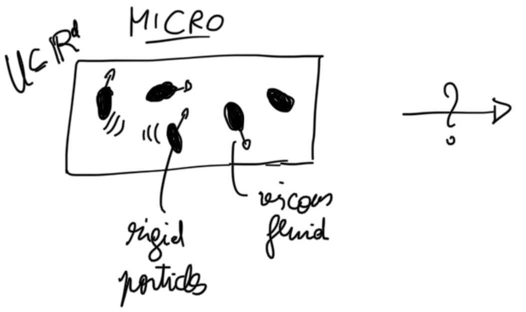
# MATHEMATICS OF PARTICLE SUSPENSIONS — A TALE FROM MICRO TO MACRO

## 0 Introduction.

Motivation : Particle suspensions — from micro to macro




MACRO

complex non-Newtonian properties
(nonlinearity, memory)
e.g. quicksand (shear-thinning)
ketchup (same + memory)

Micro system :

- rigid particles $\{I^n_{\varepsilon,N}\}_{1\le n\le N}$ $\begin{cases} \text{diam } \varepsilon \ll 1 \\ \text{number } N \gg 1 \end{cases}$

  center $X^n_{\varepsilon N} \in U$,
  orientation $R^n_{\varepsilon N} \in S^{d-1}$ (resp: ellipsoids of revolution)

- Stokes fluid :

$$\begin{cases} -\Delta u_{\varepsilon,N} + \nabla p_{\varepsilon,N} = \overset{\text{internal force}}{h} & \text{in } U \setminus \bigcup_n I^n_{\varepsilon,N} \\ \operatorname{div} u_{\varepsilon,N} = 0, \quad u_{\varepsilon,N} \in H^1_0(U) \end{cases}$$

  \+ rigid BCs (no-slip) : $\quad u_{\varepsilon,N}|_{I^n_{\varepsilon,N}} = V^n_{\varepsilon,N} + W^n_{\varepsilon,N} \times (x - X^n_{\varepsilon,N})$

  for some $V^n_{\varepsilon,N}, W^n_{\varepsilon,N} \in \mathbb{R}^3$ (unknown)

  ie $Du_{\varepsilon,N}|_{I^n_{\varepsilon,N}} = 0 \quad \forall n.$ $\qquad \left(Du \equiv \frac{1}{2}\left((\nabla u) + (\nabla u)^T\right)\right)$

- neglect particle inertia : Newton's eqn $\to$ balance of forces / torques

$$\begin{cases} \underbrace{\int_{\partial I^n_{\varepsilon,N}} \sigma(u_{\varepsilon,N}, p_{\varepsilon,N})\,\nu}_{\text{force exerted by fluid}} + \underbrace{|I^n_{\varepsilon,N}|\, e}_{\text{sedimentation}} = 0 \\ \int_{\partial I^n_{\varepsilon,N}} (x - X^n_{\varepsilon,N}) \times \sigma(u_{\varepsilon,N}, p_{\varepsilon,N})\,\nu = 0 \end{cases}$$

  , $\sigma(u,p) = 2Du - p\,\mathrm{Id}$ Cauchy stress tensor.
  (note $-\operatorname{div} \sigma(u,p) = -\Delta u + \nabla p$)

- particles' motion : $\begin{cases} \partial_t X^n_{\varepsilon,N} = V^n_{\varepsilon,N} \\ \partial_t R^n_{\varepsilon N} = W^n_{\varepsilon,N} \times R^n_{\varepsilon,N} \end{cases} \quad \forall n.$

$\Rightarrow$ quasistatic evolution: ① given particles' positions $\{I_{\varepsilon N}^n\}_n \equiv \{X_{\varepsilon N}^n, R_{\varepsilon N}^n\}_n$

$\exists !$ fluid velocity $u_{\varepsilon N} \in H_0^1(U)$:

$$\begin{cases} -\Delta u_{\varepsilon N} + \nabla p_{\varepsilon N} = h, \quad \operatorname{div} u_{\varepsilon N} = 0, \quad U \setminus \bigcup_n I_{\varepsilon N}^n \\ D u_{\varepsilon N} = 0 \quad \text{in } \bigcup_n I_{\varepsilon N}^n \\ \int_{\partial I_{\varepsilon N}^n} \sigma_{\varepsilon N} \nu + |I_{\varepsilon N}| e = 0, \quad \int_{\partial I_{\varepsilon N}^n} (x - X_{\varepsilon N}^n) \times \sigma_{\varepsilon N} \nu = 0 \quad \forall n. \end{cases}$$

exercise: this system is well-posed
(see below for corresponding scalar problem)

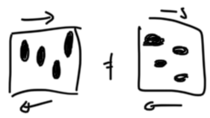
② update particle positions:

$$\begin{cases} \partial_t X_{\varepsilon N}^n = V_{\varepsilon N}^n, \quad \partial_t R_{\varepsilon N}^n = W_{\varepsilon N}^n \times R_{\varepsilon N}^n \\ \text{where } V_{\varepsilon N}^n + W_{\varepsilon N}^n \times (x - X_{\varepsilon N}^n) \equiv u_{\varepsilon N}|_{I_{\varepsilon N}^n} \quad \forall n. \end{cases}$$

Macro limit:
- averaging / "homogenization" for fluid flow given microstructure

i.e. $u_\varepsilon \sim \bar{u}$ : $\begin{cases} -\operatorname{div}(\bar{B} \nabla \bar{u}) = h & \text{in } U. \\ \operatorname{div}(\bar{u}) = 0 \end{cases}$

$\bar{B}$ = effective viscosity (depending on microstructure)
complicated...



- microstructure evolves in time, adapting to external forces

$\Rightarrow$ expect non-Newtonian behavior: $\begin{cases} \text{nonlinear response} \\ \text{memory.} \end{cases}$ (e.g. ketchup)

Plan of course: Ⅰ homogenization, effective viscosity

Ⅱ dilute expansions: simplifying dependence on microstructure
'effective medium expansions'

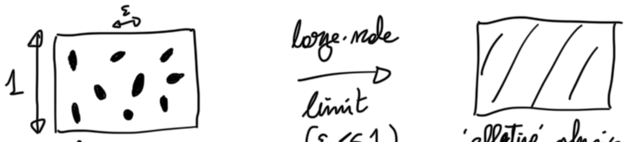
} focus on fluid flow (given microstructure)

Ⅲ coupling to dynamics in dilute regime –
mean-field limits, hydro limit.

} back to dynamics $\rightarrow$ non-Newt.

Difficulties: hydro interactions are $\begin{cases} \text{multi-body } (\rightarrow \text{nontrivial homog.}) \\ \text{non-explicit} \\ \text{long-range } (\rightarrow \text{convergence issues}) \\ \text{singular at short distance } (\rightarrow \text{make MF difficult}) \end{cases}$ unusual for MF

# (I) Homogenization and effective viscosity.




physics eqn in heterog medium (e.g. Stokes with rigid inclusions)

$L(\frac{\cdot}{\varepsilon}, \nabla)\, u_\varepsilon = h$

'effective' physics eqn ... as if homog medium (e.g. Stokes with effective viscosity)

$\bar{L}(\nabla)\, \bar{u} = h$

≈ law of large numbers for heterog media

DIFFICULTY: solution operator is nonlinear wrt heterogeneities.

## I.1) Probabilistic description of microstructure

- let $\{I_n\}_n$ family of random sets $\iff$ $\{(X_n, R_n)\}_n$ decorated point process (constructed on some proba space $(\Omega, \mathbb{P})$)

$I_n = I(X_n, R_n)$ (e.g. ellipsoids of revolution)

- disjointness assumption: $I_n \cap I_m = \emptyset \quad \forall n \neq m$ a.s.
- simplest proba setting for averaging:

* spatially homogeneous statistics ("stationarity") for $\mathcal{I} = \bigcup_n I_n$

ie law of $\mathcal{I} + x$ = law of $\mathcal{I}$ $\quad \forall x \in \mathbb{R}^d$

ie $\mathbb{P}[x_1 \in \mathcal{I} + x, \ldots, x_n \in \mathcal{I} + x]$ is independent of $x$ $\forall x_1 \ldots x_n, n$

terminology: - $\psi$ "random field" if $\psi : \mathbb{R}^d \times \Omega \to \mathbb{R}$ $\sigma(\mathcal{I})$-measurable (think $\psi = \psi(\cdot\,; \mathcal{I})$)

- $\psi$ "stationary" if $\psi = \psi(\cdot\,;\mathcal{I})$ satisfies $\psi(x+y; \mathcal{I}) = \psi(x; \mathcal{I} - y)$ a.s. $\forall x, y$

[rem: for $\psi$ stat, we have $\mathbb{E}\,\psi(x) = \mathbb{E}\,\psi$ indep of $x$!]

* ergodicity for $\mathcal{I}$ (probabilistic "minimality")

ie $\begin{cases} \text{if a stationary random field } \psi \text{ is translation-invariant (ie } \psi(x+y) = \psi(x) \text{ a.s. } \forall x,y \text{, ie } \psi(x, \mathcal{I}-y) = \psi(x, \mathcal{I})) \\ \text{then } \exists c \in \mathbb{R},\ \psi = c \text{ a.s.} \end{cases}$

**Reminder**: **Ergodic theorem** (Birkhoff-Khinchine) (generalization of LLN):

Under ergodicity assumption,
For any $\sigma(\mathcal{Y})$-meas. stationary random field $\psi$ with $\mathbb{E}|\psi|<\infty$,
we have: $\underbrace{⨍_{B_R} \psi}_{\text{spatial average}} \xrightarrow{R\uparrow\infty} \underbrace{\mathbb{E}\psi}_{\text{ensemble average}}$ a.s.

& $\psi(\cdot/\varepsilon) \rightharpoonup \mathbb{E}\psi$ in $L^1_{loc}(\mathbb{R}^d)$ a.s

**Application**: volume fraction $\lambda := \lim_{R\uparrow\infty} \dfrac{|B_R \cap \mathcal{Y}|}{|B_R|}$ exists

$= \mathbb{E}\, \mathbb{1}_{\mathcal{Y}}$ a.s.

## I.2) Simplified setting (for convenience)

We shall focus on corresponding $\begin{cases} \text{scalar problem (no pressure!)} \\ \text{in } \mathbb{R}^d \text{ (no boundary issue)} \end{cases}$

$$\begin{cases} -\Delta u_\varepsilon = h^{\in C^\infty_c(\mathbb{R}^d)} & \text{in } \mathbb{R}^d \setminus \varepsilon\mathcal{Y} \qquad (\mathcal{Y} := \bigcup_n I_n) \\ u_\varepsilon \in \dot{H}^1(\mathbb{R}^d) \\ \nabla u_\varepsilon|_{\varepsilon\mathcal{Y}} = 0 \\ \int_{\varepsilon\partial I_n} \partial_\nu u_\varepsilon = 0 \quad \forall n \end{cases}$$

"stiff inclusions" (conducting spheres in electric field)

**Exercise**

* weak formulation is:

$$\begin{cases} u_\varepsilon \in \dot{H}^1(\mathbb{R}^d),\ \nabla u_\varepsilon|_{\varepsilon\mathcal{Y}} = 0 \\ \int_{\mathbb{R}^d} \nabla v \cdot \nabla u_\varepsilon = \int_{\mathbb{R}^d\setminus\varepsilon\mathcal{Y}} h\, u_\varepsilon \quad \forall v \in H^1(\mathbb{R}^d) \text{ with } \nabla v|_{\varepsilon\mathcal{Y}} = 0. \end{cases}$$

* a priori estimate: $v = u_\varepsilon \rightsquigarrow \|\nabla u_\varepsilon\|_{L^2} \lesssim \|h\|_{L^{2d/(d+2)}}$
* this problem is well-posed (Lax-Milgram) (a.s. – provided inclusions are disjoint)
* Consider orthogonal projection $\pi_\varepsilon : \dot{H}^1(\mathbb{R}^d) \to \{ v \in \dot{H}^1(\mathbb{R}^d) : \nabla v|_{\varepsilon\mathcal{Y}} = 0 \}$.

  Show that we have: $\begin{cases} u_\varepsilon = \pi_\varepsilon v_\varepsilon \\ -\Delta v_\varepsilon = h\, \mathbb{1}_{\mathbb{R}^d\setminus\varepsilon\mathcal{Y}} \quad \text{in } \mathbb{R}^d. \end{cases}$

  (direct consequence of weak formulation)

**Observation:** (exercise)

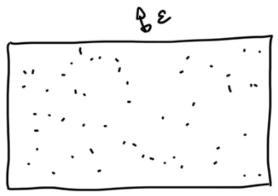
For $k \geq 1$, consider solution $u_\varepsilon^k \in H^1(\mathbb{R}^d)$

of $-\mathrm{div}\Big(\underbrace{(1+k\mathbb{1}_{\varepsilon\mathcal{Y}})}_{(1+k\mathbb{1}_{\mathcal{Y}})(\cdot/\varepsilon)\,=\,a_k(\cdot/\varepsilon)}\nabla u_\varepsilon^k\Big) = h\,\mathbb{1}_{\mathbb{R}^d\setminus\varepsilon\mathcal{Y}}$ in $\mathbb{R}^d$

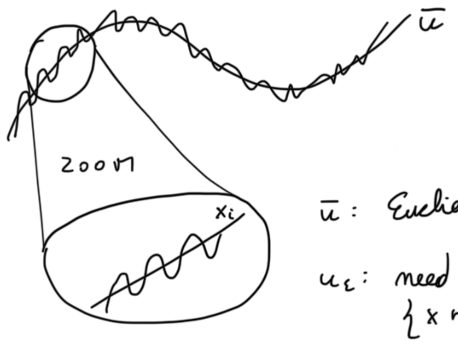
Then $u_\varepsilon^k \rightharpoonup u_\varepsilon$ in $\dot H^1$ as $k \uparrow \infty$.

i.e. stiff inclusions correspond to $\infty$ conductivity (degenerate elliptic problem)

## I.3) Heuristics for homogenization.




$$\begin{cases} -\Delta u_\varepsilon = h, & \varepsilon\mathcal{Y}^c \\ + \text{ stiff BC} \end{cases}$$




$\begin{cases} u_\varepsilon \text{ oscillates on scale } O(\varepsilon) \\ \text{expect } u_\varepsilon \sim \bar u \text{ effective profile} \end{cases}$

$\bar u$: Euclidean coordinates $\{x \mapsto x_i\}_i$, $-\Delta x_i = 0$

$u_\varepsilon$: need corrected coordinate $\{x \mapsto x_i + \underbrace{\varphi_i(x)}_{\text{"corrector"}}\}_i$

$$\begin{cases} -\Delta(x_i+\varphi_i) = 0, & \mathcal{Y}^c \\ + \text{ stiff BC} \end{cases}$$

i.e.
$$\begin{cases} -\Delta\varphi_i = 0, & \mathcal{Y}^c \\ \nabla\varphi_i + e_i = 0, & \mathcal{Y} \\ \int_{\partial I_n} \partial_\nu\varphi_i = 0 & \forall n \end{cases}$$

difficulty: in what space can we construct $\varphi_i$?
- $\varphi_i$ not stationary in general
- want at least $|\varphi_i(x)| \ll |x|$ as $|x| \uparrow \infty$
- can in fact take $\nabla\varphi_i$ stationary

$\rightsquigarrow$ "2-scale expansion" = intrinsic Taylor expansion of effective profile

expect $u_\varepsilon \sim \bar u + \varepsilon\,\varphi_i(\frac{\cdot}{\varepsilon})\nabla_i \bar u$ in $\dot H^1$

Remaining question: what is effective profile $\bar u$?

energy $\int \frac12 |\nabla u_\varepsilon|^2 - \mathbb{1}_{\mathbb{R}^d\setminus\varepsilon\mathcal{Y}}\, h\, u_\varepsilon \sim \int \frac12 \big|(e_i+\nabla\varphi_i)(\frac{\cdot}{\varepsilon})\nabla_i\bar u\big|^2 - (1-\mathbb{1}_{\mathcal{Y}})(\frac{\cdot}{\varepsilon})\, h\,\bar u$

$= \int \frac12 \nabla_i\bar u\, \nabla_j \bar u\, \big[(e_i+\nabla\varphi_i)\cdot(e_j+\nabla\varphi_j)\big](\frac{\cdot}{\varepsilon}) - h\bar u\,(1-\mathbb{1}_{\mathcal{Y}})(\frac{\cdot}{\varepsilon})$

$\to \int \frac12 \nabla\bar u \cdot \bar B\, \nabla \bar u - (1-\lambda)\, h\,\bar u$ by ergodic theorem

where $\begin{cases} \bar B_{ij} = \mathbb{E}\,(e_i+\nabla\varphi_i)\cdot(e_j+\nabla\varphi_j) \\ \lambda = \mathbb{E}\,\mathbb{1}_{\mathcal{Y}} \end{cases}$

$\Rightarrow$ expect $-\nabla\cdot\bar{B}\nabla\bar{u} = (1-\lambda)h$
effective equation

## 1.4) Homogenization theorem.

**Proposition.** under $(H^2_{\text{cluster}})$

$\exists !$ random field $\varphi_i$ s.t. $\begin{cases} -\Delta\varphi_i = 0, & \mathcal{Y}^c \\ \nabla\varphi_i + e_i = 0, & \mathcal{Y} \\ \int_{\partial I_n} \partial_\nu \varphi_i = 0 & \forall n \end{cases}$

with $\nabla\varphi_i$ stationary, $\mathbb{E}\nabla\varphi_i = 0$, $\mathbb{E}|\nabla\varphi_i|^2 < \infty$.

$\rightarrow$ can define $\bar{B}_{ij} := \mathbb{E}(e_i+\nabla\varphi_i)\cdot(e_j+\nabla\varphi_j) = \delta_{ij} + \mathbb{E}\nabla\varphi_i\cdot\nabla\varphi_j$
& $\bar{B} \geq Id$

**Rem**: • no control on $\varphi_i$ itself in general cf $\fint_{B_R}|\varphi_i - \fint\varphi_i|^2 \leq \underbrace{R^2}_{\to\infty} \underbrace{\fint_{B_R}|\nabla\varphi_i|^2}_{\to \mathbb{E}|\nabla\varphi_i|^2}$ (Poincaré disappears as $R\uparrow\infty$)

exercise: $\nexists C$ s.t. $\mathbb{E}|\varphi - \mathbb{E}\varphi|^2 \leq C\,\mathbb{E}|\nabla\varphi|^2$ $\forall \varphi$ stationary! (// BM: non-stationary vs white noise: stationary)

• but we have: $\begin{cases} \nabla\varphi_i \text{ stat} \\ \mathbb{E}\nabla\varphi_i = 0 \\ \mathbb{E}|\nabla\varphi_i|^2<\infty \end{cases}$ $\Rightarrow$ $R^{-2}\fint_{B_R}|\varphi_i|^2 \xrightarrow{R\uparrow\infty} 0$ a.s

& $\varepsilon\varphi_i(\frac{\cdot}{\varepsilon}) \xrightarrow{\varepsilon\downarrow 0} 0$ in $H^1$ a.s.

("$|\varphi_i(x)| \ll |x|$ as $|x|\uparrow\infty$") sublinearity at $\infty$

exercise! as corollary of ergodic theorem

**Theorem.** under $(H^\gamma_{\text{cluster}})$ with $\gamma > d$

$u_\varepsilon - \bar{u} - \varepsilon\varphi_i(\frac{\cdot}{\varepsilon})\nabla_i\bar{u} \longrightarrow 0$ in $\dot{H}^1$ a.s

ie $\nabla u_\varepsilon \rightharpoonup \nabla\bar{u}$ and $\nabla u_\varepsilon - (e_i+\nabla\varphi_i)(\frac{\cdot}{\varepsilon})\nabla_i\bar{u} \longrightarrow 0$ in $L^2$ a.s

BUT this cannot be true in general: cf. stiff clusters

stiff direction $e$

$e\cdot\bar{B}e = \infty$ ... ?!

$\Rightarrow$ in that case: $\nexists\, \nabla\varphi_e$ stat. with $\mathbb{E}\nabla\varphi_e = 0$, $\mathbb{E}|\nabla\varphi_e|^2<\infty$
& $(\nabla\varphi_e + e)|_{\mathcal{Y}} = 0$ a.s.

Geometric assumption:

$(H_{\text{unif}})$ [ $\exists\delta>0$: $\text{dist}(I_n, I_m) \geq \delta$ $\forall n\neq m$ a.s.


... not super physical!

$(H^{\gamma}_{\text{cluster}})$ $\left[\exists\, \delta > 0 :\right.$ the $\delta$-clusters $\{K_\alpha\}_\alpha$ satisfy

$$\mathbb{E} \sum_\alpha \mathbb{1}_{K_\alpha} (\operatorname{diam} K_\alpha)^{\gamma} < \infty$$

ie $\{K_\alpha\}_\alpha$ are the connected components of $\{I_n + \delta B\}_n$.

<u>Observation</u>: $(H^{\gamma=2}_{\text{cluster}}) \Rightarrow \exists\, \varphi_0 : \nabla\varphi_0$ stat, $\mathbb{E}\nabla\varphi_0 = 0$, $\mathbb{E}|\nabla\varphi_0|^2 < \infty$, $(\nabla\varphi_0 + e)|_{\mathcal{I}} = 0$

<u>exercise</u>: choose $\varphi_0(x) = -\sum_\alpha e\cdot(x - z_\alpha)\, \chi_\alpha(x)$, $\chi_\alpha = \begin{cases} 0 & \text{in } (K_\alpha + \delta B)^c \\ 1 & \text{in } K_\alpha \end{cases}$

& check conditions $\qquad |\nabla\chi_\alpha| \lesssim \delta^{-1}$

<u>Exercise</u>: proof of Proposition — using Lax-Milgram in suitable space

- consider $\begin{cases} \mathcal{L}^2 = \{ \Psi \in L^2_{loc}(\mathbb{R}^d \times \Omega) : \Psi \text{ stationary}, \mathbb{E}\Psi = 0, \mathbb{E}|\Psi|^2 < \infty, \nabla \times \Psi = 0 \text{ weakly} \} \\ \mathcal{H}_e = \{ \Psi \in \mathcal{L}^2 : (\Psi + e)|_{\mathcal{I}} = 0 \} \text{ convex} \end{cases}$

  $\Updownarrow$ $\Psi = \nabla\phi$ for some $\phi \in H^1_{loc}(\mathbb{R}^d; S\ldots)$

  by corrector eqn: $\left[\begin{array}{l} \forall \chi \in C^\infty_c(\mathbb{R}^d, L^\infty(\Omega)) \text{ with } \nabla\chi|_{\mathcal{I}} = 0, \\ \forall\, \Psi = \nabla\phi \in \mathcal{H}_0, \\ \int \chi\, \nabla\phi\cdot\nabla\varphi_i + \int \phi\, \nabla\chi\cdot\nabla\varphi_i = 0 \end{array}\right.$

  choosing $\chi \xrightarrow[R\uparrow\infty]{} R^{-d}\chi(\cdot/R)$, we get $\begin{cases} \mathbb{E}\, \Psi\cdot\nabla\varphi_i = 0 & \forall \Psi \in \mathcal{H}_0 \\ \nabla\varphi_i \in \mathcal{H}_{e_i} \end{cases}$

  = weak formulation of corrector eqn

  given $\theta_i \in \mathcal{H}_{e_i}$, this means: $\nabla\varphi_i = \theta_i + \hat{\Phi}_i$

  where $\begin{cases} \mathbb{E}\, \Psi\cdot\hat{\Phi}_i = 0 \;\; \forall \Psi \in \mathcal{H}_0 \\ \hat{\Phi}_i \in \mathcal{H}_0 \end{cases}$ $\Rightarrow$ can use Lax-Milgram for this problem

- missing key point: $\mathcal{H}_{e_i}$ is not empty — cf $(H_{\text{cluster}})$! $\triangleleft$

<u>Proof of qualitative homogenization (Theorem)</u>:

- consider for simplicity unif. elliptic problem $\begin{cases} -\nabla\cdot \overbrace{a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon}^{p_\varepsilon} = h \\ u_\varepsilon \in H^1(\mathbb{R}^d) \end{cases}$

  $|e|^2 \le e\cdot a(x)e \le K|e|^2$

  a priori estimate: $\|\nabla u_\varepsilon\|_{L^2} \le \|h\|_{L^{2d/d+2}}$

  $\Rightarrow$ up to subs: $\begin{cases} u_\varepsilon \rightharpoonup \bar{u}, & L^2 \\ a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon \rightharpoonup \bar{p}, & L^2 \end{cases}$ $\Rightarrow \begin{cases} -\nabla\cdot\bar{p} = h \\ \text{question: } \bar{p} \sim \bar{u}\, ? \end{cases}$

  write $\underbrace{(e_i + \nabla\varphi_i)(\tfrac{\cdot}{\varepsilon})}_{\xrightarrow{L^2_{loc}} e_i} \cdot \underbrace{a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon}_{\xrightarrow{L^2} \bar{p}} = \underbrace{[a(e_i + \nabla\varphi_i)](\tfrac{\cdot}{\varepsilon})}_{\xrightarrow{L^2_{loc}} \mathbb{E}\, a(e_i + \nabla\varphi_i) = \bar{a}e_i} \cdot \underbrace{\nabla u_\varepsilon}_{\xrightarrow{L^2} \nabla\bar{u}}$ .. weak-weak!

compensated compactness: e.g. $\int \chi \; \nabla\varphi(\tfrac{\cdot}{\varepsilon}) \cdot a(\tfrac{\cdot}{\varepsilon}) \nabla u_\varepsilon$
(or div/curl)

$$= - \int \underbrace{\varepsilon\varphi(\tfrac{\cdot}{\varepsilon})}_{\to 0} \nabla\chi \cdot \underbrace{a(\tfrac{\cdot}{\varepsilon}) \nabla u_\varepsilon}_{\to \bar{p}} + \int \chi \, \varepsilon\varphi(\tfrac{\cdot}{\varepsilon}) \, h \to 0 \, !$$

get $\bar{p} = \bar{a} \nabla \bar{u}$

$\Rightarrow -\nabla \cdot \bar{a} \nabla \bar{u} = h$ !

- exercise: how does this argument change for $a = 1 + \kappa \mathbb{1}_{\mathcal{I}}$, $\kappa \uparrow \infty$?

  (focus for simplicity of ($H_{unif}$) – uniform separation)

  $\hookrightarrow$ key is to define a suitable notion of flux $\begin{cases} \operatorname{div} p_i = 0 \text{ in } \mathbb{R}^d, & p_i \text{ stationary} \\ p_i \mathbb{1}_{\mathcal{I}^c} = (\nabla\varphi_i + e_i) \end{cases}$

  & necessarily $\mathbb{E} p_i = \bar{B} e_i$ !

  ie $\quad p_i = (\nabla\varphi_i + e_i) + \sum_n \mathbb{1}_{I_n} \nabla \psi_i^n$

  with $\begin{cases} -\Delta \psi_i^n = 0, & I_n \\ \partial_\nu \psi_i^n = \partial_\nu \varphi_i, & \partial I_n \end{cases}$

  & necessarily $\bar{B}_{ij} = \mathbb{E}\, \underbrace{(\nabla\varphi_i + e_i)}_{= 0 \text{ in } \mathcal{I}} \cdot (\nabla\varphi_j + e_j)$

  $= \mathbb{E}\, (\nabla\varphi_i + e_i) \cdot p_j$

  $= e_i \cdot \mathbb{E}\, p_j \quad$ as $\nabla \cdot p_j = 0$ !

  $\hookrightarrow$ can repeat div-curl with this ! $\quad \triangle$