This last homework sheet is optional and will not be graded.

Homework 8

Exercise 1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function, and let S_n be a random variable having a binomial distribution with parameters n, x. Considering the expectation of $f(x) - f(\frac{1}{n}S_n)$, and distinguishing between the event $\{|\frac{1}{n}S_n - x| > \delta\}$ and its complement, show that

$$\lim_{n\uparrow\infty} \sup_{0\le x\le 1} \left| f(x) - \sum_{k=0}^{n} f(\tfrac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k} \right| = 0.$$

This is Weierstrass' approximation theorem, stating that every continuous function on $[0,1]$ may be approximated by a polynomial uniformly over the interval.

Exercise 2. A sequence $(X_n)_n$ of random variables is said to be completely convergent to X if

$$\sum_n \mathbb{P}\left[|X_n - X| > \varepsilon\right] < \infty \qquad \text{for all } \varepsilon > 0.$$

Show that for sequences of independent random variables complete convergence is equivalent to a.s. convergence. Find a sequence of (dependent) random variables that converges a.s. but not completely.

Exercise 3. Let $\{X_n\}_n$ be a sequence of independent standard normal random variables. Show that

$$\mathbb{P}\left[\limsup_{n\uparrow\infty} \frac{|X_n|}{\sqrt{2\log n}} = 1\right] = 1.$$

Exercise 4. Let $(X_n)_n$ be independent random variables such that

$$\mathbb{P}\left[X_n = n\right] = \mathbb{P}\left[X_n = -n\right] = \tfrac{1}{2n\log n}, \qquad \mathbb{P}\left[X_n = 0\right] = 1 - \tfrac{1}{n\log n}.$$

Show that this sequence obeys the weak law but not the strong law of large numbers: more precisely, $\frac{1}{n}\sum_{i=1}^n X_i$ converges to 0 in probability but not almost surely.

Exercise 5. Let X_n and Y_m be independent random variables having Poisson distribution with parameters n and m, respectively. Show that

$$\frac{(X_n - n) - (Y_m - m)}{\sqrt{X_n + Y_m}} \xrightarrow{d} \mathcal{N}(0,1) \qquad \text{as } n, m \uparrow \infty.$$

Exercise 6. Let $\{X_n\}_n$ be a sequence of iid random variables with $\mathbb{P}\left[X_n = 1\right] = \mathbb{P}\left[X_n = -1\right] = \frac{1}{2}$. Does the random harmonic sum $\sum_{r=1}^n \frac{1}{r} X_r$ converge a.s. as $n \uparrow \infty$?

Exercise 7. Let $(X_n)_n$ be iid random variables with uniform distribution on $(0,1)$ and let $Z_n = \max\{X_1, \ldots, X_n\}$. As $n \uparrow \infty$, show that

$$Z_n \xrightarrow{\mathbb{P}} 1, \qquad \sqrt{Z_n} \xrightarrow{\mathbb{P}} 1, \qquad n(1 - Z_n) \xrightarrow{d} \mathrm{Exp}(1).$$

Exercise 8. Let $(X_n)_n$ be iid random variables with distribution function F and pdf f. The order statistics $X_{(1)}, \ldots, X_{(n)}$ of the subsequence $X_1, \ldots, X_n$ are obtained by rearranging the values of the

X_i in non-decreasing order. In particular, $X_{(1)} = \min\{X_1, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, \ldots, X_n\}$. The sample median Y_n of the sequence $X_1, \ldots, X_n$ is the "middle value", defined by

$$Y_n = \begin{cases} X_{(r+1)} & : \; n = 2r+1 \text{ odd}, \\ \frac{1}{2}(X_{(r)} + X_{(r+1)}) & : \; n = 2r \text{ even}. \end{cases}$$

Assuming that $n = 2r+1$ is odd, show that Y_n has pdf

$$f_{Y_n}(y) = (r+1)\binom{n}{r} F(y)^r (1 - F(y))^r f(y).$$

Deduce that, if F has median m, then

$$2n^{\frac{1}{2}} f(m)(Y_n - m) \xrightarrow{d} \mathcal{N}(0, 1).$$