Please provide complete and well-written solutions to the following exercises.
Due on February 12th before noon.

Homework 4

Exercise 1. Each toss of a coin results in heads with probability p. If $m(r)$ is the mean number of tosses up to and including the rth head, show that for $r \geq 1$,

$$m(r) = p(1 + m(r-1)) + (1-p)(1 + m(r)).$$

Deduce the value of $m(r)$.

Exercise 2. An ambidextrous student has a left and a right pocket, each initially containing n candies. Each time he feels hungry, he puts a hand into one of his pockets and, if it is not empty, he takes a candie from it and eats it. On each occasion, he is equally likely to choose either the left or right pocket. When he first puts his hand into an empty pocket, the other pocket contains X candies. Find the expected value of X.

Exercise 3. Consider a simple random walk $(S_n)_n$ starting at $S_0 = 0$: at each integer time $n \geq 1$, the walk S_n goes either up (i.e. $S_n := S_{n-1} + 1$) or down (i.e. $S_n := S_{n-1} - 1$), both with equal probabilities. Define return times inductively as follows: let $T_0 := 0$ and for all $k \geq 1$, given that $T_{k-1} < \infty$, define T_k as the time of the first return to 0 after T_{k-1},

$$T_k := \min\{n > T_{k-1} : S_n = 0\}.$$

(i) Compute $\mathbb{P}[S_{2n} = 0]$.

(ii) Prove that $\sum_m \mathbb{P}[S_m = 0] = \infty$. Can the Borel–Cantelli lemma be applied?

(iii) Prove that $\sum_m \mathbb{P}[S_m = 0] = \sum_k \mathbb{P}[T_k < \infty]$.

(iv) Prove that, given that $T_{k-1} < \infty$, the difference $T_k - T_{k-1}$ is independent of T_{k-1}.

(v) Prove that $\mathbb{P}[T_k < \infty] = \mathbb{P}[T_1 < \infty]^k$.

(vi) Deduce $\mathbb{P}[T_1 < \infty] = 1$.

(vii) Conclude that the walk is recurrent: it crosses 0 infinitely often with probability 1.

Exercise 4. Consider a simple random walk $(S_n)_n$ on $\mathbb{Z}^2$ starting at $S_0 = (0,0)$: at each integer time $n \geq 1$, the walk S_n goes in each possible direction $(1,0)$, $(0,1)$, $(-1,0)$, or $(0,-1)$ with equal probabilities. As in the previous exercise, show that the walk is recurrent: it visits $(0,0)$ infinitely often with probability 1.

Exercise 5. Consider a simple random walk $(S_n)_n$ on $\mathbb{Z}^3$ starting at $S_0 = (0,0,0)$: at each integer time $n \geq 1$, the walk S_n goes in each possible direction $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,0,0)$, $(0,-1,0)$, or $(0,0,-1)$ with equal probabilities.

(i) Show that $\mathbb{P}[S_{2n} = 0] \leq Cn^{-\frac{3}{2}}$ for a suitable constant C.

(ii) Conclude that the walk is transient: it visits $(0,0,0)$ at most a finite number of times with probability 1.

Exercise 6. The pair of discrete random variables (X, Y) has joint mass function $\mathbb{P}[X = i, Y = j] = \theta^{i+j+1}$ if $i, j = 0, 1, 2$, and $\mathbb{P}[X = i, Y = j] = 0$ otherwise, for some value θ.

(i) Is any value of θ possible?

(ii) Find the marginal mass function of X.

(iii) Compute $\mathbb{E}[X]$ and $\mathbb{E}[XY]$.

Exercise 7. Let X and Y be independent Poisson random variables with parameters λ and μ, respectively. Show that $X+Y$ has the Poisson distribution with parameter $\lambda+\mu$. Give an example to show that the conclusion is not generally true if X and Y are dependent.

Exercise 8. Consider a sequence of independent random events, each having success probability p. Compute the probability that the first success occurs on the kth trial given that the second success occurs on the nth trial.

Exercise 9. Let $X_1, \ldots, X_n$ be independent random variables that are uniformly distributed on $\{1, \ldots, N\}$. Find the distribution of $\min\{X_1, \ldots, X_n\}$ and $\max\{X_1, \ldots, X_n\}$.

Exercise 10. Hugo's bowl of spaghetti contains n strands. He selects two ends at random and joins them. He does this until no ends are left. What is the expected number of spaghetti hoops in his bowl?

Exercise 11. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables, and let $S_n = X_1 + \ldots + X_n$.

(i) Show that $\mathbb{E}\left[\frac{S_m}{S_n}\right] = \frac{m}{n}$ for $0 \leq m \leq n$.

(ii) Show that $\mathbb{E}\left[\frac{S_m}{S_n}\right] = 1 + \frac{m-n}{n}\mathbb{E}[X_1]\,\mathbb{E}\left[\frac{1}{S_n}\right]$ for $m > n$.

Exercise 12. We are provided with a coin which comes up heads with probability p at each toss. Let $v_1, v_2, \ldots, v_n$ be n distinct points on a unit circle. We examine each unordered pair (v_i, v_j) in turn and toss the coin; if it comes up heads, we join v_i and v_j by a line (called an edge), otherwise we do nothing. The resulting network is called a random graph.

(i) Compute the expected number of edges in the random graph.

(ii) Compute the expected number of triangles.