Final Exam — March 17th

Please provide complete and well-written solutions to the following exercises. All answers must be properly explained and justified.

By taking this exam, you implicitly agree to solve the exercises and write your answers on your own without using any means of cheating.

Exercise 1. (/1) Let X be a random variable and assume that there exist $a, b \in \mathbb{R}$ such that $\mathbb{P}[a \le X \le b] = 1$. Prove that

$$\mathrm{Var}[X] \le \tfrac{1}{4}(b-a)^2.$$

Exercise 2. (/3) Let $(\pi_1, \ldots, \pi_n)$ be a random permutation of the set of numbers $\{1, \ldots, n\}$ (all realizations being equally likely), and let N_n be the number of fixed points of this permutation, that is, the number of i's such that $\pi_i = i$.

(i) Compute the expectation and variance of N_n.

(ii) Compute the probability $\mathbb{P}[N_n = k]$ for all $0 \le k \le n$, and deduce that N_n converges in distribution to a Poisson random variable with parameter 1.

Exercise 3. (/3) Given a sequence $(X_i)_i$ of iid continuous random variables, consider the validity of oscillations of the form

$$X_1 > X_2 < X_3 > X_4 < X_5 > \ldots$$

and let N be the smallest index that violates this rule of oscillations. We have for instance $N = 2$ if $X_1 < X_2$, and $N = 4$ if $X_1 > X_2 < X_3 < X_4$.

(i) Prove the following recurrence identity for the cdf of N,

$$\mathbb{P}[N > k+1] = \frac{1}{2(k+1)} \sum_{j=0}^{k} \mathbb{P}[N > j]\, \mathbb{P}[N > k-j]\,.$$

Hint: Consider the random index J such that $X_{J+1} = \min\{X_i : 1 \le i \le k+1\}$.

(ii) Prove that the generating function $G_N(t) = \sum_{i=0}^{\infty} t^i \mathbb{P}[N > i]$ satisfies $2G_N'(t) = 1 + G_N(t)^2$, and infer that $G_N(t) = \tan(\frac{t}{2} + \frac{\pi}{4})$.

(iii) Compute $\mathbb{E}[N]$ and $\mathrm{Var}[N]$.

Exercise 4. (/2) Let P be uniformly distributed in the interval $(0, 1)$. Given $P = p$, consider a sequence of independent Bernoulli experiments with success probability p.

(i) Let $0 \le k \le n$. Compute the conditional pdf of P given that there have been k successes in n independent Bernoulli experiments. Deduce an explicit expression for the expectation of P given that there have been k successes in n independent Bernoulli experiments.

(ii) Assume that the rising of the sun is a random event that occurs independently each morning with unknown probability. Alice is 20 years old today and has watched the sunrise every morning since she was born. Based on this, how can she calculate the probability that the sun will also rise tomorrow?

Exercise 5. (/4) Given $a, b > 0$, let X, Y be independent random variables with $\Gamma(a, 1)$ and $\Gamma(b, 1)$ distributions, respectively.

(i) Show that $X + Y$ and $\frac{X}{X+Y}$ are independent and find their distribution.

(ii) Show that $\mathbb{E}\left[\frac{X}{X+Y}\right] = \frac{\mathbb{E}[X]}{\mathbb{E}[X]+\mathbb{E}[Y]}$.

Exercise 6. (/5) Given a sequence $(X_i)_i$ of iid random variables with expectation μ and variance σ^2, consider

$$S_n^2 := \frac{1}{n-1} \sum_{i=1}^{n} \Big(X_i - \frac{1}{n} \sum_{l=1}^{n} X_l \Big)^2 .$$

(i) Show that $\mathbb{E}\left[S_n^2\right] = \sigma^2$.

(ii) Provided that $\mathbb{E}\left[X_1^4\right]$ exists, compute $\mathrm{Var}\left[S_n^2\right]$ and infer $S_n^2 \to \sigma^2$ in L^2.

(iii) Show that actually $S_n^2 \to \sigma^2$ almost surely.

(iv) If a sequence Z_n converges to Z in distribution and if a sequence Y_n converges to a constant $c \neq 0$ in probability, show that Z_n/Y_n converges to Z/c in distribution.

(v) Using (iii) and (iv), show that

$$W_n := \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{nS_n^2}}$$

converges in distribution to a standard Gaussian. Briefly describe how this result can be used to design asymptotic confidence intervals for the parameter μ.

Exercise 7. (/2) Let $\{X_n\}_n$ be a sequence of independent standard normal random variables. Using the Borel–Cantelli lemma, prove that

$$\mathbb{P}\left[\limsup_{n\uparrow\infty} \frac{|X_n|}{\sqrt{2\log n}} = 1\right] = 1.$$