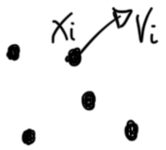
A TOUR IN CLASSICAL MEAN FIELD

Part I : INTRODUCTION

Consider N classical particles in $\mathbb{R}^d$ with pairwise long-range interactions

Let the force kernel $K : \mathbb{R}^d \to \mathbb{R}^d$ satisfy $K(x-x') = -K(x'-x)$
(action - reaction)

Particles' trajectories are described by Newton's equations:



$$\begin{cases} \partial_t X_i = V_i \\ \partial_t V_i = \frac{1}{N} \sum_{\substack{1 \le j \le N \\ j \ne i}} K(X_i - X_j) \end{cases} \qquad \forall\, 1 \le i \le N$$

$\frac{1}{N} \to$ "mean-field regime": total force $= O(1)$.

I.1 Different formulations of mean field

Mean-field question : [instead of a complicated many-body problem, what simplified description can we get as $N \nearrow \infty$ say for the particle density ?

In terms of empirical measure $\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_i, V_i)} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$,

we expect the following : If $\mu_N|_{t=0} \to F^0$ in some sense
(e.g. for random data $(X_i, V_i)|_{t=0}$ iid $\sim F^0$ we have $\mu_N|_{t=0} \overset{*}{\rightharpoonup} F^0$ in $\mathcal{P}$ a.s.) } law of large numbers

Then $\mu_N \to F$ in a corresponding sense,
where the "macro density" F satisfies the Vlasov eqn:
$$\partial_t F + v \cdot \nabla_x F + K * F \cdot \nabla_v F = 0, \quad F|_{t=0} = F^0.$$

This is viewed as propagating a law of large numbers along particle dynamics.

Statistical perspective

- Instead of individual particles' trajectories, let's consider the evolution of random ensembles of particles. More precisely, let $F_N \equiv$ N-particle density $\in \mathcal{P}((\mathbb{R}^d \times \mathbb{R}^d)^N)$

$$= \text{law of } (\underbrace{X_1, V_1}_{Z_1}, \dots, \underbrace{X_N, V_N}_{Z_N})$$

We consider exchangeable particles, so $F_N \in \mathcal{P}_{sym}((\mathbb{R}^d \times \mathbb{R}^d)^N)$.

- In terms of characteristics:

<u>Newton</u>'s equations for particles

$\iff$ <u>Liouville</u>'s equation for density:

$$\partial_t F_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} F_N + \frac{1}{N} \sum_{i \neq j}^{N} K(x_i - x_j) \cdot \nabla_{v_i} F_N = 0.$$

- As $N \uparrow \infty$, the full density F_N contains too much information and we rather aim at describing finite subsets of "typical particles"

<u>ie</u> <u>1-particle density</u> $F_{N,1}(z) := \displaystyle\int_{(\mathbb{R}^d \times \mathbb{R}^d)^{N-1}} F_N(z, z_2, \dots, z_N)\, dz_2 \dots dz_N$

(1st marginal)

<u>k-particle density</u> $F_{N,k}(z_1 \dots z_k) := \displaystyle\int_{(\mathbb{R}^d \times \mathbb{R}^d)^{N-k}} F_N(z_1 \dots z_N)\, dz_{k+1} \dots dz_N$

(kth marginal)

Reformulation of mean field in this setting:

If initial particles are approximately independent ("chaotic")

i.e. $F_{N,k}|_{t=0} \xrightarrow[N\to\infty]{} (F^0)^{\otimes k}$ $\forall k \geq 1$, for some $F^0 \in \mathcal{P} \cap L^\infty$

Then they remain so over time: $F_{N,k} \xrightarrow[N\to\infty]{} F^{\otimes k}$ $\forall k \geq 1$,

("propagation of chaos")

where the limit 1-particle density F satisfies Vlasov.

Reformulation follows from:

Grünbaum's lemma. Let $(Z_1, \dots, Z_N) \sim F_N \in \mathcal{P}_{sym}((\mathbb{R}^d \times \mathbb{R}^d)^N)$ $\forall N$ and $F \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$.

The following are equivalent:

(i) $F_{N,k} \xrightarrow[N\to\infty]{*} F^{\otimes k}$ in $\mathcal{P}((\mathbb{R}^d \times \mathbb{R}^d)^k)$ $\forall k \geq 1$

(ii) $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{Z_i} \xrightarrow[N\to\infty]{*} F$ in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ in proba

i.e. $\mathbb{P}\left[\left|\int \varphi(\mu_N - F)\right| > \varepsilon\right] \xrightarrow[N\to\infty]{} 0$ $\forall \varepsilon > 0$ $\forall \varphi \in C^\infty_c$.

Proof. • (i) $\Rightarrow$ (ii): $\mathbb{E}\left|\int \varphi(\mu_N - F)\right|^2 = \frac{N-1}{N}\int \varphi\otimes\varphi\, F_{N,2} + \frac{1}{N}\int \varphi^2 F_{N,1} + \left(\int \varphi F\right)^2$

$- 2\left(\int \varphi F\right)\left(\int \varphi F_{N,1}\right) \xrightarrow{N\to\infty} 0$ by (i).

• (ii) $\Rightarrow$ (i): we note that $\mathbb{E}\int \varphi \mu_N = \int \varphi F_{N,1}$

$$\mathbb{E}\left(\int \varphi \mu_N\right)^2 = \frac{N-1}{N}\int \varphi\otimes\varphi\, F_{N,2} + \frac{1}{N}\int \varphi^2 F_{N,1}$$

& more generally: $\left|\mathbb{E}\left(\int \varphi \mu_N\right)^k - \int \varphi^{\otimes k} F_{N,k}\right| \leq C_k \|\varphi\|_{L^\infty}^k N^{-1}$

$\Rightarrow$ (i) follows from (ii). $\square$

Rem: $\begin{cases} \text{item (ii) is refered to as } \underline{\text{Klimontovich's perspective}} \text{ to mean field.} \\ \text{item (i) is refered to as } \underline{\text{Bogolyubov's perspective}}. \end{cases}$

I.2) Classical proof of mean field: Klimontovich approach

Key observation (Klimontovich, 1960s).

[If $K \in C_b(\mathbb{R}^d)$, then μ_N is a weak solution of Vlasov.

Proof. Note that $K(0) = -K(0) = 0$ as $K \in C_b$ by action-reaction.

Then $\forall \varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^d)$:

$$\frac{d}{dt}\int \varphi \mu_N = \frac{d}{dt}\frac{1}{N}\sum_{i=1}^N \varphi(X_i, V_i)$$

$$= \frac{1}{N}\sum_{i=1}^N V_i \cdot \nabla_x \varphi(X_i,V_i) + \underbrace{\frac{1}{N^2}\sum_{i\neq j}^N K(X_i - X_j)\cdot \nabla_v \varphi(X_i,V_i)}_{= \frac{1}{N^2}\sum_{i,j=1}^N K(X_i-X_j)\cdot\nabla_v\varphi(X_i,V_i) \text{ as } K(0)=0.}$$

$$= \int \left(v\cdot\nabla_x\varphi + K*\mu_N \cdot \nabla_v\varphi\right)\mu_N. \qquad \square$$

Consequence. [Mean field amounts to stability for weak solutions of Vlasov!
— if $K \in C_b$.

Theorem (Neunzert '77, Braun-Hepp '77, Dobrushin '79, Maslov '79).

[Let $K \in W^{1,\infty}$ and $F^\circ \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\int |z|\, dF_0(z) < \infty$

If $\mu_N|_{t=0} \overset{*}{\rightharpoonup} F^\circ$ in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$

Then $\mu_N \to F$ is $C(\mathbb{R}^+, w*\text{-}\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d))$

where F is the unique weak solution of Vlasov with data F°.

Proof. We split the proof into 2 steps.

Step 1 : Compactness :

> If $K \in C_b$ and $\mu_N|_{t=0} \xrightarrow{*} F^0$ in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$
> Then μ_N is rel. compact in $C(\mathbb{R}^+, w*\text{-}\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d))$
> & any limit point is a weak sol of Vlasov with initial data F^0.

As μ_N is bounded in $L^\infty(\mathbb{R}^+, \mathcal{P})$ and as the maps $t \mapsto \int \varphi \mu_N(t)$ are bounded and equicontinuous as $N \uparrow \infty$, $\forall \varphi \in C_c(\mathbb{R}^d \times \mathbb{R}^d)$, we deduce by Arzela-Ascoli that $\{\mu_N\}_N$ is rel. compact in $C(\mathbb{R}^+, w*\text{-}\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$ (with the topology of vague convergence). Further note that the Vlasov equation for μ_N and its initial convergence ensure that $\{\mu_N(t)\}_N$ is tight $\forall t$. Hence, we find that $\{\mu_N\}_N$ is in fact rel. compact in $C(\mathbb{R}^+, w*\text{-}\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d))$.

If F is a limit point, we can pass to the limit in the weak formulation

$$\int \varphi(0)\mu_N^0 + \int_0^\infty \int \mu_N \left(\partial_t \varphi + v \cdot \nabla_x \varphi + K * \mu_N \cdot \nabla_v \varphi\right) = 0 \quad \forall \varphi \in C_c^1(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d),$$

and we deduce that F is a weak solution of Vlasov with initial data F^0.

Step 2 : Uniqueness :

> If $K \in W^{1,\infty}$ and $\int |z| \, dF^0(z) < \infty$,
> Then $\exists!$ weak Vlasov solution $F \in L^\infty(\mathbb{R}^+, \mathcal{P})$.

Using characteristics, we first note that $F \in L^\infty(\mathbb{R}^+, \mathcal{P})$ is a weak Vlasov solution

$$\Longleftrightarrow \quad \left[\begin{array}{l} F(t) = Z(t, \cdot)_* F^0 \\ \text{where } Z = (X, V) \text{ satisfies: } \forall z_0 \in \mathbb{R}^d \times \mathbb{R}^d, \\ \left\{\begin{array}{ll} \partial_t X(t, z_0) = V(t, z_0) & \\ \partial_t V(t, z_0) = (K * F(t))(X(t, z_0)), & t \geq 0 \\ Z(0, z_0) = z_0 & \end{array}\right\} \begin{array}{l} \exists! \text{ solution} \\ \text{by Cauchy-Lipschitz} \\ \text{as } K \in W^{1,\infty} \end{array} \end{array}\right.$$

In these terms, if we have 2 solutions (F, Z) and (F', Z') (with same data F^0),

then we can compute: $|Z(t, z_0) - Z'(t, z_0)|$

$$\leq \int_0^t ds \left(|V(s, z_0) - V'(s, z_0)| + |k * F(s, X(s, z_0)) - k * F'(s, X'(s, z_0))| \right)$$

$$\leq (1 + \|\nabla k\|_{L^\infty}) \int_0^t ds \, |Z(s, z_0) - Z'(s, z_0)|$$

$$+ \int_0^t ds \, |\underbrace{k * (F - F')(s, X(s, z_0))}|$$

$$= \int \Big(k(X(s, z_0) - X(s, y_0)) - k(X(s, z_0) - X'(s, y_0)) \Big) dF^0(y_0)$$

$$\leq \|\nabla k\|_{L^\infty} \int |X(s, y_0) - X'(s, y_0)| \, dF^0(y_0).$$

$\Rightarrow$ integrating wrt z_0, we get:

$$\int |Z(t, \cdot) - Z'(t, \cdot)| \, dF^0 \leq (1 + 2\|\nabla k\|_{L^\infty}) \int_0^t ds \int |Z(s, \cdot) - Z'(s, \cdot)| \, dF^0.$$

$\Rightarrow$ by Grönwall: $Z(t, \cdot) = Z'(t, \cdot)$ F^0-ae $\forall t$

<u>ie</u> $F = F'$! $\square$

<u>Remark</u>: In fact, the above proof also yields <u>Dobrushin's stability estimate</u>:

> Let $k \in W^{1,\infty}$.
> If $F, G \in L^\infty(\mathbb{R}^+; \mathcal{P})$ are weak solutions of Vlasov with data F^0, G^0,
> Then: $W_1(F(t), G(t)) \leq e^{(1 + 2\|\nabla k\|_{L^\infty}) t} \, W_1(F^0, G^0) \quad \forall t \geq 0.$
>
> Note this can be directly applied to $G = \mu_N$, yielding a quantitative mean-field estimate.

<u>OPEN PROBLEM #1</u>: What happens for k singular?
e.g. for k Coulomb, <u>ie</u> $k = -\nabla V$, $V(x) = \begin{cases} -\log|x| & : d = 2 \\ |x|^{2-d} & : d \geq 3. \end{cases}$

In that case, μ_N is <u>no longer</u> an exact weak solution of Vlasov:

we have $\partial_t \mu_N + v\cdot\nabla_x \mu_N + K \tilde{*} \mu_N \cdot \nabla_v \mu_N = 0$ in $\mathcal{D}'$

where now $K \tilde{*} \mu_N (x) = \int_{\mathbb{R}^d \setminus \{x\}} K(x-y)\, d\mu_N(y)$

$\hookrightarrow$ with diagonal removed!

... and this is FAR from innocent!

<u>Literature</u>: Mean field is only known:

- if $\begin{cases} |K(x)| \lesssim |x|^{-\gamma} \\ |\nabla K(x)| \lesssim |x|^{-\gamma-1} \end{cases}$ with some $\gamma < 1$ (see Hauray-Jabin '04)

- if $K \in L^\infty$ (see Jabin-Wang '16, relative entropy method)

More recently, we have shown it $\forall K \in L^2_{loc}$ (Bresch-D-Jabin '24), which is the <u>first</u> complete result for "genuinely" singular interactions. This is the topic of <u>Part III</u> — though still not covering Coulomb!

<u>Remark</u>: - for related results for interactions with N-dependent cut-off: see Pickl's work

- for the simpler setting of <u>Brownian</u> particles, see e.g. Bresch-Jabin-Soler '23.

<u>Remark</u>: the mean-field problem is MUCH simpler for <u>inertialess particles</u>

<u>ie</u> for <u>1st-order dynamics</u>: $\partial_t X_i = \frac{1}{N} \sum_{j(\neq i)}^{N} K(X_i - X_j)$

(such dynamics appears eg as overdamped limit, or for vortex dynamics, etc.)

$\hookrightarrow$ see <u>Part II</u> for details!

I.3) <u>Bogolyubov perspective</u>

Starting from Liouville's equation $\partial_t F_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} F_N + \frac{1}{N} \sum_{i\neq j}^{N} K(x_i - x_j)\cdot \nabla_{v_i} F_N = 0$, we recall Bogolyubov's perspective to mean field:

$$\left[\begin{array}{l} F_N|_{t=0} = (F^0)^{\otimes N} \\ \Rightarrow F_{N,k} \xrightarrow[N\to\infty]{\mathcal{D}'} F^{\otimes k} \quad \forall k \geq 1 \\ \text{where } F \text{ is a weak solution of Vlasov} \end{array}\right.$$

Natural approach: <u>BBGKY hierarchy</u> of equations for $\{F_{N,k}\}_{1\leq k\leq N}$:

integrating out Liouville wrt subsets of variables, we find (exercise):

$$\left[\begin{array}{l} \forall 1 \leq k \leq N: \\ \partial_t F_{N,k} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} F_{N,k} + \overbrace{\frac{1}{N} \sum_{i\neq j}^{k} K(x_i - x_j) \cdot \nabla_{v_i} F_{N,k}}^{O(k^2/N) \text{ creates correlations}} \\ \qquad + \underbrace{\frac{N-k}{N}}_{\xrightarrow[N\to\infty]{} 1} \sum_{i=1}^{k} \int K(x_i - x_{k+1}) \cdot \underbrace{\nabla_{v_i}}_{\text{loss of } v\text{-derivative}} \underbrace{F_{N,k+1}}_{\text{next marginal}}(z_1 \dots z_{k+1})\, dz_{k+1} = 0. \end{array}\right.$$

<u>PRO</u>: can read where chaoticity gets destroyed, <u>ie</u> where correlations are created.

<u>CON</u>: hierarchy of equations with loss of v-derivative, which makes it useless for estimates...

<u>Remark</u>: for $K \in W^{1,\infty}$, mean field can be proven as follows:

1) Compactness: up to extraction, $F_{N,k} \xrightarrow[N\to\infty]{*} F_k \quad \forall k \geq 1$

$$\left[\begin{array}{l} \text{where } \{F_k\}_k \text{ satisfies in the weak sense the limit hierarchy:} \\ \partial_t F_k + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} F_k + \sum_{i=1}^{k} \int K(x_i - x_{k+1}) \cdot \nabla_{v_i} F_{k+1}(z_1 \dots z_{k+1})\, dz_{k+1} \\ \qquad = 0. \end{array}\right.$$

2) Uniqueness: $\exists!$ sol to limit hierarchy if $K \in W^{1,\infty}$

(see Spohn '81: abstract argument and reduction to Vlasov stability)

OPEN PROBLEM #2 : [Estimate error in propagation of chaos as expected from BBGKY:

ie $F_{N,k} - F_{N,1}^{\otimes k} = O(N^{-1})$

e.g. pair correlations $G_{N,2} = F_{N,2} - F_{N,1}^{\otimes 2} = O(N^{-1})$]

Rem : in terms of the empirical measure μ_N, this means $\mu_N = F + O(N^{-1/2})$

since $\mathrm{Var}\left[\int \varphi \mu_N\right] = \frac{N-1}{N} \int \varphi^{\otimes 2} G_{N,2} + \frac{1}{N} \int \left(\varphi - \int \varphi F_{N,1}\right)^2 F_{N,1}$.

Literature : only known for $K \in W^{1,\infty}$ (D'20) [but again much more is known for inertialess particles]

— this is the topic of Part IV

Exercise : [investigate how correlation estimates follow easily from BBGKY hierarchy

- in the corresponding quantum setting (no ∇_v - loss!)
  (see Paul-Pulvirenti-Simonella '17)
- in case of Brownian particles, which amounts to adding diffusion in v, and allows to compensate for ∇_v - loss!
  (see Bresch - Jabin - Soler '23)

... but such arguments are not available in our setting!]