M285K - course #7

II.5 Qualitative corrector result

$$\begin{cases} \nabla u_\varepsilon \rightharpoonup \nabla \bar u \\ a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon \rightharpoonup \bar a \nabla \bar u \end{cases} \text{ in } L^2 \quad \text{but expect} \quad \nabla\Big[u_\varepsilon - \bar u - \varepsilon \varphi_i(\tfrac{\cdot}{\varepsilon})\nabla_i \bar u\Big] \to 0 \text{ in } L^2$$

Theorem (Papanicolaou-Varadhan & Kozlov '79).

For all $f \in L^2(\mathbb{R}^d)^d$, given $\begin{cases} -\nabla\cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon = \nabla\cdot f \\ -\nabla\cdot \bar a \nabla \bar u = \nabla \cdot f \end{cases}$ in $\mathbb{R}^d$

there holds $\|\nabla u_\varepsilon - (\nabla\varphi_i(\tfrac{\cdot}{\varepsilon}) + e_i)\nabla_i \bar u\|_{L^2(\mathbb{R}^d\times\Omega)} \xrightarrow{\varepsilon\downarrow 0} 0$.

In addition, for $f \in C_c^\infty(\mathbb{R}^d)^d$:

$$\|\nabla u_\varepsilon - (\nabla\varphi_i(\tfrac{\cdot}{\varepsilon}) + e_i)\nabla_i \bar u\|_{L^2(\mathbb{R}^d)} \xrightarrow{\varepsilon\downarrow 0} 0 \quad \text{a.s.}$$

Proof. Idea: convergence of energies

$$\underbrace{\int_{\mathbb{R}^d} \nabla u_\varepsilon \cdot a\left(\tfrac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon}_{= -\int_{\mathbb{R}^d} f \cdot \nabla u_\varepsilon} \xrightarrow[a.s.]{\varepsilon \downarrow 0} \underbrace{\int_{\mathbb{R}^d} \nabla \bar{u} \cdot \bar{a} \nabla \bar{u}}_{= -\int f \cdot \nabla \bar{u}}$$

$$(\nabla u_\varepsilon \rightharpoonup \nabla \bar{u})$$

Let $f \in C_c^\infty(\mathbb{R}^d)^d$.

$$\alpha \int_{\mathbb{R}^d} \left| \nabla \left(u_\varepsilon - \bar{u} - \varepsilon \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right) \partial_i \bar{u} \right) \right|^2$$

$$\leq \int_{\mathbb{R}^d} \nabla \left(u_\varepsilon - \bar{u} - \varepsilon \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right) \partial_i \bar{u} \right) \cdot a\left(\tfrac{\cdot}{\varepsilon}\right) \nabla \left(u_\varepsilon - \bar{u} - \varepsilon \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right) \partial_i \bar{u} \right)$$

$$= \underbrace{\int_{\mathbb{R}^d} \nabla u_\varepsilon \cdot a\left(\tfrac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon}_{} \xrightarrow{e.s} \underbrace{\int \nabla \bar u \cdot \bar a \nabla \bar u}$$

$$+ \int_{\mathbb{R}^d} \nabla_i \bar u \, \nabla_j \bar u \, \underbrace{\left(\nabla\varphi_i\left(\tfrac{\cdot}{\varepsilon}\right) + e_i\right) \cdot a\left(\tfrac{\cdot}{\varepsilon}\right) \left(\nabla\varphi_j\left(\tfrac{\cdot}{\varepsilon}\right) + e_j\right)}_{\xrightarrow{L^2_{loc}} \mathbb{E}[(\nabla\varphi_i + e_i)\cdot a(\nabla\varphi_j + e_j)] = e_i \cdot \bar a e_j} \quad \xrightarrow{e.s.} \underbrace{\int \nabla \bar u \cdot \bar a \nabla \bar u}$$

$$+ \int_{\mathbb{R}^d} \underbrace{\varepsilon\varphi_i\left(\tfrac{\cdot}{\varepsilon}\right)}_{\xrightarrow{L^2_{loc}} 0} \underbrace{\nabla_j \bar u \quad \nabla\nabla_i \bar u}_{lmd} \cdot \underbrace{a\left(\tfrac{\cdot}{\varepsilon}\right) \left(\nabla\varphi_j\left(\tfrac{\cdot}{\varepsilon}\right) + e_j\right)}_{\xrightarrow{L^2_{loc}} \bar a e_j} \quad + \underbrace{sym.}_{\to 0}$$

$$+ \int_{\mathbb{R}^d} \varepsilon\varphi_i\left(\tfrac{\cdot}{\varepsilon}\right) \varepsilon\varphi_j\left(\tfrac{\cdot}{\varepsilon}\right) \nabla\nabla_i \bar u \cdot a\left(\tfrac{\cdot}{\varepsilon}\right) \nabla\nabla_j \bar u \quad \xrightarrow{e.s} 0$$

$$\underbrace{- \int_{\mathbb{R}^d} \nabla \underbrace{\left(\bar u + \varepsilon\varphi_i\left(\tfrac{\cdot}{\varepsilon}\right)\nabla_i \bar u\right)}_{} \cdot a\left(\tfrac{\cdot}{\varepsilon}\right) \underbrace{\nabla u_\varepsilon}_{}} \qquad \underbrace{- \, sym}$$

$$= \int_{\mathbb{R}^d} \underbrace{\nabla\left(\bar u + \varepsilon\varphi_i\left(\tfrac{\cdot}{\varepsilon}\right)\nabla_i \bar u\right)}_{\nabla_i \bar u \left(\underbrace{\nabla\varphi_i\left(\tfrac{\cdot}{\varepsilon}\right)}_{\xrightarrow{(L^2_{loc})} 0} + e_i\right) + \underbrace{\varepsilon\varphi_i\left(\tfrac{\cdot}{\varepsilon}\right)}_{\xrightarrow{L^2_{loc}} 0} \nabla\nabla_i \bar u} \cdot f \qquad \to \; - \int_{\mathbb{R}^d} \nabla \bar u \cdot \bar a \nabla \bar u$$

$$\longrightarrow \int_{\mathbb{R}^d} \nabla \bar{u} \cdot f = -\int_{\mathbb{R}^d} \nabla \bar{u} \cdot \bar{a} \nabla \bar{u}$$

$$\underbrace{\int_{\mathbb{R}^d} \partial_i \bar{u} \, \underbrace{\nabla \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right)}_{\xrightarrow[L^2_{loc}]{} 0 \text{ a.s}} \cdot f}_{\longrightarrow 0 \text{ a.s}}$$

$\xrightarrow{\varepsilon \downarrow 0}$ 0 a.s

Next: convergence in $L^2(\mathbb{R}^d \times \Omega)$ $\forall f \in L^2(\mathbb{R}^d)^d$.

$\rightarrow$ argue by density.

$\forall f \in L^2(\mathbb{R}^d)^d \;\forall \delta > 0 \;\exists f' \in C_c^\infty(\mathbb{R}^d)^d : \|f - f'\|_{L^2(\mathbb{R}^d)} \leq \delta$

$\rightarrow$ energy estimate:

$$\begin{cases} \|\nabla u_\varepsilon - \nabla u_\varepsilon'\|_{L^2(\mathbb{R}^d)} \leq C\delta \\ \|\nabla \bar{u} - \nabla \bar{u}'\|_{L^2(\mathbb{R}^d)} \leq C\delta \end{cases}$$

$$\rightarrow \boxed{\left\| \nabla u_\varepsilon - \underbrace{\left(\nabla \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right) + e_i\right) \partial_i \bar{u}}_{} \right\|_{L^2(\mathbb{R}^d \times \Omega)}}$$

$\leq \| \nabla u_\varepsilon' - (\nabla \varphi_i(\tfrac{\cdot}{\varepsilon}) + e_i) \partial_i \bar{u}' \| \ldots \longrightarrow 0$

$$\leq \|\nabla u_\varepsilon - (\nabla\varphi_i(\tfrac{\cdot}{\varepsilon}) + e_i)\nabla_i \bar u\|_{L^2(\mathbb{R}^d\times\Omega)} \longrightarrow 0$$

by the previous proof

$$+ \|\nabla u_\varepsilon^\delta - \nabla u_\varepsilon\|_{L^2(\mathbb{R}^d\times\Omega)}$$

$$+ \|\underline{\nabla\varphi_i(\tfrac{\cdot}{\varepsilon}) + e_i}\|_{L^2(\Omega)}\, \|\nabla\bar u - \nabla\bar u^\delta\|_{L^2(\mathbb{R}^d)} \Bigg\} O(\delta)$$

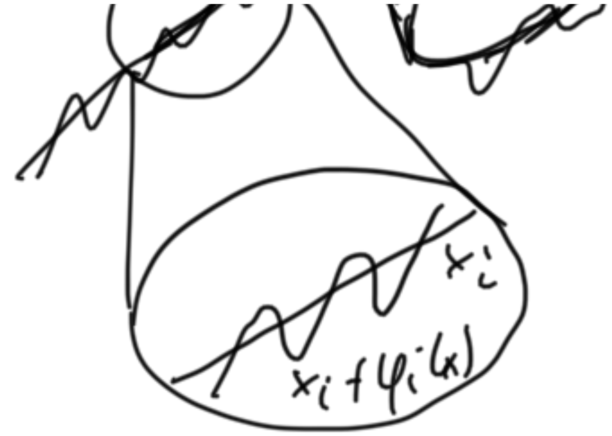
$\square$

II.6 Towards the quantitative theory.

There are 2 occurrences of the ergodic thm that must be quantified:

① $\underbrace{\nabla\varphi(\tfrac{\cdot}{\varepsilon})}_{=\nabla[\varepsilon\varphi(\frac{\cdot}{\varepsilon})]} \xrightarrow[a.s]{L^2_{loc}} 0$)

$= \nabla[\varepsilon\varphi(\tfrac{\cdot}{\varepsilon})]$ $\Rightarrow$ need a quant. sublinearity estimate for the potential $\boxed{\varphi}$.



gluing error
$= O(\varepsilon\, \varphi(\tfrac{\cdot}{\varepsilon}))$.

② $q_i(\tfrac{\cdot}{\varepsilon}) = \underbrace{a(\tfrac{\cdot}{\varepsilon})\big(\nabla\varphi_i(\tfrac{\cdot}{\varepsilon}) + e_i\big) - \bar{a}\, e_i}_{\text{flux of the corrector.}} \xrightarrow[a.s.]{L^2_{loc}} 0$

Recall corrector eqn $\nabla \cdot q_i = 0$

3D notation: $q_i = \nabla \times \sigma_i$ for some "vector potential" σ_i

σ_i is defined only up to a grad field

Coulomb gauge: $\{ -\Delta \sigma_i = \nabla \times q_i$

(indeed recover: $-\Delta(\nabla\times\sigma_i) = \nabla\times\nabla\times q_i = -\Delta q_i$)

Higher dimension: $q_i = \nabla \cdot \sigma_i$ for some skew-symmetric matrix field

$$\sigma_i = (\sigma_{ijk})_{1\le j,k\le d}$$

$$\sigma_{ijk} = -\sigma_{ikj}.$$

$$(\nabla\cdot\sigma_i)_j = \underline{\nabla_k}\,\sigma_{ij\underset{\smile}{k}}$$

$$\left(\text{indeed } \nabla\cdot q_i = \nabla\cdot\nabla\cdot\sigma_i = \underbrace{\nabla^2_{jk}}_{\text{sym}}\underbrace{\sigma_{ijk}}_{\text{skew}} = 0\right)$$

Coulomb gauge : $-\Delta\sigma_{ijk} = \nabla_j(q_i)_k - \nabla_k(q_i)_j$

__Lemma__ (flux corrector).

$\exists!\ \sigma_i = (\sigma_{ijk})_{1\le j,k\le d} \in H^1_{loc}(\mathbb{R}^d, L^2(\Omega)^{d\times d})$

__s.t__ * $-\Delta\underbrace{\sigma_{ijk}} = \underbrace{\nabla_j(q_i)_k} - \underbrace{\nabla_k(q_i)_j}$ in $\mathbb{R}^d$ (weak, a.s)

* $\nabla\sigma$ stationary, $\mathbb{E}|\nabla\sigma|^2 < \infty$, $\mathbb{E}\nabla\sigma = 0$

* anchoring $\int_B \sigma_{ijk} = 0$.

In particular, we get $\begin{cases} \sigma_{ijk} = -\sigma_{ikj} \\ \nabla\cdot\sigma_i = q_i \quad \left(\equiv a(\nabla\varphi_i + e_i) - \bar{a}\, e_i\right) \end{cases}$

In addition, $\underbrace{\varepsilon\,\sigma\left(\frac{\cdot}{\varepsilon}\right)}\to 0$ in $L^q_{loc}(\mathbb{R}^d)$ a.s. $\forall q < \frac{2d}{d-2}$

(qualitative sublinearity)

Proof. * Existence & uniqueness: exercise (copy the proof for φ)

* Skew-sym: trivial

* Relation $\nabla\cdot\sigma_i = q_i$:

Take derivative in the eqn for σ_i:

$$-\Delta \underbrace{\nabla_k \sigma_{ijk}}_{(\operatorname{div}\sigma_i)_j} = \underbrace{\nabla_k \nabla_j (q_i)_k}_{\nabla_j (\operatorname{div} q_i) = 0} - \underbrace{\nabla_k \nabla_k (q_i)_j}_{\Delta (q_i)_j}$$

$$\Rightarrow \left(-\Delta \left(\underbrace{q_i - \operatorname{div}\sigma_i}_{\in L^2_\tau(\mathbb{R}^d \times \Omega)}\right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d) \text{ a.s.}\right.$$

$$\Rightarrow \forall \chi \in C^\infty_c(\mathbb{R}^d): \quad -\Delta \underbrace{\chi * (q_i - \operatorname{div}\sigma_i)}_{\in H^1_\tau(\mathbb{R}^d \times \Omega)} = 0$$

$$\Rightarrow \nabla \chi * (q_i - \operatorname{div}\sigma_i) = 0$$

$$\Rightarrow \chi * (q_i - \operatorname{div}\sigma_i) = cst = 0$$

$$\left.\begin{aligned} &\mathbb{E}[q] = 0 \\ &\mathbb{E}[\operatorname{div}\sigma_i] = 0 \end{aligned}\right\}$$

$$\Rightarrow q_i = \operatorname{div}\sigma_i$$

$\square$

<u>Proposition.</u>

For all $f \in C_c^\infty(\mathbb{R}^d)^d$, given $\begin{cases} -\nabla\cdot a(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon = \nabla\cdot f \\ -\nabla\cdot \bar{a}\nabla\bar{u} = \nabla\cdot f \end{cases}$ in $\mathbb{R}^d$

there holds
$$-\nabla\cdot a\left(\frac{\cdot}{\varepsilon}\right)\nabla\left[u_\varepsilon - \bar{u} - \varepsilon\varphi_i\left(\frac{\cdot}{\varepsilon}\right)\nabla_i\bar{u}\right] = \nabla\cdot\left[\underbrace{\varepsilon(a\varphi_i - \sigma_i)\left(\frac{\cdot}{\varepsilon}\right)}\ \underbrace{\nabla\nabla_i\bar{u}}\right]$$

In particular:
$$\left\|\nabla\left(u_\varepsilon - \bar{u} - \varepsilon\varphi_i\left(\frac{\cdot}{\varepsilon}\right)\nabla_i\bar{u}\right)\right\|_{L^2(\mathbb{R}^d)} \lesssim \left\|\varepsilon(\varphi,\sigma)\left(\frac{\cdot}{\varepsilon}\right)\nabla^2\bar{u}\right\|_{L^2(\mathbb{R}^d)}$$
$$\|\nabla u_\varepsilon - (\nabla\varphi_i(\tfrac{\cdot}{\varepsilon}) + e_i)\nabla_i\bar{u}\|_{L^2(\mathbb{R}^d)} \lesssim \underbrace{\left\|\varepsilon(\varphi,\sigma)\left(\frac{\cdot}{\varepsilon}\right)\nabla^2\bar{u}\right\|_{L^2(\mathbb{R}^d)}}_{\to 0 \text{ a.s.}}$$

Proof. $\nabla\cdot a\left(\frac{\cdot}{\varepsilon}\right)\nabla\left(u_\varepsilon - \bar{u} - \varepsilon\varphi_i\left(\frac{\cdot}{\varepsilon}\right)\nabla_i\bar{u}\right)$

Proof. $\quad -\nabla\cdot a(\tfrac{\cdot}{\varepsilon})\nabla\left(u_\varepsilon - \bar{u} - \varepsilon\varphi_i(\tfrac{\cdot}{\varepsilon})\nabla_i\bar{u}\right)$

$$= \underbrace{-\nabla\cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon}_{\substack{=\nabla\cdot f \\ = -\nabla\cdot \bar{a}\nabla\bar{u}}} + \nabla\cdot\left[a(\tfrac{\cdot}{\varepsilon})(\nabla\varphi_i(\tfrac{\cdot}{\varepsilon})+e_i)\nabla_i\bar{u}\right] + \nabla\cdot\left[\varepsilon a\varphi_i(\tfrac{\cdot}{\varepsilon})\nabla\nabla_i\bar{u}\right]$$

$$= \nabla\cdot\Big[\big(\underbrace{a(\tfrac{\cdot}{\varepsilon})(\nabla\varphi_i(\tfrac{\cdot}{\varepsilon})+e_i) - \bar{a}e_i}_{\substack{=q_i(\tfrac{\cdot}{\varepsilon}) \\ =(\nabla\cdot\sigma_i)(\tfrac{\cdot}{\varepsilon})}}\big)\nabla_i\bar{u}\Big]$$

$$= \frac{1}{\varepsilon}\underbrace{\cancel{(\nabla\cdot\nabla\cdot\sigma_i)}}_{0}(\tfrac{\cdot}{\varepsilon}) + (\nabla\cdot\sigma_i)(\tfrac{\cdot}{\varepsilon})\cdot\nabla\nabla_i\bar{u}$$

$$= -\nabla\cdot\left[\varepsilon\sigma_i(\tfrac{\cdot}{\varepsilon})\nabla\nabla_i\bar{u}\right]$$

Indeed: $\quad \nabla\cdot\left[\varepsilon\sigma_i(\tfrac{\cdot}{\varepsilon})\nabla\nabla_i\bar{u}\right]$

$$= \nabla_j\left(\varepsilon\sigma_{ijk}(\tfrac{\cdot}{\varepsilon})\nabla^2_{ik}\bar{u}\right)$$

$$= \underbrace{\left(\nabla_j \sigma_{ijk}\right)}_{= -\nabla_j \sigma_{ikj} = -(\nabla\cdot\sigma_i)_k}\left(\tfrac{\cdot}{\varepsilon}\right) \nabla^2_{ik}\bar{u} + \underbrace{\varepsilon\, \underbrace{\sigma_{ijk}}_{\text{skew}}\left(\tfrac{\cdot}{\varepsilon}\right) \underbrace{\nabla^3_{ijk}\bar{u}}_{\text{sym}}}_{=0}$$

$$= -\left(\nabla\cdot\sigma_i\right)\left(\tfrac{\cdot}{\varepsilon}\right)\cdot\nabla\nabla_i\bar{u}\,. \qquad \square$$