Modelling the interactions between a cloud of particles and a viscous fluid

Collective behavior of particles in fluids
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Context

Issue :
To derive "macroscopic" models for the interactions between a viscous fluid and many "small" particles

Examples :

- (Navier)-Stokes+Vlasov equations :
  Hamdache, '98
  Boudin *et al '09*,

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + 6\pi \, \mathrm{div}_v[(u-v)f] = 0 \\ \rho_f(\partial_t u + u \cdot \nabla_x u) - \Delta_x u + \nabla_x p = -6\pi \displaystyle\int_{\mathbb{R}^3} f(u-v)\,\mathrm{d}v \\ \mathrm{div}_x u = 0 \end{cases}$$

- Sedimentation model :
  R. Höfer & R. Schubert '20

$$\begin{cases} -\mathrm{div}\,([2+5\phi\rho]D(u) + p\mathbb{I}) = \rho g \\ \mathrm{div} u = 0 \\ \partial_t \rho + u \cdot \nabla \rho = 0 \end{cases}$$

Main principle

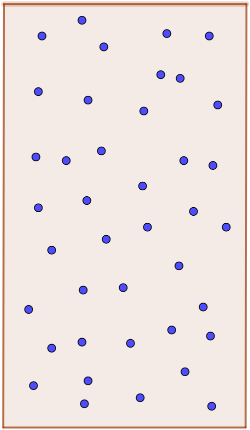


FIGURE – Flow with particles

- Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = \mathrm{div}\Sigma(u,p) \\ \mathrm{div} u = 0 \end{cases} \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} B_i(t)$$

$$\begin{cases} u_{|B_i} = V_i + \omega_i \times (x - x_i) \\ u_{|\infty} = Ax \end{cases}$$

- Particle dynamics $B_i = B(x_i, R_N)$

$$\dot{x}_i = V_i$$

$$m_N \ddot{x}_i = -\int_{\partial B_i} \Sigma(u,p) n \mathrm{d}\sigma + F$$

$$J_N \dot{\omega}_i = -\int_{\partial B_i} (x - x_i) \times \Sigma(u,p) n \mathrm{d}\sigma$$

$$\Sigma(u,p) = \mu(\nabla u + \nabla^\top u) - p\mathbb{I}_3 = 2\mu D(u) - p\mathbb{I}_3.$$

Two crucial phenomena
1 - Stokes drag

Proposition. $\exists!$ classical solution $(U_R[\delta V] + V_\infty, P_R[\delta V])$ to :

$$\left\{ \begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \mathbb{R}^3 \setminus B(0,R) \\ \operatorname{div} u & = & 0, & \text{in } \mathbb{R}^3 \setminus B(0,R) \end{array} \right. \qquad \left\{ \begin{array}{rcll} u & = & V_\infty + \delta V, & \text{on } B(0,R) \\ u & = & V_\infty, & \text{at infinity} \end{array} \right.$$

Applications.

- Computation of forces and torques :

$$\int_{\partial B(0,R)} \Sigma(u,p) n \mathrm{d}\sigma = 6\pi R \delta V, \qquad \int_{\partial B(0,R)} y \times \Sigma(u,p) n \mathrm{d}\sigma = 0.$$

- Extended weak-formulation

$$2 \int_{\mathbb{R}^3} D(u) : D(w) = 6\pi R \delta V \cdot w(0) + O(R^2)$$

for arbitrary divergence free $w \in C_c^\infty(\mathbb{R}^3)$

Two crucial phenomena

2 - Volumic rigidity

Proposition. Given a trace-free $A \in \mathcal{M}_3(\mathbb{R})$, there exists a unique classical solution to

$$\left\{ \begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \mathbb{R}^3 \setminus B(0,R) \\ \text{div}\, u & = & 0, & \text{in } \mathbb{R}^3 \setminus B(0,R) \end{array} \right. \qquad \left\{ \begin{array}{rcll} u & = & V, & \text{on } B(0,R) \\ u & = & Ax, & \text{at infinity} \end{array} \right.$$

with :

$$\int_{\partial B(0,R)} \Sigma(u,p) n \mathrm{d}\sigma = 0.$$

Applications.

- Computation of strain/torque :

$$\int_{\partial B(0,R)} y \otimes \Sigma(u,p) n \mathrm{d}\sigma = \frac{20\pi}{3} R^3 D(A) + 8\pi R^3 S(A)$$

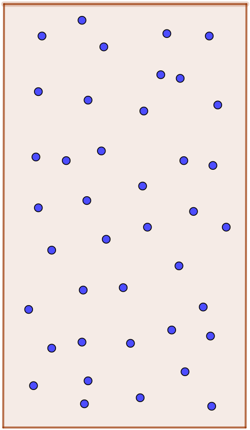
- Extended weak formulation

$$2 \int_{\mathbb{R}^3} D(u) : D(w) = \frac{20\pi}{3} R^3 D(A) : \nabla w(0) + 8\pi R^3 S(A) : \nabla w(0) + O(R^4)$$

for arbitrary divergence free $w \in C_c^\infty(\mathbb{R}^3)$

On the discrete problem

Reminder



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$$\Sigma(u,p) = \mu(\nabla u + \nabla^\top u) - p\mathbb{I}_3 = 2\mu D(u) - p\mathbb{I}_3.$$

References

Lagrangian approach

- Local-in-time existence and uniqueness of classical solutions
  [C. Grandmont & Y. Maday '98, M.D. Gunzburger, H.-C. Lee & G. A. Seregin '00]
- Existence and uniqueness of classical solutions up to contact between bodies [T. Takahashi '03 ; M. Tucsnak & T. Takahashi '04]

Eulerian approach

- Existence of weak solutions
  - up to contact between rigid bodies
    [B. Desjardins & M. Esteban 99' ; B. Desjardins & M. Esteban, '00]
  - Global-in-time (2D and 3D)
    [K.H. Hoffmann & V. Starovoitov '99 ; J. A. San Martin, V. Starovoitov & M. Tucsnak '02 ; E. Feireisl '03]
- Uniqueness of weak solutions [O. Glass & F. Sueur '13 ; V. Starovoitov '05]

Contact issue

- Blow-up of solutions [V. Starovoitov '03]
- Existence of finite-time contact
  [M.H. '07 ; M.H. & T. Takahashi '09'10'20 ; D. Gérard-Varet & M.H. '10'15 ; L. Sabbagh '19 ; S. Filippas & A. Tersenov '21]

Distance estimate

Simplified model

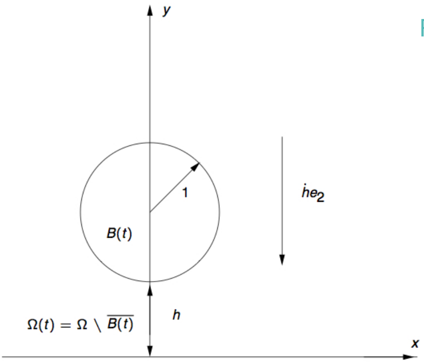

Figure 3. Symmetric configuration

Reduced system

$$\left.\begin{array}{rcl} \mu \Delta u_f - \nabla p_f &=& 0 \\ \mathrm{div} u_f &=& 0 \end{array}\right\} \quad \text{in } \Omega(t)$$

$$\begin{array}{rcl} u_f &=& 0 \quad \text{on } \partial\Omega \\ u_f &=& \dot{h} e_2 \quad \text{on } \partial B(t) \end{array}$$

$$m\ddot{h} = -\int_{\partial B(t)} \Sigma n d\sigma \cdot e_2$$

References

- Lubrication approximation
  [R.G. Cox '67, H.G. Leal '07]
- Explicit computations
  [M. Cooley & M. O'Neill '69]

Analysis of the drag

$$\Longrightarrow m\ddot{h} = -\mu K \frac{\dot{h}}{h^{\frac{3}{2}}} \, .$$

Alternative Computation of the drag

[M.H., A. Lozinski, M. Szopos, '11 ; D. Gérard-Varet & M.H. '12]

Lemma. Let $h > 0$ and denote $B^h = B((0, 1+h), 1)$.
The "unique" pair $(w, q) \in H^2(\mathbb{R}^2_+ \setminus \overline{B^h}) \times H^1(\mathbb{R}^2_+ \setminus \overline{B^h})$ solution to :

$$\left\{ \begin{array}{rcl} \mu\Delta w - \nabla q & = & 0 \\ \nabla \cdot w & = & 0 \end{array} \right. \text{in } \Omega \setminus \overline{B^h}, \quad \left\{ \begin{array}{rcl} w_{|\partial\Omega} & = & 0 \\ w_{|_{\partial B^h}} & = & e_2 \end{array} \right.$$

is characterised by :

$$E[w] = \text{argmin} \left\{ \int_{\mathbb{R}^2_+} |D(u)|^2, \; u \in C^\infty_c(\mathbb{R}^2_+) \text{ with } div\, u = 0 \text{ and } u|_{B^h} = e_2 \right\}.$$

Moreover :

$$\int_{\partial B^h} \Sigma(w, q) n d\sigma \cdot e_2 = 2\mu \int_{\Omega \setminus \overline{B^h}} |D(w)|^2.$$

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Moreover :

$$\int_{\partial B^h} \Sigma(w, q) n \mathrm{d}\sigma \cdot e_2 = 2\mu \int_{\Omega \setminus \overline{B^h}} |D(w)|^2.$$

Analysis of the drag

[D.Gérard-Varet & M.H. '12]

Application

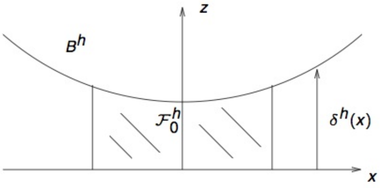


FIGURE – Geometry of the gap

Any $u \in Y_h$ reads $u = \nabla^\perp \phi$ with :

$$(BC_\phi) \quad \begin{cases} \phi(x, \delta_h(x)) & = & x + C, \\ \phi(x, 0) & = & 0 \end{cases} \qquad \forall\, x$$

Energy to minimize :

$$\mathcal{E}_h := \mu \int_{\mathcal{F}^h} |\nabla u(x,y)|^2 \mathrm{d}x\mathrm{d}y$$

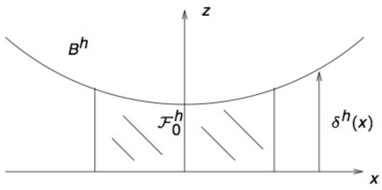
Reduced functional :

$$\tilde{Y}_h := \left\{ \phi \in C^\infty(\bar{\mathcal{F}}_0^h) \ \text{ s.t. } \ (BC_\phi) \right\} \qquad \tilde{\mathcal{E}}_h := \mu \int_{\mathcal{F}_0^h} |\partial_{22}\phi(x,y)|^2 \mathrm{d}x\mathrm{d}y$$

Remark : Extension to the 3D case [M.H. & T. Kelai '15]

Analysis of the drag
[D.Gérard-Varet & M.H. '12]

Application



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Energy to minimize :

$$\mathcal{E}_h \geq \mu \int_{\mathcal{F}_0^h} |\nabla^2 \phi(x,y)|^2 \mathrm{d}x\mathrm{d}y$$

Reduced functional :

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Remark : Extension to the 3D case [M.H. & T. Kelai '15]

Extensions

Other boundary conditions and 3D case

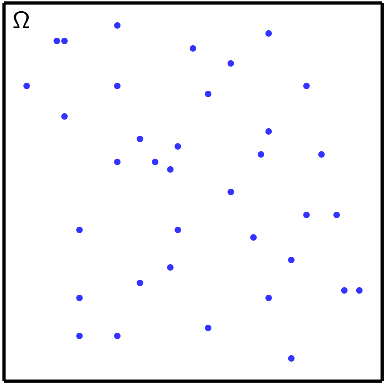
- Navier wall laws [D. Gérard-Varet, M.H., C. Wang '15]
- Tresca boundary conditions [M.H. & T. Takahashi '20]

Other fluid models

- Inviscid fluids
  [A. Munnier & J.G Houot '08, A. Munnier & K. Ramdani '15, M.H., D. Seck & L. Sokhna '18, D. Coutand '19,]
- Non-newtonian fluids [E. Feireisl, M.H., S. Nečasovà '08]

Multi-body problems [M.H. & T. Takahashi '10, L. Sabbagh '19]

On the asymptotics $N \to \infty$



A N-obstacle configuration

Geometrical data :

$$N \in \mathbb{N}, \quad |h_i - h_j| > 10R,$$

$$\mathcal{F} = \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} \overline{B(h_i, R)}$$

$$(h_1, \ldots, h_N) \in [\mathbb{R}^3]^N$$

Problem to be solved :

$$\begin{cases} -\Delta u + \nabla p = 0, & \text{in } \mathcal{F} \\ \operatorname{div} u = 0, & \text{in } \mathcal{F} \end{cases}$$

$$\begin{cases} u = v_i, & \text{on } \partial B_i \\ \lim_{|x| \to \infty} u(x) = 0. & \end{cases}$$

Still particles ($v_i = 0 \quad \forall\, i$)

Rubinstein '86,
Allaire '90,
Höfer-Velázquez '18,
Gérard-Varet '19,
Giunti-Höfer '19,
Höfer-Jansen '20,

"Moving" particles ($(v_i)_{i=1,\ldots,N}$ given)

H. Brinkman '47,
J. Luke '89,
L. Desvillettes-F. Golse-V. Ricci '08,
M.H.-A.Mecherbet '20,
M.H. '18,
M.H.-A. Moussa-F. Sueur '19,
K. Carrapatoso-M.H. '20

Theorem [L. Desvillettes, F. Golse & V. Ricci '08]

Consider the case :

(H0) $N \to \infty$ with $R = 1/N$.

Assume further that :

(H1) $d_{min} = \min\{\mathrm{dist}(h_i, h_j), \mathrm{dist}(h_i, \partial\Omega)\}$ satisfies $d_{min} \geq 2/N^{\frac{1}{3}}$.

(H2) we have the uniform bound :

$$\frac{1}{N}\sum_{i=1}^{N} |v_i|^2 \leq C_0$$

(H3) the sequence of empirical measures converges :

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{h_i} \rightharpoonup \rho(x)\mathrm{d}x \in L^\infty \qquad \frac{1}{N}\sum_{i=1}^{N} v_i \delta_{h_i} \rightharpoonup j(x)\mathrm{d}x \in L^2 .$$

Then $u \rightharpoonup \bar{u}$ in $H^1_0(\Omega) - w$ sol. to :

$$\left\{ \begin{array}{rcl} -\Delta \bar{u} + \nabla \bar{p} & = & 6\pi(j - \rho \bar{u}) , \\ \mathrm{div} \bar{u} & = & 0 \end{array} \right. \quad \text{on } \Omega$$

Theorem [M.H. '18]

Consider that (H0), (H2) and (H3) hold true.

Assume further that :

(H'1) $d_{min} >> 1/N$

(H"1) we can choose $\lambda << \min(|d_{min}|^{1/3}, N^{-1/6})$ such that :

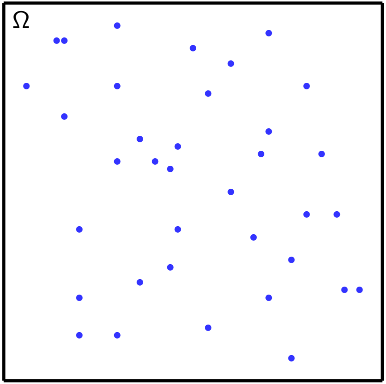
$$\frac{1}{N|\lambda|^3} \sup_{x \in \Omega} \sharp \{i, h_i \in B(x, \lambda)\} \text{ bounded .}$$

Then $u \rightharpoonup \bar{u}$ in $H^1_0(\Omega) - w$ solution to :

$$\left\{ \begin{array}{rcl} -\Delta \bar{u} + \nabla \bar{p} & = & 6\pi(j - \rho\bar{u}) \, , \\ \mathrm{div} \bar{u} & = & 0 \end{array} \right. \quad \text{on } \Omega$$

Stokes problem in a perforated domain

Einstein problem



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$$\mathcal{F} = \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} \overline{B(h_i, R)}$$

$$(h_1, \ldots, h_N) \in [\mathbb{R}^3]^N$$

Problem to be solved :

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$$\left\{ \begin{array}{l} u = V_i + \omega_i \times (x - h_i), \text{ on } \partial B_i \\ \int_{\partial B_i} (x - h_i) \times \Sigma(u,p) n \mathrm{d}\sigma = 0 \\ \int_{\partial B_i} \Sigma(u,p) n \mathrm{d}\sigma = 0. \end{array} \right.$$

$$\lim_{|x| \to \infty} |u(x) - Ax| = 0.$$

References

Einstein problem

Computation of first order correction in $\phi \sim NR^3$

Einstein '06
Sánchez-Palencia '85
Haines & Mazzucatto '12
Niethammer & Schubert '20,
Gérard-Varet '19
Gérard-Varet & Höfer '20

Keller & Rubenfeld '67
Almog & Brenner '03

M.H.-Wu '20

Computation of next orders (mostly second order)

Periodic case

Saito '50
Zuzovsky, Adler & Brenner '83
Nunan & Keller '84
Ammari, Garapon, Kang & Lee '12

Random case

Batchelor & Green '72, Hinch '77
Gérard-Varet & M.H. '20
Gérard-Varet '20
Gérard-Varet & A. Mecherbet '20
Duerinckx-Gloria '20, Duerinckx '20

Example of result

Theorem [M.H. & D. Wu '20]

Let $K \Subset \mathbb{R}^3$ and assume :

(A0) $B_i \subset K$ for $i = 1, \ldots, N$ and $d_{min} := \min_{i \neq j} |h_i - h_j| > 4R$

There exists $\varepsilon_0 > 0$ such that :

(A1) $R^3/d^3 < \varepsilon_0$

(A2) $\phi \in L^\infty(K)$ satisfies $\|\phi\|_{L^\infty(K)} < \varepsilon_0$

Denoting $u_c \in Ax + \dot{H}^1(\mathbb{R}^3)$ the unique weak solution to :

$$\begin{cases} -\mathrm{div}((2 + 5\phi)D(u_c)) + \nabla p = 0 \\ \mathrm{div} u = 0 \\ \lim_{|x| \to \infty} |u_c(x) - Ax| = 0 \end{cases}$$

and $\phi_d = \sum_{i=1}^N \mathbf{1}_{B_i}$, there holds :

$$\|u - u_c\|_{L^p_{loc}} \lesssim |A| [\|\phi_d - \phi\|_{\dot{H}^{-1}(\mathbb{R}^3)} + \left(\frac{R^3}{d^3}\right)^{1+} + \|\phi\|^2_{L^\infty(K)}] \quad \forall\, p < 3/2.$$

Key steps in the proof – Brinkman case
[M.H. '18]

- $u^N \rightharpoonup \bar{u}$ in $H^1_0(\Omega) - w$.

$$\bar{\mathcal{L}}[w] := \int_\Omega D(\bar{u}) : D(w) \quad (w \in C^\infty_c(\Omega) \text{ s.t. } div\ w = 0)$$
$$= \lim_{N\to\infty} \mathcal{L}^N[w]$$

- $(T^N_k)_{k\in\mathcal{K}^N}$ partition of Ω with cubes of width λ_N

$$\mathcal{L}^N[w] = \sum_{k\in\mathcal{K}^N} \int_{T^N_k} D(u^N) : D(\tilde{w}_k) + err^N$$

where :

$$\tilde{w}_k = \sum_{i | h_i \in T^N_k} U_R[w(h_i)](x - h_i)$$

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$$\mathcal{L}^N[w] = \sum_{k\in\mathcal{K}^N}\left(\sum_{i|h_i\in T^N_k} \frac{6\pi}{N} w(h_i)\cdot V_i - \int_{\partial T^N_k} \Sigma(\tilde{w}_k,\tilde{p}_k)\cdot u^N \mathrm{d}\sigma\right) + err^N$$

where :

$$\tilde{w}_k = \sum_{i|h_i\in T^N_k} U_R[w(h_i)](x-h_i)$$

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- $(T^N_k)_{k\in\mathcal{K}^N}$ partition of Ω with cubes of width λ_N

$$\mathcal{L}^N[w] = \sum_{k\in\mathcal{K}^N}\left(\frac{1}{N}\sum_{i|h_i\in T^N_k} 6\pi w(h_i)\cdot V_i - \frac{1}{N|T^N_k|}\int_{T^N_k} u^N \cdot \sum_{i|h_i\in T^N_k} 6\pi w(h_i)\right) + \widetilde{err}^N$$

where :

$$\tilde{w}_k = \sum_{i|h_i\in T^N_k} U_R[w(h_i)](x-h_i)$$

- Finally :

$$\bar{\mathcal{L}}[w] = 6\pi\left(\int_\Omega j\cdot w - \int_\Omega \rho\bar{u}\cdot w\right)$$

Thank you for your attention