Midterm — February 7th

Please provide complete and well-written solutions to the following exercises. All answers must be properly explained and justified.

Exercise 1. (/1) Let $(X_n)_n$ be a sequence of random variables having the same expectation $\mathbb{E}[X_n] = \mu$, let N be an integer-valued random variable with finite expectation, and assume that N is independent of X_n for all n. Show that

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}[N]\,\mathbb{E}[X_1].$$

Exercise 2. (/1.5) Suppose that a tree drops X seeds. Assume that X is a Poisson random variable with parameter 2. Each seed becomes a new tree with probability $\frac{1}{2}$.

(i) What is the expected number of seeds dropped?

(ii) Find the pmf of the random variable S that is the number of new trees.

(iii) What is the expected number of new trees?

Exercise 3. (/1.5) Let X, Y, Z be continuous random variables, and let $a := \mathbb{P}[X > Y]$, $b := \mathbb{P}[Y > Z]$, and $c := \mathbb{P}[Z > X]$.

(i) Show that $\min\{a, b, c\} \le \frac{2}{3}$.

(ii) If X, Y, Z are independent and have the same distribution, show that $a = b = c = \frac{1}{2}$.

Exercise 4. (/3) Let X be an absolutely continuous random variable. Recall that its median m_X is defined as the smallest value m such that $F_X(m) \ge \frac{1}{2}$.

(i) Compute the set of values b that minimize the function $b \mapsto \mathbb{E}[|X - b|]$. Show that m_X coincides with the minimum of this set.

(ii) Show the following analogue of Jensen's inequality for medians: if $\phi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, then $\phi(m_X) \le m_{\phi(X)}$.

(iii) Using an integration by parts, show that

$$m_X - \mathbb{E}[X] = \int_0^\infty \big(F_X(m_X - t) + F_X(m_X + t) - 1\big)\, dt.$$

Deduce that the inequality $m_X \ge \mathbb{E}[X]$ holds provided we have $F_X(m_X - t) + F_X(m_X + t) \ge 1$ for all $t \in \mathbb{R}$.

(iv) Compute the expectation and the median of a Bernoulli random variable and deduce that the three cases $m_X < \mathbb{E}[X]$, $m_X = \mathbb{E}[X]$, and $m_X > \mathbb{E}[X]$ are possible.

Exercise 5. (/3) You are lost in a national park and you are trying to find a ranger to ask for directions. However, there are two types of rangers: Two-thirds of them will give you a correct answer with probability $\frac{3}{4}$, independently each time you ask. The remaining third of the rangers will always give you the wrong answer.

(i) You find a ranger, without knowing to which type they belong. You ask whether the exit from the park is East or West, and their answer is East. What is the probability that it is correct?

(ii) You ask the same ranger again and receive the same answer. Show that the probability that it is correct is $\frac{1}{2}$.
Hint: Let T be the event that the answer is correct and let E_2 be the event that you get twice the same answer. Use Bayes' rule to compute $\mathbb{P}[T|E_2]$, assuming that East and West are equally likely to be actually the correct answer.

(iii) You ask the same person again for the third time and receive the same answer. Now what is the probability that it is correct?

(iv) You ask the same person again for the fourth time and receive the same answer. Now what is the probability that it is correct?

(v) Had the fourth answer been West instead, what is the probability that East was the correct answer?