Please provide complete and well-written solutions to the following exercises.
Due on January 22th before noon.

Homework 2

Exercise 1 (Bertrand's ballot theorem)**.** Consider an election where candidate A receives n votes and candidate B receives m votes, with $n > m$, and assume that votes are counted sequentially at random.

(a) The sequential count can be viewed as a path in $\mathbb{Z}^2$, in the following way: start from $(0,0)$, and add $(1,1)$ in case of a vote for A, or add $(1,-1)$ in case of a vote for B. If the point (x,y) is on the path, y represents the difference between the number of votes for A and that for B after the xth vote is counted. The last point of the path is $(n+m, n-m)$. In these terms, describe paths corresponding to the event that A is strictly ahead of B throughout the count.

(b) Given $a < a'$ and $b, b' \in \mathbb{Z}$, compute the total number of paths from (a,b) to (a',b').

(c) Given $a < a'$ and $b, b' \in \mathbb{Z}$, compute the number of paths from (a,b) to (a',b') that cross the axis $x = 0$.
Hint: Use and justify the reflection method: the number of paths from (a,b) to (a',b') that cross the axis $x = 0$ is the same as the total number of paths from (a,b) to $(a',-b')$.

(d) Using the previous results, deduce the probability that A is strictly ahead of B throughout the count.

Exercise 2. Let D_1, D_2, D_3 be three 4-sided dice whose sides have been labeled as follows:

$$D_1 : 0,3,3,3, \qquad D_2 : 2,2,2,5, \qquad D_3 : 1,1,4,6.$$

The three dice are rolled at random. Let A be the event that the outcome of D_1 is larger than D_2, let B the event that the outcome of D_2 is larger than D_3, and let C the event that the outcome of D_3 is larger than the outcome of D_1. Compute $\mathbb{P}[A], \mathbb{P}[B], \mathbb{P}[C]$. What is the best die?

Exercise 3. Throw a fair 6-sided die many times in a row and consider the successive results. What is the probability to obtain a 1 or a 2 before getting a 6?

Exercise 4. A biased coin shows heads with probability p whenever it is tossed. Compute the probability that no two heads occur successively in n tosses. For that purpose, denoting by u_n this probability, show that the following recurrence relation holds for all $n \geq 1$,

$$u_{n+2} = (1-p)u_{n+1} + p(1-p)u_n,$$

and then deduce the value of u_n.

Exercise 5. From the set $\{1,2,3,\ldots,n\}$, k distinct integers are selected at random and arranged in numerical order (from lowest to highest). Let $P(i,r,k,n)$ denote the probability that integer i is in position r. For example, observe that $P(1,2,k,n) = 0$, as it is impossible for the number 1 to be in the second position after ordering. Find a general formula for $P(i,r,k,n)$.

Exercise 6. You are travelling on a train with your sister. Neither of you has a valid ticket, and the inspector has caught you both. He is authorized to administer a special punishment for this offense. He holds a box containing nine apparently identical chocolates, three of which are contaminated with a deadly poison. He makes each of you, in turn, choose and immediately eat a single chocolate.

(a) If you choose before your sister, what is the probability that you will survive?

(b) If you choose first and survive, what is the probability that your sister survives?

(c) If you choose first and die, what is the probability that your sister survives?

(d) Is it in your best interests to persuade your sister to choose first?

(e) If you choose first, what is the probability that you survive, given that your sister survives?

Exercise 7. One day, Alice decides that she will start looking for a potential life partner on a dating app. She decides that every day she will pick a guy uniformly at random from among the male members of the app and date him. What Alice does not know, is that her neighbor Bob is interested in dating her. Being of a shy disposition, Bob decides that he will not ask Alice out himself. Instead, he decides that he will go out on a date with Alice only on the days that she happens to pick him from the dating app, of which he is already a member.

(i) What is the probability that Alice and Bob meet infinitely many times in this way?

(ii) Now assume that the number of male members of the app increases by 1% everyday. How does the answer to (i) change? Is there a risk for Bob never dating Alice?

Exercise 8. There are n socks in a drawer, three of which are red and the rest black. John chooses his socks by selecting two at random from the drawer, and puts them on.

(a) Given that he is three times more likely to wear socks of different colors than to wear matching red socks, determine the value of n.

(b) For this value of n, what is the probability that John wears matching black socks?

Exercise 9. Let a random experiment be the casting of three fair 6-sided dice.

(a) Find the distribution of the sum of the three outcomes.

(b) Find the distribution of the minimum of the three outcomes.

(c) Find the distribution of the maximum of the three outcomes.

(d) Find the distribution of the difference between the maximum and the minimum of the three outcomes.

Exercise 10. Determine for what values of c and α the following functions are probability mass functions:

(a) $p(k) = \frac{c}{k(k+1)}$ for $k \in \{1, 2, 3, \ldots\}$;

(b) $p(k) = ck^{\alpha}$ for $k \in \{1, 2, 3, \ldots\}$.

Exercise 11. Let X be a random variable with geometric distribution with parameter p. What is the probability that X is larger than k?