EFFECTIVE VISCOSITY OF RANDOM SUSPENSIONS

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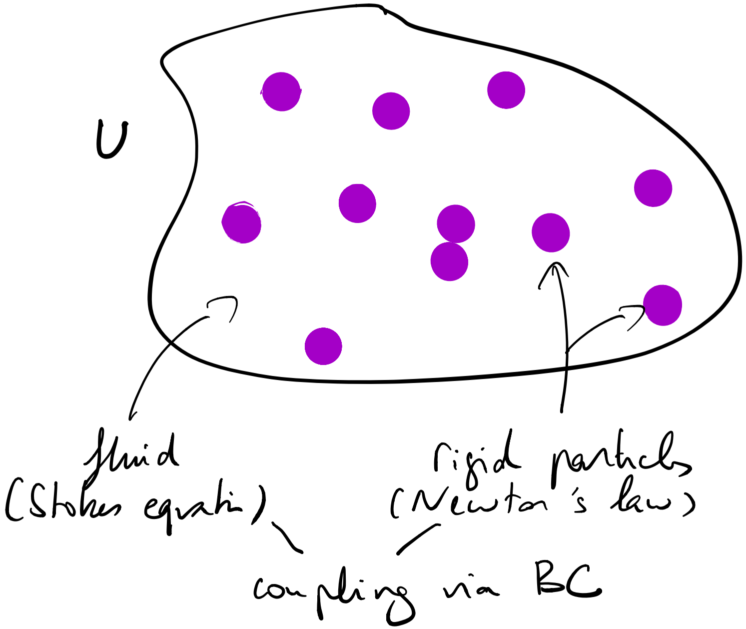
Collective behavior of particles in fluids, Dec 14-17, 2020

Setting of the problem



<u>Question</u> : What happens to the system {fluid + particles} when the number N of particles I_i is such that $\frac{1}{N|U|}\sum_i |I_i| \to \lambda \in (0,1)$?

<u>Dynamics</u> : Unclear beyond mean field limit ($\lambda = 0$).

cf. R. Höfer, A. Mecherbet

<u>Statics</u> : + velocity field when positions are given

+ invariant measure ?

Einstein contribution (1905)

Aim : measure the Avogadro number using sugar dissolved in water

based on
- Einstein's viscosity formula
- Einstein's relation in kinetic theory

Einstein's effective viscosity formula :

- Assumption 1 : water with dissolved sugar is a fluid with some viscosity $\overline{B}$ that can be measured
- Assumption 2 : sugar is so dilute that sugar molecules interact with water as if they were isolated

$$\text{“} \; \overline{B} \simeq \left(1 + \frac{5}{2}\lambda\right) \mathrm{Id} \; \text{”}$$

Batchelor's correction :

- Assumption 2' : first correction due to pairwise interactions between sugar molecules via the fluid

$$\text{“} \; \overline{B} \simeq \left(1 + \frac{5}{2}\lambda + \alpha_{Bat}\,\lambda^2\right) \mathrm{Id} \; \text{”}$$

need renormalization
(cf. E. Guazzelli II)

Mathematical analysis

Many recent contributions on the subject (cf. R. Höfer and M. Hillairet)

Approach:
- system "fluid + particles" + forcing term → solution u
- approximate u using the method of reflections, get $\tilde{u}$
- show that $\tilde{u}$ solves at first order a Stokes equation with Einstein's viscosity.

Advantage: deterministic approach, can be combined with mean field limit for sedimentation.

Drawback: treat Assumption 1 and Assumption 2 alike for $\lambda \ll 1$ and use "explicit formulas"

With M. Duerinckx, we proposed another approach closer to Einstein's and Batchelor's points of view:

quantification

1- Define a clean notion of effective viscosity for an effective fluid for any λ.

2- Analyze the effective viscosity in the regime $\lambda \ll 1$.

Outline of the talk

Part 1 Definition of the notion of effective viscosity

Assumption 1 (and more)

1.1 Model

1.2 Qualitative homogenization : effective viscosity

1.3 Quantitative homogenization : convergence rates

1.4 The case of sedimenting particles

Part 2 Expansion of the effective viscosity at low density

Assumption 2 (and more)

2.1 Einstein's formula & Batchelor's corrections

2.2 Main results

2.3 Ingredients to the proof

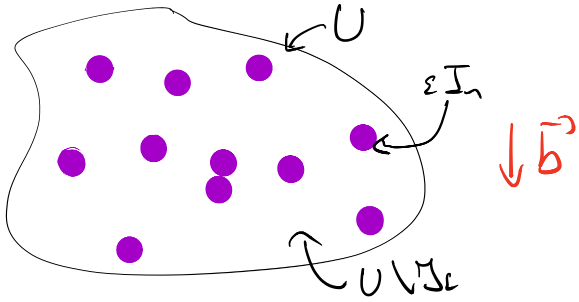
Part 1 : Homogenization of the system

[math version of Landau - Lifschitz and Batchelor - Green, cf. E. Guazzelli II]

1.1 The model

particles are rescaled : εI_n, centered at εx_n, $\{x_n\} = \mathcal{P}$ (point set)

$\mathcal{J}_\varepsilon = \bigcup \{ \varepsilon I_n \mid \varepsilon B(x_n) \subset U,\ x_n \in \mathcal{P} \}$ is the set of ε-rescaled particles



Bulk of the fluid :

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & U \setminus \mathcal{J}_\varepsilon \\ \operatorname{div} u_\varepsilon = 0 & U \setminus \mathcal{J}_\varepsilon \\ \int_{U \setminus \mathcal{J}_\varepsilon} P_\varepsilon = 0 & \\ u_\varepsilon = 0 & \partial U \end{cases}$$

Inclusions and coupling :

$$\int_{\varepsilon \partial I_n} \sigma(u_\varepsilon, P_\varepsilon)\nu = -\varepsilon^{d-1} b |I_n|$$

$$\int_{\varepsilon \partial I_n} \Theta (x - \varepsilon x_n) \cdot \sigma(u_\varepsilon, P_\varepsilon)\nu = 0, \quad \forall\, \Theta \in M^{\text{skew}}$$

$$D(u_\varepsilon) = 0 \quad \text{in } \varepsilon I_n$$

$$u_\varepsilon(x) = V_{\varepsilon,n} + \Theta_{\varepsilon,n}(x - x_{\varepsilon,n})$$

Lagrange multipliers of the two constraints

where : $\sigma(u_\varepsilon, P_\varepsilon) = 2 \underbrace{D(u_\varepsilon)}_{= \frac{1}{2}(Du_\varepsilon + Du_\varepsilon^T)} - P_\varepsilon \, \mathrm{Id}$

b is the buoyancy : $\begin{cases} b = 0 & \rightsquigarrow \text{colloidal} \quad \text{(parts 1.2 \& 1.3)} \\ b \neq 0 & \rightsquigarrow \text{sedimenting} \quad \text{(part 1.4)} \end{cases}$

1.2 Qualitative homogenization: effective viscosity $(b=0)$

Assumptions: - P is stationary and ergodic [locally would do]
- $\rho = \inf_{n \neq m} |x_n - x_m| > 0$ [can be relaxed, cf. Duerinckx]

theorem: There exists an effective viscosity $\bar{B}$ such that $(u_\varepsilon, p_\varepsilon) \to (\bar{u}, \bar{p})$ weakly in $H^1(U) \times L^2(U)$, solution of

$$\begin{cases} -\operatorname{div} 2\bar{B}\, D(\bar{u}) + \nabla \bar{p} = (1-\lambda) f & \text{in } U \\ \operatorname{div} \bar{u} = 0 & \text{in } U \\ \bar{u} = 0 & \text{on } \partial U \end{cases}$$

Formula for $\bar{B}$: For all trace-free symmetric matrices E,

$E \cdot \bar{B} E = \mathbb{E}\left[|D(\psi_E) + E|^2 \right]$, where (ψ_E, Σ_E) solves

$$\left.\begin{aligned} -\Delta \psi_E + \nabla \Sigma_E &= 0 \quad \text{in } \mathbb{R}^d \setminus \mathcal{I} \\ \operatorname{div} \psi_E &= 0 \quad \text{in } \mathbb{R}^d \setminus \mathcal{I} \\ D(\psi_E + E \cdot x) &= 0 \quad \text{in } \mathcal{I} \\ \int_{\partial I_n} \sigma(\psi_E + E\cdot x, \Sigma_E)\nu &= 0 \quad \forall n \\ \int_{\partial I_n} \Theta(x - x_n) \cdot \sigma(\psi_E + E \cdot x, \Sigma_E)\nu &= 0 \quad \forall n \end{aligned}\right\} \text{ corrector equation}$$

and $\nabla \psi_E$ and Σ_E are stationary fields

$\left[(\nabla \psi_E, \Sigma_E)(x, P + z) = (\nabla \psi_E, \Sigma_E)(x + z, P) \right]$

1.3 Quantitative homogenization: rates of convergence

Since $\mathbb{P}$ is random, $(u_\varepsilon, p_\varepsilon)$ are random fields
- oscillate (as in periodic setting)
- fluctuate (variance is not zero)

Quantitative results require quantitative assumptions on randomness
ex: functional inequalities [Talk 1]

Theorem [growth of correctors]: the correctors (φ_E, Σ_E) satisfies
└ with $\varphi_E(0)=0$

- $\mathbb{E}\left[|\nabla\varphi_E|^p + |\Sigma_E|^p\right]^{1/p} \lesssim_p 1$
- $\mathbb{E}\left[|\varphi_E(x)|^p\right]^{1/p} \lesssim_p \begin{cases} 1+\sqrt{|x|} & d=1 \\ \log(2+|x|)^{1/2} & d=2 \\ 1 & d>2 \end{cases}$

Remark: based on this, one can prove the quenched and annealed CZ estimates used in [Talk 1] for the sedimentation speed

Proof: Inspired by the analysis of $-\nabla\cdot a_\varepsilon\nabla$. Difference: two-scale expansion is more subtle (due to rigid body motions), and pressure has to be treated.

[NOT THE AIM OF THE TALK]

1.3 Quantitative homogenization: rates of convergence

This yields convergence rates for $(u_\varepsilon, p_\varepsilon) \longrightarrow (\bar{u}, \bar{p})$

<u>theorem</u>: Define $\bar{b} \in M_0^{sym}$ via $\bar{b} \cdot E = \frac{1}{d} \mathbb{E}\left[\sum_n \frac{\mathbb{1}_{I_n}}{|I_n|} \int_{\partial I_n} (x - x_n) \cdot \sigma(\psi_E + Ex, \Sigma_E)\nu\right]$

Then, if $f \in L^p(U)$ for some $p > d$, we have:

$$\left\| u_\varepsilon - \bar{u} - \varepsilon \sum_{E \in \mathcal{E}} \psi_E\left(\tfrac{\cdot}{\varepsilon}\right) \nabla_E \bar{u} \right\|_{H^1(U)}$$

$$+ \inf_{\kappa} \Big\| p_\varepsilon - \bar{p} \underbrace{- \bar{b} : D(\bar{u}) - \sum_{E \in \mathcal{E}} \left(\Sigma_E \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}\right)\left(\tfrac{\cdot}{\varepsilon}\right) \nabla_E \bar{u}}_{\to 0 \text{ weakly}} - \kappa \Big\|_{L^2(U \setminus \mathcal{I}_\varepsilon)} \leq C_d(p) \underbrace{\sqrt{\varepsilon}}_{\text{due to boundary layers}}$$

where $\mathcal{E}$ is a basis of M_0^{sym}.

<u>Proof</u>: + write equation satisfied by 2-scale expansion
+ energy estimate (on large-scale CZ)
+ bounds on (extended) correctors.

<u>Interpretation</u>: error between $(u_\varepsilon, p_\varepsilon)$ and its limit $(\bar{u}, \bar{p})$ is quantified in terms of the size ε of the particles (and not the volume fraction λ).

QUANTIFICATION OF EINSTEIN'S "ASSUMPTION 1"

1.4 The case of sedimenting particles $(b \neq 0)$

Assume
- weak correlations for $d > 2$
- weak correlations and hyperuniformity for $d \geq 1$

theorem
- $u_\varepsilon \to \bar{u}$ weakly in $H^1(U)$ as before
- pressure: needs to subtract the backflow

$$\left(p_\varepsilon - \frac{1}{\varepsilon}\lambda b \cdot x - \kappa_\varepsilon\right) \mathbb{1}_{U \setminus I_\varepsilon(U)} \to \bar{p} - \bar{\kappa} \text{ weak } L^2(U)$$

Reconstruction of oscillations: with (φ, Π) the sedimentation solution

- velocity: $$\left\| u_\varepsilon - \bar{u} - \varepsilon(1-\lambda)\varphi\left(\frac{\cdot}{\varepsilon}\right) - \varepsilon \sum_{E \in \mathcal{E}} \psi_E\left(\frac{\cdot}{\varepsilon}\right) \nabla_E \bar{u} \right\|_{H^1(U)} \to 0$$
- pressure: $$\inf_{\kappa} \Big\| p_\varepsilon - \frac{1}{\varepsilon}\lambda b \cdot e - \bar{p} - (1-\lambda)\Pi \mathbb{1}_{\mathbb{R}^d \setminus I}\left(\frac{\cdot}{\varepsilon}\right) - \bar{b} : D(\bar{u}) - \sum_{E \in \mathcal{E}} \Sigma_E \mathbb{1}_{\mathbb{R}^d \setminus I}\left(\frac{\cdot}{\varepsilon}\right) \nabla_E \bar{u} - \kappa \Big\|_{L^2(U)} \to 0$$

In particular: we have both
- sedimentation
- effective viscosity

$\Big\}$ λ not small

FIRST RESULT IN NON DIAGONAL REGIME

Part 2 Expansion of the effective viscosity at low density

[math version of Einstein and Batchelor-Green, cf. E. Gerarelli II]

2.1 Einstein's formula & Batchelor's correction

Recall $\mathcal{P}$ is the point process of particles (stationary and ergodic)

$$\lambda(\mathcal{P}) = \mathbb{E}[\,|\mathcal{P} \cap [0,1)^d|\,] = \text{intensity of the point process}$$

[= volume fraction of particles if $|I_n| = 1$]

Effective viscosity $\bar{B}$ defined by homogenization

What can one say on $\bar{B}$ when $\lambda(\mathcal{P}) \ll 1$

- Einstein : $\bar{B} \approx (1 + \frac{d+2}{2}\lambda)\,\mathrm{Id}$ (particles isolated)
- Batchelor's correction : $\bar{B} \approx (1 + \frac{d+2}{2}\lambda + \alpha\lambda^2)\,\mathrm{Id}$ (pair interactions)

Rephrase the question as :

Can we get a Taylor-expansion of $\bar{B}(\lambda)$ in terms of λ?

Remark : two distinct approximations

$$u_\varepsilon \xrightarrow[\text{homogenization}]{\varepsilon \to 0} \bar{u} \xrightarrow[\text{Taylor expansion of } \bar{B}]{\lambda \ll 1} \tilde{u}$$

Our results: optimal control of both errors

2.2 Main results

Einstein : consider particles as isolated

Batchelor : consider pair interactions between particles

Cluster expansion up to order n : consider interactions between n-uples of particles

Expect : $\bar{B} = \bar{B}^1 + \frac{1}{2!}\bar{B}^2 + \frac{1}{3!}\bar{B}^3 + \dots$

$\bar{B}^1$: isolated particles; $\bar{B}^2$: 2-particle interactions; $\bar{B}^3$: 3 particle interactions

For a Poisson process $\lambda_j(P) = \lambda_1(P)^j$

What scaling of $\bar{B}^1$, $\bar{B}^2$, $\bar{B}^3$ etc ?

- $\bar{B}^1$: particles isolated $\rightarrow$ linear : $\bar{B}^1 \sim \lambda$ (intensity)

- $\bar{B}^2$: 2-particle interactions. If interaction was short-range (it is not : cf. E. Guazzelli)

then $\bar{B}^2 \sim \lambda_2(P) = \sup\limits_{z_1, z_2} \mathbb{E}\left[\sum\limits_{n_1 \neq n_2} \mathbb{1}_{Q(z_1)}(x_{n_1}) \mathbb{1}_{Q(z_2)}(x_{n_2})\right]$

- $\bar{B}^3$: 3-particle interactions. If short-range interactions,

then $\bar{B}^3 \sim \lambda_3(P) = \sup\limits_{z_1, z_2, z_3} \mathbb{E}\left[\sum\limits_{\substack{n_1, n_2, n_3 \\ \text{distinct}}} \mathbb{1}_{Q(z_1)}(x_{n_1}) \mathbb{1}_{Q(z_2)}(x_{n_2}) \mathbb{1}_{Q(z_3)}(x_{n_3})\right]$

Justification of Einstein's formula

+ All the upcoming results hold for polydisperse suspensions of arbitrary shapes (for scalings).

+ All the results assume $\rho = \inf_{n \neq m} |x_n - x_m| > 0$

<u>Theorem</u> (Einstein's formula): Let P be ergodic. For all $0 < \alpha \leq 1$, there exists $C > 0$ and $\beta > 0$ such that if $\lambda_2(P) \leq \lambda_1(P)^{1+\alpha}$, then,

$$|\bar{B} - (Id + \bar{B}^1)| \leq C \lambda_1(P)^{1+\beta} \quad , \quad |\bar{B}^1| \sim \lambda_1(P)$$

and $\bar{B}^1$ satisfies the Einstein formula if inclusions are balls

<u>Rk</u>: + The result is optimal, assumption on ρ can be weakened

+ Also obtained independently by R. Höfer and D. Gérard-Varet

Justification of Batchelor's formula

Batchelor's contribution : propose a way to renormalize the formula for $\overline{B}^2$, which yields otherwise logarithmic divergence

<u>Proposition</u> (Batchelor) The second term of the expansion can be written as

$$E : \overline{B}^2 E = \int_{\mathbb{R}^d} \mathbb{E}\left[\int_{\partial I^0} \left(\psi_E^z - \psi_E^z(0) - D\psi_E^z(0)x\right) \cdot \sigma_E^{0,z} \nu\right] f_2(0,z)\,dz$$

$$+ \int_{\mathbb{R}^d} \mathbb{E}\left[\int_{\partial I^0} D\psi_E^z(0)x \cdot (\sigma_E^{0,z} - \sigma_E^0)\nu\right] f_2(0,z)\,dz$$

$$+ \mathbb{E}\left[\int_{\partial I^0} \sigma^0 \nu \otimes x\right] : \underbrace{\lim_{L \uparrow \infty}}_{\text{exists!}} L^{-d} \iint_{Q_L \times Q_L} \mathbb{E}\left[D\psi_{E,L}^z(y)\right] \underbrace{h_2}_{= f_2 - \lambda_1^2}(y,z)\,dy\,dz$$

two point density function

<u>Rk</u> : + Most general setting possible! (Issue : renormalization)

+ Formula can be simplified if more assumptions are made, and the limit made explicit if h_L has Dini decay.

+ scaling depends on f_2 and h_L :

<u>NEW</u>
- in general : $|\overline{B}_2| \lesssim \lambda_2(P)\,|\log \lambda_2(P)|$
- if P isotropic : $|\overline{B}_2| \lesssim \lambda_2(P)$

Beyond Einstein's formula

__Theorem__: Assume that $\mathbb{P}$ is α-mixing (with algebraic decay of the mixing coefficient at infinity). Then for all $j \in \mathbb{N}$, we have

$$|\bar{B}^j| \leq \lambda_j(\mathbb{P}) |\log \lambda_j(\mathbb{P})|^{j-1}.$$

there exists $0<s<1$ such that: given $n \geq 1$, if there exists $k(n) > n$ such that $\lambda_{k(n)}(\mathbb{P}) \leq \lambda_n(\mathbb{P})^{1+s}$, then

$$\left|\bar{B} - \left(\mathrm{Id} + \sum_{j=1}^{n} \frac{1}{j!} \bar{B}^j\right)\right| \lesssim_{n,k(n),s} \lambda_{n+1}(\mathbb{P}) \, |\log \lambda_{n+1}(\mathbb{P})|^{k(n)-1}.$$

__Rk__:
- the log correction is optimal (at least for $j=2$), related to lack of continuity of Helmholtz projection in L^∞.
- the quantitative α-mixing assumption is very mild (and essentially necessary for any explicit formula)
- the condition on intensities is also mild.

ONLY RESULT FOR $n \geq 2$ WITHOUT "STRUCTURAL" ASSUMPTIONS ON DILUTION OF $\mathbb{P}$.

Simplification under structural assumptions

$\mathcal{P} = \{x_i\}$ given with $\lambda \sim 1$, dilute $\mathcal{P}$ in two ways:

- random deletion : $\{b_i\}_{i \in \mathbb{N}}$ iid Bernoulli (law $= p\,\delta_1 + (1-p)\,\delta_0$)

$0 < p \leq 1$

$$\mathcal{P}^{(p)} = \{ x_i \mid b_i = 1 \}$$

$$\lambda_j(\mathcal{P}^{(p)}) = \lambda_j(\mathcal{P})\, p^j$$

$\rightarrow$ add independence to the system

- geometric dilation : scale $\ell \geq 1$

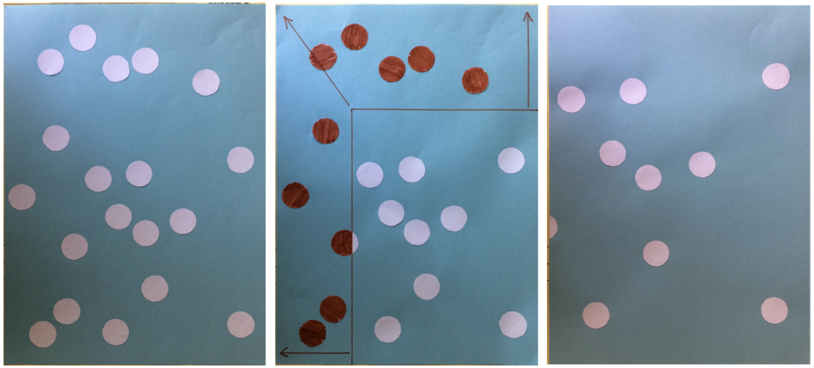
$$\mathcal{P}_\ell = \{ \ell x_i \}$$

$$\lambda_j(\mathcal{P}_\ell) \lesssim \ell^{-d j} \lambda_j(\mathcal{P})$$

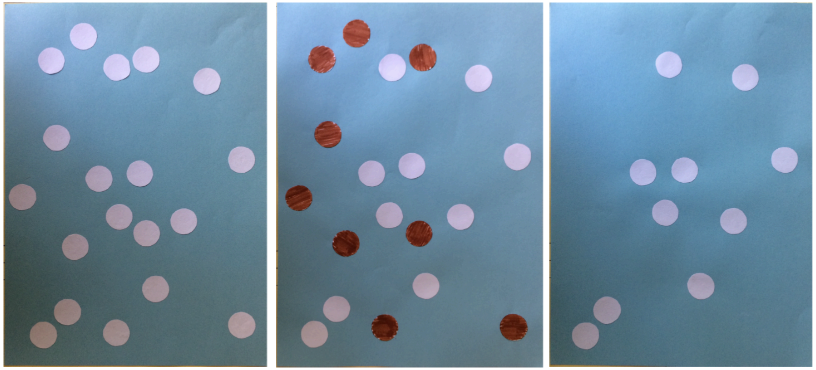
(upon modifying a bit definition of λ_j)

In these two specific settings, we can prove the summability of the cluster expansion !

Geometric dilation versus random deletion



Geometric dilation



Random deletion

Summability of the cluster expansion

<u>Theorem</u> Let $\mathbb{P}$ be ergodic. Let $\bar{B}_\ell^{(p)}$ be the effective viscosity of $\mathbb{P}^{(p)}$ and $\bar{B}_\ell$ the effective viscosity of $\mathbb{P}_\ell$. Then:

- random deletion : $\bar{B}^{(p)} = \sum_{j=0}^{\infty} p^j \frac{1}{j!} \bar{B}^j(\mathbb{P})$, $|\bar{B}^j(\mathbb{P})| \leq j!\, C^j$

  $\rightarrow$ summable for $p < 1/C$

- geometric dilation : $\bar{B}_\ell = \sum_{j=1}^{\infty} \frac{1}{j!} \bar{B}_\ell^j$, $|\bar{B}_\ell^j| \leq j!\, (C\ell^{-d})^j$

  $\rightarrow$ summable for $C\ell^{-d} < 1$

<u>Rk</u> : $\bar{B}_\ell^j$ can be Taylor-expanded wrt ℓ at all orders,

cf. J. Pertinand PhD thesis

2.3 Ingredients to the proof (1/2)

1 Cluster expansion of $\bar{B}_L$ (replace P by a periodized version P_L)

$$\bar{B}_L = \sum_{j=0}^{n} \frac{1}{j!} \bar{B}^j_L + \bar{R}^{n+1}_L$$

("explicit" formulas)

2 Control of the terms and of the remainder

2.1 Direct approach by Green's functions

→ good scaling in λ_j
→ divergence in $(\log L)^j$

$$\begin{cases} |\bar{B}^j_L| \lesssim_j \lambda_j(P) \log^{j-1} L \\ |\bar{R}^{j+1}_L| \lesssim_j \lambda_{j+1}(P) \log^j L \end{cases}$$

2.2 Subtle energy estimates

→ $|\bar{B}^j_L| \le j!\, C^j$ (subtle combinatorics!)

→ probabilistic argument

$\bar{B}^j_L(P^\rho) = \rho^j \bar{B}^j_L(P) \Rightarrow \bar{B}^j_L \xrightarrow[L\uparrow\infty]{} \bar{B}^j$ (convergence of approximation)

→ summability of cluster expansion

for geometric dilation: involved to make scaling λ^{-dj} appear

→ via elliptic regularity

2.3 Ingredients to the proof (2/2)

3 Quantitative homogenization

$\to$ α-mixing $\Rightarrow$ $|\bar{B}_L - \bar{B}| \lesssim L^{-\beta}$ [à la Armstrong Smart]

approx by periodization

$\to$ probabilistic argument $\bar{B}_L^j(P^{(P)}) = p^j \bar{B}_L^j(P)$

$\Rightarrow$ $|\bar{B}_L^j - \bar{B}^j| \lesssim L^{-\beta 2^{-j}}$

4 Optimisation

Combine $|\bar{B}_L - \bar{B}| \lesssim L^{-\beta}$ and $|\bar{B}_L^j - \bar{B}^j| \lesssim L^{-\beta 2^{-j}}$

with $|\bar{B}_L^j| \lesssim_j \lambda_j(P) \log^{j-1} L$ and $|\bar{R}_L^{j+1}| \lesssim_j \lambda_{j+1}(P) \log^j L$

Hence $\dashv$ $|\bar{B}^j| \lesssim \lambda_j |\log \lambda_j|^{j-1}$

$\dashv$ $|\bar{B} - \sum_{j=0}^{n} \frac{1}{j!} \bar{B}^j| \le |\bar{B} - \bar{B}_L| + |\bar{B}_L - \sum_{j=0}^{n} \frac{1}{j!} \bar{B}_L^j|$

$+ |\sum_{j=0}^{n} \bar{B}_L^j - \bar{B}^j|$

... optimize L wrt λ_j ...

— Cluster expansion —

Let $\chi : P(\mathbb{N}) \to S$, $E \mapsto \chi^E$, $\quad P(\mathbb{N})$: parts of $\mathbb{N}$, E : subset of $\mathbb{N}$, S : a space (e.g. $\mathbb{R}$)

Example :

$$\begin{aligned}\chi^{\{1,2,3\}} &= \chi^{\emptyset} + \left[\chi^{\{1\}} - \chi^{\emptyset} + \chi^{\{2\}} - \chi^{\emptyset} + \chi^{\{3\}} - \chi^{\emptyset}\right] \\ &+ \Big[\left(\chi^{\{1,2\}} - \chi^{\{1\}} - \chi^{\{2\}} + \chi^{\emptyset}\right) + \left(\chi^{\{2,3\}} - \chi^{\{2\}} - \chi^{\{3\}} + \chi^{\emptyset}\right) \\ &\qquad + \left(\chi^{\{1,3\}} - \chi^{\{1\}} - \chi^{\{3\}} + \chi^{\emptyset}\right)\Big] \\ &+ \left[\chi^{\{1,2,3\}} - \chi^{\{2,3\}} - \chi^{\{1,3\}} - \chi^{\{2,1\}} + \chi^{\{1\}} + \chi^{\{2\}} + \chi^{\{3\}} - \chi^{\emptyset}\right]\end{aligned}$$

With the notation $\delta^{\{n\}}\chi^E := \chi^{E \cup \{n\}} - \chi^E$, $\quad \delta^{F \cup F'}\chi^E := \delta^F \delta^{F'} \chi^E$

$$\chi^{\{1,2,3\}} = \chi^{\emptyset} + \sum_n \delta^{\{n\}}\chi^{\emptyset} + \frac{1}{2}\sum_{n_1 \neq n_2} \delta^{\{n_1,n_2\}}\chi^{\emptyset} + \frac{1}{3!}\sum_{\substack{n_1,n_2,n_3 \\ \text{distinct}}} \delta^{\{n_1,n_2,n_3\}}\chi^{\emptyset}$$

If χ is such that $\chi^{\mathbb{N}} = \chi^E$ for some finite set E, then

$$\chi^{\mathbb{N}} = \chi^{\emptyset} + \sum_{j=1}^{\infty} \sum_{|F|=j} \delta^F \chi^{\emptyset}$$ is a finite sum

Cluster expansion applied to $\bar{B}_L$

Proposition: the periodic approximation $\bar{B}_L$ of $\bar{B}$ satisfies

$$\bar{B}^{(p)}_{L,\ell} = \mathrm{Id} + \sum_{j=1}^{k} \frac{p^j}{j!} \bar{B}^j_{L,\ell} + p^{k+1} R^{(p),k+1}_{L,\ell} \,,$$

where

$$E : \bar{B}^j_{L,\ell} E = j! \sum_{|F|=j} \mathbb{E}\left[\fint_{Q_L} \delta^F \left(|D(\varphi^{\phi}_{E,L,\ell}) + |E|^2 \right) \right],$$

and

$$E : R^{(p),k+1}_{L,\ell} = \frac{1}{2} L^{-d} \sum_{|F|=k+1} \sum_{n \in F} \mathbb{E}\left[\int_{\partial I_{n,L,\ell}} \delta^{F \setminus \{n\}} \varphi^{f}_{E,L,\ell} \cdot \sigma^{E^{(p)} \cup F}_{E,L,\ell} \nu \right].$$

Rk: better to apply cluster expansion to $\bar{B}_L$ than to a solution of the Stokes problem because, as an average,

$$E : \bar{B}_L E = \mathbb{E}\left[\fint_{Q_L} |E + D(\varphi^L_E)|^2 \right]$$

is expected to be more regular than expansion of $D(\varphi_E)$ [on top of that $S = \mathbb{R}$ versus $L^2(\Omega)$]

Scalings by Green's functions

Thinking of Stokeslets, we essentially have

Have : $\sum_{|F|=j} \mathbb{E}\left[\int_{Q_L} \delta^F \left(|D(\psi^\phi_{E,L,e})| + |E|^2 \right) \right]$

Work $\lesssim \sum_{|F|=j} L^{-d}\, \mathbb{E}\left[\prod_{n_k \neq n_k'} (1+|x_{n_k} - x_{n_k'}|)^{-d} \right]$

$\lesssim \lambda_j \left(\int_{Q_L} (1+|x|)^{-d} \right)^{j-1} \sim \lambda_j (\log L)^{j-1}$

[Recall $\lambda_j = \sup_{z_1,\dots,z_j} \mathbb{E}\left[\sum_{\substack{n_1 \dots n_j \\ \text{distinct}}} \mathbb{1}_{Q(z_1)}(x_{n_1}) \cdots \mathbb{1}_{Q(z_j)}(x_{n_j}) \right]$

$= \sup_{z_1,\dots,z_j} \int_{Q(z_1)\times\dots\times Q(z_j)} \rho_j$

and $\mathbb{E}\left[\sum_{\substack{n_1 \dots n_j \\ \text{distinct}}} \zeta(x_{n_1},\dots,x_{n_j}) \right] = \int_{(\mathbb{R}^d)^j} \zeta\, \rho_j, \quad \zeta \in C_c^\infty$]

Subtle energy estimates

Proposition: For all $H \subset \mathbb{N}$, all $L, \ell, j, k \geq 0$ we have

$$S^H_{L,\ell}(k,j) \leq (C\ell^{-d})^{2(k+j)}, \quad T^H_{L,\ell}(k,j) \leq (C\ell^{-d})^{2(k+1+j)}$$

where
$$S^H_{L,\ell}(k,j) = \sum_{|G|=k} \int_{Q_L} \Big| \sum_{\substack{|F|=j \\ F\cap G=\emptyset}} \nabla \delta^{F\cup G} \varphi^H_{L,\ell} \Big|^2$$

$$T^H_{L,\ell}(k,j) = L^{-d} \sum_{|G|=k} \sum_{n\notin G\cup H} \int_{I_{n,L,\ell}} \Big| \sum_{\substack{|F|=j \\ F\cap(G\cup\{n\})=\emptyset}} \nabla \delta^{F\cup G} \varphi^H_{L,\ell} \Big|^2$$

Rk:
- subtle "interpolating" $\ell^1 - \ell^2$ estimates
- proof:
  - "two-scale" coupled induction argument on S and T
  - to take one sum out of the square: write differently the equation [play between perturbed / unperturbed operators]
  - scaling in ℓ is hard. Rough idea

$$-\Delta u = 0 \text{ in } B_\ell(0) \Rightarrow \int_{B(0)} |\nabla u|^2 \leq \ell^{-d} \int_{B_\ell(0)} |\nabla u|^2$$

Take-home messages

→ Einstein formula :

- distinguish separation of scales (⇝ homogenization)
- dilute regime (⇝ $\lambda \ll 1$) (⇝ cluster expansion)

→ Higher-order corrections :

- relevant parameters : $\lambda_j(\mathbb{P})$
- most natural and powerful approach : cluster expansion
  - → scaling via Green's functions
  - → implicit renormalization via ℓ^1-ℓ^2 estimate
  - → combination via quantitative homogenization [not needed for first two orders]

→ Variety of tools :

- PDE analysis
- Combinatorics
- Probability
- Calculus of variations

tools from stochastic homogenization