Please provide complete and well-written solutions to the following exercises.
Due on February 19th before noon.

Homework 5

Exercise 1. Let X, Y, Z be continuous random variables with the property that their values are distinct with probability 1. Let $a = \mathbb{P}[X > Y]$, $b = \mathbb{P}[Y > Z]$, $c = \mathbb{P}[Z > X]$. You showed in the Midterm that $\min\{a, b, c\} \le \frac{2}{3}$. This is known as Condorcet's voting paradox: in an election, it is possible for more than half of the voters to prefer candidate A to candidate B, more than half B to C, and at the same time more than half C to A. Now construct an example where this bound $\min\{a, b, c\} \le \frac{2}{3}$ is attained.

Exercise 2. Let X be a non-negative absolutely continuous random variable with distribution F_X. Show that for any $r \ge 1$,

$$\mathbb{E}[X^r] = \int_0^\infty r t^{r-1}(1 - F_X(t))\, dt,$$

provided the expectation is fininte.

Exercise 3.

(i) If X has a standard normal distribution, show that for all bounded functions $\phi \in C^1(\mathbb{R})$ there holds

$$\mathbb{E}[X\phi(X)] \,=\, \mathbb{E}\left[\phi'(X)\right]. \tag{$\star$}$$

(ii) Conversely, if X is an absolutely continuous random variable with C^1 density f_X, and if it satisfies ($\star$) for all bounded functions $\phi \in C^1(\mathbb{R})$, show that X has a standard normal distribution.

(iii) For fixed $z \in \mathbb{R}$, given a random variable Z with standard normal distribution, show that there is a unique bounded solution $\phi_z \in C^1(\mathbb{R})$ of the equation

$$\phi_z'(t) - t\phi_z(t) \,=\, \mathbb{1}_{t \le z} - \mathbb{P}[Z \le z]\,.$$

Hint: Solve this equation for ϕ_z explicitly. For that purpose, you may start by multiplying both sides by the standard normal density $\frac{1}{\sqrt{2\pi}} e^{-t^2/2}$.

(iv) Using (iii), show that the conclusion of (ii) holds even without assuming that X has a C^1 density: if X is an absolutely continuous random variable that satisfies ($\star$) for all bounded functions $\phi \in C^1(\mathbb{R})$, then X has a standard normal distribution. This is known as Stein's characterization.

Exercise 4.

(i) Let X be a continuous random variable with distribution function F_X. Show that the random variable $Y = F_X(X)$ is uniformly distributed on $(0, 1)$.

(ii) Let F be a continuous distribution function and let Y be a random variable that is uniformly distributed on $(0, 1)$. Let F^{-1} denote the pseudo-inverse of F, which is defined by $F^{-1}(y) := \inf\{x : F(x) \ge y\}$. Show that the random variable $X = F^{-1}(Y)$ has distribution F. Deduce a way to generate with a computer (pseudo)random numbers from any given distribution F.

Exercise 5. William Tell places a small green apple on top of a straight wall which stretches to infinity in both directions. He then takes up position at a distance of one perch from the apple, so that his line of sight to the target is perpendicular to the wall. He now selects an angle uniformly at random from his entire field of view and shoots his arrow in this direction. Assuming that his arrow hits the wall somewhere, what is the distribution function of the horizontal distance (measured in perches) between the apple and the point which the arrow strikes?

Exercise 6. If the random variable X is uniformly distributed on $[0,1]$, find the distribution and probability density of $Y = \frac{3X}{1-X}$.

Exercise 7. Construct two *dependent* continuous random variables X, Y that are uncorrelated, that is, such that $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$.

Exercise 8. Let $T_1 \sim \text{Exp}(\lambda_1)$ and $T_2 \sim \text{Exp}(\lambda_2)$ be independent.

(i) Show that the minimum is also exponential: $\min\{T_1, T_2\} \sim \text{Exp}(\lambda_1 + \lambda_2)$.

(ii) Compute the expectation and the variance of $\min\{T_1, T_2\}$, $\max\{T_1, T_2\}$, and $|T_1 - T_2|$.

(iii) Show that

$$\mathbb{P}[T_1 < T_2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

(iv) Let I be the random variable defined by

$$I := \begin{cases} 1, & \text{if } T_1 \leq T_2; \\ 2, & \text{if } T_1 > T_2; \end{cases}$$

and show that I and $\min\{T_1, T_2\}$ are independent.

(v) Find the conditional density of T_1 given that $T_1 + T_2 = a$. What does it become if $\lambda_1 = \lambda_2$?

(vi) Alice and Bob enter a beauty parlor simultaneously: Alice to get a haircut and Bob to get a manicure. Suppose that the haircut (resp. the manicure) is exponentially distributed with mean 30 minutes (resp. 20 minutes). What is the probability Alice gets done first? What is the expected amount of time until Alice and Bob are both done?

Exercise 9. Let $X_1, \ldots, X_n$ be independent absolutely continuous random variables with common distribution F and density f. Consider the random variables $U = \min\{X_1, \ldots, X_n\}$ and $V = \max\{X_1, \ldots, X_n\}$. Find the marginal densities of U and V and show that their joint density is given by

$$f_{U,V}(u,v) = \begin{cases} n(n-1)f(u)f(v)\big(F(v) - F(u)\big)^{n-2} & : \; u < v, \\ 0 & : \; u < v. \end{cases}$$

Exercise 10. Let $(X_n)_n$ be a sequence of independent, identically distributed, and continuous random variables. Define N as the index such that

$$X_1 \geq X_2 \geq \ldots \geq X_{N-1} \qquad \text{and} \qquad X_{N-1} < X_N.$$

Show that $\mathbb{P}[N = k] = \frac{k-1}{k!}$ and that $\mathbb{E}[N] = e$.

Exercise 11. If X, Y be independent standard normal random variables.

(i) Compute the mean and variance of $|X|$.

(ii) Find the joint density of $U = X - Y$ and $V = X + Y$. Are they independent?

(iii) Let (R, Θ) be the polar coordinates of the random vector (X, Y), that is, $X := R\cos\Theta$ and $Y = R\sin\Theta$. Show that Z and Θ are independent and find their distribution.

(iv) Compute $\mathbb{E}\left[\frac{X^2}{X^2+Y^2}\right]$, $\mathbb{E}\left[\frac{\min\{|X|,|Y|\}}{\max\{|X|,|Y|\}}\right]$, and $\mathbb{E}\left[\frac{\max\{|X|,|Y|\}}{\min\{|X|,|Y|\}}\right]$.

(v) Find the distribution of Y/X and show that its expectation does not exist.