + \widetilde{\xi_0(x,\cdot)} \right) = 0$$

$\equiv$ corrector eqn: $\boxed{\xi_0(x,\omega) = \nabla \varphi_i(\omega) \, \nabla_i u_0(x)}$

(5) $$\mathbb{E} \int \nabla \chi \cdot a\left(\tfrac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon = - \mathbb{E} \int \nabla \chi \cdot f$$

$$\downarrow$$

$$\mathbb{E} \int \nabla \chi(x) \cdot a^b \left(\nabla u_0(x) + (\nabla \varphi_i)^b \nabla_i u_0(x) \right) = - \int \nabla \chi \cdot f$$

$$= \int \nabla \chi \cdot \underbrace{\mathbb{E}\left[a^b \left(e_i + (\nabla \varphi_i)^b \right) \right]}_{\bar{a} e_i} \nabla_i u_0(x)$$

$$\Rightarrow \; -\nabla \cdot \bar{a} \nabla u_0 = \nabla \cdot f. \quad \square$$