Recap: Effective viscosity $e \cdot \overline{B} e = \mathbb{E}\, |D\varphi_e + e|^2$

$= |e|^2 + \mathbb{E}\, |D\varphi_e|^2 \geq |e|^2$

(larger than solvent viscosity as particles hinder fluid flow)

- defined in term of corrector φ $\Rightarrow$ depends on the law of microstructure $\{I_n\}_n$

... and non-explicit!

- needs to be coupled with microstructure dynamics

  $\hookrightarrow$ fully open problem $<$ fast micro dynamics / infinite particle system ...

- simplification: $\boxed{\text{dilute regime}}$ $\Rightarrow$ only reduced info on microstructure matters & get (semi) explicit perturbative description of $\overline{B}$

Dilute means: volume fraction $\lambda := \mathbb{E}\, \mathbb{1}_{\mathcal{I}} = \lim_{R \uparrow \infty} \fint_{B_R} \mathbb{1}_{\mathcal{I}} = \lim_{R \uparrow \infty} \frac{|\mathcal{I} \cap B_R|}{|B_R|}$ a.s.

$\ll 1$.

(II) Dilute expansions.

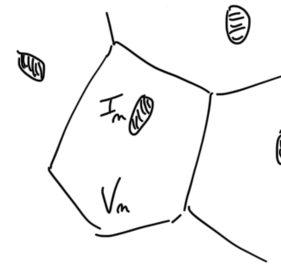
Long history: 19th century, macro effect of impurities in physical systems (Clausius, Mossotti, Maxwell, Rayleigh, Faraday ...)

* Einstein 1905: $\overline{B} = \underbrace{Id}_{\text{pure solvent}} + \underbrace{\frac{5}{2} \lambda\, Id}_{O(\lambda) \text{ contribution from summing 1-particle effects}} + o(\lambda)$ (3D, Stokes) spheres.

* next orders: summing 2-particle effects, etc "cluster expansion"
BUT divergent! (long-range interactions)
e.g. Burgers '40s ...
$\hookrightarrow$ solved to 2nd order: Batchelor-Green '72 — renormalization.

II.1) Einstein's formula.

Heuristics:



dilute regime $\to$ particles as isolated...

$\varphi_e|_{V_n} \approx \varphi_e^n$ solution of single-particle problem

$\hookrightarrow \varphi_e \approx \sum_n \mathbb{1}_{V_n} \varphi_e^n$

$\hookrightarrow e\cdot\bar{B}e \approx \mathrm{Id} + \mathbb{E}\sum_n \mathbb{1}_{V_n}|\nabla\varphi_e^n|^2 \simeq \mathrm{Id} + \mathbb{E}\sum_n \frac{\mathbb{1}_{I_n}}{|I_n|}\int_{\mathbb{R}^d}|\nabla\varphi_e^n|^2$

Theorem.

For simplicity: $\begin{cases} \text{- scalar setting} \\ \text{- assume } (H_{unif}): \operatorname{dist}(I_n, I_m) \geq \delta > 0 \ \forall n \neq m \text{ a.s.} \end{cases}$

Then $\bar{B} = \mathrm{Id} + \bar{B}^{(1)} + O\left(\lambda_2 \log(2 + \tfrac{1}{\lambda_2})\right)$

$\uparrow$ think λ^2 for Poisson (hardcore) - see later!

where $e\cdot\bar{B}^{(1)}e = \mathbb{E}\sum_n \frac{\mathbb{1}_{I_n}}{|I_n|}\int_{\mathbb{R}^d}|\nabla\varphi_e^n|^2$, $\quad \begin{cases} -\Delta\varphi_e^n = 0, & I_n^c \\ \nabla\varphi_e^n + e = 0, & I_n \\ \int_{\partial I_n}\partial_\nu\varphi_e^n = 0 \end{cases}$

$= O(\lambda)$

In particular:

- if $\{X_n\}_n \perp\!\!\!\perp$ iid $\{R_n\} \sim R_0$, $I_n = I(X_n, R_n)$,

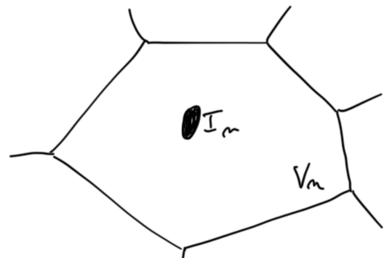
  then $e\cdot\bar{B}^{(1)}e = \frac{\lambda}{|I|}\mathbb{E}_{R_0}\int_{\mathbb{R}^d}|\nabla\varphi_e^\circ|^2$, $\quad \begin{cases} -\Delta\varphi_e^\circ = 0, & I(0,R_0)^c \\ \nabla\varphi_e^\circ + e = 0, & I(0,R_0) \\ \int_{\partial I(0,R_0)}\partial_\nu\varphi_e^\circ = 0 \end{cases}$

- if $I_n = B(X_n)$, then $\bar{B}^{(1)} = \lambda d\,\mathrm{Id}$

  (Stokes $\to \lambda\frac{d+2}{2}\mathrm{Id}$: Einstein formula)

$\boxed{\text{Proof}}$: can do PDE proof, but variational proof is most natural.

Start from: $e\cdot\bar{B}e = \mathbb{E}|\nabla\varphi_e + e|^2 = \inf\Big\{ \underbrace{\mathbb{E}|\nabla\psi + e|^2}_{= |e|^2 + \mathbb{E}|\nabla\psi|^2} : \nabla\psi \text{ stat.}, \nabla\psi + e|_{\mathcal{I}} = 0, \mathbb{E}\nabla\psi = 0 \Big\}$.



Voronoi cells: $V_n = \{x \in \mathbb{R}^d : |x - x_n| \leq |x - x_m| \ \forall m \neq n\}$.

Single-particle problems:

1) whole-space: $\begin{cases} -\Delta\varphi_e^n = 0, & I_n^c \\ \nabla\varphi_e^n + e|_{I_n} = 0 \\ \int_{\partial I_n}\partial_\nu\varphi_e^n = 0, & \varphi_e^n \in \dot{H}^1(\mathbb{R}^d) \end{cases}$

ie $\varphi_e^n = \operatorname{argmin}\Big\{ \int_{\mathbb{R}^d}|\nabla\psi|^2 : \psi \in \dot{H}^1(\mathbb{R}^d), \nabla\psi + e|_{I_n} = 0 \Big\}$

2) Dirichlet :

$$\varphi_{eD}^{n} = \operatorname{argmin}\left\{ \int_{V_n} |\nabla\varphi|^2 : \varphi \in H_0^1(V_n),\ \nabla\varphi + e|_{I_n} = 0 \right\}$$

3) Neumann :

$$\varphi_{eN}^{n} = \operatorname{argmin}\left\{ \int_{V_n} |\nabla\varphi|^2 : \varphi \in H^1(V_n),\ \nabla\varphi + e|_{I_n} = 0,\ \fint_{V_n} \varphi = 0 \right\}$$

$\boxed{\text{Step 1}}$ Energy comparison :

$$\begin{cases} e\cdot\bar{B}e \leq |e|^2 + \mathbb{E} \sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \int_{V_n} |\nabla\varphi_{eD}^n|^2 \\ e\cdot\bar{B}e \geq |e|^2 + \mathbb{E} \sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \int_{V_n} |\nabla\varphi_{eN}^n|^2 \end{cases}$$

Exercise : averaging over Voronoi cells $\to$ $\left[\begin{array}{l} \forall \zeta \text{ stationary},\ \mathbb{E}|\zeta| < \infty, \\ \text{we have } \mathbb{E}\zeta = \mathbb{E} \sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \int_{V_n} \zeta . \end{array} \right.$

($\leq$) Consider $\varphi_{e,D} \equiv \sum_n \varphi_{e,D}^n$ stationary

Get $\left| \begin{array}{l} \mathbb{E}\nabla\varphi_{e,D} = \mathbb{E} \sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \underbrace{\int_{V_n} \nabla\varphi_{e,D}^n}_{=0} = 0 \ ! \\ \mathbb{E}|\nabla\varphi_{eD}|^2 = \mathbb{E} \sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \underbrace{\int_{V_n} |\nabla\varphi_{e,D}^n|^2}_{\lesssim 1} \lesssim \lambda \end{array} \right.$

$\Rightarrow$ by energy comparison : $e\cdot\bar{B}e \leq |e|^2 + \mathbb{E}|\nabla\varphi_{eD}|^2$

$$= |e|^2 + \mathbb{E}\sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \int_{V_n} |\nabla\varphi_{e,D}^n|^2 .$$

($\geq$) $e\cdot\bar{B}e = |e|^2 + \sum_n \mathbb{E} \frac{\mathbb{1}_{I_n}}{|I_n|} \underbrace{\int_{V_n} |\nabla\varphi_e|^2}_{\geq \int_{V_n} |\nabla\varphi_{eN}^n|^2 \ !}$

$\boxed{\text{Step 2}}$ $\int_{V_n} |\nabla\varphi_{eN}^n|^2 \leq \int_{V_n} |\nabla\varphi_e^n|^2 \leq \int_{\mathbb{R}^d} |\nabla\varphi_e^n|^2 \leq \int_{\mathbb{R}^d} |\nabla\varphi_{eD}^n|^2 = \int_{V_n} |\nabla\varphi_{eD}^n|^2 .$

$$\hookrightarrow \left| e\cdot\bar{B}e - |e|^2 - \mathbb{E}\sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \int_{\mathbb{R}^d} |\nabla\varphi_e^n|^2 \right| \leq \mathbb{E}\sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \boxed{\int_{V_n} |\nabla\varphi_{eD}^n|^2 - |\nabla\varphi_{eN}^n|^2} .$$

$\boxed{\text{Step 3}}$ $\left[\begin{array}{l} \int_{V_n} |\nabla\varphi_{eD}^n|^2 - |\nabla\varphi_{eN}^n|^2 \lesssim (\rho_n)^{-d}, \\ \text{by elliptic regularity!} \end{array} \right.$ $\qquad \begin{array}{l} \rho_n = \max\{r : B_r(x_n) \subset V_n\} \\ \quad = \frac{1}{2} \min_{m : m \neq n} |x_n - x_m| \end{array}$

- $$\int_{V_m} |\nabla \varphi_{eD}^m|^2 - |\nabla \varphi_{eN}^m|^2 = \int_{V_m \setminus I_m} \operatorname{div}(\varphi_{eD}^m \nabla \varphi_{eD}^m) - \operatorname{div}(\varphi_{eN}^m \nabla \varphi_{eN}^m)$$
$$\overset{BC}{=} \int_{\partial I_m} e\cdot(x-x_m)\, \partial_\nu (\varphi_{eD}^m - \varphi_{eN}^m)$$
$$\lesssim \left(\int_{I_m+B} |\nabla(\varphi_{eD}^m - \varphi_{eN}^m)|^2\right)^{1/2} \quad \text{by Exercise 1 (trace)}$$
$$\lesssim \rho_m^{-d/2}\left(\int_{V_m} |\nabla(\varphi_{eD}^m - \varphi_{eN}^m)|^2\right)^{1/2}. \quad \text{by Exercise 2 (MVP)}$$

- $$\int_{V_m} |\nabla(\varphi_{eD}^m - \varphi_{eN}^m)|^2 = \int_{V_m\setminus I_m} \operatorname{div}(\varphi_{eD}^m \nabla \varphi_{eD}^m) + \operatorname{div}(\varphi_{eN}^m \nabla \varphi_{eN}^m) - 2\operatorname{div}(\varphi_{eD}^m \nabla \varphi_{eN}^m)$$
$$= \int_{\partial I_m} e\cdot(x-x_m)\, \partial_\nu(\varphi_{eD}^m - \varphi_{eN}^m)$$
$$\Rightarrow \int_{V_m} |\nabla(\varphi_{eD}^m - \varphi_{eN}^m)|^2 = \int_{V_m} |\nabla \varphi_{eD}^m|^2 - |\nabla \varphi_{eN}^m|^2$$
& claim follows!

Exercise 1: trace estimate
$$\begin{cases} -\Delta \varphi = 0, & 3B\setminus B \\ \nabla\varphi = 0, & B \end{cases}$$
$$\Rightarrow \int_{\partial B} |\nabla\varphi|^2 \lesssim \int_{3B\setminus B} |\nabla\varphi|^2 \; !$$

indeed: $\int_{\partial B} |\nabla\varphi|^2 \lesssim \int_{2B\setminus B} |\langle D\rangle^{1/2} \nabla\varphi|^2$ by usual trace estimate $H^{1/2}(2B\setminus B) \hookrightarrow L^2(\partial B)$
$$\lesssim \int_{3B\setminus B} |\nabla\varphi|^2 + \underbrace{\int_{\partial B} |\langle D\rangle (\varphi - \kappa)|_B|^2}_{= 0 \text{ for suitable } \kappa \text{ as } \varphi|_B \text{ is cst}}$$
(can subtract any constant) by elliptic boundary regularity

Exercise 2: mean-value property (MVP)
recall for harmonic fcts: $\begin{cases} -\Delta\varphi = 0, \; B_R \\ \Rightarrow |\nabla\varphi(0)|^2 \lesssim R^{-d} \int_{B_R} |\nabla\varphi|^2 \end{cases}$

deduce that we have similarly
$$\begin{cases} -\Delta\varphi = 0, \; B_R\setminus B, \quad R\geq 2 \\ \nabla\varphi = 0, \; B \\ \int_{\partial B} \partial_\nu\varphi = 0 \end{cases} \Rightarrow \int_{B_2} |\nabla\varphi|^2 \lesssim R^{-d} \int_{B_R} |\nabla\varphi|^2$$

indeed consider $\begin{cases} -\Delta\hat\varphi = 0, & B_R \\ \hat\varphi = \varphi, & \partial B_R \end{cases}$ harmonic approximation

standard MVP: $\int_{2B} |\nabla\hat\varphi|^2 \lesssim R^{-d} \int_{B_R} |\nabla\hat\varphi|^2$

energy $\int_{B_R} |\nabla \hat{\varphi}|^2 = \int_{\partial B_R} \varphi\, \partial_\nu \hat{\varphi} = \int_{B_R} \nabla\varphi \cdot \nabla\hat{\varphi}$

$$\Rightarrow \int_{B_R} |\nabla\hat{\varphi}|^2 \lesssim \int_{B_R} |\nabla\varphi|^2$$

approx error $\int_{B_R} |\nabla(\varphi - \hat{\varphi})|^2 = \int_{\partial B} (\varphi - \hat{\varphi})\, \partial_\nu \varphi$

$$\lesssim \left(\int_B |\nabla\hat{\varphi}|^2\right)^{1/2} \left(\int_{2B} |\nabla\varphi|^2\right)^{1/2}$$

$$\Rightarrow \int_{B_R} |\nabla(\varphi - \hat{\varphi})|^2 \lesssim \int_{2B} |\nabla\hat{\varphi}|^2$$

$\boxed{\text{Step 4}}$ Remains to estimate error $\equiv \mathbb{E} \sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \rho_n^{-d}$.

Preliminary definitions: $\{X_n\}_n$ stationary point process

- $\mathbb{E} \sum_n h(X_n) = \int h f_1$, f_1 = "1-pt. density" & $f_1(x) = \lambda_1$ cst by stationarity "intensity" $(\lambda_1 |I_0| = \lambda)$
- $\mathbb{E} \sum_{n \neq m} h(X_n, X_m) = \iint h f_2$, f_2 = "2-pt density" & set $\lambda_2 \equiv \|f_2\|_{L^\infty}$ "2-pt. intensity" e.g. Poisson process: $f_2 = f_1 \otimes f_1 \Rightarrow \lambda_2 = (\lambda_1)^2$.

Let's show error $\lesssim \lambda_2 \log(2 + \lambda_1/\lambda_2)$

Recall $I_n = I(X_n, R_n)$, $\{X_n\}_n \perp\!\!\!\perp$ iid $\{R_n\}_n$

We compute:

$$\mathbb{E} \sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \rho_n^{-d} \lesssim \mathbb{E} \sum_n \mathbb{1}_{|X_n| \leq 1} \underbrace{\rho_n^{-d}}_{\int_{\rho_n}^\infty t^{-d-1} dt} \qquad (\text{say } I_n \subset B(X_n))$$

$$\lesssim \int_1^\infty dt\; t^{-d-1}\; \underbrace{\mathbb{E} \sum_n \mathbb{1}_{|X_n| \leq 1} \underbrace{\mathbb{1}_{\rho_n \leq t}}_{\mathbb{1}_{\exists m \neq n : |X_m - X_n| \leq t}}}_{\begin{cases} \leq \mathbb{E} \sum_n \mathbb{1}_{|X_n| \leq 1} \lesssim \lambda_1 \\ \leq \mathbb{E} \sum_{n \neq m} \mathbb{1}_{|X_n| \leq 1} \mathbb{1}_{|X_n - X_m| \leq t} \lesssim \lambda_2 t^d \end{cases}}$$

$$\lesssim \int_1^\infty dt\, \frac{1}{t} \left((\lambda_1 t^{-d}) \wedge \lambda_2 \right) \lesssim \lambda_2 \log(2 + \lambda_1/\lambda_2) \qquad \square$$